MATH 2101 LINEAR ALGEBRA I, FALL SEMESTER 2023

1. Matrix representations of linear transformations

As mentioned before, a central tool in the linear algebra is matrices. Our first goal is to transfer the problems of linear transformations to matrices. For example, one can then find the rank and nullity of a linear transformation, as well as determining the invertibility of a linear transformation.

Key concepts:

- Matrix representations for a linear transformation
- Dictionary between operations on a matrix and operations on linear transformations
- Change of coordinate matrix and similar matrices

1.1. **Ordered basis and matrix representations.**

Definition 1.1. An **ordered basis** is a basis with a given specific *order*. When we write an order basis $\beta = \{v_1, \dots, v_r\}$ for a vector space V, we mean v_1 to be the first one, v_2 to be the second one, v_r to be the last one, etc.

Definition 1.2. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Let β, β' be the standard basis for $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. We shall call $[T]_\beta^{\beta'}$ the **standard matrix representation** of T.

We now convert elements in a vector space to some column matrices:

Definition 1.3. Let $\beta = \{v_1, \dots, v_m\}$ be an ordered basis for a vector space V. For $v \in V$,

$$v = a_1 v_1 + \ldots + a_m v_m$$

for some unique saclars $a_1, \dots, a_m \in \mathbb{R}$. Define the **coordinate vector of v with respect to β** as:

$$[v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}.$$

Example 1.4. Let $\beta = \left\{(1,1)^T, (1,-1)^T\right\}$. Find $[(2,4)^T]_\beta$.
Sol: $(2,4)^T = a(1,1)^T + b(1,-1)^T$ for some $a, b \in \mathbb{R}$. We have to solve the equation:

$$a + b = 2$$
$$a - b = 4$$

Thus, $a = 3$ and $b = -1$. Then

$$\left[\begin{pmatrix} 2 \\ 4 \end{pmatrix}\right]_\beta = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Example 1.5. Let $P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3 (with coefficients in $\mathbb{R}$). Let

$$\beta = \left\{1, 1+x, x^2, x^2+x^3\right\}.$$

be a basis for $P_3(\mathbb{R})$. Find

$$[1 + x + x^2 + x^3]_\beta.$$

Sol: $1 + x + x^2 + x^3 = 0(1) + (1)(1+x) + (0)(x^2) + (1)(x^2+x^3)$. Hence

$$[1 + x + x^2 + x^3]_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We now convert linear transformations to matrices:

Definition 1.6. Let $T : V \to W$ be a linear transformation. Let $\beta = \{v_1, \ldots, v_m\}$ be an ordered basis for V and let $\gamma = \{w_1, \ldots, w_n\}$ be an ordered basis for W. For each v_j,

$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{nj}w_n.$$

Define the matrix representation of T relative to β and γ as:

$$[T]_\beta^\gamma = (a_{ij}),$$

an $n \times m$-matrix.

If $\beta = \gamma$, we shall sometimes write $[T]_\beta$ for $[T]_\beta^\gamma$.

Example 1.7. Find the standard matrix representation of the linear transformation:

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_3, x_2 + x_3, x_1 + x_4).$$

Sol: Let β, β' be standard basis for $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. Then

$$[T]_\beta^{\beta'} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Example 1.8. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$. Let

$$T(x_1, x_2) = (x_1, x_1 + 2x_2, -x_1 + x_2, 3x_2)$$

Let β be the standard basis for $\mathbb{R}^2$ and let

$$\gamma = \{(1\ 1\ -1\ 0), (1\ 3\ 0\ 3), (0\ 0\ 1\ 0), (0\ 0\ 0\ 1)\}$$

be a basis for $\mathbb{R}^4$. Then

$$T(1, 0) = (1\ 1\ -1\ 0), \quad T(0, 1) = (0, 2, 1, 3).$$

It is clear that

$$(1\ 1\ -1\ 0) = (1\ 1\ -1\ 0) + 0(1\ 3\ 0\ 3) + 0(0\ 0\ 1\ 0) + 0(0\ 0\ 0\ 1),$$

$$(0\ 2\ 1\ 3) = -(1\ 1\ -1\ 0) + (1\ 3\ 0\ 3) + 0(0\ 0\ 1\ 0) + 0(0\ 0\ 0\ 1)$$

(*You sometimes need to solve some equations to find out linear combinations.*)

Then,

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrix representation uses *matrix multiplication* to compute a linear transformation:

Theorem 1.9. *Let $T : V \to W$ be a linear transformation. Let β, γ be ordered bases for V and W respectively. Then*

$$[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta.$$

Proof. This is an exercise of comparing definitions. For example, let $\beta = \{v_1, \dots, v_m\}$ and $\gamma = \{w_1, \dots, w_n\}$. We have $[T(v_i)]_\gamma$ is the i-th column of $[T]_\beta^\gamma$; and on the other hand, $[v_i]_\beta = \mathbf{e}_i$ in $\mathbb{R}^m$ and so $[T]_\beta^\gamma \mathbf{e}_i$ is the i-th column of $[T]_\beta^\gamma$. $\square$

Example 1.10. For example, in Example 1.8, if we do the multiplication:

$$[T(x_1, x_2)]_\gamma = [T]_\beta^\gamma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix}.$$

Then $T(x_1, x_2) = (x_1 - x_2)(1\ 1\ -1\ 0) + x_2(1\ 3\ 0\ 3)$, which recovers the orginal formula in Example 1.8.

One application of matrix representations is to find the nullity and rank of a matrix:

Theorem 1.11. *Let T be a linear transformation from V to W. Let β and γ be ordered bases for V and W respectively. Then*

$$\mathrm{rank}(T) = \mathrm{rank}([T]_\beta^\gamma).$$

The main idea of the proof of Theorem 1.11 is to find a suitable basis for $R(T)$. We shall not go into the details.

Example 1.12. Find the nullity and rank of T in Example 1.8.
Sol: The reduced row echelon form of the standard matrix representation of T $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 1 \\ 0 & 3 \end{pmatrix}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, the rank of A is 2. By Theorem 1.11, $\mathrm{rank}(T) = 2$ and by the dimension formula, $\mathrm{nullity}(T) = 2 - 2 = 0$.

1.2. **Additions of linear transformations and additions of matrices.**

Definition 1.13. Let $T : V \to W$ and $T' : V \to W$ be two linear transformations. Define $T + T'$ be a function from V to W given by

$$(T + T')(v) = T(v) + T'(v).$$

Example 1.14. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{pmatrix}$ and let $B = \begin{pmatrix} 0 & 4 & -1 \\ 1 & 0 & 2 \end{pmatrix}$. Then

$$(L_A + L_B)(v) = L_A(v) + L_B(v) = Av + Bv = (A + B)v = L_{A+B}(v).$$

(Justify all equalities!)

Exercise 1.15. Show that $T + T'$ is a linear transformation from V to W.
Sol: (Addition) For $v_1, v_2 \in V$, $(T + T')(v_1 + v_2) = T(v_1 + v_2) + T'(v_1 + v_2) = T(v_1) + T'(v_1) + T(v_2) + T'(v_2) = (T + T')(v_1) + (T + T')(v_2)$.
(Scalar multiplication) For $v \in V$ and $c \in \mathbb{R}$, $(T + T')(cv) = T(cv) + T'(cv) = cT(v) + cT'(v) = c(T(v) + T'(v)) = c(T + T')(v)$.

Theorem 1.16. *Let $T, T' : V \to W$ be two linear transformations. Let β and γ be ordered bases for V and W respectively. Then*

$$[T + T']_\beta^\gamma = [T]_\beta^\gamma + [T']_\beta^\gamma$$

1.3. **Compositions of linear transformations and matrix multiplications.**

Definition 1.17. Let $T : V \to W$ and $T' : W \to X$ be linear transformations. Let β, γ and α be ordered bases for V, W and X respectively. Define

$$(T' \circ T)(v) = T'(T(v)).$$

Exercise 1.18. Show that $T' \circ T$ is a linear transformation.

Theorem 1.19. *Let $T : V \to W$ and $T' : W \to X$ be linear transformations. Let β, γ and α be ordered bases for V, W and X respectively. Then*

$$[T' \circ T]_\beta^\alpha = [T']_\gamma^\alpha [T]_\beta^\gamma,$$

where the RHS is the matrix multiplication.

Proof. Let $v \in V$. By Theorem 1.9, we have

$$[T' \circ T]_\beta^\alpha [v]_\beta = [T' \circ T(v)]_\alpha.$$

We also have:

$$[T']_\gamma^\alpha [T]_\beta^\gamma [v]_\beta = [T']_\gamma^\alpha [T(v)]_\gamma = [T'(T(v))]_\alpha = [T' \circ T(v)]_\alpha.$$

Thus, the two expressions agree for any v and so the two matrices are the same. □

Example 1.20. Let $T : \mathbb{R}^4 \to \mathbb{R}$ given by $T(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4$. Let $U : \mathbb{R} \to \mathbb{R}^4$ given by $U(x) = (x, 2x, 3x, 4x)$. Let β and γ be the standard ordered bases for $\mathbb{R}^4$ and $\mathbb{R}$ respectively. Use matrix multiplication to find $U \circ T$.
Sol: Then

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \quad [U]_\gamma^\beta = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Note that $(U \circ T)(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4, 2x_1 + 2x_2 + 2x_3 + 2x_4, 3x_1 + 3x_2 + 3x_3 + 3x_4, 4x_1 + 4x_2 + 4x_3 + 4x_4)$. On the other hand,

$$[U \circ T]_\beta^\beta = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = [U]_\gamma^\beta [T]_\beta^\gamma.$$

Hence,

$$U \circ T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4, 2x_1 + 2x_2 + 2x_3 + 2x_4, 3x_1 + 3x_2 + 3x_3 + 3x_4, 4x_1 + 4x_2 + 4x_3 + 4x_4).$$

Exercise 1.21. Let $T : V \to W$, $T' : W \to X$ be linear transformations. Show that $\operatorname{rank}(T' \circ T) \leq \operatorname{rank}(T)$.
Sol: Let $\beta = \{v_1, \ldots, v_r\}$ be a basis for $R(T)$. Then, one has that $R(T' \circ T) = \operatorname{span}(\{T(v_1), \ldots, T(v_r)\}$. Since, $R(T' \circ T)$ can be spanned by r vectors, we can find

a linearly independent subset of $\{T(v_1), \ldots, T(v_r)\}$ to span $R(T' \circ T)$. In other words, $\dim(R(T' \circ T)) \leq r = \dim R(T)$.

1.4. **Invertibility.**

Definition 1.22.
- Let $T : V \to W$ be a linear transformation. We say that T is **invertible** if there exists a function $T^{-1} : W \to V$ such that $T^{-1} \circ T = \mathrm{Id}_V$ and $T \circ T^{-1} = \mathrm{Id}_W$. We shall call T^{-1} to be the **inverse** of T.
- An invertible linear transformation $T : V \to W$ is sometimes called an **isomorphism**. We say that V is **isomorphic** to W.

Exercise 1.23. The inverse of a linear transformation is still a linear transformation.

We first have some properties for an invertible linear transformation:

Theorem 1.24. *Let $T : V \to W$ be an invertible linear transformation. Then*
- $\dim(V) = \dim(W)$*;*
- T *is injective;*
- T *is surjective.*

Proof. Since T has an inverse, T is injective and surjective. By surjectivity of T, $R(T) = W$ and so $\mathrm{rank}(T) = \dim(W)$. By injectivity of T, $N(T) = \{0\}$ and so by the dimension formula, $\mathrm{rank}(T) = \dim(V)$. Hence $\dim(V) = \dim(W)$. $\square$

Example 1.25. (First check on the invertibility of a linear transformation by using dimensions) Determine if the following linear transformation

$$T : \mathbb{R}^3 \to \mathbb{R}^2, \quad T(x, y, z) = (x + y, y - z)$$

is invertible.
Sol: Since $\dim(\mathbb{R}^3) \neq \dim(\mathbb{R}^2)$, T cannot be invertible.

Example 1.26. Determine if the following linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad T(x, y) = (x, y, 0)$$

is invertible.
Sol: Since $\dim(\mathbb{R}^2) \neq \dim(\mathbb{R}^3)$, T cannot be invertible.

Theorem 1.27. *Let $T : V \to W$ be an invertible linear transformation. Let β, γ be ordered bases for V and W respectively. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible.*

Proof. Let $m = \dim(V)$. Suppose T is invertible. Then $m = \dim(W)$ (by the previous theorem). Then $[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = [T^{-1} \circ T]_\beta^\beta = [\mathrm{Id}_V]_\beta^\beta = I_m$. Similarly, we have $[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [\mathrm{Id}_W]_\gamma^\gamma = I_m$.

Conversely, suppose $[T]_\beta^\gamma$ is invertible. Let $A = [T]_\beta^\gamma$. For $w \in W$,

$$A^{-1}[w]_\gamma = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

for some $a_1, \ldots, a_m \in \mathbb{R}$. Write the ordered basis $\beta = \{v_1, \ldots, v_m\}$. Define

$$U(w) = a_1 v_1 + \ldots + a_m v_m.$$

Then $[U]_\gamma^\beta = A^{-1}$ and so $[U \circ T]_\beta^\beta = A^{-1}A = I_m$ and hence $U \circ T = \mathrm{Id}_V$. Checking the right inverse is similar. □

The above proof also gives the following:

Corollary 1.28. *Let $T : V \to W$ be an invertible linear transformation. Let α, β be bases for V and W respectively. Then*

$$[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}.$$

Example 1.29. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (x + y - z, y + z, x + z).$$

Let β be the standard basis for $\mathbb{R}^3$. Then, we have to compute the matrix representation

$$[T]_\beta$$

Note that

$$\begin{aligned} [T(\mathbf{e}_1)]_\beta &= [(1, 0, 1)]_\beta = \mathbf{e}_1 + \mathbf{e}_3, \\ [T(\mathbf{e}_2)]_\beta &= [(1, 1, 0)]_\beta = \mathbf{e}_1 + \mathbf{e}_2, \\ [T(\mathbf{e}_3)]_\beta &= [(-1, 1, 1)]_\beta = -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3. \end{aligned}$$

Thus, $[T]_\beta = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Now, we compute the determinant of $[T]_\beta$. Thus,

$$\begin{aligned} \det([T]_\beta) &= \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \\ &= (-1)^{1+1}(1)\det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + (-1)^{1+3}(1)\det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= 1 + 2 \\ &= 3 \neq 0 \end{aligned}$$

Hence, by Theorem 1.27, T is invertible.

Exercise 1.30. Find $T^{-1}(x, y, z)$ in Example 1.29.
Sol: Note that
$$[T^{-1}]_\beta = [T]_\beta^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$
Thus $T^{-1}(x, y, z) = \frac{1}{3}(x - y + 2z, x + 2y - z, -x + y + z)$.

1.5. **Change of coordinate matrix.** An example: Given an equation:
$$2x^2 + 3xy + 5y^2 = 1,$$
we can change to the new coordinate given by:
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$
In this new coordinates, we have $x'^2 + y'^2 = 1$ and we can visualize as a circle. We would like to discuss this concept properly using the concepts of a basis.

Definition 1.31. Let β, β' be two ordered bases for a linear vector space V. The matrix $[\mathrm{Id}_V]_\beta^{\beta'}$ is called a **change of coordinate matrix** from β to β'.

Example 1.32. Let $\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\beta' = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\}$. Then $[\mathrm{Id}_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Example 1.33. Let $\beta = \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$, and let $\beta' = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$. Find the change of coordinate matrix from β to β'.
Sol: We have
$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
Similarly,
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
Then, the change of coordinate matrix from β to β' is
$$\begin{pmatrix} 0 & \frac{1}{3} \\ -1 & -\frac{1}{3} \end{pmatrix}$$

Theorem 1.34. *Let β, β' be two ordered bases for a vector space V. Let $Q = [\mathrm{Id}_V]_{\beta'}^{\beta}$. Then*

- *Q is invertible;*
- *For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.*

Proof. Let $m = \dim(V)$. Let $P = [\mathrm{Id}_V]_\beta^{\beta'}$. Then $PQ = [\mathrm{Id}_V \circ \mathrm{Id}_V]_{\beta'}^{\beta} = I_m$. Similarly, $QP = [\mathrm{Id}_V \circ \mathrm{Id}_V]_{\beta'}^{\beta'} = I_m$. Hence, P is invertible.

For any $v \in V$, $Q[v]_{\beta'} = [\mathrm{Id}_V(v)]_\beta = [v]_\beta$. $\square$

Example 1.35. From the previous example, we have

$$[\mathrm{Id}_{\mathbb{R}^2}]_{\beta}^{\beta'} = ([\mathrm{Id}_{\mathbb{R}^2}]_{\beta'}^{\beta})^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}.$$

One can also change the matrix representation form one basis to another one by the following theorem:

Theorem 1.36. *Let β, β' be two ordered bases for a vector space V. Let $T : V \to V$ be a linear transformation. Let $Q = [\mathrm{Id}_V]_{\beta'}^{\beta}$. Then*

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$$

Proof. Let $v \in V$. Let $Q = [T]_{\beta'}^{\beta}$. Then $[T]_{\beta'}[v]_{\beta'} = [T(v)]_{\beta'}$. On the other hand,

$$Q^{-1}[T]_{\beta}Q[v]_{\beta'} = Q^{-1}[T]_{\beta}[v]_{\beta} = Q^{-1}[T(v)]_{\beta} = [T(v)]_{\beta'}.$$

□

Definition 1.37. Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix Q such that $Q^{-1}AQ = B$.

Similar matrices share some common properties. One is the following:

Exercise 1.38. Let β, β' be two ordered bases for a vector space V. Let T be a linear transformation from V to V. Prove that

$$\det([T]_{\beta}) = \det([T]_{\beta'}).$$

Sol: By Theorem 1.36, $[T]_{\beta'} = Q^{-1}[T]_{\beta}Q$ for $Q = [\mathrm{Id}_V]_{\beta'}^{\beta}$. Then

$$\begin{aligned}\det([T]_{\beta'}) &= \det(Q^{-1}[T]_{\beta}Q) = \det(Q^{-1})\det([T]_{\beta})\det(Q) \\ &= \det(Q)^{-1}\det([T]_{\beta})\det(Q) = \det([T]_{\beta}).\end{aligned}$$