Online Self-Assessment for Complex Analysis

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The math questions in this document are from `https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/complex.html`. I have provided my solutions and proofs in here. The latest version of this document is available at here.

Question 1

Which of the following functions is holomorphic on $\mathbb{C} \setminus \{0\}$?

1. $\frac{\overline{z}}{|z|^2}$

   Solution. Yes. We have $\frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{1}{z}$. Since $\frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, then $\frac{\overline{z}}{|z|^2}$ is holomorphic.

2. $\frac{\overline{z}}{1+|z|^2}$

   Solution. No. Let $z = x + iy$, then $\frac{\overline{z}}{1+|z|^2} = \frac{x}{(1+x^2+y^2)} - i\frac{y}{(1+x^2+y^2)}$. Moreover, $u(x,y) = \frac{x}{(1+x^2+y^2)}$ and $v(x,y) = \frac{y}{(1+x^2+y^2)}$. Furthermore, $u_x = \frac{1-x^2+y^2}{(1+x^2+y^2)^2}$ and $v_y = -\frac{1+x^2-y^2}{(1+x^2+y^2)^2}$, which does not satisfy the Cauchy-Riemann equations.

3. $\frac{\overline{z}}{z}$

   Solution. No. Let $z = x + iy$, then $\frac{\overline{z}}{z} = \frac{x-iy}{x+iy} = \frac{x^2-y^2-i2xy}{x^2+y^2}$. Moreover, $u(x,y) = \frac{x^2-y^2}{x^2+y^2}$ and $v(x,y) = -\frac{2xy}{x^2+y^2}$. Furthermore, $u_x = \frac{4xy^2}{(x^2+y^2)^2}$ and $v_y = \frac{2xy^2-2x^3}{(x^2+y^2)^2}$, which does not satisfy the Cauchy-Riemann equations.

4. $\frac{1+z}{z}$

   Solution. Yes. Because $\frac{1+z}{z}$ is the quotient of two holomorphic funtions on $\mathbb{C} \setminus \{0\}$.

Question 2

Compute the complex path integral $\int_\gamma f(z)\,dz$ for the following choices of f and γ. In each case, the path γ is parameterised as a function $[0,1] \to \mathbb{C}$.

1. $f(z) = \frac{i\overline{z}}{\pi}, \quad \gamma(t) = e^{2\pi it}$

Solution.

$$\int_\gamma \frac{i\overline{z}}{\pi}dz = \int_0^1 \frac{ie^{-2\pi it}}{\pi}de^{2\pi it} \tag{1}$$

$$= \int_0^1 \frac{ie^{-2\pi it}}{\pi}e^{2\pi it}2\pi i dt \tag{2}$$

$$= \int_0^1 2i^2 dt = \int_0^1 -2dt = -2. \tag{3}$$

2. $f(z) = z^3, \quad \gamma(t) = te^{it}\cos\left(\frac{\pi}{2}t\right) + (1-t)e^{t^2}$

   Solution. Since $f(z) = z^3$ is holomorphic on $\mathbb{C}$, by the Cauchy's theorem, the claimed integral is path independent. Therefore, $\gamma(t)$ can be replaced by $\tilde{\gamma}(t) = 1 - t$ with $t \in [0, 1]$, which guarantees both curves have common endpoints, namely $\tilde{\gamma}(0) = \gamma(0) = 1$ and $\tilde{\gamma}(1) = \gamma(1) = 0$. Hence,

$$\int_\gamma z^3 dz = \int_{\tilde{\gamma}} z^3 dz = \int_0^1 (1-t)^3 d(1-t) \tag{4}$$

$$= \int_1^0 u^3 du = \frac{u^4}{4}\Big|_1^0 = -\frac{1}{4}. \tag{5}$$

3. $f(z) = \frac{i\cos(z^2)}{\pi z^5}, \quad \gamma(t) = e^{2\pi it}$

   Solution. Since $\frac{i\cos(z^2)}{\pi}$ is holomorphic on $\mathbb{C}$ and the path γ is a unit circle centered at the origin, by the corollary of the Cauchy integral formula, i.e., $\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi = \frac{2\pi i}{n!}f^{(n)}(z)$ with z being inside of the circle γ, we have

$$\int_\gamma \frac{i\cos(\xi^2)}{\pi\xi^5}d\xi = \int_\gamma \frac{(i\cos(\xi^2))/\pi}{(\xi - 0)^{4+1}}d\xi \tag{6}$$

$$= \frac{2\pi i}{4!}\left(\frac{i\cos(z^2)}{\pi}\right)^{(4)}\Bigg|_{z=0} \tag{7}$$

$$= -\frac{2}{4!}\left(\cos(z^2)\right)^{(4)}\Bigg|_{z=0} = 1. \tag{8}$$

A simpler method is employing the residue theorem. Specifically,

$$\frac{\cos(z^2)}{z^5} = \frac{1 - \frac{(z^2)^2}{2!} + \frac{(z^2)^4}{4!} - \cdots}{z^5} \tag{9}$$

$$= \frac{1 - \frac{z^4}{2} + \frac{z^8}{4!} - \cdots}{z^5} \tag{10}$$

$$= z^{-5} - \frac{z^{-1}}{2} + \frac{z^3}{4!} - \cdots. \tag{11}$$

So, $c_{-1} = -\frac{1}{2}$. Thus,

$$\begin{aligned}
\int_\gamma \frac{i\cos(\xi^2)}{\pi\xi^5} d\xi =& \frac{i}{\pi} \cdot \int_\gamma \frac{\cos(\xi^2)}{\xi^5} d\xi && (12)\\
=& \frac{i}{\pi} \cdot 2\pi i \cdot c_{-1} && (13)\\
=& \frac{i}{\pi} \cdot 2\pi i \cdot (-\frac{1}{2}) = -i^2 = 1. && (14)
\end{aligned}$$

Question 3

We consider the function $f : \mathbb{C} \to \mathbb{C}$ with

$$f(a+bi) = \sin a + bi.$$

1. Determine the set of points $z \in \mathbb{C}$ at which f is complex differentiable. Where is it holomorphic?

   Solution. Let $u(a,b) = \sin a$ and $v(a,b) = b$, then

   $$\begin{aligned}
   u'_a = \cos a, \quad u'_b = 0 && (15)\\
   v'_a = 0, \quad v'_b = 1. && (16)
   \end{aligned}$$

   According to Cauchy-Riemann equations, $\cos a = 1$, which gives $a = 2k\pi$. Therefore, f is complex differentiable on $\{(2k\pi + bi) | k \in \mathbb{Z}, b \in \mathbb{R}\}$. However, these points are isolated, implying that f is not holomorphic everywhere which requires f to be complex differentiable on an open set.

2. Calculate $\int_\gamma f(z)\, dz$ with $\gamma(t) = t - it^2$, $t \in [0, \pi]$.

   Solution.

   $$\begin{aligned}
   \int_\gamma f(z)\, dz =& \int_0^1 (\sin t - it^2) d(t - it^2) && (17)\\
   =& \int_0^1 (\sin t - it^2)(1 - i2t) dt && (18)\\
   =& \int_0^1 \big((\sin t - 2t^3) - i(2t \sin t + t^2)\big)\, dt && (19)\\
   =& \int_0^1 (\sin t - 2t^3) dt - i \int_0^1 (2t\sin t + t^2) dt && (20)\\
   =& \left(-\cos t - \frac{t^4}{2}\right)\Bigg|_0^\pi - i\left(2\sin t - 2t\cos t + \frac{t^3}{3}\right)\Bigg|_0^\pi && (21)\\
   =& (2 - \frac{\pi^4}{2}) - i(2\pi + \frac{\pi^3}{3}). && (22)
   \end{aligned}$$