Notes on Metric and Normed Spaces

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1 Metrics and norms

1. $d(x, y) \geq 0$
2. $d(x, y) = d(y, x)$

Claim The unit ball in any normed linear space is convex.

Proof Let X be a linear space with norm $\|\cdot\|$. As usual, denote by $\overline{B}$ the unit ball

$$\overline{B} = \{x \in X \ : \ \|x\| \leq 1\}.$$

To demonstrate convexity, fix arbitrary $x, y \in \overline{B}$ and $t \in [0, 1]$. We claim that

$$tx + (1-t)y \in \overline{B}$$

or equivalently that

$$\|tx + (1-t)y\| \leq 1.$$

Indeed,

$$\begin{aligned} \|tx + (1-t)y\| &\leq \|tx\| + \|(1-t)y\| \\ &= t\|x\| + (1-t)\|y\| \\ &\leq t \cdot 1 + (1-t) \cdot 1 \\ &= 1. \end{aligned}$$

QED

Claim If $(X, \|\cdot\|)$ is a normed linear space, then

$$d(x, y) = \frac{\|x - y\|}{1 + \|x - y\|}$$

defined a nonhomogenous, translation-invariant metric on X.

Proof We proof that $d : X \times X \to \mathbb{R}$ is a metric one property at a time. Since the top and bottom of our fraction are always nonnegative, we have that $d(x, y) \geq 0$ for all $x, y \in X$. And the only time $d(x, y) = 0$ is when $\|x - y\| = 0$, which occurs only when $x = y$, so $d(x, y) = 0 \iff x = y$. Symmetry ($d(x, y) = d(y, x)$) is clear by inspection.

The triangle inequality is harder. Fix $x, y, z \in X$. We wish to show that

$$\frac{\|x - y\|}{1 + \|x - y\|} + \frac{\|y - z\|}{1 + \|y - z\|} \geq \frac{\|x - z\|}{1 + \|x - z\|}.$$

2 Convergence

Definition 1.12 A sequence (x_n) in a metric space (X, d) is *Cauchy* if $\forall \epsilon > 0, \exists N$ such that $m, n \geq N \in \mathbb{N} \implies d(x_n, x_m) < \epsilon$.

Definition 1.16 (Usual definition of convergence in a metric space)

Claim In a general metric space, every convergent sequence in is Cauchy.

Proof Let $(X, \|\cdot\|)$ be a metric space and let $\{x_n\}$ be a sequence in X converging to $x \in X$. Fix $\epsilon > 0$. So there exists N such that $n \geq N \implies d(x, x_n) < \epsilon/2$. Then for all $m, n \geq N$ the triangle inequality implies

$$\begin{aligned} d(x_n, x_m) &\leq d(x, x_n) + d(x, x_m) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

QED

Definition 1.17 A metric space (X, d) is *complete* if every Cauchy sequence in X converges to a limit in X. A subset Y is *complete* if the metric subspace $(Y, d|_Y)$ is complete. A *Banach space* is a normed linear space which is complete with respect to the norm-induced metric.

3 Upper and lower bounds

Definition 1.20 (defitions of *upper bound, lower bound, bounded from above, bounded from below* for subsets of $\mathbb{R}$)

Definition 1.121 (definitions of *supremum / least upper bound* and *infimum / greatest lower bound*)

Note that Hunter uses *monotone increasing* to mean *non-decreasing* ($n > m \implies x_n \geq x_m$) and likewise for *monotone decreasing.*

Given a sequence (x_n) in $\mathbb{R}$, we define

$$\limsup x_n = \lim_{n \to \infty} \left[\sup\{x_k \mid k \geq n\}\right].$$

Notice that the sequence (y_n) on the inside of the RHS given by

$$y_n = \sup\{x_k \mid k \geq n\}$$

is monotone increasing (i.e., never decreasing). We similarly define

$$\liminf x_n = \lim_{n\to\infty} \left[\inf\{x_k \mid k \geq n\}\right].$$

Both of these values always exist, as long as we allow $\pm\infty$ in addition to real values. Observe the useful sandwiching

$$\sup\{x_k \mid k \geq 1\} \geq \sup\{x_k \mid k \geq n\} \geq x_n \geq \inf\{x_k \mid k \geq n\} \geq \inf\{x_k \mid k \geq 1\}.$$

Notice that (x_n) is convergent if and only if $\liminf x_n = \limsup x_n$, in which case it converges to their common value.

4 Continuity

Definition 1.26 (continuity in a metric space) $f : X \to Y$ is *continuous* at $x_0 \in X$ if $\forall\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon.$$

$f : X \to Y$ is *continuous on* X if it is continuous at every points in X.

Example 1.27 (distance is continuous) Fix $a \in X$ and define $f : X \to \mathbb{R}$ by $f(x) = d(x, a)$. Then f is continuous on A.

Proof Per the premise, let (X, d) be a metric space with some $a \in X$ and define $f : X \to \mathbb{R}$ by the rule $f(x) = d(x, a)$. To show continuity, fix arbitrary $x_0 \in X$ and $\epsilon > 0$. Being carefully about which metrics are used where, we wish to find $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_{\mathbb{R}}(f(x), f(x_0)) < \epsilon.$$

From the definition of f, this is equivalent to the condition

$$d_X(x, x_0) < \delta \implies |d_X(x, a) - d_X(x_0, a)| < \epsilon.$$

But according to the following corrolary of the triangle inequality:

$$d(x_1, x_2) \geq |d(x_1, x_3) - d(x_2, x_3)|.$$

It suffices to set $\delta = \epsilon$. QED

Definition 1.30 (uniform continuity in a metric space) A function $f : X \to Y$ is *uniformly continuous* on X if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

for all $x, y \in X$. This differs from regular continuity in that δ is independent of $x, y \in X$.

Example 1.32 A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *affine* if

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) \; \forall x, y \in \mathbb{R}^n \text{ and } t \in [0, 1].$$

Every affine function is uniformly continuous and has the form $x \mapsto Ax + b$ for some constant matrix M and constant vector b.

Definition 1.33 A function $f : X \to Y$ is *sequentially continuous* at $x \in X$ if, for every sequence (x_n) converging to $x \in X$, the sequence $(f(x_n))$ converges to $f(x) \in Y$.

Proposition 1.34 Let X, Y be metric spaces. Then $f : X \to Y$ is continuous at $x \in X$ if and only if it is sequentially continuous at that point.

5 Open and closed sets

Open and closed balls in a metric space (X, d):

$$B_r(a) = \{x \in X \mid d(x, a) < r\},$$
$$\overline{B}_r(a) = \{x \in X \mid d(x, a) \geq r\}.$$

Definition 1.36 (open and closed sets in a metric space) Let (X, d) be a metric space. A subset $G \subset X$ is *open* if $\forall x \in G$, there exists $r > 0$ such that $B_r(x) \subset G$. A subset $F \subset X$ is *closed* if its complement $F^c = X - F$ is open.

Example 1.39 (rationals have Lebesgue measure zero) Let $\{q_n \mid n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$ and fix $\epsilon > 0$. For each $n \in N$, consider the open interval

$$I_n = \left(q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n}\right).$$

The union $\cup_{n\in\mathbb{N}} I_n$ is very interesting: it covers $\mathbb{Q}$, and yet the sum of the lengths of the intervals is 2ϵ, where ϵ can be made as small as we want. A subset of $\mathbb{R}$ has *Lebesgue measure zero* if $\forall \epsilon > 0$, there exists a countable collection of open intervals whose unions contains the subset and such that the sum of the lengths of the intervals is less than ϵ.

Proposition 1.41 (closed sets in terms of sequences) A subset F of a metric space is closed if and only if every convergent sequence of elements of F has its limit in F. That is, if $x_n \to x$ and $x_n \in F$ $\forall n \in \mathbb{N}$, then $x \in F$.

Definition (closure) In general topology, the closure $\overline{A}$ of a subset $A \subset X$ is the intersection of all closed sets containing A (a sort of minimal closed superset of A). In a metric space, we can construct this set by appending to A every point in X that can be reached as a limit of points in A.

$$\overline{A} = \{x \in X \mid \text{there exists a sequence } (a_n) \text{ in } X \text{ with } a_n \to x\}.$$

Definition 1.43 (dense subset) A subset A of a metric space X is *dense* in X if $\overline{A} = X$.

Definition 1.44 A metric space is *separable* if it contains a countable dense subset.

(more definitions to come)

6 The completion of a metric space

In this section we discuss how to produce a complete metric space from an incomplete one.

Definition 1.49 A map $\iota : X \to Y$ is called an *isometry* or an *isometric embedding* of X into Y if it satisfies

$$d_Y(\iota(x_1), \iota(x_2)) = d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. If $\iota : X \to Y$ is onto (and thus bijective) we call it a *metric space isomorphism* or more briefly an *isomorphism* if the metric structure is clear from context. As usual, if such an ι exists we say that X and Y are *isomorphic*.

Note that an isometry is automatically continuous (simply assign $\delta = \epsilon$ in the definition of continuity).

Example The map $\iota : \mathbb{C} \to \mathbb{R}^2$ with $x + iy \mapsto (x, y)$ is a metric spac isomorphism.

Definition 1.51 A metric space $(\tilde{X}, \tilde{d})$ is called the *completion* of the metric space (X, d) if the following conditions hold:

- There exists an isometric embedding $\iota : X \to \tilde{X}$.
- The image $\iota(X) \subset \tilde{X}$ is dense in X.
- $(\tilde{X}, \tilde{d})$ is complete.

Theorem 1.52 Every metric space has a completion. Furthermore, the completion is unique up to isomorphism.

Proof of Theorem 1.52 This proof is looong. Read the proof in the book, I won't write it here.

7 Compactness

Definition 1.54 A subset K of a metric space X is *sequentially compact* if every sequence in K has a convergent subsequence whose limit belongs to K.

Theorem 1.56 (Heine-Borel) A subset of $\mathbb{R}^n$ is sequentially compact iff it is closed and bounded.

Proof of 1.56 The proof in one direction is easy: suppose some subset $X \subset \mathbb{R}^n$ is closed and bounded and let (x_n) be a sequence in X. By Bolzano-Weierstrass (proved next), (x_n) has some convergence subsequence (y_n) (call the limit $y \in \mathbb{R}^n$). Since X is closed, we have $y \in X$. **Theorem 1.57 (Bolzano Weierstrass)** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof of 1.57 Recall the telescoping technique used to prove that every sequence contained in $[a,b] \subset \mathbb{R}$ contains a convergent subsequence. This technique generalizes naturally to $[a_1,b_1] \times [a_2,b_2] \subset \mathbb{R}^2$ if we use quadrants rather than halves, and likewise to higher dimensions. So given any bounded sequence in $\mathbb{R}^n$, pick any bounded hyper-rectangle and apply this telescoping technique to furnish a convergent subsequence. QED

Definition (ϵ-nets) Let X be a metric space with subspace A. Let $\epsilon > 0$. A collect of subsets $\{x_\alpha\}$ of X is called an *ϵ-net* of A if the family of open balls $\{x_\alpha\}$ covers A.

Definition 1.58 A subset of a metric space is *totally bounded* if it has a finite ϵ-net for every $\epsilon > 0$.

8 Maxima and minima

9 Exercises

Exercise 1.2 Give an $\epsilon - \delta$ proof that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

when $|x| < 1$.

Solution We begin by obtaining a nice expression for the nth partial sum

$$s_n(x) = \sum_{n=0}^{k} x^k = 1 + x + x^2 + \cdots + x^k.$$

To do so, observe that

$$\begin{aligned} xs_n(x) &= x \sum_{n=0}^{k} x^k \\ &= x + x^2 + \cdots + x^{k+1} \\ &= s_n(x) + x^{k+1} - 1. \end{aligned}$$

Solving for $s_n(x)$ gives

$$s_n(x) = \frac{1 - x^{k+1}}{1 - x}.$$

In other words,

$$\begin{aligned} s_n(x) &= \frac{1 - x^{k+1}}{1 - x} \\ &= \frac{1}{1 - x} - \frac{x^{k+1}}{1 - x}. \end{aligned}$$

So it finally suffices to show that $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$. For $\epsilon > 0$, we need to find $N \in \mathbb{N}$ such that $n \geq N$ implies $|x^n| < \epsilon$. To do so, simply set $N > \log_x(\epsilon)$. QED

Exercise 1.3 (Nov 25) If x, y, z are points in a metric space (X, d), show that

$$d(x, y) \geq |d(x, z) - d(y, z)|$$

Solution The triangle inequality tells us that

$$d(x, z) \geq d(x, y) + d(y, z)$$

and

$$d(y, z) \geq d(x, y) + d(x, z).$$

Isolating $d(x, y)$ in both inequalities gives

$$\begin{aligned} d(x, y) &\leq d(x, z) - d(y, z), \\ d(x, y) &\leq -((d, z) - d(y, z)). \end{aligned}$$

Equivalently,

$$d(x, y) \leq |d(x, z) - d(y, z)|.$$

QED

Exercise 1.4 (Nov 26) Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Prove that the Cartesian product $Z = X \times Y$ is a metric space with metric $d : Z \times Z \to \mathbb{R}$ defined by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

where $z_i = (x_i, y_i)$.

Solution Without any actual calculation we have

$$d(z_1, z_2) = d(z_2, d_1) \text{ (symmetry)}.$$

Since d_X and d_Y only give nonnegative values, so does d.

$$d(z_1, z_2) \geq 0.$$

And for the same reason, d vanishes if and only if d_X and d_Y do

$$d(z_1, z_2) = 0 \text{ iff } z_1 = z_2.$$

As for the triangle inequality, we already know that

$$\begin{aligned} d_X(x_1, x_3) &\leq d_X(x_1, x_2) + d_X(x_2, x_3), \\ d_Y(y_1, y_3) &\leq d_Y(y_1, y_2) + d_Y(y_2, y_3). \end{aligned}$$

Adding these two equations gives

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3).$$

QED

Exercise 1.8 (Nov 27) Let (x_n) be a bounded sequence of real numbers.

Part (a) Prove that for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there are $n_1, n_2 \geq N$, such that

$$\limsup x_n \leq x_{n_1} + \epsilon, \;\; x_{n_2} - \epsilon \leq \liminf x_n.$$

Solution to (a) Notice that we only need to prove the statement about $\limsup$, since the proof for $\liminf$ is logically identical. We introduce the nicer notation

$$\sup_n = \sup\{x_k \mid k \geq n\}$$

so that

$$\limsup x_n = \lim_{n\to\infty} \sup_n$$

and similarly for $\inf_n$ and $\liminf$. Now fix $\epsilon > 0$ and $N \in \mathbb{N}$. We want to prove that there exists $n_1 \geq N$ such that

$$\limsup x_n \leq n_1 + \epsilon.$$

Suppose that were false. Then we would have

$$\limsup x_n - \epsilon > x_{n_1}$$

for all $n_1 \geq N$. In other words, all terms past the Nth term would be bounded above by $\limsup x_n - \epsilon$. But then we would have $\sup_n$ and we would have

$$\sup_{n_1} \leq \limsup x_n - \epsilon$$

for all $n_1 \geq N$. This contraditions the fact that $\sup_{n_1} \to \limsup x_n$ as $n_1 \to \infty$, which must be true by definition, so our supposition is false. QED

Part (b) Prove that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$x_m \leq \limsup x_n + \epsilon,$$
$$x_m \geq \liminf x_n - \epsilon$$

for all $m \geq N$.

Solution to (b) Again by symmetry we only need to prove the first statement. So fix $\epsilon > 0$. From the definition

$$\limsup x_n = \lim_{m\to\infty} \sup_m = \lim_{m\to\infty} \sup\{x_k \mid k \geq m\}$$

and the definition of a limit, there must exist $N \in \mathbb{N}$ such that

$$m \geq N \implies |\sup_m - \limsup x_n| < \epsilon.$$

Since $(\sup_m)$ is monotic decreasing, the absolute value bars are superfluous:

$$m \geq N \implies \sup_m - \limsup x_n < \epsilon.$$

And by definition we have

$$x_m \leq \sup_m.$$

So we conclude that, for all $m \geq N$,

$$x_m - \limsup x_n < \epsilon.$$

QED

Part (c) Prove that (x_n) converges if and only if

$$\liminf x_n = \limsup x_n$$

Solution to (c) One direction of this equivalence amounts to the sandwich thereom: Assume $\liminf x_n = \limsup x_n$ and call the common limit $L \in \mathbb{R}$. Observe the convenient bounding

$$\inf_n \leq x_n \leq \sup_n \text{ for all } n \in N.$$

Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies L - \epsilon < \inf_n \leq x_n \leq \sup_n \leq L + \epsilon$$

and we have

$$\lim_{n\to\infty} x_n = L.$$

The opposite direction follows immediately from parts (a) and (b). QED

Exercise 1.11 (Nov 28) If $(x_n), (a_n), (b_n)$ are sequences of real numbers such that

$$\lim_{n\to\infty} x_n = x$$

and such that

$$a_n \leq x_n \leq b_n \; \forall n \in \mathbb{N}$$

prove that

$$\limsup a_n \leq x \leq \liminf b_n.$$

Solution First, observe that (x_n) converges and is thus bounded, so (a_n) is bounded above and (b_n) is bounded below. This still allows for the possibility that $\limsup a_n = -\infty$ or $\liminf b_n = \infty$, but in those cases our claimed inequalities are trivially satisfied. So without loss of generality, we assume that $\limsup a_n \in \mathbb{R}$ and $\liminf b_n \in \mathbb{R}$, or equivalently that (a_n) is also bounded below and (b_n) above.

From the inequality

$$a_n \leq x_n \; \forall n \in \mathbb{N}$$

we see that

$$\sup\{a_k \mid k \geq n\} \leq \sup\{x_k \mid k \geq n\}$$

which holds in the limit $n \to \infty$.

$$\limsup a_n \leq \limsup x_n.$$

But since (x_n) is a convergent series we know its lim sup agrees with its limit:

$$\limsup x_n = x.$$

So we conclude

$$\limsup a_n \leq x.$$

By identical reasoning, we have

$$x \leq \liminf b_n.$$

QED

Exercise 1.12 (Nov 28) Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Show that the composition

$$h = g \circ f : X \to Z$$

is also continuous.

Solution This problem is messy if we use the $\epsilon - \delta$ criterion for continuity but easy if we use the topological criterion: a function is continuous iff its preimages of open sets are open. So let $U \subset Z$ be open and let $V \subset X$ be its preimage under h. We wish to show that V is open. But we have

$$\begin{aligned} V &= h^{-1}(U) \\ &= (g \circ f)^{-1}(U) \\ &= f^{-1}(g^{-1}(U)) \end{aligned}$$

g is continuous, so $g^{-1}(U) \subset Y$ is open in Y. And f is continuous, so $f^{-1}(g^{-1}(U)) \subset X$ is open in X. QED