# Bootstrap (Part 3)

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# Overview

- So far we used three different bootstraps:
  - Nonparametric bootstrap on the rows (e.g. regression, PCA with random rows and columns)
  - Nonparametric bootstrap on the residuals (e.g. regression)
  - Parametric bootstrap (e.g. PCA with fixed rows and columns)
- Today, we will look at some tricks to improve the bootstrap for confidence intervals:
  - Studentized bootstrap

# Introduction

- A statistics is (asymptotically) pivotal if its limiting distribution does not depend on unknown quantities
- For example, with observations $X_1, \ldots, X_n$ from a normal distribution with unknown mean and variance, a pivotal quantity is

$$T(X_1, \ldots, X_n) = \sqrt{n} \left( \frac{\theta - \hat{\theta}}{\hat{\sigma}} \right)$$

with unbiased estimates for sample mean and variance

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \hat{\theta})^2$$

- Then $T(X_1, \ldots, X_n)$ is a pivot following the Student's t-distribution with $\nu = n - 1$ degrees of freedom
- Because the distribution of $T(X_1, \ldots, X_n)$ does not depend on $\mu$ or $\sigma^2$

# Introduction

- The bootstrap is better at estimating the distribution of a pivotal statistics than at a nonpivotal statistics
- We will see an asymptotic argument using Edgeworth expansions
- But first, let us look at an example

# Motivation

- Take $n = 20$ random exponential variables with mean 3

```
x = rexp(n,rate=1/3)
```

- Generate $B = 1000$ bootstrap samples of $x$, and calculate the mean for each bootstrap sample

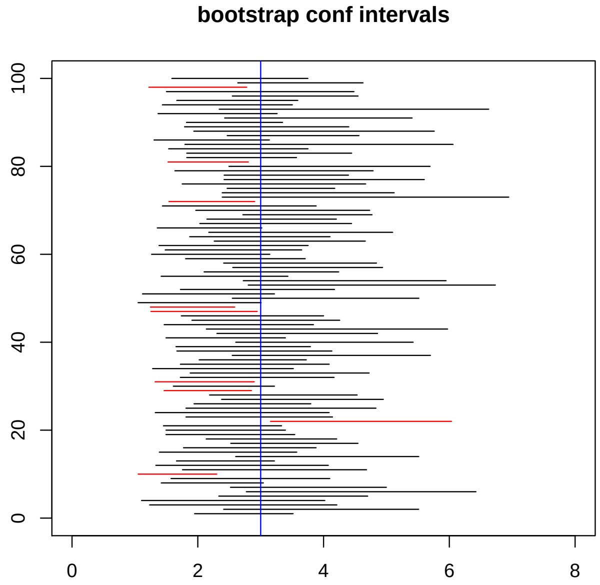
```
s = numeric(B)
for (j in 1:B) {  boot = sample(n,replace=TRUE)
                  s[j] = mean(x[boot]) }
```

- Form confidence interval from bootstrap samples using quantiles ($\alpha = .025$)

```
simple.ci = quantile(s,c(.025,.975))
```

- Repeat this process 100 times
- Check how often the intervals actually contains the true mean

# Motivation



# Motivation

- Another way is to calculate a pivotal quantity as the bootstrapped statistic
- Calculate the mean and standard deviation

```
x = rexp(n,rate=1/3)
mean.x = mean(x)
sd.x = sd(x)
```

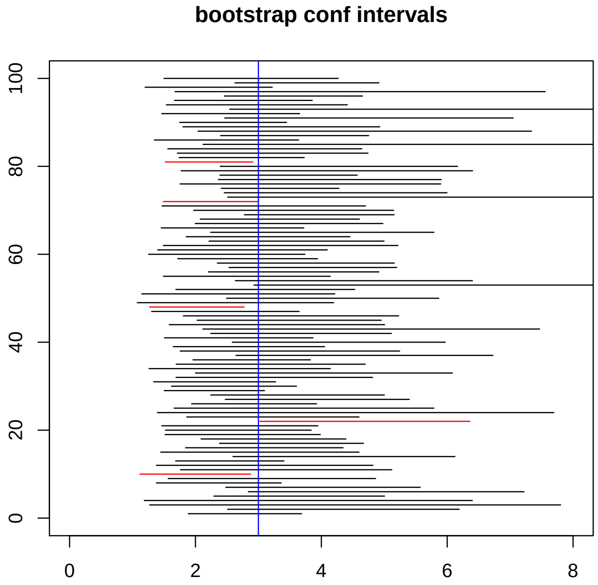
- For each bootstrap sample, calculate

```
z = numeric(B)
for (j in 1:B) {
  boot = sample(n,replace=TRUE)
  z[j] = (mean.x - mean(x[boot]))/sd(x[boot]) }
```

- Form a confidence interval like this

```
pivot.ci = mean.x + sd.x*quantile(z,c(.025,.975))
```

# Motivation



# Studentized Bootstrap

- Consider $X_1, \ldots, X_n$ from $F$
- Let $\hat{\theta}$ be an estimate of some $\theta$
- Let $\hat{\sigma}^2$ be a standard error for $\hat{\theta}$ estimated using the bootstrap
- Most of the time as $n$ grows

$$\frac{\hat{\theta} - \theta}{\hat{\sigma}} \dot{\sim} N(0, 1)$$

- Let $z^{(\alpha)}$ be the $100 \cdot \alpha$th percentile of $N(0, 1)$
- Then a standard confidence interval with coverage probability $1 - 2\alpha$ is

$$\hat{\theta} \pm z^{(1-\alpha)} \cdot \hat{\sigma}$$

- As $n \to \infty$, the bootstrap and standard intervals converge

# Studentized Bootstrap

- How can we improve the standard confidence interval?
- These intervals are valid under assumption that

$$Z = \frac{\hat{\theta} - \theta}{\hat{\sigma}} \dot{\sim} N(0, 1)$$

- But this is only valid as $n \to \infty$
- And are approximate for finite $n$
- When $\hat{\theta}$ is the sample mean, a better approximation is

$$Z = \frac{\hat{\theta} - \theta}{\hat{\sigma}} \dot{\sim} t_{n-1}$$

and $t_{n-1}$ is the Student's $t$ distribution with $n - 1$ degrees of freedom

# Studentized Bootstrap

- With this new approximation, we have

$$\hat{\theta} \pm t_{n-1}^{(1-\alpha)} \cdot \hat{\sigma}$$

- As $n$ grows the $t$ distribution converges to the normal distribution
- Intuitively, it widens the interval to account for unknown standard error
- But, for instance, it does not account for skewness in the underlying population
- This can happen when $\hat{\theta}$ is not the sample mean
- The Studentized bootstrap can adjust for such errors

# Studentized Bootstrap

- We estimate the distribution of

$$Z = \frac{\hat{\theta} - \theta}{\hat{\sigma}} \dot{\sim} \quad ?$$

- by generating $B$ bootstrap samples $X^{*1}, X^{*2}, \ldots, X^{*B}$
- and computing

$$Z^{*b} = \frac{\hat{\theta}^{*b} - \hat{\theta}}{\hat{\sigma}^{*b}}$$

- Then the $\alpha$th percentile of $Z^{*b}$ is estimated by the value $\hat{t}^{(\alpha)}$ such that

$$\frac{\#\{Z^{*b} \leq \hat{t}^{(\alpha)}\}}{B} = \alpha$$

- Which yields the studentized bootstrap interval

$$(\hat{\theta} - \hat{t}^{(1-\alpha)} \cdot \hat{\sigma}, \hat{\theta} - \hat{t}^{(\alpha)} \cdot \hat{\sigma})$$

# Asymptotic Argument in Favor of Pivoting

- Consider parameter $\theta$ estimated by $\hat{\theta}$ with variance $\frac{1}{n}\sigma^2$
- Take the pivotal statistics

$$S = \sqrt{n}\left(\frac{\hat{\theta} - \theta}{\hat{\sigma}}\right)$$

with estimate $\hat{\theta}$ and asymptotic variance estimate $\hat{\sigma}^2$

- Then, we can use Edgeworth expansions

$$P(S \leq x) = \Phi(X) + \sqrt{n}q(x)\phi(x) + O(\sqrt{n})$$

with
$\Phi$ standard normal distribution,
$\phi$ standard normal density, and
$q$ even polynomials of degree 2

# Asymptotic Argument in Favor of Pivoting

- Bootstrap estimates are

$$S = \sqrt{n}\left(\frac{\hat{\theta}^* - \hat{\theta}}{\hat{\sigma}^*}\right)$$

- Then, we can use Edgeworth expansions

$$P(S^* \leq x | X_1, \ldots, X_n) = \Phi(X) + \sqrt{n}\hat{q}(x)\phi(x) + O(\sqrt{n})$$

- $\hat{q}$ is obtain by replacing unknowns in $q$ with bootstrap estimates
- Asymptotically, we further have

$$\hat{q} - q = O(\sqrt{n})$$

# Asymptotic Argument in Favor of Pivoting

- Then, the bootstrap approximation to the distribution of $S$ is

$$P(S \leq x) - P(S^* \leq x | X_1, \ldots, X_n) =$$

$$\left(\Phi(X) + \sqrt{n} q(x) \phi(x) + O(\sqrt{n})\right) - \left(\Phi(X) + \sqrt{n} \hat{q}(x) \phi(x) + O(\sqrt{n})\right)$$

$$= O\left(\frac{1}{n}\right)$$

- Compared to the normal approximation $\sqrt{n}$
- Which the same as the error when using standard bootstrap (can be shown with the same argument)

# Studentized Bootstrap

- These pivotal intervals are more accurate in large samples than that of standard intervals and $t$ intervals
- Accuracy comes at the cost of generality
  - standard normal tables apply to all samples and all samples sizes
  - $t$ tables apply to all samples of fixed $n$
  - studentized bootstrap tables apply only to given sample
- The studentized bootstrap can be asymmetric
- It can be used for simple statistics, like mean, median, trimmed mean, and sample percentile
- But for more general statistics like the correlation coefficients, there are some problems:
  - Interval can fall outside of allowable range
  - Computational issues if both parameter and standard error have to be bootstrapped

# Studentized Bootstrap

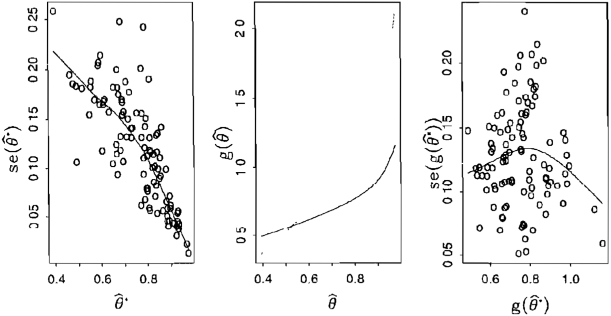
- The Studentized bootstrap works better for variance stabilized parameters
- Consider a random variable $X$ with mean $\theta$ and standard deviation $s(\theta)$ that varies as a function of $\theta$
- Using the delta method and solving an ordinary differential equation, we can show that

$$g(x) = \int^{x} \frac{1}{s(u)} du$$

  will make the variance of $g(X)$ constant
- Usually $s(u)$ is unknown
- So we need to estimate $s(u) = \mathrm{se}(\hat{\theta}|\theta = u)$ using the bootstrap

# Studentized Bootstrap

1. First bootstrap $\hat{\theta}$, second bootstrap $\hat{\text{se}}(\hat{\theta})$ from $\hat{\theta}^*$
2. Fit curve through points $(\hat{\theta}^{*1}, \hat{\text{se}}(\hat{\theta}^{*1})), \ldots, (\hat{\theta}^{*B}, \hat{\text{se}}(\hat{\theta}^{*B}))$
3. Variance stabilization $g(\hat{\theta})$ by numerical integration
4. Studentized bootstrap using $g(\hat{\theta}^*) - g(\hat{\theta})$
   (no denominator, since variance is now approximately one)
5. Map back through transformation $g^{-1}$




Source: Efron and Tibshirani (1994)

# Studentized Bootstrap in R

```
library(boot)
mean.fun = function(d, i) {
  m = mean(d$hours[i])
  n = length(i)
  v = (n-1)*var(d$hours[i])/n^2
  c(m, v) }
air.boot <- boot(aircondit, mean.fun, R = 999)
results = boot.ci(air.boot, type = c("basic", "stud"))
```

# Studentized Bootstrap in R

```
results
```

```
## BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
## Based on 999 bootstrap replicates
##
## CALL :
## boot.ci(boot.out = air.boot, type = c("basic", "stud"))
##
## Intervals :
## Level      Basic              Studentized
## 95%   ( 22.2, 171.2 )   ( 49.0, 303.0 )
## Calculations and Intervals on Original Scale
```

# References


- Efron (1987). Better Bootstrap Confidence Intervals
- Hall (1992). The Bootstrap and Edgeworth Expansion
- Efron and Tibshirani (1994). An Introduction to the Bootstrap
- Love (2010). Bootstrap-$t$ Confidence Intervals (Link to blog entry)