Online Self-Assessment for Linear Algebra

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The math questions in this document are from `https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/linalg.html`. I have provided my solutions and proofs in here. The latest version of this document is available at here.

Question 1

Let F be a field and $U_1, U_2 \subseteq V$ two linear subspaces of the F-vector space V. Which of the following are equivalent to the assertion that V is the direct sum of U_1 and U_2?

1. Every $v \in V$ has a unique representation as a sum $v = u_1 + u_2$ with $u_1 \in U_1, u_2 \in U_2$.

   Solution. Yes. Suppose $v \in U_1 \cap U_2$, then v has a unique representation $v = u_1 + u_2$ with $u_1 \in U_1, u_2 \in U_2$. Furthermore, $\mathbf{0} = (u_1 - v) + u_2$. Since $\mathbf{0} \in V$ and its representation is unique by the given hypotheses, then $u_1 - v = \mathbf{0}$ and $u_2 = \mathbf{0}$ due to the fact that $\mathbf{0} = \mathbf{0} + \mathbf{0}$ with $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$. This gives $u_1 = v$ and $u_2 = 0$. Likewise, $\mathbf{0} = u_2 + (u_1 - v)$. Then, we have $u_1 = 0$ and $u_2 = v$. Therefore, $v = u_1 = u_2 = \mathbf{0}$. Thus, $\mathbf{0} = U_1 \cap U_2$. Hence, $V = U_1 \oplus U_2$. For the other direction, given $V = U_1 \oplus U_2$, for any $v \in V$, we have $v = u_1 + u_2$ with $u_1 \in U_1$ and $u_2 \in U_2$. We only need to show the uniqueness of u_1 and u_2. Suppose that there exist $u_1' \neq u_1$ and $u_2' \neq u_2$ such that $v = u_1' + u_2'$, then $u_1 - u_1' = u_2' - u_2$. Obviously, the linear combination $u_1 - u_1' \in U_1$ and $u_2 - u_2' \in U_2$ hold. Therefore, $u_1 - u_1' \in U_1 \cap U_2$ and $u_2' - u_2 \in U_1 \cap U_2$. Since $\{\mathbf{0}\} = U_1 \cap U_2$, then we get $u_1' = u_1$ and $u_2' = u_2$ as desired.

2. Every affine subspace of V has non-empty intersections with both U_1 and U_2.

   Solution. No. A counterexample is $U_1 = V$ and $U_2 = V$. By definition, an affine subspace of V is a subset of V of the form

   $$a + V_1 = \{a + v_1 \mid v_1 \in V_1\} \tag{1}$$

   where a is a point in V, and V_1 is a linear subspace of V. It is easy to see that every affine subspace of V has non-empty intersections with $V = U_1 = U_2$, but V is not the direct sum of U_1 and U_2.

3. There is a projection $\pi : V \to V$ such that $\ker(\pi) = U_1$ and $\text{image}(\pi) = U_2$.

   Solution. Yes. If $V = U_1 \oplus U_2$, we can choose π to be the identity on U_2 and $\mathbf{0}$ on U_1.

   On the other hand, by definition, a projection is idempotent, namely $\pi^2 = \pi$. For any $v \in V$, let $u_1 = v - \pi(v)$ and $u_2 = \pi(v)$. Then, $\pi(u_1) = \pi(v) - \pi^2(v) = \mathbf{0}$, implying $u_1 \in \ker(\pi) = U_1$. Plus, $u_2 \in \text{image}(\pi) = U_2$. Therefore, $v = u_1 + u_2$ with $u_1 \in \ker(\pi)$ and $u_2 \in \text{image}(\pi)$.

Now we show the decomposition is unique. If $v = u_1 + u_2 = u_1' + u_2'$ with $u_1, u_1' \in U_1$ and $u_2, u_2' \in U_2$, applying π to v gives $\pi(v) = u_2 = u_2'$. Furthermore, $v - u_2 = u_1 = u_1' = v - u_2'$.

Let $u \in U_1 \cap U_2$. We have shown that u can be represented as $u_1 + u_2$ uniquely with $u_1 \in U_1$ and $u_2 \in U_2$. Then $\pi(u) = \pi(u_1) + \pi(u_2) = \mathbf{0} + u_2$. Since $u \in U_1 \cap U_2$, then $\mathbf{0} = \pi(u) = u$. Furthermore, from $\mathbf{0} + u_2 = 0$, we get $u_2 = \mathbf{0}$. Then we have $u_1 = u - u_2 = \mathbf{0} - \mathbf{0} = \mathbf{0}$. Thus, $U_1 \cap U_2 = \{\mathbf{0}\}$.

4. For every pair of linear maps $\varphi_1 \in \mathrm{Hom}(U_1, V), \varphi_2 \in \mathrm{Hom}(U_2, V)$, there is a $\varphi \in \mathrm{Hom}(V, V)$ that extends both φ_1 and φ_2 in the sense that $\varphi(u_i) = \varphi_i(u_i)$ for $i = 1, 2$ and $u_i \in U_i$.

   Solution. No. Any two subspaces U_1 and U_2 of V with $U_1 \cap U_2 = \{\mathbf{0}\}$ satisfy the stated property, but $V = U_1 + U_2$ does not necessarily hold. For example, U_1 and U_2 can be taken with $\dim(U_1) + \dim(U_2) < \dim(V)$.

5. Every basis B_1 of U_1 can be extended to a basis B for V using only vectors from U_2 in $B \setminus B_1$.

   Solution. No. A counterexample is $U_2 = V$ in which $U_1 \cap U_2 = \{\mathbf{0}\}$ is not guaranteed because U_1 can be any nontrivial subspace of V.

6. For every basis B of V, $B \cap U_i$ forms a basis of U_i for $i = 1, 2$.

   Solution. No. Let $V = U_1 \oplus U_2$ where $U_1 = \{c(1,1) | c \in \mathbb{R}\}$ and $U_2 = \{c(1,-1) | c \in \mathbb{R}\}$, then for a basis B of V, say $(1,0)$ and $(0,1)$, we have $B \cap U_1 = \{\mathbf{0}\}$, which does not form a basis of U_1.

7. Every union $B = B_1 \cup B_2$ of bases B_i of U_i for $i = 1, 2$ is a basis for V.

   Solution. Personally, this question is a little strange because, by the explanation on the website, if V is a one-dimensional vector space, then there is only one basis for V, which makes this question is no longer interesting. You can never express a one-dimensional vector space as a direct sum of two nontrivial subspaces. In addition, "$i = 1, 2$" appears in the question, so we are supposed to talk about a space that has a dimension of at least 2, right? After excluding this special case, I think that the statement is true.

8. For every $v \in V$ there is a $u \in U_1$ such that $(v - u) \in U_2$.

   Solution. No. A counterexample is either $U_1 = V$ or $U_2 = V$.

Question 2

Consider the standard 3-dimensional vector space $V = (\mathbb{F}_5)^3$ over the 5-element field $\mathbb{F}_5$ with addition and multiplication modulo 5. Specify:

1. The number of points in an affine subspace of dimension 1.

   Solution. 5. A linear subspace of dimension 1 in $V = (\mathbb{F}_5)^3$ consists of all scalar multiples of a non-zero vector. Since $\mathbb{F}_5$ has 5 elements, for a chosen vector v, the multiples of v give us five distinct points:

$$\{0, v, 2v, 3v, 4v\}.$$

If we choose a point a not at the origin and consider the affine subspace parallel to a linear subspace of dimension 1, it also contains 5 points, obtained by translating the linear subspace by a:

$$\{a, a+v, a+2v, a+3v, a+4v\}.$$

2. The number of points in a linear subspace of dimension 2.

   Solution. 25. A linear subspace of dimension 1 in $V = (\mathbb{F}_5)^3$ consists of $5 \times 5 = 25$ distinct points. So the affine space also contains 25 points.

3. The number of points in the quotient space V/U for a linear subspace of dimension 2.

   Solution. 5. It is clear that $\dim(V/U) = \dim V - \dim U = 3-2 = 1$. Thus, in $\mathbb{F}_5$, the quotient space V/U has 5 points.

4. The number of affine subspaces associated to a fixed linear subspace of dimension 1.

   Solution. 25. For a fixed linear subspace W of dimension 1, it can be described as $W = \{kv | k \in \mathbb{F}_5\}$ for a fixed vector $v \in V$. For any $u \in V$, it gives an affine subspace $u+W$. When $u \in W$, the resulting affine space will be W itself. So we only need to count those u that reside in the complement of W. Thus, since the dimension of its complement is 2, then the number of u is 25, which gives 25 affine subspaces. Note that since $\mathbf{0}$ is in the complement of W, then W itself has been actually counted as an affine subspace. So we do not need to count it again.

5. The number of distinct linear complements of a fixed linear subspace of dimension 2.

   Solution. 25. A linear subspace of dimension 2 leaves $125-25 = 100$ points in its complement. Since scaling any one-dimensional complement results in the same line, each 1-dimensional complement is counted 4 times. Thus, the number of distinct linear complements is $100/4 = 25$.

Question 3

Classify the following matrices up to similarity in $\mathbb{R}^{3\times 3}$. Hint: You can use invariants such as the trace, the determinant, the characteristic polynomial and the minimal polynomial.

$$A_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 4 & 6 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 5 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 20 & -30 & -15 \\ 5 & -10 & -5 \\ 6 & -6 & -4 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

1. A_1 and A_2

   Solution. No. A_1 and A_2 do not have identical eigenvalues.

2. A_1 and A_3

   Solution. No. $\mathrm{tr}(A_3) = 7 \neq \mathrm{tr}(A_1) = 8$. So A_1 and A_3 do not identical eigenvalues.

3. A_1 and A_4

   Solution. No. A_1 and A_4 do not have identical eigenvalues.

4. A_1 and A_5

   Solution. No. $\operatorname{tr}(A_5) = 6 \neq \operatorname{tr}(A_1) = 8$.

5. A_1 and A_6

   Solution. No. The dimension of the eigenspace of A_1 corresponding to the eigenvalue 2 is 1, but the counterpart for A_6 is 2.

6. A_2 and A_3

   Solution. No. $\operatorname{tr}(A_3) = 7 \neq \operatorname{tr}(A_2) = 8$. So A_2 and A_3 do not identical eigenvalues.

7. A_2 and A_4

   Solution. No. A_2 and A_4 do not identical eigenvalues, 4, 1, 3 and 3, 3, 1 respectively.

8. A_2 and A_5

   Solution. No. $\operatorname{tr}(A_2) = 8 \neq \operatorname{tr}(A_5) = 6$. So A_2 and A_5 do not identical eigenvalues.

9. A_2 and A_6

   Solution. No. $\det(A_2) = 12 \neq \det(A_6) = 16$. So A_2 and A_6 do not identical eigenvalues.

10. A_3 and A_4

    Solution. Yes. The characteristic polynomial of A_3 is $(\lambda-1)(\lambda-3)^3$ which is the same as A_4's characteristic polynomial. Also, they have the same dimension of eigenspace corresponding to 3, i.e. 1.

11. A_3 and A_5

    Solution. No. $\operatorname{tr}(A_3) = 7 \neq \operatorname{tr}(A_5) = 6$. So A_3 and A_5 do not have identical eigenvalues.

12. A_3 and A_6

    Solution. No. $\operatorname{tr}(A_3) = 7 \neq \operatorname{tr}(A_6) = 8$. So A_3 and A_6 do not have identical eigenvalues.

13. A_4 and A_5

    Solution. No. $\operatorname{tr}(A_4) = 7 \neq \operatorname{tr}(A_5) = 6$. So A_4 and A_5 do not have identical eigenvalues.

14. A_4 and A_6

    Solution. No. $\operatorname{tr}(A_4) = 7 \neq \operatorname{tr}(A_6) = 8$. So A_4 and A_6 do not have identical eigenvalues.

15. A_5 and A_6

    Solution. No. $\operatorname{tr}(A_5) = 6 \neq \operatorname{tr}(A_6) = 8$. So A_5 and A_6 do not have identical eigenvalues.