Notes on Fourier Series

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First draft: December 5, 2023 Last update: December 27, 2023

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1 Orthogonal Functions

To describe orthogonal functions, we follow the definition in (Apostol, 1974, page 306). First, we denote by $L(I)$ the set of Lebesgue-integrable functions on an interval I. Then we denote by $L^2(I)$ the set of all complex-valued functions f which are measurable on I and are such that $|f|^2 \in L(I)$. The inner product (f, g) of two such functions, defined by

$$(f, g) = \int_I f(x)\overline{g(x)}\mathrm{d}x, \tag{1.1}$$

always exists.

Definition 1.1 (orthogonal systems). *Let $S = \{\phi_0, \phi_1, \phi_2, \cdots\}$ be a collection of functions in $L^2(I)$. If*

$$(\phi_n, \phi_m) = 0 \qquad \textit{whenever } m \neq n, \tag{1.2}$$

the collection S is said to be an orthogonal system on I. If, in addition, each ϕ_n has norm 1, then S is said to be orthonormal on I.

The following orthogonal system is fundamental in the field of Fourier analysis.

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots, \sin nx, \cos nx, \cdots\} \tag{1.3}$$

More specifically, for $m, n \in \mathbb{N}^+$, on any interval with the length of 2π, we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \mathrm{d}x = \int_{-\pi}^{\pi} \sin mx \sin nx \mathrm{d}x = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}. \tag{1.4}$$

Particularly, we have

$$\int_{-\pi}^{\pi} 1 \cdot \cos mx \mathrm{d}x = \int_{-\pi}^{\pi} 1 \cdot \sin mx \mathrm{d}x = 0, \qquad m = 1, 2, \cdots. \tag{1.5}$$

2 Fourier Series

2.1 Definition

Suppose $f(x)$ can be represented as the following series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

which means the right-hand side converges to $f(x)$. Now we compute the coefficients a_n and b_n using the trigonometric orthogonality discussed earlier. Assume the right-hand side of (2.1) can be integrated term by term, then multiplying both sides by $\cos mx (m = 0, 1, 2, \cdots)$ and integrating both sides over $[-\pi, \pi]$ gives

$$\int_{-\pi}^{\pi} f(x) \cos mx \mathrm{d}x = \int_{-\pi}^{\pi} f(x) \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx \mathrm{d}x \tag{2.2}$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx \mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \mathrm{d}x + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \mathrm{d}x \tag{2.3}$$

$$= a_m \pi \tag{2.4}$$

which implies

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \mathrm{d}x, \qquad n = 0, 1, 2, \cdots. \tag{2.5}$$

Likewise, we get

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \mathrm{d}x, \qquad n = 1, 2, \cdots. \tag{2.6}$$

(2.5) and (2.6) are called Euler formulas for *Fourier coefficients*.

Definition 2.1 (Fourier series). *Given $f(x)$ is 2π-periodic, Riemann integrable, and absolutely integrable on $[-\pi, \pi]$, the Fourier series is defined by*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{2.7}$$

where a_n and b_n are computed by (2.5) and (2.6), respectively, which are called Fourier coefficients.

Note. A trigonometric series is not necessarily a Fourier series. For example,

$$f(x) = \sum_{n=2}^{\infty} \frac{\sin nx}{\ln n} \tag{2.8}$$

is uniformly convergent on any closed interval residing in $(0, 2\pi)$, which follows from the Dirichlet's test for uniform convergence. However, it is not a Fourier series because it does not satisfy the definition of Fourier series.

2.2 Some useful results for computing Fourier coefficients

$$\int_0^{\pi} \sin nx \mathrm{d}x = -\int_{-\pi}^{0} \sin nx \mathrm{d}x = \frac{1-(-1)^n}{n} = \frac{2}{2k-1}, \tag{2.9}$$

$$\int_0^{\pi} x\cos nx \mathrm{d}x = -\int_{-\pi}^{0} x\cos nx \mathrm{d}x = \frac{(-1)^n-1}{n^2} = -\frac{2}{(2k-1)^2}, \tag{2.10}$$

$$\int_0^{\pi} x\sin nx \mathrm{d}x = \int_{-\pi}^{0} x\sin nx \mathrm{d}x = \frac{(-1)^{n+1}}{n}\pi, \tag{2.11}$$

$$\int_0^{\pi} x^2\cos nx \mathrm{d}x = \int_{-\pi}^{0} x^2\cos nx \mathrm{d}x = \frac{2(-1)^n}{n^2}\pi, \tag{2.12}$$

$$\int_0^{\pi} e^x\cos nx \mathrm{d}x = \frac{(-1)^n e^{\pi}-1}{n^2+1}, \tag{2.13}$$

$$\int_0^{2\pi} x\sin nx \mathrm{d}x = -\frac{2\pi}{n}, \tag{2.14}$$

$$\int_0^{2\pi} x\cos nx \mathrm{d}x = 0, \tag{2.15}$$

where $n, k \in \mathbb{N}^+$.

$$\int_0^{\pi/2} \cos x\cos nx \mathrm{d}x = -\int_{\pi/2}^{\pi} \cos x\cos nx \mathrm{d}x = -\frac{\cos\frac{n\pi}{2}}{n^2-1} = \frac{(-1)^k}{4k^2-1} \tag{2.16}$$

where $n, k \in \mathbb{N}^+$.

3 Fourier Sine and Cosine Series

It is easy to observe that when $f(x)$ is an odd function, the Fourier coefficients a_n vanish. In this case, the Fourier series is called Fourier sine series since it is comprised of sine functions as follows.

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \tag{3.1}$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \mathrm{d}x, \qquad n = 1, 2, \cdots. \tag{3.2}$$

When $f(x)$ is an even function, the Fourier coefficients b_n vanish. In this case, the Fourier series is called Fourier cosine series since it is comprised of cosine functions as follows.

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos nx, \tag{3.3}$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \mathrm{d}x, \qquad n = 0, 1, 2, \cdots. \tag{3.4}$$

4 More results on coefficients of Fourier Series

4.1 f defined on $[a, a + 2\pi]$

When $f(x)$ is defined on $(a, a + 2\pi)$, the coefficients a_n and b_n can be obtained in the same way as on $(-\pi, \pi)$ as follows:

$$\int_a^{a+2\pi} f(x)\cos mx \mathrm{d}x = \int_a^{a+2\pi} f(x)\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx \mathrm{d}x \tag{4.1}$$

$$= \frac{a_0}{2}\int_a^{a+2\pi}\cos mx \mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_a^{a+2\pi}\cos nx \cos mx \mathrm{d}x + \sum_{n=1}^{\infty} b_n \int_a^{a+2\pi}\sin nx \sin mx \mathrm{d}x \tag{4.2}$$

$$= a_m \pi \tag{4.3}$$

which implies

$$a_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos nx \mathrm{d}x, \qquad n = 0, 1, 2, \cdots. \tag{4.4}$$

Likewise, we get

$$b_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin nx \mathrm{d}x, \qquad n = 1, 2, \cdots. \tag{4.5}$$

4.2 f defined on $[-T, T]$

If $f(x)$ is $2T$-periodic, let $x = \frac{T}{\pi}t$ where $t \in [-\pi, \pi]$, then

$$\phi(t) = f(\frac{T}{\pi}t) = f(x) \tag{4.6}$$

is periodic with period 2π. Thus, with the results obtained in section 2.1, we have

$$\phi(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \tag{4.7}$$

and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi}{T}x + b_n \sin \frac{n\pi}{T}x), \tag{4.8}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(t)\cos nt \mathrm{d}t = \frac{1}{T}\int_{-T}^{T} f(x)\cos \frac{n\pi}{T}x \mathrm{d}x, \qquad n = 0, 1, 2, \cdots, \tag{4.9}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(t)\sin nt \mathrm{d}t = \frac{1}{T}\int_{-T}^{T} f(x)\sin \frac{n\pi}{T}x \mathrm{d}x, \qquad n = 1, 2, \cdots. \tag{4.10}$$

4.3 f defined on $[0, T]$

If $f(x)$ is defined on $[0, T]$, then we can take advantage of (4.4) and (4.5) with $a = 0$. Also, we need the trick of change of variables as performed in (4.6). Let $x = \frac{T}{2\pi}t$ where $t \in [0, 2\pi]$, then

$$f(x) = f(\frac{T}{2\pi}t) = \phi(t). \tag{4.11}$$

Combining this with (4.4) and (4.5) gives

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \phi(t)\cos nt\mathrm{d}t = \frac{2}{T}\int_0^T f(x)\cos(\frac{2n\pi}{T}x)\mathrm{d}x, \qquad n = 0, 1, 2, \cdots. \tag{4.12}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \phi(t)\sin nt\mathrm{d}t = \frac{2}{T}\int_0^T f(x)\sin(\frac{2n\pi}{T}x)\mathrm{d}x, \qquad n = 1, 2, \cdots. \tag{4.13}$$

Finally, the Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(\frac{2n\pi}{T}x) + b_n\sin(\frac{2n\pi}{T}x)\right), \tag{4.14}$$

4.4 Special cases for vanishing coefficients

In some cases, we only want to see $\cos x, \sin x, \cos 3x, \sin 3x, \cdots$ in a Fourier series. In other words, the components of trigonometric functions of $2kx$ vanish. What functions have this kind of Fourier series? The following proposition gives the answer.

Proposition 4.1. *Given $f(x)$ is Riemann integrable or absolutely integrable on $[-\pi, \pi]$, then*

1. *if $f(x) = f(x + \pi)$ for $x \in [-\pi, \pi]$, then $a_{2n-1} = b_{2n-1} = 0$;*
2. *if $f(x) = -f(x + \pi)$ for $x \in [-\pi, \pi]$, then $a_{2n} = b_{2n} = 0$.*

Proof. 1.

$$\begin{aligned}
a_{2n-1} &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2n-1)x\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(2n-1)x\mathrm{d}x + \frac{1}{\pi}\int_0^{\pi} f(x)\cos(2n-1)x\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x+\pi)\cos(2n-1)x\mathrm{d}x + \frac{1}{\pi}\int_0^{\pi} f(x)\cos(2n-1)x\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} f(t)\cos(2n-1)(t-\pi)\mathrm{d}t + \frac{1}{\pi}\int_0^{\pi} f(x)\cos(2n-1)x\mathrm{d}x \qquad (t = x + \pi) \\
&= -\frac{1}{\pi}\int_0^{\pi} f(t)\cos(2n-1)t\mathrm{d}t + \frac{1}{\pi}\int_0^{\pi} f(x)\cos(2n-1)x\mathrm{d}x \\
&= 0
\end{aligned}$$

$$\begin{aligned}
b_{2n-1} &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2n-1)x\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(2n-1)x\mathrm{d}x + \frac{1}{\pi}\int_0^{\pi} f(x)\sin(2n-1)x\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x+\pi)\sin(2n-1)x\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(2n-1)x\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin(2n-1)(t-\pi)\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(2n-1)x\mathrm{d}x \qquad (t = x+\pi) \\
&= -\frac{1}{\pi}\int_{0}^{\pi} f(t)\sin(2n-1)t\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(2n-1)x\mathrm{d}x \\
&= 0
\end{aligned}
$$

2.

$$
\begin{aligned}
a_{2n} &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2nx\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos 2nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos 2nx\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -f(x+\pi)\cos 2nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos 2nx\mathrm{d}x \\
&= -\frac{1}{\pi}\int_{0}^{\pi} f(t)\cos 2n(t-\pi)\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos 2nx\mathrm{d}x \qquad (t = x+\pi) \\
&= -\frac{1}{\pi}\int_{0}^{\pi} f(t)\cos 2nt\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos 2nx\mathrm{d}x \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_{2n} &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2nx\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin 2nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin 2nx\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -f(x+\pi)\sin 2nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin 2nx\mathrm{d}x \\
&= -\frac{1}{\pi}\int_{0}^{\pi} f(t)\sin 2n(t-\pi)\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin 2nx\mathrm{d}x \qquad (t = x+\pi) \\
&= -\frac{1}{\pi}\int_{0}^{\pi} f(t)\sin 2nt\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin 2nx\mathrm{d}x \\
&= 0
\end{aligned}
$$

□

5 Dirichlet's kernel

The Dirichlet's kernel is defined by

$$
D_n(t) = \frac{1}{2} + \sum_{k=1}^{n}\cos kt = \begin{cases} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}} & \text{if } t \neq 2m\pi,\ \text{where } m \in \mathbb{Z}, \\ n+\frac{1}{2} & \text{if } t = 2m\pi,\ \text{where } m \in \mathbb{Z}. \end{cases} \tag{5.1}
$$

It is easy to show the case when $t = 2m\pi$. Now we show the case when $t \neq 2m\pi$.

$$
2\sin\frac{t}{2}\cdot\left(\frac{1}{2} + \sum_{k=1}^{n}\cos kt\right) = \sin\frac{t}{2} + 2\sum_{k=1}^{n}\sin\frac{t}{2}\cos kt
$$

$$
\begin{aligned}
&= \sin\frac{t}{2} + 2\sin\frac{t}{2}\cos t + 2\sin\frac{t}{2}\cos 2t + \cdots + 2\sin\frac{t}{2}\cos nt \\
&= \sin\frac{t}{2} + \sin(\frac{t}{2} + t) + \sin(\frac{t}{2} - t) + \sin(\frac{t}{2} + 2t) + \sin(\frac{t}{2} - 2t) \\
&\quad + \cdots + \sin(\frac{t}{2} + nt) + \sin(\frac{t}{2} - nt) \\
&= \sin\frac{t}{2} - \sin\frac{t}{2} + \sin\frac{3t}{2} - \sin\frac{3t}{2} + \sin\frac{5t}{2} \\
&\quad + \cdots - \sin\frac{(2n-1)t}{2} + \sin\frac{(2n+1)t}{2} \\
&= \sin\frac{(2n+1)t}{2}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\sum_{k=1}^{n}\cos kt &= \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}} - \frac{1}{2} \\
&= \frac{\sin(n+\frac{1}{2})t - \sin\frac{t}{2}}{2\sin\frac{t}{2}} \\
&= \frac{\sin(\frac{n+1}{2}+\frac{n}{2})t - \sin(\frac{n+1}{2}-\frac{n}{2})t}{2\sin\frac{t}{2}} \\
&= \frac{2\cos(\frac{n+1}{2})t\sin\frac{nt}{2}}{2\sin\frac{t}{2}} = \frac{\sin\frac{nt}{2}\cos(\frac{n+1}{2})t}{\sin\frac{t}{2}}.
\end{aligned}
$$

Bibliography

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