# Complex Analysis

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## 1 Basics

Let $\mathbb{C}$ be a set of complex numbers with a distance (metric space). We normally choose the absolute value, defined by $|a|= \sqrt{\alpha^2 + \beta^2}$ for $a = \alpha + \beta i \in \mathbb{C}$, as the distance.

The following three statements are equivalent:

1. A sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is *convergent* to $a \in \mathbb{C}$.
2. the sequence $(|z_n - a|)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is convergent to 0.
3. $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$: $|z_n - a|< \epsilon$.

An $\epsilon$-ball around $a \in \mathbb{C}$ is defined as

$$B_\epsilon(a) := \{w \in \mathbb{C} \mid |w - a|< \epsilon\}. \tag{1}$$

A function $f : \mathbb{C} \to \mathbb{C}$ is *continuous* at $z_0 \in \mathbb{C}$ if for all sequences $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ satisfying $\lim_{n\to\infty} z_n = z_0$, then $\lim_{n\to\infty} f(z_n) = f(z_0)$.

The domain of a complex-valued function $f : \mathbb{C} \to \mathbb{C}$ is supposed to be an open set. A set $U \subseteq \mathbb{C}$ is called open if $\forall u \in U$, $\exists\epsilon > 0$: $B_\epsilon(z) \subseteq U$.

Given an open set $U \subseteq \mathbb{C}$ and $z_0 \in U$, $f : U \to \mathbb{C}$ is called (complex) *differentiable* if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2}$$

exists. This limit, denoted $f'(z_0)$, is called the (complex) *derivative* of $f$ at $z_0$.

**Example:** For the function $f(z) = mz + c$, $m, z, c \in \mathbb{C}$, its derivative at $z_0$ is given by $f'(z_0) = m$. □

**Example:** Not all functions are differentiable, such as $f(z) = \bar{z}$. To see this, for $z_0 = 0$, the limit

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\bar{z}}{z} \tag{3}$$

does not exist. □

**Definition 1.** Given an open set $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$ is *holomorphic* on $U$ if $f$ is differentiable at every $z_0 \in \mathbb{C}$. If $U = \mathbb{C}$, then the holomorphic function $f$ is called *entire*.

The holomorphic functions have some nice properties as follows:

1. $f$ is holomorphic $\Longrightarrow$ $f$ is continuous.

2. $f$ and $g$ are holomorphic $\Longrightarrow$ $f + g$ and $f \cdot g$ are holomorphic.

3. the sum rule, product rule, quotient rule and chain rule for derivatives hold.

**Example:**

1. A polynomial is an entire function. More specifically, $f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ with $a_m, \cdots, a_0 \in \mathbb{C}$. Its first derivative is $f'(z) = m a_m z^{m-1} + \cdots + a_1$.

2. $f : \mathbb{C} \backslash \{0\} \to \mathbb{C}$, $f(z) = \frac{1}{z}$ is holomorphic.

3. Let $S = \{z \in \mathbb{C} \mid q(z) = 0\}$, then $f(z) = \frac{p(z)}{q(z)}$ is defined on $\mathbb{C} \backslash S$ where $p(z)$ and $q(z)$ are polynomials. Then $f$ is holomorphic.

$\square$