Real Analysis

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1 Sequences

A sequence of real numbers, denoted $(a_n)_{n\in\mathbb{N}}$, is a map from natural numbers $\mathbb{N}$ to real numbers. Note that the starting index can be any nonnegative integers, so a more general notation for a sequence is $(a_n)_{n=m}^{\infty}$ where $m \geq 0, m \in \mathbb{Z}$.

1.1 Cauchy sequence

Definition 1 (**Cauchy sequence of reals**)**.** A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is a Cauchy sequence if, for every real $\epsilon > 0$, there exists an $N \geq m$ such that $|a_n - a_{n'}| \leq \epsilon$ for all $n, n' \geq N$.

Definition 2 (bounded sequences)**.** A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is bounded by a real number M iff we have $|a_n| \leq M$ for all $n \geq m$.

Proposition 1 (**Cauchy sequences are bounded**)**.** *If a sequence (s_n) is Cauchy, then (s_n) is bounded.*

Proof. Since (s_n) is Cauchy, there is an N such that for a given $\epsilon > 0$, $|s_n - s_m| \leq \epsilon$ for all $m, n > N$. Hence, $|s_n| \leq \epsilon + |s_m|$. Let $M := \max\{|s_1|, |s_2|, \cdots, |s_N|, \epsilon + |s_m|\}$, then we have two cases.

- Case 1: when $n \leq N$, we have $|s_n| \leq \max\{|s_1|, |s_1|, \cdots, |s_N|\} \leq M$;
- Case 2: when $n \geq N + 1$, we have $|s_n| \leq \epsilon + |s_m| \leq M$.

Thus, (s_n) is bounded. □

1.2 Convergence and limit

Definition 3 (**Convergence of sequences**)**.** A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is convergent if and only if, for a real number L and every real $\epsilon > 0$, there exists an $N \geq m$ such that $|a_n - L| \leq \epsilon$ for all $n \geq N$.

See Exercise 6.1.2 in Tao's Analysis I.

Proposition 2 (**Uniqueness of limits**). *Let $(a_n)_{n=m}^{\infty}$ be a real sequence starting at some integer index m, and let $L \neq L'$ be two distinct real numbers. Then it is not possible for $(a_n)_{n=m}^{\infty}$ to converge to L while also converging to L'.*

Proof. Suppose for sake of contradiction that $(a_n)_{n=m}^{\infty}$ was converging to both L and L'. Let $\epsilon = |L - L'|/3$; note that ϵ is positive since $L \neq L'$. Since $(a_n)_{n=m}^{\infty}$ converges to L, there exists an $N \geq m$ such that $|a_n - L| \leq \epsilon$ for all $n \geq N$. Similarly, there is an $M \geq m$ such that $|a_n - L'| \leq \epsilon$ for all $n \geq M$. If we set $n := \max\{N, M\}$, then we have $|a_n - L| \leq \epsilon$ and $|a_n - L'| \leq \epsilon$. Hence, by the triangle inequality, $|L - L'| \leq |a_n - L| + |a_n - L'| \leq 2\epsilon = 2|L - L'|/3$, which contradicts the fact that $|L - L'| > 0$. Thus it is not possible to converge to both L and L'. ☐

Definition 4 (**Limits of sequences**). If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L, we say that $(a_n)_{n=m}^{\infty}$ is **convergent** and that its **limit** is L; we write

$$L = \lim_{n \to \infty} a_n$$

to denote this fact. If a sequence $(a_n)_{n=m}^{\infty}$ is not converging to any real number L, we say that the sequence $(a_n)_{n=m}^{\infty}$ is **divergent** and we leave $\lim_{n \to \infty} a_n$ undefined.

Remark 1. Note that, convergence means that all the terms are eventually close to **a fixed number**, whereas Cauchy means that all the terms are eventually close to **each other**.

Definition 5 (Subsequences). Let $(n_k)_{k \in \mathbb{N}}$ be a sequence of natural numbers that is strictly increasing, then $(a_{n_k})_{k \in \mathbb{N}}$ is called a subsequence of $(a_n)_{n \in \mathbb{N}}$.

Definition 6 (Limit points, accumulation points). x is a limit point (an accumulation point) of $(a_n)_{n=m}^{\infty}$ if, for every ϵ and every $N \geq m$, there exists an $n \geq N$ such that $|a_n - x| \leq \epsilon$.

Proposition 3. *$a \in \mathbf{R}$ is an accumulation point of $(a_n)_{n \in \mathbb{N}}$ if and only if for all $\epsilon > 0$, the ϵ-neighborhood of a contains infinitely many sequence members of $(a_n)_{n \in \mathbb{N}}$.*

Proposition 4 (**Convergent sequences are Cauchy**). *If (s_n) converges to s, (s_n) is Cauchy.*

Proof. For a given $\epsilon > 0$, there exists an N such that $|s_n - s| \leq \epsilon/2$ for all $n \geq N$. Also, we have $|s_m - s| \leq \epsilon/2$ for all $m \geq N$. By the triangle inequality, $|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m| \leq \epsilon$. Thus, (s_n) is Cauchy. ☐

Corollary 1 (**Convergent sequences are bounded.**). *Every convergent sequence of real numbers is bounded.*

Proof. By Proposition 4, convergent sequences are Cauchy. By Proposition 1, Cauchy sequences are bounded. Thus, convergent sequences are bounded. ☐

Theorem 1 (Completeness of the reals). *A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is a Cauchy sequence if and only if it is convergent.*

The following proof is largely taken from Terence Tao's Analysis I.

Proof. Proposition 4 has shown that every convergent sequence is Cauchy, so it suffices to prove that every Cauchy sequence is convergent.

Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence. We know from Proposition 1 that the sequence $(a_n)_{n=1}^{\infty}$ is bounded, which implies that $L^- := \liminf_{n \to \infty} a_n$ and $L^+ := \limsup_{n \to \infty} a_n$ of the sequence are both finite. To show that the sequence converges, it will suffice to show that $L^- = L^+$.

Now let $\epsilon > 0$ be any real number. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists an $N \geq 1$ such that $a_N - \epsilon \leq a_n \leq a_N + \epsilon$ for all $n \geq N$. This implies that

$$a_N - \epsilon \leq \inf(a_n)_{n=N}^{\infty} \leq \sup(a_n)_{n=N}^{\infty} \leq a_N + \epsilon \tag{1}$$

and hence by the definition of L^- and L^+

$$a_N - \epsilon \leq L^- \leq L^+ \leq a_N + \epsilon. \tag{2}$$

Thus we get

$$0 \leq L^+ - L^- \leq 2\epsilon \tag{3}$$

which is true for all $\epsilon > 0$. Since L^- and L^+ do not depend on ϵ, then we must have $L^+ = L^-$. Thus, $(a_n)_{n=1}^{\infty}$ is convergent. □

Remark 2. Theorem 1 tells us Cauchy sequences and convergent sequences are equivalent. More straightforwardly,

$$\textbf{Cauchy sequences} \iff \textbf{convergent sequences}. \tag{4}$$

Theorem 2 (Limit Laws)**.** *Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$.*

(a) The sequence $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$; in other words,

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n. \tag{5}$$

(b) The sequence $(a_n b_n)_{n=m}^{\infty}$ converges to xy; in other words,

$$\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n). \tag{6}$$

(c) For any real number c, the sequence $(ca_n)_{n=m}^{\infty}$ converges to cx; in other words,

$$\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty} a_n. \tag{7}$$

(d) The sequence $(a_n - b_n)_{n=m}^{\infty}$ converges to $x - y$; in other words,

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n. \tag{8}$$

(e) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^{\infty}$ converges to y^{-1}; in other words,

$$\lim_{n\to\infty} b_n^{-1} = (\lim_{n\to\infty} b_n)^{-1}. \tag{9}$$

(f) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y; in other words,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}. \tag{10}$$

(g) The sequence $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$; in other words,

$$\lim_{n\to\infty} \max(a_n, b_n) = \max(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n). \tag{11}$$

(h) The sequence $(\min(a_n, b_n))_{n=m}^{\infty}$ converges to $\min(x, y)$; in other words,

$$\lim_{n\to\infty} \min(a_n, b_n) = \min(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n). \tag{12}$$

Proof. (a) Since $(a_n)_{n=m}^{\infty}$ converges to x, then for all $\epsilon > 0$, there exists a positive integer $N_1 > m$ such that for any $n > N_1$, $|a_n - x| \leq \frac{\epsilon}{2}$. Similarly, for all $\epsilon > 0$, there exists a positive integer $N_2 > m$ such that for any $n > N_2$, $|b_n - y| \leq \frac{\epsilon}{2}$. Then for any $n > \max(N_1, N_2)$, we have

$$|a_n + b_n - x - y| = |a_n - x + b_n - y| \leq |a_n - x| + |b_n - y| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{13}$$

which implies $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$, as desired.

(b) Since $(a_n)_{n=m}^{\infty}$ converges to x, then for all $\epsilon > 0$, there exists a positive integer $N_1 > m$ such that for any $n > N_1$, $|a_n - x| \leq \epsilon$. Also, by Corollary 1, $|a_n| \leq M \in \mathbf{R}$ for all $n \geq m$. Similarly, for all $\epsilon > 0$, there exists a positive integer $N_2 > m$ such that for any $n > N_2$, $|b_n - y| \leq \epsilon$. Then for any $n > \max(N_1, N_2)$, we have

$$\begin{aligned} |a_n b_n - xy| &= |a_n b_n - y a_n + y a_n - xy| && (14)\\ &= |a_n(b_n - y) + y(a_n - x)| && (15)\\ &\leq |a_n(b_n - y)| + |y(a_n - x)| && (16)\\ &\leq (M + |y|)\epsilon && (17) \end{aligned}$$

which implies $(a_n b_n)_{n=m}^{\infty}$ converges to xy since M and y are constants.

(c) This is the special case when $b_n = c$ for all $n \geq m$.

(d) It follows from (a) and c due to the fact $a_n - b_n = a_n + (-1) \cdot b_n$.

□

Theorem 3 (squeeze test, sandwich theorem)**.** *Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, and $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \geq m$. Suppose also that $(a_n)_{n=m}^{\infty}$ and $(a_n)_{n=m}^{\infty}$ both converge to the same limit L. Then $(b_n)_{n=m}^{\infty}$ is also convergent to L.*

Proof. Let $(d_n)_{n=m}^{\infty}$ be a sequence with $d_n = b_n - a_n$, then $0 \leq d_n \leq c_n - a_n$. Since both $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ converge to L, according to the limit laws, $\lim_{n\to\infty}(c_n - a_n) = L - L = 0$. Therefore, for all $\epsilon > 0$, there exists a positive integer $N > m$ such that for all $n > N$, $|d_n| = |d_n - 0| \leq |c_n| \leq \epsilon$. This indicates that $(d_n)_{n=m}^{\infty}$ is convergent with $\lim_{n\to\infty} d_n = 0$. Furthermore,

$$\lim_{n\to\infty}(d_n + a_n) = \lim_{n\to\infty} d_n + \lim_{n\to\infty} a_n = 0 + L = L. \tag{18}$$

Since $b_n = d_n + a_n$, then $\lim_{n\to\infty} b_n = L$ as desired. □

1.3 Upper bound and supremum

Definition 7 (**Upper bound**)**.** Let E be a subset of $\mathbf{R}$, and let M be a real number. We say that M is an **upper bound** for E, iff we have $x \leq M$ for every element x in E.

Definition 8 (**Least upper bound**)**.** Let E be a subset of $\mathbf{R}$, and let M be a real number. We say that M is a **least upper bound** for E, iff (a) M is an upper bound for E, and also (b) any other upper bound M' for E must be larger than or equal to M.

Proposition 5 (**Uniqueness of least upper bound**)**.** *Let E be a subset of $\mathbf{R}$. Then E can have at most one least upper bound.*

Proof. Let M_1 and M_2 be two least upper bounds. Since M_1 is a least upper bound and M_2 is an upper bound, then by definition of least upper bound we have $M_2 \geq M_1$. Since M_2 is a least upper bound and M_1 is an upper bound, we similarly have $M_1 \geq M_2$. Thus $M_1 = M_2$. Thus there is at most one least upper bound. □

Now we come to an important property of the real numbers:

Theorem 4 (**Existence of least upper bound**). *Let E be a nonempty subset of $\mathbf{R}$. If E has an upper bound, (i.e., E has some upper bound M), then it must have exactly one least upper bound.*

Proof. This theorem will take quite a bit of effort to prove, see Page 118 of Terence Tao's "Analysis 1, 3rd Edition". □

Definition 9 (**Supremum**). Let E be a subset of the real numbers. If E is non-empty and has some upper bound, **we define** $\sup(E)$ **to be the least upper bound of** E. If E is non-empty and has no upper bound, we set $\sup(E) := +\infty$; if E is empty, we set $\sup(E) := -\infty$. We refer to $\sup(E)$ as the supremum of E, and also denote it by $\sup E$.

Similarly, we can define greatest lower bound and infimum.

Remark 3. We can think of Theorem 4 as saying "$\sup(E)$ always exists". Because either E is bounded above (in which case $\sup(E)$ exists), or E is unbounded(in which case $\sup(E) = \infty$). This is a fundamental theorem of analysis. Also, by Proposition 5, $\sup(E)$ or $\inf(E)$ is unique.

Important fact: $\inf(S) = -\sup(-S)$.

Theorem 5. *Let E be a subset of $\mathbf{R}^*$. Then the following statements are true.*

1. *For every $x \in E$ we have $x \leq \sup(E)$ and $x \in \inf(E)$.*
2. *Suppose that $M \in \mathbf{R}^*$ is an upper bound for E, i.e., $x \leq M$ for all $x \in E$. Then we have $\sup(E) \leq M$.*
3. *Suppose that $M \in \mathbf{R}^*$ is a lower bound for E, i.e., $x \geq M$ for all $x \in E$. Then we have $\inf(E) \geq M$.*

Proof. □

Proposition 6 (least upper bound property). *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be the extended real number $x := \sup(a_n)_{n=m}^{\infty}$. Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in \mathbf{R}^*$ is an upper bound for a_n (i.e., $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which $y < a_n \leq x$.*

Proof. □

Proposition 7 (Monotone bounded sequences converge). *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite upper bound $M \in \mathbf{R}$, and which is also increasing (i.e., $a_{n+1} \geq a_n$ for all $n \leq m$). Then $(a_n)_{n=m}^{\infty}$ is convergent, and in fact*

$$\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty} \leq M.$$

Proof. □

Example: The sequence $(a_n)_{n=1}^{\infty}$ given by $a_n = (1 + \frac{1}{n})^n$ is convergent.

Proof. First, we show that $(a_n)_{n=1}^{\infty}$ is an increasing sequence. In order to do this, we employ GM-Am inequality to get

$$a_n = (1 + \frac{1}{n})^n \cdot 1 \leq \left(\frac{n(1+\frac{1}{n})+1}{n+1}\right)^{n+1} = \left(\frac{n+1+1}{n+1}\right)^{n+1} = a_{n+1} \tag{19}$$

which indicates $(a_n)_{n=1}^{\infty}$ is an increasing sequence. Now we show that $(a_n)_{n=1}^{\infty}$ is bounded from above as follows.

$$\begin{aligned} a_n =&(1+\frac{1}{n})^n && (20)\\ =&\sum_{k=0}^{n}\binom{n}{k}\cdot 1^{n-k}\cdot(\frac{1}{n})^k && (21)\\ =&\binom{n}{0}\cdot 1^n\cdot(\frac{1}{n})^0+\binom{n}{1}\cdot 1^{n-1}\cdot(\frac{1}{n})^1+\sum_{k=2}^{n}\binom{n}{k}\cdot 1^{n-k}\cdot(\frac{1}{n})^k && (22)\\ =&1+1+\sum_{k=2}^{n}\binom{n}{k}\cdot\frac{1}{n^k} && (23)\\ =&2+\sum_{k=2}^{n}\frac{n(n-1)(n-2)\cdots(n-k+1)}{n\cdot n\cdot n\cdot\cdots\cdot n}\cdot\frac{1}{k!} && (24)\\ \leq&2+\sum_{k=2}^{n}\frac{1}{k!}\leq 2+\sum_{k=2}^{n}\frac{1}{k(k-1)} && (25)\\ =&2+1-\frac{1}{n}<3. && (26) \end{aligned}$$

Hence, $(a_n)_{n=1}^{\infty}$ is increasing and bounded above. By Proposition 7, it converges. □

□

Remark 4. By convention we use e to denote the limit of the above sequence, namely

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e = 2.718281828459\cdots \tag{27}$$

which is an irrational number.

1.4 Bolzano-Weirstrass Theorem

Theorem 6 (Bolzano-Weierstrass theorem: every bounded sequence has a convergent subsequence)**.** *Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence (i.e., there exists a real number $M>0$ such that $|a_n|\leq M$ for all $n\in\mathbb{N}$). Then there is at least one subsequence of $(a_n)_{n=0}^{\infty}$ which converges.*

Proof. Since $(a_n)_{n=0}^{\infty}$ is a bounded sequence, we can find an interval $[c_0,d_0]\subset[-M,M]$ such that every member of $(a_n)_{n=0}^{\infty}$ resides in $[c_0,d_0]$. Now we bisect $[c_0,d_0]$ at $(c_0+d_0)/2$. If the left half contains infinitely many members of $(a_n)_{n=0}^{\infty}$, let $c_1:=c_0$ and $d_1:=(c_0+d_0)/2$, otherwise let $c_1:=(c_0+d_0)/2$ and $d_1:=d_0$. In this way, we can construct the following nested intervals.

$$[c_0,d_0]\supset[c_1,d_1]\supset\cdots\supset[c_n,d_n]\supset\cdots \tag{28}$$

which yields two monotone bounded sequences $(c_n)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$. Specifically, $(c_n)_{n=0}^{\infty}$ is increasing and bounded above, and $(d_n)_{n=0}^{\infty}$ is decreasing and bounded below. Therefore, by Proposition 7, $(c_n)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$ are both convergent. Since $d_n-c_n=\frac{1}{2^n}(d_0-c_0)$,

$$\lim_{n\to\infty}(d_n-c_n)=\lim_{n\to\infty}\frac{1}{2^n}(d_0-c_0)=0, \tag{29}$$

which implies $\lim_{n\to\infty}d_n=\lim_{n\to\infty}c_n=0$. For each interval $[c_n,d_n]$, we choose a point b_n which is a member of $(a_n)_{n=0}^{\infty}$. Thus, we get a subsequence $(b_n)_{n=0}^{\infty}$ where $c_n\leq b_n\leq d_n$. By the sandwich theorem, $(b_n)_{n=0}^{\infty}$ is a convergent subsequence which converges to $\lim_{n\to\infty}d_n$. □

1.5 The upper limit and lower limit

Let E denote the set of limit points,

$$E = \{\xi | \xi \text{ is a limit point of } \{x_n\}\}.$$

$H = \max E$ is called the upper limit of the sequence $\{x_n\}$, denoted as

$$H = \overline{\lim}_{n\to\infty} x_n;$$

$H = \min E$ is called the lower limit of the sequence $\{x_n\}$, denoted as

$$H = \underline{\lim}_{n\to\infty} x_n.$$

Note that the upper limit and lower limit are called the limit superior and the limit inferior, respectively, which are defined as follows:

$$\overline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geq n} x_n = \inf_n \sup_{k\geq n} x_k; \tag{30}$$

$$\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{k\geq n} x_n = \sup_n \inf_{k\geq n} x_k. \tag{31}$$

Proposition 8 (Limits are limit points)**.** *Let $(a_n)_{n=m}^{\infty}$ be a sequence which converges to a real number c. Then c is a limit point of $(a_n)_{n=m}^{\infty}$, and in fact it is the only limit point of $(a_n)_{n=m}^{\infty}$.*

Proof. See Exercise 6.4.1 in Tao's Analysis I. □

References