Complex Analysis

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1 Basics

Let $\mathbb{C}$ be a set of complex numbers with a distance (metric space). We normally choose the absolute value, defined by $|a|= \sqrt{\alpha^2 + \beta^2}$ for $a = \alpha + i\beta \in \mathbb{C}$, as the distance.

The following three statements are equivalent:

1. A sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is *convergent* to $a \in \mathbb{C}$.
2. the sequence $(|z_n - a|)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is convergent to 0.
3. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$: $|z_n - a|< \epsilon$.

An ϵ-ball around $a \in \mathbb{C}$ is defined as

$$B_\epsilon(a) := \{w \in \mathbb{C} \mid |w - a|< \epsilon\}. \tag{1}$$

A function $f : \mathbb{C} \to \mathbb{C}$ is *continuous* at $z_0 \in \mathbb{C}$ if for all sequences $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ satisfying $\lim_{n\to\infty} z_n = z_0$, then $\lim_{n\to\infty} f(z_n) = f(z_0)$.

The domain of a complex-valued function $f : \mathbb{C} \to \mathbb{C}$ is supposed to be an open set. A set $U \subseteq \mathbb{C}$ is called open if $\forall u \in U$, $\exists \epsilon > 0$: $B_\epsilon(z) \subseteq U$.

Given an open set $U \subseteq \mathbb{C}$ and $z_0 \in U$, $f : U \to \mathbb{C}$ is called (complex) *differentiable* if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2}$$

exists. This limit, denoted $f'(z_0)$, is called the (complex) *derivative* of f at z_0.

Example: For the function $f(z) = mz + c$, $m, z, c \in \mathbb{C}$, its derivative at z_0 is given by $f'(z_0) = m$. $\square$

Example: Not all functions are differentiable, such as $f(z) = \bar{z}$. To see this, for $z_0 = 0$, the limit

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\bar{z}}{z} \tag{3}$$

does not exist. $\square$

Definition 1. Given an open set $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$ is *holomorphic* on U if f is differentiable at every $z_0 \in \mathbb{C}$. If $U = \mathbb{C}$, then the holomorphic function f is called *entire*.

The holomorphic functions have some nice properties as follows:

1. f is holomorphic $\Longrightarrow$ f is continuous.
2. f and g are holomorphic $\Longrightarrow$ $f + g$ and $f \cdot g$ are holomorphic.
3. the sum rule, product rule, quotient rule and chain rule for derivatives hold.

Example:

1. A polynomial is an entire function. More specifically, $f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ with $a_m, \cdots, a_0 \in \mathbb{C}$. Its first derivative is $f'(z) = m a_m z^{m-1} + \cdots + a_1$.
2. $f : \mathbb{C} \backslash \{0\} \to \mathbb{C}$, $f(z) = \frac{1}{z}$ is holomorphic.
3. Let $S = \{z \in \mathbb{C} \mid q(z) = 0\}$, then $f(z) = \frac{p(z)}{q(z)}$ is defined on $\mathbb{C} \backslash S$ where $p(z)$ and $q(z)$ are polynomials. Then f is holomorphic.

$\square$

2 Total Differentiability in $\mathbb{R}^2$

A complex plane can be interpreted as a vector space of $\mathbb{R}^2$. Specifically, a map $f : \mathbb{C} \to \mathbb{C}$ induces a map $f_R : \mathbb{R}^2 \to \mathbb{R}^2$. For example, $f(z) = z^2, z = x + iy \in \mathbb{C}$, due to the fact $z^2 = x^2 - y^2 + i(2xy)$, then we get

$$f_R\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}. \tag{4}$$

Definition 2. A map $f_R : \mathbb{R}^2 \to \mathbb{R}^2$ is called (totally) differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ if there is a matrix $J \in \mathbb{R}^2$ with

$$f_R\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f_R\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + J\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) + \rho\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right). \tag{5}$$

where a map $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} \frac{\rho\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|} = \mathbf{0}. \tag{6}$$

J is called the Jacobian matrix of f_R at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ which is defined by

$$J = \begin{pmatrix} \frac{\partial f_R\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)}{\partial x_0} & \frac{\partial f_R\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)}{\partial y_0} \end{pmatrix}. \tag{7}$$

For example, the Jacobian matrix of (4) is given by

$$J = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}. \tag{8}$$

2.1 Cauchy-Riemann Equations

Now let us connect the Jacobian of the vector-valued function f_R on $\mathbb{R}^2$ with the derivative of f. Given a complex-valued function $f(z) : \mathbb{C} \to \mathbb{C}$, it is differentiable if

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z - z_0) \tag{9}$$

where $\rho(z - z_0)$ tend to 0 more rapidly than $z - z_0$ in the sense that $\rho(z - z_0)/(z - z_0) \to 0$ for $z - z_0 \to 0$. By comparison with (5), the second terms on the right-hand side of both equations should have the same implication. In other words, the matrix-vector multiplication corresponds to $f'(z_0)(z - z_0)$. Let $f'(z_0) = a + ib$ and $z - z_0 = x + iy$, then $f'(z_0)(z - z_0) = (ax - by) + i(bx + ay)$. Furthermore, we rewrite it in the form of matrix-vector multiplication as follows:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}. \tag{10}$$

Let $f(z) = u(x, y) + iv(x, y)$ and $f_R(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$ where $u(x, y)$ and $v(x, y)$ are real-valued functions, then

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{11}$$

Combining this with (10) yields

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{12}$$

which are called Cauchy-Riemann equations. If u and v are real differentiable functions of the real variables, then f is complex differentiable at a complex point if and only if the partial derivatives of u and v satisfy the Cauchy–Riemann equations at that point.