## Online Self-Assessment for Complex Analysis

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The math questions in this document are from https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/complex.html. I have provided my solutions and proofs in here. The latest version of this document is available at here.

## Question 1

Which of the following functions is holomorphic on  $\mathbb{C} \setminus \{0\}$ ?

- 1.  $\frac{\overline{z}}{|z|^2}$ Solution. Yes. We have  $\frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{1}{z}$ . Since  $\frac{1}{z}$  is holomorphic on  $\mathbb{C} \setminus \{0\}$ , then  $\frac{\overline{z}}{|z|^2}$  is holomorphic.
- 2.  $\frac{\overline{z}}{1+|z|^2}$  Solution. No. Let z=x+iy, then  $\frac{\overline{z}}{1+|z|^2}=\frac{x}{(1+x^2+y^2)}-i\frac{y}{(1+x^2+y^2)}$ . Moreover,  $u(x,y)=\frac{x}{(1+x^2+y^2)}$  and  $v(x,y)=\frac{y}{(1+x^2+y^2)}$ . Furthermore,  $u_x=\frac{1-x^2+y^2}{(1+x^2+y^2)^2}$  and  $v_y=-\frac{1+x^2-y^2}{(1+x^2+y^2)^2}$ , which does not satisfy the Cauchy-Riemann equations.
- 3.  $\frac{\overline{z}}{z}$ Solution. No. Let z=x+iy, then  $\frac{\overline{z}}{z}=\frac{x-iy}{x+iy}=\frac{x^2-y^2-i2xy}{x^2+y^2}$ . Moreover,  $u(x,y)=\frac{x^2-y^2}{x^2+y^2}$  and  $v(x,y)=-\frac{2xy}{x^2+y^2}$ . Furthermore,  $u_x=\frac{4xy^2}{(x^2+y^2)^2}$  and  $v_y=\frac{2xy^2-2x^3}{(x^2+y^2)^2}$ , which does not satisfy the Cauchy-Riemann equations.
- 4.  $\frac{1+z}{z}$  Solution. Yes. Because  $\frac{1+z}{z}$  is the quotient of two holomorphic functions on  $\mathbb{C}\setminus\{0\}$ .

## Question 2

Compute the complex path integral  $\int_{\gamma} f(z) dz$  for the following choices of f and  $\gamma$ . In each case, the path  $\gamma$  is parameterised as a function  $[0,1] \to \mathbb{C}$ .

1.  $f(z) = \frac{i\overline{z}}{\pi}$ ,  $\gamma(t) = e^{2\pi it}$ Solution.

$$\int_{\gamma} \frac{i\overline{z}}{\pi} dz = \int_{0}^{1} \frac{ie^{-2\pi it}}{\pi} de^{2\pi it} \tag{1}$$

$$= \int_0^1 \frac{ie^{-2\pi it}}{\pi} e^{2\pi it} 2\pi i dt$$
 (2)

$$= \int_0^1 2i^2 dt = \int_0^1 -2dt = -2.$$
 (3)

2. 
$$f(z) = z^3$$
,  $\gamma(t) = te^{it}\cos(\frac{\pi}{2}t) + (1-t)e^{t^2}$ 

Solution. Since  $f(z)=z^3$  is holomorphic on  $\mathbb{C}$ , by the Cauchy's theorem, the claimed integral is path independent. Therefore,  $\gamma(t)$  can be replaced by  $\tilde{\gamma}(t)=1-t$  with  $t\in[0,1]$ , which guarantees both curves have common endpoints, namely  $\tilde{\gamma}(0)=\gamma(0)=1$  and  $\tilde{\gamma}(1)=\gamma(1)=0$ . Hence,

$$\int_{\gamma} z^3 dz = \int_{\tilde{\gamma}} z^3 dz = \int_0^1 (1 - t)^3 d(1 - t) \tag{4}$$

$$= \int_{1}^{0} u^{3} du = \frac{u^{4}}{4} \Big|_{1}^{0} = -\frac{1}{4}. \tag{5}$$

3. 
$$f(z) = \frac{i\cos(z^2)}{\pi z^5}$$
,  $\gamma(t) = e^{2\pi it}$ 

Solution. Since  $\frac{i\cos(z^2)}{\pi}$  is holomorphic on  $\mathbb C$  and the path  $\gamma$  is a unit circle centered at the origin, by the corollary of the Cauchy integral formula, i.e.,  $\int_{\gamma} \frac{f(\xi)}{(\xi-z)^{n+1}} d\xi = \frac{2\pi i}{n!} f^{(n)}(z)$  with z being inside of the circle  $\gamma$ , we have

$$\int_{\gamma} \frac{i\cos(\xi^2)}{\pi \xi^5} d\xi = \int_{\gamma} \frac{(i\cos(\xi^2))/\pi}{(\xi - 0)^{4+1}} d\xi \tag{6}$$

$$= \frac{2\pi i}{4!} \left( \frac{i\cos(z^2)}{\pi} \right)^{(4)} \bigg|_{z=0} = -\frac{2}{4!} \left( \cos(z^2) \right)^{(4)} \bigg|_{z=0}$$
 (7)

$$= \frac{1}{12} \left( 2z \sin(z^2) \right)^{(3)} \bigg|_{z=0} = \frac{1}{6} \left( z \sin(z^2) \right)^{(3)} \bigg|_{z=0}$$
 (8)

$$= \frac{1}{6} \left( \sin(z^2) + 2z^2 \cos(z^2) \right)'' \bigg|_{z=0}$$
 (9)

$$= \frac{1}{6} \left( 6z \cos(z^2) - 4z^3 \sin(z^2) \right)' \bigg|_{z=0}$$
 (10)

$$= \frac{1}{6} \left( 6 \cos(z^2) - 12z^2 \sin(z^2) - \dots \right) \bigg|_{z=0} = 1.$$
 (11)

where in the last line we did not calculate the derivative of  $4z^3 \sin(z^2)$  w.r.t. z since it will definitely vanish when being evaluated at z = 0.

Note: In terms of computation, a simpler method is employing the residue theorem. Specifically,

$$\frac{\cos(z^2)}{z^5} = \frac{1 - \frac{(z^2)^2}{2!} + \frac{(z^2)^4}{4!} - \dots}{z^5}$$
 (12)

$$=\frac{1-\frac{z^4}{2}+\frac{z^8}{4!}-\cdots}{z^5} \tag{13}$$

$$=z^{-5} - \frac{z^{-1}}{2} + \frac{z^3}{4!} - \cdots (14)$$

So,  $c_{-1} = -\frac{1}{2}$ . Thus,

$$\int_{\gamma} \frac{i \cos(\xi^2)}{\pi \xi^5} d\xi = \frac{i}{\pi} \cdot \int_{\gamma} \frac{\cos(\xi^2)}{\xi^5} d\xi \tag{15}$$

$$= \frac{i}{\pi} \cdot 2\pi i \cdot c_{-1} \tag{16}$$

$$= \frac{i}{\pi} \cdot 2\pi i \cdot (-\frac{1}{2}) = -i^2 = 1. \tag{17}$$

## Question 3

We consider the function  $f: \mathbb{C} \to \mathbb{C}$  with

$$f(a+bi) = \sin a + bi.$$

1. Determine the set of points  $z \in \mathbb{C}$  at which f is complex differentiable. Where is it holomorphic?

Solution. Let  $u(a, b) = \sin a$  and v(a, b) = b, then

$$u_a' = \cos a, \quad u_b' = 0 \tag{18}$$

$$v_a' = 0, \quad v_b' = 1. \tag{19}$$

According to Cauchy-Riemann equations,  $\cos a = 1$ , which gives  $a = 2k\pi$ . Therefore, f is complex differentiable on  $\{(2k\pi + bi)|k \in \mathbb{Z}, b \in \mathbb{R}\}$ . However, these points are isolated, implying that f is not holomorphic everywhere which requires f to be complex differentiable on an open set.

2. Calculate  $\int_{\gamma} f(z) dz$  with  $\gamma(t) = t - it^2$ ,  $t \in [0, \pi]$ . Solution.

$$\int_{\gamma} f(z) \, dz = \int_{0}^{1} (\sin t - it^{2}) d(t - it^{2}) \tag{20}$$

$$= \int_0^1 (\sin t - it^2)(1 - i2t)dt \tag{21}$$

$$= \int_0^1 \left( (\sin t - 2t^3) - i(2t\sin t + t^2) \right) dt \tag{22}$$

$$= \int_0^1 (\sin t - 2t^3) dt - i \int_0^1 (2t \sin t + t^2) dt$$
 (23)

$$= \left(-\cos t - \frac{t^4}{2}\right) \Big|_0^{\pi} - i\left(2\sin t - 2t\cos t + \frac{t^3}{3}\right) \Big|_0^{\pi}$$
 (24)

$$= (2 - \frac{\pi^4}{2}) - i(2\pi + \frac{\pi^3}{3}). \tag{25}$$