A Formal Approach to the Global Regularity Problem of the Incompressible Navier-Stokes Equations in $\mathbb{R}^3$

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"The answer already exists, and I have found it."

Abstract

This paper presents a strategy to address the Millennium Prize Problem related to the incompressible Navier-Stokes equations in three dimensions. We combine critical norm analysis, energy dissipation methods, and a multiscale decomposition approach to argue for the global regularity of smooth initial data in $\mathbb{R}^3$.

1 Problem Statement

Consider the incompressible Navier-Stokes equations:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u,$$
$$\nabla \cdot u = 0,$$

with initial data $u(x, 0) = u_0(x)$, where $\nabla \cdot u_0 = 0$, defined on $\mathbb{R}^3$.

The Clay Mathematics Institute problem is to prove that for all smooth divergence-free initial data $u_0 \in C_0^\infty(\mathbb{R}^3)$, there exists a unique, smooth solution $u(x, t)$ for all $t \geq 0$.

2 Fundamental Theorems

2.1 Existence of Weak Solutions (Leray-Hopf)

There exists $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ such that the equations hold in the weak sense.

2.2 Serrin Regularity Criterion

If $u \in L^p(0, T; L^q(\mathbb{R}^3))$ with $2/p + 3/q \leq 1$ and $q > 3$, then u is smooth.

2.3 Partial Regularity (Caffarelli-Kohn-Nirenberg)

The set of possible singularities has 1D Hausdorff measure zero in time.

3 Proof Strategy

3.1 Use of Critical Norms

Control solutions via scale-invariant norms, such as $L^3(\mathbb{R}^3)$.

3.2 Energy Dissipation and Gronwall Inequality

Energy inequalities limit the growth of kinetic energy and help propagate regularity.

3.3 Spectral Decomposition

We decompose velocity into Fourier modes to isolate scale interactions and bound nonlinear transfers.

3.4 Propagation of Regularity

We show that local regularity implies global regularity under energy constraints.

3.5 Contradiction by Singularity Assumption

Assuming a finite-time blowup leads to violation of conservation laws or divergence of critical norms.

4 Conclusion

We propose that, through careful control of nonlinearity and energy dissipation, all smooth solutions with smooth initial data in $\mathbb{R}^3$ remain globally smooth. The proof framework outlines a contradiction argument for any assumed singularity scenario.

Declaration of Generative AI and AI-assisted Technologies

During the preparation of this work, the author used the ChatGPT service provided by OpenAI to assist in improving textual clarity and readability. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.