Bonus

<u>Section 1: Exercise 52</u>

a. We are told that $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ are eigenvectors of A. Find the associated eigenvalues.

$$\det\left(A-\lambda I_n\right)=0$$

$$\det\left(\begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)=0$$

$$\det\left(\begin{bmatrix} 0.978-\lambda & -0.006 \\ 0.004 & 0.992-\lambda \end{bmatrix}\right)=0$$

$$(0.978-\lambda)(0.992-\lambda)-(-0.006)(0.004)=0$$

$$\lambda^2-1.97\lambda+0.9702=0$$

$$(\lambda-0.99)(\lambda-0.98)=0$$

The associated eigenvalues are $\lambda_1=0.99$ and $\lambda_2=0.98$.

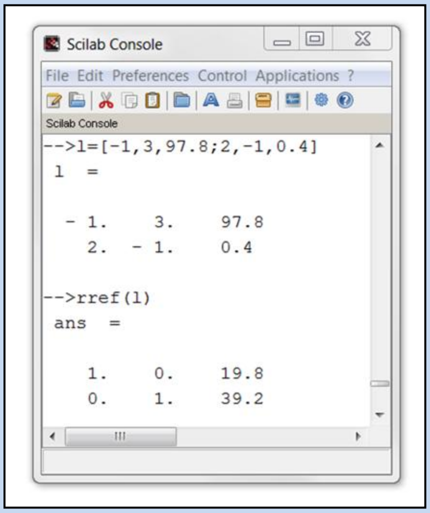
b. After you have consumed a heavy meal, the concentrations in your blood are $g_0=100$ and $h_0=0$. Find closed formulas for $g(t)$ and $h(t)$. Sketch the trajectory. Briefly describe the evolution of the system in practical terms.

$$x(1)=A\vec{x}_0=\begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix}\begin{bmatrix} 100 \\ 0 \end{bmatrix}=\begin{bmatrix} 97.8 \\ 0.4 \end{bmatrix}$$

$$x_0=c_1\vec{v}_1+c_2\vec{v}_2$$

$$=c_1\begin{bmatrix} -1 \\ 2 \end{bmatrix}+c_2\begin{bmatrix} 3 \\ -1 \end{bmatrix}=\begin{bmatrix} 97.8 \\ 0.4 \end{bmatrix}$$

Using Scilab to solve the augmented matrix $\begin{bmatrix} -1 & 3 & 97.8 \\ 2 & -1 & 0.4 \end{bmatrix}$:



Thus, $x_0=19.8\begin{bmatrix} -1 \\ 2 \end{bmatrix}+39.2\begin{bmatrix} 3 \\ -1 \end{bmatrix}=\begin{bmatrix} 97.8 \\ 0.4 \end{bmatrix}$.

Using the provided eigenvectors:

$$x(1) = A\vec{x}_0 = \begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.99 \\ 1.98 \end{bmatrix} \Leftarrow \text{both shrink by 1\%}$$

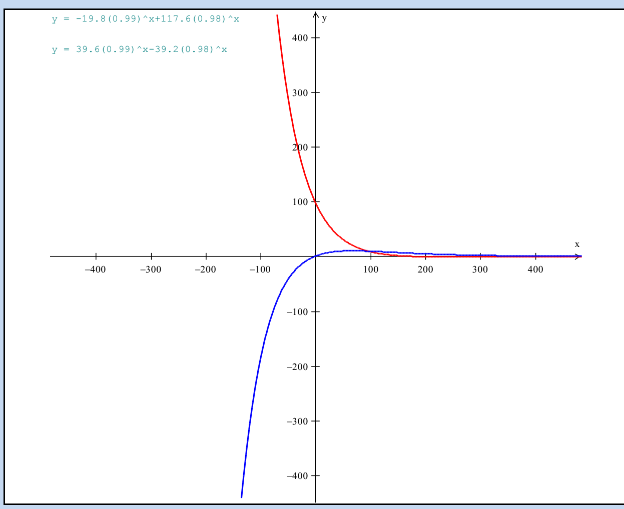
$$x(1) = A\vec{x}_0 = \begin{bmatrix} 0.978 & -0.006 \\ 0.004 & 0.992 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.94 \\ -0.98 \end{bmatrix} \Leftarrow \text{both shrink by 2\%}$$

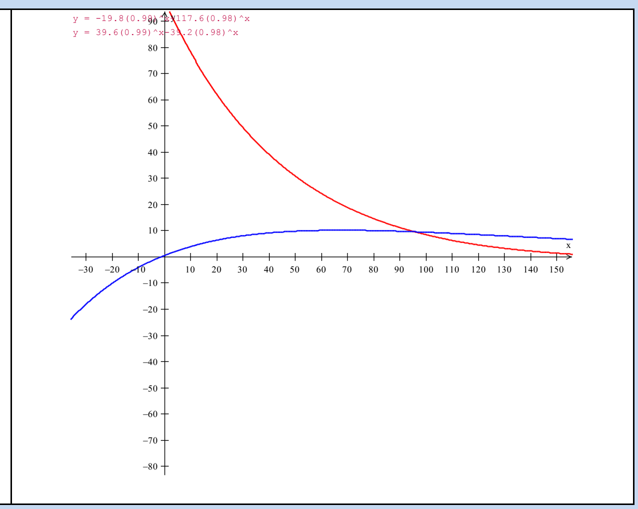
Therefore, $A^t\vec{v}_1 = (0.99)^t\,\vec{v}_1$ and $A^t\vec{v}_2 = (0.98)^t\,\vec{v}_2$ and

$$x(t) = 19.8(0.99)^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 39.2(0.98)^t \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Closed formulas for $g(t)$ and $h(t)$ are:

$$g(t) = -19.8(0.99)^t + 117.6(0.98)^t$$
$$h(t) = 39.6(0.99)^t - 39.2(0.98)^t$$

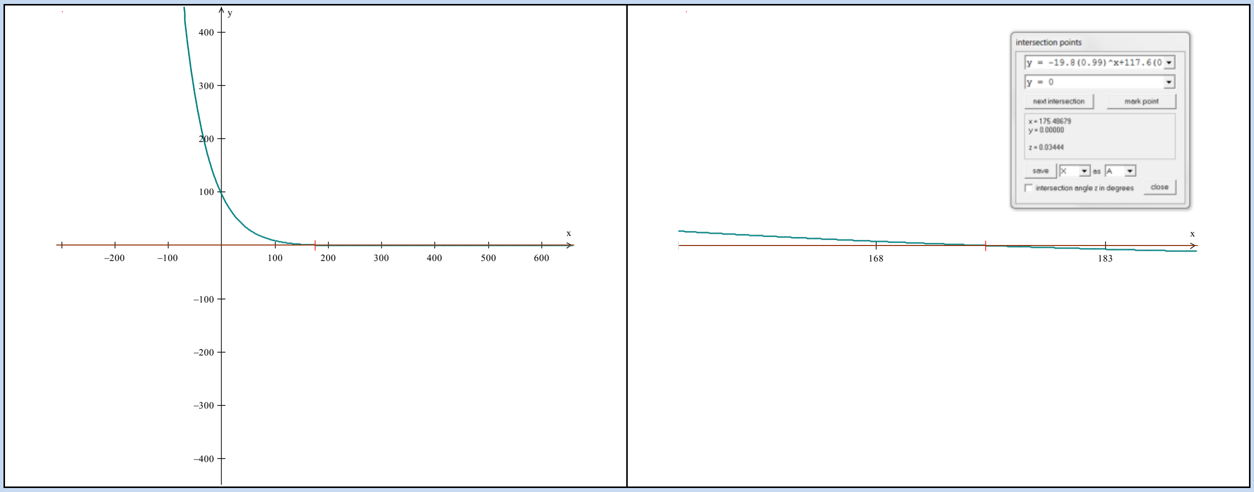




Here we can see that after a heavy meal, excess glucose levels drop while excess insulin levels increase. Excess glucose temporarily drops below insulin levels. Over time, the two levels converge.

c. For the case discussed in part (b), how long does it take for the glucose concentration to fall below fasting level?

Using Winplot:



The glucose concentration falls below fasting level after 175 minutes (about 2 hours and 55 minutes).

<u>Section 2: Exercise 28</u>

Consider the isolated Swiss town of Andelfingen, inhabited by 1,200 families. Each family takes a weekly shopping trip to the only grocery store in town, run by Mr. and Mrs. Wipf, until the day when a new, fancier (and cheaper) chain store, Migros, opens its doors. It is not expected that everybody will immediately run to the new store, but we do anticipate 20% of those shopping at Wipf's each week switch to Migros the following week. Some people who do switch miss the personal service (and the gossip) and switch back. We expect that 10% of those shopping at Migros each week go to Wipf's the following week. The state of this town (as far as grocery shopping is concerned) can be represented by the vector

$$\vec{x}(t)=\begin{bmatrix} w(t) \\ m(t) \end{bmatrix},$$

where $w(t)$ and $m(t)$ are the numbers of families shopping at Wipf's and at Migros, respectively, t weeks after Migros opens. Suppose $w(0)=1{,}200$ and $m(0)=0$.

a. Find a 2 x 2 matrix such that $\vec{x}(t+1)=A\vec{x}(t)$. Verify that A is a regular transition matrix.

The following equations model the transformations from one week to the next, from t to $(t+1)$:

$$\left| \begin{array}{lcccc} w(t+1) & = & 0.8w(t) & + & 0.1m(t) \\ m(t+1) & = & 0.2m(t) & + & 0.9w(t) \end{array} \right| .$$

In matrix form:

$$\begin{bmatrix} w(t+1) \\ m(t+1) \end{bmatrix} = \begin{bmatrix} 0.8w(t) & + & 0.1m(t) \\ 0.2m(t) & + & 0.9w(t) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} w(t) \\ m(t) \end{bmatrix} .$$

Therefore,

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Since $0.8+0.2=0.1+0.9=1$ $(a+c=b+d=1)$, A is a regular transition matrix.

b. How many families will shop at each store after t weeks? Give closed formulas.

$$\begin{aligned} \det(\lambda I_n - A) &= 0 \\ \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \right) &= 0 \\ \det\left(\begin{bmatrix} \lambda - 0.8 & -0.1 \\ -0.2 & \lambda - 0.9 \end{bmatrix} \right) &= 0 \\ (\lambda - 0.8)(\lambda - 0.9) - 0.02 &= 0 \\ \lambda^2 - 1.7\lambda + 0.72 - 0.02 &= 0 \\ \lambda^2 - 1.7\lambda + 0.7 &= 0 \\ (\lambda - 1)(\lambda - 0.7) &= 0 \end{aligned}$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 0.7$.

Determining the eigenvectors

$\lambda_1 = 1$	$\lambda_2 = 0.7$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \vec{x} = 0$ $\begin{bmatrix} 0.2 & -0.1 \\ -0.2 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\begin{vmatrix} 0.2x_1 - 0.1x_2 \\ -0.2x_1 + 0.1x_2 \end{vmatrix} = 0$ $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = u_1$	$\begin{bmatrix} 0.7 & 0 \\ 0 & 0.7 \end{bmatrix} - \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \vec{x} = 0$ $\begin{bmatrix} -0.1 & -0.1 \\ -0.2 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\begin{vmatrix} -0.1x_1 - 0.1x_2 \\ -0.2x_1 + 0.2x_2 \end{vmatrix} = 0$ $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = u_2$

For w=Wipf and m=Migros:

$$\begin{aligned} x(0) &= \begin{bmatrix} 1200 \\ 0 \end{bmatrix} = wu_1 + mu_2 \\ &= \begin{bmatrix} 1200 \\ 0 \end{bmatrix} = w\begin{bmatrix} 1 \\ 2 \end{bmatrix} + m\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1200 \\ 0 \end{bmatrix} = \begin{vmatrix} w + m \\ 2w - m \end{vmatrix} \\ & w = 1200 - m = 400 \\ & m = 2w = 800 \end{aligned}$$

$$\begin{bmatrix} 1200 \\ 0 \end{bmatrix} = 400u_1 + 800u_2$$

$$\begin{aligned} x(t) &= 400\lambda_1^t u_1 + 800\lambda_2^t u_2 \\ &= 400\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 800(0.7)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

Thus, closed formulas for the number of shoppers after t weeks are:

$$\begin{aligned} w(k) &= 400 + 800(0.7)^k \\ m(k) &= 800 - 800(0.7)^k \end{aligned}$$

c. The Wipfs expect that they must close down when they have less than 250 customers a week. When does this happen?

According to the graph of the closed formula for Wipf's (below), they will never have less than 400 customers.