

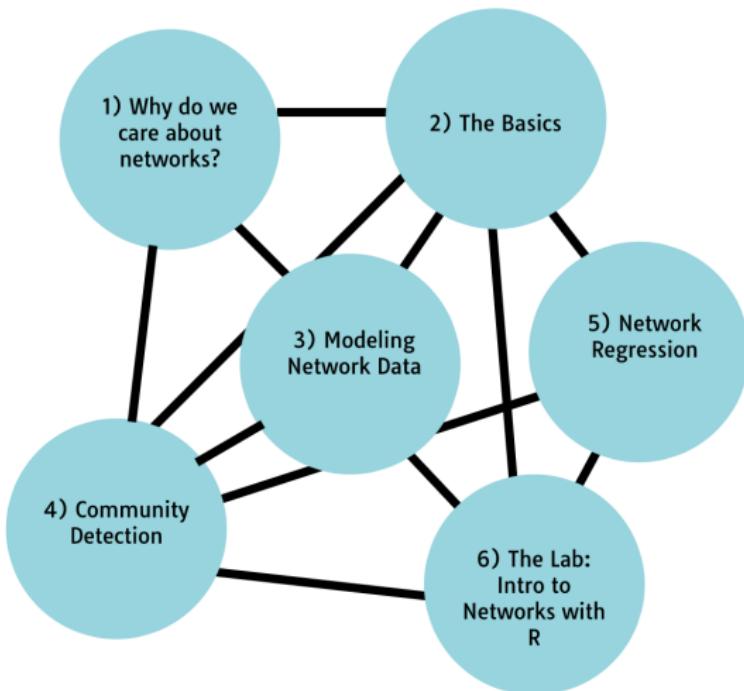
# Introduction to Network Analysis

Heather Mathews

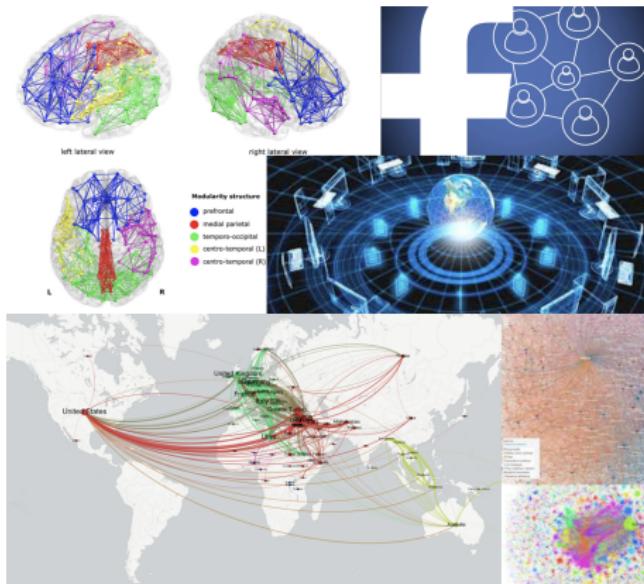
Duke University

November 6, 2019

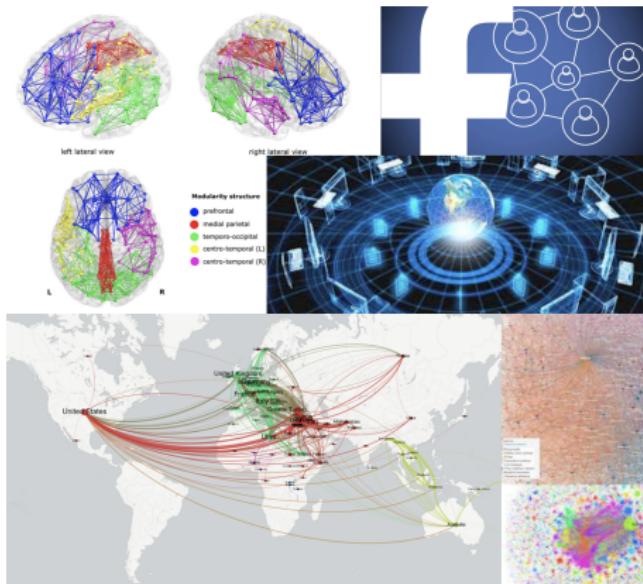
# Outline



# Who uses networks?

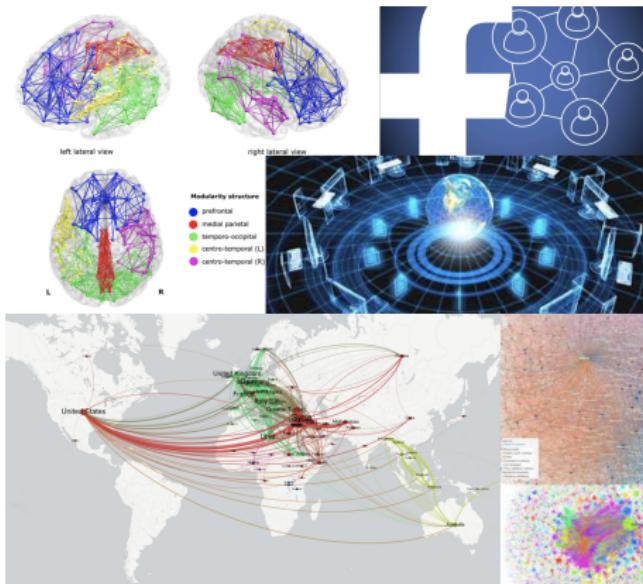


# Who uses networks?



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- Brain networks, social networks, computer networks, traffic networks, trade networks...

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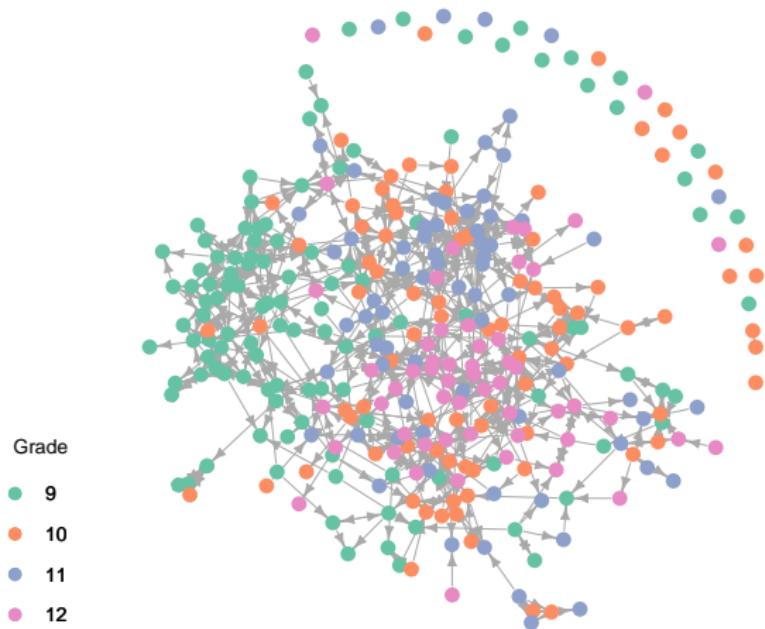
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- Covariate information on students was collected including grade, smoking status, drinking habits, club involvement, GPA, and sport involvement

# AddHealth: Visualizing our Network

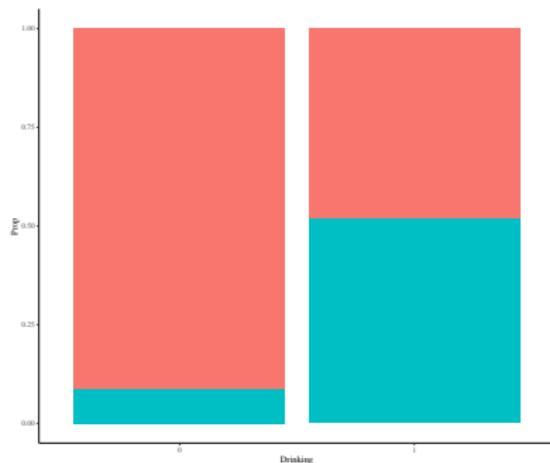


## AddHealth

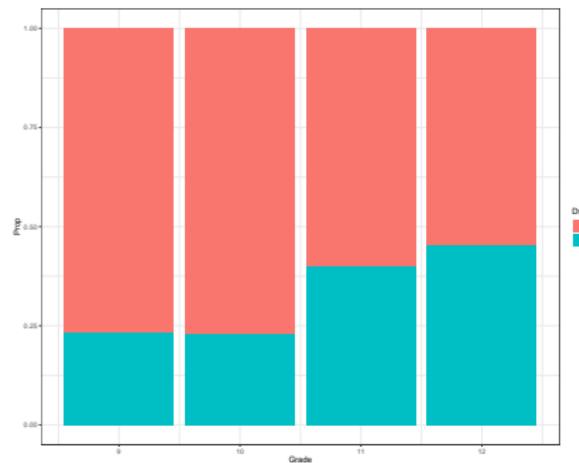
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Smoking and drinking



Drinking by grade

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- Connect covariate estimation and latent communities
  - ▶ Higher GPA might increase popularity within the 'nerdy' clique but decrease popularity within the 'popular' clique



and I got that  
*red lip*  
CLASSIC

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red lip  
**CLASSIC**

## Karate Club Data

### Network Scientists with Karate Trophies



5 MONTHS AGO  
#NETWORKSCIENCE  
#KARATECLUB  
#TROPHY



The first scientist at any conference on networks who uses Zachary's karate club as an example is inducted into the Zachary Karate Club Club, and awarded a prize. This tumblr records those moments.

RSS  
 ARCHIVE

## Motivating Example: Karate Club

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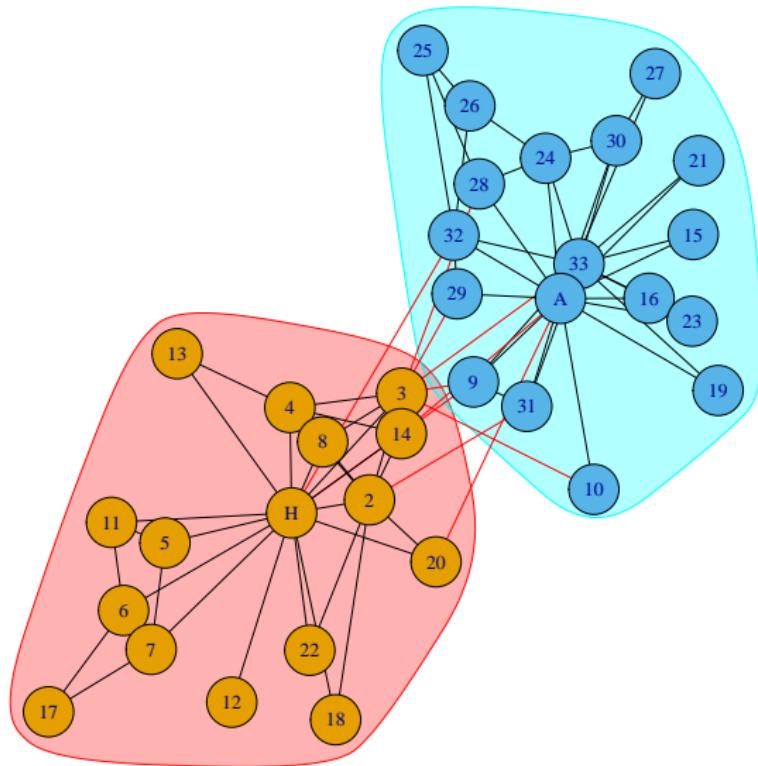
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- He was able to classify 33/34 of the members into either John's or Mr. Hi's club
- Goal: Investigating possible fission in small community setting and interested in how information flows between the 2 clubs
- This gives us ground truth for communities! Which has made it very popular for testing community detection algorithms

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- We can view a pass as a connection in a network
- Really, I am mainly using this example to brag about Duke (sorry Dr. Bailer)
- But in all seriousness, network analysis is popular in sports analytics!

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- **Sociomatrix:**  $n \times n$  matrix,  $A$ , to represent relationships between nodes. If binary entries, this is called an **adjacency matrix**

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

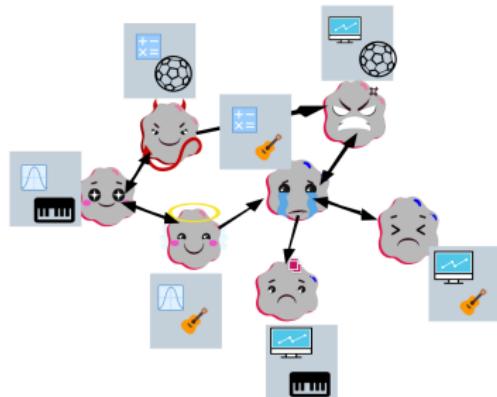
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Adjacency matrix,  $A$



Graph of  $A$

# Formally Defining a Network

- **Graph:**  $G = (V, E)$  with  $V = \{1, \dots, n\}$  and  $E = \{(i, j) \mid 1 \leq j \neq i \leq n\}$ .  $E \subset \mathcal{E}$  (if  $E = \mathcal{E}$ , fully connected graph)
  - ▶ Number of nodes:  $n = |V|$
  - ▶ Number of edges:  $m = |E|$

From	To
1	2
1	3
2	1
2	3
3	1

Example Edgelist

## Directed vs Undirected

- Sometime edges between individuals are reciprocated (that is, if  $i$  is friends with  $j$ ,  $j$  is friends with  $i$ )
- If all edges in our graph are reciprocated, then we have an **undirected network**

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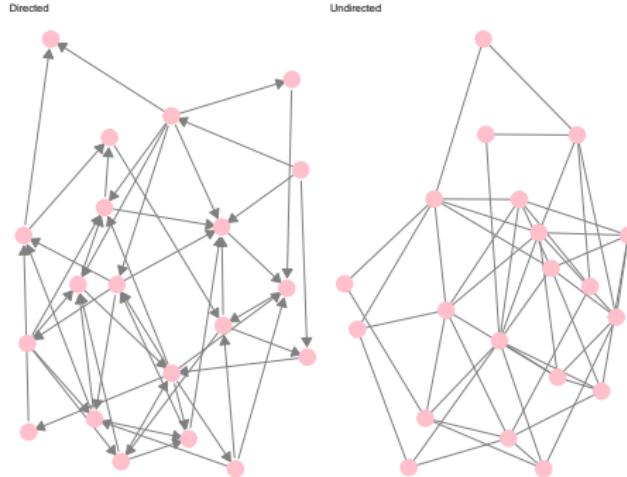
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# Directed vs. Undirected



Directed: Asymmetric adjacency matrix, Undirected: Symmetric adjacency matrix

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**Degree:** Number of nodes a person is connected to

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<b>Degree</b>	$d_i^{out} = \sum_{j:i \neq j} A_{i,j}$ (out) $d_i^{in} = \sum_{j:i \neq j} A_{j,i}$ (in)	$d_i = \sum_{j:i \neq j} A_{i,j}$
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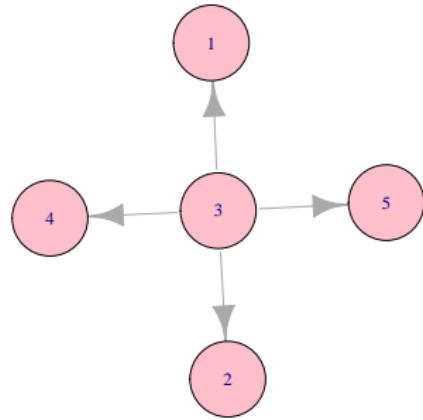
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Undirected matrix,  $A$

# Sociability vs Popularity

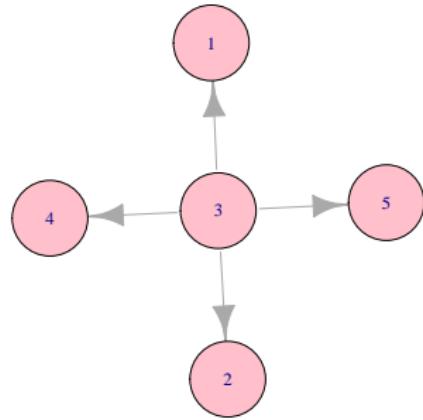
# Sociability vs Popularity

**Node 3 is Social**



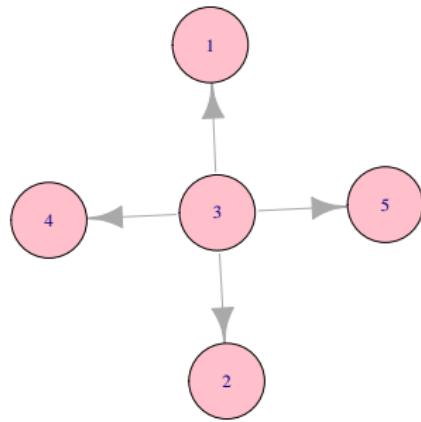
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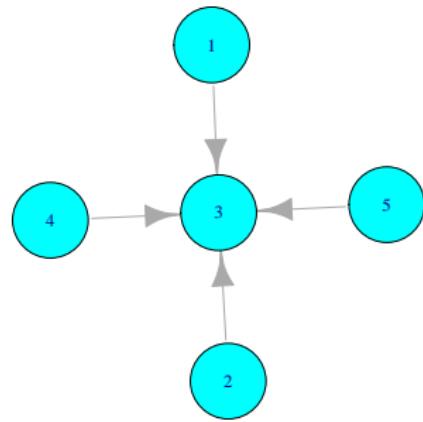


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- **Homophily:** More likely to connect with people who are similar to you

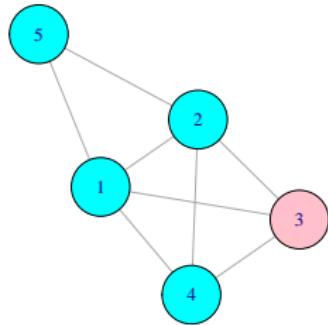
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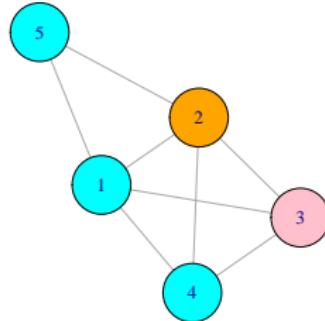
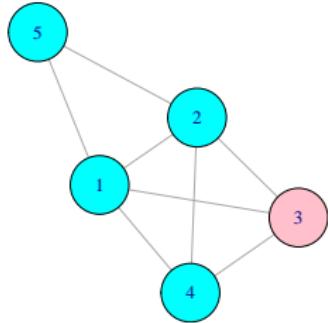


- **Homophily:** More likely to connect with people who are similar to you
- **Transitivity:** Friends of friends have higher probability of being friends

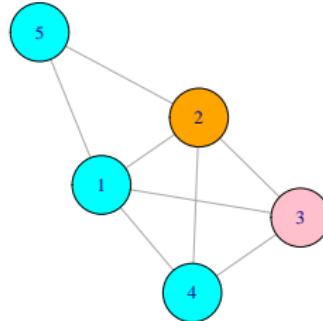
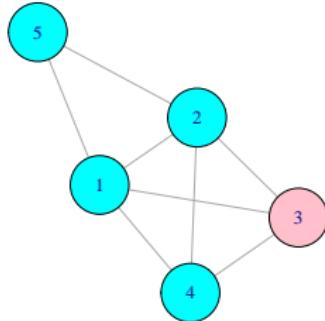
Sorry to say, but your friends have more friends than you...



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	1	2	3	4	5
Degree	4.00	4.00	3.00	3.00	2.00
Avg. Deg. of Friends	3.00	3.00	3.67	3.67	4.00

# Notions of Centrality

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- **Betweenness Centrality:** Looks at how many times a node is part of the shortest path between other nodes

$g_{j,k}$  = # of shortest paths to get from  $j$  to  $k$

$g_{j,k}(i)$  = # of shortest paths from  $j$  to  $k$  that go through  $i$

$$c_i = \sum_{j < k} g_{j,k}(i) / g_{j,k}$$

## Notions of Centrality

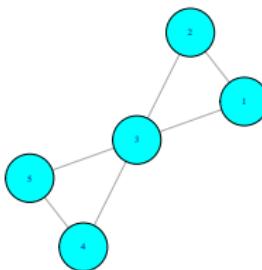
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- ▶ This is useful for finding individuals that are like bridges (flow of info)



Betweenness Centrality: 0 0 4 0 0

## Notions of Centrality

- **Closeness Centrality:** Measures how close one node is to all other nodes in the network. Define  $d_{i,j}$  as the minimum path length from  $i$  to  $j$ .

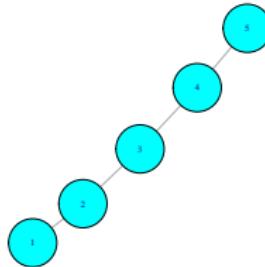
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- ▶ Sum up the shortest paths between all nodes (good for looking at who influences spread of info)



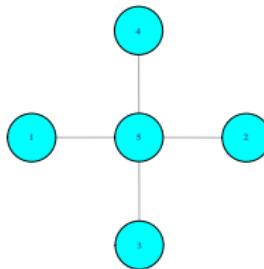
Closeness Centrality: 0.40 0.57 0.67 0.57 0.40

# Notions of Centrality

- **Degree Centrality:** Importance based on number of connections a node has

$$c_i = \sum_{j:j \neq i} A_{i,j}$$

- ▶ Useful for revealing direct connections and locating popular nodes



Degree Centrality 1 1 1 1 4

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- **EigenCentrality:** Centrality of each node is proportional to the sum of its neighbor's centralities

$$c_i = \frac{1}{\lambda} \sum_{j:j \neq i} A_{i,j} c_j$$

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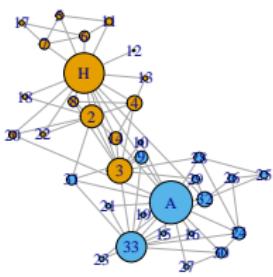
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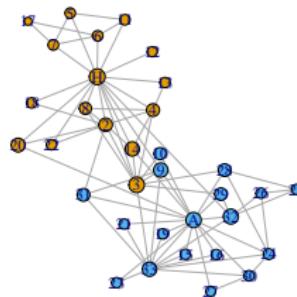
- ▶  $\lambda$  corresponds to the greatest eigenvalue of  $A$  and  $c$  corresponds to the top eigenvector. The  $i^{th}$  component of  $c$  gives the relative centrality score of vertex  $i$
- ▶ Central nodes are connected to other central nodes (very similar to degree centrality)
- ▶ Basis for Google's PageRank
- ▶ For graph on previous slide, eigen centrality is: 0.5 0.5 0.5 0.5 1.0

# Example: Karate Club

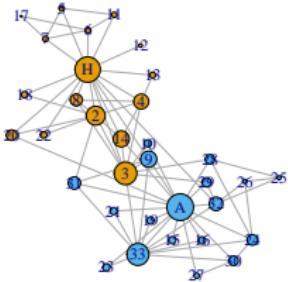
Degree



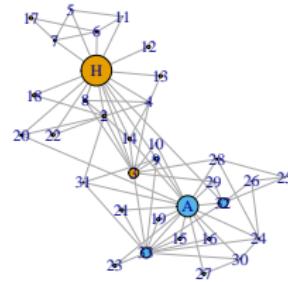
Closeness



Eigen

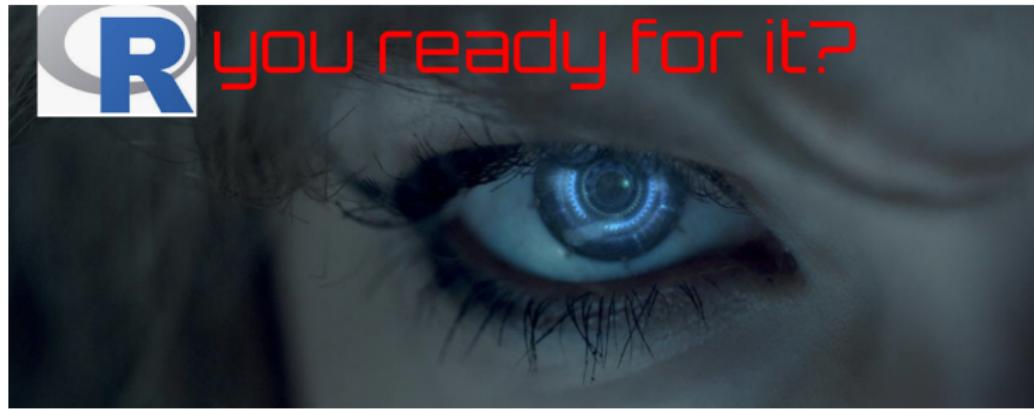


Betweenness



Now off to the lab...

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# Modeling and Generating Networks

- Erdos-Renyi Graph (ER)
- Exponential Random Graph Models (ERGMS)
- Stochastic Block Models (SBM)
- Additive and Multiplicative Effects Network (AMEN)

# Erdos-Renyi Graph

Generating graphs:

- **G(n,p)**: ER graph where edges are drawn independently with probability,  $p$

$$P(G_0) = P(G = G_0) = p^m(1 - p)^{(N-m)}$$
$$E(\# \text{ of Edges}) = \binom{n}{2}p$$

## Relating ER to the Adj. Matrix

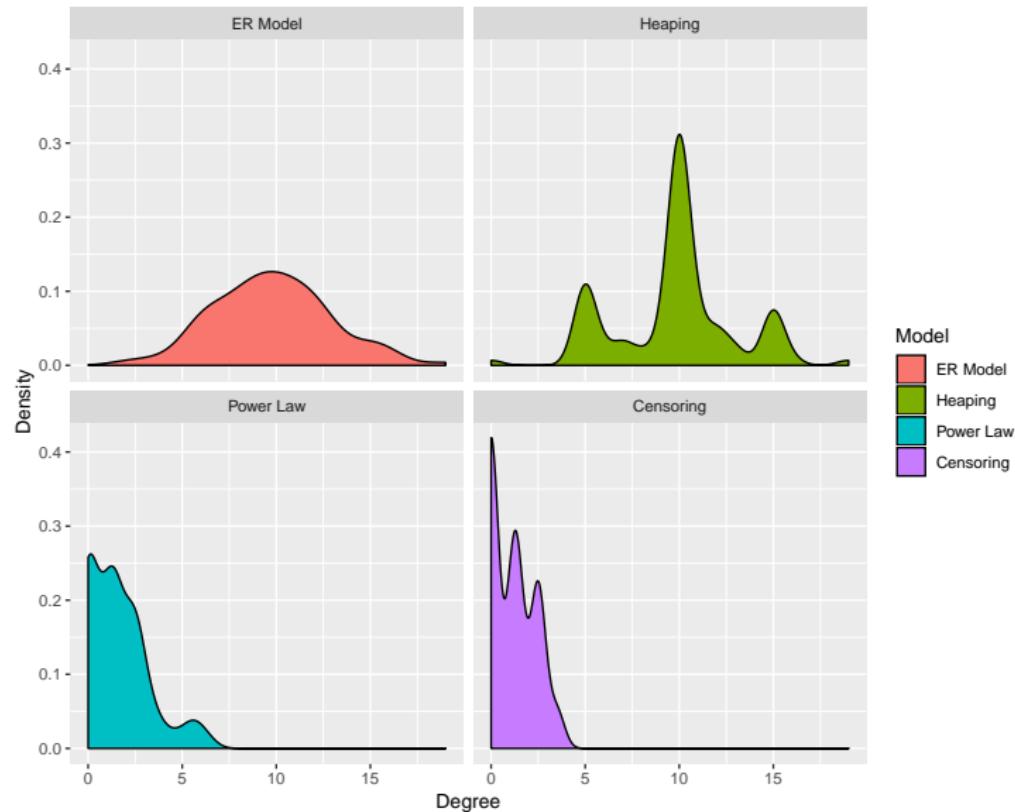
How do the models on the previous slide relate to the adjacency matrix,  $A$ ?

## Relating ER to the Adj. Matrix

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$$A_{i,j}|p \sim Bern(p)$$

# Degree Distribution... and why we care



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- Bin( $n-1, p$ )

Back to the lab...

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# Exponential Random Graph Models: ERGM

Basic generative model for networks that is based off of sufficient statistics

$$P_{\theta}(G_n) = \exp(\theta^T t_n(G_n) - \Psi_n(\theta))$$

where

- $\theta$ : Parameter (or vector of parameters) we want to estimate
- $t_n$ : Sufficient statistic
- $\Psi_n$ : Normalizing constant

# Examples of ERGM

The ER Model:

- In this model, each edge is sampled iid Bernoulli with some probability  $p$ . For this model,  $t_n = \sum_{i < j} A_{i,j}$

$$P_\theta(A_n) \propto \exp\left\{\theta \sum_{i < j} A_{ij}\right\}$$

## ER as an ERGM

We have seen the likelihood for the  $ER(p)$ :

$$\begin{aligned} p^m(1-p)^{N-m} &= \exp\{\log(p^m(1-p)^{N-m})\} \\ &= \exp\{m \times \log(p) + (N-m) \times \log(1-p)\} \\ &= \exp\left\{m \times \log\left(\frac{p}{1-p}\right) + N \times \log(1-p)\right\} \end{aligned}$$

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Recall the ERGM

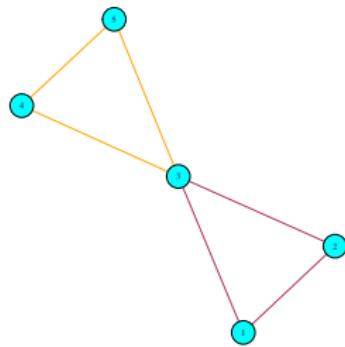
$$P_\theta(A_n) \propto \exp(\theta t_n(A_n)) \quad (1)$$

where

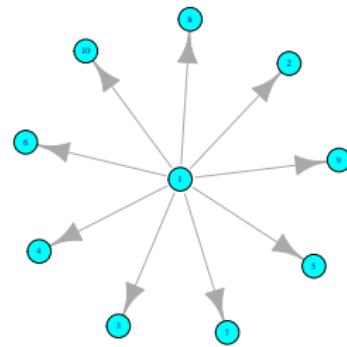
- $\theta = \log\left(\frac{p}{1-p}\right)$
- $t_n = m = \sum_{i < j} A_{i,j}$

## ERGM Cont.

We can also consider some other sufficient statistics to include in our model such as # of stars, # of triangles, # of edges, etc.



Triangles



Star

## What kinds of questions can ERGMs help answer?

- We can look at expectation of degree, edges, degree distribution
- We can test what model fits a new network best (were 2 graphs generated from the same model?)
- HOWEVER, not always consistent estimators :/

Back to the lab...

Back to the lab...

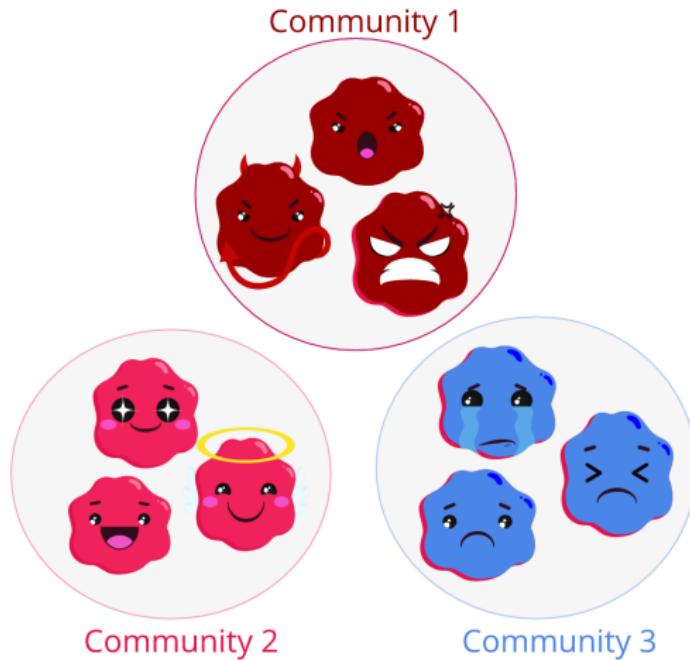


# The Stochastic Block Model

Now we consider if nodes come from **communities** (Holland et al, 1983)

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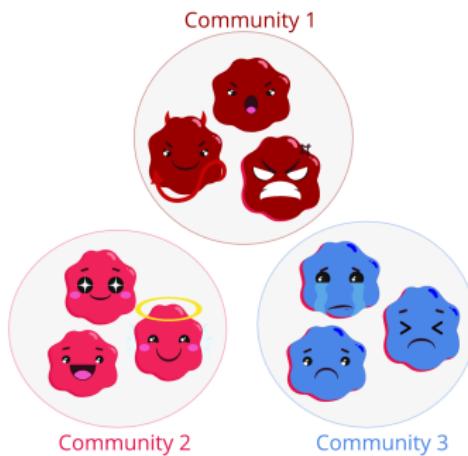
$$\theta = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}$$

## Understanding the SBM Visually

$$B = \begin{pmatrix} 0.4 & 0.05 & 0.3 \\ 0.05 & 0.4 & 0.1 \\ 0.3 & 0.1 & 0.4 \end{pmatrix}$$

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$$P(\text{Angel} \rightarrow \text{Devil} | \text{Angel}, \text{Devil}) = 0.05$$

## Another Generative Model: The Stochastic Block Model

- $\theta = [\theta_1, \theta_2, \dots, \theta_K]$  is a vector containing probabilities that a person belongs to a particular community,  $k \in \{1, \dots, K\}$

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- $B \in \mathbb{R}^{K \times K}$  is a preference matrix that describes the probability of connection of nodes based solely on an individual's membership
- $Z$  indicates which community a person belongs to

$$\begin{aligned} P(Z_i = k) &= \theta_k \\ A_{i,j} | Z_i, Z_j &\sim \text{Bern}(B_{Z_i, Z_j}) \\ P(A, Z, \theta, B) &= \prod_K \theta_k^{\sum 1_{z_i=k}} \prod_{i,j} B_{Z_i, Z_j}^{A_{ij}} (1 - B_{Z_i, Z_j})^{1-A_{ij}} \end{aligned}$$

# Balanced Multi-Label Propagation for Overlapping Community Detection in Social Networks

Authors

Authors and affiliations

Zhi-Hao Wu , You-Fang Lin, Steve Gregory, Huai-Yu Wan, Sheng-Feng Tian

## Mutually Enhancing Community Detection and Sentiment Analysis on Twitter Networks

William Deitrick, Wei Hu\*

## Bayesian Inference and Testing of Group Differences in Brain Networks

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Identifying functional urban regions within traffic flow  
Ed Marley

Pages 40-42 | Received 30 Jan 2014, Accepted 03 Feb 2014, Published online: 12 Mar 2014

Daniele Durante\* and David B. Dunson†

## Promoting Small and Medium Enterprises with a Clustering Approach: A Policy Experience from Indonesia by Tulus Tambunan

Community Detection in General Stochastic Block models: Fundamental Limits and Efficient Algorithms for Recovery

Publisher: IEEE

2 Author(s)

Emmanuel Abbe ; Colin Sandon

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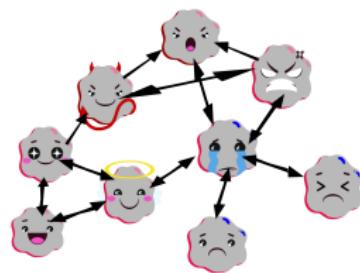
## Social selection and peer influence in an online social network

## Community Detection

- Maybe we believe our network,  $A$ , came from an SBM, and we care about finding community labels

# Community Detection

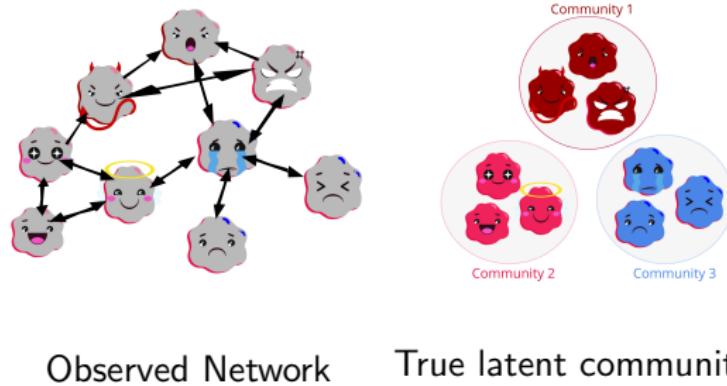
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Observed Network

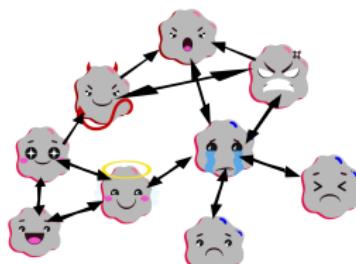
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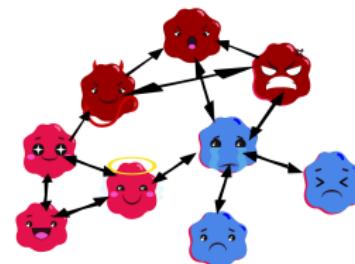
- Maybe we believe our network,  $A$ , came from an SBM, and we care about finding community labels



Observed Network



True latent communities



Goal

# Methods for Community Detection

- Spectral methods
- Gaussian Mixture Models
- Centrality based approaches

# Spectral Clustering

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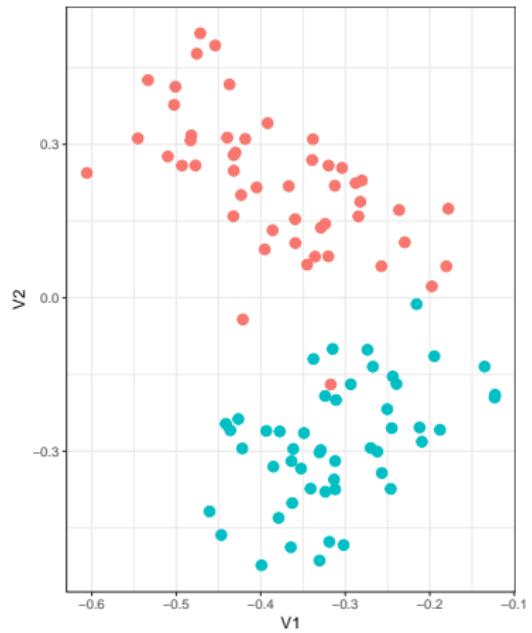
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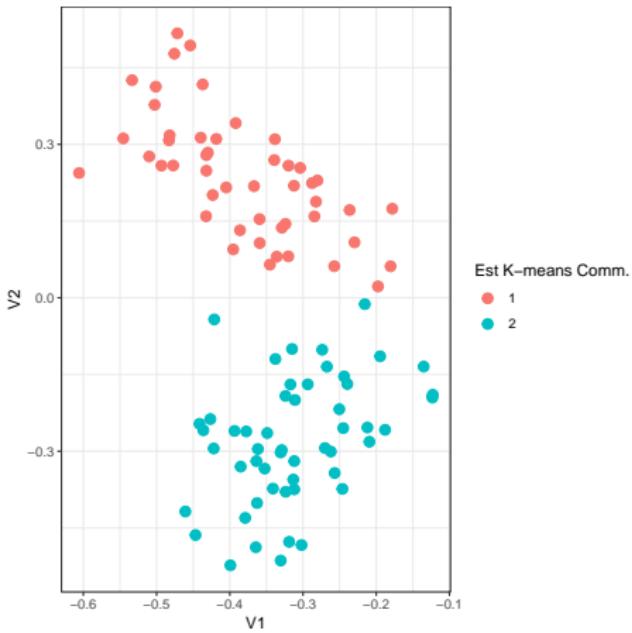
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- If community structure exists, it should presumably show up in a lower dimensional representation of  $A$
- Consider taking the top  $K$  eigenvectors of  $A$  where  $K$  is the number of hypothesized communities that exist

# Looking at our eigenvectors



True Comm.  
● 1  
● 2



Est K-means Comm.  
● 1  
● 2

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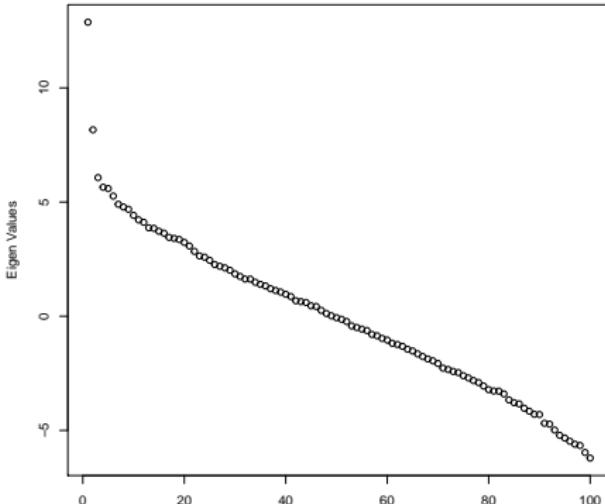
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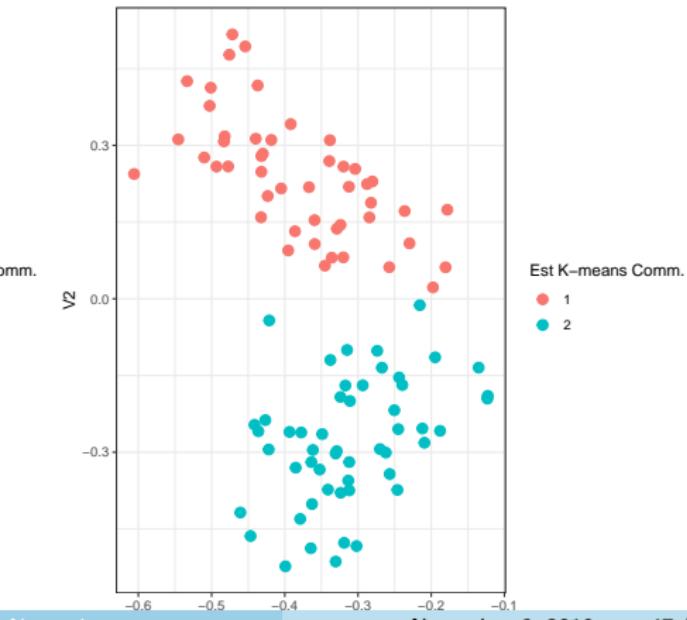
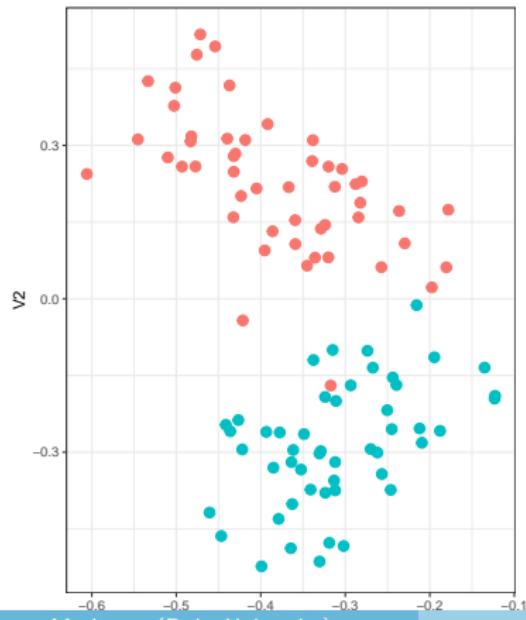
## Once we have our number of communities... [Rohe et al., 2012]

- Run a clustering algorithm on the top  $K$  eigenvectors corresponding to the top  $K$  eigenvalues (in magnitude),  $V_{:,1:K}$
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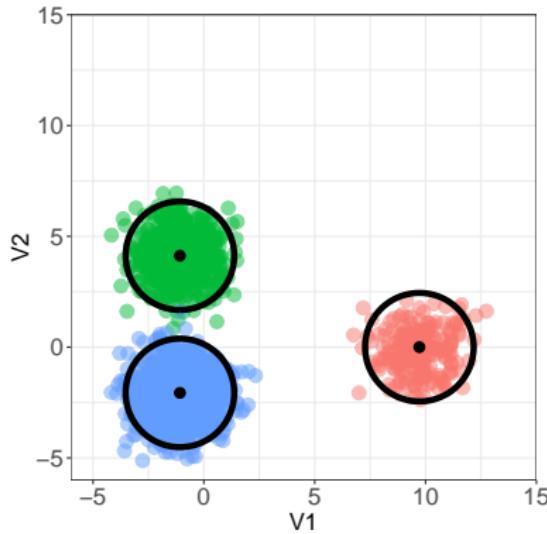
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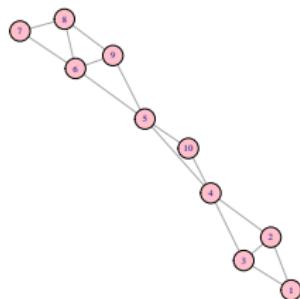
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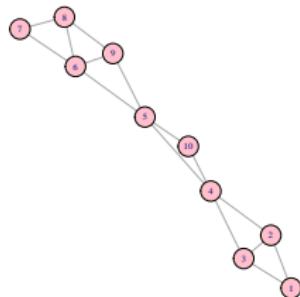


Step 1

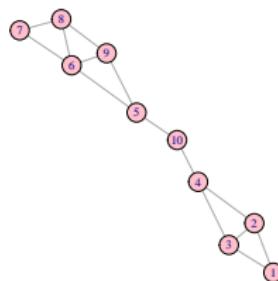
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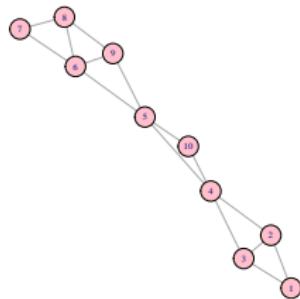


Step 2

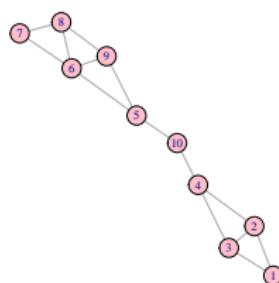
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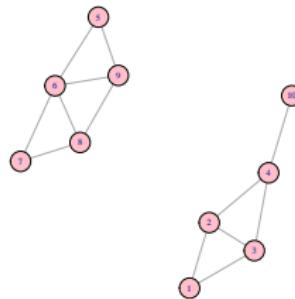
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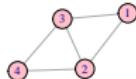
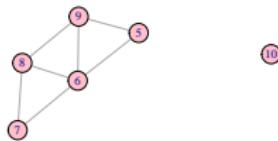
Step 2



Step 3

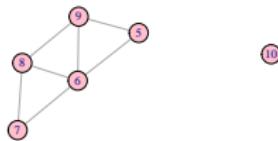
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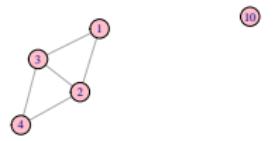


Step 4

# Edge Betweenness Algorithm



Step 4



Step 5

Back to the lab...

Back to the lab...



# Moving to Regression: Connecting to things we know

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- Consider a typical probit regression where we observe a binary response and covariates
- We want to estimate  $\beta$  so we consider probit regression where  $\epsilon$  are assumed to be iid

$$A = \mathbb{1}_{Z>0}$$

$$Z = X\beta + \epsilon$$

## Connecting to things we know

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Remember that dyadic dependence...



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However, with a network, it is not reasonable to assume that all of these entries are independent

# Social Relations Model [Warner et al., 1979]

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- $\beta_0$ : Overall global mean
- $c$ : Individual row (sociability/sender behavior) random effects
- $d$ : Individual column (popularity/reciever behavior) random effects

$$\begin{pmatrix} c_i \\ d_i \end{pmatrix} \stackrel{i.i.d}{\sim} N(0, \Sigma_{cd}) \text{ where } \Sigma_{cd} = \begin{bmatrix} \sigma_c^2 & \sigma_{cd} \\ \sigma_{cd} & \sigma_d^2 \end{bmatrix}$$

$$\begin{pmatrix} \epsilon_{i,j} \\ \epsilon_{j,i} \end{pmatrix} \stackrel{i.i.d}{\sim} N(0, \Sigma_e) \text{ where } \Sigma_e = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

# Adding in covariate information

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$$a_{i,j} = \mathbb{1}_{z_{i,j} > 0}$$
$$z_{i,j} =$$
$$\beta_0 + \sum_{p=1}^{P_r} (\textcolor{blue}{x_{r,p,i}} \beta_{r,p}) + \sum_{p=1}^{P_c} (\textcolor{blue}{x_{c,p,j}} \beta_{c,p}) + \sum_{p=1}^{P_d} (\textcolor{blue}{x_{d,p,i,j}} \beta_{d,p}) + c_i + d_j + \epsilon_{i,j}$$

- $\textcolor{blue}{X_r}$ : Observed covariate information for row covariates
- $\textcolor{blue}{X_c}$ : Observed covariate information for column covariates
- $\textcolor{blue}{X_d}$ : Observed covariate information for dyadic covariates
- $\beta$ : Coefficients of interest estimating covariate effects on connections

## Putting this into matrix form

$$\begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix} = \begin{bmatrix} x_{r,1} & x_{r,1} & x_{r,1} \\ x_{r,2} & x_{r,2} & x_{r,2} \\ x_{r,3} & x_{r,3} & x_{r,3} \end{bmatrix} \beta_r + \begin{bmatrix} x_{c,1} & x_{c,2} & x_{c,3} \\ x_{c,1} & x_{c,2} & x_{c,3} \\ x_{c,1} & x_{c,2} & x_{c,3} \end{bmatrix} \beta_c + \dots$$

# Better Models...

## Better Models...



## Adding in multiplicative effects (AMEN [Hoff, 2018])

- Sometimes there are higher order latent dependencies between nodes
  - ▶ **Example:** Nodes may be *homophilous*. Individuals who are similar to one another are more likely to be friends (clustering)

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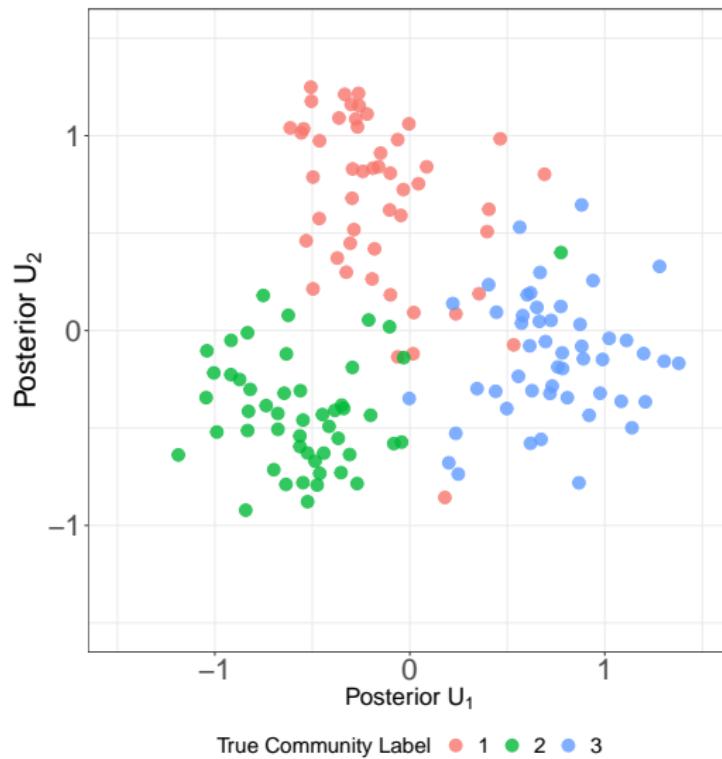
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- $\textcolor{blue}{U}, \textcolor{teal}{V}$ : Latent factor matrices of rank  $R$  ( $n \times R$  matrices)
- $\textcolor{blue}{u_i}$  gives us information about a node as a sender
- $\textcolor{teal}{v_j}$  gives us information about a node as a receiver
- They can describe notions of stochastic equivalence (if  $\textcolor{blue}{u_i}$  is similar to  $\textcolor{blue}{u_j}$ , then they may share similar behaviors)

## But what if we have latent community structure?

- Latent multiplicative effects can capture latent community structure



## How do we use this model?

- The standard AMEN model is implemented using a Markov Chain Monte Carlo (MCMC) algorithm
- For the standard model, this is provided in the ‘amen’ R package

**BRACE YOURSELVES**



# Brief overview of Bayesian Methods

- In frequentist methods, we have a parameter that we want to estimate,  $\theta$ , that is considered to be fixed, but unknown

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- In Bayesian methods, we are still interested in estimating  $\theta$ , however we believe it is an unknown, **random** quantity
- Rather than come up with a point estimate, we want to derive a posterior distribution for  $\theta$ . What does that mean exactly?

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- We then have a *sampling model*,  $p(y|\theta)$  that describes our beliefs about  $y$  had we known the true  $\theta$
- Our goal is then to get  $p(\theta|y)$  which describes our beliefs for the true value of  $\theta$  after observing our data  $y$

# Bayes Rule

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int_{\tilde{\theta} \in \Theta} p(y|\tilde{\theta})p(\tilde{\theta})d\tilde{\theta}}$$

## Back to AMEN: How do we implement this model?

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$$S(A) = \{Z : a_{i,j} > 0 \Rightarrow z_{i,j} > 0, a_{i,j} = 0 \Rightarrow z_{i,j} \leq 0\}$$

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- We can then use a Gibbs Sampler with MH to approximate  $P(\theta|Z \in S(A))$

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- Up to this point, we have focused on adjacency matrices/ binary networks
- However, networks can have many different forms
- AMEN allows us to easily consider some of these
  - ▶ **Censored Binary:** Still 1 or 0, but not all connections are observed
  - ▶ **Ranked:** Connections are ranked by some level of importance
  - ▶ **Fixed Rank Nomination:** A node can only have a FIXED number of connections and they are ranked

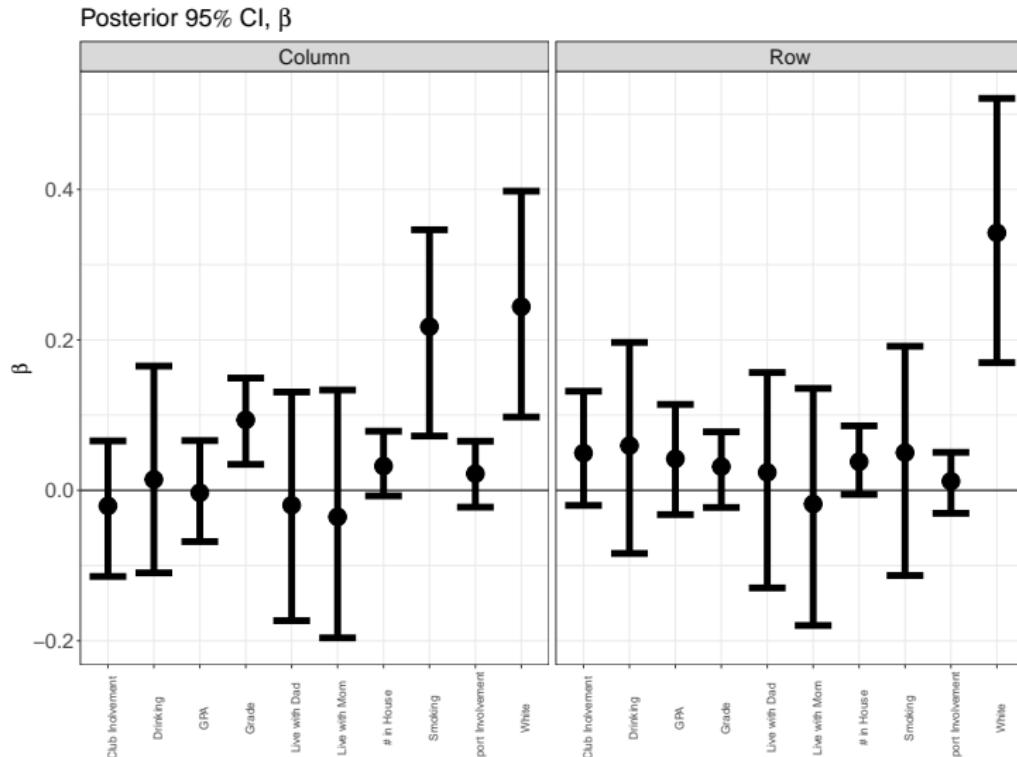
# Fixed Rank Nomination

## AddHealth

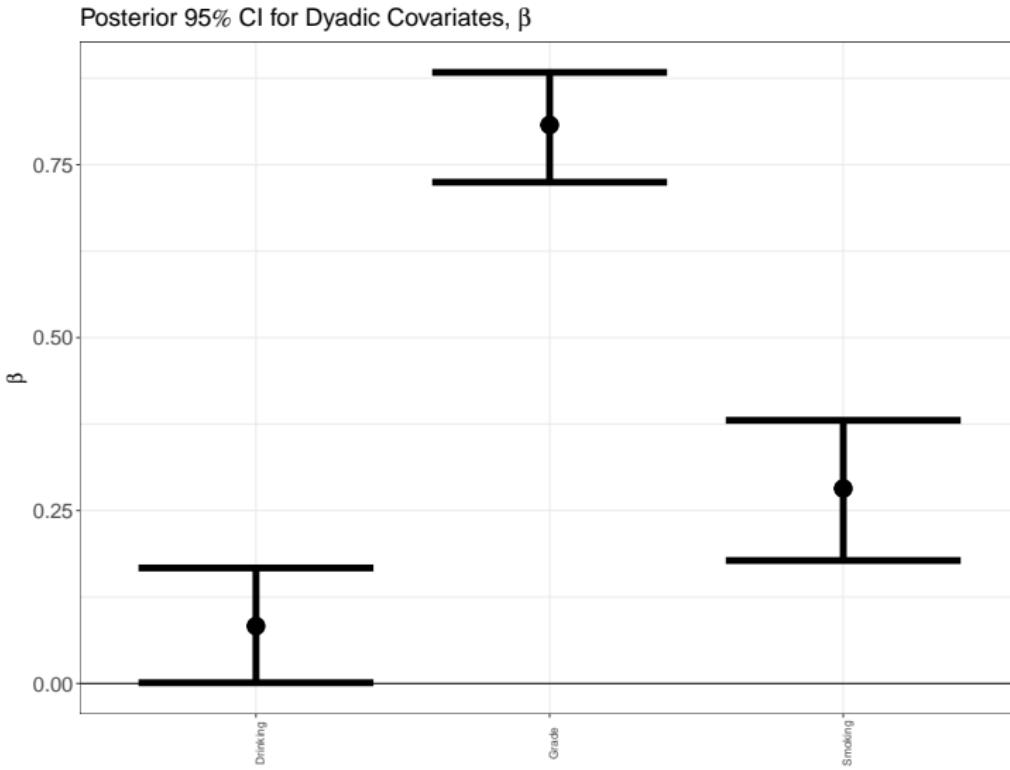
- Each person could rank up to 5 male friends and 5 female friends
- This introduces a *censorship* issue.
  - If person  $i$  ranks the 5 people, are they friends with the 6<sup>th</sup> and just didn't have room to rank them? or do they not like that person?
- If someone ranks less than 5 people, then we assume that they are not friends with person 6
- Another issue... perhaps person  $i$  just has never met person 6 but they would be great friends if they had

# Back to AddHealth: Regression Results

- 95% CI for  $\beta$  estimates when fitting AMEN model with  $R = 2$  on AddHealth network



# Standard AMEN R=2



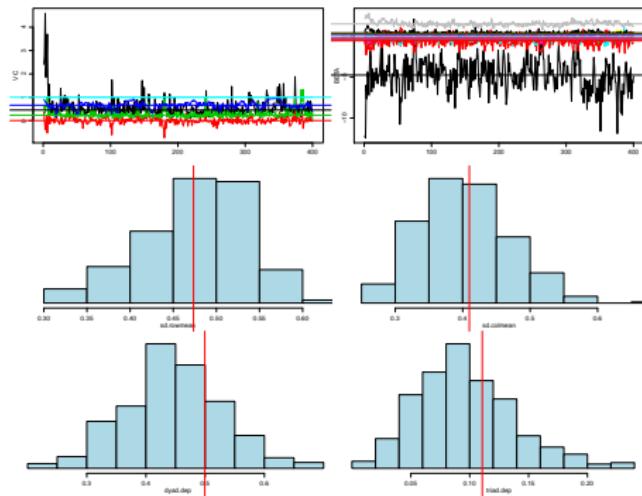
## What appears to be important?

- Column: Grade, Smoking, White
- Row: White
- Dyadic: Sharing same drinking behavior, grade, smoking behavior

# Diagnostics: Goodness of Fit Statistics

AMEN provides us with some posterior predictive goodness of fit statistics:

- ① Empirical standard deviation for row means
- ② Empirical standard deviation for column means
- ③ Empirical within-dyad correlation
- ④ Normalized measure of triadic dependence



And back to the lab...

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<https://igraph.org/r/doc/igraph.pdf> (igraph)

<https://cran.r-project.org/web/packages/ergm/ergm.pdf> (ergm)

<https://arxiv.org/pdf/1506.08237.pdf> (amen)



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