Notes on Metric and Normed Spaces

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1 Metrics and norms

Claim The unit ball in any normed linear space is convex.

Proof Let X be a linear space with norm $\|\cdot\|$. As usual, denote by $\overline{B}$ the unit ball

$$\overline{B} = \{x \in X \;:\; \|x\| \leq 1\}.$$

To demonstrate convexity, fix arbitrary $x, y \in \overline{B}$ and $t \in [0,1]$. We claim that

$$tx + (1-t)y \in \overline{B}$$

or equivalently that

$$\|tx + (1-t)y\| \leq 1.$$

Indeed,

$$\begin{aligned}
\|tx + (1-t)y\| &\leq \|tx\| + \|(1-t)y\| \\
&= t\,\|x\| + (1-t)\,\|y\| \\
&\leq t \cdot 1 + (1-t) \cdot 1 \\
&= 1.
\end{aligned}$$

QED

Claim If $(X, \|\cdot\|)$ is a normed linear space, then

$$d(x,y) = \frac{\|x-y\|}{1 + \|x-y\|}$$

defined a nonhomogenous, translation-invariant metric on X.

Proof We proof that $d : X \times X \to \mathbb{R}$ is a metric one property at a time. Since the top and bottom of our fraction are always nonnegative, we have that $d(x,y) \geq 0$ for all $x, y \in X$. And the only time $d(x,y) = 0$ is when $\|x-y\| = 0$, which occurs only when $x = y$, so $d(x,y) = 0 \iff x = y$. Symmetry ($d(x,y) = d(y,x)$) is clear by inspection.

The triangle inequality is harder. Fix $x, y, z \in X$. We wish to show that

$$\frac{\|x-y\|}{1+\|x-y\|} + \frac{\|y-z\|}{1+\|y-z\|} \geq \frac{\|x-z\|}{1+\|x-z\|}.$$

2 Convergence

Definition 1.12 A sequence (x_n) in a metric space (X,d) is *Cauchy* if $\forall \epsilon > 0, \exists N$ such that $m, n \geq N \in \mathbb{N} \implies d(x_n, x_m) < \epsilon$.

Definition 1.16 (Usual definition of convergence in a metric space)

Claim In a general metric space, every convergent sequence in is Cauchy.

Proof Let $(X, \|\cdot\|)$ be a metric space and let $\{x_n\}$ be a sequence in X converging to $x \in X$. Fix $\epsilon > 0$. So there exists N such that $n \geq N \implies d(x, x_n) < \epsilon/2$. Then for all $m, n \geq N$ the triangle inequality implies

$$\begin{aligned} d(x_n, x_m) &\leq d(x, x_n) + d(x, x_m) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

QED

Definition 1.17 A metric space (X, d) is *complete* if every Cauchy sequence in X converges to a limit in X. A subset Y is *complete* if the metric subspace $(Y, d|_Y)$ is complete. A *Banach space* is a normed linear space which is complete with respect to the norm-induced metric.

3 Upper and lower bounds

Definition 1.20 (defitions of *upper bound, lower bound, bounded from above, bounded from below* for subsets of $\mathbb{R}$)

Definition 1.121 (definitions of *supremum / least upper bound* and *infimum / greatest lower bound*)

Note that Hunter uses *monotone increasing* to mean *non-decreasing* ($n > m \implies x_n \geq x_m$) and likewise for *monotone decreasing.*

Given a sequence (x_n) in $\mathbb{R}$, we define

$$\text{lim sup } x_n = \lim_{n\to\infty} \left[\sup\{x_k \mid k \geq n\}\right].$$

Notice that the sequence (y_n) on the inside of the RHS given by

$$y_n = \sup\{x_k \mid k \geq n\}$$

is monotone increasing (i.e., never decreasing). We similarly define

$$\liminf x_n = \lim_{n\to\infty} \left[\inf\{x_k \mid k \geq n\}\right].$$

Both of these values always exist, as long as we allow $\pm\infty$ in addition to real values. Observe the useful sandwiching

$$\sup\{x_k \mid k \geq 1\} \geq \sup\{x_k \mid k \geq n\} \geq x_n \geq \inf\{x_k \mid k \geq n\} \geq \inf\{x_k \mid k \geq 1\}.$$

Notice that (x_n) is convergent if and only if $\liminf x_n = \limsup x_n$, in which case it converges to their common value.

4 Continuity

Definition 1.26 (continuity in a metric space) $f : X \to Y$ is *continuous* at $x_0 \in X$ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon.$$

$f : X \to Y$ is *continuous on* X if it is continuous at every points in X.

Example 1.27 (distance is continuous) Fix $a \in X$ and define $f : X \to \mathbb{R}$ by $f(x) = d(x, a)$. Then f is continuous on A.

Proof Per the premise, let (X, d) be a metric space with some $a \in X$ and define $f : X \to \mathbb{R}$ by the rule $f(x) = d(x, a)$. To show continuity, fix arbitrary $x_0 \in X$ and $\epsilon > 0$. Being carefully about which metrics are used where, we wish to find $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_{\mathbb{R}}(f(x), f(x_0)) < \epsilon.$$

From the definition of f, this is equivalent to the condition

$$d_X(x, x_0) < \delta \implies |d_X(x, a) - d_X(x_0, a)| < \epsilon.$$

But according to the following corrolary of the triangle inequality:

$$d(x_1, x_2) \geq |d(x_1, x_3) - d(x_2, x_3)|.$$

It suffices to set $\delta = \epsilon$. QED

Definition 1.30 (uniform continuity in a metric space) A function $f : X \to Y$ is *uniformly continuous* on X if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

for all $x, y \in X$. This differs from regular continuity in that δ is independent of $x, y \in X$.

Example 1.32 A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *affine* if

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) \ \forall x, y \in \mathbb{R}^n \text{ and } t \in [0, 1].$$

Every affine function is uniformly continuous and has the form $x \mapsto Ax + b$ for some constant matrix M and constant vector b.

Definition 1.33 A function $f : X \to Y$ is *sequentially continuous* at $x \in X$ if, for every sequence (x_n) converging to $x \in X$, the sequence $(f(x_n))$ converges to $f(x) \in Y$.

Proposition 1.34 Let X, Y be metric spaces. Then $f : X \to Y$ is continuous at $x \in X$ if and only if it is sequentially continuous at that point.

5 Open and closed sets

Open and closed balls in a metric space (X, d):

$$B_r(a) = \{x \in X \mid d(x, a) < r\},$$
$$\overline{B}_r(a) = \{x \in X \mid d(x, a) \geq r\}.$$

Definition 1.36 (open and closed sets in a metric space) Let (X, d) be a metric space. A subset $G \subset X$ is *open* if $\forall x \in G$, there exists $r > 0$ such that $B_r(x) \subset G$. A subset $F \subset X$ is *closed* if its complement $F^c = X - F$ is open.

Example 1.39 (rationals have Lebesgue measure zero) Let $\{q_n \mid n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$ and fix $\epsilon > 0$. For each $n \in N$, consider the open interval

$$I_n = \left(q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n}\right).$$

The union $\cup_{n\in\mathbb{N}} I_n$ is very interesting: it covers $\mathbb{Q}$, and yet the sum of the lengths of the intervals is 2ϵ, where ϵ can be made as small as we want. A subset of $\mathbb{R}$ has *Lebesgue measure zero* if $\forall\epsilon > 0$, there exists a countable collection of open intervals whose unions contains the subset and such that the sum of the lengths of the intervals is less than ϵ.

6 The completion of a metric space

7 Compactness

8 Maxima and minima

9 Exercises

Exercise 1.2 Give an $\epsilon - \delta$ proof that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

when $|x| < 1$.

Solution We begin by obtaining a nice expression for the nth partial sum

$$s_n(x) = \sum_{n=0}^{k} x^k = 1 + x + x^2 + \cdots + x^k.$$

To do so, observe that

$$\begin{aligned} xs_n(x) &= x\sum_{n=0}^{k} x^k \\ &= x + x^2 + \cdots + x^{k+1} \\ &= s_n(x) + x^{k+1} - 1. \end{aligned}$$

Solving for $s_n(x)$ gives

$$s_n(x) = \frac{1 - x^{k+1}}{1-x}.$$

In other words,

$$
\begin{aligned}
s_n(x) &= \frac{1 - x^{k+1}}{1 - x} \\
&= \frac{1}{1 - x} - \frac{x^{k+1}}{1 - x}.
\end{aligned}
$$

So it finally suffices to show that $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$. For $\epsilon > 0$, we need to find $N \in \mathbb{N}$ such that $n \geq N$ implies $|x^n| < \epsilon$. To do so, simply set $N > \log_x(\epsilon)$. QED

Exercise 1.3 If x, y, z are points in a metric space (X, d), show that

$$d(x, y) \geq |d(x, z) - d(y, z)|$$

Solution The triangle inequality tells us that

$$d(x, z) \geq d(x, y) + d(y, z)$$

and

$$d(y, z) \geq d(x, y) + d(x, z).$$

Isolating $d(x, y)$ in both inequalities gives

$$
\begin{aligned}
d(x, y) &\leq d(x, z) - d(y, z), \\
d(x, y) &\leq -((d, z) - d(y, z)).
\end{aligned}
$$

Equivalently,

$$d(x, y) \leq |d(x, z) - d(y, z)|.$$

QED