Solutions to Exercises in *Introduction to Manifolds* by Tu

Mathew Calkins
mathewpcalkins@gmail.com

November 16, 2018

1 Smooth Functions on a Euclidean Space

1.1 A function that is C^2 but not C^3

Let $g : \mathbb{R} \to \mathbb{R}$ be the function in Example 1.2(iii). Show that the function $h(x) = \int_0^x g(t)dt$ is C^2 but not C^3 at $x = 0$.

Solution In Example 1.2(iii) we defined

$$f(x) = x^{1/3}$$

and

$$g(x) = \int_0^x f(t)dt = \frac{3}{4}x^{4/3}$$

. Notice that

$$h''(x) = g'(x) = f(x) = x^{1/3}.$$

Since the 2nd derivative of h exists and is continuous at $x = 0$, h is C^2 at $x = 0$. However, $h'''(x) = f'(x)$ is does not exist at $x = 0$ so h is not C^3 at $x = 0$. QED

1.2 A function very flat at 0

Let $f(x)$ be the function on $\mathbb{R}$ defined in example 1.3.

(a) Show by induction that for $x > 0$ and $k \geq 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.

Solution For $x > 0$ (and we only concern ourselvers with that case for this problem) recall that $f(x) = e^{1/x}$. So our claim is true for $k = 0$ with $p_1(y) = 1$. By standard calculus we have

$$f'(x) = -\frac{1}{x^2}e^{1/x}.$$

In other words,

$$f^{(1)}(x) = p_{2\cdot 1}(1/x)e^{1/x}$$

where

$$p_2(y) = -y^2.$$

So our claim is also true for $k = 1$. So suppose that we have

$$f^{(k)}(x) = p_{2k}(1/x)e^{1/x}$$

for some $k \geq 1$. Then the chain rule gives

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left(p_{2k}(1/x)e^{1/x}\right) \\ &= -\frac{p'_{2k}(1/x)}{x^2}e^{1/x} - p_{2k}(1/x)\frac{e^{1/x}}{x^2} \\ &= \left(-\frac{p_{2k}(1/x) + p'_{2k}(1/x)}{x^2}\right)e^{1/x}. \end{aligned}$$

Since the term inside the parentheses has order $2(k+1)$ when thought over as a polynomial in $1/x$, we see conclude our induction step and write

$$f^{(k+1)}(x) = p_{2(k+1)}(1/x)e^{1/x}.$$

(b) Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.

Solution Recall that, for each $k \geq 0$, we have (to be continued)

2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

3 The Exterior Algebra of Multicovectors

4 Differential Forms on $\mathbb{R}^n$

4.1 A 1-form on $\mathbb{R}^3$

Let ω be the 1-form $zdx - dz$ and let X be the vector field $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

Solution

From linearity and the identity

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$$

we have

$$\begin{aligned}
\omega(X) &= (zdx - dz)\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\
&= zydx\left(\frac{\partial}{\partial x}\right) + \text{ terms that vanish by orthogonality} \\
&= zy.
\end{aligned}$$

Next we write

$$\omega = \omega_i dx^i$$

where $(\omega_1, \omega_2, \omega_3) = (z, 0, -1)$. Then the general rule for differentating 1-forms states that

$$\begin{aligned}
d\omega &= d\omega_i \wedge dx^i \\
&= d\omega_1 \wedge dx^1 + d\omega_2 \wedge dx^2 + d\omega_3 \wedge dx^3 \\
&= dz \wedge dx + 0 \wedge dy + 0 \wedge dz \\
&= dz \wedge dx.
\end{aligned}$$

5 Manifolds

The real line with two origins

Let A and B be two points not on the real line $\mathbb{R}$. Consider the set $S = (R - \{0\}) \cup A, B$. For any two positive real number c, d, define

$$I_A(-c, d) =]-c, 0[\cup \{A\} \cup]0, d[$$

and similarly for $I_B(-c, d)$ with B instead of A. Define a topology on S as follows: on $(\mathbb{R} - \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis of neighborhoods at A is the set

$$\{I_A(-c, d) \mid c, d > 0\}$$

and likewise at B.

(a) Prove that the map $h : I_A(-c, d) \to]-c, 0[\cup]0, d[$ defined by

$$\begin{aligned} h(x) &= 0 \quad \text{for } x \in]-c, 0[\cup]0, d[, \\ h(A) &= 0 \end{aligned}$$

is a homeomorphism.

6 Smooth Maps on a Manifold

7 Quotients