Introduction to Network Analysis

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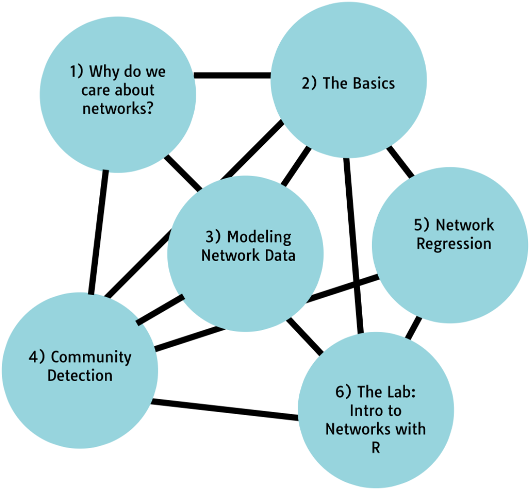
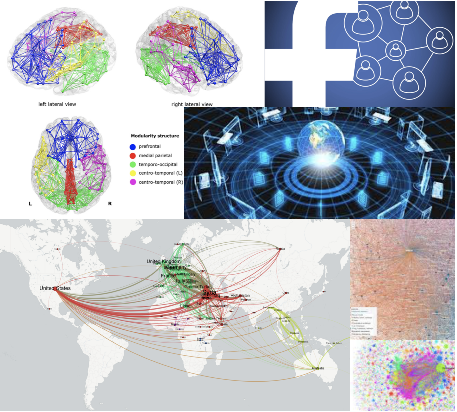
Outline


1) Why do we care about networks?
2) The Basics
3) Modeling Network Data
4) Community Detection
5) Network Regression
6) The Lab: Intro to Networks with R

Who uses networks?

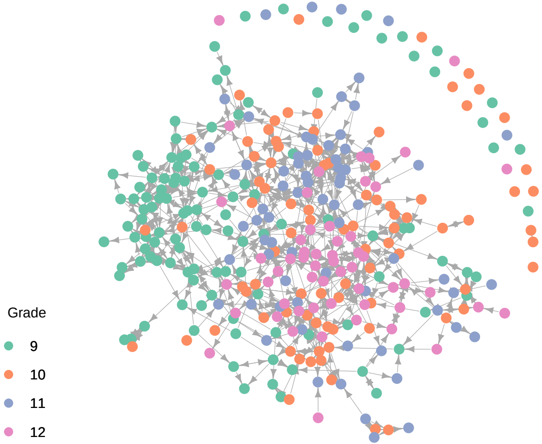


- Almost everyone! From economists to sociologists to biologists
- Brain networks, social networks, computer networks, traffic networks, trade networks...

Motivating Example: National Longitudinal Study of Adolescent to Adult Health (AddHealth)

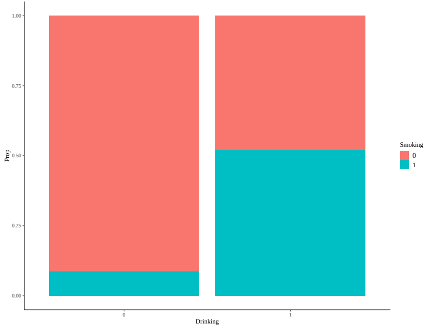
- Study was conducted due to a Congressional mandate to study factors influencing health behaviors of adolescents
- Collected data on approximately 20,000 high school students across the United States during the 1994-1995 school year
- Network Data: In each high school, each student was asked to nominate their top 5 male and female friends
- Covariate information on students was collected including grade, smoking status, drinking habits, club involvement, GPA, and sport involvement

AddHealth: Visualizing our Network

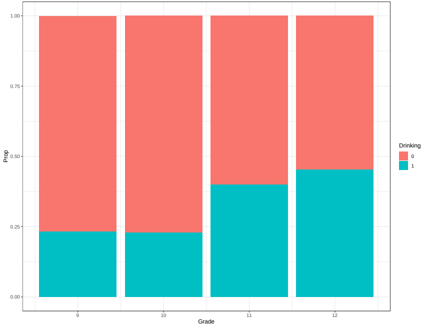


AddHealth

- Fairly uniform grade distribution
- Approximately 84% of male students are white
- About 22% of students are smokers, 30% drink alcohol
- On average, 4 individuals live in a household



Smoking and drinking



Drinking by grade

Goals of AddHealth

- Investigate which covariates might influence the probability of a friendship
- Identify possible clustering within our network. Do latent communities exist?
- Connect covariate estimation and latent communities
  - Higher GPA might increase popularity within the 'nerdy' clique but decrease popularity within the 'popular' clique

and I got that

red lip

CLASSIC

Karate Club Data

Network Scientists with Karate Trophies

5 MONTHS AGO
#NETWORKSCIENCE
#KARATECLUB
#TROPHY

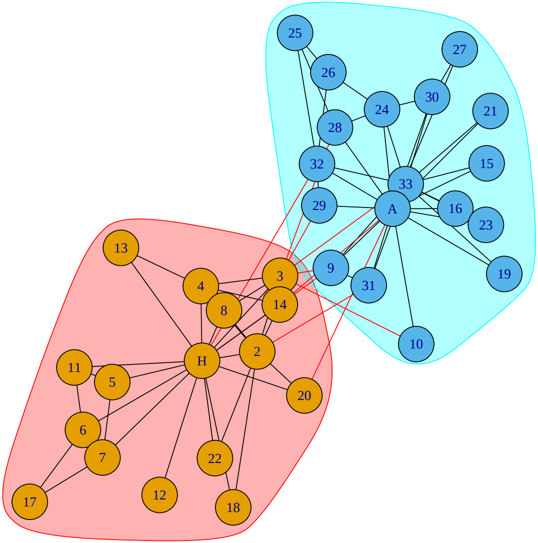
The first scientist at any conference on networks who uses Zachary's karate club as an example is inducted into the Zachary Karate Club Club, and awarded a prize. This tumblr records those moments.

RSS
ARCHIVE

Motivating Example: Karate Club

- Over three years (1970-1972), Zachary studied 34 individuals who once belonged to one karate club
- During that time, there was a conflict between John A. (adminstrator) and Mr. Hi (instructor) that resulted in the one club splitting into 2
- Interactions outside of the karate clubs were observed by Zachary
- He was able to classify 33/34 of the members into either John's or Mr. Hi's club
- Goal: Investigating possible fission in small community setting and interested in how information flows between the 2 clubs
- This gives us ground truth for communities! Which has made it very popular for testing community detection algorithms

Motivating Example: Karate Club

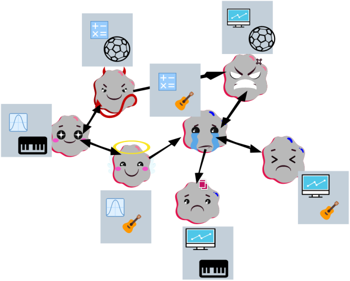


Motivating Example: Sports Analytics

What is a network?

- A network consists of **relational data** (data that describes relationships between actors)
- **Sociomatrix**: $n \times n$ matrix, A, to represent relationships between nodes. If binary entries, this is called an **adjacency matrix**

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$



Adjacency matrix, A

Graph of A

Formally Defining a Network

- **Graph**: $G = (V, E)$ with $V = \{1, ..., n\}$ and $\mathcal{E} = \{(i, j)\ \forall 1 \leq j \neq i \leq n\}$. $E \subset \mathcal{E}$ (if $E = \mathcal{E}$, fully connected graph)
  - Number of nodes: $n = |V|$
  - Number of edges: $m = |E|$

From	To
1	2
1	3
2	1
2	3
3	1

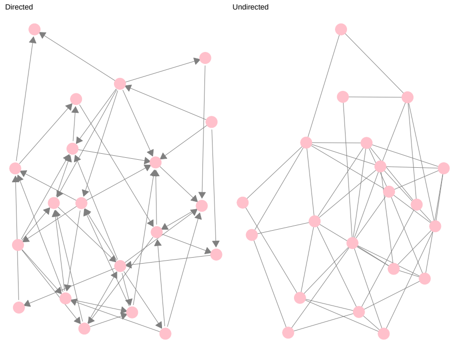
Example Edgelist

Directed vs Undirected

- Sometime edges between individuals are reciprocated (that is, if i is friends with j, j is friends with i)
- If all edges in our graph are reciprocated, then we have an **undirected network**

- However, if i is connected to j and j does not reciprocate that connection, then we have a directed edge from i to j. This leads to a **directed network**

Directed vs. Undirected



Directed: Asymmetric adjacency matrix, Undirected: Symmetric adjacency matrix

Directed vs. Undirected

Degree: Number of nodes a person is connected to

Density: Proportion of edges in graph over maximum possible number of edges

	Directed	**Undirected**
Max Possible # of Edges	$n^2 - n = n(n-1)$	$n(n-1)/2$
Degree	$d_i^{out} = \sum_{j:i \neq j} A_{i,j}$ (out) $d_i^{in} = \sum_{j:i \neq j} A_{j,i}$ (in)	$d_i = \sum_{j:i \neq j} A_{i,j}$
Density	$m/(n(n-1))$	$2m/(n(n-1))$

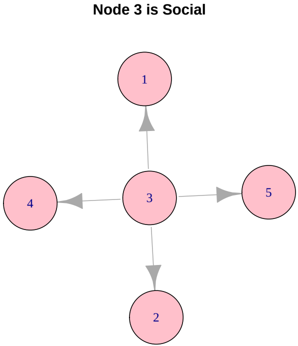
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

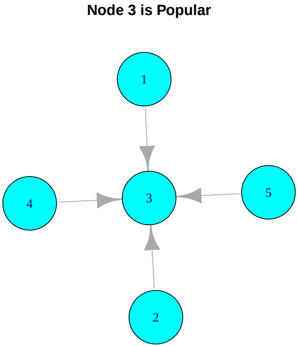
Directed matrix, A

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Undirected matrix, A

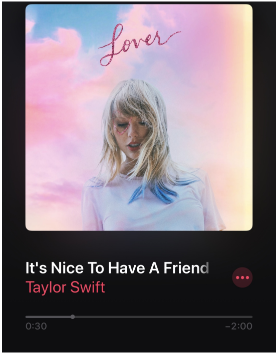
Sociability vs Popularity





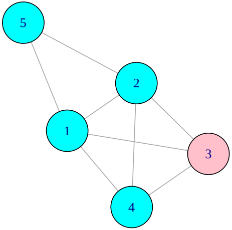
Some Relevant Definitions

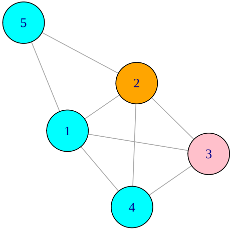
- **Reciprocity**: If person 1 is connected to person 2, person 2 is connected to person 1



- **Homophily**: More likely to connect with people who are similar to you
- **Transitivity**: Friends of friends have higher probability of being friends

Sorry to say, but your friends have more friends than you...





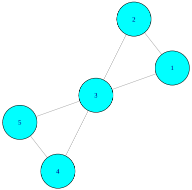
	1	2	3	4	5
Degree	4.00	4.00	3.00	3.00	2.00
Avg. Deg. of Friends	3.00	3.00	3.67	3.67	4.00

Notions of Centrality

- **Betweenness Centrality:** Looks at how many times a node is part of the shortest path between other nodes

$$g_{j,k} = \# \text{ of shortest paths to get from } j \text{ to } k$$
$$g_{j,k}(i) = \# \text{ of shortest paths from } j \text{ to } k \text{ that go through i}$$
$$c_i = \sum_{j<k} g_{j,k}(i)/g_{j,k}$$

  - This is useful for finding indiviudals that are like bridges (flow of info)



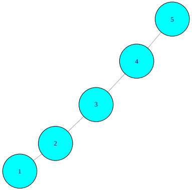
Betweenness Centrality: 0 0 4 0 0

Notions of Centrality

- **Closeness Centrality:** Measures how close one node is to all other nodes in the network.Define $d_{i,j}$ as the minimum path length from i to j.

$$c_i = \frac{1}{\sum_{j:j \neq i} d_{i,j}}$$

  - Sum up the shortest paths between all nodes (good for looking at who influences spread of info)



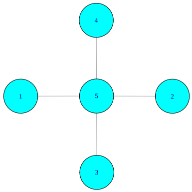
Closeness Centrality: 0.40 0.57 0.67 0.57 0.40

Notions of Centrality

- **Degree Centrality:** Importance based on number of connections a node has

$$c_i = \sum_{j:j \neq i} A_{i,j}$$

  - Useful for revealing direct connections and locating popular nodes



Degree Centrality 1 1 1 1 4

Notions of Centrality

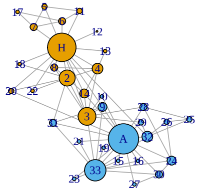
- **EigenCentrality:** Centrality of each node is proportional to the sum of its neighbor's centralities

$$c_i = \frac{1}{\lambda} \sum_{j:j \neq i} A_{i,j} c_j$$

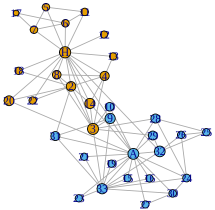
  - λ corresponds to the greatest eigenvalue of A and c corresponds to the top eigenvector. The i^{th} component of c gives the relative centrality score of vertex i
  - Central nodes are connected to other central nodes (very similar to degree centrality)
  - Basis for Google's PageRank
  - For graph on previous slide, eigen centrality is: 0.5 0.5 0.5 0.5 1.0

Example: Karate Club

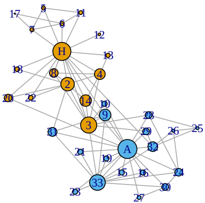
Degree



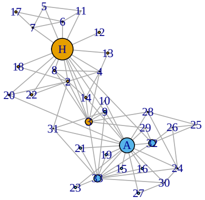
Closeness



Eigen



Betweenness



Now off to the lab...

Modeling and Generating Networks

- Erdos-Renyi Graph (ER)
- Exponential Random Graph Models (ERGMS)
- Stochastic Block Models (SBM)
- Additive and Multiplicative Effects Network (AMEN)

Erdos-Renyi Graph

Generating graphs:

- **G(n,p)**: ER graph where edges are drawn independently with probability, p

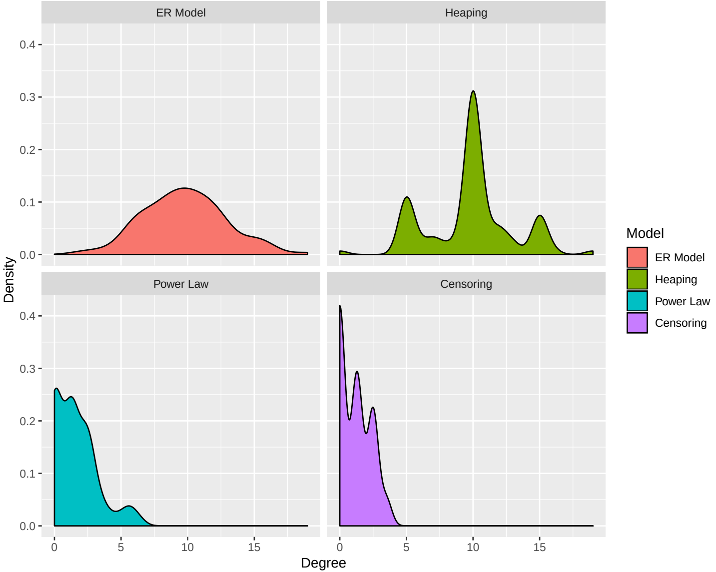
$$P(G_0) = P(G = G_0) = p^m(1-p)^{(N-m)}$$
$$E(\#\text{ of Edges}) = \binom{n}{2}p$$

Relating ER to the Adj. Matrix

How do the models on the previous slide relate to the adjacency matrix, A?

$$A_{i,j}|p \sim Bern(p)$$

Degree Distribution... and why we care



Degree Distribution

- What is the degree distribution of an ER model?
- ER(p) has (n-1) possible friends
- Let person i have k friends, thus $\binom{n-1}{k}$ possibilities and probability of a friend is p:
$$\binom{n-1}{k} p^k (1-p)^{n-1-k}$$
- Bin(n-1,p)

Back to the lab...

Exponential Random Graph Models: ERGM

Basic generative model for networks that is based off of sufficient statistics

$$P_\theta(G_n) = exp(\theta^T t_n(G_n) - \Psi_n(\theta))$$

where

- θ: Parameter (or vector of parameters) we want to estimate
- t_n: Sufficient statistic
- Ψ_n: Normalizing constant

Examples of ERGM

The ER Model:

- In this model, each edge is sampled iid Bernoulli with some probability p. For this model, $t_n = \sum_{i<j} A_{i,j}$

$$P_\theta(A_n) \propto exp\{\theta \sum_{i<j} A_{ij}\}$$

ER as an ERGM

We have seen the likelihood for the $ER(p)$:

$$
\begin{aligned}
p^m(1-p)^{N-m} &= exp\{log(p^m(1-p)^{N-m})\} \\
&= exp\{m \times log(p) + (N-m) \times log(1-p)\} \\
&= exp\{m \times log(\frac{p}{1-p}) + N \times log(1-p)\}
\end{aligned}
$$

Recall the ERGM

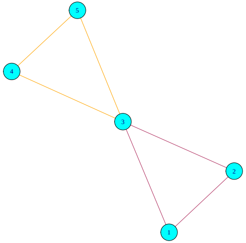
$$
P_\theta(A_n) \propto exp(\theta t_n(A_n)) \tag{1}
$$

where

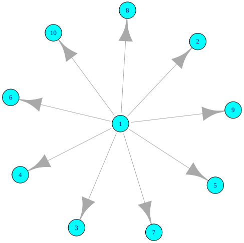
- $\theta = \log(\frac{p}{1-p})$
- $t_n = m = \sum_{i<j} A_{i,j}$

ERGM Cont.

We can also consider some other sufficient statistics to include in our model such as # of stars, # of triangles, # of edges, etc.



Triangles

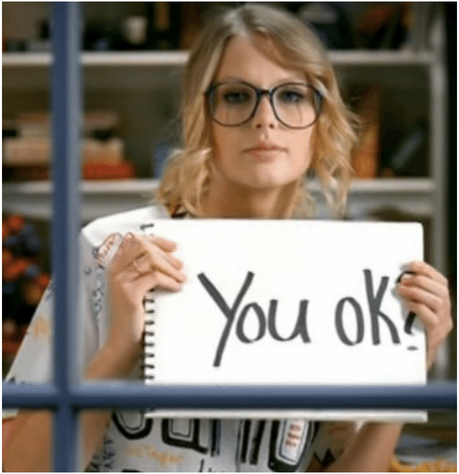


Star

What kinds of questions can ERGMs help answer?

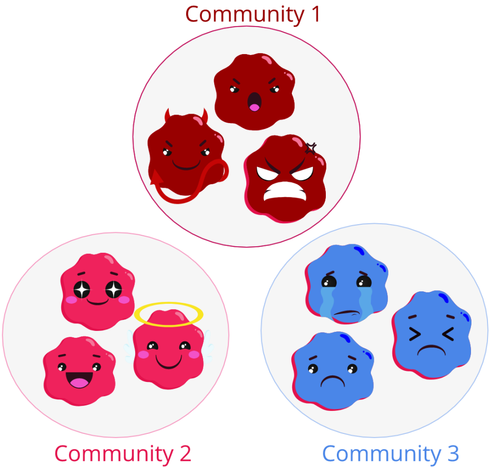
- We can look at expectation of degree, edges, degree distribution
- We can test what model fits a new network best (were 2 graphs generated from the same model?)
- HOWEVER, not always consistent estimators :/

Back to the lab...



The Stochastic Block Model

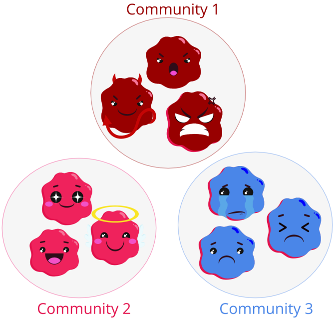
Now we consider if nodes come from **communities** (Holland et al, 1983)



$$\theta = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}$$

Understanding the SBM Visually

$$B = \begin{pmatrix} 0.4 & 0.05 & 0.3 \\ 0.05 & 0.4 & 0.1 \\ 0.3 & 0.1 & 0.4 \end{pmatrix}$$



Another Generative Model: The Stochastic Block Model

- $\theta = [\theta_1, \theta_2, .., \theta_K]$ is a vector containing probabilities that a person belongs to a particular community, $k \in \{1, ..., K\}$
- $B \in \mathbb{R}^{K \times K}$ is a preference matrix that describes the probability of connection of nodes based soley on an individual's membership
- Z indicates which community a person belongs to

$$P(Z_i = k) = \theta_k$$
$$A_{i,j} | Z_i, Z_j \sim Bern(B_{z_i, z_j})$$
$$P(A, Z, \theta, B) = \prod_K \theta_k^{\sum 1_{z_i = k}} \prod_{i,j} B_{Z_i, Z_j}^{A_{ij}} (1 - B_{Z_i, Z_j})^{1 - A_{i,j}}$$

Balanced Multi-Label Propagation for Overlapping Community Detection in Social Networks

Authors   Authors and affiliations

Zhi-Hao Wu, You-Fang Lin, Steve Gregory, Huai-Yu Wan, Sheng-Feng Tian

Mutually Enhancing Community Detection and Sentiment Analysis on Twitter Networks

William Deitrick, Wei Hu*

Bayesian Inference and Testing of Group Differences in Brain Networks

Daniele Durante* and David B. Dunson†

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Identifying functional urban regions within traffic flow

Ed Manley

Pages 40-42 | Received 30 Jan 2014, Accepted 03 Feb 2014, Published online: 12 Mar 2014

Promoting Small and Medium Enterprises with a Clustering Approach: A Policy Experience from Indonesia

by Tulus Tambunan

Community Detection in General Stochastic Block models: Fundamental Limits and Efficient Algorithms for Recovery

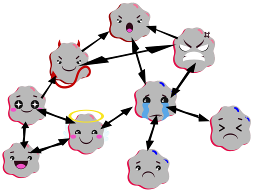
Publisher: IEEE

2 Author(s) Emmanuel Abbe ; Colin Sandon View All Authors

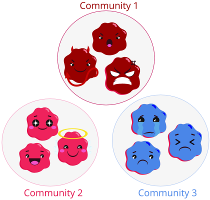
Social selection and peer influence in an online social network

Community Detection

- Maybe we believe our network, A, came from an SBM, and we care about finding community labels



Observed Network



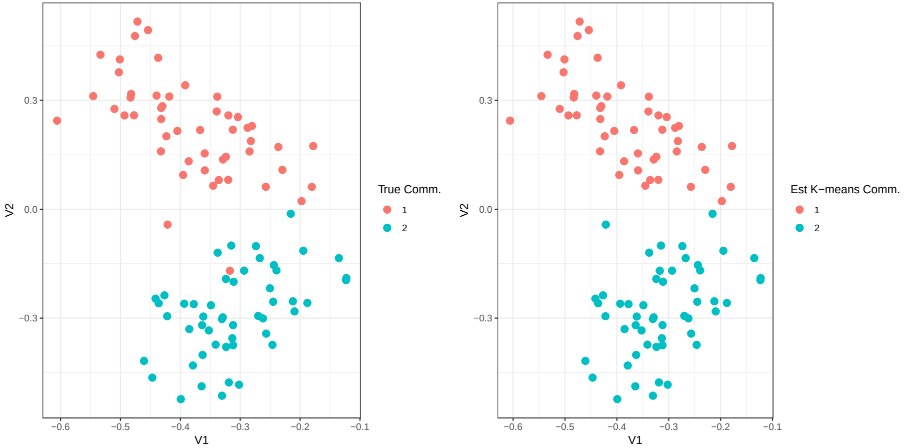
Methods for Community Detection

- Spectral methods
- Gaussian Mixture Models
- Centrality based approaches

Spectral Clustering

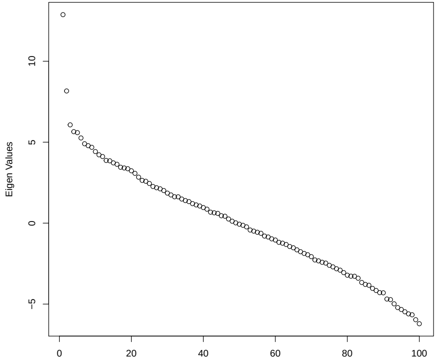
- Consider taking a *spectral decomposition* of A (eigendecomposition, singular value decomposition)
- If we take the eigendecomposition,
$$A = V \Lambda V^T$$
where $\Lambda = Diag(\lambda_1, \ldots, \lambda_n)$ contains the eigenvalues in decreasing order (in magnitude), and $V = (V_1, \ldots, V_n)^T$ is a matrix containing the orthonormal columns corresponding to the eigenvectors
- If community structure exists, it should presumably show up in a lower dimensional representation of A
- Consider taking the top K eigenvectors of A where K is the number of hypothesized communities that exist

Looking at our eigenvectors



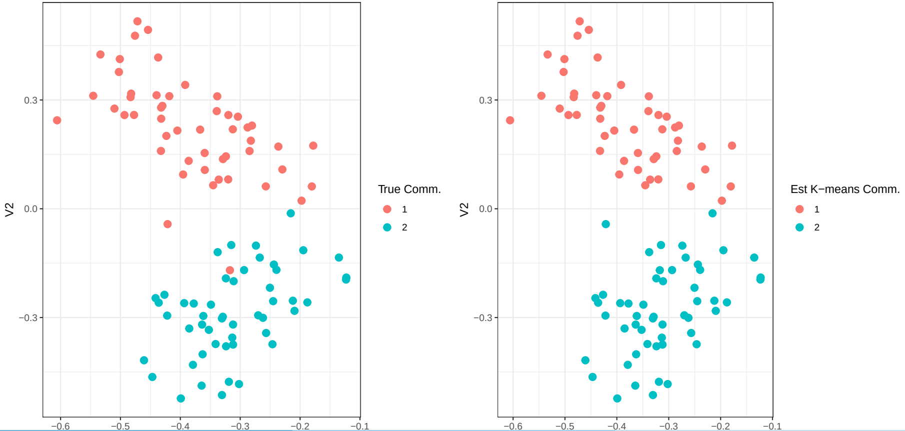
The Ultimate Question: How do we pick the number of communities?

- The answer to this is not super clear
- Elbow plots, look for eigen values that escape the bulk
- Prior information
- Try a few different options



Once we have our number of communities... [Rohe et al., 2012]

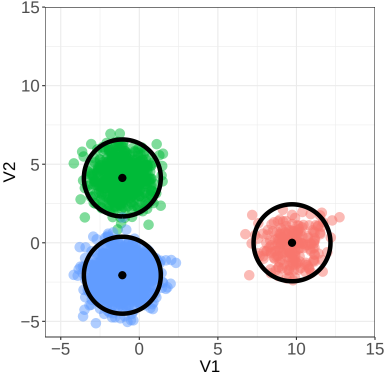
- Run a clustering algorithm on the top K eigenvectors corresponding to the top K eigenvalues (in magnitude), $V_{,1:K}$
- Can also cluster on $V_{,1:K}\Lambda^{1/2}_{1:K,1:K}$



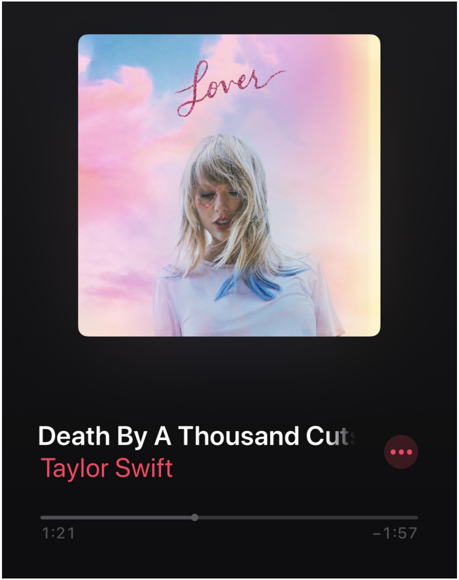
Gaussian Mixture Models

- Consider the latent positions that we can obtain from spectral decompositions of A
- It can actually be shown that these latent positions can be modeled using Gaussian Mixture Models:

$$P(V_i = v_i) = \prod_{i=1}^{n} \sum_{k=1}^{K} \pi_k N(v_i; \mu_k, \sigma_k^2)$$

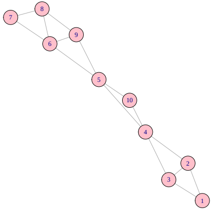


Moving to Centrality Based Example...

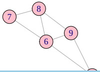


Edge Betweenness Algorithm

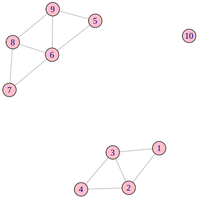
- Basic Idea:
  - Find the edge with maximum edge betweenness centrality and delete it



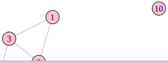
Step 1



Edge Betweenness Algorithm



Step 4



Back to the lab...

Moving to Regression: Connecting to things we know

- Consider a typical probit regression where we observe a binary response and covariates
- We want to estimate β so we consider probit regression where ϵ are assumed to be iid

$$
\begin{aligned}
A &= \mathbb{1}_{Z>0} \\
Z &= X\beta + \epsilon
\end{aligned}
$$

Connecting to things we know

A is a type of response variable, but it is a matrix. To get something more familiar, we could vectorize A such that:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow vec(A) = \begin{pmatrix} A_{,1} \\ A_{,2} \\ A_{,3} \\ A_{,4} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{pmatrix}$$

Remember that dyadic dependence...

However, with a network, it is not reasonable to assume that all of these entries are independent

Social Relations Model [Warner et al., 1979]

- We are interested in modeling the variability in A

$$a_{i,j} = \mathbb{1}_{z_{i,j}>0}$$
$$z_{i,j} = \beta_0 + c_i + d_j + \epsilon_{i,j}$$

- β_0: Overall global mean
- c: Individual row (sociability/sender behavior) random effects
- d: Individual column (popularity/reciever behavior) random effects

$$\begin{pmatrix} c_i \\ d_i \end{pmatrix} \overset{i.i.d}{\sim} N(0, \Sigma_{cd}) \text{ where } \Sigma_{cd} = \begin{bmatrix} \sigma_c^2 & \sigma_{cd} \\ \sigma_{cd} & \sigma_d^2 \end{bmatrix}$$
$$\begin{pmatrix} \epsilon_{i,j} \\ \epsilon_{j,i} \end{pmatrix} \overset{i.i.d}{\sim} N(0, \Sigma_e) \text{ where } \Sigma_e = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Adding in covariate information

$$a_{i,j} = \mathbb{1}_{z_{i,j}>0}$$
$$z_{i,j} =$$
$$\beta_0 + \sum_{p=1}^{P_r}(x_{r,p,i}\beta_{r,p}) + \sum_{p=1}^{P_c}(x_{c,p,j}\beta_{c,p}) + \sum_{p=1}^{P_d}(x_{d,p,i,j}\beta_{d,p}) + c_i + d_j + \epsilon_{i,j}$$

- X_r: Observed covariate information for row covariates
- X_c: Observed covariate information for column covariates
- X_d: Observed covariate information for dyadic covariates
- β: Coefficients of interest estimating covariate effects on connections

Putting this into matrix form

$$\begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix} = \begin{bmatrix} x_{r,1} & x_{r,1} & x_{r,1} \\ x_{r,2} & x_{r,2} & x_{r,2} \\ x_{r,3} & x_{r,3} & x_{r,3} \end{bmatrix} \beta_r + \begin{bmatrix} x_{c,1} & x_{c,2} & x_{c,3} \\ x_{c,1} & x_{c,2} & x_{c,3} \\ x_{c,1} & x_{c,2} & x_{c,3} \end{bmatrix} \beta_c + \ldots$$

Better Models...

Adding in multiplicative effects (AMEN [Hoff, 2018])

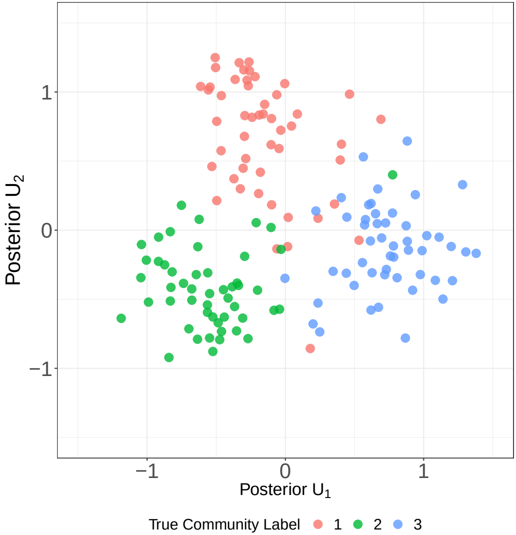
- Sometimes there are higher order latent dependencies between nodes
  - **Example:** Nodes may be *homophilous*. Individuals who are similar to one another are more likely to be friends (clustering)

$$a_{i,j} = \mathbb{1}_{z_{i,j}>0}$$
$$z_{i,j} = \beta_0 + \sum_{p=1}^{P_r}(x_{r,p,i}\beta_{r,p}) + \sum_{p=1}^{P_c}(x_{c,p,j}\beta_{c,p}) + \sum_{p=1}^{P_d}(x_{d,p,i,j}\beta_{d,p}) + c_i + d_j + u_i^T v_j + \epsilon_{i,j}$$

- U, V: Latent factor matrices of rank R ($n \times R$ matrices)
- u_i gives us information about a node as a sender
- v_j gives us information about a node as a reciever
- They can describe notions of stochastic equivalence (if u_i is similar to u_j, then they may share similar behaviors)

But what if we have latent community structure?

- Latent multiplicative effects can capture latent community structure



How do we use this model?

- The standard AMEN model is implemented using a Markov Chain Monte Carlo (MCMC) algorithm
- For the standard model, this is provided in the 'amen' R package


BRACE YOURSELVES
BAYESIAN ANALYSIS IS COMING

Brief overview of Bayesian Methods

- In frequentist methods, we have a parameter that we want to estimate, θ, that is considered to be fixed, but unknown
- In Bayesian methods, we are still interested in estimating θ, however we believe it is an unknown, **random** quantity
- Rather than come up with a point estimate, we want to derive a posterior distribution for θ. What does that mean exactly?

Bayesian Methods

- We observe a dataset, y, which comes from a sample space, $\mathcal{Y}$ where $\mathcal{Y}$ represents all possible datasets that y could come from
- We are interested in estimating a population parameter, θ, that comes from Θ (parameter space of θ)
- In a Bayesian setting, we put a *prior* distribution on our parameter, $\theta \in \Theta$. This describes our beliefs about the true population parameter ($p(\theta)$)
- We then have a *sampling model*, $p(y|\theta)$ that describes our beliefs about y had we known the true θ
- Our goal is then to get $p(\theta|y)$ which describes our beliefs for the true value of θ after observing our data y

Bayes Rule

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int_{\tilde{\theta}\in\Theta} p(y|\tilde{\theta})p(\tilde{\theta})d\tilde{\theta}}$$

Back to AMEN: How do we implement this model?

- We have a model: $Z \sim P(Z|\theta)$, $\theta \in \Theta$
- When Y is binary,

$$S(A) = \{Z : a_{i,j} > 0 \Rightarrow z_{i,j} > 0, a_{i,j} = 0 \Rightarrow z_{i,j} \leq 0\}$$

- Likelihood is:

$$L_B(\theta, Z) = Pr(Z \in S(A)|\theta) = \int_{S(A)} P(Z|\theta) d\mu(Z)$$

- We can then use a Gibbs Sampler with MH to approximate $P(\theta|Z \in S(A))$

Why AMEN? Different likelihoods!

- Up to this point, we have focused on adjacency matrices/ binary networks
- However, networks can have many different forms
- AMEN allows us to easily consider some of these
  - **Censored Binary**: Still 1 or 0, but not all connections are observed
  - **Ranked**: Connections are ranked by some level of importance
  - **Fixed Rank Nomination**: A node can only have a FIXED number of connections and they are ranked

Fixed Rank Nomination

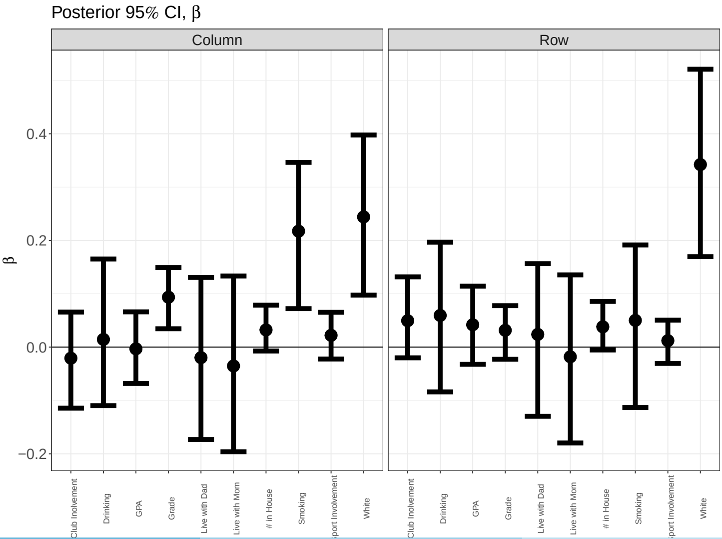
AddHealth

- Each person could rank up to 5 male friends and 5 female friends
- This introduces a *censorship* issue.

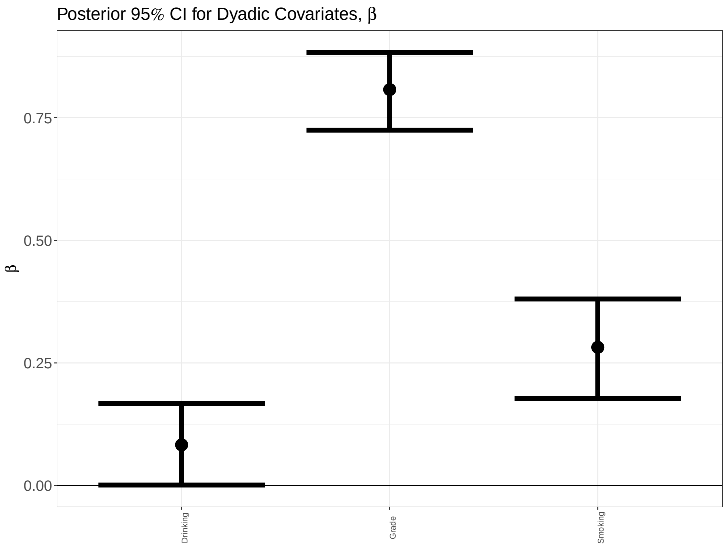
  If person i ranks the 5 people, are they friends with the 6^{th} and just didn't have room to rank them? or do they not like that person?
- If someone ranks less than 5 people, then we assume that they are not friends with person 6
- Another issue... perhaps person i just has never met person 6 but they would be great friends if they had

Back to AddHealth: Regression Results

- 95% CI for β estimates when fitting AMEN model with $R = 2$ on AddHealth network



Standard AMEN R=2



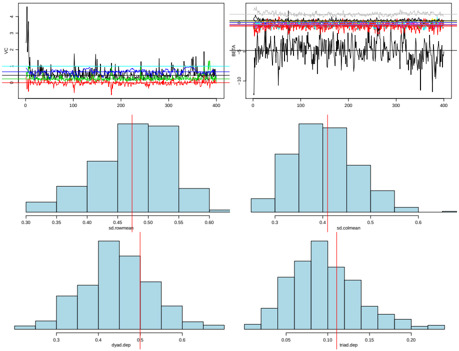
What appears to be important?

- Column: Grade, Smoking, White
- Row: White
- Dyadic: Sharing same drinking behavior, grade, smoking behavior

Diagnostics: Goodness of Fit Statistics

AMEN provides us with some posterior predictive goodness of fit statistics:

1. Empirical standard deviation for row means
2. Empirical standard deviation for column means
3. Empirical within-dyad correlation
4. Normalized measure of triadic dependence



And back to the lab...

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