Notes on Fourier Series

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1 Orthogonal Functions

To describe orthogonal functions, we follow the definition in (Apostol, 1974, page 306). First, we denote by $L(I)$ the set of Lebesgue-integrable functions on an interval I. Then we denote by $L^2(I)$ the set of all complex-valued functions f which are measurable on I and are such that $|f|^2 \in L(I)$. The inner product (f, g) of two such functions, defined by

$$(f,g) = \int_I f(x)\overline{g(x)}\mathrm{d}x, \tag{1.1}$$

always exists.

Definition 1.1 (orthogonal systems). *Let $S = \{\phi_0, \phi_1, \phi_2, \cdots\}$ be a collection of functions in $L^2(I)$. If*

$$(\phi_n, \phi_m) = 0 \qquad \text{whenever } m \neq n, \tag{1.2}$$

the collection S is said to be an orthogonal system on I. If, in addition, each ϕ_n has norm 1, then S is said to be orthonormal on I.

The following orthogonal system is fundamental in the field of Fourier analysis.

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots, \sin nx, \cos nx, \cdots\} \tag{1.3}$$

More specifically, for $m, n \in \mathbb{N}^+$, on any interval with the length of 2π, we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \mathrm{d}x = \int_{-\pi}^{\pi} \sin mx \sin nx \mathrm{d}x = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}. \tag{1.4}$$

Particularly, we have

$$\int_{-\pi}^{\pi} 1 \cdot \cos mx \mathrm{d}x = \int_{-\pi}^{\pi} 1 \cdot \sin mx \mathrm{d}x = 0, \qquad m = 1, 2, \cdots. \tag{1.5}$$

2 Fourier Series

2.1 Definition

Suppose $f(x)$ can be represented as the following series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2.1}$$

which means the right-hand side converges to $f(x)$. Now we compute the coefficients a_n and b_n using the trigonometric orthogonality discussed earlier. Assume the right-hand side of (2.1) can be integrated term by term, then multiplying both sides by $\cos mx (m = 0, 1, 2, \cdots)$ and integrating both sides over $[-\pi, \pi]$ gives

$$\int_{-\pi}^{\pi} f(x) \cos mx \mathrm{d}x = \int_{-\pi}^{\pi} f(x) \left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right] \cos mx \mathrm{d}x \tag{2.2}$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx \mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \mathrm{d}x + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \mathrm{d}x \tag{2.3}$$

$$= a_m \pi \tag{2.4}$$

which implies

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \mathrm{d}x, \qquad n = 0, 1, 2, \cdots. \tag{2.5}$$

Likewise, we get

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \mathrm{d}x, \qquad n = 1, 2, \cdots. \tag{2.6}$$

(2.5) and (2.6) are called Euler formulas for *Fourier coefficients*.

Definition 2.1 (Fourier series). *Given $f(x)$ is 2π-periodic, Riemann integrable, and absolutely integrable on $[-\pi, \pi]$, the Fourier series is defined by*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{2.7}$$

where a_n and b_n are computed by (2.5) and (2.6), respectively, which are called Fourier coefficients.

Note. A trigonometric series is not necessarily a Fourier series. For example,

$$f(x) = \sum_{n=2}^{\infty} \frac{\sin nx}{\ln n} \tag{2.8}$$

is uniformly convergent on any closed interval residing in $(0, 2\pi)$, which follows from the Dirichlet's test for uniform convergence. However, it is not a Fourier series because it does not satisfy the definition of Fourier series.

2.2 Some useful results for computing Fourier coefficients

$$\int_{0}^{\pi} \sin nx \mathrm{d}x = -\int_{-\pi}^{0} \sin nx \mathrm{d}x = \frac{1-(-1)^n}{n} = \frac{2}{2k-1} \tag{2.9}$$

where $n, k \in \mathbb{N}^+$.

$$\int_0^{\pi/2} \cos x \cos nx \mathrm{d}x = -\int_{\pi/2}^{\pi} \cos x \cos nx \mathrm{d}x = -\frac{\cos \frac{n\pi}{2}}{n^2 - 1} = \frac{(-1)^k}{4k^2 - 1} \tag{2.10}$$

where $n, k \in \mathbb{N}^+$.

Bibliography

Apostol, T. M. (1974). *Mathematical Analysis, Second Edition.* Pearson Education.