Online Self-Assessment for Analysis

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The math questions in this document are from `https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/analysis.html`. I have provided my solutions and proofs in here. The latest version of this document is available at here.

Question 1

1. Every sequence $\{x_n\}_{n\geq 1}$ of real numbers that satisfies $|x_n - x_{2n}| \to 0$ is convergent. Yes or no? Justify your answer.

   Solution. No. A counterexample is

   $$x_n = \begin{cases} 1, & n = 1, 2, 2^2, 2^3, \cdots, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

   This means that if $n = 2^k$ with k being a non-negative integer, then $x_n = x_{2n} = 1$. On the other hand, $x_n = x_{2n} = 0$ for $n \neq 2^k$. Both cases give $|x_n - x_{2n}| \to 0$, but $\{x_n\}_{n\geq 1}$ is not convergent, because it has two subsequences which converge to different limits. More generally, given that $\phi(n)$ is a strictly increasing function and $\phi(n) - n$ is unbounded, we can always find a divergent sequence $\{x_n\}_{n\geq 1}$ with $|x_{\phi(n)} - x_n| \to 0$ by constructing it as follows.

   $$x_n = \begin{cases} 1, & n = k, \phi(k), \phi(\phi(k)), \cdots, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

   where k is the smallest positive interger such that $\phi(k) > k$.

2. For every sequence $\{x_n\}_{n\geq 1}$ of real numbers the sequence $\{y_n\}_{n\geq 1}$ with

   $$y_n := \frac{1}{1 + x_n^2}$$

   has a convergent subsequence. Yes or no? Justify your answer.

   Solution. Yes. Since x_n is a real number, then $x_n^2 \geq 0$. Then we have $0 \leq 1/(1 + x_n^2) \leq 1$, which implies that y_n is bounded. According to the fact that a bounded sequence must contain a convergent subsequence, the claim is true.

3. For every set $A \subseteq \mathbb{R}$ we let $\exp(A) := \{e^a | a \in A\}$. Then $\exp(A)$ has a finite infimum for every nonempty A.

   Solution. Yes. For any real number a, we have $e^a > 0$. Therefore, $\exp A$ is bounded below by 0 for every nonempty set A. Then it must have exactly one greatest lower bound, i.e. infimum, which is supposed to be no less than 0.

4. A set is closed if, and only if, it is not open.

   Solution. No. For example, $[0, 1)$ is not open, but it is not closed, either.

5. Let $K_n \subseteq \mathbb{R}$ be compact for every $n \geq 1$. Then the intersection $\bigcap_{n\geq 1} K_n$ is compact as well.

   Solution. Yes. In $\mathbb{R}$, a set is compact if and only if it is bounded and closed. Since K_n is compact, then K_n is closed and bounded. For boundedness, for any K_n, there exists $M_n > 0$ such that $K_n \subseteq [-M_n, M_n]$. Then $\bigcap_{n\geq 1} K_n$ is closed because the intersection of any collection of any closed sets is closed. In addition, $\bigcap_{n\geq 1} K_n$ is bounded due to $\bigcap_{n\geq 1} K_n \subseteq [-M_1, M_1]$. Hence, $\bigcap_{n\geq 1} K_n$ is closed and bounded. Thus, it is compact as well.

6. Let $f : (0, \infty) \to \mathbb{R}$ be continuous. Then

$$x \mapsto \frac{1}{1 + f(x)^2}$$

   has a limit for $x \to 0^+$ (i.e. x tending to 0 from the right).

   Solution. No. A counterexample is $f(x) = \sin(1/x)$ which oscillates between -1 and 1 as $x \to 0^+$, resulting in an oscillation between $1/2$ and 1 of the above mapping.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then the function $x \mapsto |f(x)|$ is continuously differentiable on $\mathbb{R}$.

   Solution. No. A counterexample is $f(x) = x$ which is twice continuously differentiable. However, the derivative of $|f(x)|$ does not exist at $x = 0$. Because $|f(x)|' = -1$ for all $x < 0$ and $|f(x)|' = 1$ for all $x > 0$.

8. Let $\{f_n\}_{n\geq 1}$ be a uniformly convergent sequence of real functions on $[0, 1]$. Then the sequence $\{|f_n|\}_{n\geq 1}$ is uniformly convergent as well.

   Solution. Yes. Since $f_n(x)$ is uniformly convergent to $f(x)$ on $[0, 1]$, then for all $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$

$$|f_n(x) - f(x)| < \epsilon, \quad \forall x \in [0, 1] \tag{3}$$

   holds. By the triangle inequality, for all $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$

$$||f_n(x)| - |f(x)|| \leq |f_n(x) - f(x)| < \epsilon, \quad \forall x \in [0, 1] \tag{4}$$

   which implies that the sequence $\{|f_n|\}_{n\geq 1}$ is uniformly convergent as well.

Question 2

1. Check the following series for convergence and determine its limit, if it exists:

$$\sum_{n=1}^{\infty} \frac{n^2 + 3n}{n^3 + n^2 - n + 5}$$

Solution. The above series is divergent. To see this,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3+n^2-n+5} &> \sum_{n=1}^{\infty} \frac{n^2}{n^3+n^2-n+5n} && (5)\\ &> \sum_{n=1}^{\infty} \frac{n^2}{n^3+n^2+4n} && (6)\\ &> \sum_{n=1}^{\infty} \frac{n}{n^2+n+4} && (7)\\ &> \sum_{n=1}^{\infty} \frac{n}{n^2+n+4n} = \sum_{n=1}^{\infty} \frac{n}{n^2+5n} && (8)\\ &= \sum_{n=1}^{\infty} \frac{1}{n+5} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{5} \frac{1}{n} && (9)\\ & && (10)\end{aligned}$$

which indicates that the series is divergent as it is greater than the harmonic series subtracted by a finite constant which is a divergent series.

2. Determine the set of $x \in \mathbb{R}$, for which

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(3k)!}(x-5)^{2k}$$

converges.

Solution. Let $y = (x-5)^2$, then we need to determine the radius of convergence for the following power series.

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(3k)!} y^k. \tag{11}$$

We employ the ratio test as follows.

$$\begin{aligned}\beta = \limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \limsup_{k\to\infty} \frac{((k+1)!)^2}{(3(k+1))!} \frac{(3k)!}{(k!)^2} && (12)\\ &= \limsup_{k\to\infty} \frac{(k+1)^2}{(3k+1)(3k+2)(3k+3)} && (13)\\ &= 0 < 1. && (14)\end{aligned}$$

Therefore, the radius of convergence is

$$R = \frac{1}{\beta} = +\infty. \tag{15}$$

Furthermore, the power series converges for all $y \geq 0$, i.e., $(x-5)^2 \geq 0$. This implies that the radius of convergence of the original power series is $+\infty$. Hence, it is convergent for any $x \in \mathbb{R}$.

Question 3

Let the function $f_n : [0, \infty) \to \mathbb{R}$ for $n \geq 1$ be defined as

$$f_n(x) := \int_0^x e^{-\frac{t^2}{n}}\, dt.$$

1. Show that f_n is continuously differentiable on $(0, \infty)$ for each n.

   Solution. By definition, we have

$$\begin{aligned}
f_n'(x) &= \lim_{h\to 0} \frac{\int_0^{x+h} e^{-\frac{t^2}{n}}\, dt - \int_0^x e^{-\frac{t^2}{n}}\, dt}{h} && (16)\\
&= \lim_{h\to 0} \frac{\int_x^{x+h} e^{-\frac{t^2}{n}}\, dt}{h} && (17)\\
&= \lim_{h\to 0} \frac{e^{-\frac{c^2}{n}} \int_x^{x+h} dt}{h}, \quad \text{where } c \in [x, x+h] && (18)\\
&= \lim_{h\to 0} \frac{e^{-\frac{c^2}{n}} h}{h} && (19)\\
&= \lim_{h\to 0} e^{-\frac{c^2}{n}} = e^{-\frac{x^2}{n}} && (20)
\end{aligned}$$

   Since the function $g(x) = e^{-\frac{x^2}{n}}$ is an exponential function composed with a polynomial function, both of which are continuous on $\mathbb{R}$, then $g(x)$ is continuous. Thus, f_n is continuously differentiable on $(0, \infty)$ for each n with $f_n'(x) = g(x)$.

2. Show that for every $x \geq 0$ the limit $\lim_{n\to\infty} f_n(x)$ exists and determine its value.

   Solution. Obviously, for any fixed $t \geq 0$, we have

$$\lim_{n\to\infty} e^{-\frac{t^2}{n}} = 1, \tag{21}$$

   which shows the pointwise convergence of the integrand. Note that $0 \leq e^{-\frac{t^2}{n}} \leq 1$ for all $t \geq 0$ and $n \geq 1$. Also, $F(t) = 1$ is integrable on $(0, \infty)$. By the Dominated Convergence Theorem, we can exchange the limit and the integral as follows.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_0^x e^{-\frac{t^2}{n}}\, dt = \int_0^x \lim_{n\to\infty} e^{-\frac{t^2}{n}}\, dt = \int_0^x 1 dt = x. \tag{22}$$