

Notes on Metric and Normed Spaces

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1 Metrics and norms

Claim The unit ball in any normed linear space is convex.

Proof Let X be a linear space with norm $\|\cdot\|$. As usual, denote by \overline{B} the unit ball

$$\overline{B} = \{x \in X : \|x\| \leq 1\}.$$

To demonstrate convexity, fix arbitrary $x, y \in \overline{B}$ and $t \in [0, 1]$. We claim that

$$tx + (1 - t)y \in \overline{B}$$

or equivalently that

$$\|tx + (1 - t)y\| \leq 1.$$

Indeed,

$$\begin{aligned} \|tx + (1 - t)y\| &\leq \|tx\| + \|(1 - t)y\| \\ &= t\|x\| + (1 - t)\|y\| \\ &\leq t \cdot 1 + (1 - t) \cdot 1 \\ &= 1. \end{aligned}$$

QED

Claim If $(X, \|\cdot\|)$ is a normed linear space, then

$$d(x, y) = \frac{\|x - y\|}{1 + \|x - y\|}$$

defined a nonhomogenous, translation-invariant metric on X .

Proof We proof that $d : X \times X \rightarrow \mathbb{R}$ is a metric one property at a time. Since the top and bottom of our fraction are always nonnegative, we have that $d(x, y) \geq 0$ for all $x, y \in X$. And the only time $d(x, y) = 0$ is when $\|x - y\| = 0$, which occurs only when $x = y$, so $d(x, y) = 0 \iff x = y$. Symmetry ($d(x, y) = d(y, x)$) is clear by inspection.

The triangle inequality is harder. Fix $x, y, z \in X$. We wish to show that

$$\frac{\|x - y\|}{1 + \|x - y\|} + \frac{\|y - z\|}{1 + \|y - z\|} \geq \frac{\|x - z\|}{1 + \|x - z\|}.$$

2 Convergence

Definition 1.12 A sequence (x_n) in a metric space (X, d) is *Cauchy* if $\forall \epsilon > 0, \exists N$ such that $m, n \geq N \in \mathbb{N} \implies d(x_n, x_m) < \epsilon$.

Definition 1.16 (Usual definition of convergence in a metric space)

Claim In a general metric space, every convergent sequence is Cauchy.

Proof Let $(X, \|\cdot\|)$ be a metric space and let $\{x_n\}$ be a sequence in X converging to $x \in X$. Fix $\epsilon > 0$. So there exists N such that $n \geq N \implies d(x, x_n) < \epsilon/2$. Then for all $m, n \geq N$ the triangle inequality implies

$$\begin{aligned} d(x_n, x_m) &\leq d(x, x_n) + d(x, x_m) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

QED

Definition 1.17 A metric space (X, d) is *complete* if every Cauchy sequence in X converges to a limit in X . A subset Y is *complete* if the metric subspace $(Y, d|_Y)$ is complete. A *Banach space* is a normed linear space which is complete with respect to the norm-induced metric.

3 Upper and lower bounds

Definition 1.20 (definitions of *upper bound*, *lower bound*, *bounded from above*, *bounded from below* for subsets of \mathbb{R})

Definition 1.121 (definitions of *supremum* / *least upper bound* and *infimum* / *greatest lower bound*)

Note that Hunter uses *monotone increasing* to mean *non-decreasing* ($n > m \implies x_n \geq x_m$) and likewise for *monotone decreasing*.

Given a sequence (x_n) in \mathbb{R} , we define

$$\limsup x_n = \lim_{n \rightarrow \infty} [\sup\{x_k \mid k \geq n\}].$$

Notice that the sequence (y_n) on the inside of the RHS given by

$$y_n = \sup\{x_k \mid k \geq n\}$$

is monotone increasing (i.e., never decreasing). We similarly define

$$\liminf x_n = \lim_{n \rightarrow \infty} [\inf\{x_k \mid k \geq n\}].$$

Both of these values always exist, as long as we allow $\pm\infty$ in addition to real values. Observe the useful sandwiching

$$\sup\{x_k \mid k \geq 1\} \geq \sup\{x_k \mid k \geq n\} \geq x_n \geq \inf\{x_k \mid k \geq n\} \geq \inf\{x_k \mid k \geq 1\}.$$

Notice that (x_n) is convergent if and only if $\liminf x_n = \limsup x_n$, in which case it converges to their common value.

4 Continuity

Definition 1.26 (continuity in a metric space) $f : X \rightarrow Y$ is *continuous* at $x_0 \in X$ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon.$$

$f : X \rightarrow Y$ is *continuous on* X if it is continuous at every points in X .

Example 1.27 (distance is continuous) Fix $a \in X$ and define $f : X \rightarrow \mathbb{R}$ by $f(x) = d(x, a)$. Then f is continuous on A .

Proof Per the premise, let (X, d) be a metric space with some $a \in X$ and define $f : X \rightarrow \mathbb{R}$ by the rule $f(x) = d(x, a)$. To show continuity, fix arbitrary $x_0 \in X$ and $\epsilon > 0$. Being carefully about which metrics are used where, we wish to find $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_{\mathbb{R}}(f(x), f(x_0)) < \epsilon.$$

From the definition of f , this is equivalent to the condition

$$d_X(x, x_0) < \delta \implies |d_X(x, a) - d_X(x_0, a)| < \epsilon.$$

But according to the following corollary of the triangle inequality:

$$d(x_1, x_2) \geq |d(x_1, x_3) - d(x_2, x_3)|.$$

It suffices to set $\delta = \epsilon$. QED

Definition 1.30 (uniform continuity in a metric space) A function $f : X \rightarrow Y$ is *uniformly continuous* on X if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

for all $x, y \in X$. This differs from regular continuity in that δ is independent of $x, y \in X$.

Example 1.32 A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *affine* if

$$f(tx + (1 - t)y) = tf(x) + (1 - t)f(y) \quad \forall x, y \in \mathbb{R}^n \text{ and } t \in [0, 1].$$

Every affine function is uniformly continuous and has the form $x \mapsto Ax + b$ for some constant matrix M and constant vector b .

Definition 1.33 A function $f : X \rightarrow Y$ is *sequentially continuous* at $x \in X$ if, for every sequence (x_n) converging to $x \in X$, the sequence $(f(x_n))$ converges to $f(x) \in Y$.

Proposition 1.34 Let X, Y be metric spaces. Then $f : X \rightarrow Y$ is continuous at $x \in X$ if and only if it is sequentially continuous at that point.

5 Open and closed sets

Open and closed balls in a metric space (X, d) :

$$\begin{aligned} B_r(a) &= \{x \in X \mid d(x, a) < r\}, \\ \overline{B}_r(a) &= \{x \in X \mid d(x, a) \leq r\}. \end{aligned}$$

Definition 1.36 (open and closed sets in a metric space) Let (X, d) be a metric space. A subset $G \subset X$ is *open* if $\forall x \in G$, there exists $r > 0$ such that $B_r(x) \subset G$. A subset $F \subset X$ is *closed* if its complement $F^c = X - F$ is open.

Example 1.39 (rationals have Lebesgue measure zero) Let $\{q_n \mid n \in \mathbb{N}\}$ be an enumeration of \mathbb{Q} and fix $\epsilon > 0$. For each $n \in \mathbb{N}$, consider the open interval

$$I_n = \left(q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n} \right).$$

The union $\cup_{n \in \mathbb{N}} I_n$ is very interesting: it covers \mathbb{Q} , and yet the sum of the lengths of the intervals is 2ϵ , where ϵ can be made as small as we want. A subset of \mathbb{R} has *Lebesgue measure zero* if $\forall \epsilon > 0$, there exists a countable collection of open intervals whose unions contains the subset and such that the sum of the lengths of the intervals is less than ϵ .

Proposition 1.41 (closed sets in terms of sequences) A subset F of a metric space is closed if and only if every convergent sequence of elements of F has its limit in F . That is, if $x_n \rightarrow x$ and $x_n \in F \forall n \in \mathbb{N}$, then $x \in F$.

Definition (closure) In general topology, the closure \overline{A} of a subset $A \subset X$ is the intersection of all closed sets containing A (a sort of minimal closed superset of A). In a metric space, we can construct this set by appending to A every point in X that can be reached as a limit of points in A .

$$\overline{A} = \{x \in X \mid \text{there exists a sequence } (a_n) \text{ in } A \text{ with } a_n \rightarrow x\}.$$

Definition 1.43 (dense subset) A subset A of a metric space X is *dense* in X if $\overline{A} = X$.

Definition 1.44 A metric space is *separable* if it contains a countable dense subset.

(more definitions to come)

6 The completion of a metric space

In this section we discuss how to produce a complete metric space from an incomplete one.

Definition 1.49 A map $\iota : X \rightarrow Y$ is called an *isometry* or an *isometric embedding* of X into Y if it satisfies

$$d_Y(\iota(x_1), \iota(x_2)) = d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. If $\iota : X \rightarrow Y$ is onto (and thus bijective) we call it a *metric space isomorphism* or more briefly an *isomorphism* if the metric structure is clear from context. As usual, if such an ι exists we say that X and Y are

isomorphic.

Note that an isometry is automatically continuous (simply assign $\delta = \epsilon$ in the definition of continuity).

Example The map $\iota : \mathbb{C} \rightarrow \mathbb{R}^2$ with $x + iy \mapsto (x, y)$ is a metric space isomorphism.

Definition 1.51 A metric space (\tilde{X}, \tilde{d}) is called the *completion* of the metric space (X, d) if the following conditions hold:

- There exists an isometric embedding $\iota : X \rightarrow \tilde{X}$.
- The image $\iota(X) \subset \tilde{X}$ is dense in \tilde{X} .
- (\tilde{X}, \tilde{d}) is complete.

Theorem 1.52 Every metric space has a completion. Furthermore, the completion is unique up to isomorphism.

Proof of Theorem 1.52 This proof is looong. Read the proof in the book, I won't write it here.

7 Compactness

8 Maxima and minima

9 Exercises

Exercise 1.2 Give an $\epsilon - \delta$ proof that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

when $|x| < 1$.

Solution We begin by obtaining a nice expression for the n th partial sum

$$s_n(x) = \sum_{k=0}^n x^k = 1 + x + x^2 + \cdots + x^n.$$

To do so, observe that

$$\begin{aligned} xs_n(x) &= x \sum_{n=0}^k x^n \\ &= x + x^2 + \cdots + x^{k+1} \\ &= s_n(x) + x^{k+1} - 1. \end{aligned}$$

Solving for $s_n(x)$ gives

$$s_n(x) = \frac{1 - x^{k+1}}{1 - x}.$$

In other words,

$$\begin{aligned} s_n(x) &= \frac{1 - x^{k+1}}{1 - x} \\ &= \frac{1}{1 - x} - \frac{x^{k+1}}{1 - x}. \end{aligned}$$

So it finally suffices to show that $\lim_{n \rightarrow \infty} x^n = 0$ for $|x| < 1$. For $\epsilon > 0$, we need to find $N \in \mathbb{N}$ such that $n \geq N$ implies $|x^n| < \epsilon$. To do so, simply set $N > \log_x(\epsilon)$. QED

Exercise 1.3 (Nov 25) If x, y, z are points in a metric space (X, d) , show that

$$d(x, y) \geq |d(x, z) - d(y, z)|$$

Solution The triangle inequality tells us that

$$d(x, z) \geq d(x, y) + d(y, z)$$

and

$$d(y, z) \geq d(x, y) + d(x, z).$$

Isolating $d(x, y)$ in both inequalities gives

$$\begin{aligned} d(x, y) &\leq d(x, z) - d(y, z), \\ d(x, y) &\leq -((d, z) - d(y, z)). \end{aligned}$$

Equivalently,

$$d(x, y) \leq |d(x, z) - d(y, z)|.$$

QED

Exercise 1.4 (Nov 26) Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Prove that the Cartesian product $Z = X \times Y$ is a metric space with metric $d : Z \times Z \rightarrow \mathbb{R}$ defined by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

where $z_i = (x_i, y_i)$.

Solution Without any actual calculation we have

$$d(z_1, z_2) = d(z_2, z_1) \text{ (symmetry).}$$

Since d_X and d_Y only give nonnegative values, so does d .

$$d(z_1, z_2) \geq 0.$$

And for the same reason, d vanishes if and only if d_X and d_Y do

$$d(z_1, z_2) = 0 \text{ iff } z_1 = z_2.$$

As for the triangle inequality, we already know that

$$\begin{aligned} d_X(x_1, x_3) &\leq d_X(x_1, x_2) + d_X(x_2, x_3), \\ d_Y(y_1, y_3) &\leq d_Y(y_1, y_2) + d_Y(y_2, y_3). \end{aligned}$$

Adding these two equations gives

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3).$$

QED

Exercise 1.8 (Nov 27) Let (x_n) be a bounded sequence of real numbers.

Part (a) Prove that for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there are $n_1, n_2 \geq N$, such that

$$\limsup x_n \leq x_{n_1} + \epsilon, \quad x_{n_2} - \epsilon \leq \liminf x_n.$$

Solution to (a) Notice that we only need to prove the statement about \limsup , since the proof for \liminf is logically identical. We introduce the nicer notation

$$\sup_n = \sup\{x_k \mid k \geq n\}$$

so that

$$\limsup x_n = \lim_{n \rightarrow \infty} \sup_n$$

and similarly for \inf_n and \liminf . Now fix $\epsilon > 0$ and $N \in \mathbb{N}$. We want to prove that there exists $n_1 \geq N$ such that

$$\limsup x_n \leq n_1 + \epsilon.$$

Suppose that were false. Then we would have

$$\limsup x_n - \epsilon > x_{n_1}$$

for all $n_1 \geq N$. In other words, all terms past the N th term would be bounded above by $\limsup x_n - \epsilon$. But then we would have \sup_n and we would have

$$\sup_{n_1} \leq \limsup x_n - \epsilon$$

for all $n_1 \geq N$. This contradicts the fact that $\sup_{n_1} \rightarrow \limsup x_n$ as $n_1 \rightarrow \infty$, which must be true by definition, so our supposition is false. QED

Part (b) Prove that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} x_m &\leq \limsup x_n + \epsilon, \\ x_m &\geq \liminf x_n - \epsilon \end{aligned}$$

for all $m \geq N$.

Solution to (b) Again by symmetry we only need to prove the first statement. So fix $\epsilon > 0$. From the definition

$$\limsup x_n = \lim_{m \rightarrow \infty} \sup_m = \lim_{m \rightarrow \infty} \sup\{x_k \mid k \geq m\}$$

and the definition of a limit, there must exist $N \in \mathbb{N}$ such that

$$m \geq N \implies |\sup_m - \limsup x_n| < \epsilon.$$

Since (\sup_m) is monotonic decreasing, the absolute value bars are superfluous:

$$m \geq N \implies \sup_m - \limsup x_n < \epsilon.$$

And by definition we have

$$x_m \leq \sup_m.$$

So we conclude that, for all $m \geq N$,

$$x_m - \limsup x_n < \epsilon.$$

QED

Part (c) Prove that (x_n) converges if and only if

$$\liminf x_n = \limsup x_n$$

Solution to (c) One direction of this equivalence amounts to the sandwich theorem: Assume $\liminf x_n = \limsup x_n$ and call the common limit $L \in \mathbb{R}$. Observe the convenient bounding

$$\inf_n x_n \leq x_n \leq \sup_n x_n \text{ for all } n \in \mathbb{N}.$$

Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies L - \epsilon < \inf_n x_n \leq x_n \leq \sup_n x_n < L + \epsilon$$

and we have

$$\lim_{n \rightarrow \infty} x_n = L.$$

The opposite direction follows immediately from parts (a) and (b). QED

Exercise 1.11 (Nov 28) If $(x_n), (a_n), (b_n)$ are sequences of real numbers such that

$$\lim_{n \rightarrow \infty} x_n = x$$

and such that

$$a_n \leq x_n \leq b_n \quad \forall n \in \mathbb{N}$$

prove that

$$\limsup a_n \leq x \leq \liminf b_n.$$

Solution First, observe that (x_n) converges and is thus bounded, so (a_n) is bounded above and (b_n) is bounded below. This still allows for the possibility that $\limsup a_n = -\infty$ or $\liminf b_n = \infty$, but in those cases our claimed inequalities are trivially satisfied. So without loss of generality, we assume that $\limsup a_n \in \mathbb{R}$ and $\liminf b_n \in \mathbb{R}$, or equivalently that (a_n) is also bounded below and (b_n) above.

From the inequality

$$a_n \leq x_n \quad \forall n \in \mathbb{N}$$

we see that

$$\sup\{a_k \mid k \geq n\} \leq \sup\{x_k \mid k \geq n\}$$

which holds in the limit $n \rightarrow \infty$.

$$\limsup a_n \leq \limsup x_n.$$

But since (x_n) is a convergent series we know its \limsup agrees with its limit:

$$\limsup x_n = x.$$

So we conclude

$$\limsup a_n \leq x.$$

By identical reasoning, we have

$$x \leq \liminf b_n.$$

QED

Exercise 1.12 (Nov 28) Let (X, d_X) , (Y, d_Y) , and (Z, d_Z) be metric spaces and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous functions. Show that the composition

$$h = g \circ f : X \rightarrow Z$$

is also continuous.

Solution This problem is messy if we use the $\epsilon - \delta$ criterion for continuity but easy if we use the topological criterion: a function is continuous iff its preimages of open sets are open. So let $U \subset Z$ be open and let $V \subset X$ be its preimage under h . We wish to show that V is open. But we have

$$\begin{aligned} V &= h^{-1}(U) \\ &= (g \circ f)^{-1}(U) \\ &= f^{-1}(g^{-1}(U)) \end{aligned}$$

g is continuous, so $g^{-1}(U) \subset Y$ is open in Y . And f is continuous, so $f^{-1}(g^{-1}(U)) \subset X$ is open in X . QED