

Online Self-Assessment for Real Analysis

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The math questions in this document are from <https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/analysis.html>. I have provided my solutions and proofs in here. The latest version of this document is available at here.

Question 1

1. Every sequence $\{x_n\}_{n \geq 1}$ of real numbers that satisfies $|x_n - x_{2n}| \rightarrow 0$ is convergent. Yes or no? Justify your answer.

Solution. No. The counterexample is

$$x_n = \begin{cases} 1, & n = 1, 2, 2^2, 2^3, \dots, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

This means that if $n = 2^k$ with k being a non-negative integer, then $x_n = x_{2n} = 1$. On the other hand, $x_n = x_{2n} = 0$ for $n \neq 2^k$. Both cases give $|x_n - x_{2n}| \rightarrow 0$, but $\{x_n\}_{n \geq 1}$ is not convergent, because it has two subsequences which converge to different limits. More generally, given that $\phi(n)$ is a strictly increasing function and $\phi(n) - n$ is unbounded, we can always find a divergent sequence $\{x_n\}_{n \geq 1}$ with $|x_{\phi(n)} - x_n| \rightarrow 0$ by constructing it as follows.

$$x_n = \begin{cases} 1, & n = k, \phi(k), \phi(\phi(k)), \dots, \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

where k is the smallest positive integer such that $\phi(k) > k$.

2. For every sequence $\{x_n\}_{n \geq 1}$ of real numbers the sequence $\{y_n\}_{n \geq 1}$ with

$$y_n := \frac{1}{1 + x_n^2}$$

has a convergent subsequence. Yes or no? Justify your answer.

Solution. Yes. Since x_n is a real number, then $x_n^2 \geq 0$. Then we have $0 \leq 1/(1 + x_n^2) \leq 1$, which implies that y_n is bounded. According to the fact that a bounded sequence must contain a convergent subsequence, the claim is true.

3. For every set $A \subseteq \mathbb{R}$ we let $\exp(A) := \{e^a | a \in A\}$. Then $\exp(A)$ has a finite infimum for every nonempty A .

Solution. Yes. For any real number a , we have $e^a > 0$. Therefore, $\exp A$ is bounded below by 0 for every nonempty set A . Then it must have exactly one greatest lower bound, i.e. infimum, which is supposed to be no less than 0.

4. A set is closed if, and only if, it is not open.

Solution. No. For example, $[0, 1)$ is not open, but it is not closed, either.

5. Let $K_n \subseteq \mathbb{R}$ be compact for every $n \geq 1$. Then the intersection $\bigcap_{n \geq 1} K_n$ is compact as well.

Solution. Yes. In \mathbb{R} , a set is compact if and only if it is bounded and closed. Since K_n is compact, the K_n is closed and bounded. For boundedness, for any K_n , there M_n such that $K_n \subseteq [M_n, M_n]$. Then $\bigcap_{n \geq 1} K_n$ is because the intersection of any collection of any closed sets is closed. In addition, $\bigcap_{n \geq 1} K_n$ is bounded due to $\bigcap_{n \geq 1} K_n \subseteq [M_1, M_1]$. Hence, $\bigcap_{n \geq 1} K_n$ is closed and bounded. Thus, it is compact as well.

6. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be continuous. Then

$$x \mapsto \frac{1}{1 + f(x)^2}$$

has a limit for $x \rightarrow 0^+$ (i.e. x tending to 0 from the right).

Solution. No. A counterexample is $f(x) = \sin(1/x)$ which oscillates between -1 and 1 as $x \rightarrow 0^+$, resulting in an oscillation between $1/2$ and 1 of the above mapping.

7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable. Then the function $x \mapsto |f(x)|$ is continuously differentiable on \mathbb{R} .

Solution. No. A counterexample is $f(x) = x$ which is twice continuously differentiable. However, the derivative of $|f(x)|$ does not exist at $x = 0$. Because $|f(x)|' = -1$ for all $x < 0$ and $|f(x)|' = 1$ for all $x > 0$.

8. Let $\{f_n\}_{n \geq 1}$ be a uniformly convergent sequence of real functions on $[0, 1]$. Then the sequence $\{|f_n|\}_{n \geq 1}$ is uniformly convergent as well.

Solution. Yes. Since $f_n(x)$ is uniformly convergent to $f(x)$ on $[0, 1]$, then for all $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$

$$|f_n(x) - f(x)| < \epsilon, \quad \forall x \in [0, 1] \quad (3)$$

holds. By the triangle inequality, for all $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$

$$||f_n(x)| - |f(x)|| \leq |f_n(x) - f(x)| < \epsilon, \quad \forall x \in [0, 1] \quad (4)$$

which implies that the sequence $\{|f_n|\}_{n \geq 1}$ is uniformly convergent as well.

Question 2

1. Check the following series for convergence and determine its limit, if it exists:

$$\sum_{n=1}^{\infty} \frac{n^2 + 3n}{n^3 + n^2 - n + 5}$$

Solution. The above series is divergent. To see this,

$$\sum_{n=1}^{\infty} \frac{n^2 + 3n}{n^3 + n^2 - n + 5} > \sum_{n=1}^{\infty} \frac{n^2}{n^3 + n^2 - n + 5n} \quad (5)$$

$$> \sum_{n=1}^{\infty} \frac{n^2}{n^3 + n^2 + 4n} \quad (6)$$

$$> \sum_{n=1}^{\infty} \frac{n}{n^2 + n + 4} \quad (7)$$

$$> \sum_{n=1}^{\infty} \frac{n}{n^2 + n + 4n} = \sum_{n=1}^{\infty} \frac{1}{n + 5} \quad (8)$$

which demonstrates that the series is divergent.

2. Determine the set of $x \in \mathbb{R}$, for which

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(3k)!} (x - 5)^2 k$$

converges.

Solution. Since $\sum_{k=1}^{\infty} \frac{(k!)^2}{(3k)!} (x - 5)^2 k = (x - 5)^2 \sum_{k=1}^{\infty} \frac{(k!)^2}{(3k)!} k$, we only need to check the convergence of $\sum_{k=1}^{\infty} \frac{(k!)^2}{(3k)!} k$. We employ the ratio test as follows.

$$\limsup_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = \limsup_{k \rightarrow \infty} \frac{((k+1)!)^2}{(3(k+1))!} (k+1) \frac{(3k)!}{(k!)^2 k} \quad (9)$$

$$= \limsup_{k \rightarrow \infty} \frac{(k+1)^3}{(3k+1)(3k+2)(3k+3)k} \quad (10)$$

$$= 0 < 1 \quad (11)$$

which shows that the series is convergent for all $x \in \mathbb{R}$.

Question 3

Let the function $f_n : [0, \infty) \rightarrow \mathbb{R}$ for $n \geq 1$ be defined as

$$f_n(x) := \int_0^x e^{-\frac{t^2}{n}} dt.$$

1. Show that f_n is continuously differentiable on $(0, \infty)$ for each n .

Solution. By definition, we have

$$f'_n(x) = \lim_{h \rightarrow 0} \frac{\int_0^{x+h} e^{-\frac{t^2}{n}} dt - \int_0^x e^{-\frac{t^2}{n}} dt}{h} \quad (12)$$

$$= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} e^{-\frac{t^2}{n}} dt}{h} \quad (13)$$

$$= \lim_{h \rightarrow 0} \frac{e^{-\frac{c^2}{n}} \int_x^{x+h} dt}{h}, \quad \text{where } c \in [x, x+h] \quad (14)$$

$$= \lim_{h \rightarrow 0} \frac{e^{-\frac{c^2}{n}} h}{h} \quad (15)$$

$$= \lim_{h \rightarrow 0} e^{-\frac{c^2}{n}} = e^{-\frac{x^2}{n}} \quad (16)$$

Since the function $g(x) = e^{-\frac{x^2}{n}}$ is an exponential function composed with a polynomial function, both of which are continuous on \mathbb{R} , then $g(x)$ is continuous. Thus, f_n is continuously differentiable on $(0, \infty)$ for each n with $f'_n(x) = g(x)$.

2. Show that for every $x \geq 0$ the limit $\lim_{n \rightarrow \infty} f_n(x)$ exists and determine its value.

Solution. Obviously, for any fixed $t \geq 0$, we have

$$\lim_{n \rightarrow \infty} e^{-\frac{t^2}{n}} = 1, \quad (17)$$

which shows the pointwise convergence of the integrand. Note that $0 \leq e^{-\frac{t^2}{n}} \leq 1$ for all $t \geq 0$ and $n \geq 1$. Also, $F(t) = 1$ is integrable on $(0, \infty)$. By the Dominated Convergence Theorem, we can exchange the limit and the integral as follows.

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \int_0^x e^{-\frac{t^2}{n}} dt = \int_0^x \lim_{n \rightarrow \infty} e^{-\frac{t^2}{n}} dt = \int_0^x 1 dt = x. \quad (18)$$