Online Self-Assessment for Complex Analysis

Youming Zhao

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The math questions in this document are from `https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/complex.html`. I have provided my solutions and proofs in here. The latest version of this document is available at here.

Question 1

Which of the following functions is holomorphic on $\mathbb{C} \setminus \{0\}$?

1. $\frac{\overline{z}}{|z|^2}$

   Solution. Yes. We have $\frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{1}{z}$. Since $\frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, then $\frac{\overline{z}}{|z|^2}$ is holomorphic.

2. $\frac{\overline{z}}{1+|z|^2}$

   Solution. No. Let $z = x + iy$, then $\frac{\overline{z}}{1+|z|^2} = \frac{x}{(1+x^2+y^2)} - i\frac{y}{(1+x^2+y^2)}$. Moreover, $u(x, y) = \frac{x}{(1+x^2+y^2)}$ and $v(x, y) = \frac{y}{(1+x^2+y^2)}$. Furthermore, $u_x = \frac{1-x^2+y^2}{(1+x^2+y^2)^2}$ and $v_y = -\frac{1+x^2-y^2}{(1+x^2+y^2)^2}$, which does not satisfy the Cauchy-Riemann equations.

3. $\frac{\overline{z}}{z}$

   Solution. No. Let $z = x + iy$, then $\frac{\overline{z}}{z} = \frac{x-iy}{x+iy} = \frac{x^2-y^2-i2xy}{x^2+y^2}$. Moreover, $u(x, y) = \frac{x^2-y^2}{x^2+y^2}$ and $v(x, y) = -\frac{2xy}{x^2+y^2}$. Furthermore, $u_x = \frac{4xy^2}{(x^2+y^2)^2}$ and $v_y = \frac{2xy^2-2x^3}{(x^2+y^2)^2}$, which does not satisfy the Cauchy-Riemann equations.

4. $\frac{1+z}{z}$

   Solution. Yes. Because $\frac{1+z}{z}$ is the quotient of two holomorphic functions on $\mathbb{C} \setminus \{0\}$.

Question 2

Compute the complex path integral $\int_\gamma f(z)\, dz$ for the following choices of f and γ. In each case, the path γ is parameterised as a function $[0, 1] \to \mathbb{C}$.

1. $f(z) = \frac{i\overline{z}}{\pi}, \quad \gamma(t) = e^{2\pi it}$

   Solution.

$$\int_\gamma \frac{i\overline{z}}{\pi} dz = \int_0^1 \frac{ie^{-2\pi it}}{\pi} de^{2\pi it} \tag{1}$$

$$= \int_0^1 \frac{ie^{-2\pi it}}{\pi} e^{2\pi it} 2\pi i dt \tag{2}$$

$$= \int_0^1 2i^2 dt = \int_0^1 -2dt = -2. \tag{3}$$

2. $f(z) = z^3, \quad \gamma(t) = te^{it}\cos\left(\frac{\pi}{2}t\right) + (1-t)e^{t^2}$

   Solution. Since $f(z) = z^3$ is holomorphic on $\mathbb{C}$, by the Cauchy's theorem, the claimed integral is path independent. Therefore, $\gamma(t)$ can be replaced by $\tilde{\gamma}(t) = 1 - t$ with $t \in [0, 1]$, which guarantees both curves have common endpoints, namely $\tilde{\gamma}(0) = \gamma(0) = 1$ and $\tilde{\gamma}(1) = \gamma(1) = 0$. Hence,

$$\int_\gamma z^3 dz = \int_{\tilde{\gamma}} z^3 dz = \int_0^1 (1-t)^3 d(1-t) \tag{4}$$

$$= \int_1^0 u^3 du = \left.\frac{u^4}{4}\right|_1^0 = -\frac{1}{4}. \tag{5}$$

3. $f(z) = \frac{i\cos(z^2)}{\pi z^5}, \quad \gamma(t) = e^{2\pi it}$

   Solution. Since $\frac{i\cos(z^2)}{\pi}$ is holomorphic on $\mathbb{C}$ and the path γ is a unit circle centered at the origin, by the corollary of the Cauchy integral formula, i.e., $\int_\gamma \frac{f(\xi)}{(\xi-z)^{n+1}} d\xi = \frac{2\pi i}{n!} f^{(n)}(z)$ with z being inside of the circle γ, we have

$$\int_\gamma \frac{i\cos(\xi^2)}{\pi\xi^5} d\xi = \int_\gamma \frac{(i\cos(\xi^2))/\pi}{(\xi-0)^{4+1}} d\xi \tag{6}$$

$$= \frac{2\pi i}{4!} \left(\frac{i\cos(z^2)}{\pi}\right)^{(4)} \Bigg|_{z=0} = -\frac{2}{4!} \left(\cos(z^2)\right)^{(4)} \Bigg|_{z=0} \tag{7}$$

$$= \frac{1}{12} \left(2z\sin(z^2)\right)^{(3)} \Bigg|_{z=0} = \frac{1}{6} \left(z\sin(z^2)\right)^{(3)} \Bigg|_{z=0} \tag{8}$$

$$= \frac{1}{6} \left(\sin(z^2) + 2z^2\cos(z^2)\right)'' \Bigg|_{z=0} \tag{9}$$

$$= \frac{1}{6} \left(6z\cos(z^2) - 4z^3\sin(z^2)\right)' \Bigg|_{z=0} \tag{10}$$

$$= \frac{1}{6} \left(6\cos(z^2) - 12z^2\sin(z^2) - \cdots\right) \Bigg|_{z=0} = 1. \tag{11}$$

   where in the last line we did not calculate the derivative of $4z^3\sin(z^2)$ w.r.t. z since it will definitely vanish when being evaluated at $z = 0$.

   Note: In terms of computation, a simpler method is employing the residue theorem. Specifically,

$$\frac{\cos(z^2)}{z^5} = \frac{1 - \frac{(z^2)^2}{2!} + \frac{(z^2)^4}{4!} - \cdots}{z^5} \tag{12}$$

$$=\frac{1-\frac{z^4}{2}+\frac{z^8}{4!}-\cdots}{z^5} \tag{13}$$

$$=z^{-5}-\frac{z^{-1}}{2}+\frac{z^3}{4!}-\cdots. \tag{14}$$

So, $c_{-1}=-\frac{1}{2}$. Thus,

$$\int_\gamma \frac{i\cos(\xi^2)}{\pi\xi^5}d\xi=\frac{i}{\pi}\cdot\int_\gamma\frac{\cos(\xi^2)}{\xi^5}d\xi \tag{15}$$

$$=\frac{i}{\pi}\cdot 2\pi i\cdot c_{-1} \tag{16}$$

$$=\frac{i}{\pi}\cdot 2\pi i\cdot(-\frac{1}{2})=-i^2=1. \tag{17}$$

Question 3

We consider the function $f:\mathbb{C}\to\mathbb{C}$ with

$$f(a+bi)=\sin a+bi.$$

1. Determine the set of points $z\in\mathbb{C}$ at which f is complex differentiable. Where is it holomorphic?

   Solution. Let $u(a,b)=\sin a$ and $v(a,b)=b$, then

   $$u'_a=\cos a,\quad u'_b=0 \tag{18}$$

   $$v'_a=0,\quad v'_b=1. \tag{19}$$

   According to Cauchy-Riemann equations, $\cos a=1$, which gives $a=2k\pi$. Therefore, f is complex differentiable on $\{(2k\pi+bi)|k\in\mathbb{Z},b\in\mathbb{R}\}$. However, these points are isolated, implying that f is not holomorphic everywhere which requires f to be complex differentiable on an open set.

2. Calculate $\int_\gamma f(z)\,dz$ with $\gamma(t)=t-it^2$, $t\in[0,\pi]$.

   Solution.

   $$\int_\gamma f(z)\,dz=\int_0^1(\sin t-it^2)d(t-it^2) \tag{20}$$

   $$=\int_0^1(\sin t-it^2)(1-i2t)dt \tag{21}$$

   $$=\int_0^1\left((\sin t-2t^3)-i(2t\sin t+t^2)\right)dt \tag{22}$$

   $$=\int_0^1(\sin t-2t^3)dt-i\int_0^1(2t\sin t+t^2)dt \tag{23}$$

   $$=\left(-\cos t-\frac{t^4}{2}\right)\Big|_0^\pi-i\left(2\sin t-2t\cos t+\frac{t^3}{3}\right)\Big|_0^\pi \tag{24}$$

   $$=(2-\frac{\pi^4}{2})-i(2\pi+\frac{\pi^3}{3}). \tag{25}$$