AC274 - Homework #2

Matheus C. Fernandes

September 22, 2015

Problem 1: Compute shear and vorticity for a) Poiseuille, b) Couette and c) Elongational flow

From: Lecture 3

Recalling from lecture, the deformation tensor D_{ij} can be split up into two separate parts, namely

$$D_{ij} = \partial_i u_j = \underbrace{\frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)}_{S_{ij}} + \underbrace{\frac{1}{2}\left(\partial_i u_j - \partial_j u_i\right)}_{\Omega_{ij}},$$

where S_{ij} is the symmetric shear rate part, and Ω_{ij} is the anti-symmetric part giving us the vorticity. We know that for a), the Poiseuille flow is described by

$$\boldsymbol{u} = \begin{bmatrix} \frac{\Delta P}{2\mu L}\left[I^2 - y^2\right] \\ 0 \\ 0 \end{bmatrix}$$

So therefore, for he Poiseuille flow the shear part for this becomes

$$S_{ij} = \begin{bmatrix} 0 & -\frac{y\Delta P}{2\mu L} & 0 \\ -\frac{y\Delta P}{2\mu L} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now the vorticity becomes

$$\Omega_{ij} = \begin{bmatrix} 0 & -\frac{y\Delta P}{2\mu L} & 0 \\ \frac{y\Delta P}{2\mu L} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now for b) the Couette flow we know that the velocity is described by:

$$\boldsymbol{u} = \begin{bmatrix} \frac{yV}{I} \\ 0 \\ 0 \end{bmatrix},$$

where, V is the velocity of the top plate, and I is the distance from one plate to the other.

Thus for a Couette flow, the shear and vorticity are:

$$S_{ij} = \begin{bmatrix} 0 & \frac{V}{2I} & 0 \\ \frac{V}{2I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Omega_{ij} = \begin{bmatrix} 0 & \frac{V}{2I} & 0 \\ -\frac{V}{2I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For the elongational flow, we know that the flow field is described by

$$\boldsymbol{u} = \begin{bmatrix} \epsilon x \\ -\epsilon y \\ 0 \end{bmatrix}$$

Thus the shear and vorticity are as follows

$$S_{ij} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Omega_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 2: Show that 2D flows have no vortex stretching

From: Lecture 3

The definition of vortex stretching is given as

$$\boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u},$$

so that our goal is to show that this quantity is equivalent to 0. The vorticity $\boldsymbol{w}$ is defined as

$$\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{u}.$$

For a 2D flow we can write the velocity vector $\boldsymbol{u}$ as

$$\boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix}$$

Thus, the vorticity, or the curl of the velocity vector, yields:

$$\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{u} = \begin{bmatrix} 0 & 0 & \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \end{bmatrix}.$$

Now for the gradient of velocity, we obtain:

$$\boldsymbol{\nabla} \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Therefore, the vortex stretching becomes:

$$\boldsymbol{w}\cdot\boldsymbol{\nabla u}=\begin{bmatrix}0 & 0 & \frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z}\\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z}\\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z}\end{bmatrix}=\mathbf{0}$$

Which means that the non-zero term in the left matrix will multiply only z terms in the right matrix, making the vortex stretching exactly 0.

Problem 3

September 22, 2015

1 Problem 3: Write a computer program to track the deformation of a square within a Couette flow

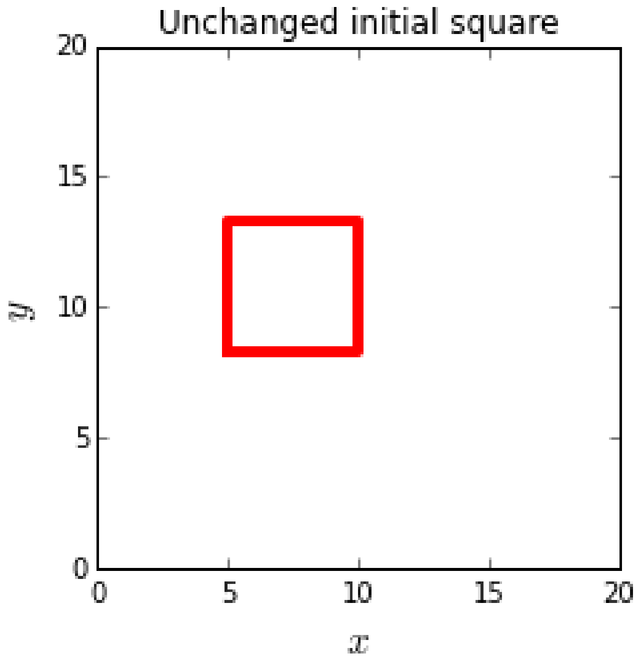
The couette flow can be described by the velocity vector

$$\mathbf{u} = \begin{bmatrix} \frac{yv}{I} \\ 0 \\ 0 \end{bmatrix}$$

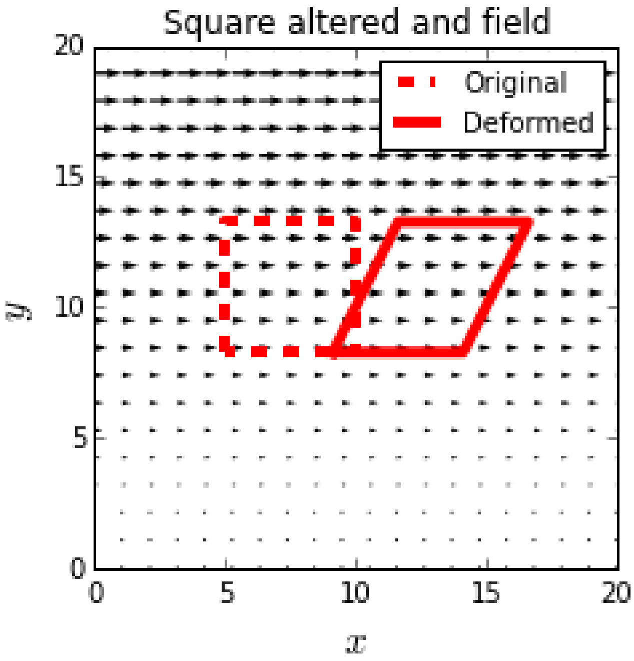
Were v is the velocity of the top plate assuming that the bottom plate is stationary.

```
In [1]: %matplotlib inline

        import numpy as np
        import matplotlib.pyplot as plt
        import math
        from pylab import *

        # Paramters
        v=10.0 #upper boundary velocity boundary condtion
        I=20.0 #thickness of the flow tube
        L=20.0#length of the flow tube

        squareIX=5.0 # inital X position of square
        squareIY=8.25 #initial Y poistion of square
        squareHeight=5.0 # initial height of a square
        squareWidth=5.0 #initial width of the square
```

```
In [2]: xdomain=np.linspace(0,L,20)
        ydomain=np.linspace(0,I,20)
        x,y=np.meshgrid(xdomain,ydomain)

        u=v/I

        fig = plt.figure(figsize=(3.5,3.5))
        squareX=np.array([squareIX, squareIX, squareIX+squareHeight, squareIX+squareHeight, squareIX])
        squareY=np.array([squareIY , squareIY+squareWidth, squareIY+squareWidth, squareIY, squareIY])
        #plotting unchanged square
        plt.axis([0,L,0,I])
        plt.plot(squareX,squareY,'r',linewidth=4)
        plt.xlabel('$x$',fontsize=15)
        plt.ylabel('$y$',fontsize=15)
        plt.title('Unchanged initial square',fontsize=12)

        plt.show()
```



```
In [3]: # plotting deformed square
        squareXdef=squareX+squareY*u
        squareYdef=squareY

        fig = plt.figure(figsize=(3.5,3.5))
        plt.axis([0,L,0,I])
        plt.plot(squareX,squareY,'--r',linewidth=4,label='Original')
        plt.plot(squareXdef,squareYdef,'r',linewidth=4,label='Deformed')
        plt.quiver(x,y,y*u,x*0,width=0.006)
        plt.xlabel('$x$',fontsize=15)
        plt.ylabel('$y$',fontsize=15)
        plt.title('Square altered and field',fontsize=12)
        plt.legend(fontsize=10)

        plt.show()
```


Square altered and field
Original
Deformed
y
x
0
5
10
15
20
0
5
10
15
20