Continuous Functions

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Contents

1 Convergence of functions

Given a metric space (X, d) and a collection of functions $f_n : X \to \mathbb{R}$, we distinguish between two notions of convergence to $f : X \to \mathbb{R}$.

- Pointwise convergence: for all $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$.
- Metric convergence: $\lim_{n\to\infty} \|f_n - f\| = 0$ for some metric $\|\cdot\|$ on the space of functions from which we draw each f_n.

By considering the sequence of maps $f_n : [0, 1] \to \mathbb{R}$ given by $x \mapsto x^n$, we see that a pointwise-convergent sequence of continuous functions may have a discontinuous limit. So this is a bad notion of convergence if we want to restrict ourselves to continuous functions.

For continuous functions $f : X \to \mathbb{R}$, a natural norm on spaces of continuous functions is the *uniform* or sup norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Imagining the behavior of the L^p norm for large p, this is sometimes written $\|\cdot\|_\infty$.

Definition 2.2 A sequence of bounded functions $f_n : X \to \mathbb{R}$ *converges uniformly* to a function f if

$$\lim_{n \to \infty} \|f_n - f\| = 0.$$

Theorem 2.3 Let (f_n) be a sequence of bounded, continuous, real-valued functions on a metric space (X, d). If $f_n \to f$ uniformly, then f is continuous.

Proof To prove this claim we don't need the full machinery of a sequence of function converging uniformly to f. We work with a weaker premise which better captures the essence of why our theorem is true: the difference $\|f - g\|$ can be made arbitrarily small by appropriate choice of continuous $g : X \to \mathbb{R}$. Morally, our premise is that we have a function $f : X \to \mathbb{R}$ which can be approximated arbitrarily well (if we measure approximations using the uniform norm) by continuous functions.

Morals aside, we return to our claim that $f : X \to \mathbb{R}$ is continous. Fixing $\epsilon > 0$, we want to find $\delta > 0$ such that

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon.$$

From the premise, there must exist continuous $g : X \to \mathbb{R}$ such that

$$\|f - g\| = \sup_{x \in X} |f(x) - g(x)| < \epsilon/3.$$

Pick such a function g. From the continuity of g, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |g(x) - g(y)| < \epsilon/3.$$

In that case, when $d(x,y) < \delta$ we have

$$\begin{aligned}
|f(x) - f(y)| &= |(f(x) - g(x)) + (g(x) - g(y)) + (g(y) - f(y))| \\
&\leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}$$

QED

2 Spaces of continuous functions

Given a metric space (X, d), we denote by $C(X)$ the space of continuous functions $f : X \to \mathbb{R}$. This is a real linear space under the obvious operations.

Theorem 2.4 Let (K, d) be a compact metric space. Then $C(K)$ is complete.

3 Approximation by polynomials

4 Compact subsets of $C(K)$

5 Ordinary differential equations

6 Exercises

Exercise 2.2 (November 29) Let $f_n \in C([a,b])$ be a sequence of functions converging uniformly to a function f. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Give a counterexample to show that the pointwise convergence of continuous functions f_n to a continuous function f does not imply the convergence of the corresponding integrals.

Solution First we recall the generic bound

$$\left|\int_a^b g(x)dx\right| \leq \int_a^b |g(x)|dx$$

for $g \in C([a,b])$ (i.e., for continous $g : [a,b] \to \mathbb{R}$). To show that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

we equivalently show that

$$\lim_{n\to\infty} \left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| = 0.$$

To do so, observe the sequence of upper bounds

$$\begin{aligned}
\lim_{n\to\infty} \left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &= \lim_{n\to\infty} \left|\int_a^b \left(f_n(x) - f(x)\right) dx\right| \\
&\leq \lim_{n\to\infty} \int_a^b \left|f_n(x) - f(x)\right| dx \\
&\leq \lim_{n\to\infty} \left((b-a) \cdot \sup_{x\in[a,b]} |f_n(x) - f(x)|\right) \\
&= (b-a) \cdot \lim_{n\to\infty} \left(\sup_{x\in[a,b]} |f_n(x) - f(x)|\right)
\end{aligned}$$

So it suffices to show that

$$\lim_{n\to\infty} \left(\sup_{x\in[a,b]} |f_n(x) - f(x)|\right).$$

But that is precisely the definition of the statement "$f_n \to f$ uniformly" so we are done. QED

(still thinking about the counterexample for the other claim in the problem statement)