Continuous Functions

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1 Convergence of functions

Given a metric space (X, d) and a collection of functions $f_n : X \to \mathbb{R}$, we distinguish between two notions of convergence to $f : X \to \mathbb{R}$.

- Pointwise convergence: for all $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$.
- Metric convergence: $\lim_{n\to\infty} \|f_n - f\| = 0$ for some metric $\|\cdot\|$ on the space of functions from which we draw each f_n.

By considering the sequence of maps $f_n : [0, 1] \to \mathbb{R}$ given by $x \mapsto x^n$, we see that a pointwise-convergent sequence of continuous functions may have a discontinuous limit. So this is a bad notion of convergence if we want to restrict ourselves to continuous functions.

For continuous functions $f : X \to \mathbb{R}$, a natural norm on spaces of continuous functions is the *uniform* or sup norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

Imagining the behavior of the L^p norm for large p, this is sometimes written $\|\cdot\|_\infty$.

2 Spaces of continuous functions

3 Approximation by polynomials

4 Compact subsets of $C(K)$

5 Ordinary differential equations