

Complex Analysis

Youming Zhao

Email: yoming0.zhao@gmail.com

First draft: January 1, 2022 Last update: December 27, 2023

Contents

1	Basics	1
2	The exponential representation of complex numbers	2
3	Total Differentiability in \mathbb{R}^2	3
3.1	Cauchy-Riemann Equations	3

1 Basics

Let \mathbb{C} be a set of complex numbers with a distance (metric space). We normally choose the absolute value, defined by $|a| = \sqrt{\alpha^2 + \beta^2}$ for $a = \alpha + i\beta \in \mathbb{C}$, as the distance.

The following three statements are equivalent:

1. A sequence $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}$ is *convergent* to $a \in \mathbb{C}$.
2. the sequence $(|z_n - a|)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is convergent to 0.
3. $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N: |z_n - a| < \epsilon$.

An ϵ -ball around $a \in \mathbb{C}$ is defined as

$$B_\epsilon(a) := \{w \in \mathbb{C} \mid |w - a| < \epsilon\}. \quad (1)$$

A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is *continuous* at $z_0 \in \mathbb{C}$ if for all sequences $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}$ satisfying $\lim_{n \rightarrow \infty} z_n = z_0$, then $\lim_{n \rightarrow \infty} f(z_n) = f(z_0)$.

The domain of a complex-valued function $f : \mathbb{C} \rightarrow \mathbb{C}$ is supposed to be an open set. A set $U \subseteq \mathbb{C}$ is called open if $\forall u \in U, \exists \epsilon > 0: B_\epsilon(u) \subseteq U$.

Given an open set $U \subseteq \mathbb{C}$ and $z_0 \in U$, $f : U \rightarrow \mathbb{C}$ is called (complex) *differentiable* if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad (2)$$

exists. This limit, denoted $f'(z_0)$, is called the (complex) *derivative* of f at z_0 .

Example: For the function $f(z) = mz + c$, $m, z, c \in \mathbb{C}$, its derivative at z_0 is given by $f'(z_0) = m$.

□

Example: Not all functions are differentiable, such as $f(z) = \bar{z}$. To see this, for $z_0 = 0$, the limit

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \rightarrow 0} \frac{\bar{z}}{z} \quad (3)$$

does not exist. \square

Definition 1. Given an open set $U \subseteq \mathbb{C}$, $f : U \rightarrow \mathbb{C}$ is *holomorphic* on U if f is differentiable at every $z_0 \in \mathbb{C}$. If $U = \mathbb{C}$, then the holomorphic function f is called *entire*.

The holomorphic functions have some nice properties as follows:

1. f is holomorphic $\implies f$ is continuous.
2. f and g are holomorphic $\implies f + g$ and $f \cdot g$ are holomorphic.
3. the sum rule, product rule, quotient rule and chain rule for derivatives hold.

Example:

1. A polynomial is an entire function. More specifically, $f(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$ with $a_m, \dots, a_0 \in \mathbb{C}$. Its first derivative is $f'(z) = m a_m z^{m-1} + \dots + a_1$.
2. $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$, $f(z) = \frac{1}{z}$ is holomorphic.
3. Let $S = \{z \in \mathbb{C} \mid q(z) = 0\}$, then $f(z) = \frac{p(z)}{q(z)}$ is defined on $\mathbb{C} \setminus S$ where $p(z)$ and $q(z)$ are polynomials. Then f is holomorphic.

\square

2 The exponential representation of complex numbers

The exponential representation of complex numbers only with an imaginary part is given by Euler's formula as follows:

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (4)$$

Then a complex number z can be written as $z = r e^{i\theta}$. Furthermore,

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \quad (5)$$

$$\frac{z_1}{z_2} = r_1 e^{i\theta_1} \cdot \frac{1}{r_2} e^{-i\theta_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, \quad z_2 \neq 0. \quad (6)$$

This representation is powerful in some applications. For example,

$$\begin{aligned} & [\cos \phi + \cos 2\phi + \dots + \cos n\phi] + i[\sin \phi + \sin 2\phi + \dots + \sin n\phi] \\ &= e^{i\phi} + e^{i2\phi} + \dots + e^{in\phi} = \frac{e^{i\phi}(1 - e^{in\phi})}{1 - e^{i\phi}} \\ &= \frac{e^{i\phi/2}(1 - e^{in\phi})}{e^{-i\phi/2} - e^{i\phi/2}} = \frac{e^{i\phi/2} \cdot e^{in\phi/2}(e^{-in\phi/2} - e^{in\phi/2})}{e^{-i\phi/2} - e^{i\phi/2}} \\ &= e^{i\frac{n+1}{2}\phi} \cdot \frac{-2i \sin \frac{n\phi}{2}}{-2i \sin \frac{\phi}{2}} = \left(\cos \frac{n+1}{2}\phi + i \sin \frac{n+1}{2}\phi \right) \frac{\sin \frac{n\phi}{2}}{\sin \frac{\phi}{2}} \\ &= \frac{\sin \frac{n\phi}{2} \cos \frac{n+1}{2}\phi}{\sin \frac{\phi}{2}} + i \frac{\sin \frac{n\phi}{2} \sin \frac{n+1}{2}\phi}{\sin \frac{\phi}{2}} \end{aligned}$$

which implies

$$\cos \phi + \cos 2\phi + \cdots + \cos n\phi = \frac{\sin \frac{n\phi}{2} \cos \frac{n+1}{2}\phi}{\sin \frac{\phi}{2}} \quad (7)$$

$$\sin \phi + \sin 2\phi + \cdots + \sin n\phi = \frac{\sin \frac{n\phi}{2} \sin \frac{n+1}{2}\phi}{\sin \frac{\phi}{2}}. \quad (8)$$

Similarly, we get

$$\cos \phi + \cos 3\phi + \cdots + \cos(2n-1)\phi = \frac{1 - \cos 2n\phi}{2 \sin \phi} \quad (9)$$

$$\sin \phi + \sin 3\phi + \cdots + \sin(2n-1)\phi = \frac{\sin 2n\phi}{2 \sin \phi}. \quad (10)$$

3 Total Differentiability in \mathbb{R}^2

A complex plane can be interpreted as a vector space of \mathbb{R}^2 . Specifically, a map $f : \mathbb{C} \rightarrow \mathbb{C}$ induces a map $f_R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. For example, $f(z) = z^2, z = x + iy \in \mathbb{C}$, due to the fact $z^2 = x^2 - y^2 + i(2xy)$, then we get

$$f_R \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}. \quad (11)$$

Definition 2. A map $f_R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is called (totally) differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2$ if there is a matrix $J \in \mathbb{R}^2$ with

$$f_R \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = f_R \left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) + J \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right) + \rho \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right). \quad (12)$$

where a map $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\lim_{\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} \frac{\rho \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right)}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|} = \mathbf{0}. \quad (13)$$

J is called the Jacobian matrix of f_R at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ which is defined by

$$J = \left(\frac{\partial f_R \left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right)}{\partial x_0} \quad \frac{\partial f_R \left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right)}{\partial y_0} \right). \quad (14)$$

For example, the Jacobian matrix of (11) is given by

$$J = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}. \quad (15)$$

3.1 Cauchy-Riemann Equations

Now let us connect the Jacobian of the vector-valued function f_R on \mathbb{R}^2 with the derivative of f . Given a complex-valued function $f(z) : \mathbb{C} \rightarrow \mathbb{C}$, it is differentiable if

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z - z_0) \quad (16)$$

where $\rho(z - z_0)$ tend to 0 more rapidly than $z - z_0$ in the sense that $\rho(z - z_0)/(z - z_0) \rightarrow 0$ for $z - z_0 \rightarrow 0$. By comparison with (12), the second terms on the right-hand side of both equations should have the same implication. In other words, the matrix-vector multiplication corresponds to $f'(z_0)(z - z_0)$. Let $f'(z_0) = a + ib$ and $z - z_0 = x + iy$, then $f'(z_0)(z - z_0) = (ax - by) + i(bx + ay)$. Furthermore, we rewrite it in the form of matrix-vector multiplication as follows:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}. \quad (17)$$

Let $f(z) = u(x, y) + iv(x, y)$ and $f_R\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$ where $u(x, y)$ and $v(x, y)$ are real-valued functions, then

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \quad (18)$$

Combining this with (17) yields

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (19)$$

which are called Cauchy-Riemann equations. If u and v are real differentiable functions of the real variables, then f is complex differentiable at a complex point if and only if the partial derivatives of u and v satisfy the Cauchy-Riemann equations at that point.