Online Self-Assessment for Ordinary Differential Equations

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August 14, 2024

The math questions in this document are from `https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/ode.html`. I have provided my solutions and proofs in here. The latest version of this document is available at here.

Question 1

Solve the following system of linear differential equations:

$$y'(t) = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix} y(t) + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} \quad \text{for } t \geq 0 \text{ with } y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solution. Let $A = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix}$ and $F(t) = \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix}$. We need to find the eigenvalues and eigenvectors of matrix A. The eigenvalues are the roots of the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$. Then we compute the determinant

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 \\ -6 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda + 1). \tag{1}$$

Therefore, the roots are $\lambda_1 = -1$ and $\lambda_1 = 4$. Furthermore, the associated eigenvectors can be obtained by solving $(-I - A)\boldsymbol{v}_1 = \boldsymbol{0}$ and $(4I - A)\boldsymbol{v}_2 = \boldsymbol{0}$, which gives

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{2}$$

Therefore, the general solution of the corresponding homogeneous differential equation is

$$y(t) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{4t}, \quad c_1, c_2 \in \mathbb{R}. \tag{3}$$

This general solution can be written more compactly as $y(t) = \Phi(t)\boldsymbol{c}$ with

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{4}$$

To get the general solution of the nonhomogeneous differential equation, we apply variation of parameters to it. Specifically, let $\boldsymbol{c}(t)$ be a vector function instead of a constant vector and then we have $y(t) = \Phi(t)\boldsymbol{c}(t)$. By differentiating both sides w.r.t. t, we get

$$y'(t) = A\Phi(t)\boldsymbol{c}(t) + F(t) = \Phi'(t)\boldsymbol{c}(t) + \Phi(t)\boldsymbol{c}'(t). \tag{5}$$

Since $\Phi(t)$ is the fundamental solution of the corresponding homogeneous equation, then $\Phi'(t) = A\Phi(t)$ and we substitute this into (5). Thus, we get $F(t) = \Phi(t)\boldsymbol{c}'(t)$. Furthermore, $\boldsymbol{c}'(t) = \Phi^{-1}(t)F(t)$. By integrating both sides w.r.t. t, we can get

$$\boldsymbol{c}(t) = \boldsymbol{c}(t_0) + \int_{t_0}^{t} \Phi^{-1}(s)F(s)ds. \tag{6}$$

Substituting this into $y(t) = \Phi(t)\boldsymbol{c}(t)$, we have

$$y(t) = \Phi(t)\boldsymbol{c}(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)F(s)ds. \tag{7}$$

Let $t = t_0 = 0$, we get

$$y(0) = \Phi(0)\boldsymbol{c}(0) + \mathbf{0} \tag{8}$$

$$\boldsymbol{c}(0) = \Phi^{-1}(0)y(0) \tag{9}$$

$$\boldsymbol{c}(0) = \begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{10}$$

Finally, substituting $\Phi(t)$, $\boldsymbol{c}(0)$ and $t_0 = 0$ into (7) yields

$$y(t) = \frac{1}{5}\begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}\int_0^t \begin{pmatrix} e^{-s} & e^{4s} \\ -3e^{-s} & 2e^{4s} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ e^{2s} \end{pmatrix} ds \tag{11}$$

$$= \frac{1}{5}\begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}\int_0^t \frac{1}{5e^{3s}}\begin{pmatrix} 2e^{4s} & -e^{4s} \\ 3e^{-s} & e^{-s} \end{pmatrix}\begin{pmatrix} 0 \\ e^{2s} \end{pmatrix} ds \tag{12}$$

$$= \frac{1}{5}\begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}\int_0^t \frac{1}{5}\begin{pmatrix} 2e^{4s} & -e^{4s} \\ 3e^{-s} & e^{-s} \end{pmatrix}\begin{pmatrix} 0 \\ e^{-s} \end{pmatrix} ds \tag{13}$$

$$= \frac{1}{5}\begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \frac{1}{5}\begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}\int_0^t \begin{pmatrix} -e^{3s} \\ e^{-2s} \end{pmatrix} ds \tag{14}$$

$$= \frac{1}{5}\begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \frac{1}{5}\begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}\begin{pmatrix} -\frac{1}{3}e^{3s} \\ -\frac{1}{2}e^{-2s} \end{pmatrix}\Bigg|_0^t \tag{15}$$

$$= \frac{1}{5}\begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \frac{1}{5}\begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}\begin{pmatrix} -\frac{1}{3}(e^{3t} - 1) \\ -\frac{1}{2}(e^{-2t} - 1) \end{pmatrix} \tag{16}$$

$$= \frac{1}{5}\begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \frac{1}{5}\begin{pmatrix} \frac{1}{3}e^{-t} - \frac{5}{6}e^{2t} + \frac{1}{2}e^{4t} \\ -e^{-t} + e^{4t} \end{pmatrix} \tag{17}$$

$$= \begin{pmatrix} \frac{7}{15}e^{-t} - \frac{1}{6}e^{2t} + \frac{7}{10}e^{4t} \\ -\frac{7}{5}e^{-t} + \frac{7}{5}e^{4t} \end{pmatrix} \tag{18}$$

Note: Perhaps it is simpler to use the method of undetermined coefficients since 2 is not an eigenvalue of the matrix A. However, variation of parameters is a more general method.

Question 2

Show that every solution of the differential equation

$$y''(t) = y'(t) + \sin(y(t)) \tag{19}$$

is smooth (i.e., it has derivatives of all orders).

Proof. Since $y(t)$ satisfies the second-order differential equation $y''(t) = y'(t) + \sin(y(t))$, then $y(t)$ has at least second-order derivative. In other words, $y(t)$ is C^2. Now we rewrite (19) as $y''(t) = f(t, y(t), y'(t))$ with $f(t, y(t), y'(t)) = y'(t) + \sin(y(t))$ which is continuously differentiable in y and y'. This implies that $f(t, y(t), y'(t))$ is Lipschitz continuous in y and y'. By the Picard-Lindelöf theorem, the existence of $y(t)$ for some initial condition $y(t_0) = y_0$ and $y'(t_0) = y'_0$ is guaranteed.

Now we prove by induction on k that every solution of (19) has derivatives of all orders. First, $y''(t)$ is continuously differentiable due to $y''(t) = y'(t) + \sin(y(t))$. Then suppose inductively that $y(t)$ is C^k, meaning that $y(t)$ has continuous derivatives up to order k. We need to show that $y(t)$ is C^{k+1}. Now we differentiate both sides of (19) w.r.t. t to get

$$y^{(3)}(t) = y''(t) + y'(t)\cos(y(t)). \tag{20}$$

Since the right hand side is continuous, $y(t)$ is C^3. By repeating this process, we can get an expression for $y^{(k+1)}(t)$ which equals to $g(t, y(t), y'(t), \cdots, y^{(k)}(t))$. Moreover, g only contains additions and multiplications between its variables, and $\sin y(t)$ and $\cos y(t)$. By the inductive hypothesis, $y(t)$ is C^{k+1}. This closes the induction. □

Question 3

Decide which of the following statements are correct and briefly justify your answer or give a counterexample.

1. Every solution of the differential equation

$$y'(t) = 1 + t^2 + \cos(y(t))$$

is monotonically increasing.

Solution. Yes. Since $\cos(y(t)) \geq -1$ and $1 + t^2 \geq 1$, then $y'(t) = 1 + t^2 + \cos(y(t)) \geq 0$.

2. The function

$$t \mapsto e^{tA} \cdot \mathbf{v}$$

with matrix $A \in \mathbb{R}^{3\times 3}$ and vector $\mathbf{v} = (1, -1, -1)^T \in \mathbb{R}^3$ satisfies $y'(t) = Ay(t)$.

Solution. Yes. The matrix exponential e^{tA} is defined as

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \tag{21}$$

which is well-defined for any square matrix A and real or complex t. Then

$$\begin{aligned} y'(t) = \left(e^{tA} \cdot \mathbf{v}\right)' &= \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} t^n\right)' \cdot \mathbf{v} && (22)\\ &= \left(\sum_{n=1}^{\infty} \frac{A^n}{(n-1)!} t^{n-1}\right) \cdot \mathbf{v} && (23)\\ &= A\left(\sum_{n=1}^{\infty} \frac{A^{n-1}}{(n-1)!} t^{n-1}\right) \cdot \mathbf{v} && (24)\\ &= A\left(\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}\right) \cdot \mathbf{v} = Ae^{tA} \cdot \mathbf{v} = Ay(t). && (25) \end{aligned}$$

3. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $u : \mathbb{R} \to \mathbb{R}$ is a solution to $y'(t) = f(t, y(t))$ with $y(1) = 0$, then

$$u(t) = \int_1^t f(s, u(s))\, ds$$

for all $t \in \mathbb{R}$.

Solution. Yes. According to the Fundamental Theorem of Calculus, we have

$$u(t) = u(1) + \int_1^t f(s, u(s)) ds, \quad \forall t \in \mathbb{R}. \tag{26}$$

Given that $u(1) = 0$, we get

$$u(t) = \int_1^t f(s, u(s)) ds, \quad \forall t \in \mathbb{R}. \tag{27}$$

We can verify the solution by differentiating both sides w.r.t. t as follows.

$$u'(t) = f(t, u(t)), \quad \forall t \in \mathbb{R}, \tag{28}$$

which is exactly the original differential equation.