A First Amplitude

Mathew Calkins
mathewpcalkins@gmail.com

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Abstract

We work through an elementary S-matrix calculation in scalar Yukawa theory following the popular exposition by David Tong. For pedagogical purposes, we proceed slowly and begin with a self-contained review of the necessary background in QFT.

Contents

0 Review of prequisite facts

0.1 Free Fields

We are interested in the dynamics of a real massive scalar field ϕ and a complex massive scalar field ψ, whose corresponding respective particles we'll call mesons and nucleons. In its simplest free theory, the dynamics of ϕ are governed by the *Klein Gordon* Lagrangian

$$\mathcal{L}_{\mathrm{KG}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

The complex scalar field behaves with the natural complex general generalization of those dynamics.

$$\mathcal{L}_{\mathrm{KG},\mathbb{C}} = \partial_\mu\psi^*\partial^\mu\psi - M^2\psi^*\psi.$$

Canonical quantization of ϕ in the Schrödinger picture gives the operator-valued scalar field

$$\phi_S(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

where $E_{\mathbf{p}}$ is shorthand for

$$E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2},$$

$a^\dagger_{\mathbf{p}}$ is a meson-producing creation operator

$$a^\dagger_{\mathbf{p}}\left|\mathbf{p}_1, \ldots, \mathbf{p}_n\right\rangle = \left|\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{p}\right\rangle$$

and $a_{\mathbf{p}}$ is the corresponding annihilation operator. Crucially,

$$a_{\mathbf{p}}\left|\mathbf{p}_1, \ldots, \mathbf{p}_n\right\rangle = 0$$

if none of the $\mathbf{p}_i$ are equal to $\mathbf{p}$. In particular, annihilation operators all annihilate the vacuum.

$$a_{\mathbf{p}}\left|0\right\rangle = 0.$$

A slightly different story holds for ψ. Canonical quantization gives

$$\psi_S(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + c^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right).$$

Here b is an annihilation operator for nucleons and $c^\dagger$ is a creation operator for *anti*-nucleons. To create nucleons and annihilate anti-nucleons we apply the conjugate field

$$\psi^\dagger_S(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + c_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right).$$

Notice that when we quantize a complex field we replace superscripts as $* \to \dagger$.

To keep track of which fields create and annihilate which sorts of particles, we briefly record that

$$\begin{aligned} \phi &\sim a + a^\dagger, \\ \psi &\sim b + c^\dagger, \\ \psi^\dagger &\sim b^\dagger + c. \end{aligned}$$

0.2 Basis states

In a universe of mesons, nucleons, and anti-nucleons, every possible state can be built up from the creation operators above. We have a single 0-particle basis vector

$$|0\rangle ,$$

three collections of 1-particle basis vectors parametrized by momentum

$$a_{\mathbf{p}} |0\rangle , b_{\mathbf{p}} |0\rangle , c_{\mathbf{p}} |0\rangle ,$$

six collections of 2-particle basis states parametrized the same way

$$a_{\mathbf{p}} a_{\mathbf{q}} |0\rangle , a_{\mathbf{p}} b_{\mathbf{q}} |0\rangle , a_{\mathbf{p}} c_{\mathbf{q}} |0\rangle , b_{\mathbf{p}} b_{\mathbf{q}} |0\rangle , b_{\mathbf{p}} c_{\mathbf{q}} |0\rangle , c_{\mathbf{p}} c_{\mathbf{q}} |0\rangle$$

and so on. Tthe commutation relations

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = [c_{\mathbf{p}}, c_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}),$$

tell us that basis states with identical particle content are Dirac-orthonormal up to a factor of $(2\pi)^3$. For example, let $|\mathbf{p}\rangle$ and $|\mathbf{q}\rangle$ be basis states each with a single meson. Then we have

$$\begin{aligned} \langle 0| [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] |0\rangle &= \langle 0| (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle \\ \langle 0| a_{\mathbf{p}} a_{\mathbf{q}}^\dagger |0\rangle - \langle 0| a_{\mathbf{q}}^\dagger a_{\mathbf{p}} |0\rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \langle 0 |0\rangle \\ \langle \mathbf{p} |\mathbf{q}\rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned}$$

All other commutators of creation and annihilation operators vanish, which tells us that basis states that differ in their numbers of mesons, nucleons, or anti-nucleons are always orthogonal to one another.

0.3 The Interaction Picture

Frequently we are interested in systems with Hamiltonians of the form

$$H = H_0 + H_{\text{int}}$$

where H_0 corresponds to a free theory with known spectrum. In that case it is helpful to transition into the *interaction picture*, defined in terms of the Schrödinger picture as follows. First, H_0 is defined to agree with the Schrödinger picture:

$$(H_0)_I = (H_0)_S.$$

Operators in general time evolve as

$$\mathcal{O}_I(t) = e^{iH_0t}\mathcal{O}_s e^{-iH_0t}.$$

States evolve as

$$|\psi(t)\rangle_I = e^{iH_0t}\, |\psi(t)\rangle_S\, .$$

The dynamics of the states are then governed by the interaction Hamiltonian:

$$i\frac{d\,|\psi(t)\rangle_I}{dt} = H_I(t)\, |\psi(t)\rangle_I$$

where we use the shorthand

$$H_I(t) \equiv (H_{\text{int}})_I(t).$$

0.4 Dyson's formula

It's useful to write the dynamics of a theory in terms of a unitary time-evolution operator U which acts as

$$|\psi(t)\rangle_I = U(t, t_0)\, |\psi(t_0)\rangle_I\, .$$

This is soluble in terms of the familiar time-ordering operator as

$$U(t, t_o) = T\exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right).$$

We define the *S-matrix* by declaring its matrix elements $\langle f|\, S\, |i\rangle$ to be

$$\langle f|\, S\, |i\rangle = \lim_{t_\pm\to\pm\infty} \langle f|\, U(t_+, t_-)\, |i\rangle\, .$$

0.5 Scalar Yukawa Theory

We work with in an interacting meson + nucleon theory built by combining the free theories we described earlier and adding an interaction term:

$$\begin{aligned}
\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi & \qquad\qquad \text{(kinetic terms)} \\
- M^2\psi^*\psi - \frac{1}{2}m^2\phi^2 & \qquad\qquad \text{(free potential terms)} \\
- g\psi^*\psi\phi & \qquad\qquad \text{(interaction term)}
\end{aligned}$$

The corresponding Hamiltonian is

$$H = H_0 + H_{\text{int}}$$

where

$$\begin{aligned}
H_0 = \int d^3x \bigg(&\partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\
&+ M^2\psi^*\psi + \frac{1}{2}m^2\phi^2\bigg).
\end{aligned}$$

and

$$H_{\text{int}} = -g\int d^3x\ \psi^\dagger\psi\phi.$$

The time evolution rule for operators in the interaction picture then gives

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)$$

and

$$\psi_I(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{ip\cdot x} + c_{\mathbf{p}}^\dagger e^{ip\cdot x}\right).$$

But we aren't done yet as we don't have a nice closed-form description of state evolution. The next section computes this approximately

1 Meson Decay

1.1 Setting up the integral

We are interested in the amplitude for a meson to decay into a nucleon + anti-nucleon pair, computed out to linear order in the coupling constant g. We abbreviate this process as

$$\phi \to \psi\overline{\psi}$$

For simplicity we use initial and final states of well-defined momenta. After applying the appropriately relativistic normalizations, our initial and final states are

$$\begin{aligned}
|i\rangle &= \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}}\,|0\rangle\,, \\
|f\rangle &= \sqrt{4E_{\mathbf{q}_1}E_{\mathbf{q}_2}}b^\dagger_{\mathbf{q}_1}c^\dagger_{\mathbf{q}_2}\,|0\rangle\,.
\end{aligned}$$

The quantity we are interested in is the S-matrix element

$$\langle f|\,S\,|i\rangle\,.$$

As usual we will interchange limits as convenient without worrying about analytic subtleties.

By definition this matrix element is

$$\langle f|\,S\,|i\rangle = \lim_{t_\pm\to\pm\infty}\langle f|\,U(t_+,t_-)\,|i\rangle\,.$$

Dyson's formula gives

$$\begin{aligned}
\langle f|\,S\,|i\rangle &= \lim_{t_\pm\to\pm\infty}\langle f|\left[T\exp\left(-i\int_{t_-}^{t_+}H_I(t')dt'\right)\right]|i\rangle \\
&= \langle f|\left[T\exp\left(-i\int_{-\infty}^{\infty}H_I(t')dt'\right)\right]|i\rangle \\
&= \langle f|\left[T\exp\left(-ig\int d^4x\ \psi_I^\dagger(x)\psi_I(x)\phi_I(x)\right)\right]|i\rangle\,.
\end{aligned}$$

Going out to linear order in g means going out to the linear term in the exponential.

$$T\exp\left(\cdots\right) = 1 - ig\int d^4x\ \psi_I^\dagger(x)\psi_I(x)\phi_I(x) + o(g^2).$$

So our S-matrix element is

$$\begin{aligned}
\langle f|\,S\,|i\rangle &= \langle f|\left[1 - ig\int d^4x\ \psi_I^\dagger(x)\psi_I(x)\phi_I(x)\right]|i\rangle \\
&= \langle f\,|i\rangle - ig\int d^4x\ \ \langle i|\,\psi_I^\dagger(x)\psi_I(x)\phi_I(x)\,|f\rangle\,.
\end{aligned}$$

Our initial and final states have different particle content, so

$$\langle f|\,S\,|i\rangle = ig\int d^4x\ \ \langle i|\,\psi_I^\dagger(x)\psi_I(x)\phi_I(x)\,|f\rangle\,.$$