Complex Analysis

Youming Zhao
Email: youming0.zhao@gmail.com

First draft: January 1, 2022 Last update: December 31, 2023

Contents

1 Basics

Let $\mathbb{C}$ be a set of complex numbers with a distance (metric space). We normally choose the absolute value, defined by $|a|=\sqrt{\alpha^2+\beta^2}$ for $a=\alpha+i\beta\in\mathbb{C}$, as the distance.

The following three statements are equivalent:

1. A sequence $(z_n)_{n\in\mathbb{N}}\subseteq\mathbb{C}$ is *convergent* to $a\in\mathbb{C}$.
2. the sequence $(|z_n-a|)_{n\in\mathbb{N}}\subseteq\mathbb{R}$ is convergent to 0.
3. $\forall\epsilon>0$, $\exists N\in\mathbb{N}$, $\forall n\geq N$: $|z_n-a|<\epsilon$.

An ϵ-ball around $a\in\mathbb{C}$ is defined as

$$B_\epsilon(a):=\{w\in\mathbb{C}\mid |w-a|<\epsilon\}. \tag{1}$$

A function $f:\mathbb{C}\to\mathbb{C}$ is *continuous* at $z_0\in\mathbb{C}$ if for all sequences $(z_n)_{n\in\mathbb{N}}\subseteq\mathbb{C}$ satisfying $\lim_{n\to\infty}z_n=z_0$, then $\lim_{n\to\infty}f(z_n)=f(z_0)$.

The domain of a complex-valued function $f:\mathbb{C}\to\mathbb{C}$ is supposed to be an open set. A set $U\subseteq\mathbb{C}$ is called open if $\forall u\in U$, $\exists\epsilon>0$: $B_\epsilon(z)\subseteq U$.

Given an open set $U\subseteq\mathbb{C}$ and $z_0\in U$, $f:U\to\mathbb{C}$ is called (complex) *differentiable* if

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \tag{2}$$

exists. This limit, denoted $f'(z_0)$, is called the (complex) *derivative* of f at z_0.

Example: For the function $f(z)=mz+c$, $m,z,c\in\mathbb{C}$, its derivative at z_0 is given by $f'(z_0)=m$. $\square$

Example: Not all functions are differentiable, such as $f(z) = \bar{z}$. To see this, for $z_0 = 0$, the limit

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\bar{z}}{z} \tag{3}$$

does not exist. □

Definition 1. Given an open set $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$ is *holomorphic* on U if f is differentiable at every $z_0 \in \mathbb{C}$. If $U = \mathbb{C}$, then the holomorphic function f is called *entire*.

The holomorphic functions have some nice properties as follows:

1. f is holomorphic $\Longrightarrow$ f is continuous.
2. f and g are holomorphic $\Longrightarrow$ $f + g$ and $f \cdot g$ are holomorphic.
3. the sum rule, product rule, quotient rule and chain rule for derivatives hold.

Example:

1. A polynomial is an entire function. More specifically, $f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ with $a_m, \cdots, a_0 \in \mathbb{C}$. Its first derivative is $f'(z) = m a_m z^{m-1} + \cdots + a_1$.
2. $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$, $f(z) = \frac{1}{z}$ is holomorphic.
3. Let $S = \{z \in \mathbb{C} \mid q(z) = 0\}$, then $f(z) = \frac{p(z)}{q(z)}$ is defined on $\mathbb{C}\backslash S$ where $p(z)$ and $q(z)$ are polynomials. Then f is holomorphic.

□

2 The exponential representation of complex numbers

The exponential representation of complex numbers only with an imaginary part is given by Euler's formula as follows:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{4}$$

Then a complex number z can be written as $z = re^{i\theta}$. Furthermore,

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}, \tag{5}$$

$$\frac{z_1}{z_2} = r_1 e^{i\theta_1} \cdot \frac{1}{r_2} e^{-i\theta_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}, \quad z_2 \neq 0. \tag{6}$$

This representation is powerful in some applications. For example,

$$\begin{aligned}
&[\cos\phi + \cos 2\phi + \cdots + \cos n\phi] + i[\sin\phi + \sin 2\phi + \cdots + \sin n\phi] \\
=& e^{i\phi} + e^{i2\phi} + \cdots + e^{in\phi} = \frac{e^{i\phi}(1 - e^{in\phi})}{1 - e^{i\phi}} \\
=& \frac{e^{i\phi/2}(1 - e^{in\phi})}{e^{-i\phi/2} - e^{i\phi/2}} = \frac{e^{i\phi/2} \cdot e^{in\phi/2}(e^{-in\phi/2} - e^{in\phi/2})}{e^{-i\phi/2} - e^{i\phi/2}} \\
=& e^{i\frac{n+1}{2}\phi} \cdot \frac{-2i\sin\frac{n\phi}{2}}{-2i\sin\frac{\phi}{2}} = (\cos\frac{n+1}{2}\phi + i\sin\frac{n+1}{2}\phi)\frac{\sin\frac{n\phi}{2}}{\sin\frac{\phi}{2}} \\
=& \frac{\sin\frac{n\phi}{2}\cos\frac{n+1}{2}\phi}{\sin\frac{\phi}{2}} + i\frac{\sin\frac{n\phi}{2}\sin\frac{n+1}{2}\phi}{\sin\frac{\phi}{2}}
\end{aligned}$$

which implies

$$\cos\phi + \cos 2\phi + \cdots + \cos n\phi = \frac{\sin\frac{n\phi}{2}\cos\frac{n+1}{2}\phi}{\sin\frac{\phi}{2}} \tag{7}$$

$$\sin\phi + \sin 2\phi + \cdots + \sin n\phi = \frac{\sin\frac{n\phi}{2}\sin\frac{n+1}{2}\phi}{\sin\frac{\phi}{2}}. \tag{8}$$

Similarly, we get

$$\cos\phi + \cos 3\phi + \cdots + \cos(2n-1)\phi = \frac{1-\cos 2n\phi}{2\sin\phi} \tag{9}$$

$$\sin\phi + \sin 3\phi + \cdots + \sin(2n-1)\phi = \frac{\sin 2n\phi}{2\sin\phi}. \tag{10}$$

3 Total Differentiability in $\mathbb{R}^2$

A complex plane can be interpreted as a vector space of $\mathbb{R}^2$. Specifically, a map $f:\mathbb{C}\to\mathbb{C}$ induces a map $f_R:\mathbb{R}^2\to\mathbb{R}^2$. For example, $f(z)=z^2, z=x+iy\in\mathbb{C}$, due to the fact $z^2=x^2-y^2+i(2xy)$, then we get

$$f_R\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x^2-y^2\\2xy\end{pmatrix}. \tag{11}$$

Definition 2. A map $f_R:\mathbb{R}^2\to\mathbb{R}^2$ is called (totally) differentiable at $\begin{pmatrix}x_0\\y_0\end{pmatrix}\in\mathbb{R}^2$ if there is a matrix $J\in\mathbb{R}^2$ with

$$f_R\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = f_R\left(\begin{pmatrix}x_0\\y_0\end{pmatrix}\right) + J\left(\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right) + \rho\left(\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right). \tag{12}$$

where a map $\rho:\mathbb{R}^2\to\mathbb{R}^2$ such that

$$\lim_{\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x_0\\y_0\end{pmatrix}} \frac{\rho\left(\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right)}{\left\|\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}x_0\\y_0\end{pmatrix}\right\|} = \mathbf{0}. \tag{13}$$

J is called the Jacobian matrix of f_R at $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ which is defined by

$$J = \begin{pmatrix}\frac{\partial f_R\left(\begin{pmatrix}x_0\\y_0\end{pmatrix}\right)}{\partial x_0} & \frac{\partial f_R\left(\begin{pmatrix}x_0\\y_0\end{pmatrix}\right)}{\partial y_0}\end{pmatrix}. \tag{14}$$

For example, the Jacobian matrix of (11) is given by

$$J = \begin{pmatrix}2x & -2y\\2y & 2x\end{pmatrix}. \tag{15}$$

3.1 Cauchy-Riemann Equations

Now let us connect the Jacobian of the vector-valued function f_R on $\mathbb{R}^2$ with the derivative of f. Given a complex-valued function $f(z):\mathbb{C}\to\mathbb{C}$, it is differentiable if

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \rho(z-z_0) \tag{16}$$

where $\rho(z-z_0)$ tend to 0 more rapidly than $z-z_0$ in the sense that $\rho(z-z_0)/(z-z_0) \to 0$ for $z-z_0 \to 0$. By comparison with (12), the second terms on the right-hand side of both equations should have the same implication. In other words, the matrix-vector multiplication corresponds to $f'(z_0)(z-z_0)$. Let $f'(z_0) = a+ib$ and $z-z_0 = x+iy$, then $f'(z_0)\cdot(z-z_0) = (ax-by)+i(bx+ay)$. Furthermore, we rewrite it in the form of matrix-vector multiplication as follows:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix}. \tag{17}$$

Let $f(z) = u(x,y) + iv(x,y)$ and $f_R(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ where $u(x,y)$ and $v(x,y)$ are real-valued functions, then

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{18}$$

Combining this with (17) yields

$$a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{19}$$

which are called Cauchy-Riemann equations. Since $f'(z_0) = a+ib$, then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{20}$$

We have the very important result as follows: If u and v are real differentiable functions of the real variables, then f is complex differentiable at a complex point if and only if the partial derivatives of u and v satisfy the Cauchy–Riemann equations at that point.

3.2 Examples

The following examples how to use Cauchy-Riemann equations to check if a function is holomorphic. Given a function $f(z) = z^2 + iz$ with $z \in \mathbb{C}$, let $z = x+iy$, then $f(x+iy) = (x+iy)^2 + i(x+iy) = (x^2-2y^2) + i(2xy-x)$. Furthermore, $u(x,y) = x^2 - y^2 - y$ and $v(x,y) = 2xy + x$. Thus,

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y - 1$$

$$\frac{\partial v}{\partial x} = 2y + 1, \quad \frac{\partial v}{\partial y} = 2x$$

which satisfy the Cauchy-Riemann equations. Thus, $f(z) = z^2 + iz$ is holomorphic.

3.3 Wirtinger derivatives

We first present the definition and then explain why it is defined that way. Recall that $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, then

$$
\begin{aligned}
\frac{\partial f}{\partial z} =& \frac{1}{2}(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}) \\
=& \frac{1}{2}(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}) \\
=& \frac{1}{2}\left(\frac{\partial (u+iv)}{\partial x} - i(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\right) \\
=& \frac{1}{2}\left(\frac{\partial (u+iv)}{\partial x} - i\frac{\partial (u+iv)}{\partial y}\right) \\
=& \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).
\end{aligned}
$$

Note that strictly speaking, the notation $\frac{\partial f}{\partial z}$ is supposed to be $\frac{\mathrm{d}\, f}{\mathrm{d}\, z}$. Furthermore, given a smooth function $f(z) : \mathbb{C} \to \mathbb{C}$, the Wirtinger derivatives are defined by

$$
\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \tag{21}
$$

$$
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{22}
$$

The second equation is derived from rewriting $u(x,y)$ as $u(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i})$. To see this,

$$
\begin{aligned}
\frac{\partial f(z,\bar{z})}{\partial \bar{z}} =& \frac{\partial (u+iv)}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial (u+iv)}{\partial y}\frac{\partial y}{\partial \bar{z}} \\
=& \frac{1}{2}\frac{\partial (u+iv)}{\partial x} - \frac{1}{2i}\frac{\partial (u+iv)}{\partial y} \\
=& \frac{1}{2}\frac{\partial (u+iv)}{\partial x} + \frac{i}{2}\frac{\partial (u+iv)}{\partial y} \\
=& \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})
\end{aligned}
$$

where the second last line, given f is holomorphic, gives

$$
\begin{aligned}
\frac{\partial f(z,\bar{z})}{\partial \bar{z}} =& \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\right) \\
=& \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x})\right) \\
=& \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right) = 0
\end{aligned}
$$

where the Cauchy-Riemann equations were used on the second last line. This shows that the equivalent equations of Cauchy-Riemann equations is

$$
\frac{\partial f(z,\bar{z})}{\partial \bar{z}} = 0. \tag{23}
$$