

# Online Self-Assessment for Ordinary Differential Equations

Youming Zhao

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The math questions in this document are from <https://www2.mathematik.tu-darmstadt.de/~eickmeyer/OSA/ode.html>. I have provided my solutions and proofs in here. The latest version of this document is available at here.

## Question 1

Solve the following system of linear differential equations:

$$y'(t) = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix} y(t) + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} \quad \text{for } t \geq 0 \text{ with } y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

*Solution.* Let  $A = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix}$  and  $F(t) = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$ . We need to find the eigenvalues and eigenvectors of matrix  $A$ . The eigenvalues are the roots of the characteristic polynomial  $p(\lambda) = \det(\lambda I - A)$ . Then we compute the determinant

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 \\ -6 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda + 1). \quad (1)$$

Therefore, the roots are  $\lambda_1 = -1$  and  $\lambda_2 = 4$ . Furthermore, the associated eigenvectors can be obtained by solving  $(-I - A)\mathbf{v}_1 = \mathbf{0}$  and  $(4I - A)\mathbf{v}_2 = \mathbf{0}$ , which gives

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \quad (2)$$

Therefore, the general solution of the corresponding homogeneous differential equation is

$$y(t) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{4t}, \quad c_1, c_2 \in \mathbb{R}. \quad (3)$$

This general solution can be written more compactly as  $y(t) = \Phi(t)\mathbf{c}$  with

$$\Phi(t) = \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \quad (4)$$

To get the general solution of the nonhomogeneous differential equation, we apply variation of parameters to it. Specifically, let  $\mathbf{c}(t)$  be a vector function instead of a constant vector and then we have  $y(t) = \Phi(t)\mathbf{c}(t)$ . By differentiating both sides w.r.t.  $t$ , we get

$$A\Phi(t)\mathbf{c}(t) + F(t) = \Phi'(t)\mathbf{c}(t) + \Phi(t)\mathbf{c}'(t). \quad (5)$$

Since  $\Phi(t)$  is the fundamental solution of the corresponding homogeneous equation, then  $\Phi'(t) = A\Phi(t)$  and we substitute this into (5). Thus, we get  $F(t) = \Phi(t)\mathbf{c}'(t)$ . Furthermore,  $\mathbf{c}'(t) = \Phi^{-1}(t)F(t)$ . By integrating both sides w.r.t.  $t$ , we can get

$$\mathbf{c}(t) = \mathbf{c}(t_0) + \int_{t_0}^t \Phi^{-1}(s)F(s)ds. \quad (6)$$

Substituting this into  $y(t) = \Phi(t)\mathbf{c}(t)$ , we have

$$y(t) = \Phi(t)\mathbf{c}(t_0) + \Phi(t) \int_{t_0}^t \Phi^{-1}(s)F(s)ds. \quad (7)$$

Let  $t = t_0 = 0$ , we get

$$y(0) = \Phi(0)\mathbf{c}(0) + \mathbf{0} \quad (8)$$

$$\mathbf{c}(0) = \Phi^{-1}(0)y(0) \quad (9)$$

$$\mathbf{c}(0) = \begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \quad (10)$$

Finally,

$$y(t) = \frac{1}{5} \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-s} & e^{4s} \\ -3e^{-s} & 2e^{4s} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ e^{2s} \end{pmatrix} ds \quad (11)$$

$$= \frac{1}{5} \begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix} \int_0^t \frac{1}{5e^{3s}} \begin{pmatrix} 2e^{4s} & -e^{4s} \\ 3e^{-s} & e^{-s} \end{pmatrix} \begin{pmatrix} 0 \\ e^{2s} \end{pmatrix} ds \quad (12)$$

$$= \frac{1}{5} \begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix} \int_0^t \frac{1}{5} \begin{pmatrix} 2e^{4s} & -e^{4s} \\ 3e^{-s} & e^{-s} \end{pmatrix} \begin{pmatrix} 0 \\ e^{-s} \end{pmatrix} ds \quad (13)$$

$$= \frac{1}{5} \begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \frac{1}{5} \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix} \int_0^t \begin{pmatrix} -e^{3s} \\ e^{-2s} \end{pmatrix} ds \quad (14)$$

$$= \frac{1}{5} \begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \frac{1}{5} \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix} \begin{pmatrix} -\frac{1}{3}e^{3s} \\ -\frac{1}{2}e^{-2s} \end{pmatrix} \Big|_0^t \quad (15)$$

$$= \frac{1}{5} \begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \frac{1}{5} \begin{pmatrix} e^{-t} & e^{4t} \\ -3e^{-t} & 2e^{4t} \end{pmatrix} \begin{pmatrix} -\frac{1}{3}(e^{3t} - 1) \\ -\frac{1}{2}(e^{-2t} - 1) \end{pmatrix} \quad (16)$$

$$= \frac{1}{5} \begin{pmatrix} 2e^{-t} + 3e^{4t} \\ -6e^{-t} + 6e^{4t} \end{pmatrix} + \frac{1}{5} \begin{pmatrix} \frac{1}{3}e^{-t} - \frac{5}{6}e^{2t} + \frac{1}{2}e^{4t} \\ -e^{-t} + e^{4t} \end{pmatrix} \quad (17)$$

$$= \begin{pmatrix} \frac{7}{15}e^{-t} - \frac{1}{6}e^{2t} + \frac{7}{10}e^{4t} \\ -\frac{7}{5}e^{-t} + \frac{7}{5}e^{4t} \end{pmatrix} \quad (18)$$

Note: Perhaps it is simpler to use the method of undetermined coefficients since 2 is not an eigenvalue of the matrix  $A$ . However, variation of parameters is a more general method.

## Question 2

Show that every solution of the differential equation

$$y''(t) = y'(t) + \sin(y(t)) \quad (19)$$

is smooth (i.e., it has derivatives of all orders).

*Proof.* Since  $y(t)$  satisfies the second-order differential equation  $y''(t) = y'(t) + \sin(y(t))$ , then  $y(t)$  has at least second-order derivative. In other words,  $y(t)$  is  $C^2$ . Now we rewrite (19) as  $y''(t) = f(t, y(t), y'(t))$  with  $f(t, y(t), y'(t)) = y'(t) + \sin(y(t))$  which is continuously differentiable in  $y$  and  $y'$ . This implies that  $f(t, y(t), y'(t))$  is Lipschitz continuous in  $y$  and  $y'$ . By the Picard-Lindelöf theorem, the existence of  $y(t)$  for some initial condition  $y(t_0) = y_0$  and  $y'(t_0) = y'_0$  is guaranteed.

Now we prove by induction on  $k$  that every solution of (19) has derivatives of all orders. First,  $y''(t)$  is continuously differentiable due to  $y''(t) = y'(t) + \sin(y(t))$ . Then suppose inductively that  $y(t)$  is  $C^k$ , meaning that  $y(t)$  has continuous derivatives up to order  $k$ . We need to show that  $y(t)$  is  $C^{k+1}$ . Now we differentiate both sides of (19) w.r.t.  $t$  to get

$$y^{(3)}(t) = y''(t) + y'(t) \cos(y(t)). \quad (20)$$

Since the right hand side is continuous,  $y(t)$  is  $C^3$ . By repeating this process, we can get an expression for  $y^{(k+1)}(t)$  which equals to  $g(t, y(t), y'(t), \dots, y^{(k)}(t))$ . Moreover,  $g$  only contains additions and multiplications between its variables, and  $\sin y(t)$  and  $\cos y(t)$ . By the inductive hypothesis,  $y(t)$  is  $C^{k+1}$ . This closes the induction.  $\square$

### Question 3

Decide which of the following statements are correct and briefly justify your answer or give a counterexample.

1. Every solution of the differential equation

$$y'(t) = 1 + t^2 + \cos(y(t))$$

is monotonically increasing.

*Solution.* Yes. Since  $\cos(y(t)) \geq -1$  and  $1 + t^2 \geq 1$ , then  $y'(t) = 1 + t^2 + \cos(y(t)) \geq 0$ .

2. The function

$$t \mapsto e^{tA} \cdot \mathbf{v}$$

with matrix  $A \in \mathbb{R}^{3 \times 3}$  and vector  $\mathbf{v} = (1, -1, -1)^T \in \mathbb{R}^3$  satisfies  $y'(t) = Ay(t)$ .

*Solution.* Yes. The matrix exponential  $e^{tA}$  is defined as

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \quad (21)$$

which is well-defined for any square matrix  $A$  and real or complex  $t$ . Then

$$y'(t) = (e^{tA} \cdot \mathbf{v})' = \left( \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n \right)' \cdot \mathbf{v} \quad (22)$$

$$= \left( \sum_{n=1}^{\infty} \frac{A^n}{(n-1)!} t^{n-1} \right) \cdot \mathbf{v} \quad (23)$$

$$= A \left( \sum_{n=1}^{\infty} \frac{A^{n-1}}{(n-1)!} t^{n-1} \right) \cdot \mathbf{v} \quad (24)$$

$$= A \left( \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n \right) \cdot \mathbf{v} = Ae^{tA} \cdot \mathbf{v} = Ay(t). \quad (25)$$

3. If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a continuous function and  $u : \mathbb{R} \rightarrow \mathbb{R}$  is a solution to  $y'(t) = f(t, y(t))$  with  $y(1) = 0$ , then

$$u(t) = \int_1^t f(s, u(s)) ds$$

for all  $t \in \mathbb{R}$ .

*Solution.* Yes. According to the Fundamental Theorem of Calculus, we have

$$u(t) = u(1) + \int_1^t f(s, u(s)) ds, \quad \forall t \in \mathbb{R}. \quad (26)$$

Given that  $u(1) = 0$ , we get

$$u(t) = \int_1^t f(s, u(s)) ds, \quad \forall t \in \mathbb{R}. \quad (27)$$

We can verify the solution by differentiating both sides w.r.t.  $t$  as follows.

$$u'(t) = f(t, u(t)), \quad \forall t \in \mathbb{R}, \quad (28)$$

which is the original differential equation.