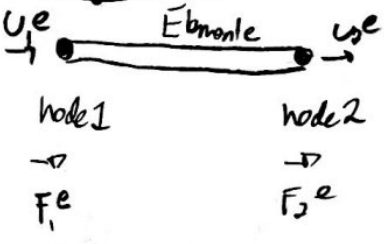
Lecture 1

ES 128 - Finite Element Analysis

1/28/2019

Single bar



Governing Eqns. (Linear Elastic Materials)

1) Equilibrium: $F_1^e + F_2^e = 0$

2) Constitutive Relation: $\sigma_e = E\,\varepsilon_e$

$\hookrightarrow \varepsilon_e = \dfrac{\delta^e}{L^e} = \dfrac{u_2^e - u_1^e}{L^e}$

$\hookrightarrow \sigma_e = \dfrac{F_2^e}{A^e}$

3) ~~3~~

$$F_2^e = AE\frac{(u_2^e - u_1^e)}{L^e} = \underbrace{\left(\frac{AE}{L}\right)^e}_{k^e}(u_2^e - u_1^e)$$

$\hookrightarrow$ do the same for F_1^e

Write down in Matrix form

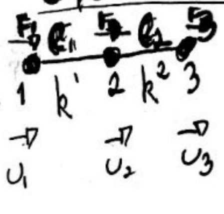
$$\begin{vmatrix} F_1^e \\ F_2^e \end{vmatrix} = \begin{vmatrix} k^e & -k^e \\ -k^e & k^e \end{vmatrix} \begin{vmatrix} u_1^e \\ u_2^e \end{vmatrix}$$

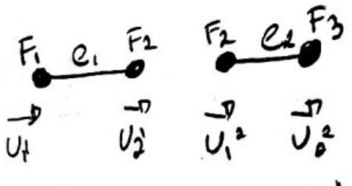
$\hookrightarrow$ stiffness matrix is always symmetric, determinant of this matrix is 0, its singular: this is because there are no B.C. specified

$$\underline{F}^e = \underline{\underline{k}}^e \, \underline{d}^e$$

$\underline{F}^e$: element force vector; $\underline{\underline{k}}^e$: element stiffness; $\underline{d}^e$: element disp vector

System of bars





$$\begin{vmatrix} F_1^1 \\ F_2^1 \end{vmatrix} = k^1 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} u_1^1 \\ u_2^1 \end{vmatrix} ; \quad \begin{vmatrix} F_1^2 \\ F_2^2 \end{vmatrix} = k^2 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} u_1^2 \\ u_2^2 \end{vmatrix}$$

$$\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} \qquad \begin{vmatrix} u_2 \\ u_3 \end{vmatrix}$$

$\rightarrow$ create link

$u_1 = u_1^1$

$u_2 = u_2^1 = u_1^2$

$u_3 = u_2^2$

Element Connectivity Chart

Elem	node1	node2
1	1	2
2	2	3

$$\begin{vmatrix} F_1^1 \\ F_2^1 \\ 0 \end{vmatrix} = k^1 \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}$$

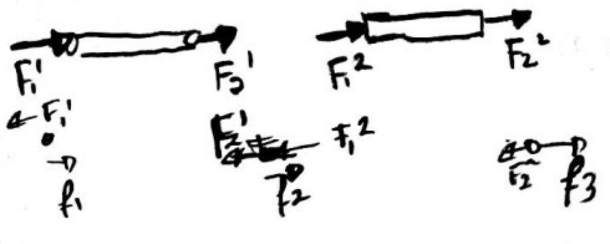
expanded stiffness matrix

$$\begin{vmatrix} 0 \\ F_1^2 \\ F_2^2 \end{vmatrix} = k^2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}$$

↳ global displacement vector

We have 2 equations and now we need to put them together

Free body diagram



$f_1 = F_1^1$ $\quad f_2 = F_2^1 + F_2^2$ $\quad f_3 = F_2^2$

$$\underline{F} = \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} = \begin{vmatrix} F_1^1 \\ F_2^1 + F_2^2 \\ F_2^2 \end{vmatrix}$$

↳ global force vector

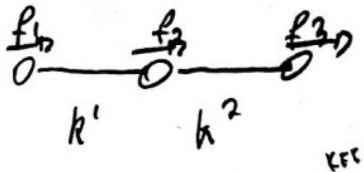
$$\underline{F} = \underbrace{\left(k^1 \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} + k^2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} \right)}_{\underline{\underline{K}} \text{ - global stiffness matrix}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

↳ global displacement vector - $\underline{d}$

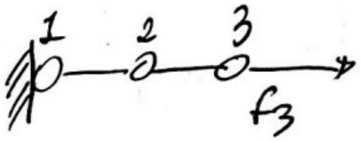
$$\underline{F} = \underline{\underline{K}}\, \underline{d} \Rightarrow \underline{\underline{K}} = \sum K^e_{expanded}$$

$$\underline{\underline{K}} = \sum \underline{\underline{K}}^{i}_{\text{expanden}} = \begin{pmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1+k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{pmatrix}$$

01/30/2019 Lecture 2



$$\begin{array}{ccccc} & & K_{EE} & K_{EF} & \\ \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} & = & \left|\begin{array}{c|cc} k^1 & -k^1 & 0 \\ \hline -k^1 & k^1+k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{array}\right| & & \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} \\ & & K_{FE} & K_{FF} & \end{array}$$



$\Rightarrow \underline{F} = \underline{\underline{K}}\,\underline{d}$

2 Types of nodes:

- Essential node: displacement is known
  E-nodes
- Free nodes
  F-nodes: displacement is unknown

$\underline{d_E} = |u_1| \quad \underline{d_F} = \begin{vmatrix} u_2 \\ u_3 \end{vmatrix}$

$F_E = |f_1| \quad F_F = \begin{vmatrix} f_2 \\ f_3 \end{vmatrix}$

$$\begin{vmatrix} F_E \\ F_F \end{vmatrix} = \begin{vmatrix} K_{EE} & K_{EF} \\ K_{FE} & K_{FF} \end{vmatrix} \begin{vmatrix} \underline{d_E} \\ \underline{d_F} \end{vmatrix}$$

we want to determine $\underline{d_F}$ which is unknown.

$$\underline{F_F} = \underline{\underline{K_{FE}}}\,\underline{d_E} + \underline{\underline{K_{FF}}}\,\underline{d_F}$$

$$\underline{d_F} = \underline{\underline{K_{FF}}}^{-1}\left(\underline{F_F} - \underline{\underline{K_{FE}}}\,\underline{d_E}\right)$$

$$\underline{F_E} = \underline{\underline{K_{EE}}}\,\underline{d_E} + \underline{\underline{K_{EF}}}\,\underline{d_F}$$

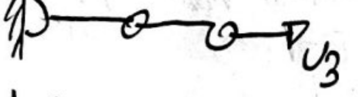
$$\underline{\underline{K}}_{FF} = \begin{vmatrix} k^1+k^2 & -k^2 \\ -k^2 & k^2 \end{vmatrix}$$

$$\underline{\underline{k}}_{FE} = \begin{vmatrix} -k^1 \\ 0 \end{vmatrix} \; ; \; \underline{F_F} = \begin{vmatrix} 0 \\ f_3 \end{vmatrix}$$

$$\underline{\underline{d}}_F = \begin{vmatrix} \frac{f_3}{k_1} \\ f_3\left(\frac{1}{k_1}+\frac{1}{k_2}\right) \end{vmatrix}$$

$\Rightarrow$ you can now use this to solve for reaction forces. This will give that reaction force $f_1 = -f_3$ from force balance.

New problem



$$\underline{d_E} = \begin{vmatrix} U_1 \\ U_3 \end{vmatrix} \quad d_F = |U_2|$$

$$\underline{F_E} = \begin{vmatrix} F_1 \\ F_3 \end{vmatrix} \quad F_F = |F_2| = |0|$$

K_{FE} K_{FF}

$$\begin{vmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1+k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{vmatrix}$$

K_{EF}

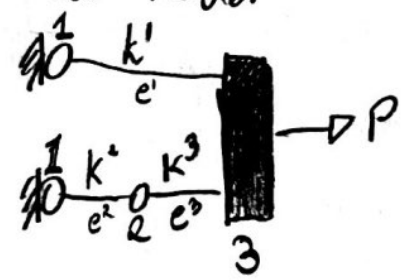
k_{EE}

K_{ij} force on i due to displacement of node j

$$d_F = U_2 = \frac{1}{k_1+k_2}\left(0 - \begin{bmatrix} -k_1 & -k_2 \end{bmatrix}\begin{bmatrix} 0 \\ U_3 \end{bmatrix}\right.$$

$$= \frac{k^2 \bar{u}_3}{k_1+k_2}$$

New Problem



$$\underline{K}^{(1)} = k_1 \begin{array}{c|cc|c} & 1 & 3 & \\ & 1 & -1 & 1 \\ & -1 & 1 & 3 \end{array}$$

$$\underline{K}^{(2)} = k_2 \begin{array}{c|cc|c} & 1 & 2 & \\ & 1 & -1 & 1 \\ & -1 & 1 & 2 \end{array}$$

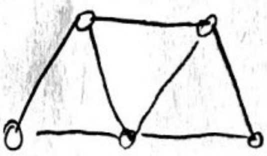
$$\underline{K}^{(3)} = k_3 \begin{array}{c|cc|c} & 2 & 3 & \\ & 1 & -1 & 2 \\ & -1 & 1 & 3 \end{array}$$

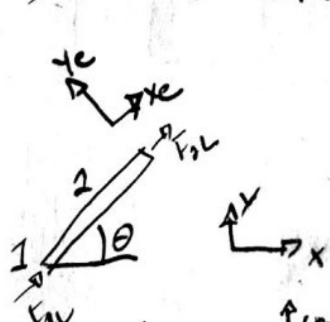
$$\underline{\underline{K}} = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{ccc|c} 1 \; K_{EE} & 2 \; K_{EF} & 3 & \\ \left| k_1+k_2 \right. & -k_2 & -k_1 & 1 \\ -k^2 & k^2+k_3 & -k_3 & 2 \\ -k_1 & -k_3 & K_1+k_3 & 3 \\ K_{FE} & K_{FF} & & \end{array}$$

$$\underline{d}_E = |0| \qquad \underline{d}_F = \begin{vmatrix} U_2 \\ U_3 \end{vmatrix}$$

$$\underline{F}_E = |f_1| \qquad \underline{F}_F = \begin{vmatrix} 0 \\ P \end{vmatrix}$$

$$\underline{d}_F = \underline{\underline{K_{FF}}}^{-1}\left(\underline{F}_F - \cancel{\underline{\underline{K}}_{FE}\underline{d}_E}\right) = \frac{1}{K_1K_2 + k_2k_3 + k_1k_3}\begin{bmatrix} k_3 P \\ k_2 + k_3 P \end{bmatrix}$$





$$\underline{d}^e = \begin{vmatrix} u_{1x}^e \\ u_{1y}^e \\ u_{2x}^e \\ u_{2y}^e \end{vmatrix} \qquad \underline{F}^e = \begin{vmatrix} F_{1x}^e \\ F_{1y}^e \\ F_{2x}^e \\ F_{2y}^e \end{vmatrix}$$

$$\begin{vmatrix} F_{1L}^e \\ F_{2L}^e \end{vmatrix} = K_2 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} u_{1L}^e \\ u_{2L}^e \end{vmatrix}$$

$$\underline{F}_L^e = \underline{\underline{k}}_L^e \, \underline{d}_L^e$$

$$u_{1L}^e = u_{1x}^e \cos\theta + u_{1y}^e \sin\theta$$

$$u_{2L}^e = u_{2x}^e \cos\theta + u_{2y}^e \sin\theta$$

$\underline{\underline{T}}$

$$\begin{vmatrix} U_{1L}^e \\ U_{2L}^e \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{vmatrix} \begin{vmatrix} U_{1x}^e \\ U_{1y}^e \\ U_{2x}^e \\ U_{2y}^e \end{vmatrix}$$

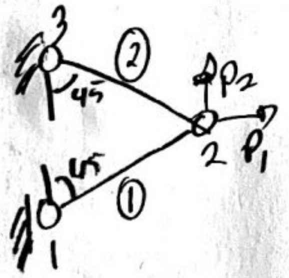
$$\underline{\underline{T}}\,\underline{F}^e = \underline{\underline{K}}_L^e\,\underline{\underline{T}}\,\underline{d}^e$$

$$\underline{F}^e = \underline{\underline{T}}^{-1}\,\underline{\underline{K}}_L^e\,\underline{\underline{T}}\,\underline{d}^e$$

the same

$$\underline{\underline{T}}^T \quad k_e \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$k^e \begin{vmatrix} c^2 & c^2 & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{vmatrix}$$



Elem	node 1	node 2	θ
1	1	2	45°
2	2	3	135°

$$\underline{\underline{K}}^{(1)} = \frac{k}{2} \begin{array}{c} \begin{array}{cccc} 1x & 1y & 2x & 2y \end{array} \\ \begin{vmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{vmatrix} \end{array} \begin{array}{c} 1x \\ 1y \\ 2x \\ 2y \end{array}$$

$$\underline{\underline{K}}^{(2)} = \frac{k}{2} \begin{array}{c} \begin{array}{cccc} 2x & 2y & 3x & 3y \end{array} \\ \begin{vmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} \end{array} \begin{array}{c} 2x \\ 2y \\ 3x \\ 3y \end{array}$$

$$\underline{\underline{K}} = \frac{k}{2} \begin{array}{cccccc|c} 1x & 1y & 2x & 2y & 3x & 3y & \\ \end{array}$$

$$\begin{array}{c} \\ \underline{\underline{K}} = \dfrac{k}{2} \end{array} \left|\begin{array}{cccccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1^{\;K_{FF}} & 0 & 0 \\ -1 & -1 & 1+1 & 1-1 & -1 & 1 \\ -1 & -1 & -1 & 1+1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right| \begin{array}{c} 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \end{array}$$

$$\underline{d_E} = \begin{vmatrix} u_{1x} \\ u_{1y} \\ u_{3x} \\ u_{3y} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \quad \underline{d_F} = \begin{vmatrix} u_{2x} \\ u_{2y} \end{vmatrix}$$

$$\underline{F_E} = \begin{vmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{vmatrix} \quad F_F = \begin{vmatrix} P_1 \\ P_2 \end{vmatrix}$$

$$\underline{d_F} = \underline{\underline{K_{FF}}}^{-1} \left(\underline{F_F} - \underline{\underline{K_{FE}}}\,\underline{d_E} \right)$$

$$= \begin{vmatrix} u_{2x} \\ u_{2y} \end{vmatrix} = \begin{vmatrix} P_1/k \\ P_2/k \end{vmatrix}$$