

Online Self-Assessment for Complex Analysis

Youming Zhao

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The math questions in this document are from <https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/complex.html>. I have provided my solutions and proofs in here. The latest version of this document is available at here.

Question 1

Which of the following functions is holomorphic on $\mathbb{C} \setminus \{0\}$?

1. $\frac{\bar{z}}{|z|^2}$

Solution. Yes. We have $\frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{1}{z}$. Since $\frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, then $\frac{\bar{z}}{|z|^2}$ is holomorphic.

2. $\frac{\bar{z}}{1+|z|^2}$

Solution. No. Let $z = x + iy$, then $\frac{\bar{z}}{1+|z|^2} = \frac{x}{(1+x^2+y^2)} - i \frac{y}{(1+x^2+y^2)}$. Moreover, $u(x, y) = \frac{x}{(1+x^2+y^2)}$ and $v(x, y) = -\frac{y}{(1+x^2+y^2)}$. Furthermore, $u_x = \frac{1-x^2+y^2}{(1+x^2+y^2)^2}$ and $v_y = -\frac{1+x^2-y^2}{(1+x^2+y^2)^2}$, which does not satisfy the Cauchy-Riemann equations.

3. $\frac{\bar{z}}{z}$

Solution. No. Let $z = x + iy$, then $\frac{\bar{z}}{z} = \frac{x-iy}{x+iy} = \frac{x^2-y^2-i2xy}{x^2+y^2}$. Moreover, $u(x, y) = \frac{x^2-y^2}{x^2+y^2}$ and $v(x, y) = -\frac{2xy}{x^2+y^2}$. Furthermore, $u_x = \frac{4xy^2}{(x^2+y^2)^2}$ and $v_y = \frac{2xy^2-2x^3}{(x^2+y^2)^2}$, which does not satisfy the Cauchy-Riemann equations.

4. $\frac{1+z}{z}$

Solution. Yes. $\frac{1+z}{z}$ is the quotient of two holomorphic functions on $\mathbb{C} \setminus \{0\}$.

Question 2

Compute the complex path integral $\int_{\gamma} f(z) dz$ for the following choices of f and γ . In each case, the path γ is parameterised as a function $[0, 1] \rightarrow \mathbb{C}$.

1. $f(z) = \frac{i\bar{z}}{\pi}$, $\gamma(t) = e^{2\pi it}$

Solution.

$$\int_{\gamma} \frac{i\bar{z}}{\pi} dz = \int_0^1 \frac{ie^{-2\pi it}}{\pi} de^{2\pi it} \quad (1)$$

$$= \int_0^1 \frac{ie^{-2\pi it}}{\pi} e^{2\pi it} 2\pi i dt \quad (2)$$

$$= \int_0^1 -2 dt = -2. \quad (3)$$

2. $f(z) = z^3$, $\gamma(t) = te^{it} \cos\left(\frac{\pi}{2}t\right) + (1-t)e^{it^2}$

Solution. $f(z) = z^3$ is holomorphic on \mathbb{C} . Since $\gamma(0) = 1$ and $\gamma(1) = 0$, by Cauchy's theorem, $\gamma(t)$ can be replaced by $\gamma(t) = 1 - t$ with $t \in [0, 1]$. Hence,

$$\int_{\gamma} z^3 dz = \int_0^1 (1-t)^3 d(1-t) \quad (4)$$

$$= \int_1^0 u^3 du = \frac{u^4}{4} \Big|_1^0 = -\frac{1}{4}. \quad (5)$$

3. $f(z) = \frac{i \cos(z^2)}{\pi z^5}$, $\gamma(t) = e^{2\pi it}$

Solution. $i \cos(z^2)$ is holomorphic on \mathbb{C} . By the corollary of the Cauchy integral formula, we have

$$\int_{\gamma} \frac{i \cos(\xi^2)}{\pi \xi^5} d\xi = \frac{2\pi i}{4!} \left(\frac{i \cos(z^2)}{\pi} \right)^{(4)} \Big|_{z=0} = 1. \quad (6)$$

A simpler method is employing the residue theorem. Specifically,

$$\frac{\cos(z^2)}{\pi z^5} = \frac{1 - \frac{z^4}{2} + \frac{z^8}{4!} + \dots}{z^5} = z^{-5} - \frac{z^{-1}}{2} + \frac{z^3}{4!} + \dots. \quad (7)$$

So, $c_{-1} = -\frac{1}{2}$. Thus,

$$\int_{\gamma} \frac{i \cos(\xi^2)}{\pi \xi^5} d\xi = 2\pi i \cdot \frac{i}{\pi} \cdot \left(-\frac{1}{2}\right) = 1. \quad (8)$$

Question 3

We consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ with

$$f(a + bi) = \sin a + bi.$$

1. Determine the set of points $z \in \mathbb{C}$ at which f is complex differentiable. Where is it holomorphic?

Solution.

$$u(a, b) = \sin a, \quad v(a, b) = b \quad (9)$$

$$u_a = \cos a, \quad u_b = 0 \quad (10)$$

$$v_a = 0, \quad v_b = 1 \quad (11)$$

According to Cauchy-Riemann equations, $\cos a = 1$, which gives $a = 2k\pi$. Therefore, f is complex differentiable on $\{(2k\pi + bi) | k \in \mathbb{Z}, b \in \mathbb{R}\}$. However, these points are isolated, implying that f is not holomorphic everywhere which requires f to be complex differentiable on an open set.

2. Calculate $\int_{\gamma} f(z) dz$ with $\gamma(t) = t - it^2$, $t \in [0, \pi]$.

Solution.

$$\int_{\gamma} f(z) dz = \int_0^1 (\sin t - it^2) d(t - it^2) \quad (12)$$

$$= \int_0^1 (\sin t - it^2)(1 - i2t) dt \quad (13)$$

$$= \int_0^1 (\sin t - 2t^3) - i(2t \sin t + t^2) dt \quad (14)$$

$$= \int_0^1 (\sin t - 2t^3) dt - i \int_0^1 (2t \sin t + t^2) dt \quad (15)$$

$$= \left(-\cos t - \frac{t^4}{2}\right) \Big|_0^{\pi} - i \left(2 \sin t - 2t \cos t + \frac{t^3}{3}\right) \Big|_0^{\pi} \quad (16)$$

$$= \left(2 - \frac{\pi^4}{2}\right) - i \left(2\pi + \frac{\pi^3}{3}\right). \quad (17)$$