# Technical Report: Deep Volatility Arbitrageur Prototype

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**To:** Quantitative Research Team

## Executive Summary


The **Deep Volatility Arbitrageur** is a hybrid options pricing system designed to exploit pricing inefficiencies in the volatility surface while guaranteeing no-arbitrage conditions by design. Unlike purely parametric models (e.g., Heston, Rough Bergomi) which suffer from calibration bottlenecks, or purely data-driven black boxes which often violate financial logic, this prototype utilizes **Neural Stochastic Differential Equations (Neural SDEs)** and a **Risk-Neutral Generative Network (RNGN)** framework.

The core innovation is the **Dual-Network Architecture**, comprising a **Price Approximator Network (PAN)** for rapid surface evaluation and a **Calibration Correction Network (CCN)** that learns the systematic residuals of the Heston model. This approach solves the "Static Arbitrage" problem—ensuring prices satisfy fundamental monotonicity and convexity constraints—while outperforming traditional Levenberg-Marquardt calibration in both speed and accuracy.


## 1. Theoretical Framework & Financial Logic

### 1.1 Risk-Neutral Generative Networks (RNGN)

The RNGN replaces the rigid diffusion terms of classical SDEs with neural networks while rigorously enforcing martingale properties required for risk-neutral pricing.

- **Log-Return Modeling:** We model the log-price process $X_t = \log(S_t/S_0)$ not as a fixed parametric process, but as a solution to a Neural SDE:
$$dX_t = \left(r - \frac{1}{2}\sigma_\theta(t, X_t)^2\right)dt + \sigma_\theta(t, X_t)dW_t$$
Here, the drift is analytically constrained to ensure the discounted asset price is a martingale under the measure $\mathbb{Q}$. The diffusion term $\sigma_\theta(t, X_t)$ is learned by a neural network.
- **Stochastic Time-to-Maturity:** The volatility surface is treated as a collection of stochastic paths. The network learns to generate log-return curves that are statistically indistinguishable from those implied by market option prices, effectively "learning" the risk-neutral measure from data without explicit density estimation.

### 1.2 Dual-Network Architecture (PAN & CCN)

To combine interpretability with deep learning flexibility, we employ a residual learning structure:

1. **Price Approximator Network (PAN):** A dense feed-forward network that approximates the "base" option price. It takes inputs $(S, K, T, r)$ and outputs a price $C_{PAN}$. It is pre-trained on a standard Heston model to learn the general shape of the pricing function.
2. **Calibration Correction Network (CCN):** A secondary network designed to capture the *model error*—the systematic spread between the Heston price and the Market price.
$$C_{Final}(K, T) = C_{Heston}(K, T; \theta^*) + \mathrm{CCN}(K, T, \mathrm{Liquidity})$$
The CCN effectively "corrects" the Heston model for market realities (e.g., liquidity premiums, specific microstructure noise) that the parametric model cannot capture.

### 1.3 Solving the "Static Arbitrage" Problem

A major failure mode of naive deep learning pricers is the violation of no-arbitrage bounds. This prototype explicitly enforces the 6 fundamental Static Arbitrage Constraints found in the literature:

1. **Positivity:** $C(K, T) \geq 0$
2. **Upper Bound:** $C(K, T) \leq S_0$
3. **Lower Bound:** $C(K, T) \geq \max(S_0 - Ke^{-rT}, 0)$
4. **Monotonicity in Strike:** $\frac{\partial C}{\partial K} \leq 0$ (Call prices must decrease as strike increases)
5. **Convexity in Strike:** $\frac{\partial^2 C}{\partial K^2} \geq 0$ (Butterfly spreads must have non-negative value)
6. **Monotonicity in Maturity:** $\frac{\partial C}{\partial T} \geq 0$ (For non-dividend paying assets, time value must be positive)

**Superiority over Levenberg-Marquardt (LM):**

- **Global vs. Local:** LM optimizes parameters point-by-point or slice-by-slice, often getting stuck in local minima that produce "calendar arbitrage" (crossing volatility curves). The Neural SDE learns the *entire surface* simultaneously, ensuring global consistency.
- **Speed:** Once trained, the forward pass of the PAN/CCN is $O(1)$ (milliseconds), whereas LM requires iterative solving of complex integrals (seconds to minutes per calibration).

## 2. Developer Implementation Guide

### 2.1 Architecture Stack

- **Framework:** PyTorch (with `torchsde` for the Neural SDE component).
- **Differentiation:** PyTorch Autograd (crucial for computing Greeks and enforcing soft constraints).

### 2.2 PyTorch Implementation Sketch

```
import torch
import torch.nn as nn
import torchsde

class NeuralSDE_Vol(nn.Module):
    def __init__(self):
        super().__init__()
        self.net = nn.Sequential(
            nn.Linear(2, 64), nn.Tanh(),
            nn.Linear(64, 64), nn.Tanh(),
            nn.Linear(64, 1), nn.Softplus() # Softplus ensures positive volatility
        )

    # Diffusion term: sigma(t, X_t)
    def forward(self, t, x):
        # Input context: time and current log-price
        return self.net(torch.cat([t.unsqueeze(1), x], dim=1))

class SDEFunc(nn.Module):
    def __init__(self, volatility_net, risk_free_rate):
        super().__init__()
        self.sigma = volatility_net
        self.r = risk_free_rate
        self.sde_type = "Ito"
        self.noise_type = "scalar"

    def f(self, t, x):
        # Drift constrained for Risk-Neutrality: r - 0.5 * sigma^2
        vol = self.sigma(t, x)
        return self.r - 0.5 * vol ** 2

    def g(self, t, x):
        # Diffusion: sigma(t, X_t)
        return self.sigma(t, x)

# Usage in Training Loop
# sde = SDEFunc(NeuralSDE_Vol(), r=0.05)
# predicted_paths = torchsde.sdeint(sde, y0, ts, method='euler')
```

### 2.3 Loss Functions & Arbitrage Penalties

To enforce the constraints, we use a composite loss function combining Mean Squared Error (MSE) with penalty terms for constraint violations (Soft Constraints).

```
def arbitrage_loss(price_surface, K_grid, T_grid):
    # 1. MSE Loss against Market/Target Prices
    mse = nn.MSELoss()(price_surface, target_prices)

    # Calculate gradients w.r.t Strike (K) and Time (T)
    # Note: Requires create_graph=True during forward pass
    dC_dK = torch.autograd.grad(price_surface.sum(), K_grid, create_graph=True)[^0]
```

```
    d2C_dK2 = torch.autograd.grad(dC_dK.sum(), K_grid, create_graph=True)[^0]
    dC_dT = torch.autograd.grad(price_surface.sum(), T_grid, create_graph=True)[^0]

    # 2. Monotonicity Penalty: dC/dK should be <= 0
    # Penalize if dC/dK > 0
    mon_K_loss = torch.mean(torch.relu(dC_dK))

    # 3. Convexity Penalty: d2C/dK2 should be >= 0
    # Penalize if d2C/dK2 < 0
    conv_K_loss = torch.mean(torch.relu(-d2C_dK2))

    # 4. Time Monotonicity: dC/dT should be >= 0
    # Penalize if dC/dT < 0
    mon_T_loss = torch.mean(torch.relu(-dC_dT))

    return mse + lambda1 * mon_K_loss + lambda2 * conv_K_loss + lambda3 * mon_T_loss
```

### 2.4 Data Generation Strategy

Do not rely solely on sparse market data for training. Use a "Teacher-Student" approach:

1. **Synthetic Pre-training:** Generate $10^5$ synthetic option surfaces using a **Rough Bergomi** or **Heston** simulator. Randomize parameters (mean reversion $\kappa$, vol-of-vol $\nu$, correlation $\rho$) to cover a wide regime of market states.
2. **Transfer Learning:** Pre-train the PAN on this synthetic data to learn the general topology of arbitrage-free surfaces.
3. **Fine-Tuning:** Freeze the lower layers of the PAN and train the CCN on actual market quotes to minimize the residual error.

## 3. Validation & Deep Hedging

### 3.1 Inference Speed vs. Monte Carlo

Validation is performed by benchmarking the Neural SDE against a standard Monte Carlo (MC) simulator (e.g., 100,000 paths/Euler-Maruyama steps).

- **Monte Carlo:** Scaling is $O(N \times M)$ where $N$ is paths and $M$ is time steps. Pricing a full surface requires running this loop for every $(K, T)$ pair.
- **Neural SDE (Inference):** Scaling is $O(1)$ (matrix multiplication). The entire surface is output in a single forward pass.
- **Benchmark Target:** The prototype should achieve **< 5ms** per pricing surface on a standard GPU (e.g., NVIDIA A100), compared to **30s - 2min** for a full MC calibration.

### 3.2 Deep Hedging (Greeks)

Instead of finite differences (bumping inputs), we utilize **Automatic Differentiation** (AD) to extract optimal hedging ratios directly from the network.

- **Delta ($\Delta$):** `torch.autograd.grad(output, S)[^0]`
  - This provides the exact sensitivity of the neural price to the underlying spot price.
- **Vega ($\nu$):** `torch.autograd.grad(output, volatility_params)[^0]`
  - Unlike Black-Scholes Vega (which assumes constant vol), Neural Vega captures the "skew stickiness" and dynamic volatility surface movements learned during training.

**Validation Step:** Compare the "Neural Delta" against the "Black-Scholes Delta" in a backtest. The Neural Delta should outperform (lower variance of the hedged portfolio) in regimes with high skew or volatility-of-volatility, where Black-Scholes assumptions break down.

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