# A brief survey on Basel problem and a journey to the closed form expression of Apéry's constant

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**Abstract**

In this note, we collect various proofs for the famous Basel problem. Our final goal is to find out the closed form expression for the Apéry's constant. Luckily, we show that this closed form solution for the following series is given by

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

To the best of our knowledge, this is the first closed form which is independent with any other functions and is related to the Apéry's constant.

## 1 Apéry's constant

The Apéry constant is an irrational number which is equal to

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^2} + \cdots . \tag{1}$$

Indeed, $\zeta(3)$ has been shown an irrational number by Apéry, which is also known as Apéry's theorem[2]. From a post on Quora [4], I noticed that one nice result can be found on MathOverflow with proof [5]. Specifically, in terms of the derivative of $\zeta$ at $-2k$, $\zeta(2k+1)$ is given by

$$\zeta(2k+1) = \frac{(-1)^k 2^{2k+1}}{(2k)!} \pi^{2k} \zeta'(-2k). \tag{2}$$

## 2 Basel problem

### 2.1 A proof using Taylor's expansion

I found the following simple proof from `https://www.bilibili.com/video/BV1X441117t6?vd_source=b5354f4326756884a1a9d4e2863e8a6b`. It is said that

Euler proved the result in this way. Let's check it out. The Taylor's expansion of $\sin(x)$ is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{3}$$

$$= x(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots) = xp(x) \tag{4}$$

where $p(x)$ is defined as

$$p(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots . \tag{5}$$

From the fact that for any $m \in \mathbb{Z}$, $\sin(m\pi) = 0$, we have $p(m\pi) = 0$ for any $0 \neq m \in \mathbb{Z}$ and $p(0) = 1$. Let $q(x)$ be

$$q(x) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots \tag{6}$$

Expanding the right-hand side yields

$$q(x) = 1 - \frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)x^2 + \cdots . \tag{7}$$

Comparing the coefficients of $x^2$ between $p(x)$ and $q(x)$ gives

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) \tag{8}$$

which gives the desired result

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}. \tag{9}$$

## 2.2 A proof using Fourier series

I found this proof when I was reading a comment written by langdaooudishen (a bilibili user), in the comments section of a video on bilibili [3].

Consider the function

$$f(x) = \begin{cases} 0, & x \in [-1, 0) \\ x^2, & x \in [0, 1) \end{cases} . \tag{10}$$

Now we derive the Fourier series of this function.

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\mathrm{d}x = \int_{-1}^{1} f(x)\mathrm{d}x = \int_0^1 x^2\mathrm{d}x = \frac{1}{3} \tag{11}$$

where $T = 1$ in this case. For $n = 1, 2, \cdots$, we have the Fourier coefficients

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi}{T}x\mathrm{d}x = \int_0^1 x^2\cos n\pi x\mathrm{d}x = \frac{2\cdot(-1)^n}{n^2\pi^2}, \tag{12}$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi}{T}x\mathrm{d}x = \int_0^1 x^2 \sin n\pi x\mathrm{d}x = \frac{(-1)^{n+1}}{n} + \frac{2\cdot[(-1)^n - 1]}{n^3\pi^3}, \tag{13}$$

which yield the Fourier series of $f(x)$ as follows.

$$f(x) \sim \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n} + \frac{2\cdot[(-1)^n - 1]}{n^3\pi^2}\right]\sin n\pi x. \tag{14}$$

According to the Dirichlet–Jordan test, the above Fourier series converges to

$$\begin{cases} 0, & x \in (-1,0) \\ x^2, & x \in [0,1) \\ \frac{1}{2}, & x = \pm 1 \end{cases} \tag{15}$$

Thus, when $x = 1$, we have

$$\frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2} \tag{16}$$

which gives the desired result

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}. \tag{17}$$

Moreover, when $x = -1$, we have

$$\frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{1}{6} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = 0 \tag{18}$$

which gives

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}. \tag{19}$$

((17)+(19))/2 yields

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}. \tag{20}$$

## 2.3 A proof with geometry

In the perspective of geometry, [6] provides a simple proof for the Basel problem. The YouTube vblogger 3Blue1Blown visualized this idea [1]. I may make notes about this idea later.

# 3 My derivation on the closed-form expression for Apéry's constant

In this section, I will provide two approaches to deriving the closed form of the following series via Fourier series.

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}. \tag{21}$$

In my view, this result is possibly important to obtain the final closed form for the Apéry's constant. Because it is very similar to the following series which is equal to seven eighth of the Apéry's constant. To see this,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} &= 1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \cdots && (22)\\
&= \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots\right) - \left(\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} + \cdots\right) && (23)\\
&= \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots\right) - \frac{1}{2^3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right) && (24)\\
&= \frac{7}{8}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots\right) && (25)\\
&= \frac{7}{8}\sum_{k=1}^{\infty} \frac{1}{k^3}. && (26)
\end{aligned}$$

## 3.1 Approach 1

Inspired by the proof with Fourier series for the Basel problem, I have the following derivations. Let's first take a closer look at (14). We can observe that there is an $n^3$ on the denominator of the right hand side of (14). This kind of form is exactly what we are looking for. We can let $x = 1/2$ such that $\sin n\pi x$ does not vanish and then $1/n^3$ not vanish either. In the following derivations, we do not consider the constant coefficients and then put them back at last to simplify the deriving process. Now let us analyze term by term. When $x = 1/2$, for the first infinite sum, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi}{2} &= -\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \cdots && (27)\\
&= -\frac{1}{4}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right) && (28)\\
&= -\frac{1}{4} \cdot \frac{\pi^2}{12} = -\frac{\pi^2}{48}. && (29)
\end{aligned}$$

where the last line follows from (19). For the third infinite sum, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots \tag{30}$$

$$= \frac{\pi}{4} \tag{31}$$

where the last line follows from the following derivation.

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \int_0^1 \mathrm{d}x - \int_0^1 x^2 \mathrm{d}x + \int_0^1 x^4 \mathrm{d}x - \cdots \tag{32}$$

$$= \int_0^1 (1 - x^2 + x^4 - \cdots) \mathrm{d}x \tag{33}$$

$$= \int_0^1 \frac{1}{1 - (-x^2)} \mathrm{d}x \tag{34}$$

$$= \int_0^1 \frac{1}{1 + x^2} \mathrm{d}x \tag{35}$$

$$= \arctan 1 - \arctan 0 = \frac{\pi}{4} \tag{36}$$

where the third line follows from $x^2 < 1$. For the third infinite sum, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^3} \sin \frac{n\pi}{2} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n^3} \sin \frac{n\pi}{2} \tag{37}$$

$$= -\left(2 - \frac{2}{3^3} + \frac{2}{5^3} - \cdots\right) \tag{38}$$

$$= -2\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right) \tag{39}$$

Putting the above results together, since (14) converges to $1/4$ when $x = 1/2$, we get

$$\frac{1}{4} = \frac{1}{6} - \frac{2}{\pi^2} \cdot \frac{\pi^2}{48} + \frac{1}{\pi} \cdot \frac{\pi}{4} - \frac{1}{\pi} \cdot \frac{2}{\pi^2} \cdot 2\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right) \tag{40}$$

$$= \frac{1}{6} - \frac{1}{24} + \frac{1}{4} - \frac{4}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right) \tag{41}$$

which gives

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32} \tag{42}$$

## 3.2 Approach 2

Given the successful application of Fourier series in the previous subsection, I found out another Fourier series which can be useful to the derivation of the

Apéry's constant. This Fourier series is given by

$$x^3 \sim 2\sum_{n=1}^{\infty} \frac{(-1)^n(6-\pi^2 n^2)}{n^3} \sin nx,\ \ x \in (-\pi, \pi). \tag{43}$$

Since $f(x) = x^3$ is an odd function, then its Fourier series only consists of components of sine functions. To not miss out some useful intermediate results, we derive (43) as follows:

$$\begin{align}
b_n =& \frac{2}{\pi}\int_0^\pi x^3 \sin nx \mathrm{d}x \tag{44}\\
=& -\frac{2}{n\pi}\int_0^\pi x^3 \mathrm{d}\cos nx \tag{45}\\
=& -\frac{2}{n\pi}\left[(-1)^n\pi^3 - 3\int_0^\pi x^2\cos nx \mathrm{d}x\right] \tag{46}\\
=& -\frac{2(-1)^n\pi^2}{n} + \frac{6}{n\pi}\int_0^\pi x^2 \cos nx \mathrm{d}x \tag{47}\\
=& -\frac{2(-1)^n\pi^2}{n} + \frac{6}{n^2\pi}\int_0^\pi x^2 \mathrm{d}\sin nx \tag{48}\\
=& -\frac{2(-1)^n\pi^2}{n} - \frac{12}{n^2\pi}\int_0^\pi x \sin nx \mathrm{d}x \tag{49}\\
=& -\frac{2(-1)^n\pi^2}{n} + \frac{12}{n^3\pi}\int_0^\pi x \mathrm{d}\cos nx \tag{50}\\
=& -\frac{2(-1)^n\pi^2}{n} + \frac{12}{n^3\pi}\left[(-1)^n\pi - \int_0^\pi \cos nx \mathrm{d}x\right] \tag{51}\\
=& -\frac{2(-1)^n\pi^2}{n} + \frac{12(-1)^n}{n^3}. \tag{52}
\end{align}$$

Likewise, we analyze term by term. When $x = \pi/2$, for the first sum, we have

$$\begin{align}
\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi}{2} &= -1 + \frac{1}{3^3} - \frac{1}{5^3} + \cdots \tag{53}\\
&= -\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right). \tag{54}
\end{align}$$

For the second term, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3} \sin\frac{n\pi}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi}{2} \tag{55}$$

$$= -1 + \frac{1}{3} - \frac{1}{5} + \cdots \tag{56}$$

$$= -\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) \tag{57}$$

$$= -\frac{\pi}{4} \tag{58}$$

where the last line follows from (36). Combining these together,

$$\frac{\pi^3}{8} = -2 \cdot 6 \cdot \left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right) - 2\pi^2(-\frac{\pi}{4}) \tag{59}$$

$$= -12\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right) + \frac{\pi^3}{2}. \tag{60}$$

Then, we get the same result as the previous subsection as follows.

$$12\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right) = \frac{\pi^3}{2} - \frac{\pi^3}{8} = \frac{3\pi^3}{8} \tag{61}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{1}{12} \cdot \frac{3\pi^3}{8} = \frac{\pi^3}{32} \tag{62}$$

Also, I explored the case of $x = \pi/4$ a little bit on paper. I got some results but they might not be as straightforward as the above results.

# References

[1] 3Blue1Blown. *Summing inverse squares by euclidean geometry.* `https://www.youtube.com/watch?v=d-o3eB9sfls`. [Online; accessed 10-Nov-2023]. 2018.

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