Notes on Metric and Normed Spaces

Mathew Calkins
mathewpcalkins@gmail.com

November 19, 2018

1 Metrics and norms

Claim The unit ball in any normed linear space is convex.

Proof Let X be a linear space with norm $\|\cdot\|$. As usual, denote by $\overline{B}$ the unit ball

$$\overline{B} = \{x \in X \ : \ \|x\| \leq 1\}.$$

To demonstrate convexity, fix arbitrary $x, y \in \overline{B}$ and $t \in [0,1]$. We claim that

$$tx + (1-t)y \in \overline{B}$$

or equivalently that

$$\|tx + (1-t)y\| \leq 1.$$

Indeed,

$$\begin{aligned}
\|tx + (1-t)y\| &\leq \|tx\| + \|(1-t)y\| \\
&= t\,\|x\| + (1-t)\,\|y\| \\
&\leq t \cdot 1 + (1-t) \cdot 1 \\
&1.
\end{aligned}$$

QED

Claim If $(X, \|\cdot\|)$ is a normed linear space, then

$$d(x,y) = \frac{\|x-y\|}{1 + \|x-y\|}$$

defined a nonhomogenous, translation-invariant metric on X.

Proof We proof that $d : X \times X \to \mathbb{R}$ is a metric one property at a time. Since the top and bottom of our fraction are always nonnegative, we have that $d(x, y) \geq 0$ for all $x, y \in X$. And the only time $d(x, y) = 0$ is when $\|x - y\| = 0$, which occurs only when $x = y$, so $d(x, y) = 0 \iff x = y$. Symmetry ($d(x, y) = d(y, x)$) is clear by inspection.

The triangle inequality is harder. Fix $x, y, z \in X$. We wish to show that

$$\frac{\|x-y\|}{1+\|x-y\|} + \frac{\|y-z\|}{1+\|y-z\|} \geq \frac{\|x-z\|}{1+\|x-z\|}.$$

2 Convergence

Definition 1.12 A sequence (x_n) in a metric space (X, d) is *Cauchy* if $\forall \epsilon > 0, \exists N$ such that $m, n \geq N \in \mathbb{N} \implies d(x_n, x_m) < \epsilon$.

Definition 1.16 (Usual definition of convergence in a metric space)

Claim In a general metric space, every convergent sequence in is Cauchy.

Proof Let $(X, \|\cdot\|)$ be a metric space and let $\{x_n\}$ be a sequence in X converging to $x \in X$. Fix $\epsilon > 0$. So there exists N such that $n \geq N \implies d(x, x_n) < \epsilon/2$. Then for all $m, n \geq N$ the triangle inequality implies

$$\begin{aligned} d(x_n, x_m) &\leq d(x, x_n) + d(x, x_m) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

QED

Definition 1.17 A metric space (X, d) is *complete* if every Cauchy sequence in X converges to a limit in X. A subset Y is *complete* if the metric subspace $(Y, d|_Y)$ is complete. A *Banach space* is a normed linear space which is complete with respect to the norm-induced metric.

3 Upper and lower bounds

Definition 1.20 (defitions of *upper bound, lower bound, bounded from above, bounded from below* for subsets of $\mathbb{R}$)

Definition 1.121 (definitions of *supremum / least upper bound* and *infimum / greatest lower bound*)

Note that Hunter uses *monotone increasing* to mean *non-decreasing* ($n > m \implies x_n \geq x_m$) and likewise for *monotone decreasing*.

Given a sequence (x_n) in $\mathbb{R}$, we define

$$\limsup x_n = \lim_{n\to\infty} \left[\sup\{x_k \mid k \geq n\}\right].$$

Notice that the sequence (y_n) on the inside of the RHS given by

$$y_n = \sup\{x_k \mid k \geq n\}$$

is monotone increasing (i.e., never decreasing). We similarly define

$$\liminf x_n = \lim_{n\to\infty} \left[\inf\{x_k \mid k \geq n\}\right].$$

Both of these values always exist, as long as we allow $\pm\infty$ in addition to real values. Observe that

$$\sup\{x_k \mid k \geq n\} \geq \inf\{x_k \mid k \geq n\}$$

and that the LHS is a monotone Notice that (x_n) is convergent if and only if $\liminf x_n = \limsup x_n$, in which case it converges to their common value.