MATH 2101 LINEAR ALGEBRA I, FALL SEMESTER 2023

1. Linear transformations

Key concepts in this section:

- What is a linear transformation?
- What are examples of linear transformations?
- How to construct linear transformations?
- Concepts of null spaces, range, nullity and rank.
- Dimension formula and some applications on the formula
- Injectivity for a linear transformation
- Matrix multiplication as a linear transformation

1.1. Definitions.

As mentioned before, the important things in vectors spaces are the addition and the scalar multiplication. We wish to have functions that also preserve the addition and scalar multiplication structure.

Definition 1.1. Let V and W be vector spaces. Then a function $T : V \to W$ is a *linear transformation from V to W* if for any $x, y \in V$ and a scalar $c \in \mathbb{R}$,

(1) (addition) $T(x + y) = T(x) + T(y)$;
(2) (scalar multiplication) $T(cx) = cT(x)$.

We now give several examples:

Example 1.2. Let $T : \mathbb{R} \to \mathbb{R}$ given by

$$T(x) = 5x.$$

Then T is a transformation.

The following is the most typical example of a linear transformation:

Example 1.3. Let A be an $n \times m$-matrix. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ given by:

$$T(x) = Ax.$$

Then it is a linear transformation.
Sol:

- Check addition: For $x_1, x_2 \in \mathbb{R}^m$, $T(x_1 + x_2) = A(x_1 + x_2) = Ax_1 + Ax_2 = T(x_1) + T(x_2)$.

- Check scalar multiplication: For a scalar $c \in \mathbb{R}$ and $x \in \mathbb{R}^m$, $T(cx) = A(cx) = c(Ax) = cT(x)$.

The followings are basic properties of linear transformations:

Theorem 1.4. *Let $T : V \to W$ be a linear transformation. Then*
- $T(0) = 0$;
- *for any $a_1, \ldots, a_r \in \mathbb{R}$ and any $x_1, \ldots, x_r \in V$,*

$$T(a_1x_1 + \ldots + a_rx_r) = a_1T(x_1) + \ldots + a_rT(x_r).$$

Proof. Exercises. *Sol:* For the first bullet, $T(0) = T(0v) = 0T(v) = 0$. For the second bullet,

$$T(a_1x_1 + \ldots + a_rx_r) = T(a_1x_1) + \ldots + T(a_rx_r) = a_1T(x_1) + \ldots + a_rT(x_r),$$

where the first bullet follows the addition property of a linear transformation and the second bullet follows from the scalar multiplication of a linear transformation. □

The first thing to check whether a function is a linear transformation is to check $T(0)$.

Example 1.5. (Non-example) Let $T : \mathbb{R} \to \mathbb{R}$ given by

$$T(x) = 3x + 2.$$

Determine if T is a linear transformation.
Sol: Note that $T(0) = 2 \neq 0$, and so T is **not** a linear transformation.

Example 1.6. (Non-example) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix}$ and let $b = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x) = Ax + b$. Determine if T is a linear transformation.
Sol: Note that $T(0) = b \neq 0$. so T is **not a linear transformation**.

Example 1.7. (Example not in matrix form) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - 2x_3).$$

To see that T is a linear transformation, let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$,

$$\begin{aligned} T(x + y) &= ((x_1 + y_1) + (x_2 + y_2), x_2 + y_2 - 2(x_3 + y_3)) \\ &= (x_1 + x_2, x_2 - 2x_3) + (y_1 + y_2, y_2 - 2y_3) \\ &= T(x) + T(y), \end{aligned}$$

where the second equality follows from addition of two vectors. And, for $c \in \mathbb{R}$,

$$\begin{aligned} T(cx) &= (cx_1 + cx_2, cx_2 - 2cx_3) \\ &= c(x_1 + x_2, x_2 - x_3), \end{aligned}$$

where the second equality follows from the scalar multiplication on a vector.

Example 1.8. (Non-example) Let $T : \mathbb{R}^2 \to \mathbb{R}$ given by

$$T(x, y) = xy$$

Note that $T(1,1) = 1$ and $T(2,2) = 4 \neq 2T(1,1)$. Hence, T is not a linear transformation.

The following are more elegnat examples:

Example 1.9. (Example from vector spaces of functions from $\mathbb{R}$ to $\mathbb{R}$) Let V be the vector space of *differentiable* functions from $\mathbb{R}$ to $\mathbb{R}$. Then the differentiation is a linear transformation from V to V. In order to see this, it suffices to see that the differentation preserves the addition and scalar multiplication: for $f_1, f_2 \in V$,

$$(f_1 + f_2)'(x) = f_1'(x) + f_2'(x) = (f_1' + f_2')(x)$$

and for $c \in \mathbb{R}$ and for $f \in V$,

$$(cf)'(x) = c(f'(x)) = (c(f'))(x).$$

Example 1.10. (Example from vector space of matrices) Let V be the vector space of $n \times m$-matrics. Let P be a $n \times n$ matrix and let Q be a $m \times m$ matrix. Let

$$T : V \to V, \quad T(A) = PAQ.$$

Then T is a linear transformation.
Sol: We have to check additions and scalar multiplications.

- For $A, B \in V$, $T(A + B) = P(A + B)Q = P(AQ + BQ) = PAQ + PBQ$.
- For $A \in V$ and a scalar $c \in \mathbb{R}$, $T(cA) = P(cA)Q = cPAQ = cT(A)$.

Example 1.11. (non-example) Let V be the vector space of $n \times n$-matrices. Let

$$T : V \to V, \quad T(A) = A^2.$$

Determine if T is a linear transformation.
Sol: $T(I_n) = I_n$. On the other hand, $T(cI_n) = c^2 I_n \neq cT(I_n)$. Hence, it is not a linear transformation.

1.2. Identity and zero transformations.

Definition 1.12. Let V, W be vector spaces.

(1) The **identity transformation** $\mathrm{Id}_V : V \to V$ is the function $\mathrm{Id}_V(v) = v$.
(2) The **zero transformation** $T_0 : V \to W$ is the function $T_0(v) = 0$.

Proposition 1.13. *Both identity transformation and zero transformation are linear transformations.*

1.3. **Construct linear transformations.** We have the following theorem of constructing a linear transformation:

Theorem 1.14. *Let V and W be vector spaces. Let $\{v_1, \dots, v_n\}$ be a basis for V and let $w_1, \dots, w_n \in W$ Then there* **exists** *a* **unique** *linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all i.*

Remark 1.15. We emphasis that $w_1, \dots, w_n$ are arbitrary and do not have to form a basis.

Proof. The key is that for any $v \in V$, v can be uniquely written as $a_1 v_1 + \dots + a_n v_n$ for some $a_1, \dots, a_n \in \mathbb{R}$. We define

$$(*) \quad T(v) = a_1 w_1 + \dots + a_n w_n.$$

One has to check it is linear:

(a) (Addition) For $v, v' \in V$, write as:

$$v = a_1 v_1 + \dots + a_n v_n, \quad v' = a_1' v_1 + \dots + a_n' v_n$$

for some unique $a_1, \dots, a_n, a_1', \dots, a_n'$. Note that

$$v + v' = (a_1 + a_1')v_1 + \dots + (a_n + a_n')v_n$$

and so $T(v + v') = (a_1 + a_1')w_1 + \dots + (a_n + a_n')w_n$ from (*). On the other hand,

$$T(v)+T(v') = a_1 w_1+\dots+a_n w_n+a_1' w_1+\dots+a_n' w_n = (a_1+a_1')w_1+\dots+(a_n+a_n')w_n.$$

Hence $T(v + v') = T(v) + T(v')$.

(b) (Scalar multiplication) For $v \in V$ and $c \in \mathbb{R}$, write:

$$v = a_1 v_1 + \dots + a_n v_n.$$

Then $cv = (ca_1)v_1 + \dots + (ca_n)v_n$ and so

$$T(cv) = (ca_1)w_1 + \dots (ca_n)w_n.$$

On the other hand, $cT(v) = c(a_1 w_1 + \dots + a_n w_n) = (ca_1)w_1 + \dots + (ca_n)w_n$. Hence,

$$T(cv) = cT(v).$$

$\square$

Example 1.16. Let $v_1 = (1, 3)$ and $v_2 = (2, 4)$ in $\mathbb{R}^2$. Let $w_1 = (2, 3, 4)$ and $w_2 = (1, 0, 0)$. Since $\{v_1, v_2\}$ forms a basis for $\mathbb{R}^2$, there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Find $T((1, 1))$.
Sol: Note that $(1, 1) = (2, 4) - (1, 3)$.

$$T((1, 1)) = T((2, 4) - (1, 3)) = T((2, 4)) - T((1, 3)) = (2, 3, 4) - (1, 0, 0) = (1, 3, 4).$$

Example 1.17. Let $v_1 = (1,3)$ and $v_2 = (2,6)$ in $\mathbb{R}^2$. Let $w_1 = (2,3,4)$ and $w_2 = (1,0,0)$. Does there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$?

Sol: No. (Note that v_1 and v_2 are linearly dependent. So they do not form a basis and we cannot use the theorem.) Suppose we have such linear transformation. Then $T((1,3)) = (1,3,4)$ and by scalar multiplication,

$$T((2,6)) = T(2v_1) = 2T(v_1) = (4,6,8) \neq w_2.$$

This gives a contradiction.

Example 1.18. Determine if there is a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying the followings:

$$T((1\ 0\ -1)) = (1\ 0), \quad T((0\ 1\ 1)) = (0\ 1), \quad T((1\ 1\ 1)) = (1\ 1), \quad T((0\ 0\ -1)) = 0.$$

Sol: Note that

$$\det\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\right) = 1 \neq 0$$

and so the solution for the equation

$$a_1(1\ 0\ -1) + a_2(0\ 1\ 1) + a_3(1\ 1\ 1) = 0$$

has only the trivial solution. Then $\{(1\ 0\ -1), (0\ 1\ 1), (1\ 1\ 1)\}$ is linearly independent and so forms a basis for $\mathbb{R}^3$. Thus there exists a unique linear transformation T' satisfying

$$T'((1\ 0\ -1)) = (1\ 0), \quad T'((0\ 1\ 1)) = (0\ 1), \quad T'((1\ 1\ 1)) = (1\ 1).$$

Now we check if the remaining condition is also satisifed for T'.

Since

$$(0\ 0\ -1) = (1\ 0\ -1) - (1\ 1\ 1) + (0\ 1\ 1),$$

$T'((0\ 0\ -1)) = T'((1\ 0\ -1) - (1\ 1\ 1) + (0\ 1\ 1)) = (1\ 0) - (1\ 1) + (0\ 1) = 0$. Hence, such linear transformation satisfying the four conditions exists.

1.4. **Null spaces and ranges.**

Definition 1.19. Let $T : V \to W$ be a linear transformation.

(1) The **null space** $N(T)$ is the set of all vectors x in V such that $T(x) = 0$. In other words, $N(T) = \{x \in V : T(x) = 0\}$.

(2) The **range** $R(T)$ is the subset of W containing all the images of T. In other words, $R(T) = \{T(x) \in W : x \in V\}$.

Example 1.20. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (0,0)$. Then $N(T) = \mathbb{R}^2$ and $R(T) = 0$.

Example 1.21. Let V be a vector space. Then $N(\mathrm{Id}_V) = 0$ and $R(\mathrm{Id}_V) = V$.

Example 1.22. Let $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$T(v) = \begin{pmatrix} I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix} v.$$

Then

$$N(T) = \left\{(0, \dots, 0, v_{n+1}, \dots, v_{2n})^T : v_{n+1}, \dots, v_{2n} \in \mathbb{R}\right\},$$
$$R(T) = \left\{(v_1, \dots, v_n, 0, \dots, 0)^T : v_1, \dots, v_n \in \mathbb{R}\right\}.$$

Theorem 1.23. *Let $T : V \to W$ be a linear transformation. Then $N(T)$ and $R(T)$ are linear subspaces of V and W respectively.*

Proof. For $v, w \in N(T)$, $T(v) = T(w) = 0$ and so $T(v + w) = T(v) + T(w) = 0$. Thus $v + w \in N(T)$. For $c \in \mathbb{R}$ and $v \in V$, $T(cv) = cT(v) = 0$ and so $cv \in N(T)$. These check the conditions for subspaces.

For $v', w' \in R(T)$, $v' = T(v)$ and $w' = T(w)$ for some $v, w \in V$. Hence, $v' + w' = T(v) + T(w) = T(v + w) \in R(T)$. For $c \in \mathbb{R}$ and $v' = T(v) \in R(T)$, $cT(v) = T(cv) \in R(T)$. These check the conditions for subspaces. □

We have the following results useful for computing the range (image) of a linear transformation:

Theorem 1.24. *Let $T : V \to W$ be a linear transformation. Let $\beta = \{v_1, \dots, v_m\}$ be a basis for V. Then*

$$R(T) = \operatorname{span}\{T(v_1), \dots, T(v_m)\}.$$

Proof. For any $v' \in \operatorname{span}\{T(v_1), \dots, T(v_m)\}$,

$$v' = a_1T(v_1) + \dots + a_mT(v_m)$$

for some $a_1, \dots, a_m \in \mathbb{R}$. Then, by Theorem 1.4,

$$v' = T(a_1v_1 + \dots + a_mv_m)$$

and so is in $R(T)$. This shows the inclusion $\supseteq$.

For $v' \in R(T)$, $v' = T(v)$ for some $v \in V$. By the definition of a basis, we have

$$v = b_1v_1 + \dots + b_mv_m$$

for some $b_1, \dots, b_m \in \mathbb{R}$. Thus,

$$v' = T(b_1v_1 + \dots + b_mv_m) = b_1T(v_1) + \dots + b_mT(v_m) \in \operatorname{span}\{T(v_1), \dots, T(v_m)\}.$$

This shows another inclusion $\subseteq$. □

Example 1.25. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x, y, z) = (x - y, y - z)$. To compute $N(T) = 0$, it is to solve the system of linear equations:

$$x - y = 0$$
$$y - z = 0$$

and so $N(T) = \{(t, t, t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$.

We also have that $R(T) = \mathbb{R}^2$. To see this, we consider the standard basis $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Then, by Theorem 1.24,

$$R(T) = \mathrm{span}(\{T((1, 0, 0)), T((0, 1, 0)), T((0, 0, 1))\}) = \mathrm{span}(\{(1, 0), (-1, 1), (0, 1)\}) = \mathbb{R}^2.$$

Exercise 1.26. Let $T : V \to W$ be a linear transformation. Prove that $\dim(R(T)) \leq \dim(V)$.

Sol: Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V. Since $R(T) = \mathrm{span}(\{T(v_1), \ldots, T(v_n)\})$, we can find a subset of linearly independent vectors in $\{T(v_1), \ldots, T(v_n)\}$ to span $R(T)$. Thus $\dim(R(T)) \leq \dim(V)$.

We shall prove a stronger dimension formula soon.

Remark 1.27. Let $T : V \to W$ be a linear transformation. Let $\{v_1, \ldots, v_n\}$ be a basis for V. In general, $\{T(v_1), \ldots, T(v_n)\}$ is **not** a basis for $R(T)$.

- For example, take T to be the zero transformation and $V \neq 0$.
- Let w be any vector in W. By Theorem 1.14, there exists a linear transformation $T : V \to W$ such that $T(v_1) = \ldots = T(v_n) = w$. When $n \geq 2$, $\{T(v_1), \ldots, T(v_n)\}$ is not a basis for $R(T)$.

1.5. **Dimension formula.**

Definition 1.28. Let $T : V \to W$ be a linear transformation. Then the nullity of T, denoted by $\mathrm{nullity}(T)$, is the dimension of $N(T)$ and the rank of T, denoted by $\mathrm{rank}(T)$, is the dimension of $R(T)$.

Theorem 1.29. *Let $T : V \to W$ be a linear transformation. Then*

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V)$$

Remark 1.30. We will first understand this theorem in some special cases:

- $T = \mathrm{id}_V : V \to V$. In such case, $N(T) = 0$, $R(T) = V$.
- $T_0 : V \to W$ (the zero transformation). In such case, $N(T_0) = V$, $R(T_0) = 0$.
- Let $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x, y) = x$. Then

Proof. Let $r = \mathrm{nullity}(T)$ and let $m = \dim(V)$. Let $\{v_1, \ldots, v_r\}$ be a basis for $N(T)$. We find vectors $w_1, \ldots, w_{m-r}$ so that $v_1, \ldots, v_r, w_1, \ldots, w_{m-r}$ form a basis for V. Then, by Theorem 1.24,

$$R(T) = \operatorname{span}\{T(v_1), \dots, T(v_r), T(w_1), \dots, T(w_{m-r})\} = \operatorname{span}\{T(w_1), \dots, T(w_{m-r})\}.$$

Suppose

$$a_1 T(w_1) + \dots + a_{m-r} T(w_{m-r}) = 0$$

and so $T(a_1 w_1 + \dots + a_{m-r} w_{m-r}) = 0$ and so $a_1 w_1 + \dots + a_{m-r} w_{m-r} \in \operatorname{null}(T)$. Hence,

$$a_1 w_1 + \dots + a_{m-r} w_{m-r} = b_1 v_1 + \dots + b_r v_r$$

for some $b_1, \dots, b_r \in \mathbb{R}$. Since $\{v_1, \dots, v_r, w_1, \dots, w_{m-r}\}$ forms a basis for V, we have that

$$a_1 = \dots = a_r = 0$$

This shows that $\{T(w_1), \dots, T(w_{m-r})\}$ is a set of linearly independent vectors. Hence, $\{T(w_1), \dots, T(w_{m-r})\}$ is a basis for $R(T)$ and so $\operatorname{rank}(T) = m - r$.

Hence $\operatorname{nullity}(T) + \operatorname{rank}(T) = r + (m - r) = \dim(V)$. □

Example 1.31. (Method of finding range by Theorem 1.24) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ given by

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_4, x_2 + x_5, x_3 + x_4).$$

Find $\operatorname{nullity}(T)$ and $\operatorname{rank}(T)$.

Sol: We can guess the image by using Theorem 1.14. Consider $T(1,0,0,0,0) = (1,0,0)$, $T(0,1,0,0,0) = (0,1,0)$ and $T(0,0,1,0,0) = (0,0,1)$, $T(0,0,0,1,0) = (1,0,1)$, $T(0,0,0,0,1) = (0,1,0)$. Hence,

$$R(T) = \mathbb{R}^3,$$

and so $\operatorname{rank}(T) = 3$. Then $\operatorname{nullity}(T) = 5 - 3 = 2$.

Example 1.32. (Method of finding nullity by solving equations) Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2 - x_3, x_2 + x_3 + x_4)$$

Determine if T is surjective.

Sol: We shall approach by finding the nullity first. (*The nullity can be usually found by solving system of linear equations.*) We first compute the space $N(T)$:

$$\begin{aligned} x_1 + x_2 - x_3 \quad &= 0 \\ x_2 + x_3 + x_4 &= 0 \end{aligned}$$

We obtain the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right).$$

The reduced row echelon form is then given by:

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right).$$

The null space $N(T)$ is given by the solution set of the equations: set the free variables $x_3 = s$ and $x_4 = t$,

$$\{(2s+t, -s-t, s, t) : s, t \in \mathbb{R}\}.$$

Hence, a basis for $N(T)$ is given by:

$$\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, $\mathrm{nullity}(T) = 2$ and so by the dimension formula, $\mathrm{rank}(T) = 2$. Then we must have $R(T) = \mathbb{R}^2$. This shows that T is surjective.

1.6. **Injectivity.** Recall that a function $f : X \to Y$ (between two sets X and Y) is said to be injective if for any $x, y \in X$, $f(x) = f(y)$ implies $x = y$. The concept of a linear transformation T can be boiled down to the case that $T(x) = 0$:

Theorem 1.33. *Let $T : V \to W$ be a linear transformation. Then T is injective if and only if $N(T) = \{0\}$.*

Proof. Suppose T is injective. Then $T(v) = 0 = T(0)$ implies $v = 0$.

Suppose $N(T) = \{0\}$. Let $v, w \in V$ such that $T(v) = T(w) \Rightarrow T(v) - T(w) = 0 \Rightarrow T(v-w) = 0$. Hence $v - w = 0 \Rightarrow v = w$. This implies T is injective. □

From the dimension formula (Theorem 1.29), we also have:

Corollary 1.34. *Let $T : V \to W$ be a linear transformation. Then the following statements are equivalent:*

- *T is injective;*
- *$N(T) = 0$;*
- $\mathrm{rank}(T) = \dim(V)$.

Example 1.35. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_1)$$

Show that T is injective.

Sol: Let $(x_1, x_2) \in N(T)$. Then we have to solve:

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

Solving the equations, we have $x_1 = x_2 = 0$ and so $N(T) = 0$.

Exercise 1.36. Let $T : V \to W$ be a linear transformation. Prove that if $\dim(V) > \dim(W)$, then T is not injective.
Sol: Since $R(T)$ is a subspace of W, $\operatorname{rank}(T) \leq \dim(W)$. Hence,

$$\dim(V) - \operatorname{rank}(T) \geq \dim(V) - \dim(W) > 0.$$

By the dimension formula, the first term is equal to $\operatorname{nullity}(T)$ and so the inequality becomes

$$\operatorname{nullity}(T) > 0.$$

Thus $N(T) \neq 0$ and so T is not injective.

Exercise 1.37. Let $T : V \to W$ be a linear transformation. Show that T is surjective and injective if and only if $N(T) = 0$ and $\dim(V) = \dim(W)$.
Sol: Suppose T is surjective an injective. Then the injectivity part implies that $N(T) = 0$. The surjectivity implies that $R(T) = W$. Then $\operatorname{nullity}(T) = 0$ and $\operatorname{rank}(T) = \operatorname{rank}(W)$. By the dimension formula, one sees that $\dim(W) = \dim(V)$.

Now $N(T) = 0$ and $\dim(V) = \dim(W)$. The condition $N(T) = 0$ implies that T is injective. Now by $\operatorname{nullity}(T) = 0$ and the dimension formula,

$$\operatorname{rank}(T) = \dim(V) = \dim(W).$$

Hence, $R(T) = W$.

1.7. **Matrix multiplication as a linear transformation.** For an $n \times m$ matrix A, let $L_A : \mathbb{R}^m \to \mathbb{R}^n$ defined by:

$$L_A(x) = Ax,$$

where Ax is the matrix multiplication. We now try to compare some concepts of linear transformations and matrices in this content.

Theorem 1.38. *Let A be an $n \times m$ matrix. Then*

- $R(L_A) = \operatorname{col}(A)$*;*
- $N(L_A) = \{x \in \mathbb{R}^m : Ax = 0\}$*; the space is also called the* **null space** *of A.*

Proof. We first prove the first bullet. Write $A = (v_1 \dots v_m)$. For any $v \in R(L_A)$, $v = Ax$ for some $x = (x_1 \ \dots \ x_m)^T \in \mathbb{R}^m$. Then, $v = x_1 v_1 + \dots + x_m v_m$. Hence, $v \in \operatorname{col}(A)$. For another inclusion, let $v \in \operatorname{col}(A)$, $v = x_1 v_1 + \dots + x_m v_m$ for some $x = (x_1 \ \dots \ x_m)^T \in \mathbb{R}^m$. Then $v = Ax$ and so $v \in R(L_A)$.

The second bullet is more straightforward and we leave as an exercise. $\square$

Theorem 1.39. *Let A be an $n \times m$ matrix. Then,*

- *the number of leading ones in the reduced row echelon form of A is equal to* $\text{rank}(L_A)$.
- *the number of free variables in the reduced row echelon form of A is equal to* $\text{nullity}(L_A)$.

Proof. (Sketch) By Theorem 1.38, $\text{rank}(L_A) = \dim(\text{col}(A))$. The reduced row echelon form R of A is obtain by multiplying a product of (invertible) elementary row operations. Then, we have $\dim(\text{col}(A)) = \dim(\text{col}(R))$. Note that $\dim(\text{col}(R))$ is equal to the number of leading ones in R. Then (a) follows by putting all the equalites together.

The second one follows from solving the system of linear equation $Ax = 0$ by Theorem 1.38. □

Remark 1.40. The number of leading ones plus the number of free variables is equal to m. This agrees with the dimension formula:

$$\text{rank}(L_A) + \text{nullity}(L_A) = \dim(\mathbb{R}^m).$$

Definition 1.41. Let A be a nm matrix. The **rank** of A is defined as $\text{rank}(L_A)$. We denote by $\text{rank}(A)$, the rank of A.

Exercise 1.42. Let

$$B = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & -1 \\ 0 & 2 & 4 & 1 \\ 3 & 1 & -1 & 1 \end{pmatrix}.$$

Find a basis for $R(L_B)$, $N(L_B)$, and also find $\text{rank}(B)$.

Sol: The reduced row echelon form is:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then a basis for $R(L_B) = \text{col}(B)$ is

$$R(L_B) = \text{span}(\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}).$$

Hence, $\text{rank}(B) = 3$.

By using Jordan-Gaussian eliminations, we need to consider

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Set the free variables $x_3 = s$. Then the solution set for the equation $Bx = 0$ is $\{(s, -2s, s, 0) : s \in \mathbb{R}\}$. So a basis is

$$\{(1, -2, 1, 0)\}.$$