Series

Youming Zhao
Email: youming0.zhao@gmail.com

First draft: May 1, 2022   Last update: October 14, 2023

Contents

1 Notations

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \quad (k=1,2,\cdots)$$
$$\binom{\alpha}{0} = 1$$

where α is a nonzero real number. Note that in combinatorics α is usually a positive integer n, i.e., $\binom{n}{k}$ which is also denoted as C_n^k with $\mathrm{C}_n^0 = 1$. In this case, we have

$$\mathrm{C}_n^k = \frac{n!}{k!\,(n-k)!}$$

2 Preliminaries

2.1 Basic elementary functions

2.2 Elementary functions

3 Alternating series

3.1 Definitions

Definition 3.1. *A series $\sum_{n=1}^{\infty} x_n$ is called an **alternating series** if*

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n (u_n > 0).$$

*Furthermore, if the sequence $\{u_n\}$ is nonincreasing and converges to 0, then we call $\sum_{n=1}^{\infty} x_n$ **Leibniz series**.*

4 The product of two infinite series

The product of two convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is the sum of all the terms $a_i b_j (i = 1, 2, \cdots; j = 1, 2, \cdots)$. These terms can be arranged in the form of an infinite matrix as follows.

$$\begin{array}{ccccc}
a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & \cdots \\
a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & \cdots \\
a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & \cdots \\
a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Since the laws of associativity and commutativity do not hold for series, the way how these terms are arranged matters. Two common arrangements are square arrangement and diagonal arrangement. They give two kinds of products of infinite series, i.e., square product and Cauchy product, respectively.

4.1 Square product

We introduce the square product via the following infinite matrix which consists of infinite many squares.

$$\begin{array}{cccccc}
 & | & | & | & | & \\
\cdot & a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & \cdots \\
 & & | & | & | & \\
\cdot & a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & \cdots \\
 & & & | & | & \\
\cdot & a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & \cdots \\
 & & & & | & \\
\cdot & a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Let

$$
\begin{aligned}
&d_1 = a_1 b_1, \\
&d_2 = a_1 b_2 + a_2 b_2 + a_2 b_1, \\
&\cdots\cdots \\
&d_n = a_1 b_n + a_2 b_n + \cdots + a_n b_n + a_n b_{n-1} + \cdots + a_2 b_1, \\
&\cdots\cdots
\end{aligned}
$$

then $\sum_{n=1}^{\infty}$ is the product of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ through the above square arrangement. We call it **square product**.

Theorem 4.1 (Square arrangement of the product of two convergent series generates a convergent series). *Given two infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, the product of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ obtained through a square arrangement is convergent if both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and the following*

$$\sum_{n=1}^{\infty} d_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$$

Proof. From the definition of the square arrangement, we get that the partial sum

$$\sum_{n=1}^{N} d_n = \left(\sum_{n=1}^{N} a_n\right)\left(\sum_{n=1}^{N} b_n\right)$$

always hold. Since both of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, we can take $N \to \infty$ on both sides, which gives

$$\sum_{n=1}^{\infty} d_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right),$$

as desired.

□

Note. We could not prove the above claim by analyzing d_n. Specifically, d_n can be expressed as

$$d_n = \left(\sum_{i=1}^{n} a_i\right) b_n + a_n \left(\sum_{i=1}^{n-1} b_i\right)$$

Let $s_n = \alpha_n b_n$ with $\alpha_n = \sum_{i=1}^{n} a_i$ and $t_n = a_n \beta_n$ with $\beta_n = \sum_{i=1}^{n-1} b_i$, then $d_n = s_n + t_n$. You may observe that α_n is bounded since $\sum_{n=1}^{\infty} a_n$ converges. Moreover, $\sum_{n=1}^{\infty} b_n$ is convergent. You may think of Abel's test which is a method of testing for convergence of $\sum_{n=1}^{\infty} x_n y_n$. However, Abel's test requires the monotonicity and boundedness of α_n which corresponds to x_n. It is clear that α_n is bounded but not necessarily monotone. Although $\sum_{n=1}^{\infty} b_n$ which corresponds to $\sum_{n=1}^{\infty} y_n$ converge, we could not conclude that $\sum_{n=1}^{\infty} d_n$ converges.

4.1.1 A sufficient condition for the convergence of the product of two infinite series

Theorem 4.2. *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both absolutely convergent, then the series given by the sum of $a_i b_j$ over all i and $j > 0$ in any arrangement method also absolutely converge. Its sum is equal to*

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right).$$

Proof. For an arbitrary n, $a_{i_k}b_{j_k}(k=1,2,\cdots)$ is an arbitrary arrangement of $a_ib_j(i=1,2,\cdots;j=1,2,\cdots)$. Let

$$N=\max_{1\leq k\leq n}\{i_k,j_k\},$$

so

$$\sum_{k=1}^{n}|a_{i_k}b_{j_k}|\leq\sum_{i=1}^{N}|a_i|\cdot\sum_{j=1}^{N}|b_j|\leq\sum_{i=1}^{\infty}|a_i|\cdot\sum_{j=1}^{\infty}|b_j|,$$

which implies that $\sum_{k=1}^{\infty}a_{i_k}b_{j_k}$ absolutely converge. Since all permutations of an absolutely convergent real series converge to the same value, we denote by $\sum_{n=1}^{\infty}d_n$ the product of $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ via square arrangement. Then $\sum_{n=1}^{\infty}d_n$ is the series that is obtained by permuting and associating the terms of $\sum_{k=1}^{n}a_{i_k}b_{j_k}$. Thus, we have

$$\sum_{k=1}^{n}a_{i_k}b_{j_k}=\sum_{n=1}^{\infty}d_n=\left(\sum_{n=1}^{\infty}a_n\right)\left(\sum_{n=1}^{\infty}b_n\right).$$

This completes our proof. $\square$

4.2 Cauchy product

We use the following matrix to illustrate the diagonal arrangement of the aforementioned infinite matrix.

$$\begin{array}{ccccc}
a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & \cdots \\
a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & \cdots \\
a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & \cdots \\
a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Let

$$\begin{aligned}
&c_1=a_1b_1,\\
&c_2=a_1b_2+a_2b_1,\\
&\cdots\cdots\\
&c_n=\sum_{i+j=n+1}a_ib_j=a_1b_n+a_2b_{n-1}+\cdots+a_nb_1,\\
&\cdots\cdots
\end{aligned}$$

then we call

$$\sum_{n=1}^{\infty}c_n=\sum_{n=1}^{\infty}(a_1b_n+a_2b_{n-1}+\cdots+a_nb_1)$$

Cauchy product of $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$.

4.2.1 Convergence of Cauchy product

The convergence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is not sufficient for the convergence of Cauchy product. Let's look at the following example.

Example 4.3. Given $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$, both series are (conditionally) convergent. The terms of their Cauchy product are given by

$$c_n = \sum_{i+j=n+1} a_i b_j = \sum_{i+j=n+1} \frac{(-1)^{i+1}}{\sqrt{i}} \frac{(-1)^{j+1}}{\sqrt{j}} = (-1)^{n+1} \sum_{i+j=n+1} \frac{1}{\sqrt{ij}}.$$

By the AGM inequality,

$$\sqrt{ij} \leq \frac{i+j}{2} = \frac{n+1}{2} \Longleftrightarrow \frac{1}{\sqrt{ij}} \geq \frac{2}{n+1},$$

since each c_n contains n terms, then

$$|c_n| \geq n \cdot \frac{2}{n+1},$$

which implies that $c_n \nrightarrow 0$. This means that the necessary condition for the convergence of a series is not satisfied. Thus, the Cauchy product $\sum_{n=1}^{\infty} c_n$ diverges.

4.2.2 Applications of Cauchy product

Different starting indices give different forms of Cauchy product. For example,

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j,$$
$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty} \sum_{i+j=n+1} a_i b_j$$

When we apply Cauchy product to power series, the former one is preferable due to the fact that $x^i x^j = x^{i+j} = x^n$. Let's take a look at a concrete example.

Example 4.4. Find the Taylor expansion of $\frac{e^{-x}}{1-x}$ at $x_0 = 0$ and calculate its convergence region.

Solution: By Cauchy product, we have

$$\begin{aligned}
\frac{e^{-x}}{1-x} &= \left(\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n\right) \\
&= \sum_{n=0}^{\infty} \sum_{i+j=n} \frac{(-x)^i}{i!} x^j \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{(-1)^i}{i!}\right) x^n \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \cdots + \frac{(-1)^n}{n!}\right) x^n \\
&= \begin{cases} \sum_{n=0}^{\infty} \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) x^n, \\ \sum_{n=2}^{\infty} \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) x^n. \end{cases}
\end{aligned}$$

Some people may observe that the first two terms cancel out in the second last equality, then they get the last equality. In the first case, what is the first term? 0? In the second case, the constant term is

0 and the coefficient of x^1 is 0 as well. Both gives incorrect answers, because the true constant term is 1. The right way to deal with this problem is to write out the first 2 terms separately.

$$\begin{aligned}\frac{e^{-x}}{1-x} &= \left(\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)\left(\sum_{n=0}^{\infty}x^n\right)\\ &= 1+\left(\frac{(-1)}{1}+1\right)x+\sum_{n=2}^{\infty}\left(\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}\right)x^n\\ &= 1+\sum_{n=2}^{\infty}\left(\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}\right)x^n.\end{aligned}$$

Now we talk about its convergence region. Let $a_n = \frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}$. Obviously, $\sum_{n=2}^{\infty} a_n$ is an alternating series. Thus, $\frac{1}{2!}-\frac{1}{3!}\le a_n \le \frac{1}{2!}$ when $n\ge 2$. This gives $\overline{\lim}_{n\to\infty}\sqrt[n]{a_n}=1$. By the root test, the convergence radius is 1. When $x=\pm 1$,

$$|a_n x^n| = \sum_{n=2}^{\infty}\frac{(-1)^n}{n!} = \frac{1}{e} \nrightarrow 0,$$

which implies that the Taylor series of $\frac{e^{-x}}{1-x}$ does not converge at $x=\pm1$. Hence, its convergence region is $(-1,1)$. □

4.3 Infinite products

Infinite products are less famous than infinite series, but that does not mean it is not important.

Definition 4.5. *The infinite product*

$$\prod_{n=1}^{\infty} p_n$$

converges if the partial product

$$P_n = \prod_{n=1}^{m} p_n = l \neq 0 \text{ as } m \to +\infty.$$

We say that the infinite product ***diverges*** *if either the partial product does not converge or it converges to* 0*, but we usually say that* ***the infinite product diverges to*** 0 *instead for the latter case.*

Theorem 4.6. *If $\prod_{n=1}^{\infty} p_n$ converges, then*

1. $\lim_{n\to\infty} p_n = 1$

2. $\lim_{m\to\infty} \prod_{n=m+1}^{\infty} p_n = 1.$

Proof. Given $\prod_{n=1}^{\infty} p_n = l$, we denote by $\{P_n\}$ the sequence of the partial product of $\prod_{n=1}^{\infty} p_n$, then

$$\begin{aligned}\lim_{n\to\infty} p_n &= \lim_{n\to\infty}\frac{P_n}{P_{n-1}} = \frac{\lim_{n\to\infty}P_n}{\lim_{n\to\infty}P_{n-1}} = \frac{l}{l} = 1;\\ \lim_{m\to\infty}\prod_{n=m+1}^{\infty} p_n &= \lim_{m\to\infty}\frac{\prod_{n=1}^{\infty}p_n}{\prod_{n=1}^{m}p_n} = \frac{\prod_{n=1}^{\infty}p_n}{\lim_{m\to\infty}\prod_{n=1}^{m}p_n} = \frac{l}{l} = 1.\end{aligned}$$

This completes the proof. □

Remark 4.7. We often denote p_n by $1+a_n$. Then for (1) of Theorem 4.6, if $\prod_{n=1}^{\infty}(1+a_n)$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Remark 4.8. The first result of Theorem 4.6 is analogous to the necessary condition of series convergence. So, it can be used to determine the divergence of an infinite product. For example, it is obvious that for $p_n = \frac{n}{2n+1}$, $\prod_{n=1}^{\infty} p_n$ is divergent.

4.3.1 Convergence of infinite products

Since $\lim\limits_{n\to\infty} p_n = 1$, there must exist a positive integer N such that $p_n > 0$ for $n > N$. As we know that the convergence of an infinite product has nothing to do with the first N terms, we assume $p_n > 0$.

Theorem 4.9. *The infinite product $\prod_{n\to\infty} p_n$ is convergent if and only if the series $\sum_{n=1}^{\infty} \ln p_n$.*

Proof. Let

$$P_n = \prod_{k=1}^{n} p_k$$

and

$$S_n = \sum_{k=1}^{n} \ln p_k.$$

Then we have

$$P_n = \prod_{k=1}^{n} p_k = e^{S_n}.$$

Thus,

$$S_n \to S \Longleftrightarrow e_n^S \to e^S$$

which implies that the sequence $\{P_n\}$ converges to a nonzero real number if and only if the sequence $\{S_n\}$ is convergent. In particular, $\{P_n\}$ converges to 0, i.e., $\prod_{n=1}^{\infty} p_n$ diverges to 0 if and only if $\{S_n\}$ diverges to $-\infty$, namely, $\sum_{n=1}^{\infty} \ln p_n = -\infty$. □

Corollary 4.10. *Given $a_n > 0$ (or $a_n < 0$), $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ is convergent.*

Proof. All terms of $\sum_{n=1}^{\infty} \ln(1+a_n)$ and $\sum_{n=1}^{\infty} a_n$ are positive (or negative). $\lim_{n\to\infty} a_n = 0$ is a necessary condition for both of them. When $\lim_{n\to\infty} a_n = 0$, we have

$$\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = 1,$$

By the limit comparison test, $\sum_{n=1}^{\infty} \ln(1+a_n)$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge. From Theorem 4.9, we also have that $\prod_{n=1}^{\infty}(1+a_n)$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge. This completes the proof. □

Note. The limit comparison test only works for nonnegative series, see Mirko (2019) and Longo and Valori (2003).

When the signs of a_n do not remain constant, the convergence of $\sum_{n=1}^{\infty} a_n$ does not guarantee the convergence of $\prod_{n=1}^{\infty}(1+a_n)$. Here is an example.

Example 4.11. Consider $\sum_{n=1}^{\infty} a_n$ where $a_{2k-1} = \frac{1}{\sqrt{k+1}}$ and $a_{2k} = \frac{-1}{\sqrt{k+1}}$ with $k \geq 1$. Then we have

$$\begin{aligned}\prod_{n=1}^{\infty}(1+a_n) &= (1+\frac{1}{\sqrt{2}})(1-\frac{1}{\sqrt{2}})(1+\frac{1}{\sqrt{3}})(1-\frac{1}{\sqrt{3}})\cdots \\ &= (1-\frac{1}{2})(1-\frac{1}{3})\cdots \\ &= \frac{1}{2}\cdot\frac{2}{3}\cdots \to 0\end{aligned}$$

this means $\prod_{n=1}^{\infty}(1+a_n)$ diverges to 0. However, $\sum_{n=1}^{\infty} a_n$ is a Leibniz series, and hence convergent. This example demonstrates that a convergent $\sum_{n=1}^{\infty} a_n$ does not guarantee the convergence of $\prod_{n=1}^{\infty}(1+a_n)$.

We have another result for the case when the signs of a_n are not fixed.

Corollary 4.12. *Provided $\sum_{n=1}^{\infty} a_n$ is convergent, $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n^2$ is convergent.*

Proof. Since $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. From $\ln(1+a_n) \leq a_n$, we get

$$\lim_{n\to\infty}\frac{a_n - \ln(1+a_n)}{a_n^2} = \lim_{n\to\infty}\frac{\frac{1}{2}a_n^2 + o(a_n^2)}{a_n^2} = \frac{1}{2},$$

According to the limit comparison test for nonnegative series, we get that $\sum_{n=1}^{\infty}(a_n - \ln(1+a_n))$ and $\sum_{n=1}^{\infty} a_n^2$ both converge or both diverge. Since $\sum_{n=1}^{\infty} a_n$ is convergent, we obtain that $\sum_{n=1}^{\infty}\ln(1+a_n)$ and $\sum_{n=1}^{\infty} a_n^2$ both converge or both diverge. □

Corollary 4.13. *Provided that $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n^2 = +\infty$, then $\prod_{n=1}^{\infty}(1+a_n)$ diverges to 0.*

Proof. According to Corollary 4.12, $\sum_{n=1}^{\infty}(a_n - \ln(1+a_n))$ diverges to $+\infty$. Since $\sum_{n=1}^{\infty} a_n$ is convergent, $\sum_{n=1}^{\infty}\ln(1+a_n)$ diverges to $-\infty$. Thus,

$$e^{\sum_{n=1}^{\infty}\ln(1+a_n)} = 0 \Longrightarrow \prod_{n=1}^{\infty}(1+a_n) = 0,$$

as desired. □

5 Taylor expansions of some elementary functions

Note that when we are talking about Taylor expansion of $f(x)$ at x_0, we definitely have $f(x) = \sum_{n=0}^{\infty}\frac{f^n(x_0)}{n!}(x-x_0)^n$ on $O(x_0,\rho)(0<\rho\leq r)$ where r is the radius of convergence of Taylor series of $f(x)$. When x is not in the region of convergence, the equality does not hold.

5.1 Taylor expansion of $(1+x)^\alpha$

When $\alpha \neq 0$ and $\alpha \notin \mathbf{N}_+$,

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k, \begin{cases} x\in(-1,1), & \text{if } \alpha \leq -1 \\ x\in(-1,1], & \text{if } -1<\alpha<0 \\ x\in[-1,1], & \text{if } \alpha > 0. \end{cases}$$

If $\alpha = 0$, then $(1+x)^0 = 1$. When $\alpha \in \mathbf{N}_+$, say, α is a positive integer n, then

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + nx^{n-1} + x^n \\ &= \mathrm{C}_n^0 \cdot 1^n \cdot x^0 + \mathrm{C}_n^1 \cdot 1^{n-1} \cdot x^1 + \mathrm{C}_n^2 \cdot 1^{n-2} \cdot x^2 + \cdots + \mathrm{C}_n^{n-1} \cdot 1^1 \cdot x^{n-1} + \mathrm{C}_n^n \cdot 1^0 \cdot x^n \\ &= \sum_{k=0}^{n} \mathrm{C}_n^k \cdot 1^{n-k} \cdot x^k\end{aligned}$$

which is the well-known **binomial expansion formula**. When $\alpha = \frac{1}{2}, -\frac{1}{2}, -1$, their Taylor expansions are

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^3 + \cdots \\ &= 1 + \frac{1}{2}x + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}x^n, \quad (-1 \le x \le 1) \\ \frac{1}{\sqrt{1+x}} &= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 - \cdots \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}x^n, \quad (-1 < x \le 1) \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n x^n, \quad (-1 < x < 1) \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\ &= \sum_{n=0}^{\infty} x^n, \quad (-1 < x < 1)\end{aligned}$$

where the last one is obtained by substituting $-x$ for x into the third one. The last one is exactly the geometric series formula.

We may need the following fact when we explore the convergence region of this series.

$$\binom{\alpha}{k+1} \Big/ \binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!} \cdot \frac{k!}{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)} = \frac{\alpha-k}{k+1}$$

5.2 Applications of the Taylor expansion of $(1+x)^\alpha$

We can make use of the Taylor expansion of $(1+x)^\alpha$ to obtain the Taylor expansions of other functions. For example,

5.2.1 Taylor expansions of $\frac{1}{\sqrt{1-x^2}}$ and $\arcsin x$

Substituting $-x^2$ into the Taylor expansion of $\frac{1}{\sqrt{1+x}}$ for x gives

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= 1 - \frac{1}{2}(-x^2) + \frac{1\cdot 3}{2\cdot 4}(-x^2)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(-x^2)^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}(-x^2)^3 - \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 + \cdots\end{aligned}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!\,!}{(2n)!\,!} x^{2n}, \quad (-1 < x < 1)$$

where the convergence region follows from the fact that when $x = \pm 1$, $-x^2 = -1$, i.e., $-1 < -x^2 \leq 1$ is violated, which implies $1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n}$ diverges at $x = \pm 1$. Here, $(-1, 1]$ is the convergence region of the Taylor expansion of $\frac{1}{\sqrt{1+x}}$. Also, we can use Raabe's test for convergence of a series. Specifically, when $x = \pm 1$, we have $1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$. Then

$$\lim_{n\to\infty} n \left(\frac{x_n}{x_{n+1}} - 1 \right) = \lim_{n\to\infty} n \left(\frac{(2n-1)!\,!}{(2n)!\,!} \cdot \frac{(2n+2)!\,!}{(2n+1)!\,!} - 1 \right) = \lim_{n\to\infty} n \left(\frac{2n+2}{2n+1} - 1 \right) = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1.$$

so this series diverges at $x = \pm 1$. Now we compute the Taylor expansion of $\arcsin x$ from the Taylor expansion of $\frac{1}{\sqrt{1-x^2}}$.

$$\begin{aligned}
\arcsin x = \arcsin x - \arcsin 0 = \int_0^x \frac{1}{\sqrt{1-t^2}} \mathrm{d}t &= \int_0^x (1 + \sum_{n=1}^{\infty} \frac{(2n-1)!\,!}{(2n)!\,!} t^{2n}) \mathrm{d}t \\
&= x + \sum_{n=1}^{\infty} \frac{(2n-1)!\,!}{(2n)!\,!} \int_0^x t^{2n} \mathrm{d}t \\
&= x + \sum_{n=1}^{\infty} \frac{(2n-1)!\,!}{(2n)!\,!} \frac{x^{2n+1}}{2n+1}, \quad (-1 \leq x \leq 1) \\
&= x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \cdots, \quad (-1 \leq x \leq 1)
\end{aligned}$$

where the convergence region follows from Raabe's test. When $x = \pm 1$, we have $x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!(2n+1)}$.

$$\begin{aligned}
\lim_{n\to\infty} n \left(\frac{x_n}{x_{n+1}} - 1 \right) &= \lim_{n\to\infty} n \left(\frac{(2n-1)!\,!}{(2n)!\,!\,(2n+1)} \cdot \frac{(2n+2)!\,!\,(2n+3)}{(2n+1)!\,!} - 1 \right) \\
&= \lim_{n\to\infty} n \left(\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1 \right) = \lim_{n\to\infty} \frac{n(6n+5)}{(2n+1)^2} = \frac{3}{2} > 1.
\end{aligned}$$

so this series converges at $x = \pm 1$.

Let $x = 1$, then

$$\frac{\pi}{2} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!\,!}{(2n)!\,!\,(2n+1)},$$

which can be used to estimate the value of π.

5.2.2 Taylor expansions of $\frac{1}{1+x^2}$ and $\arctan x$

$$\begin{aligned}
\frac{1}{1+x^2} &= 1 - x^2 + (x^2)^2 - (x^2)^3 + \cdots \\
&= 1 - x^2 + x^4 - x^6 + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad (-1 < x < 1)
\end{aligned}$$

which follows from the Taylor expansion of $1/(1+x)$. Furthermore, we can get the Taylor expansion of $\arctan x$

$$\arctan x = \arctan x - \arctan 0 = \int_0^x \frac{1}{1+t^2} \mathrm{d}t = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} \mathrm{d}t$$

$$= (-1)^n \sum_{n=0}^{\infty} \int_0^x t^{2n} \mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad (-1 \le x \le 1)$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad (-1 \le x \le 1)$$

In this example, integration expands the convergence region with two endpoints. Normally, this kind of expansion is only to include one or two boundary points.

When $x = 1$, we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

which can be used to estimate the value of π.

5.2.3 Taylor expansions of $\ln(1+x)$ and $\ln \frac{1}{x+1}$

Let $f(x) = \ln(1+x)$, then $f'(x) = 1/(1+x)$. Since $\sum_{n=0}^{\infty}(-1)^n x^n$ converges with $x \in (-1, 1)$, then

$$\begin{aligned}
\ln(1+x) = \ln(1+x) - \ln(1+0) &= \int_0^x \frac{1}{t+1} \mathrm{d}t = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \mathrm{d}t \\
&= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad (-1 < x \le 1) \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad (-1 < x \le 1)
\end{aligned}$$

with 1 inclusive. Notice that integration expands the convergence region by making one endpoint inclusive. When $x = 1$, we get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^n \frac{1}{n+1} + \cdots$$

Also, it is easy to get

$$\begin{aligned}
\ln \frac{1}{x+1} = -\ln(1+x) &= -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n, \quad (-1 < x \le 1) \\
&= -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots, \quad (-1 < x \le 1)
\end{aligned}$$

5.2.4 Taylor expansions of $\ln(1-x)$ and $\ln \frac{1}{1-x}$

Substituting $-x$ into the Taylor expansion of $\ln(1+x)$ for x yields,

$$\begin{aligned}
\ln(1-x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (-x)^{n+1}, \quad (-1 < -x \le 1) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (-1)^{n+1} x^{n+1}, \quad (-1 \le x < 1) \\
&= -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\sum_{n=1}^{\infty} \frac{1}{n} x^n, \quad (-1 \le x < 1)
\end{aligned}$$

By this, we can get

$$\ln \frac{1}{1-x} = -\ln(1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n, \quad (-1 \leq x < 1)$$

Bibliography

Longo, M. and Valori, V. (2003). The comparison test - not just for nonnegative series. https://www.disei.unifi.it/upload/sub/pubblicazioni/repec/flo/workingpapers/storicodimad/2003/dimadwp2003-01.pdf. [Online; accessed 21-May-2022].

Mirko (2019). A discussion about limit comparison test for checking the convergence of an infinite series. https://math.stackexchange.com/questions/3448308/limit-comparison-test-for-checking-the-convergence-of-an-infinite-series. [Online; accessed 21-May-2022].