Introduction to Network Analysis

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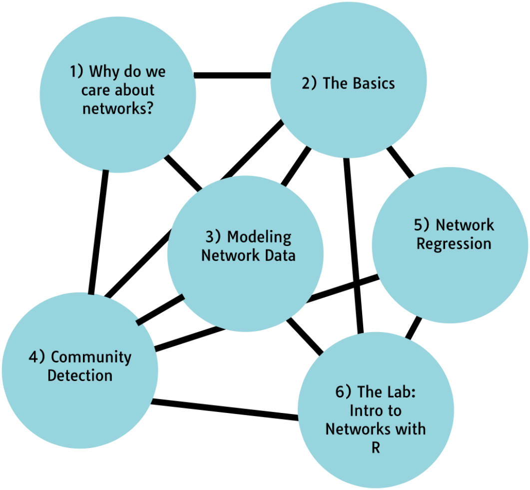
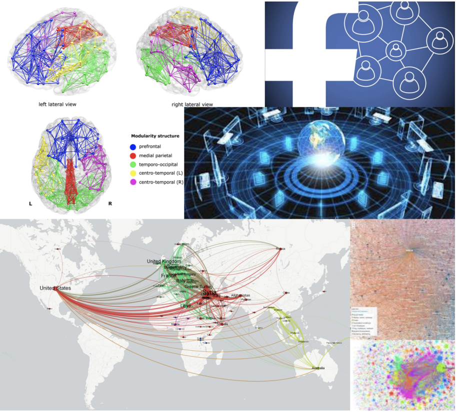
Duke University

November 7, 2019

Outline


1) Why do we care about networks?
2) The Basics
3) Modeling Network Data
4) Community Detection
5) Network Regression
6) The Lab: Intro to Networks with R

Who uses networks?



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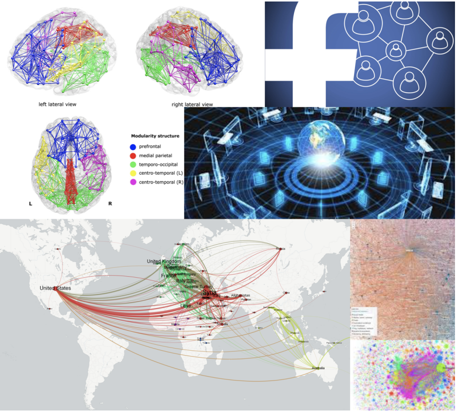

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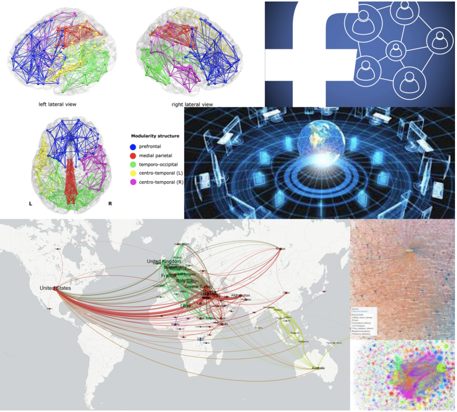

- Almost everyone! From economists to sociologists to biologists
- Brain networks, social networks, computer networks, traffic networks, trade networks...

Motivating Example: National Longitudinal Study of Adolescent to Adult Health (AddHealth)

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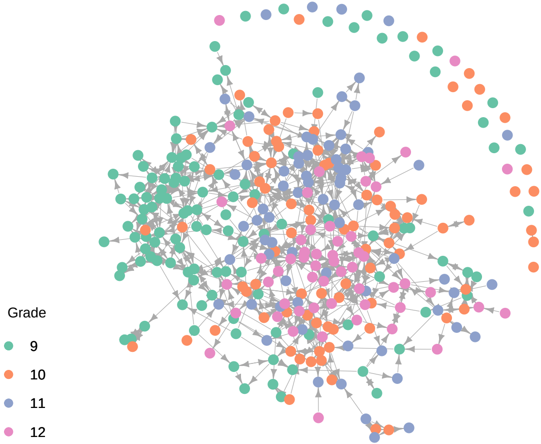
- Network Data: In each high school, each student was asked to nominate their top 5 male and female friends
- Covariate information on students was collected including grade, smoking status, drinking habits, club involvement, GPA, and sport involvement

AddHealth: Visualizing our Network



AddHealth

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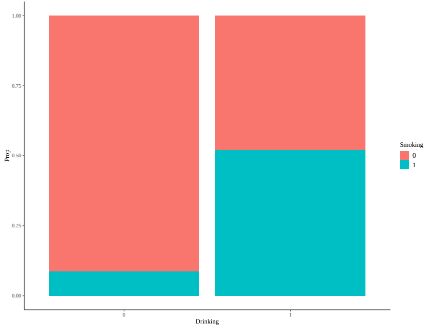
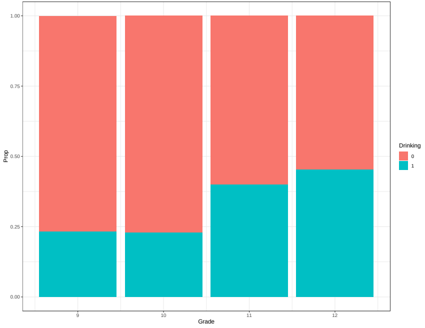


Smoking and drinking



Drinking by grade

Goals of AddHealth

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- Connect covariate estimation and latent communities
  - Higher GPA might increase popularity within the 'nerdy' clique but decrease popularity within the 'popular' clique

and I got that

red lip

CLASSIC

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red lip
CLASSIC

Karate Club Data

Network Scientists with Karate Trophies

5 MONTHS AGO
#NETWORKSCIENCE
#KARATECLUB
#TROPHY

The first scientist at any conference on networks who uses Zachary's karate club as an example is inducted into the Zachary Karate Club Club, and awarded a prize. This tumblr records those moments.

RSS
ARCHIVE

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- He was able to classify 33/34 of the members into either John's or Mr. Hi's club
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- This gives us ground truth for communities! Which has made it very popular for testing community detection algorithms

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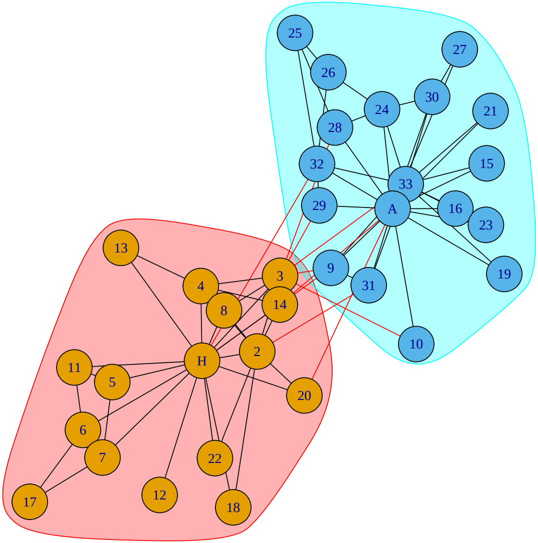

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$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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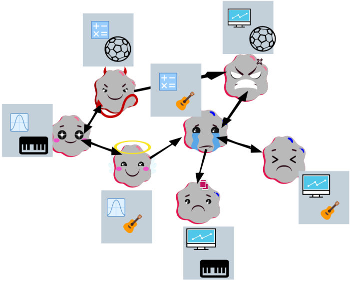

Graph of A

Formally Defining a Network

- **Graph**: $G = (V, E)$ with $V = \{1, ..., n\}$ and $\mathcal{E} = \{(i, j) \; \forall 1 \leq j \neq i \leq n\}$. $E \subset \mathcal{E}$ (if $E = \mathcal{E}$, fully connected graph)
  - Number of nodes: $n = |V|$
  - Number of edges: $m = |E|$

From	To
1	2
1	3
2	1
2	3
3	1

Example Edgelist

Directed vs Undirected

- Sometime edges between individuals are reciprocated (that is, if i is friends with j, j is friends with i)
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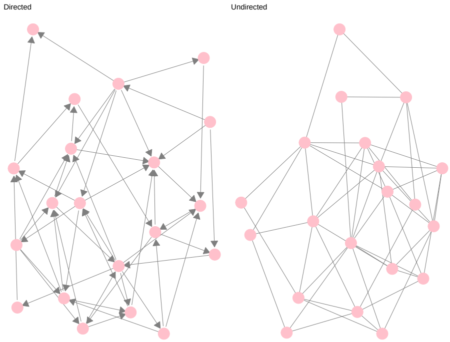

Directed: Asymmetric adjacency matrix, Undirected: Symmetric adjacency matrix

Directed vs. Undirected

Degree: Number of nodes a person is connected to

Density: Proportion of edges in graph over maximum possible number of edges

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	Directed	**Undirected**
Max Possible # of Edges	$n^2 - n = n(n-1)$	$n(n-1)/2$
Degree	$d_i^{out} = \sum_{j:i \neq j} A_{i,j}$ (out) $d_i^{in} = \sum_{j:i \neq j} A_{j,i}$ (in)	$d_i = \sum_{j:i \neq j} A_{i,j}$
Density	$m/(n(n-1))$	$2m/(n(n-1))$

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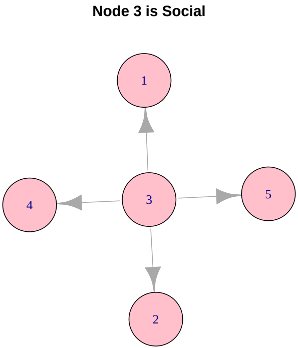

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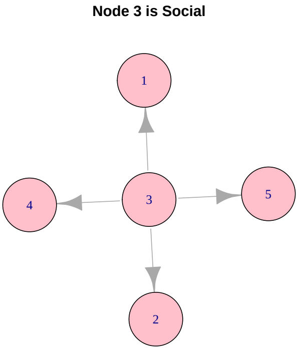

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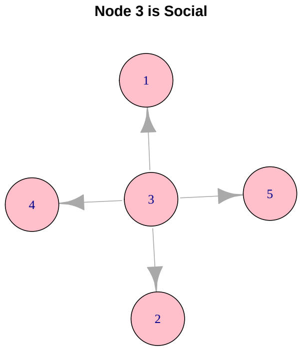

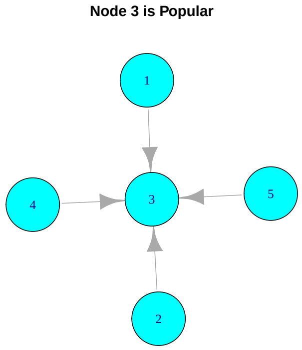

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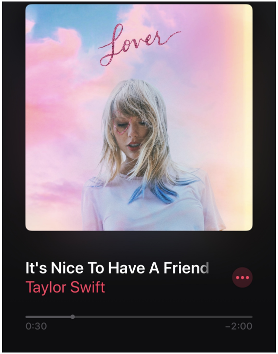

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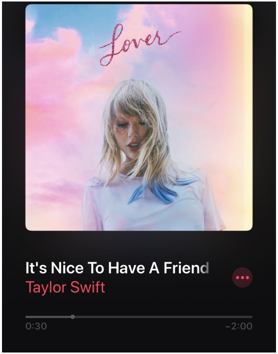

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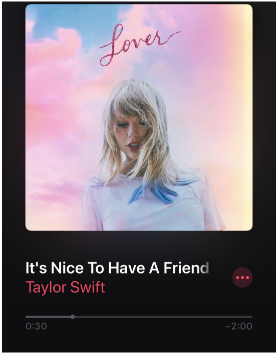
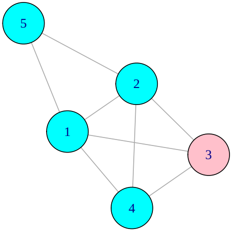


- **Homophily**: More likely to connect with people who are similar to you
- **Transitivity**: Friends of friends have higher probability of being friends

Sorry to say, but your friends have more friends than you...



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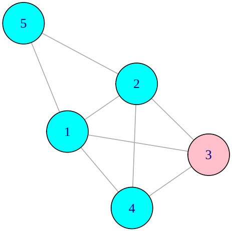

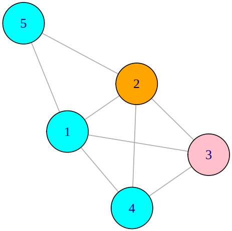

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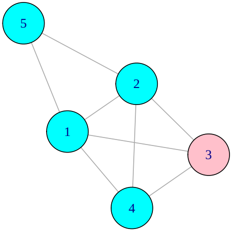

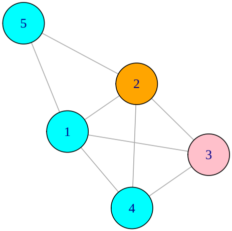

	1	2	3	4	5
Degree	4.00	4.00	3.00	3.00	2.00
Avg. Deg. of Friends	3.00	3.00	3.67	3.67	4.00

Notions of Centrality

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- **Betweenness Centrality:** Looks at how many times a node is part of the shortest path between other nodes

$$g_{j,k} = \# \text{ of shortest paths to get from } j \text{ to } k$$
$$g_{j,k}(i) = \# \text{ of shortest paths from } j \text{ to } k \text{ that go through i}$$
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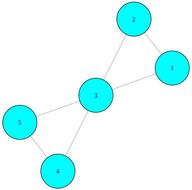
  - This is useful for finding indiviudals that are like bridges (flow of info)



Betweenness Centrality: 0 0 4 0 0

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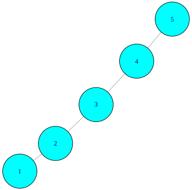
  - Sum up the shortest paths between all nodes (good for looking at who influences spread of info)



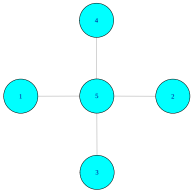
Closeness Centrality: 0.40 0.57 0.67 0.57 0.40

Notions of Centrality

- **Degree Centrality:** Importance based on number of connections a node has

$$c_i = \sum_{j:j \neq i} A_{i,j}$$

  - Useful for revealing direct connections and locating popular nodes



Degree Centrality 1 1 1 1 4

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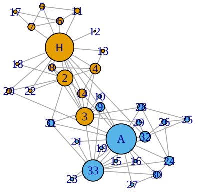
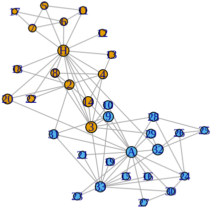
  - λ corresponds to the greatest eigenvalue of A and c corresponds to the top eigenvector. The i^{th} component of c gives the relative centrality score of vertex i
  - Central nodes are connected to other central nodes (very similar to degree centrality)
  - Basis for Google's PageRank
  - For graph on previous slide, eigen centrality is: 0.5 0.5 0.5 0.5 1.0

Example: Karate Club

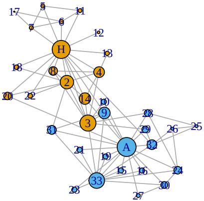
Degree



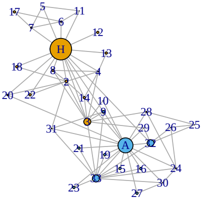
Closeness



Eigen



Betweenness



Now off to the lab...

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Modeling and Generating Networks

- Erdos-Renyi Graph (ER)
- Exponential Random Graph Models (ERGMS)
- Stochastic Block Models (SBM)
- Additive and Multiplicative Effects Network (AMEN)

Erdos-Renyi Graph

Generating graphs:

- **G(n,p)**: ER graph where edges are drawn independently with probability, p

$$P(G_0) = P(G = G_0) = p^m(1-p)^{(N-m)}$$
$$E(\# \text{ of Edges}) = \binom{n}{2}p$$

Relating ER to the Adj. Matrix

How do the models on the previous slide relate to the adjacency matrix, A?

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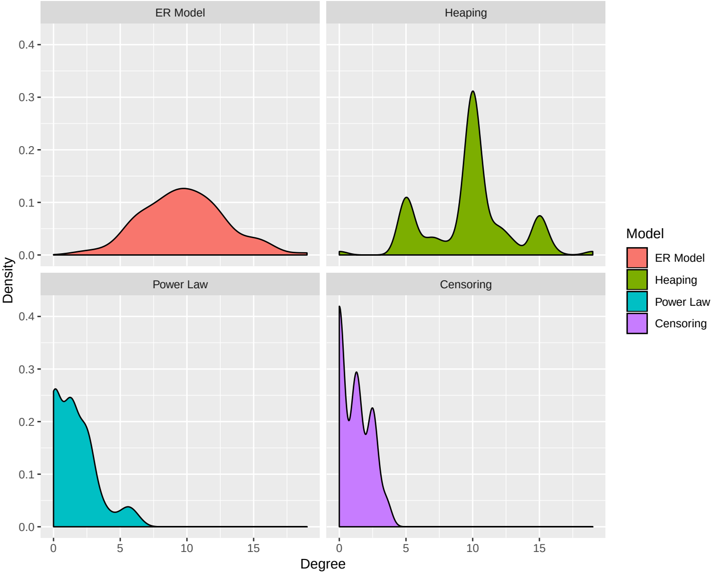
$$A_{i,j} | p \sim Bern(p)$$

Degree Distribution... and why we care



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$$\binom{n-1}{k} p^k (1-p)^{n-1-k}$$
- Bin(n-1,p)

Back to the lab...

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Exponential Random Graph Models: ERGM

Basic generative model for networks that is based off of sufficient statistics

$$P_\theta(G_n) = exp(\theta^T t_n(G_n) - \Psi_n(\theta))$$

where

- θ: Parameter (or vector of parameters) we want to estimate
- t_n: Sufficient statistic
- Ψ_n: Normalizing constant

Examples of ERGM

The ER Model:

- In this model, each edge is sampled iid Bernoulli with some probability p. For this model, $t_n = \sum_{i<j} A_{i,j}$

$$P_\theta(A_n) \propto exp\{\theta \sum_{i<j} A_{ij}\}$$

ER as an ERGM

We have seen the likelihood for the $ER(p)$:

$$\begin{aligned} p^m(1-p)^{N-m} &= exp\{log(p^m(1-p)^{N-m})\} \\ &= exp\{m \times log(p) + (N-m) \times log(1-p)\} \\ &= exp\{m \times log(\frac{p}{1-p}) + N \times log(1-p)\} \end{aligned}$$

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Recall the ERGM

ER as an ERGM

We have seen the likelihood for the $ER(p)$:

$$
\begin{aligned}
p^m(1-p)^{N-m} &= exp\{log(p^m(1-p)^{N-m})\} \\
&= exp\{m \times log(p) + (N-m) \times log(1-p)\} \\
&= exp\{m \times log(\frac{p}{1-p}) + N \times log(1-p)\}
\end{aligned}
$$

Recall the ERGM

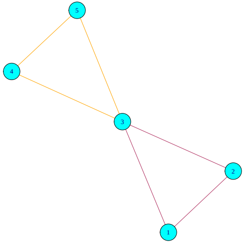
$$P_\theta(A_n) \propto exp(\theta t_n(A_n)) \tag{1}$$

where

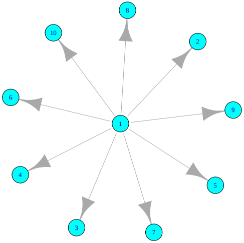
- $\theta = \log(\frac{p}{1-p})$
- $t_n = m = \sum_{i<j} A_{i,j}$

ERGM Cont.

We can also consider some other sufficient statistics to include in our model such as # of stars, # of triangles, # of edges, etc.



Triangles



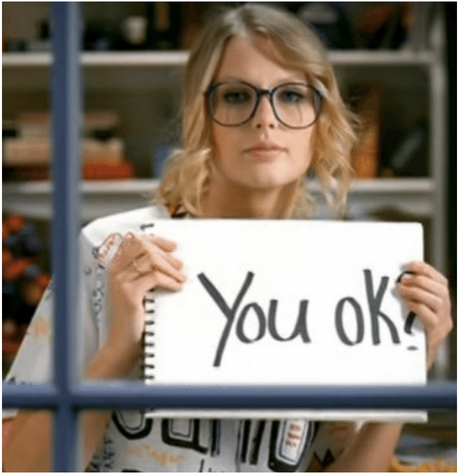
Star

What kinds of questions can ERGMs help answer?

- We can look at expectation of degree, edges, degree distribution
- We can test what model fits a new network best (were 2 graphs generated from the same model?)
- HOWEVER, not always consistent estimators :/

Back to the lab...

Back to the lab...



The Stochastic Block Model

Now we consider if nodes come from **communities** (Holland et al, 1983)

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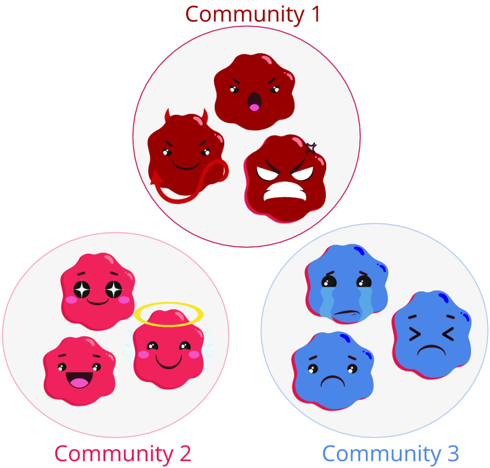

$$\theta = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Understanding the SBM Visually

$$B = \begin{pmatrix} 0.4 & 0.05 & 0.3 \\ 0.05 & 0.4 & 0.1 \\ 0.3 & 0.1 & 0.4 \end{pmatrix}$$

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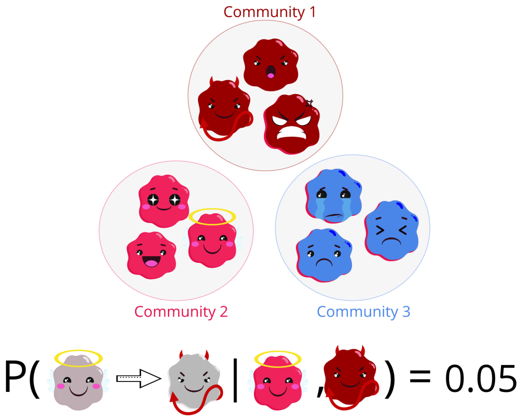

Another Generative Model: The Stochastic Block Model

- $\theta = [\theta_1, \theta_2, .., \theta_K]$ is a vector containing probabilities that a person belongs to a particular community, $k \in \{1, ..., K\}$

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- Z indicates which community a person belongs to

$$P(Z_i = k) = \theta_k$$
$$A_{i,j} | Z_i, Z_j \sim Bern(B_{z_i,z_j})$$
$$P(A, Z, \theta, B) = \prod_K \theta_k^{\sum 1_{z_i=k}} \prod_{i,j} B_{Z_i,Z_j}^{A_{ij}} (1 - B_{Z_i,Z_j})^{1-A_{i,j}}$$

Balanced Multi-Label Propagation for Overlapping Community Detection in Social Networks

Authors Authors and affiliations

Zhi-Hao Wu, You-Fang Lin, Steve Gregory, Huai-Yu Wan, Sheng-Feng Tian

Mutually Enhancing Community Detection and Sentiment Analysis on Twitter Networks

William Deitrick, Wei Hu*

Bayesian Inference and Testing of Group Differences in Brain Networks

Daniele Durante* and David B. Dunson†

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by Tulus Tambunan

Community Detection in General Stochastic Block models: Fundamental Limits and Efficient Algorithms for Recovery

Publisher: IEEE

2 Author(s) Emmanuel Abbe ; Colin Sandon View All Authors

Social selection and peer influence in an online social network

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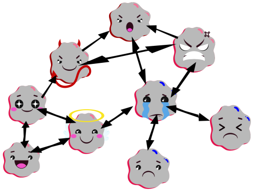

Observed Network

Community Detection

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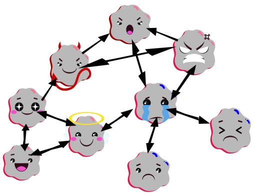

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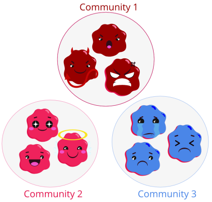

True latent communities

Community Detection

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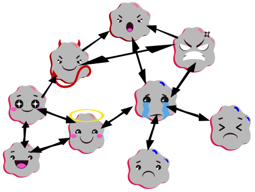

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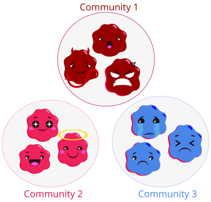

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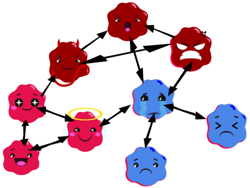

Goal

Methods for Community Detection

- Spectral methods
- Gaussian Mixture Models
- Centrality based approaches

Spectral Clustering

- Consider taking a *spectral decomposition* of A (eigendecomposition, singular value decomposition)

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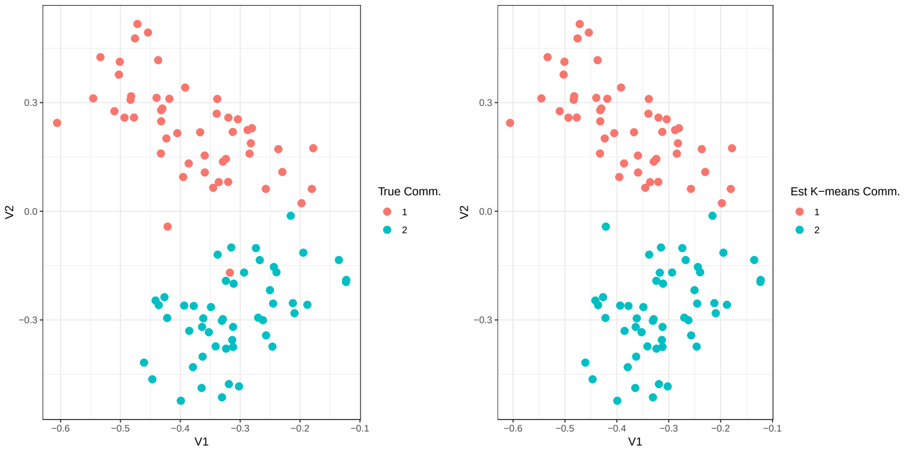
  where $\Lambda = Diag(\lambda_1, \ldots, \lambda_n)$ contains the eigenvalues in decreasing order (in magnitude), and $V = (V_1, \ldots, V_n)^T$ is a matrix containing the orthonormal columns corresponding to the eigenvectors
- If community structure exists, it should presumably show up in a lower dimensional representation of A
- Consider taking the top K eigenvectors of A where K is the number of hypothesized communities that exist

Looking at our eigenvectors



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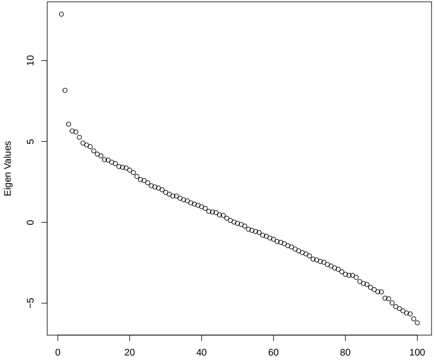

Once we have our number of communities... [Rohe et al., 2012]

- Run a clustering algorithm on the top K eigenvectors corresponding to the top K eigenvalues (in magnitude), $V_{,1:K}$
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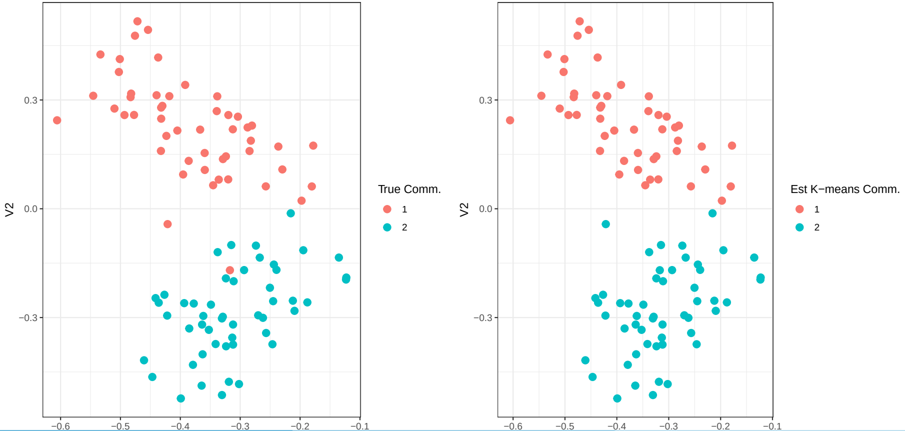

Gaussian Mixture Models

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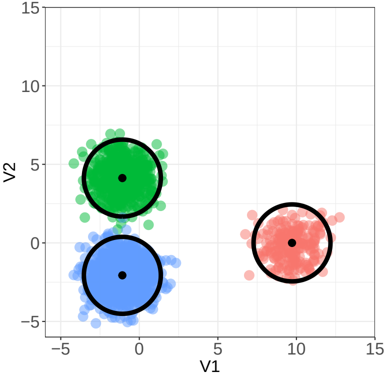

Moving to Centrality Based Example...

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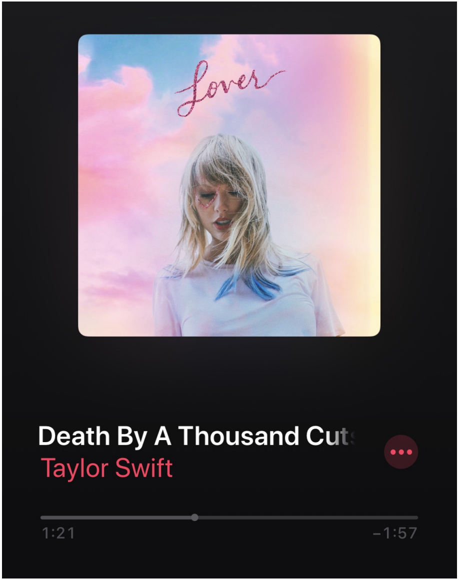

Edge Betweenness Algorithm

- Basic Idea:
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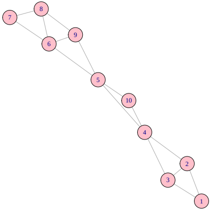

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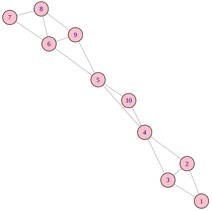

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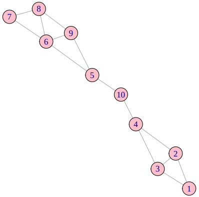

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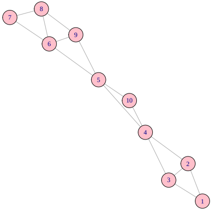

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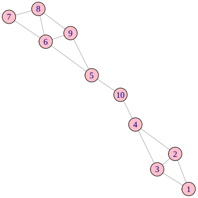

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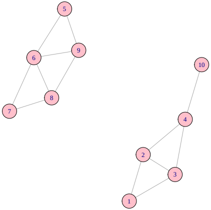

Step 3

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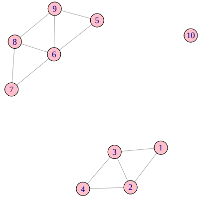

Step 4

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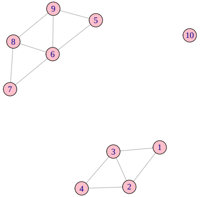

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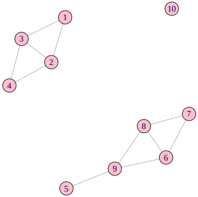

Step 5

Back to the lab...

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Moving to Regression: Connecting to things we know

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- Consider a typical probit regression where we observe a binary response and covariates

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- Consider a typical probit regression where we observe a binary response and covariates
- We want to estimate β so we consider probit regression where ϵ are assumed to be iid

$$A = \mathbb{1}_{Z>0}$$
$$Z = X\beta + \epsilon$$

Connecting to things we know

A is a type of response variable, but it is a matrix. To get something more familiar, we could vectorize A such that:

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Remember that dyadic dependence...

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However, with a network, it is not reasonable to assume that all of these entries are independent

Social Relations Model [Warner et al., 1979]

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$$

- β_0: Overall global mean
- c: Individual row (sociability/sender behavior) random effects
- d: Individual column (popularity/reciever behavior) random effects

$$
\begin{pmatrix} c_i \\ d_i \end{pmatrix} \overset{i.i.d}{\sim} N(0, \Sigma_{cd}) \text{ where } \Sigma_{cd} = \begin{bmatrix} \sigma_c^2 & \sigma_{cd} \\ \sigma_{cd} & \sigma_d^2 \end{bmatrix}
$$

$$
\begin{pmatrix} \epsilon_{i,j} \\ \epsilon_{j,i} \end{pmatrix} \overset{i.i.d}{\sim} N(0, \Sigma_e) \text{ where } \Sigma_e = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}
$$

Adding in covariate information

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$$a_{i,j} = \mathbb{1}_{z_{i,j}>0}$$

$$z_{i,j} =$$

$$\beta_0 + \sum_{p=1}^{P_r}(x_{r,p,i}\beta_{r,p}) + \sum_{p=1}^{P_c}(x_{c,p,j}\beta_{c,p}) + \sum_{p=1}^{P_d}(x_{d,p,i,j}\beta_{d,p}) + c_i + d_j + \epsilon_{i,j}$$

- X_r: Observed covariate information for row covariates
- X_c: Observed covariate information for column covariates
- X_d: Observed covariate information for dyadic covariates
- β: Coefficients of interest estimating covariate effects on connections

Putting this into matrix form

$$\begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix} = \begin{bmatrix} x_{r,1} & x_{r,1} & x_{r,1} \\ x_{r,2} & x_{r,2} & x_{r,2} \\ x_{r,3} & x_{r,3} & x_{r,3} \end{bmatrix} \beta_r + \begin{bmatrix} x_{c,1} & x_{c,2} & x_{c,3} \\ x_{c,1} & x_{c,2} & x_{c,3} \\ x_{c,1} & x_{c,2} & x_{c,3} \end{bmatrix} \beta_c + \ldots$$

Better Models...

Better Models...

Adding in multiplicative effects (AMEN [Hoff, 2018])

- Sometimes there are higher order latent dependencies between nodes
  - **Example:** Nodes may be *homophilous*. Individuals who are similar to one another are more likely to be friends (clustering)

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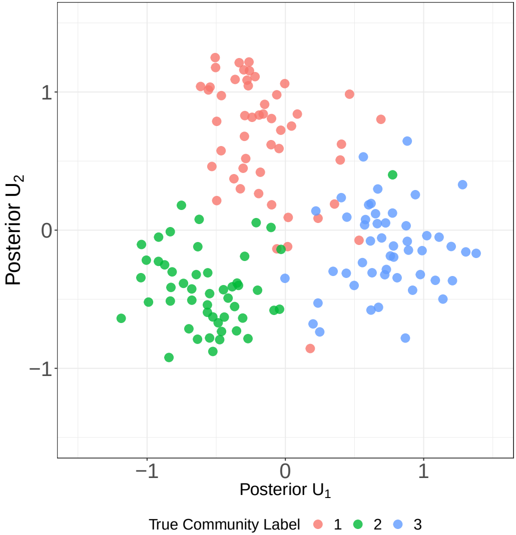
- U, V: Latent factor matrices of rank R ($n \times R$ matrices)
- u_i gives us information about a node as a sender
- v_j gives us information about a node as a reciever
- They can describe notions of stochastic equivalence (if u_i is similar to u_j, then they may share similar behaviors)

But what if we have latent community structure?

- Latent multiplicative effects can capture latent community structure



How do we use this model?

- The standard AMEN model is implemented using a Markov Chain Monte Carlo (MCMC) algorithm
- For the standard model, this is provided in the ‘amen’ R package


BRACE YOURSELVES
BAYESIAN ANALYSIS IS COMING

Brief overview of Bayesian Methods

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- In frequentist methods, we have a parameter that we want to estimate, θ, that is considered to be fixed, but unknown
- In Bayesian methods, we are still interested in estimating θ, however we believe it is an unknown, **random** quantity
- Rather than come up with a point estimate, we want to derive a posterior distribution for θ. What does that mean exactly?

Bayesian Methods

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- We then have a *sampling model*, $p(y|\theta)$ that describes our beliefs about y had we known the true θ
- Our goal is then to get $p(\theta|y)$ which describes our beliefs for the true value of θ after observing our data y

Bayes Rule

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int_{\tilde{\theta}\in\Theta} p(y|\tilde{\theta})p(\tilde{\theta})d\tilde{\theta}}$$

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- We can then use a Gibbs Sampler with MH to approximate $P(\theta|Z \in S(A))$

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- Up to this point, we have focused on adjacency matrices/ binary networks
- However, networks can have many different forms
- AMEN allows us to easily consider some of these
  - **Censored Binary**: Still 1 or 0, but not all connections are observed
  - **Ranked**: Connections are ranked by some level of importance
  - **Fixed Rank Nomination**: A node can only have a FIXED number of connections and they are ranked

Fixed Rank Nomination

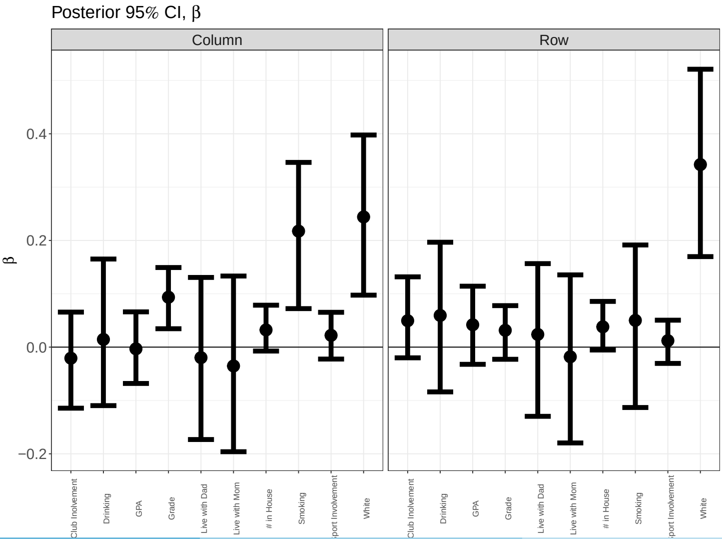
AddHealth

- Each person could rank up to 5 male friends and 5 female friends
- This introduces a *censorship* issue.

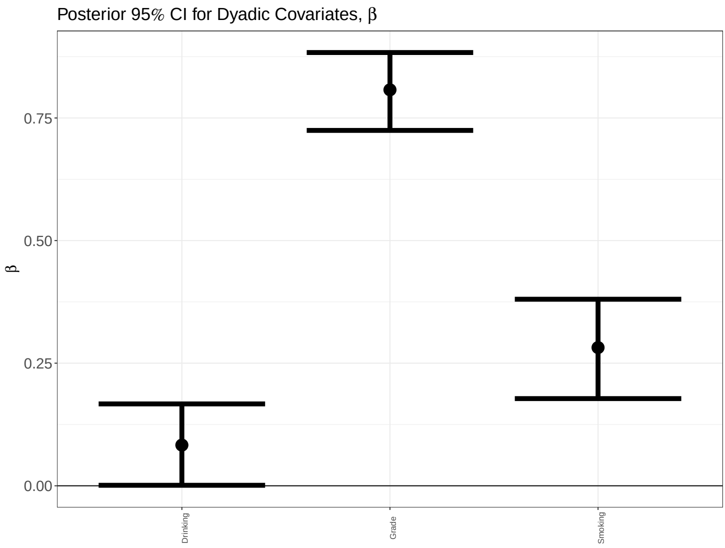
  If person i ranks the 5 people, are they friends with the 6^{th} and just didn't have room to rank them? or do they not like that person?
- If someone ranks less than 5 people, then we assume that they are not friends with person 6
- Another issue... perhaps person i just has never met person 6 but they would be great friends if they had

Back to AddHealth: Regression Results

- 95% CI for β estimates when fitting AMEN model with $R = 2$ on AddHealth network



Standard AMEN R=2



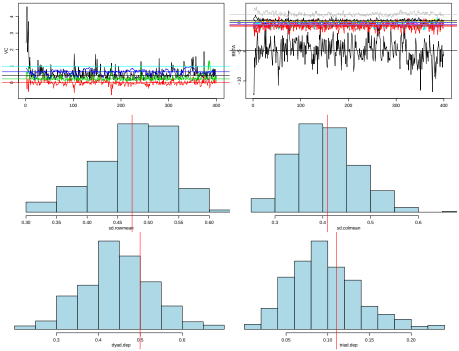
What appears to be important?

- Column: Grade, Smoking, White
- Row: White
- Dyadic: Sharing same drinking behavior, grade, smoking behavior

Diagnostics: Goodness of Fit Statistics

AMEN provides us with some posterior predictive goodness of fit statistics:

1. Empirical standard deviation for row means
2. Empirical standard deviation for column means
3. Empirical within-dyad correlation
4. Normalized measure of triadic dependence



And back to the lab...

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https://igraph.org/r/doc/igraph.pdf (igraph)
https://cran.r-project.org/web/packages/ergm/ergm.pdf (ergm)
https://arxiv.org/pdf/1506.08237.pdf (amen)

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