

Notes on Fourier Series

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1 Orthogonal Functions

To describe orthogonal functions, we follow the definition in (Apostol, 1974, page 306). First, we denote by $L(I)$ the set of Lebesgue-integrable functions on an interval I . Then we denote by $L^2(I)$ the set of all complex-valued functions f which are measurable on I and are such that $|f|^2 \in L(I)$. The inner product (f, g) of two such functions, defined by

$$(f, g) = \int_I f(x) \overline{g(x)} dx, \quad (1.1)$$

always exists.

Definition 1.1 (orthogonal systems). Let $S = \{\phi_0, \phi_1, \phi_2, \dots\}$ be a collection of functions in $L^2(I)$. If

$$(\phi_n, \phi_m) = 0 \quad \text{whenever } m \neq n, \quad (1.2)$$

the collection S is said to be an orthogonal system on I . If, in addition, each ϕ_n has norm 1, then S is said to be orthonormal on I .

The following orthogonal system is fundamental in the field of Fourier analysis.

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin nx, \cos nx, \dots\} \quad (1.3)$$

23 More specifically, for $m, n \in \mathbb{N}^+$, on any interval with the length of 2π , we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}. \quad (1.4)$$

24 Particularly, we have

$$\int_{-\pi}^{\pi} 1 \cdot \cos mx dx = \int_{-\pi}^{\pi} 1 \cdot \sin mx dx = 0, \quad m = 1, 2, \dots \quad (1.5)$$

25 2 Fourier Series

26 2.1 Definition

27 Suppose $f(x)$ can be represented as the following series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (2.1)$$

28 which means the right-hand side converges to $f(x)$. Now we compute the coefficients a_n and b_n
 29 using the trigonometric orthogonality discussed earlier. Assume the right-hand side of (2.1) can be
 30 integrated term by term, then multiplying both sides by $\cos mx$ ($m = 0, 1, 2, \dots$) and integrating both
 31 sides over $[-\pi, \pi]$ gives

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \int_{-\pi}^{\pi} f(x) \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx dx \quad (2.2)$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx dx \quad (2.3)$$

$$= a_m \pi \quad (2.4)$$

32 which implies

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n = 0, 1, 2, \dots \quad (2.5)$$

33 Likewise, we get

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 1, 2, \dots \quad (2.6)$$

34 (2.5) and (2.6) are called Euler formulas for *Fourier coefficients*.

Definition 2.1 (Fourier series). Given $f(x)$ is 2π -periodic, Riemann integrable, and absolutely integrable on $[-\pi, \pi]$, the Fourier series is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad (2.7)$$

where a_n and b_n are computed by (2.5) and (2.6), respectively, which are called *Fourier coefficients*.

36 *Note.* A trigonometric series is not necessarily a Fourier series. For example,

$$f(x) = \sum_{n=2}^{\infty} \frac{\sin nx}{\ln n} \quad (2.8)$$

37 is uniformly convergent on any closed interval residing in $(0, 2\pi)$, which follows from the Dirichlet's
 38 test for uniform convergence. However, it is not a Fourier series because it does not satisfy the
 39 definition of Fourier series.

40 2.2 Some useful results for computing Fourier coefficients

$$\int_0^{\pi} \sin nx dx = - \int_{-\pi}^0 \sin nx dx = \frac{1 - (-1)^n}{n} = \frac{2}{2k-1}, \quad (2.9)$$

$$\int_0^{\pi} x \cos nx dx = - \int_{-\pi}^0 x \cos nx dx = \frac{(-1)^n - 1}{n^2} = -\frac{2}{(2k-1)^2}, \quad (2.10)$$

$$\int_0^{\pi} x \sin nx dx = \int_{-\pi}^0 x \sin nx dx = \frac{(-1)^{n+1}}{n} \pi, \quad (2.11)$$

$$\int_0^{\pi} x^2 \cos nx dx = \int_{-\pi}^0 x^2 \cos nx dx = \frac{2(-1)^n}{n^2} \pi, \quad (2.12)$$

$$\int_0^{\pi} e^x \cos nx dx = \frac{(-1)^n e^{\pi} - 1}{n^2 + 1}, \quad (2.13)$$

$$\int_0^{2\pi} x \sin nx dx = -\frac{2\pi}{n}, \quad (2.14)$$

$$\int_0^{2\pi} x \cos nx dx = 0, \quad (2.15)$$

41 where $n, k \in \mathbb{N}^+$.

$$\int_0^{\pi/2} \cos x \cos nx dx = - \int_{\pi/2}^{\pi} \cos x \cos nx dx = -\frac{\cos \frac{n\pi}{2}}{n^2 - 1} = \frac{(-1)^k}{4k^2 - 1} \quad (2.16)$$

42 where $n, k \in \mathbb{N}^+$.

43 3 Fourier Sine and Cosine Series

44 It is easy to observe that when $f(x)$ is an odd function, the Fourier coefficients a_n vanish. In this
 45 case, the Fourier series is called Fourier sine series since it is comprised of sine functions as follows.

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \quad (3.1)$$

46 where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx, \quad n = 1, 2, \dots \quad (3.2)$$

47 When $f(x)$ is an even function, the Fourier coefficients b_n vanish. In this case, the Fourier series
 48 is called Fourier cosine series since it is comprised of cosine functions as follows.

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos nx, \quad (3.3)$$

where

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx, \quad n = 0, 1, 2, \dots \quad (3.4)$$

4 More results on coefficients of Fourier Series

4.1 f defined on $[a, a + 2\pi]$

When $f(x)$ is defined on $(a, a + 2\pi)$, the coefficients a_n and b_n can be obtained in the same way as on $(-\pi, \pi)$ as follows:

$$\int_a^{a+2\pi} f(x) \cos mx dx = \int_a^{a+2\pi} f(x) \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx dx \quad (4.1)$$

$$= \frac{a_0}{2} \int_a^{a+2\pi} \cos mx dx + \sum_{n=1}^{\infty} a_n \int_a^{a+2\pi} \cos nx \cos mx dx + \sum_{n=1}^{\infty} b_n \int_a^{a+2\pi} \sin nx \sin mx dx \quad (4.2)$$

$$= a_m \pi \quad (4.3)$$

which implies

$$a_n = \frac{1}{\pi} \int_a^{a+2\pi} f(x) \cos nx dx, \quad n = 0, 1, 2, \dots \quad (4.4)$$

Likewise, we get

$$b_n = \frac{1}{\pi} \int_a^{a+2\pi} f(x) \sin nx dx, \quad n = 1, 2, \dots \quad (4.5)$$

4.2 f defined on $[-T, T]$

If $f(x)$ is $2T$ -periodic, let $x = \frac{T}{\pi}t$ where $t \in [-\pi, \pi]$, then

$$\phi(t) = f\left(\frac{T}{\pi}t\right) = f(x) \quad (4.6)$$

is periodic with period 2π . Thus, with the results obtained in section 2.1, we have

$$\phi(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \quad (4.7)$$

and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{T}x + b_n \sin \frac{n\pi}{T}x \right), \quad (4.8)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \cos ntdt = \frac{1}{T} \int_{-T}^T f(x) \cos \frac{n\pi}{T}x dx, \quad n = 0, 1, 2, \dots, \quad (4.9)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \sin ntdt = \frac{1}{T} \int_{-T}^T f(x) \sin \frac{n\pi}{T}x dx, \quad n = 1, 2, \dots \quad (4.10)$$

62 **4.3 f defined on $[0, T]$**

63 If $f(x)$ is defined on $[0, T]$, then we can take advantage of (4.4) and (4.5) with $a = 0$. Also, we need
 64 the trick of change of variables as performed in (4.6). Let $x = \frac{T}{2\pi}t$ where $t \in [0, 2\pi]$, then

$$f(x) = f\left(\frac{T}{2\pi}t\right) = \phi(t). \quad (4.11)$$

65 Combining this with (4.4) and (4.5) gives

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos ntdt = \frac{2}{T} \int_0^1 f\left(\frac{2\pi}{T}x\right) \cos\left(\frac{2n\pi}{T}x\right)dx, \quad n = 0, 1, 2, \dots \quad (4.12)$$

$$66 \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin ntdt = \frac{2}{T} \int_0^1 f\left(\frac{2\pi}{T}x\right) \sin\left(\frac{2n\pi}{T}x\right)dx, \quad n = 1, 2, \dots \quad (4.13)$$

67 Finally, the Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2n\pi}{T}x\right) + b_n \sin\left(\frac{2n\pi}{T}x\right) \right), \quad (4.14)$$

68 **Bibliography**

69 Apostol, T. M. (1974). *Mathematical Analysis, Second Edition*. Pearson Education.