Online Self-Assessment for Theory of Integration

Youming Zhao

August 25, 2024

The math questions in this document are sourced from `https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/measuretheory.html`. I have provided my solutions and proofs here. The latest version of this document is available here.

Question 1

Decide which of the following claims are correct. **Note:** By λ_d we denote the Lebesgue measure on $\mathbb{R}^d$.

1. If $\mathcal{A}$ is a σ-algebra on a set X, and $A, B \subseteq X$ with $A, B \notin \mathcal{A}$, then $A \cup B \notin \mathcal{A}$.

   Solution. False. A counter example is that for $X = \{1, 2, 3\}$, $\mathcal{A} = \{\emptyset, \{1, 2\}, \{3\}, X\}$ is a σ-algebra on X, but $A \cup B \in \mathcal{A}$ for the subsets of X, $A = \{1\}, B = \{2\}$ with $A, B \notin \mathcal{A}$.

2. Let $\mathcal{A}$ be the powerset of $\mathbb{N}$ (the set of natural numbers) and

   $$\mu(A) := \begin{cases} 1 & \text{if } A \text{ is infinite} \\ 0 & \text{if } A \text{ is finite} \end{cases}$$

   Then μ is an outer measure on $\mathbb{N}$.

   Solution. False. Let $A_n = \{n\}$ with $n \in 1, 2, \cdots$, then each A_n is in the powerset of $\mathbb{N}$, i.e., $A_n \in \mathcal{A}$, and $\bigcup\limits_{n=1}^{\infty} A_n = \mathbb{N}$. However, $\mu(\bigcup\limits_{n=1}^{\infty} A_n) = \mu(\mathbb{N}) = 1 > 0 = 0 + 0 + \cdots = \sum_{n=1}^{\infty} \mu(A_n)$, which does not meet the countable sub-additivity, namely $\mu(\bigcup\limits_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$, that an outer measure requires.

3. Let $A \subseteq \mathbb{R}^2$ be compact for some $d \in \mathbb{N}$, and $f : \mathbb{R}^d \to \mathbb{R}^d$ continuous. Then $\lambda_d(f(A)) < \infty$. Note that I guess that the power of the first $\mathbb{R}$ should be d instead of 2.

   Solution. True. This is because the image of the compact set A under a continuous function f is still compact. Since a compact set $f(A)$ in $\mathbb{R}^d$ is closed and bounded, then it has finite Lebesgue measure.

   To be rigorous, we now show that the continuous image of compact sets is compact. Let $\{H_i\}_{i \in I}$ is an open cover of $f(A)$ with an index set I. Since f is continuous, the preimage of each H_i is open in $\mathbb{R}^d$. Define $G_i = f^{-1}(H_i)$, then $\{G_i\}_{i \in I}$ forms an open cover of A. Since A is compact, then there exists a finite subcover $\{G_{i_1}, G_{i_2}, \cdots, G_{i_k}\}$ such that

   $$A \subseteq \bigcup_{j=1}^{k} G_{i_j}. \tag{1}$$

Applying f to both sides yields

$$f(A) \subset f\left(\bigcup_{j=1}^{k} G_{i_j}\right) = \bigcup_{j=1}^{k} f(G_{i_j}) = \bigcup_{j=1}^{k} H_{i_j}. \tag{2}$$

Hence, $\{H_{i_1}, H_{i_2}, \cdots, H_{i_k}\}$ a finite subcover of $f(A)$, meaning that $f(A)$ is compact.

Note that this above proof also applies to general topological spaces. Actually, I encountered this kind of proof several times while I was learning real analysis and point-set topology.

4. Every measure is an outer measure.

   Solution. True. Every measure is indeed an outer measure because the definition of a measure is stringent than that of an outer measure. To illustrate this, we compare their definitions as follows.

   - **Measure:** A measure μ on a σ-algebra $\mathcal{A}$ of subsets of a set X is a function that assigns a non-negative real number or infinity to each set in $\mathcal{A}$ and satisfies
     - $\mu^*(\emptyset) = 0$,
     - Countable additivity: for any countable collection $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$,

       $$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$
   - **Outer Measure:** An outer measure μ^* on a set X is a function defined on all subsets of X that satisfies:
     - $\mu^*(\emptyset) = 0$,
     - Monotonicity: If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$,
     - Countable sub-additivity: For any countable collection $\{A_i\}_{i=1}^{\infty}$ of subsets of X,

       $$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

   It is straightforward to see that countable additivity implies countable sub-additivity. Moreover, from countable additivity, we can also derive monotonicity. Specifically, if $A \subseteq B$, then $A \cup (B \setminus A) = B$. By countable additivity, we get

   $$\mu(A) + \mu(B \setminus A) = \mu(B) \tag{3}$$

   which implies $\mu(A) \leq \mu(B)$. This proves that a measure also satisfies the property of monotonicity. Therefore, since a measure satisfies all the requirements of an outer measure, it follows that every measure is indeed an outer measure.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be measurable. Then

   $$\int_{\mathbb{R}^2} f(x, y)\, \mathrm{d}\lambda_2(x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)\, \mathrm{d}\lambda_1(x)\, \mathrm{d}\lambda_1(y).$$

   Note that there might be a typo on the power of the second $\mathbb{R}$. I guess that it is supposed to be $\mathbb{R}$ instead of $\mathbb{R}^2$. However, if there is no typo here, then f would be a vector-valued function, which is also possible but somewhat unusual though.

Solution. False. The condition that f is measurable is not sufficient to guarantee that the double integral can be evaluated as an iterated integral. According to Fubini's theorem, f must also be integrable(or non-negative, as in Tonelli's theorem). From Fubini's theorem, we know that it is necessary that two iterated integrals should be at least equal. Based on this point, we can construct a counterexample as follows:

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0) \end{cases} \tag{4}$$

which has only one point of discontinuity, i.e., $(0,0)$. To see this, since $f(x,y)$ is a ratio of two polynomials, then it is continuous wherever the denominator is nonzero, so we only need to consider the behavior of the function along different paths as (x,y) approaches $(0,0)$:

- Along x-axis: $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2-0^2}{(x^2+0^2)^2} = \lim_{x\to 0} \frac{1}{x^2} = +\infty$,
- Along y-axis: $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0^2-y^2}{(0^2+y^2)^2} = \lim_{y\to 0} \frac{-1}{y^2} = -\infty$,
- Along the line $y = x$: $\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2-x^2}{(x^2+x^2)^2} = \lim_{x\to 0} \frac{0}{4x^4} = 0$,

Since the limit of $f(x,y)$ depends on the path taken as (x,y) approaches $(0,0)$, and the limits are not consistent, then the limit of $f(x,y)$ does not exist as (x,y) approaches $(0,0)$.

Since continuous functions on $\mathbb{R}^2$ are measurable, and one single point of discontinuity does not affect the measurability of a function, then f is still measurable. If f is integrable on $\mathbb{R}^2$, then it must also be integrable on a square of $\mathbb{R}^2$, e.g., $(0,1)\times(0,1)$. Now let's compute the iterated integrals on the square:

$$\int_{(0,1)}\int_{(0,1)} f(x,y)\mathrm{d}\lambda_1(x)\,\mathrm{d}\lambda_1(y) = \int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\mathrm{d}x\,\mathrm{d}y \tag{5}$$

$$= \int_0^1 \left.\frac{-x}{x^2+y^2}\right|_0^1 \mathrm{d}y \tag{6}$$

$$= \int_0^1 \frac{-1}{1+y^2}\mathrm{d}y \tag{7}$$

$$= -\arctan(y)\Big|_0^1 = -\frac{\pi}{4} \tag{8}$$

$$\int_{(0,1)}\int_{(0,1)} f(x,y)\mathrm{d}\lambda_1(y)\,\mathrm{d}\lambda_1(x) = \int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\mathrm{d}y\,\mathrm{d}x \tag{9}$$

$$= \int_0^1 \left.\frac{y}{x^2+y^2}\right|_0^1 \mathrm{d}x \tag{10}$$

$$= \int_0^1 \frac{1}{1+x^2}\mathrm{d}x \tag{11}$$

$$= \arctan(x)\Big|_0^1 = \frac{\pi}{4} \tag{12}$$

where (5) and (9) follows from that λ_1 is the Lebesgue measure on $\mathbb{R}$ and f is continuous on $(0,1)\times(0,1)$. As demonstrated by the computations above, the iterated integrals are not equal to each other, indicating that we cannot safely evaluate the double integral as an iterated integral without further hypotheses, even if f is measurable.

Question 2

Compute the following limit (integration is with respect to Lebesgue measure):

$$\lim_{n\to\infty} \int_{[0,n]} e^{-\sqrt{x^2+n^{-2}}}\,\mathrm{d}x.$$

Before we proceed, we need to define the indicator/characteristic function of a set E as follows which is crucial for solving the above problem.

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

I will provide two solutions. The first one employs the Monotone Convergence Theorem(a.k.a. Levi's Theorem), while the second one uses the Dominated Convergence Theorem.

Solution. The original integral can be rewritten as

$$\int_{[0,+\infty)} \chi_{[0,\,n]}(x) e^{-\sqrt{x^2+n^{-2}}}\,\mathrm{d}x. \tag{14}$$

Since $-\sqrt{x^2+n^{-2}} < -\sqrt{x^2+(n+1)^{-2}}$ for any $x \geq 0$, then we have

$$0 \leq \chi_{[0,\,n]}(x) e^{-\sqrt{x^2+n^{-2}}}(x) \leq \chi_{[0,\,n\,+\,1]}(x) e^{-\sqrt{x^2+(n+1)^{-2}}}. \tag{15}$$

Define

$$f_n(x) = \chi_{[0,\,n]}(x) e^{-\sqrt{x^2+n^{-2}}}, \quad \text{for } x \geq 0,\ n = 1, 2, 3, \cdots \tag{16}$$

which is a measurable function since it is a product of two measurable functions. Then we have

$$0 \leq f_n(x) \leq f_{n+1}(x), \quad \text{for } x \geq 0,\ n = 1, 2, 3, \cdots. \tag{17}$$

Furthermore, $\lim_{n\to\infty} f_n(x) = e^{-x}$ for $x \geq 0$. By the Monotone Convergence Theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_{[0,n]} e^{-\sqrt{x^2+n^{-2}}}\,\mathrm{d}x &= \lim_{n\to\infty} \int_{[0,+\infty)} \chi_{[0,\,n]}(x) e^{-\sqrt{x^2+n^{-2}}}\,\mathrm{d}x && (18)\\
&= \int_{[0,+\infty)} \lim_{n\to\infty} \chi_{[0,\,n]}(x) e^{-\sqrt{x^2+n^{-2}}}\,\mathrm{d}x && (19)\\
&= \int_{[0,+\infty)} e^{-x}\,\mathrm{d}x = -e^{-x}\Big|_0^{+\infty} && (20)\\
&= -(0-1) = 1. && (21)
\end{aligned}$$

Solution. Using the same notation as in the first solution, we observe that $f_n(x)$ is dominated by e^{-x} on $[0, +\infty)$. Specifically,

$$|\chi_{[0,\,n]}(x) e^{-\sqrt{x^2+n^{-2}}}| \leq e^{-\sqrt{x^2+n^{-2}}} < e^{-x}, \quad \text{for } x \geq 0,\ n = 1, 2, 3, \cdots. \tag{22}$$

In addition, $\lim_{n\to\infty} f_n(x) = e^{-x}$ for $x \geq 0$. By the Dominated Convergence Theorem, we get

$$\begin{aligned}
\lim_{n\to\infty} \int_{[0,n]} e^{-\sqrt{x^2+n^{-2}}}\,\mathrm{d}x &= \lim_{n\to\infty} \int_{[0,+\infty)} f_n(x)\mathrm{d}x && (23)\\
&= \int_{[0,+\infty)} \lim_{n\to\infty} f_n(x)\mathrm{d}x && (24)\\
&= \int_{[0,+\infty)} e^{-x}\,\mathrm{d}x = -e^{-x}\Big|_0^{+\infty} && (25)\\
&= -(0-1) = 1. && (26)
\end{aligned}$$