Graph Pebbling - Lesson 1

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1 Introduction

Graph Pebbling is a network optimization model for the transportation of resources that are consumed in transit. Electricity, heat, or other energy may dissipate as it moves from one location to another, oil tankers may use up some of the oil it transports, or information may be lost as it travels through its medium. The central problem in this model asks whether discrete pebbles from one set of vertices can be moved to another while pebbles are lost in the process. A typical question asks how many pebbles are necessary to guarantee that, from any configuration of that many pebbles, one can move a pebble to any particular vertex.

Definition 1. *A configuration C on a graph G is a function $C : V(G) \to \mathbb{N}$. The value $C(v)$ signifies the number of pebbles at vertex v. We also write $C(S) = \sum_{v \in S} C(v)$ for a subset $S \subseteq V(G)$ of vertices.*

Definition 2. *For an edge $\{u, v\} \in E(G)$, if u has at least two pebbles on it, then a pebbling step from u to v removes two pebbles from u and places one pebble on v. That is, if C is the original configuration, then the resulting configuration C' has $C'(u) = C(u) - 2$, $C'(v) = C(v) + 1$, and $C'(x) = C(x)$ for all $x \in V(G) - \{u, v\}$*

Definition 3. *We say that a configuration C on G is r-solvable if it is possible from C to place a pebble on r via pebbling steps. It is r-unsolvable otherwise.*

Definition 4. *More generally, for a configuration D, we say that C is $D-$solvable if it is possible to perform pebbling steps from C to arrive at another configuration C' for which $C'(v) \geq D(v)$ for all $v \in V(G)$. It is D-unsolvable otherwise. We denote by $G(S)$ the directed subgraph of G induced by a set S of pebbling steps.*

Definition 5. *The size $|C|$ of a configuration C on a graph G is the total number of pebbles on G; i.e. $|C| = \sum_{v \in V(G)} C(v)$*

Definition 6. *For a graph G and a particular root vertex r, the rooted pebbling number $\pi(G, r)$ is defined to be the minimum number t so that every configuration C on G of size t is r-solvable.*

Definition 7. *The pebbling number $\pi(G)$ is defined to be the minimum number t so that every configuration C on G of size t is solvable.*