Chapter 1
Shape Properties of Tensor-Product Bernstein Polynomials

Michael S. Floater

Abstract. This paper reviews recently established shape properties of tensor-product Bernstein polynomials, emphasizing monotonicity, convexity, and sign changes. Some comparisons with other tensor-product bases are also discussed.

1. Introduction

This chapter concerns shape properties of the *tensor-product Bernstein polynomial*

$$f(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} B_{i,m}(x) B_{j,n}(y), \qquad (x,y) \in [0,1] \times [0,1], \tag{1.1}$$

with basis functions

$$B_{i,k}(t) = \binom{k}{i} t^i (1-t)^{k-i}, \quad i = 0, \ldots, k,$$

and coefficients $c_{ij} \in \mathbf{R}$.

We note that f is a bivariate *function* rather than a *parametric surface*. If one replaced the coefficients c_{ij} by *control points* $\mathbf{c}_{ij}$ in $\mathbf{R}^3$ then f would become a *Bézier surface*. Bézier surfaces (and their generalizations to *spline surfaces*) are better suited to surface *design*. One has much greater freedom to model complicated shapes in the parametric case.

On the other hand, bivariate functions such as f (and their generalizations to *spline functions*) are often better suited than parametric surfaces to *surface fitting*, especially when approximating scattered data and it is often desirable that f satisfy some shape property. This motivates the discussion, in Sections 2 and 3, of recent work on sufficient conditions on the coefficients of f which ensure that f is either monotonic or convex.

Another, less obvious use of bivariate polynomials is the design of *curves*. For, provided the coefficients c_{ij} of f satisfy certain properties, one can ensure that the set of zeros

$$Z_f = \{(x,y) \in [0,1] \times [0,1] : f(x,y) = 0\} \tag{1.2}$$

form a single curve. A designer can manipulate this curve by altering the values of the coefficients of f. We will therefore consider conditions on the coefficients which ensure that Z_f is indeed a simple curve in Section 4.

We will only be concerned in this paper with shape properties of f which are not merely trivial extensions of analogous properties of univariate polynomials. A brief discussion of monotonicity and convexity of f in the x and y directions can be found in [**G**].

2. Monotonicity

We begin in this section by showing that the tensor-product Bernstein basis preserves monotonicity in all directions, a statement we will make precise later.

Recall that a function of one variable is said to be *monotonic* if it is either purely increasing or purely decreasing. If, on the other hand, a function g of two variables is increasing in some *direction* $d \in \mathbf{R}^2$, then it must also be decreasing in the direction $-d$. Therefore, when discussing *monotonicity* of g, we will specify one or more directions in which the function is increasing. We should also note that g will not be very interesting if it increases in the directions d *and* $-d$ for then it must be *constant* in that direction.

Bearing these points in mind then, let us consider whether f in (1.1) is increasing over its domain $[0,1] \times [0,1]$ in some given direction $d = (d_1, d_2) \in \mathbf{R}^2$. This question is clearly equivalent to that of whether the directional derivative

$$D_d f = d_1 \frac{\partial f}{\partial x} + d_2 \frac{\partial f}{\partial y} = d_1 f_x + d_2 f_y \tag{2.1}$$

is non-negative over $[0,1] \times [0,1]$. Differentiating the function f of (1.1) yields the two first partial derivatives

$$f_x = m \sum_{i=0}^{m-1} \sum_{j=0}^{n} B_{i,m-1}(x) B_{j,n}(y) \Delta_{10} c_{ij}, \qquad f_y = n \sum_{i=0}^{m} \sum_{j=0}^{n-1} B_{i,m}(x) B_{j,n-1}(y) \Delta_{01} c_{ij},$$

where $\Delta_{10} c_{ij} = c_{i+1,j} - c_{ij}$ and $\Delta_{01} c_{ij} = c_{i,j+1} - c_{ij}$. Next we apply the easily derived identity

$$\sum_{i=0}^{k} B_{i,k}(t) c_i = \sum_{i=0}^{r} B_{i,r}(t) \sum_{j=0}^{k-r} B_{j,k-r}(t) c_{i+j} \tag{2.2}$$

which can be interpreted as saying that a Bernstein polynomial of degree k can be expressed as a Bernstein polynomial of degree r whose coefficients are themselves polynomials of degree $k - r$. This identity is intimately related to the well known *de Casteljau algorithm* for evaluation of Bernstein polynomials and Bézier curves.

Applying (2.2) with $k = n$ and $r = n - 1$, we deduce that

$$f_x = m \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} B_{i,m-1}(x) B_{j,n-1}(y) \sum_{\ell=0}^{1} B_{\ell,1}(y) \Delta_{10} c_{i,j+\ell}$$

and similarly,

$$f_y = n \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} B_{i,m-1}(x) B_{j,n-1}(y) \sum_{k=0}^{1} B_{k,1}(x) \Delta_{01} c_{i+k,j}.$$

Since Bernstein basis functions sum to one, it follows that

$$D_d f = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sum_{k,\ell=0}^{1} B_{i,m-1}(x) B_{j,n-1}(y) B_{k,1}(x) B_{\ell,1}(y) \big(m d_1 \Delta_{10} c_{i,j+\ell} + n d_2 \Delta_{01} c_{i+k,j} \big).$$

Thus if

$$md_1\Delta_{10}c_{i,j+\ell} + nd_2\Delta_{01}c_{i+k,j} \geq 0, \qquad k,l \in \{0,1\}, \tag{2.3}$$

for all $i = 0,\ldots,m-1$, $j = 0,\ldots,n-1$, then f is increasing in the direction d.

Is there a geometric interpretation of condition (2.3)? To answer this question it is now time to formally introduce the *control net* of f which is nothing more than the unique function $\hat{f} : [0,1] \times [0,1]$ which is bilinear over each rectangle

$$R_{ij} = [\frac{i}{m}, \frac{i+1}{m}] \times [\frac{j}{n}, \frac{j+1}{n}], \qquad i = 0,\ldots,m-1, \quad j = 0,\ldots,n-1,$$

and satisfies the interpolation conditions

$$\hat{f}(\frac{i}{m}, \frac{j}{n}) = c_{ij}, \qquad i = 0,\ldots,m, \quad j = 0,\ldots,n,.$$

A straightforward computation, carried out in [**FP**], shows that $\hat{f}$ is increasing in the direction d over the rectangle R_{ij} if and only if condition (2.3) holds. This then provides our geometric interpretation and we conclude as follows.

Theorem 2.1. *The tensor-product Bernstein basis preserves monotonicity. In other words, if $\hat{f}$ is increasing in a direction d then so is f.*

At this stage one could understandably ask 'what is so special about Bernstein polynomials'? Are there other tensor-product systems of functions that preserve monotonicity? Indeed we could consider the more general function

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}u_i(x)v_j(y), \qquad (x,y) \in [0,1] \times [0,1],$$

where $U = (u_0,\ldots,u_n)$ and $V = (v_0,\ldots,v_n)$ are any two sequences of functions on $[0,1]$. A sufficient condition for monotonicity preservation for systems of functions of one variable (such as U) is that they sum to one and are *totally positive* (see [**CP**]) but it has been established in [**FP**] that total positivity of both U and V is not enough to guarantee monotonicity preservation of the corresponding tensor-product system. Moreover it was shown in [**FP**] that a necessary condition for monotonicity preservation is that both systems U and V *preserve linearity*, that is to say that if the coefficients of f are sampled from a linear function, i.e. $c_{ij} = g(i/m, j/n)$ for some linear function $g(x,y) = Ax+By+C$, then f is itself linear. This condition rules out several known systems of functions in computer-aided design. For example, the tensor-product system of trigonometric functions formed by $U = (c_0^m,\ldots,c_m^m)$ and $V = (c_0^n,\ldots,c_n^n)$ where

$$c_j^k(t) = \binom{k}{j}\cos^{2(k-j)}(t)\sin^{2j}(t), \qquad j = 0,\ldots,k, \quad t \in [0,1],$$

does not preserve monotonicity because t is not in the span of $\{1, \cos t, \ldots, \sin t\}$, see [**FP**]. In fact this is not so surprising when one differentiates the corresponding function

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}c_i^m(x)c_j^n(y),$$

using formulas in [**P**] analogous to those of Bernstein polynomials, to yield

$$D_d f = \frac{\pi}{2} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sum_{k,\ell=0}^{1} c_i^{m-1}(x) c_j^{n-1}(y) c_k^1(x) c_\ell^1(y) \big(d_1 m \sin x \Delta_{10} c_{i,j+\ell} + d_2 n \sin y \Delta_{01} c_{i+k,j}\big).$$

However the trigonometric system does preserve monotonicity in the two axial directions $d = (1,0)$ and $d = (0,1)$, see [**FP**].

So are there any monotonicity preserving tensor-product systems other than Bernstein polynomials? Well, taking into account the fact that Bernstein polynomials are a special case of *spline* functions, it may not come as a great surprise that *tensor-product B-splines* also preserve monotonicity, albeit as long as the control net is defined to reflect the knot vectors in an appropriate way. In fact it was shown in [**FP**] that if the control net of a tensor-product spline function is defined by placing the coefficients at corresponding knot averages in each of the two variables then Theorem 2.1 also holds for spline functions.

3. Convexity

We saw in the previous section how a tensor-product polynomial f in Bernstein form is monotonically increasing in a direction d if its control net is. It is therefore natural also to suppose that f might be *convex* whenever its control net is convex. This is indeed true and was established by Cavaretta and Sharma:

Theorem 3.1 [CS]. *Tensor-product Bernstein polynomials preserve convexity.*

At first sight this result seems to be a pure analogy of monotonicity preservation. Yet, as we will see, there is a fundamental difference. In a certain sense, Theorem 3.1 is really an example of 'univariate' shape preservation. In order to appreciate this flaw, one needs to take a closer look at the control net $\hat{f}$ of f when $\hat{f}$ is convex. Since, over any rectangle R_{ij}, $\hat{f}$ is bilinear, the two second partial derivatives $\partial^2 \hat{f}/\partial x^2$ and $\partial^2 \hat{f}/\partial y^2$ are zero and so, if $\hat{f}$ is convex there, its mixed partial derivative must also be zero. This implies that

$$0 = \frac{\partial^2 \hat{f}}{\partial x \partial y} = mn(c_{ij} - c_{i+1,j} - c_{i,j+1} + c_{i+1,j+1})$$

and by induction on i and j we must have

$$c_{ij} = c_{i0} + c_{j0} - c_{00}. \tag{3.1}$$

Considering each row one can then deduce that $\hat{f}$ is convex if and only if

$$\Delta_{20} c_{i,j} \geq 0, \qquad i = 0, \ldots, m-2, \quad j = 0, \ldots, n,$$

$$\Delta_{02} c_{i,j} \geq 0, \qquad i = 0, \ldots, m, \quad j = 0, \ldots, n-2,$$

and

$$\Delta_{11} c_{i,j} = 0, \qquad i = 0, \ldots, m-1, \quad j = 0, \ldots, n-1, \tag{3.2}$$

where

$$\Delta_{20}c_{i,j} = \Delta_{10}\Delta_{10}c_{i,j} = c_{ij} - 2c_{i+1,j} + c_{i+2,j},$$
$$\Delta_{11}c_{i,j} = \Delta_{10}\Delta_{01}c_{i,j} = c_{ij} - c_{i+1,j} - c_{i,j+1} + c_{i+1,j+1},$$
$$\Delta_{02}c_{i,j} = \Delta_{01}\Delta_{01}c_{i,j} = c_{ij} - 2c_{i,j+1} + c_{i,j+2}.$$

Moreover equation (3.1) implies that

$$f(x,y) = \sum_{i=0}^{m} c_{i0}B_{i,m}(x) + \sum_{j=0}^{n} c_{0j}B_{j,n}(y) - c_{00}.$$

We now see the degeneracy in Theorem 3.1; for the control net of f to be convex, f must belong to a very small class of bivariate polynomials, namely those which are a sum of a polynomial in x and a polynomial in y. The function f in this case is said to *translational*, since any row of coefficients c_{ij} differs from any other row only by a constant.

If convexity of the control net is viewed as a sufficient *condition* for the convexity of f then what Theorem 3.1 gives us is a very *strong* convexity condition. Only translational f can hope to satisfy it.

This brings us to a second aspect of this paper. In addition to shape *preservation* we are interested in *sufficient conditions* on the control net, preferably in the form of algebraic inequalities on the coefficients, which ensure a certain shape property of f. Such conditions can be used as the starting point for *constrained* scattered data approximation. For example, one might wish to find an approximation f to given data (x_i, y_i, z_i), $i = 1, \ldots, N$, under the constraint that f be convex (which would certainly be a sensible constraint if the data were known to have been sampled from a convex function). One method of solving such a problem is constrained *least squares* approximation; see [**J**], [**F3**], [**K**]. For such a method to be effective, the convexity conditions should be as *weak* as possible, otherwise the approximation will be poor. The conditions should also preferably be *linear* inequalities, to ease computation.

For the purpose of constrained scattered data approximation, the condition that the control net is convex is clearly too strong and this has motivated the development of weaker convexity conditions. The following approach to finding convexity conditions, taken from [**F1**], follows the same general plan as for monotonicity in the previous section; we use the de Casteljau algorithm to represent partial derivatives in a uniform way. This time we compute partial derivatives of order 2:

$$f_{xx} = m(m-1)\sum_{i=0}^{m-2}\sum_{j=0}^{n} B_{i,m-2}(x)B_{j,n}(y)\Delta_{20}c_{ij},$$

$$f_{xy} = mn\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} B_{i,m-1}(x)B_{j,n-1}(y)\Delta_{11}c_{ij},$$

$$f_{yy} = n(n-1)\sum_{i=0}^{m}\sum_{j=0}^{n-2} B_{i,m}(x)B_{j,n-2}(y)\Delta_{02}c_{ij}.$$

As in Section 2 we can use (2.2) to express these three derivatives as Bernstein polynomials of degree $(m-2)\times(n-2)$ with variable coefficients. Then, recalling that the second order directional derivative of f in the direction $d=(d_1,d_2)$ is given by

$$D_d^2 f = d_1^2 f_{xx} + 2d_1 d_2 f_{xy} + d_2^2 f_{yy},$$

one can establish that

$$D_d^2 f = \sum_{i=0}^{m-2}\sum_{j=0}^{n-2} B_{i,m-2}(x)B_{j,n-2}(y)A_{i,j}(x,y),$$

where

$$\begin{aligned} A_{i,j}(x,y) = m(m-1)d_1^2 &\sum_{\ell=0}^{2} B_{\ell,2}(y)\Delta_{20}c_{i,j+\ell} \\ &+ 2mnd_1d_2 \sum_{k,\ell=0}^{1} B_{k,1}(x)B_{\ell,1}(y)\Delta_{11}c_{i+k,j+\ell} \\ &+ n(n-1)d_2^2 \sum_{k=0}^{2} B_{k,2}(x)\Delta_{02}c_{i+k,j}. \end{aligned}$$

Clearly if the quadratic polynomial $A_{i,j}(x,y)$ is nonnegative for all $x,y\in[0,1]$ and for all $i=0,\dots,m-2$, $j=0,\dots,n-2$ and $d_1,d_2\in\mathbf{R}$, then f is convex. It remains to find a discrete set of sufficient conditions, i.e. independent of x, y, d_1, and d_2. We consider two approaches.

Our first approach is to degree-raise the terms of $A_{i,j}$ in order to represent it as a biquadratic polynomial,

$$\begin{aligned} A_{i,j}(x,y) = \sum_{k=0}^{2}\sum_{\ell=0}^{2} B_{k,2}(x)B_{\ell,2}(y)\Big(& m(m-1)d_1^2\Delta_{20}c_{i,j+\ell} + \\ & 2mnd_1d_2a_{i,j,k,\ell} + n(n-1)d_2^2\Delta_{02}c_{i+k,j}\Big), \end{aligned}$$

where

$$\begin{aligned} a_{i,j,k,\ell} = \frac{1}{4}\Big(&(2-k)(2-\ell)\Delta_{11}c_{i+k,j+\ell} + k(2-\ell)\Delta_{11}c_{i+k-1,j+\ell} \\ &+ (2-k)\ell\Delta_{11}c_{i+k,j+\ell-1} + k\ell\Delta_{11}c_{i+k-1,j+\ell-1}\Big). \end{aligned} \tag{3.3}$$

Then a sufficient condition for f to be convex is the following. For $m,n\ge 2$, let

$$K_{mn} = \frac{mn}{(m-1)(n-1)}.$$

Theorem 3.2. *Suppose $m, n \geq 2$. If*

$$\Delta_{20} c_{i,j} \geq 0, \qquad i = 0, \ldots, m-2, \quad j = 0, \ldots, n,$$

$$\Delta_{02} c_{i,j} \geq 0, \qquad i = 0, \ldots, m, \quad j = 0, \ldots, n-2,$$

and for all $k, \ell \in \{0, 1, 2\}$,

$$\Delta_{20} c_{i,j+\ell} \Delta_{02} c_{i+k,j} \geq K_{mn} a_{i,j,k,\ell}^2, \qquad i = 0, \ldots, m-2, \quad j = 0, \ldots, n-2, \tag{3.4}$$

then f is convex.

A close inspection of equation (3.3) reveals that $a_{i,j,k,\ell}$ is in every case a first order difference in both i and j and there are only three types, up to symmetry:

$$\begin{aligned} a_{ij00} &= c_{i,j} - c_{i+1,j} - c_{i,j+1} + c_{i+1,j+1}, \\ a_{ij10} &= 2(c_{i,j} - c_{i+2,j} - c_{i,j+1} + c_{i+2,j+1}), \\ a_{ij11} &= 4(c_{i,j} - c_{i+2,j} - c_{i,j+2} + c_{i+2,j+2}). \end{aligned}$$

Thus ignoring symmetries, there are three kinds of inequality appearing in (3.4):

$$\Delta_{20} c_{i,j} \Delta_{02} c_{i,j} \geq K_{mn} a_{i,j,0,0}^2,$$

$$\Delta_{20} c_{i,j} \Delta_{02} c_{i+1,j} \geq K_{mn} a_{i,j,1,0}^2,$$

$$\Delta_{20} c_{i,j+1} \Delta_{02} c_{i+1,j} \geq K_{mn} a_{i,j,1,1}^2,$$

and the coefficients involved (at most nine) are indicated in Figure 3.1.

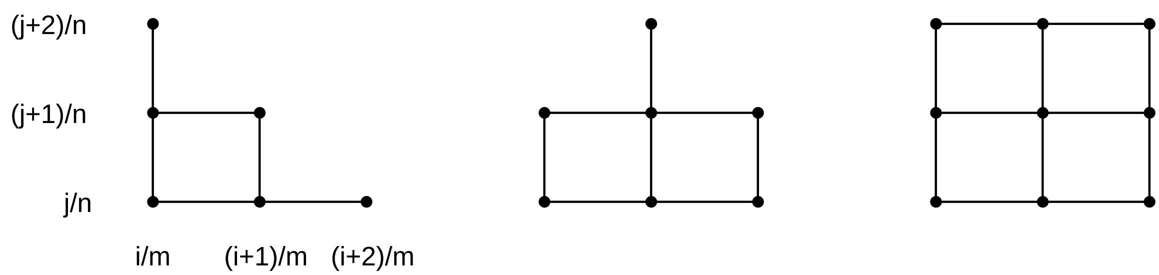

Figure 3.1. Coefficients in condition (3.4)

Our second approach is to again apply (2.2) in order to get a common representation for the second partial derivatives and it provides a set of conditions which are more homogeneous. Specifically, applying (2.2) to the first and last sums in $A_{i,j}$, with $k = 2$ and $r = 1$, we find

$$A_{i,j}(x, y) = \sum_{k,\ell,r,s=0}^{1} B_{k,1}(x) B_{\ell,1}(y) B_{r,1}(x) B_{s,1}(y) b_{i,j,k,\ell,r,s},$$

where

$$b_{i,j,k,\ell,r,s} = m(m-1) d_1^2 \Delta_{20} c_{i,j+\ell+s} + 2mn d_1 d_2 \Delta_{11} c_{i+k,j+\ell} + n(n-1) d_2^2 \Delta_{02} c_{i+k+r,j},$$

which implies the following.

Theorem 3.3 ([F1]). *Suppose $m, n \geq 2$. If*

$$\Delta_{20} c_{i,0} \geq 0, \qquad i = 0, \ldots, m-2, \quad j = 0, \ldots, n,$$

$$\Delta_{02} c_{0,j} \geq 0, \qquad i = 0, \ldots, m, \quad j = 0, \ldots, n-2,$$

and for all $k, \ell, r, s \in \{0, 1\}$,

$$\Delta_{20} c_{i,j+\ell+s} \Delta_{02} c_{i+k+r,j} \geq K_{mn} (\Delta_{11} c_{i+k,j+\ell})^2, \qquad i = 0, \ldots, m-2, \quad j = 0, \ldots, n-2, \tag{3.5}$$

then f is convex.

Before looking more closely at the conditions in this theorem, let us emphasize their usefulness by pointing out that they have a straightforward generalization to conditions for the convexity of tensor-product splines and we refer the reader to [**F1**]. The conditions of Theorem 3.3 have also been generalized to convexity conditions for *parametric* spline surfaces in [**KK**].

There are only three types of inequality, up to symmetry, appearing in (3.5):

$$\Delta_{20} c_{i,j} \Delta_{02} c_{i,j} \geq K_{mn} (\Delta_{11} c_{i,j})^2,$$

$$\Delta_{20} c_{i,j} \Delta_{02} c_{i+1,j} \geq K_{mn} (\Delta_{11} c_{i,j})^2,$$

$$\Delta_{20} c_{i,j+1} \Delta_{02} c_{i+1,j} \geq K_{mn} (\Delta_{11} c_{i,j})^2.$$

In all three cases, only six coefficients are involved and the three configurations are displayed in Figure 3.2. The inequalities in (3.5) are stronger than inequalities (3.4), so the convexity conditions of Theorem 3.3 are stronger than those of Theorem 3.2.

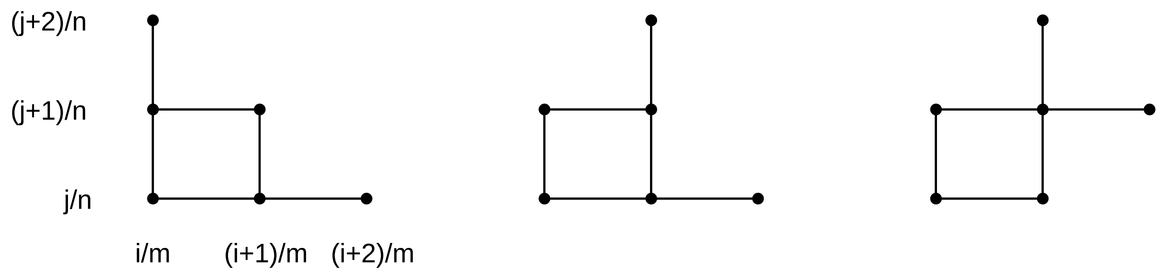

Figure 3.2. Coefficients in condition (3.5)

We have derived two different sets of sufficient conditions for f to be convex and both sets are weaker than the condition that the control net be convex (see [**F1**]). However unlike the latter condition these two new condition sets consist of *quadratic* inequalities rather than linear ones. Let us see now how one can obtain stronger conditions than (3.4) or (3.5) which are *linear* but still do not imply that f be translational.

First of all we note that $K_{mn} \geq 4$ for $m, n \geq 2$ and so a sufficient condition for the conditions of Theorem 3.3 to hold is that the 2×2 symmetric matrix

$$A = \begin{pmatrix} \Delta_{20} c_{i,j+\ell+s} & 2\Delta_{11} c_{i+k,j+\ell} \\ 2\Delta_{11} c_{i+k,j+\ell} & \Delta_{02} c_{i+k+r,j} \end{pmatrix}$$

is positive semi-definite. Since weak diagonal dominance is a sufficient condition for positive semi-definiteness we have

Theorem 3.4 ([CFP]). *The polynomial f is convex if*

$$\Delta_{20} c_{i,j+\ell} \geq 2|\Delta_{11} c_{i+k,j}|, \qquad i = 0, \ldots, m-2, \quad j = 0, \ldots, n-1,$$

$$\Delta_{02} c_{i+k,j} \geq 2|\Delta_{11} c_{i,j+\ell}|, \qquad i = 0, \ldots, m-1, \quad j = 0, \ldots, n-2,$$

for all $k, \ell \in \{0, 1\}$.

Similar to Theorems 2.1 and 3.1, the conditions in Theorem 3.4 have a geometric interpretation in terms of the control net of f. Up to symmetry there is now only *one* inequality involved, adequately represented by taking the first inequality with $k = \ell = 0$, namely

$$\Delta_{20} c_{i,j} \geq 2|\Delta_{11} c_{i,j}|$$

and this can be expressed as the two *linear* inequalities

$$\Delta_{20} c_{i,j} \geq 2\Delta_{11} c_{i,j}, \qquad \Delta_{20} c_{i,j} \geq -2\Delta_{11} c_{i,j}.$$

Furthermore, these two linear inequalities can be expressed in the form

$$\frac{2c_{i,j+1} + c_{i+2,j}}{3} \geq \frac{c_{i,j} + 2c_{i+1,j+1}}{3}, \tag{3.6}$$

$$\frac{c_{i,j} + (c_{i+2,j} + 2c_{i+1,j+1})/3}{2} \geq \frac{2c_{i+1,j} + c_{i,j+1}}{3}. \tag{3.7}$$

Now let $\alpha_{kl} = (k/m, l/n)$ and consider the quadrilateral $S \cup T$ consisting of the two triangles $S = \Delta\alpha_{ij}\alpha_{i+1,j+1}\alpha_{i,j+1}$ and $T = \Delta\alpha_{ij}\alpha_{i+2,j}\alpha_{i+1,j+1}$ depicted on the left of Figure 3.3. Further, let $q : S \cup T \to \mathbf{R}$ be the function which is linear over each of the two triangles and satisfies $q(\alpha_{kl}) = c_{kl}$ at the four vertices. Then by considering the two diagonal line segments, $[\alpha_{ij}, \alpha_{i+1,j+1}]$ and $[\alpha_{i,j+1}, \alpha_{i+2,j}]$ of the quadrilateral $S \cup T$, one can show that (3.6) is equivalent to the geometric condition that q is convex; see [**CFP**].

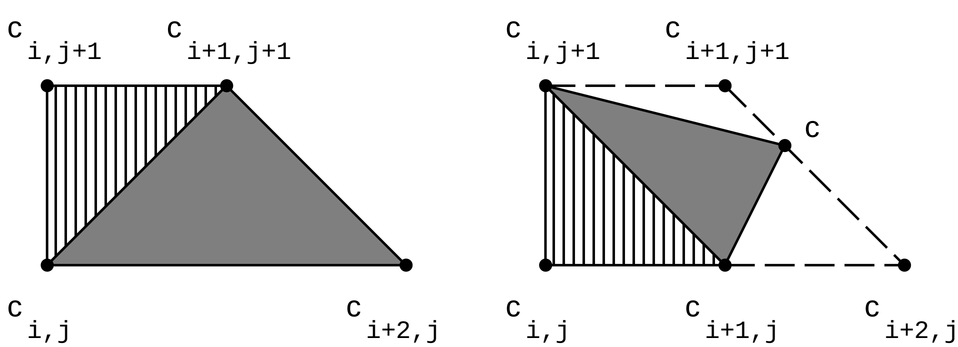

Figure 3.3. Geometric interpretation of the linear inequalities

In order to interpret (3.7) we can first define the intermediate point $\alpha = (\alpha_{i+2,j} + 2\alpha_{i+1,j+1})/3$, and corresponding 'coefficient' $c = (c_{i+2,j} + 2c_{i+1,j+1})/3$. Then (3.7) is equivalent to the convexity of the piecewise linear function over the quadrilateral $S \cup T$, where $S = \Delta\alpha_{ij}\alpha_{i+1,j}\alpha_{i+1,j}$ and $T = \Delta\alpha_{i+1,j}\alpha_{i,j+1}\alpha$; as indicated on the right of Figure 3.3.

We have thus achieved our goal of constructing linear convexity conditions which do not force f to be translational. Moreover, the conditions have a natural geometric interpretation in terms of the convexity of piecewise linear functions over quadrilaterals defined by the control net.

Yet, as pointed out in [**CFP**], it must be possible to find even weaker linear convexity conditions. Indeed, by identifying the condition of positive-semidefiniteness of a 2×2 matrix with the planar region enclosed by a parabola, one can generate weaker and weaker sufficient conditions by approximating the parabola by polygons with more and more vertices. Jüttler [**J**] has proposed an algorithm for generating arbitrarily weak convexity conditions for tensor-product spline functions and has applied them to constrained least squares approximation.

Finally we mention that establishing convexity preservation, embodied in Theorem 3.1, was a somewhat secondary result in [**CS**]. Cavaretta and Sharma's main result was to show that tensor-product polynomials satisfy a certain variation diminishing property which can be viewed as the diminution of the total variation of gradients of f (see [**G**] for a more precise interpretation and further discussion). Specifically, it was shown in [**CS**] that

$$V(f) \leq V(\hat{f})$$

where, for a C^2 function g over $[0,1]\times]0,1]$, the variation of g is defined by

$$V(g) = \int_0^1 \int_0^1 (f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2) dx dy.$$

A specific formula for $V(\hat{f})$ is given in [**CS**] because, since $\hat{f}$ is only piecewise C^2, its derivatives need to be interpreted as distributions.

4. Zero Curves

In this final section, we look at conditions on f in (1.1) which bound the number of connected components of its zero set Z_f in (1.2). The motivation for this comes from the growing interest in computer-aided design in representations of implicitly defined curves (and surfaces), an example of which is the A-spline described in [**BX**].

A classical property of *univariate* Bernstein polynomials is *variation diminution.* The Bernstein basis $B_{0,n}, B_{1,n}, \ldots, B_{n,n}$ is said to be *variation diminishing* because the number of (weak) sign changes in the interval $[0,1]$ of any Bernstein polynomial

$$g(x) = \sum_{i=0}^{n} c_i B_{i,n}(x)$$

is less than or equal to the number of (strong) sign changes in the sequence $c_0, c_1, \ldots, c_n$.

In the special case of *one* (strong) sign change in the coefficients, it is not difficult to show that there is a stronger implication: the function g must have precisely one zero x_0, say, in $[0,1]$ and moreover $g'(x_0) \neq 0$. In this section we generalize, in a certain sense, this property of zeros to tensor-product Bernstein polynomials.

Let us suppose that there is some $x_0 \in (0,1)$ such that the coefficients of f in (1.1) satisfy the condition

$$c_{ij} < 0 \qquad \text{for } i/m < x_0, \tag{4.1}$$

and

$$c_{ij} > 0 \qquad \text{for } i/m > x_0. \tag{4.2}$$

We note that if $x_0 = k/m$ for some $k = 1, \ldots, m-1$ then the values of c_{kj} are unimportant. Then, letting

$$d_i(y) = \sum_{j=0}^{n} c_{ij} B_{j,n}(y),$$

we have under this condition that for $y \in [0,1]$,

$$d_i(y) < 0 \qquad \text{for } i/m < x_0,$$

and

$$d_i(y) > 0 \qquad \text{for } i/m > x_0.$$

This implies, by the variation diminishing property of Bernstein polynomials, that

$$f(x,y) = \sum_{i=0}^{m} d_i(y) B_{i,m}(x),$$

when regarded as a function of x, has precisely one zero $p(y)$ say in $(0,1)$ and $f(x,y) < 0$ for $x < p(y)$ and $f(x,y) > 0$ for $x > p(y)$. Moreover $f_x(p(y),y) > 0$. We deduce by the Implicit Function Theorem that the function $p : [0,1] \to (0,1)$ is analytic. Moreover we have shown that the only zeros of f in its domain $[0,1] \times [0,1]$ are the points of the curve $(p(y),y)_{0 \le y \le 1}$.

For example, on the left of Figure 4.1, $m = 4$ and $n = 3$ and the coefficients of f satisfy conditions (4.1) and (4.2) provided we choose $x_0 = 1/4$. Consequently the zeros of f form a curve of the form $x = p(y)$, as suggested on the right of Figure 4.1.

$$\begin{array}{ccccc}
- & - & + & + & + \\
- & + & + & + & + \\
- & - & + & + & + \\
- & - & + & + & +
\end{array}$$

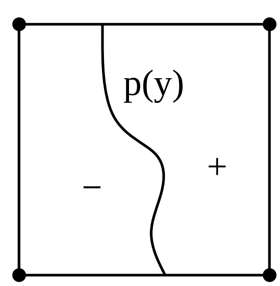

Figure 4.1. Signs of the coefficients and the zeros of f

We can interpret condition (4.1–4.2) as saying that the straight line $x = x_0$ splits the control net p of f into two halves in such a way that the coefficients c_{ij} in one half are negative and those in the other positive. Somewhat surprisingly, it turns out that a similar shape property holds for *any* straight line which splits the coefficients of the control net into negative and positive ones.

Any straight line other than one of the form $x = x_0$ can be expressed as

$$y = ax + b, \qquad a, b \in \mathbf{R}. \tag{4.3}$$

Suppose, then, that such a line intersects the interior of the domain $[0,1] \times [0,1]$ of f in such a way that all coefficients on one side of the line are negative and those on the other positive. Then, for the sake of simplicity, we may assume that

$$c_{ij} < 0 \qquad \text{for } \tfrac{j}{n} < a\tfrac{i}{m} + b, \tag{4.4}$$

and

$$c_{ij} > 0 \qquad \text{for } \tfrac{j}{n} > a\tfrac{i}{m} + b, \tag{4.5}$$

(otherwise the signs are simply reversed). The following property on the zeros of f can then be established.

Theorem 4.1 ([F2]). *If the coefficients of f in (1.1) satisfy conditions (4.4) and (4.5) then its zero set Z_f in (1.2) is a simple curve.*

To see this we consider the family of curves $(x_\alpha(y), y)_{0 \le y \le 1}$ where, for each real $\alpha > 0$,

$$x_\alpha(y) = \frac{\alpha y^\beta}{(1-y)^\beta + \alpha y^\beta}, \qquad \beta = -\frac{na}{m}. \tag{4.6}$$

Along such a curve,

$$f(x_\alpha(y), y) = \left((1-y)^\beta + \alpha y^\beta\right)^{-m} \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} \binom{m}{i} \binom{n}{j} \alpha^i y^{i\beta+j} (1-y)^{m\beta+n-(i\beta+j)}.$$

Thus the function

$$g_\alpha(y) := \left((1-y)^\beta + \alpha y^\beta\right)^m f(x_\alpha(y), y), \qquad y \in [0,1],$$

is a generalized Bernstein polynomial. Such a generalized polynomial can be shown to be variation diminishing by transforming it into a Muntz polynomial (see [**F2**]). Since from (4.4) and (4.5),

$$c_{ij} < 0 \qquad \text{for } i\beta + j < nb,$$

and

$$c_{ij} > 0 \qquad \text{for } i\beta + j > nb,$$

the coefficients of g_α have one (strong) sign change and therefore g_α has precisely one zero in $(0,1)$, which we can call $p(\alpha)$. By showing that every point in $[0,1] \times [0,1]$ belongs

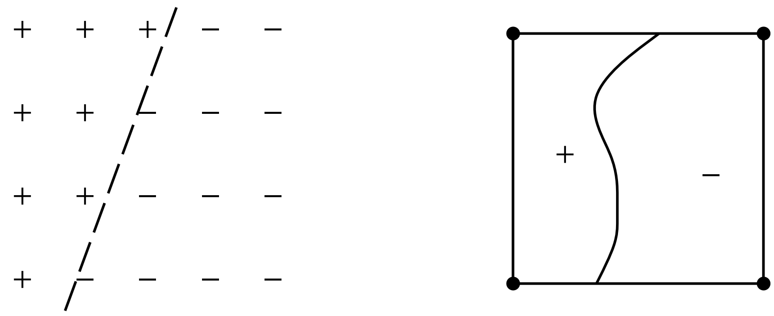

Figure 4.2. Signs of the coefficients and the zeros of f

to precisely one curve x_α (see [**F2**]), one can thus deduce that the *only* zeros of f in $[0,1] \times [0,1]$ consist of the points of the curve $(x_\alpha(p(\alpha)), p(\alpha))_{0 \le \alpha < \infty}$ which is analytic.

Figure 4.2 shows an example where the coefficients satisfy conditions (4.4–4.5) with respect to the line in (4.3) with $a = 8/3$ and $b = -1/2$ and so the zeros of f form a simple curve as suggested to the right of the figure.

The family of curves $(x_\alpha(y), y)_{0 \le y \le 1}$ turns out to have other uses. For example, let

$$N_f = \{(x, y) \in [0,1] \times [0,1] : f(x, y) < 0\}.$$

By varying β as well as α in (4.6), it was shown in [**F2**] that if only one of the coefficients c_{ij} of f is negative and the rest are positive, then if N_f is non-empty, it must be simply connected. Both this property and that of Theorem 4.1 have analogies to Bernstein polynomials over triangles ([**F2**]).

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