Online Self-Assessment for Abstract Algebra

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August 12, 2024

The math questions in this document are from `https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/algebra.html`. I have provided my solutions and proofs in here. The latest version of this document is available at here.

Question 1

Which of the following claims are correct?

1. If G is a finite group and N a normal subgroup of G, then G has a subgroup H that forms a system of representatives for the cosets of N.

   Solution. False. Consider $G = \mathbb{Z}_4$ and $N = \{0, 2\}$. It is easy to verify that N is indeed a normal subgroup of G. Furthermore, the cosets of N can be obtained as follows.

$$
\begin{aligned}
0 + N =& \{0, 2\}, && (1)\\
1 + N =& \{1, 3\}, && (2)\\
2 + N =& \{2, 0\}, && (3)\\
3 + N =& \{3, 1\} && (4)
\end{aligned}
$$

   Therefore, the distinct cosets of N are $\{0, 2\}$ and $\{1, 3\}$. There are 4 possible systems of representatives for these two cosets of N, i.e., $\{0, 1\}$, $\{0, 3\}$, $\{2, 1\}$, and $\{2, 3\}$. However, none of them is a subgroup of G.

2. For every field K, the ring $K[x, y]$ of polynomials in two indeterminates is a principal ideal ring.

   Solution. False. Assume, for the sake of contradiction, that the ideal $I = (x, y)$ is principal. This would mean there exists a polynomial $p(x, y) \in K[x, y]$ such that

$$I = (p(x, y)) = \{q(x, y) \cdot p(x, y) \mid p(x, y) \in K[x, y]\}. \tag{5}$$

   This would imply that any element in the ideal I, including x and y, can be expressed as a multiple of $p(x, y)$. In particular, we must have

$$x = i_1(x, y) \cdot p(x, y), \quad y = i_2(x, y) \cdot p(x, y) \tag{6}$$

   for some polynomials $i_1(x, y), i_2(x, y) \in K[x, y]$. According to (6), the degree of $p(x, y)$ is at most 1. Note that K is a field, a nonzero element has a multiplicative inverse. If $\deg(p(x, y)) = 0$, it is a unit (in which case I would be the entire ring, which it isn't) or zero (in which case $I = \{0\}$, which it also isn't). Hence, $p(x, y)$ must be a polynomial of degree 1. This means

$$p(x, y) = ax + by + c \tag{7}$$

for some $a, b, c \in K$. Now we substitute it into the expressions for x and y as follows.

$$x = i_1(x, y)(ax + by + c), \quad y = i_2(x, y)(ax + by + c) \tag{8}$$

which holds for all x and y, implying $b = 0$ and $a = 0$ respectively. Therefore, $p(x, y) = c$, which contradicts the previous analysis. Furthermore, this implies the assumption is wrong. Thus, (x, y) is not principal.

3. For every field K, the ring $K[x]$ of polynomials in one indeterminate is a euclidean ring.

   Solution. True. A euclidean ring is a type of integral domain that has a Euclidean function that allows for a division algorithm similar to that in the integers. Specifically, in a Euclidean ring, for any two elements a and b (with $b \neq 0$), there exist q (quotient) and r (remainder) such that

$$a = bq + r \tag{9}$$

   where r is either 0 or the Euclidean function of r is strictly smaller than that of b.

   In $K[x]$, the Euclidean function is typically taken to be the degree of the polynomial. More specifically, for any nonzero polynomial $f(x)$, the Euclidean function $\phi(f(x))$ is the degree of $f(x)$.