AC274 - Homework #3

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Problem 1: Show that in a dilute gas the mean free path is larger than the mean intermolecular distance: give actual numbers for standard air.

The mean free path is given by

$$\lambda = 1/n\sigma,$$

where n is the number of target particles per unit volume and σ is the effective cross-sectional area for collision[1]. The mean intermolecular distance is given by

$$d = (1/n)^{1/3}.$$

The first goal of the problem is to prove that $d << \lambda \Rightarrow \frac{d}{\lambda} << 1$, for a dilute gas. The diluteness parameter describing how dilute a gas is, can be described by:

$$\tilde{n} = (s/d)^3,$$

where if $\tilde{n} << 1$ it is a dilute gas and $\tilde{n} \approx 1$ it is a dense gas almost a liquid. The diluteness is defined in terms of the number of target particles as $\tilde{n} = ns^3$. Thus, for the purpose of this comparison, we can quantify the size of the mean free path to the intermolecular distance as:

$$\frac{d}{\lambda} = \frac{(1/n)^{1/3}}{1/(n\sigma)} = n^{2/3}\sigma$$

Furthermore, the effective size of a molecule can be written scaled in terms of the effective cross-sectional area of collision as $s \sim \sqrt{\sigma}$, where we can simplify the above equation as

$$\frac{d}{\lambda} = n^{2/3}s^2,$$

where replacing $n^{2/3} = (\tilde{n}s^{-3})^{2/3}$ yields

$$\frac{d}{\lambda} = \tilde{n}^{2/3}.$$

Thus, for a dilute gas where $\tilde{n} << 1$, then

$$\frac{d}{\lambda} = \tilde{n}^{2/3} << 1 \quad \text{meaning,} \quad d << \lambda.$$

To calculate using actual numbers of air, we need to figure out two properties of air: the volume number density n, and the effective size of the molecule s. To obtain n, we use the density of air, namely $\rho = 1.2922$ [kg/m^3], and the molar mass $M = 0.02896$ [kg/mol], to approximate how many particles per unit volume[2]:

$$n = \frac{\rho}{M} = 44.6202 \text{ [mol/m}^3\text{]} = 2.687 \times 10^{25} \text{ [molecules/m}^3\text{]}$$

[1]Exact defintions obtained from wikipdia page, `https://en.wikipedia.org/wiki/Mean_free_path`

[2]Numbers obtained from: `https://en.wikipedia.org/wiki/Density_of_air`

Furthermore, form the internet we obtain the air molecule diameter to be

$$s = 4 \times 10^{-10} \text{ [m]}$$ [3]

and thus the effective cross-section of collision to be

$$\sigma \sim (4 \times 10^{-10})^2 = 1.6 \times 10^{-19} [\text{m}^2].$$

We can therefore calculate the mean free path to be:

$$\lambda = 2.326 \times 10^{-7} \text{ [m]},$$

and the intermolecular distance to be

$$d = 3.3387 \times 10^{-9} \text{ [m]}.$$

Making $d << \lambda$ as 3.3387×10^{-9} [m] $<<$ 2.326×10^{-7} [m] .

Problem 2: What fraction of air molecules move faster than 1000 m/s in standard conditions?

[4]To obtain what fraction of air molecules move faster than 1000 m/s in standard conditions, we start be recalling that the probability distribution as a function of speed is :

$$f(v)dv = 4\pi v^2 A^3 e^{-3Bv^2} dv. \tag{1}$$

To find the unknowns, A and B, we must then use the definition that

$$\int_0^\infty f(v)dv = 1 \Rightarrow 4\pi A^3 \frac{1}{4B}\sqrt{\frac{\pi}{B}} = 1 \Rightarrow 4\pi A^3 = \frac{4}{\sqrt{\pi}} B^{3/2} \quad \text{exacly as found in eq. (1)}$$

and that the average kinetic energy of the particles is related to the temperature by

$$\overline{\frac{1}{2}mv^2} = \frac{3}{2}kT. \tag{2}$$

We can find the average kinetic energy per particle by using the particle distribution function such that

$$\overline{\frac{1}{2}mv^2} = \int_0^\infty \frac{1}{2}mv^2 f(v)dv = \frac{3m}{4B}. \tag{3}$$

Putting together eq. (2) and eq. (3), we obtain that B must be given by:

$$B = \frac{m}{2kT}.$$

[3]Value obtained from: http://www.nuffieldfoundation.org/practical-physics/estimate-molecular-size-more-formal-method

[4]This answer was based on the information found in the lecture notes and in the following website: http://galileo.phys.virginia.edu/classes/252/kinetic_theory.html

This means that the distribution function is

$$\boxed{f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}.} \tag{4}$$

Now looking at standard conditions for air, we know the following parameters which we must plug into eq. (4):

$$k = 1.38 \times 10^{-23}[\text{joules/K}]$$
$$T = 294.15[\text{K}]$$
$$m = 4.8 \times 10^{-26}[\text{kg}]^5.$$

Integrating the function seen in eq. (4) from 1000 to ∞ we obtain that the fraction of particles moving faster than 1000[m/s] is

$$\boxed{\int_{1000}^{\infty} f(v)dv = 0.00147634}$$

where $f(v)$ looks like the following:

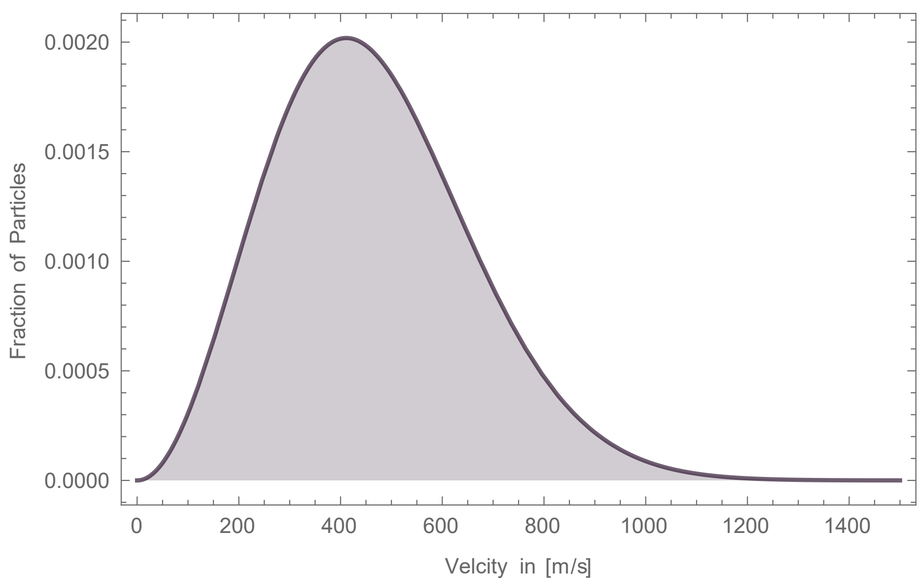

Figure 1: Plot of the velocity distribution function eq. (4).

Problem 3: Same for air molecules in a car moving at 100 km/h (in the Lab frame).

Assuming that we are in the Lab frame of the car, we notice a constant airflow everywhere of 100 [km/hr] (27.7778 [m/s]). Assuming that this velocity is in the x-direction, we will call it $u_x = 27.7778$. Furthermore, since any direction is as good as any other direction, the distribution function must only depend on the total speed of the particle, not on the separate velocity components. Therefore:

$$F(v_x)F(v_y)F(v_z) = f(v_x^2 + v_y^2 + v_z^2).$$

[5]From Wolframalpha: `http://www.wolframalpha.com/input/?i=mass+of+air+molecule`

[6]Thus, to find the change we must realte the distribution functions to allow bulk fluid flow, in which case the velocity origin is shifted as $v = (v_x - u_x)^2 + v_y^2 + v_z^2$, such that:

$$\boxed{f((v_x-u_x)^2+v_y^2+v_z^2) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}((v_x-u_x)^2+v_y^2+v_z^2)e^{-m((v_x-u_x)^2+v_y^2+v_z^2)/2kT}.} \tag{5}$$

Therefore, as we did in problem 2, we must integrate from 1000 to infinity in each direction to obtain the fraction of particles moving over 1000 [m/s].

$$\boxed{\int_{1000}^{\infty}\int_{1000}^{\infty}\int_{1000}^{\infty} f((v_x-u_x)^2+v_y^2+v_z^2)dv_x dv_y dv_z = \mathbf{0.00152025}}$$

The comparison plot between what found in problem 2 vs what found in problem 3 can be see in fig. 2.

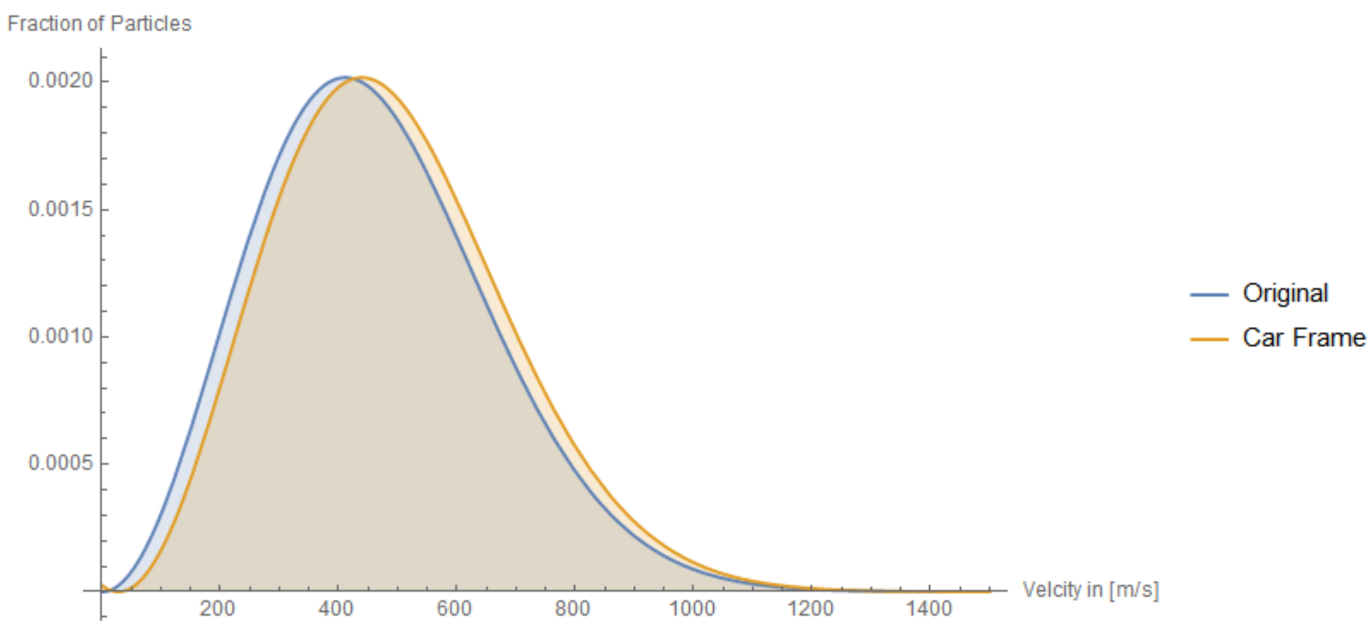

Figure 2: Plot of the velocity distribution function for the car reference frame(as described by eq. (5)) vs what we found in problem 2.

ALL CALCULATION WERE DONE IN MATHEMATICA WHICH THE NOTEBOOK USED CAN BE FOUD AT `http://fer.me/ac274hw3`

[6]Obtained from: `https://en.wikipedia.org/wiki/Distribution_function`