Chapter 11

Graph Pebbling

20:11 November 17, 2012 This is the cover page for Section 11.4: Graph Pebbling

Section 11.4
Graph Pebbling

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INTRODUCTION

Graph Pebbling is a network optimization model for the transportation of resources that are consumed in transit. Electricity, heat, or other energy may dissipate as it moves from one location to another, oil tankers may use up some of the oil it transports, or information may be lost as it travels through its medium. The central problem in this model asks whether discrete pebbles from one set of vertices can be moved to another while pebbles are lost in the process. A typical question asks how many pebbles are necessary to guarantee that, from any configuration of that many pebbles, one can move a pebble to any particular vertex. This section will describe this question and other variations of it, and will present the main results and applications in the theory. Good surveys of the subject can be found in [Hurl05, Hurl12, HurlGPP].

All graphs considered are simple and connected.

11.4.1 Solvability

Here we develop the notion of moving from one configuration of pebbles to another via pebbling steps.

NOTATION

The set of nonnegative integers is denoted by $\mathbb{N}$. We use $n = n(G)$ to denote the number of vertices of a graph G. When H is a subgraph of G, we write $G - H$ to denote the graph having vertices $V(G - H) = V(G)$ and edges $E(G - H) = E(G) - E(H)$. The eccentricity at vertex r, diameter, girth, connectivity, and domination number of a graph G are written $\mathsf{ecc}_G(r)$, $\mathsf{diam}(G)$, $\mathsf{gir}(G)$, $\kappa(G)$, and $\mathsf{dom}(G)$, respectively, while $\mathsf{dist}_G(u,v)$ denotes the distance between vertices u and v in G (we may write $\mathsf{ecc}(r)$ and $\mathsf{dist}(u,v)$ when G is understood). Also, the minimum degree of G is denoted $\delta(G)$ and we write lg for the base 2 logarithm.

DEFINITIONS

D1: A ***configuration*** C on a graph G is a function $C : V(G) \to \mathbb{N}$. The value $C(v)$ signifies the number of pebbles at vertex v. We also write $C(S) = \sum_{v \in S} C(v)$ for a subset $S \subseteq V(G)$ of vertices.

D2: For an edge $\{u,v\} \in E(G)$, if u has at least two pebbles on it, then a ***pebbling step from*** u ***to*** v removes two pebbles from u and places one pebble on v. That is, if C is the original configuration, then the resulting configuration C' has $C'(u) = C(u) - 2$, $C'(v) = C(v) + 1$, and $C'(x) = C(x)$ for all $x \in V(G) - \{u,v\}$.

D3: A pebbling step from u to v is r***-greedy*** if $\mathsf{dist}(v,r) < \mathsf{dist}(u,r)$. It is r***-semigreedy*** if $\mathsf{dist}(v,r) \le \mathsf{dist}(u,r)$.

D4: We say that a configuration C on G is r***-solvable*** if it is possible from C to place a pebble on r via pebbling steps. It is r***-unsolvable*** otherwise.

D5: More generally, for a configuration D, we say that C is D***-solvable*** if it is possible to perform pebbling steps from C to arrive at another configuration C' for which $C'(v) \ge D(v)$ for all $v \in V(G)$. It is D***-unsolvable*** otherwise. We denote by $G(S)$ the directed subgraph of G induced by a set S of pebbling steps.

D6: We say that a configuration C on G is k***-fold*** r***-solvable*** if it is possible from C to place k pebbles on r via pebbling steps.

NOTE: The k-fold r-solvability of C is the specific instance of D-solvability for which D has k pebbles on r and none elsewhere.

D7: The ***size*** $|C|$ of a configuration C on a graph G is the total number of pebbles on G; i.e. $|C| = \sum_{v \in V(G)} C(v)$.

D8: For a graph G and a particular ***root*** vertex r, the ***rooted pebbling number*** $\pi(G,r)$ is defined to be the minimum number t so that every configuration C on G of size t is r-solvable.

D9: A sequence of paths $\mathcal{P} = (P[1], \ldots, P[h])$ is a ***maximum*** r***-path partition*** of a rooted tree (T,r) if $\mathcal{P}$ forms a partition of $E(T)$, r is a leaf of $P[1]$, $T_i = \cup_{j=1}^{i} P[j]$ is a tree for all $1 \le i \le h$, and $P[i]$ is a maximum length path in $T - T_{i-1}$, among all such paths with one endpoint in T_{i-1}, for all $1 \le i \le h$.

D10: We define the function $f(T,r) = \sum_{i=1}^{h} 2^{l_i} - h + 1$, where $(l_1, \ldots, l_h)$ is the sequence of lengths $l_i = \mathsf{diam}(P[i])$ in a maximum r-path partition $\mathcal{P}$ of a rooted tree (T,r). Also, set $f_k(T,r) = f(T,r) + (k-1)2^{l_1}$.

D11: A ***thread*** in a graph G is a subpath of G whose vertices have degree two in G.

Figure 11.4.1: Two r-unsolvable configurations on the path P_7.

EXAMPLES

E1: Figure 11.4.1 shows two r-unsolvable configurations (of maximum size, right) on the path with 7 vertices.

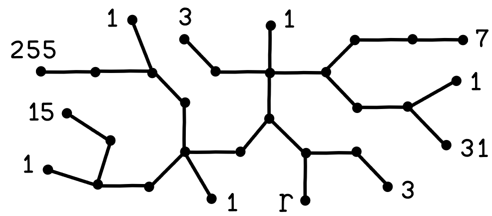

Figure 11.4.2: An r-unsolvable configuration on a tree.

E2: Figure 11.4.2 shows a maximum sized r-unsolvable configuration on a tree.

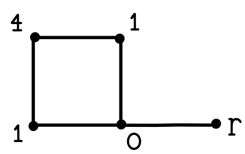

Figure 11.4.3: An r-solvable configuration.

E3: Figure 11.4.3 shows an r-solvable configuration on the 4-cycle with pendant edge.

FACTS

F1: If H is a connected, spanning subgraph of a graph G then $\pi(H, r) \geq \pi(G, r)$ every root vertex r.

F2: Every graph G on n vertices has rooted pebbling number $\pi(G, r) \geq n$ for every root vertex r.

F3: The complete graph K_n on n vertices has rooted pebbling number $\pi(K_n, r) = n$ for every root vertex r.

F4: Every graph G on n vertices has rooted pebbling number $\pi(G, r) \geq 2^{\mathsf{ecc}_G(r)}$ for every root vertex r.

F5: The path P_n on n vertices has rooted pebbling number $\pi(P_n, r) = 2^{n-1}$ when r is one of its leaves.

F6: Every graph G on n vertices has rooted pebbling number $\pi(G, r) \leq (n-1)(2^{\mathsf{ecc}_G(r)} - 1) + 1$ for every root vertex r.

F7: [Chun] If (T, r) is a tree with root r then $\pi(T, r) \geq f(T, r)$.

F8: **[No-Cycle Lemma]** [BCCMW] If a configuration C is D-solvable then there exists a D-solution S for which $G(S)$ is acyclic.

F9: **[Squishing Lemma]** [BCCMW] For every root vertex r of a graph G there is a maximum-sized r-unsolvable configuration such that, on each thread not containing r, all pebbles sit on one vertex or two adjacent vertices.

Weight Functions

Weight functions can be used to provide upper bounds on rooted pebbling numbers of graphs.

DEFINITIONS

D12: For a tree T rooted at a vertex r we define the ***parent*** of vertex $v \in V(T) - \{r\}$ to be the unique neighbor v^+ of v for which $\mathsf{dist}(v^+, r) = \mathsf{dist}(v, r) - 1$. We say also that v is a ***child*** of v^+.

D13: We say that a rooted subtree (T, r) of (G, r) is an r***-strategy*** if associated with it is a ***weight function*** $\mathsf{w} : V(G) \to \mathbb{N}$ having the properties that $\mathsf{w}(v) = 0$ for all $v \notin V(T)$ and $\mathsf{w}(v^+) \geq 2\mathsf{w}(v)$ for every vertex $v \neq r$. The r-strategy T is ***basic*** if equality holds for all such $v \in V(T)$.

D14: For a rooted graph (G, r) with r-strategy (T, w), we say that the ***weight of a vertex*** v is $\mathsf{w}(v)$ when $v \in T$ and 0 otherwise, and define the ***weight of a configuration*** C on G to be

$$\mathsf{w}(C) = \sum_{v \in V(G)} C(v)\mathsf{w}(v).$$

NOTATION: We denote by $\mathbf{J}_r$ the configuration on any rooted graph (G, r) having no pebbles on r and one pebble on every other vertex. Furthermore, let wt denote the weight function for any breadth-first search spanning tree (G, r), where $wt(r) = 1$; that is, $wt(v) = 2^{-\mathsf{dist}(v,r)}$ for all $v \in V(G)$.

D15: For a rooted graph (G, r) on n vertices, let $\mathcal{C}$ be the set of all r-unsolvable configurations on G, viewed as points in $\mathbb{N}^{n-1}$: each $C \in \mathcal{C}$ is identified with the coordinates $(C(v_2), \ldots, C(v_n))$, where $V(G) = \{r, v_2, \ldots, v_n\}$. The convex hull of $\mathcal{C}$ is called the r***-unsolvability polytope*** of G, denoted $\mathbf{U}(G, r)$. Define the r***-strategy polytope*** $\mathbf{T}(G, r)$ by the set of linear inequalities given by the Weight Function Lemma over all r-strategies.

FACTS

F10: If C is a configuration on the rooted graph (G, r) and C' is the configuration obtained from C after a pebbling step from u to v then, for any r-strategy (T, w) of (G, r) containing the edge $\{u, v\}$, we have $\mathsf{w}(C') \leq \mathsf{w}(C)$, with equality if and only if $\mathsf{w}(v) = 2\mathsf{w}(u)$ (when $\mathsf{w} = wt$ this means that the step is greedy).

F11: If C is an r-solvable configuration on G then $wt(C) \geq 1$.

F12: A configuration C on a path rooted at a leaf r is r-solvable if and only if $wt(C) \geq 1$.

F13: Every r-strategy is a conic combination of basic r-strategies; that is, for every r-strategy (T, w) of a rooted graph (G, r), there are basic r-strategies $(T_1, \mathsf{w}_1), \ldots, (T_h, \mathsf{w}_h)$ of (G, r) and nonnegative coefficients $\alpha_1, \ldots, \alpha_h$ so that, for all $v \in v(G)$, we have $\mathsf{w}(v) = \sum_{i=1}^{h} \mathsf{w}_i(v)$.

F14: **[Weight Function Lemma]** [Hurl10] Let (T, w) be an r-strategy of the rooted graph (G, r) and suppose that C is an r-unsolvable configuration on G. Then $\mathsf{w}(C) \leq \mathsf{w}(\mathbf{J}_r)$.

EXAMPLES

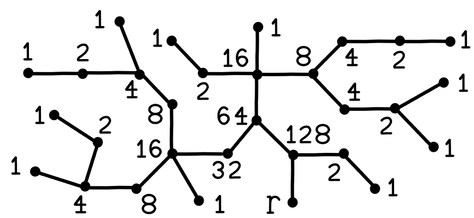

Figure 11.4.4: A rooted tree with a basic r-strategy.

E4: Figure 11.4.4 displays the upper bound for the rooted tree (T, r) of Figure 11.4.2 given by a basic r-strategy: $\pi(T, r) = 320$.

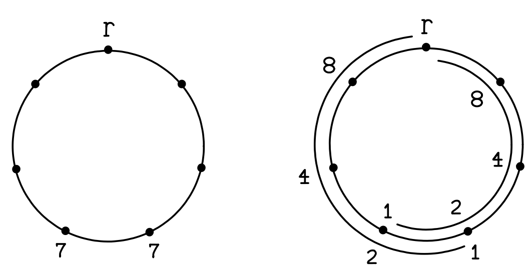

Figure 11.4.5: A rooted cycle with (left) its maximum r-unsolvable configuration and (right) two basic r-strategies.

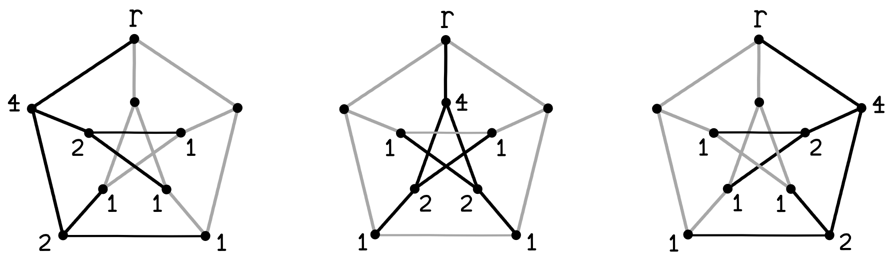

Figure 11.4.6: Three basic r-strategies of a rooted Petersen graph.

E5: Figure 11.4.5 displays the lower and upper bounds for the rooted cycle (C_7, r) given by an r-unsolvable configuration and basic r-strategy, respectively: $\pi(C_7, r) = 15$.

E6: Figure 11.4.6 displays the upper bound for a rooted Petersen graph (P, r) given by three basic r-strategies: $\pi(P, r) = 10$.

FACTS

F15: [Chun] If (T, r) is a tree with root r then $\pi(T, r) \geq f(T, r)$ (and hence $\pi(T, r) = f(T, r)$).

F16: Every rooted graph (G, r) has rooted pebbling number $\pi(G, r) \leq f(T, r)$ for any breadth-first search spanning tree T of G rooted at r.

F17: [CHHM] Let G be a graph in which each of its blocks is a clique, and suppose that T is a breadth-first search spanning tree of G rooted at r. Then $\pi(G, r) = \pi(T, r)$.

F18: [PaSnVo] For every root vertex r in the cycle C_n we have $\pi(C_{2k,r}) = 2^k$ for all $k \geq 2$ and $\pi(C_{2k+1}, r) = \lceil (2^{k+2} - 1)/3 \rceil$ for all $k \geq 1$.

F19: [Hurl10] For a polytope $\mathbf{P}$ of configurations on a rooted graph (G, r) define $z_{\mathbf{P}}(G, r) = \max_{C \in \mathbf{P}} |C|$ and $\pi_{\mathbf{P}}(G, r) = \lfloor z_{\mathbf{P}}(G, r) \rfloor + 1$. The Weight Function Lemma implies that $\mathbf{U}(G, r) \subseteq \mathbf{T}(G, r)$, and hence $\pi(G, r) = \pi_{\mathbf{U}}(G, r) \leq \pi_{\mathbf{T}}(G, r)$.

F20: [Uniform Covering Lemma] [Hurl10] Let G be a graph on n vertices. If some collection of r-strategies $\{(T_i, \mathsf{w}_i)\}_{i=1}^{k}$ has the property that there is a constant c such that, for every $v \in V(G) - \{r\}$, we have $\sum_{i=1}^{k} \mathsf{w}_i(v) = c$, then $\pi(G, r) = n$.

F21: [Chun] The d-dimensional cube Q^d has $\pi(Q^d, r) = n(Q^d) = 2^d$ for every root vertex r.

RESEARCH PROBLEMS

RP1: Is there a characterization for r-solvable configurations on trees rooted at r?

RP2: Is $\pi_{\mathbf{T}}(G, r) \leq 2\pi(G, r)$ for every rooted graph (G, r)?

RP3: Find larger classes of strategies than those arising from trees.

Complexity

Here we discuss questions such as how long it takes to decide if a particular configuration C on a graph G is D-solvable, or to calculate $\pi(G,r)$ for a rooted graph (G,r).

DEFINITIONS

D16: A graph G is a ***split*** graph if its vertices can be partitioned into a clique K and an independent set I.

D17: Let $\mathcal{H}$ be a hypergraph with vertices $V(\mathcal{H})$ and edges $E(\mathcal{H}) = \{e_1, \ldots, e_k\}$. For a given t define the ***pebbling graph*** $G = G_t(\mathcal{H})$ as follows. The vertices of G are given by $V(G) = V(\mathcal{H}) \cup E(\mathcal{H}) \cup \{u_1, \ldots, u_k\} \cup \{r, w_1, \ldots, w_t\}$. The edges of G include $ve_i \in E(G)$ for every $v \in e_i$, as well as the paths $w_t u_i e_i$ for every $i \leq k$ and the path $rw_1 \cdots w_t$.

D18: We define **SOLVABLE** to be the problem of deciding, for configurations C and D on a graph G, if C is D-solvable.

D19: We define **UPPERBOUND** to be the problem of deciding, for given k and configuration D on a graph G, if $\pi(G,D) \leq k$.

FACTS

F22: [MilCla] The configuration C is ***D-solvable on*** G if and only if there is a nonnegative integral solution to the system $\{C(u) + \sum_{v \in V}(x_{v,u} - x_{u,v}) \geq D(u)$ for all $u \in V\}$. Hence SOLVABLE $\in$ NP.

F23: [HurKie] Let $\mathcal{H}$ be a 4-uniform hypergraph on 2^{t+2} vertices with pebbling graph $G = G_t(\mathcal{H})$. Define the configuration C on G by $C(v) = 2$ for all $v \in V(\mathcal{H})$ and $C(v) = 0$ otherwise. Then C is r-solvable if and only if $\mathcal{H}$ has a perfect matching. Hence SOLVABLE is NP-complete.

F24: [CLST] When restricted to the class of diameter two graphs, SOLVABLE remains NP-complete.

F25: [CuDiLe] When restricted to the class of planar graphs, SOLVABLE remains NP-complete.

F26: [CuDiLe] When restricted to the class of diameter two planar graphs, SOLVABLE $\in$ P.

F27: [MilCla] UPPERBOUND is complete for the class of decision problems computable in polynomial time by a co-NP machine equipped with an NP-complete oracle (Π_2^{P}-complete).

F28: [BCCMW] If (T,r) is a rooted tree then $\pi(T,r)$ can be calculated in linear time. Moreover, for any configuration C, in linear time, we can find an r-solution or determine that none exist.

RESEARCH PROBLEMS

RP4: Is r-SOLVABLE $\in$ P when restricted to the class of cubes?

RP5: Is r-SOLVABLE $\in$ P when restricted to the class of split graphs?

11.4.2 Pebbling Numbers

We turn our attention now to configurations that solve every possible root.

DEFINITIONS

D20: We say that a configuration C on G is ***(k-fold) solvable*** if it is (k-fold) r-solvable for every vertex r.

D21: The ***pebbling number*** $\pi(G)$ is defined to be the minimum number t so that every configuration C on G of size t is solvable.

D22: For two graphs G_1 and G_2, define the ***cartesian product*** $G_1 \square G_2$ to be the graph with vertex set $V(G_1 \square G_2) = \{(v_1, v_2) | v_1 \in V(G_1), v_2 \in V(G_2)\}$ and edge set $E(G_1 \square G_2) = \{\{(v_1, v_2), (w_1, w_2)\} | (v_1 = w_1$ and $(v_2, w_2) \in E(G_2))$ or $(v_2 = w_2$ and $(v_1, w_1) \in E(G_1))\}$. We write $\Pi_{i=1}^k G_i$ to mean $G_1 \square \ldots \square G_k$ and set $G^k = \Pi_{i=1}^k G$.

D23: The ***support*** $s(C)$ of a configuration C on G is the set of vertices that have a pebble of C; i.e. $s(C) = \{v \in V(G) \mid C(v) > 0\}$. The size of the support is denoted $\sigma(C) = |s(C)|$.

D24: A graph G has the ***2-pebbling property*** if every configuration C of size at least $2\pi(G) - \sigma(C) + 1$ is 2-fold solvable. A ***Lemke*** graph is any graph that does not have the 2-pebbling property; the smallest of these is called ***the Lemke graph***.

D25: For $n = 2k(+1)$, the ***sun*** S_n is the split graph with perfect matching joining $I = I_k$ to $K = K_k$ (and one extra leaf when n is odd).

D26: For $m \geq 2t + 1$ the ***Kneser*** graph $K(m, t)$ has as vertices all t-subsets of $\{1, 2, \ldots, m\}$ and edges between every pair of disjoint sets. For example $K_n = K(n, 1)$ and $P = K(5, 2)$.

EXAMPLES

E7: The complete graph K_n on n vertices has $\pi(K_n) = n$.

E8: The complete graph K_n has the 2-pebbling property because the maximum number of pebbles that can be placed on σ vertices without having two vertices with at least two pebbles or one vertex with at least four pebbles is $\sigma + 2$, which is stricly less than $2n - \sigma + 1$.

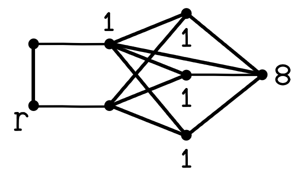

Figure 11.4.7: The Lemke graph with a 2-fold r-unsolvable configuration.

E9: The smallest graph without the 2-pebbling property is the Lemke graph L in Figure 11.4.7. We have $\pi(L) = 8$ and $|C| = 12 = 2(8) - 5 + 1$, but C cannot place two pebbles on r.

FACTS

F29: [Chun] The path P_n on n vertices has $\pi(P_n) = 2^{n-1}$. More generally, let r^* be a leaf of a longest path in a tree T. Then $\pi(T) = \pi(T, r^*)$.

F30: [Hurl99] The Petersen graph P has $\pi(P) = 10$.

F31: [PaSnVo] The cycle C_n on $n \geq 3$ vertices has $\pi(C_{2k}) = 2^k$ for all $k \geq 2$ and $\pi(C_{2k+1}) = \lceil (2^{k+2} - 1)/3 \rceil$ for all $k \geq 1$.

F32: [AlGuHu] If G is a diameter 3 split graph then $\pi(G)$ is given as follows. Let x be the number of cut vertices of G and, for a vertex r, define $\delta^*(G, r)$ to be the minimum degree of a vertex at maximum distance from r.

1. If $\mathrm{x} \geq 2$ then
$$\pi(G) = n + \mathrm{x} + 2.$$
2. If $\mathrm{x} = 1$ then
$$\pi(G) = \begin{cases} n + 5 - \delta^* & \text{if } r \text{ is a leaf with } \mathsf{ecc}(r) = 3 \text{ and } \delta^* = \delta^*(G, r) \leq 4; \\ n + 1 & \text{otherwise.} \end{cases}$$
3. If $\mathrm{x} = 0$ then
$$\pi(G) = \begin{cases} n + 4 - \delta^* & \text{if there is a cone vertex } r \text{ with } \deg(r) = 2, \\ & \quad \mathsf{ecc}(r) = 3, \text{ and } \delta^* = \delta^*(G, r) \leq 3; \\ n + 1 & \text{if no such } r \text{ exists and } G \text{ is Pereyra;} \\ n & \text{otherwise.} \end{cases}$$

F33: [Chun] The d-dimensional cube Q^d has $\pi(Q^d) = 2^d$. More generally, let $G = \Pi_{i=1}^k P_{l_i+1}$ be the cartesian product of k paths of lengths $l_i = diam(P_{l+i+1})$, with $l = \sum_{i=1}^k l_i$. Then $\pi(G) = 2^l$.

F34: [FosSne, Hers03] If G_1 and G_2 are both cycles then $\pi(G_1 \square G_2) \leq \pi(G_1)\pi(G_2)$.

F35: [Chun, FosSne] If G_1 and G_2 are both trees then $\pi(G_1 \square G_2) \leq \pi(G_1)\pi(G_2)$.

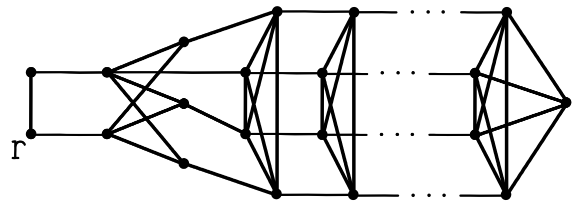

Figure 11.4.8: An infinite family of Lemke graphs.

F36: [Chun, Moew92, Hers08] If G is a tree, cycle, complete graph, or complete bipartite graph and H has the 2-pebbling property then $\pi(G\square H) \le \pi(G)\pi(H)$.

F37: [Wang] None of the graphs in Figure 11.4.8 has the 2-pebbling property.

F38: [GaoYin] If G is a bipartite graph with largest part size $s \ge 15$ and minimum degree at least $\lceil \frac{s+1}{2} \rceil$ then G is Class 0 and has the 2-pebbling property.

F39: [CHKT] If G_1 and G_2 are connected graphs on n vertices that satisfy $\delta(G_i) \ge k$ and $k \ge 2^{12n/k+15}$, then $\pi(G_1\square G_2) \le \pi(G_1)\pi(G_2)$.

F40: If G is a graph on n vertices with $\mathsf{diam}(G) = d$ then $e_\pi(G) \ge d/\lg n$.

RESEARCH PROBLEMS

RP6: Does every bipartite graph have the 2-pebbling property?

RP7: [**Graham's Conjecture**] Every pair of graphs G_1 and G_2 satisfy $\pi(G_1\square G_2) \le \pi(G_1)\pi(G_2)$.

RP8: Is $\pi(L^2) = 64$?

Diameter, Connectivity, and Class 0

In this subsection we study graphs having smallest possible pebbling number.

DEFINITIONS

D27: A graph G is of ***Class 0*** if $\pi(G) = n$.

D28: The k^{th} ***graph power*** $G^{(k)}$ of a graph G is formed from G by adding edges between every pair of vertices of distance at most k in G.

D29: The ***pyramid*** is any graph on 6 vertices isomorphic to the union of the 6-cycle (r, a, p, c, q, b) and the (inner) triangle (a, b, c). A ***near-pyramid*** is a pyramid minus one of the edges of its inner triangle (a, b, c).

D30: A graph G is ***pyramidal*** if it contains an induced (near-) pyramid, having 6-cycle C and inner (near-) triangle K, and can be drawn in the plane so that

1. the edges of K are drawn in the interior of the region bounded by C and
2. every other edge of G can be drawn inside the convex hull of exactly one of the sets $\{r, a, b\}$, $\{p, a, c\}$, $\{q, b, c\}$, or $\{a, b, c\}$.

D31: Define the ***pebbling exponent*** $e_\pi(G)$ of a graph G to be the minimum k such that $G^{(k)}$ is Class 0.

EXAMPLES

E10: Complete graphs, balanced complete bipartite graphs, cubes, and the Petersen graph are all Class 0.

E11: If u is a cut vertex of G then, for vertices r and v in different components of $G-u$, the configuration C with $C(r, u, v) = (0, 0, 3)$ and $C(w) = 1$ otherwise is r-unsolvable of size n.

E12: Let G be pyramidal with 6-cycle (r, a, p, c, q, b) and inner (near-) triangle (a, b, c). Then the configuration C with $C(r, a, p, c, q, b) = (0, 0, 3, 0, 3, 0)$ and $C(v) = 1$ otherwise is r-unsolvable of size n.

FACTS

F41: [ChaGod] Set $d = \mathsf{diam}(G)$. Then

1. $\pi(G) \le (n - d)(2^d - 1) + 1$,
2. $\pi(G) \le (n + \lfloor \frac{n-1}{d} \rfloor - 1)2^{d-1} - n + 2$, and
3. $\pi(G) \le 2^{d-1}(n + 2\mathsf{dom}(G)) - \mathsf{dom}(G) + 1$.

The inequalities in parts 1 and 2 are sharp, and the coefficient of 2 in part 3 can be reduced to 1 in the case of perfect domination.

F42: If G is Class 0 then $\kappa(G) \ge 2$.

F43: [BCFHHS] If G is Class 0 with n vertices and e edges then $e \ge \lfloor 3n/2 \rfloor$.

F44: [PaSnVo] If G is a graph with n vertices and e edges and $e \ge \binom{n-1}{2} + 2$ then G is Class 0. Because the complete graph K_{n-1} plus a pendant edge has a cut vertex, this result is tight.

F45: [PaSnVo] If $\mathsf{diam}(G) = 2$ then $\pi(G) \le n(G) + 1$.

F46: [ClHoHu] If $\mathsf{diam}(G) = 2$ and $\kappa(G) \ge 2$ then $\pi(G) = n + 1$ if and only if G is pyramidal.

F47: [ClHoHu] If $\mathsf{diam}(G) = 2$ and $\kappa(G) \ge 3$ then G is Class 0.

F48: [CHKT] There is a function $k(d) \leq 2^{2d+3}$ such that if G is a graph with $\mathsf{diam}(G) = d$ and $\kappa(G) \geq k(d)$ then G is of Class 0. Moreover, $k(d) \geq 2^d/d$.

F49: [CHKT] For any constant $c > 0$ there is an integer t_0 such that, for $t > t_0$, $s \geq c(t/\lg_2 t)^{1/2}$ and $m = 2t+s$, we have $\kappa(K(m,t)) \geq 2^{2d+3}$, where $d = \mathsf{diam}(K(m,t))$; hence $K(m,t)$ is Class 0.

F50: [CHKT] Let $G \in \mathcal{G}(n,p)$ be a random graph on n vertices with edge probability p and let $d = \mathsf{diam}(G)$. If $p \gg (n \lg_2 n)^{1/d}/n$ then $\Pr[\kappa(G) \geq 2^{2d+3}] \to 1$ as $n \to \infty$; hence $\Pr[G$ is Class $0] \to 1$ as $n \to \infty$.

F51: [AlGuHu] If G is a split graph with $\delta(G) \geq 3$ then G is Class 0.

F52: [Hurl10] The pebbling exponent of the cycle satisfies

$$\frac{n/2}{\lg n} \leq e_\pi(C_n) \leq \frac{n/2}{\lg n - \lg\lg n}.$$

F53: [PoStYe] If $\mathsf{diam}(G) = 3$ then $\pi(G) \leq \lfloor 3n/2 \rfloor + 2$, which is best possible, as shown by the sun S_n.

F54: [PoStYe] If $\mathsf{diam}(G) = 4$ then $\pi(G) \leq 3n/2 + c$, for some constant c.

F55: [Post] If $\mathsf{diam}(G) = d$ then $\pi(G) \leq (2^{\lceil d/2 \rceil} - 1)n/\lceil d/2 \rceil + c$, for some constant c.

F56: [CzyHur03] There is a constant c so that if $\delta(G_i) > cn/\lg n$ for $i \in \{1,2\}$ then $G_1 \square G_2$ is Class 0.

F57: [CzyHur06] Let $g_0(n)$ denote the maximum number g such that there exists a Class 0 graph G on at most n vertices with finite $\mathsf{gir}(G) \geq g$. Then for all $n \geq 3$ we have

$$\left\lfloor \sqrt{(\lg_2 n)/2 + 1/4} - 1/2 \right\rfloor \leq g_0(n) \leq 1 + 2\lg_2 n.$$

RESEARCH PROBLEMS

RP9: Find infinitely many Class 0 graphs with n vertices and at most $3n/2 + o(n)$ edges.

RP10: Decide if $K(m,t)$ is Class 0 for all $m = 2t + s$ with $s \in O((t/\lg_2 t)^{1/2})$.

RP11: Find the smallest $k(d)$ such that G is Class 0 for every diamter d graph G with $\kappa(G) \geq k(d)$.

Complexity

Calculating $\pi(G)$ and $\pi(G,r)$ are polynomially equivalent, but it may be possible to calculate $\pi(G)$ faster than by calculating $\pi(G,r)$ for every r.

DEFINITION

D32: We define **PebblingNumber** to be the problem of deciding if $\pi(G) \leq k$.

FACTS

F58: [HeHeHu13] Calculating $\pi(G)$ when G is a diamter two graph can be done in $O(n^4)$ time.

F59: [AlGuHu] Calculating $\pi(G)$ when G is a split graph can be done in $O(n^\beta)$ time, where $\omega \cong 2.376$ is the exponent of matrix multiplication and $\beta = 2\omega/(\omega+1) \cong 1.41$.

RESEARCH PROBLEMS

RP12: Is PebblingNumber $\in$ P when restricted to interval graphs of fixed diameter?

11.4.3 Optimal Pebbling

While pebbling can be thought of as a worst-case scenario — we give an adversary enough pebbles so that we can solve the graph no matter how he arranges them — optimal pebbling can be considered a best-case scenario — we place few pebbles carefully so as to solve the graph.

DEFINITIONS

D33: The ***optimal pebbling number*** $\pi^*(G)$ is the minimum number t for which there exists a solvable configuration of size t.

D34: Let C be a configuration on G and suppose that $\deg(v) = 2$ and $C(v) \geq 3$. A ***smoothing move*** at v removes two pebbles from v and adds one pebble to each of its neighbors. A ***smooth*** configuration has no smoothing move available; that is, C is smooth if $C(v) \leq 2$ whenever v has degree 2.

D35: For $S \subseteq V(G)$, the operation of ***collapsing*** S forms a new graph H in which S is replaced by a single vertex that is adjacent to all the neighbors of vertices of S that are in $V - S$. (Note that S need not be connected.)

EXAMPLES

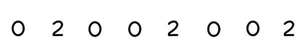

Figure 11.4.9: A minimum solvable configuration on the path P_8.

E13: Figure 11.4.9 displays the upper bound of $\pi^*(P_8) \leq \lceil 2(8)/3 \rceil$.

E14: The configuration with 2 pebbles on a single vertex can reach any other vertex of the complete graph, and so $\pi^*(K_n) = 2$ for all n.

FACTS

F60: Every graph G satisfies $\pi^*(G) \le 2\mathsf{dom}(G)$.

F61: **[Smoothing Lemma]** [BCCMW] If G has at least 3 vertices then G has a smooth minimum solvable configuration with no pebbles on leaves.

F62: **[Collapsing Lemma]** [BCCMW] If H is obtained from G by collapsing sets of vertices then $\pi^*(G) \ge \pi^*(H)$.

F63: [BCCMW] Every graph G satisfies $\pi^*(G) \le \lceil 2n/3 \rceil$, with equality for paths and cycles.

F64: [Moew98] The d-cube has $(4/3)^d \le \pi^*(Q^d) \le (4/3)^{d+O(\lg d)}$.

F65: [FuShi00, HeHeHu11] For all graphs G and H we have $\pi^*(G \square H) \le \pi^*(G)\pi^*(H)$.

F66: [BCCMW] If G has n vertices and $\delta(G) = k$, then $\pi^*(G) \le \frac{4n}{k+1}$.

F67: [BCCMW] For all $t \ge 1$, $k = 3t$ and $n \ge k+3$, there is a graph G with n vertices, $\delta(G) = k$ and $\pi^*(G) \ge (2.4 - \frac{24}{5k+15} - o(1))\frac{n}{k+1}$.

Complexity

DEFINITION

D36: We define **OptimalPebblingNumber** to be the problem of deciding if $\pi^*(G) \le k$.

FACTS

F68: [MilCla] The problem OptimalPebblingNumber is NP-complete.

RESEARCH PROBLEMS

RP13: Is there a graph G with $\pi^*(G) \ge 3n(G)/(\delta(G)+1)$?

RP14: Does $\delta(G) \ge 3$ imply that $\pi^*(G) \le \lceil n(G)/2 \rceil$?

11.4.4 Thresholds

The probabilistic model of pebbling studies the typical case; that is, small configurations are usually unsolvable and large configurations are usually solvable — at roughly how many pebbles is the transition?

We assume that all sequences $\mathcal{G} = (G_1, \ldots, G_k, \ldots)$ of graphs considered have an increasing number of vertices $n = n_k = n(G_k)$.

NOTATION

The sequences of complete graphs, stars, paths, cycles, and cubes are denoted $\mathcal{K}$, $\mathcal{S}$, $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{Q}$, respectively. The sequence of graph products is written $\mathcal{G}\square\mathcal{H} = (G_1\square H_1, \ldots, G_k\square H_k, \ldots)$, with $\mathcal{G}^2 = \mathcal{G}\square\mathcal{G}$. For sets of functions A and B on the integers we write $A \lesssim B$ to mean that $a \in O(b)$ for every $a \in A, b \in B$.

DEFINITION

D37: Let $C_k : [n] \to \mathbb{N}$ denote a configuration on $V(G_k)$ and, for a function $h : \mathbb{N} \to \mathbb{N}$ and fixed $n = n_k$, define the uniform probability space $X_{n,h}$ of all configurations C_k of size $h = h(n)$. Denote by P_n^+ the probability that C_n is solvable on G_k and let $t : \mathbb{N} \to \mathbb{N}$ be any function. We say that t is a ***pebbling threshold*** for $\mathcal{G}$, and write $\tau(\mathcal{G}) = \Theta(t)$, if $P_n^+ \to 0$ whenever $h(n) \ll t(n)$ and $P_n^+ \to 1$ whenever $h(n) \gg t(n)$.

EXAMPLE

E15: Solvability on K_k is equivalent to the labelled version of Feller's Birthday Problem (also hashing collisions in computer science). Thus $\tau(\mathcal{K}) = \Theta(\sqrt{n})$.

FACTS

F69: [BBCH] Every graph sequence $\mathcal{G}$ has nonempty threshold $\tau(\mathcal{G})$.

F70: [CEHK] Every graph sequence $\mathcal{G}$ satisfies $\tau(\mathcal{K}) \lesssim \tau(\mathcal{G}) \lesssim \tau(\mathcal{P})$.

F71: [CEHK, BBCH, GJSW, CzyHur08] For every constant $c > 1$, we have $\tau(\mathcal{P}) \subseteq \Omega\left(n2^{\sqrt{\lg n}/c}\right) \cap O\left(n2^{c\sqrt{\lg n}}\right)$.

F72: [BekHur] The squares of cliques has threshold $\tau(\mathcal{K}^2) = \Theta(\sqrt{n})$.

F73: [Alon, CzyWag] For all $\epsilon > 0$ the sequence of cubes has threshold $\tau(\mathcal{Q}) \in \Omega(n^{1-\epsilon}) \cap O(n/(\lg\lg n)^{1-\epsilon})$.

F74: If $\mathcal{G}$ is a sequence of graphs of bounded diameter then $\tau(\mathcal{G}) \subseteq O(n)$.

F75: [CzyHur08] Let t_1 and t_2 be functions satisfying $\tau(\mathcal{K}) \lesssim t_1 \ll t_2 \lesssim \Theta(n)$. Then there is some graph sequence $\mathcal{G}$ such that $t_1 \lesssim \tau(\mathcal{G}) \lesssim t_2$.

F76: [BjoHol] There exist graph sequences $\mathcal{G} = (G_1, \ldots, G_k, \ldots)$ and $\mathcal{H} = (H_1, \ldots, H_k, \ldots)$ such that $\pi(G_k) < \pi(H_k)$ for all k but $\tau(\mathcal{H}) \lesssim \tau(\mathcal{G})$.

F77: [CzyHur06] Suppose that $t \in \tau(\mathcal{P})$ and $s \in \tau(\mathcal{P}^2)$. Then $s(n) \in O\left(t\left(\sqrt{n}\right)^2\right)$.

F78: [CzyHur03] Define $\mathbf{G}(n, \delta)$ to be the set of all connected graphs on n vertices having minimum degree at least $\delta = \delta(n)$. Let $\mathcal{G}_\delta = \{G_1, \ldots, G_k, \ldots\}$ denote any sequence of graphs with each $G_k \in \mathbf{G}(k, \delta)$. For every function $n^{1/2} \ll \delta = \delta(n) \leq n-1$, $\tau(\mathcal{G}_\delta) \subseteq O(n^{3/2}/\delta)$. In particular, if in addition $\delta \in \Omega(n)$ then $\tau(\mathcal{G}_\delta) = \Theta(n^{1/2})$.

REMARKS

R1: Note the need to rescale threshold functions of products of graph sequences in terms of the new number of vertices $n(G_k \square H_k) = n(G_k) \square n(H_k)$; for example, in Fact 72 we have $\sqrt{n^2} = \sqrt{n}^2$.

RESEARCH PROBLEMS

RP15: Determine $\tau(\mathcal{P})$.

RP16: Determine $\tau(\mathcal{Q})$.

RP17: Extend Fact 75 to the range $\Omega(n) \cap \tau(\mathcal{P})$.

RP18: Suppose that $\mathcal{G}$ is any graph sequence, $t \in \tau(\mathcal{G})$ and $s \in \tau(\mathcal{G}^2)$. Is it true that $s(n) \in O\left(t\left(\sqrt{n}\right)^2\right)$?

11.4.5 Other Variations

Here we present a few variations on the pebbling theme and a taste of the main results for each.

NOTATION

We write kI_v for the configuration with k pebbles on v and 0 elsewhere, and J for the configuration with 1 pebble on each vertex. Also $k\mathscr{I}$ denotes the set of all such kI_v, and $\mathscr{C}_t$ is the set of all configurations of size t. Denote by $\mathscr{M}(G)$ the set of all configurations corresponding to dominating sets in a graph G; that is, the configuration corresponding to a dominating set has one pebble on each of its vertices and none elsewhere. Next, consider the set of all induced paths on $d+1$ vertices in G and write $\mathscr{P}_d^+(G)$ for those configurations on such paths with two pebbles on one leaf of the path, no pebbles on the other leaf, and one pebble on all other vertices of the path. Finally, for $\mathbf{d} = \langle d_1, \ldots, d_m \rangle$ let $P^{\mathbf{d}}$ denote the graph $P_{d_1+1} \square \cdots \square P_{d_m+1}$.

DEFINITIONS

D38: Let G be a ***weighted graph*** with edge weights $\mathsf{w} : E(G) \to \mathbb{N}$. For an edge $\{u, v\} \in E(G)$, if u has at least $\mathsf{w}(uv)$ pebbles on it, then a ***weighted pebbling step from*** u ***to*** v removes $\mathsf{w}(uv)$ pebbles from u and places one pebble on v. The corresponding ***weighted pebbling number*** $\pi(G_{\mathsf{w}})$ is defined to be the minimum number t so that every configuration of size t solves any r via weighted pebbling steps.

D39: For a graph G and set of configurations $\mathscr{D}$ on G the ***(optimal) pebbling number*** $\pi(G,\mathscr{D})$ (resp. $\pi^*(G,\mathscr{D})$) is the minimum t for which every (resp. some) $C \in \mathscr{C}_t$ is D-solvable for every $D \in \mathscr{D}$. The k***-fold (optimal) pebbling number*** $\pi_k(G) = \pi(G, k\mathscr{I})$ (resp. $\pi^*_k(G) = \pi^*(G, k\mathscr{I})$).

D40: For a graph G the ***fractional (optimal) pebbling number*** is defined to be $\hat{\pi}(G) = \lim_{k\to\infty} \pi_k(G)/k$ (resp. $\hat{\pi}^*(G) = \lim_{k\to\infty} \pi^*_k(G)/k$).

D41: The ***cover pebbling number*** of a graph G is defined to be $\pi(G,J)$. A configuration D is ***positive*** if $D(v) > 0$ for every vertex v. For positive D on G we define the function $s(G,D) = \max_v \sum_u D(u)2^{\mathsf{dist}(u,v)}$.

D42: For a set of configurations $\mathscr{D}$ the configuration C is ***weakly*** $\mathscr{D}$***-solvable*** if C solves some $D \in \mathscr{D}$. The ***target pebbling number*** $\pi^-(G,\mathscr{D})$ is the minimum t for which every $C \in \mathscr{C}_t$ is weakly D-solvable.

D43: The ***domination target pebbling number*** of G is defined to be $\pi^-(G,\mathscr{M})$.

D44: For given d the ***distance pebbling number*** $\vec{\pi}_d(G)$ of a graph G is defined to be the minimum t such that, for every size t configuration, there is some pebble that can move to a vertex at distance d from where it started; in other words, $\vec{\pi}_d(G) = \pi^-(G,\mathscr{P}^+_d)$.

D45: A ***rubbling step*** on a graph G is either a pebbling step or a strict rubbling step. A ***strict rubbling step*** takes one pebble from each of two neighbors u and w of a vertex v and places one pebble on v. The ***(optimal) rubbling number*** of G, denoted $\rho(G)$ (resp. $\rho^*(G)$), is defined to be the minimum t so that every (resp. some) configuration of size t can solve any root vertex r via rubbling steps.

EXAMPLES

E16: For a positive configuration D on the complete graph we have $s = s(K_n,D) = 2|D| - \min D$. If $D(v) = \min D$ then the configuration that places $s-1$ pebbles on v and none elsewhere cannot solve D.

E17: For a vertex $v \in V(G)$, the configuration that places no pebbles on v and all its neighbors and one pebble on every other vertex cannot solve any dominating set of G.

E18: The pigeonhole principle implies that $\vec{\pi}_1(G) = n(G)+1$ for every graph G.

E19: Every configuration of two pebbles on K_n solves every vertex via rubbling steps.

FACTS

F79: [Chun] Given $\mathbf{d} = \langle d_1,\dots,d_m\rangle$ we represent the vertices of $P^{\mathbf{d}}$ by coordinates $\langle v_1,\dots,v_m\rangle$ with each $0 \le v_i \le d_i$ and denote by $\mathbf{e}_i = \langle 0,\dots,1,\dots,0\rangle$ the i^{th} standard basis vector. For any $\mathbf{w} = \langle w_1,\dots,w_m\rangle$ define the weight function $\mathsf{w}(\mathbf{uv}) = w_i$ when $|\mathbf{u}-\mathbf{v}| = \mathbf{e}_i$ and write $\mathbf{w}^{\mathbf{d}} = \prod_{i=1}^m w_i^{d_i}$. Then $\pi(P^{\mathbf{d}}_{\mathsf{w}}) = \mathbf{w}^{\mathbf{d}}$.

F80: [Chun] For every root r of a tree T we have $\pi_k(T,r) = f_k(T,r)$ for all k.

F81: [HeHeHu13] If G is a diameter two graph with n vertices and m edges then $\pi_k(G) \le \pi(G) + 4(k-1)$. Furthermore, from any configuration of size at least $\pi(G) + 4(k-1)$, k pebbles can be placed on any root vertex r in at most $6n + \min\{3t, m\}$ steps.

F82: [HeHeHu13, HMOZ] Every graph G satisfies $\hat{\pi}(G) = 2^{\mathsf{diam}(G)}$.

F83: [HeHeHu13] If G is a complete graph, cycle, tree, or has $\pi(G) = 2^{\mathsf{diam}(G)}$ then $\pi(G, \mathscr{C}_k) = \pi_k(G)$.

F84: [HeHeHu13] For all $d \ge 0$, $n \ge 1$, and $k \ge 1$ we have $\hat{\pi}^*(K_n) = 2n/(n+1)$, $\hat{\pi}^*(P_n) = (n+2)/3$, $\hat{\pi}^*(C_{2k}) = k2^{k+1}/3(2^k-1)$, $\hat{\pi}^*(C_{2k+1}) = (2k+1)(2^{k-1})/(3(2^{k-1})-1)$, $\hat{\pi}^*(Q^d) = (4/3)^d$, and $\hat{\pi}^*(P) = 5/2$.

F85: **[Stacking Theorem]** [Sjos] Every positive configuration D on a graph G has $\pi(G,D) = s(G,D)$.

F86: [GGTVWY] The complete r-partite graph $K = K_{s_1,\ldots,s_t}$ has domination target pebbling number $\pi^-(K,\mathscr{M}) = 3$ if every $s_i = 2$ and $\max_i s_i$.

F87: [GGTVWY] For the path P_n on n vertices has $\pi^-(P_n,\mathscr{M}) = 2(2^n - 2^{n \bmod 3})/7 + \left\lfloor \frac{n \bmod 3}{2} \right\rfloor$.

F88: [Knap] The distance pebbling number of the cycle is $\vec{\pi}_d(C_n) = (2^d-1)\lfloor n/d \rfloor + 2^{n \bmod d}$.

F89: [BelSie] If G has n vertices and diameter d then $\rho(G) \le (n-d+1)(2^{d-1}-1)$.

F90: [KatSie] If G has n vertices and diameter 2 then $\rho(G) \le \sqrt{2n-1} + 5$. Furthermore, for all n there is a diameter 2 graph G with $\rho(G) \ge \left\lfloor \sqrt{2n-1} \right\rfloor + 2$.

F91: [KatSie] If G has n vertices and diameter d then $\lceil (d+2)/2 \rceil \le \rho^*(G) \le \lceil (n+1)/2 \rceil$.

REMARK

R2: Note that the Stacking Theorem implies that the computational complexity of calculating pebbling numbers applies only to nonpositive target configurations.

RESEARCH PROBLEMS

RP19: Is it true that $\pi(G, \mathscr{C}_k) = \pi_k(G)$ for every graph G?

RP20: Is there a constant c such that $\pi^*(G) \le c\hat{\pi}^*(G)$ for every graph G?

11.4.6 Applications

Graph pebbling arose as a method to prove a conjecture of Erdős and Lemke in combinatorial number theory. It has since produced a more general result in combinatorial group theory and another in p-adic diophantine equations.

FACTS

F92: [Chun, EllHur] Fact 79 implies that, if $g_1, \ldots, g_n$ is a sequence of elements of an abelian group $\mathbf{G}$ of size n, then there is a nonempty subsequence $(g_k)_{k\in K}$ such that $\sum_{k\in K} a_k = 0_{\mathbf{G}}$ and $\sum_{k\in K} 1/|g_k| \le 1$, where $|g|$ denotes the order of the element g in $\mathbf{G}$ and $0_{\mathbf{G}}$ is the identity element in $\mathbf{G}$.

F93: [Knap] Write $n = 2^t m$, where m is odd. Define $d = 1$ for odd n ($t = 0$) and $d = t + 2$ for even n ($t > 0$). If $s \ge \vec{\pi}(C_n, d)$ then, for all integer coefficients $a_1, \ldots, a_s$, the additive form $F(\mathbf{x}) = \sum_{i=1}^{s} a_i x_i^n$ has a nontrivial (not all zero) solution to $F(\mathbf{x}) = 0$ in the 2-adic integers.

REMARK

R3: The pebbling steps studied here have a cost in the loss of pebbles. Various no-cost rules for pebbling steps have been studied for years and have found applications in a wide array of areas. One version, dubbed *black and white* pebbling, was applied to computational complexity theory in studying time-space tradeoffs, as well as to optimal register allocation for compilers. Connections have been made also to pursuit and evasion games and graph searching. Another (*black* pebbling) is used to reorder large sparse matrices to minimize in-core storage during an out-of-core Cholesky factorization scheme. A third version yields results in computational geometry in the rigidity of graphs, matroids, and other structures.

RESEARCH PROBLEM

RP21: Prove that Fact 92 holds for all groups $\mathbf{G}$.

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