AC274 - Homework #2

Matheus C. Fernandes

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Problem 1: Compute shear and vorticity for a) Poiseuille, b) Couette and c) Elongational flow

From: Lecture 3

Recalling from lecture, the deformation tensor D_{ij} can be split up into two separate parts, namely

$$D_{ij} = \partial_i u_j = \underbrace{\frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)}_{S_{ij}} + \underbrace{\frac{1}{2}\left(\partial_i u_j - \partial_j u_i\right)}_{\Omega_{ij}},$$

where S_{ij} is the symmetric shear rate part, and Ω_{ij} is the anti-symmetric part giving us the vorticity.
We know that for a), the Poiseuille flow is described by

$$\boldsymbol{u} = \begin{bmatrix} \frac{\Delta P}{2\mu L}\left[I^2 - y^2\right] \\ 0 \\ 0 \end{bmatrix}$$

So therefore, for he Poiseuille flow the shear part for this becomes

$$S_{ij} = \begin{bmatrix} 0 & -\frac{y\Delta P}{2\mu L} & 0 \\ -\frac{y\Delta P}{2\mu L} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now the vorticity becomes

$$\Omega_{ij} = \begin{bmatrix} 0 & -\frac{y\Delta P}{2\mu L} & 0 \\ \frac{y\Delta P}{2\mu L} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now for b) the Couette flow we know that the velocity is described by:

$$\boldsymbol{u} = \begin{bmatrix} \frac{yV}{I} \\ 0 \\ 0 \end{bmatrix},$$

where, V is the velocity of the top plate, and I is the distance from one plate to the other.

Thus for a Couette flow, the shear and vorticity are:

$$S_{ij} = \begin{bmatrix} 0 & \frac{V}{2I} & 0 \\ \frac{V}{2I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Omega_{ij} = \begin{bmatrix} 0 & \frac{V}{2I} & 0 \\ -\frac{V}{2I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For the elongational flow, we know that the flow field is described by

$$\boldsymbol{u} = \begin{bmatrix} \epsilon x \\ -\epsilon y \\ 0 \end{bmatrix}$$

Thus the shear and vorticity are as follows

$$S_{ij} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Omega_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 2: Show that 2D flows have no vortex stretching

From: Lecture 3

The definition of vortex stretching is given as

$$\boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u},$$

so that our goal is to show that this quantity is equivalent to 0. The vorticity $\boldsymbol{w}$ is defined as

$$\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{u}.$$

For a 2D flow we can write the velocity vector $\boldsymbol{u}$ as

$$\boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix}$$

Thus, the vorticity, or the curl of the velocity vector, yields:

$$\boldsymbol{w} = \boldsymbol{\nabla} \times \boldsymbol{u} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \end{bmatrix}.$$

Now for the gradient of velocity, we obtain:

$$\boldsymbol{\nabla} \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Therefore, the vortex stretching becomes:

$$\boldsymbol{w} \cdot \boldsymbol{\nabla u} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} = 0$$

Which means that the non-zero term in the left matrix will multiply only z terms in the right matrix, making the vortex stretching exactly 0.

Problem 3: Write a computer program to track the deformation of a square within a Couette flow

From: Lecture 3