MATH 2101 LINEAR ALGEBRA I, FALL SEMESTER 2023

1. Eigenvalues and Eigenvectors

Key concepts in this section:

- The meaning of diagonalizability, eigenvalues, eigenvectors and their relations.
- How to compute eigenvalues from characteristic polynomials?
- The eigenspaces, algebraic multiplicity and geometric multiplicity
- Some applications: taking high power on a diagonalizable matrix

1.1. **Motivation.** The effect of a diagonal matrix is easier to understand. For example, $\operatorname{diag}(a_1, \ldots, a_n)$ sketches the elementary vectors $\mathbf{e}_i$ by the scalar a_i. Now the question is that after some change of basis, the matrix can be transformed to a diagonal one.

1.2. **Eigenvalues and eigenvectors.**

Definition 1.1. An $n \times n$ matrix A is said to be **diagonalizable** if there exists an invertible $n \times n$-matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. (In other words, a diagonalizable matrix A is *similar* to a diagonal matrix.)

Example 1.2. Any diagonal matrix D is diagonalizable Since $I_n^{-1} D I_n = D$. For example,

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is diagonalizable.

Example 1.3. Let $Q = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$. Let $D = \operatorname{diag}(3, 4)$. Since Q is invertible, we obtain a diagonalizable matrix:

$$QDQ^{-1} = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 28 & 11 \end{pmatrix}.$$

Exercise 1.4. Let A be an invertible $n \times n$ matrix. Prove that A is diagonalizable if and only if A^{-1} is diagonalizable.

Sol: Suppose A is diagonalizable. Then there exists an invertible Q such that $Q^{-1}AQ = D$ for some diagonal matrix. Then

$$(Q^{-1}AQ)^{-1} = D^{-1}$$

and so $Q^{-1}A^{-1}Q = D^{-1}$.

Producing a diagonalizable matrix, one can start with a diagonal matrix and multiplying an invertible matrix and its inverse on left and right sides respectively (as in Example 1.3). A harder question is to *determine if a matrix is diagonalizable and how to diagonalize a matrix.*

Definition 1.5. Let A be an $n \times n$ matrix. A *non-zero* vector $v \in \mathbb{R}^n$ is said to be an **eigenvector** of A if there exists a scalar $\lambda \in \mathbb{R}$ such that $Av = \lambda v$. We shall call λ to be the **eigenvalue** corresponding to the eigenvector v.

Example 1.6. Let

$$A = \begin{pmatrix} 1 & 2 \\ 6 & 5 \end{pmatrix}.$$

Then

$$A \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Then $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector of A and 7 is an eigenvalue corresponding to $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Example 1.7. (The first relation of eigenvectors to diagonal matrices) Let $D = \mathrm{diag}(a_1, \ldots, a_n)$ be a $n \times n$ diagonal matrix. Then the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the eigenvectors of D since

$$D\mathbf{e}_i = a_i \mathbf{e}_i$$

Then each a_i is an eigenvalue associated to the eigenvector $\mathbf{e}_i$.

Exercise 1.8. Let A be an invertible $n \times n$ matrix. Then v is an eigenvector of A if and only if v is aso an eigenvector of A^{-1}.

Sol: Suppose v is an eigenvector of A. Then $Av = \lambda v$ for some λ in $\mathbb{R}$. Then $v = A^{-1}(\lambda v)$ and so $A^{-1}v = \lambda^{-1}v$. By definition. v is an eigenvector for A^{-1}. This proves the only if direction.

For the if direction, the proof is similar to the only if direction by replacing A with A^{-1} in above arguments.

Example 1.9. (Eigenvalues of some diagonalizable matrices) Let $D = \mathrm{diag}(2, 4)$. Let $Q = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. What are the eigenvalues of QDQ^{-1}? (This example will suggest the relation between Q and eigenvectors.)

Sol: We have that

$$Q^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{e}_1, \quad Q^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \mathbf{e}_2.$$

Then,

$$QDQ^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = QD\mathbf{e}_1 = 2Q\mathbf{e}_1 = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

and

$$QDQ^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = QD\mathbf{e}_2 = 4Q\mathbf{e}_2 = 4\begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Thus 2 and 4 are the eigenvalues and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ corresponding to them respectively.

We shall come back to the question how to find eigenvalues and eigenvectors. We first see how to address the diagonalization problem.

Theorem 1.10. *Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if there exists a basis $\beta = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ such that each v_i is also an eigenvector of A.*

Proof. We first prove the only if direction. Suppose A is diagonalizable. Then $A = QDQ^{-1}$ for a diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and an invertible matrix Q.

We first show that $Q\mathbf{e}_1, \ldots, Q\mathbf{e}_n$ are eigenvectors. Check:

$$A(Q\mathbf{e}_i) = (QDQ^{-1})(Q\mathbf{e}_i) = QD\mathbf{e}_i = Q(\lambda_i\mathbf{e}_i) = \lambda_i(Q\mathbf{e}_i).$$

Since $Q\mathbf{e}_i \neq 0$, it is an eigenvector.

We now check those eigenvectors form a basis. Since Q is invertible and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is linearly independent, $\{Q\mathbf{e}_1, \ldots, Q\mathbf{e}_n\}$ is also linearly independent. Similarly, since Q is invertible and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$, $\{Q\mathbf{e}_1, \ldots, Q\mathbf{e}_n\}$ also spans $\mathbb{R}^n$. In other words, $\{Q\mathbf{e}_1, \ldots, Q\mathbf{e}_n\}$ forms a basis for $\mathbb{R}^n$ (see Assignment Problem!).

We now prove the if direction.Suppose there exists an ordered basis $\beta = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ such that those v_i are eigenvectors. Let λ_i be the corresponding eigenvalue of v_i. Let β' be the standard basis for $\mathbb{R}^n$. Let $Q = [\mathrm{Id}_{\mathbb{R}^n}]_{\beta}^{\beta'}$ be the change of coordinate matrix. Then

$$Q^{-1}AQ\mathbf{e}_i = Q^{-1}Av_i = Q^{-1}(\lambda_i v_i) = \lambda_i Q^{-1}v_i = \lambda_i\mathbf{e}_i$$

for all i. This implies that $Q^{-1}AQ$ is a diagonal matrix. Hence, A is diagonalizable. $\square$

The following theorem tells you how to diagonalize a matrix from a set of eigenvectors which forms a basis:

Theorem 1.11. *Let $\beta = \{v_1, \ldots, v_n\}$ be eigenvectors of a $n \times n$-matrix A. Suppose $\{v_1, \ldots, v_n\}$ is a basis. Let $Q = (v_1\ v_2\ \ldots\ v_n)$ be a $n \times n$ matrix. Then $Q^{-1}AQ$ is a diagonal matrix with the diagonal entries equal to the eigenvalues.*

Proof. One has to investigate the if part of the proof of Theorem 1.10. □

Exercise 1.12. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. Verify that $(-3, 1)^T$ and $(1, 1)^T$ are the eigenvectors of A. Then diagonalize A.
Sol: Let $Q = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}$. Then

$$Q^{-1}AQ = \frac{-1}{4} \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Exercise 1.13. Let A be an $n \times n$ matrix. Suppose A has only one eigenvalue equal to λ. Then A is diagonalizable if and only if $A = \lambda I_n$.
Sol: Suppose $A = \lambda I_n$. Then it is clear that A is diagonalizable.

Suppose A is diagonalizable. Then there exists an invertible matrix Q such that $Q^{-1}AQ = \lambda I_n$ since A has only one eigenvalue. Then, $A = Q(\lambda I_n)Q^{-1}) = \lambda I_n$.

1.3. **Finding eigenvalues.** We need to introduce characteristics polynomials:

Definition 1.14. Let A be an $n \times n$ matrix. The polynomial $\det(A - tI_n)$ in the variable t is called the *characteristics polynomial* of A.

Example 1.15. Find the characteristics polynomial of $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$.
Sol: $\det(A - tI_2) = \det \begin{pmatrix} 1-t & 3 \\ 2 & 1-t \end{pmatrix} = (1-t)^2 - 6 = t^2 - 2t - 5$.

Example 1.16. Let A be a diagonal matrix. Suppose the entries in the diagonal are $a_1, \ldots, a_n$. Then the characteristics polynomial of A is:

$$\det(A - tI_n) = (a_1 - t) \ldots (a_n - t).$$

Theorem 1.17. *(Finding an eigenvalue by solving the roots for the characteristic polynomial) Let A be an $n \times n$ matrix. Then a scalar λ is an eigenvalue of A if and only if* $\det(A - \lambda I_n) = 0$.

Proof. Suppose λ is an eigenvalue of A. Then, there exists a non-zero vector $v \in \mathbb{R}^m$ such that $(A - \lambda I_n)v = 0$. In other words, there exists a non-trivial solution for $(A - \lambda I_n)x = 0$ and so $\det(A - \lambda I_n) = 0$.

Conversely, $\det(A-\lambda I_n)=0$ and so there is a non-trivial solution $v\in\mathbb{R}^m$ such that $(A-\lambda I_n)v=0$ and so $Av=\lambda v$. Hence, v is an eigenvector and λ is its associated eigenvalue. □

Example 1.18. Let $A=\lambda I_n$. Then the characteristics polynomial of A is $(\lambda-t)^n$. So A has only one eigenvalue, which is λ.

Example 1.19. Find all the eigenvalues of the matrix

$$A=\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Sol: We first find the characteristics polynomial of A:

$$\begin{aligned}
\det(A-tI_3) &= \det\begin{pmatrix} 2-t & 0 & -1 \\ 0 & 2-t & 0 \\ -1 & 0 & 2-t \end{pmatrix} \\
&= (2-t)\det\begin{pmatrix} 2-t & -1 \\ -1 & 2-t \end{pmatrix} \\
&= (2-t)[(2-t)^2-1] \\
&= (2-t)(t^2-4t+3) \\
&= (2-t)(t-3)(t-1)
\end{aligned}$$

Hence, the roots for $\det(A-tI_3)$ are $1,2,3$. Hence, all the eigenvalues are $1,2,3$.

Example 1.20. (c.f. Exercise 1.13) Let $A=\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$. Show that A is not diagonalizable.

Sol: The characteristics polynomial of A is

$$\det(A-tI_2)=\det\begin{pmatrix} -1-t & 1 \\ -4 & 3-t \end{pmatrix}=(1-t)^2.$$

Then the only eigenvalue of A is 1. Thus if A is diagonalizable,

$$Q^{-1}AQ=I_2$$

for some invertible matrix Q. This implies that $A=I_2$, giving a contradiction.

Here is one important property for characteristics polynomials:

Theorem 1.21. *Let A be an $n\times n$ matrix. Then*

- *the characteristics polynomial of A has degree n with the leading coefficient $(-1)^n$;*
- *A has at most n eigenvalues.*

Proof. We shall not go into details. The main idea for the first one is to expand the determinant $\det(A - tI_n)$.

For the second one, if λ is an eigenvalue of A, then $\det(A - \lambda I_n) = 0$ by Theorem 1.17. In other words, λ is a root of the equation $\det(A - tI_n) = 0$. There are at most n roots for a polynomial of degree n and so there are at most n eigenvalues. $\square$

1.4. **Linear independence of eigenvectors.** We have discussed how to find eigenvalues (Theorem 1.17). Now we address some aspects of eigenvectors.

Theorem 1.22. *Let A be an $n \times n$ matrix. Suppose A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Let $v_1, \dots, v_n$ be eigenvectors corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$ respectively. Then $v_1, \dots, v_n$ are linearly independent.*

Proof. Let $v_1, \dots, v_n$ be the eigenvectors with the associated eigenvalues $\lambda_1, \dots, \lambda_n$ respectively. We shall show inductively on k that $\{v_1, \dots, v_k\}$ is linearly independent. If $k = 1$, it follows from definitions.

Now, we consider the equation:

$$a_1 v_1 + \ldots + a_k v_k = 0.$$

Multiplying A on the left, we have:

$$a_1 \lambda_1 v_1 + \ldots + a_k \lambda_k v_k = 0.$$

Then, $a_1 \lambda_1 v_1 + \ldots + a_k \lambda_k v_k = 0$. Then, subtracting with the equation:

$$a_1 \lambda_k v_1 + \ldots + a_k \lambda_k v_k = 0,$$

we have $a_1(\lambda_1 - \lambda_k)v_1 + \ldots + a_{k-1}(\lambda_1 - \lambda_k)v_{k-1} = 0$. By inductions, we have:

$$a_i(\lambda_i - \lambda_k) = 0$$

for $i \leq k-1$. Since $\lambda_i \neq \lambda_k$ for all $i \leq k-1$, we have that $a_1 = \ldots = a_{k-1} = 0$. This then forces $a_k = 0$ since $v_k \neq 0$. This shows that $v_1, \dots, v_k$ are linearly independent. $\square$

Corollary 1.23. *Let A be an $n \times n$ matrix with n distinct eigenvalues. Then A is diagonalizable.*

Proof. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues. Then, for each eigenvalue λ_i, we have an eigenvector v_i. By Theorem 1.22, $v_1, \dots, v_n$ are linearly independent. Since $\dim(\mathbb{R}^n) = n$, $v_1, \dots, v_n$ form a basis for $\mathbb{R}^n$. By Theorem 1.10, A is diagonalizable. $\square$

Example 1.24. Let $A = \begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}$. Then $\det(A - tI_2) = \det \begin{pmatrix} 5-t & -2 \\ 2 & -t \end{pmatrix} = -t(5-t) - 4 = t^2 - 5t + 4 = (t-4)(t-1)$. Then A has 2 distinct eigenvalues and so A is diagonalizable.

To find the eigenvector corresponding to 1, we solve $(A - I_2)v = 0$, that is $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} v = 0$. Then an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

To find the eigenvector corresponding to 4, we solve $(A - 4I_2)v = 0$, that is $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} v = 0$. Then an eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Let $Q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Then $Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

Example 1.25. Let $A = \begin{pmatrix} 1 & -4 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Then $\det(A - tI_3) = (1-t)(-1-t)(2-t)$. Hence, there are three eigenvalues of A, namely $1, -1, 2$. Thus, A is diagonalizable.

To find an eigenvector corresponding to 1, we solve $(A - I_3)v = 0$ and so an eigenevector is $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$.

To find an eigenvector corresponding to -1, we solve $(A + I_3)v = 0$, that is $\begin{pmatrix} 2 & -4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} v = 0$. Thus $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}^T$ is an eigenvector.

To find an eigenvector corresponding to 2, we solve $(A - 2I_3)v = 0$, that is $\begin{pmatrix} -1 & -4 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} v = 0$. Then $\begin{pmatrix} 3 & 0 & 1 \end{pmatrix}^T$ is an eigenvector.

Let $Q = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

1.5. Algebraic and geometric multiplicities.

Definition 1.26. Let A be an $n \times n$ matrix. Let λ be an eigenvalue of A. The **algebraic multiplicity** of λ is the largest integer k such that $(t - \lambda)^k$ divides $\det(A - tI_n)$.

Example 1.27. Find the algebraic multiplicity of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Sol: The characteristics polynomial of A is

$$\det(A - tI_t) = \det \begin{pmatrix} 1-t & 2 & 3 & 4 \\ 0 & 2-t & 0 & 3 \\ 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & -1-t \end{pmatrix} = (1-t)(2-t)(1-t)(-1-t).$$

The algebraic multiplicity of 1 is 2, and both the algebraic multiplicity of 2 and the algebraic multiplicity of -1 are 1.

Exercise 1.28. Let A be an $n \times n$ matrix. Let λ be an eigenvalue of A.

(a) What is the maximum possible algebraic multiplicity of λ in general? What is the minimum possible algebraic multiplicity of λ in general?
(b) Suppose we know that A has exactly three distinct eigenvalues. What is the maximum possible algebraic multiplicity of λ?

Sol: For (a), the maximum possible algebraic multiplicity of λ is n since the characteristics polynomial has degree n. For example, take $A = I_n$ and the algebraic multiplicity of 1 is n. The minimum possible algebraic multiplicity of λ is 1.
For (b), the maximum possible algebraic multiplity of λ is $n-2$.

On the other hand, we have:

Definition 1.29. Let A be an $n \times n$ matrix. Let λ be an eigenvalue of A. Define

$$E_\lambda = \{v \in \mathbb{R}^n : Av = \lambda v\}.$$

The set is called the **eigenspace** corresponding to the eigenvalue λ. (Check that E_λ is a vectors subspace of $\mathbb{R}^n$.) The dimension of E_λ is called the **geometric multiplicity** of λ. The geometric multiplicity of an eigenvalue λ is non-zero i.e. at least one (why? by definition of an eigenvalue).

Exercise 1.30. Describe a method to find the geometric multiplicity of λ using a reduced row echelon form.
Sol: Let λ be an eigenvalue of a matrix A. Note that

$$E_\lambda = \{v \in \mathbb{R}^n : (A - \lambda I_n)v = 0\}$$

Then one finds the number of free variables in the reduced row echelon form of $A - \lambda I_n$. That number is equal to $\dim E_\lambda$.

Theorem 1.31. *Let A be an $n \times n$ matrix. Then the geometric multiplicity of λ is less than or equal to the algebraic multiplicity of λ.*

We shall not go deep to the proof of Theorem 1.31.

Theorem 1.32. *Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if the algebraic multiplicity of λ is equal to the geometric multiplicity of λ for each eigenvalue λ.*

Proof. We only sketch the main idea for proving the only if direction. Suppose A is diagonalizable. Then we can find a basis for $\mathbb{R}^n$ whose vectors are eigenvectors of A. Since the sum of all algebraic multiplicities are equal to n, we must need to have the geometric multiplicity of λ equal to the algebraic multiplicity in order to achieve such basis. □

Example 1.33. (c.f. Example 1.20) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that A is not diagonalizable by Theorem 1.32.

Sol: The characteristics polynomial of A is $\det(A - tI_2) = (1-t)^2$ and so the algebraic multiplicity of 1 is 2. Moreover, $E_1 = \{v : (A - I_2)v = 0\}$ and so we have to solve:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v = 0.$$

Hence $E_1 = \left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$. Hence, the geometric multiplicity of 1 is 1. Theorem 1.32 then implies A is not diagonalizable.

Exercise 1.34. Let A be an $n \times n$ matrix. If A has distinct eigenvalues λ_1 and λ_2. Suppose $\dim E_{\lambda_1} = p$ and $\dim E_{\lambda_2} = n - p$. Show that A is diagonalizable.

Sol: Let m_1 and m_2 be the algebraic multiplicities of λ_1 and λ_2 respectively. We have the following three equations:

$$1 \leq \dim\, E_{\lambda_1} \leq m_1,$$

$$1 \leq \dim\, E_{\lambda_2} \leq m_2,$$

and $m_1 + m_2 = n$. Hence,

$$n = \dim\, E_{\lambda_1} + \dim\, E_{\lambda_2} \leq m_1 + m_2 \leq n$$

This forces the inequalites are equalities and so

$$\dim\, E_{\lambda_1} = m_1, \quad \dim\, E_{\lambda_2} = m_2.$$

By Theorem 1.32, A is diagonalizable.

We give another solution using Theorem 1.10.

Sol: We can find a basis $\{v_1, \ldots, v_p\}$ for E_{λ_1} and a basis $\{v'_1, \ldots, v'_{n-p}\}$ for E_{λ_2}. Then, by the definition of an eigenespace, $v_1, \ldots, v_p, v'_1, \ldots, v'_{n-p}$ are eigenvectors.

We now show that $v_1, \ldots, v_p, v'_1, \ldots, v'_{n-p}$ are linearly independent. Indeed, we consider

$$a_1v_1 + \ldots + a_pv_p + b_1v'_1 + \ldots + b_{n-p}v'_{n-p} = 0$$

for some $a_1, \ldots, a_p, b_1, \ldots, b_{n-p} \in \mathbb{R}$. Then,

$$a_1v_1 + \ldots + a_pv_p = -b_1v'_1 - \ldots - b_{n-p}v'_{n-p}$$

Now, by multiplying A on both sides,

$$A(a_1v_1 + \ldots + a_pv_p) = A(-b_1v'_1 - \ldots - b_{n-p}v'_{n-p}).$$

Using eigenvalues gives that

$$\lambda_1(a_1v_1 + \ldots + a_pv_p) = \lambda_2(-b_1v'_1 - \ldots - b_{n-p}v'_{n-p}).$$

Combining equations, we have:

$$\lambda_1(a_1v_1 + \ldots a_pv_p) = \lambda_2(a_1v_1 + \ldots + a_pv_p).$$

Since $\lambda_1 \neq \lambda_2$, we must have:

$$a_1v_1 + \ldots + a_pv_p = 0.$$

By the linear independence, we have $a_1 = \ldots = a_p = 0$. Similarly,

$$-b_1v'_1 - \ldots - b_{n-p}v'_{n-p} = 0.$$

The linear independence again gives $b_1 = \ldots = b_{n-p} = 0$. This shows that $v_1, \ldots, v_p, v'_1, \ldots, v'_{n-p}$ are linearly independent.

Hence, $\left\{v_1, \ldots, v_p, v'_1, \ldots, v'_{n-p}\right\}$ forms a basis for $\mathbb{R}^n$. By Theorem 1.10, A is diagonalizable.

1.6. **Some applications.** Let A be an $n \times n$-matrix. One may ask if there is an efficient way to compute A^{100}? For example, if one has a diagonal matrix:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$D^{100} = \begin{pmatrix} 2^{100} & 0 \\ 0 & (-1)^{100} \end{pmatrix}.$$

In general, if one has a diagonalizable matrix A, one can first diagonalize to get a diagonal matrix $D = Q^{-1}AQ$ for some invertible matrix Q. Then, one takes $D^{100} = Q^{-1}A^{100}Q$. This then gives $A^{100} = QD^{100}Q^{-1}$.

Example 1.35. Let

$$A = \begin{pmatrix} 8 & -6 \\ 9 & -7 \end{pmatrix}.$$

Find A^{50}.

Sol: We first find the eigenvalues of A. The characteristics polynomial of A is

$$\det(A - xI_2) = \det\begin{pmatrix} 8-x & -6 \\ 9 & -7-x \end{pmatrix} = (8-x)(-7-x) + 54 = x^2 - x - 2.$$

Hence, the eigenvalues are 2 and -1.

We now find an eigenvector for 2: Solve

$$(A - 2I_2)x = 0$$

and so

$$\begin{pmatrix} 6 & -6 \\ 9 & -9 \end{pmatrix} x = 0.$$

Then $(1,1)^T$ is an eigenvector.

We now find an eigenvector for -1: Solve

$$(A + I_2)x = 0$$

and so

$$\begin{pmatrix} 9 & -6 \\ 9 & -6 \end{pmatrix} x = 0.$$

Then $(2,3)^T$ is an eigenvector.

Let $Q = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. Then

$$Q^{-1}AQ = \mathrm{diag}(2, -1)$$

and

$$A = Q\mathrm{diag}(2, -1)Q^{-1}.$$

Then $A^{50} = Q\mathrm{diag}(2^{50}, 1)Q^{-1}$. Thus

$$A^{50} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2^{50} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3\cdot 2^{50} - 2 & -2\cdot 2^{50} + 2 \\ 3\cdot 2^{50} - 3 & -2\cdot 2^{50} + 3 \end{pmatrix}$$

We now consider another problem. For a $n \times n$ matrix A, define

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \dots$$

If A is a 1×1 matrix i.e. A is a number, then e^A is the usual exponential function.

Example 1.36. Let $A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$. Compute e^A.

Sol: Then $e^A = \begin{pmatrix} e^3 & 0 \\ 0 & e^2 \end{pmatrix}$.

Example 1.37. Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. Compute e^A.

Sol: We first need to diagonalize A.

- Eigenvalues: Solve $\det(A - tI_2) = 0$. Note
$$\det(A - tI_2) = \det \begin{pmatrix} 1-t & 2 \\ -1 & 4-t \end{pmatrix} = (1-t)(4-t) + 2 = t^2 - 5t + 6.$$
Solving $t^2 - 5t + 5 = 0$ gives $t = 2$ or 3, and so 2 and 3 are the eigenvalues of A.
- Eigenvectors: For $\lambda = 2$, solve the system
$$(A - 2I_2)x = 0.$$
This gives
$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} x = 0$$
and so an eigenvector is $(2,1)^T$.
For $\lambda = 3$, solve the system
$$(A - 3I_2)x = 0.$$
This gives
$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} x = 0$$
and so an eigenvector is $(1,1)^T$.
Let $Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then
$$Q^{-1}AQ = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$
and so
$$A = Q \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} Q^{-1}.$$
Let $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. We then have that:
$$e^A = Qe^DQ^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2e^2 - e^3 & -2e^2 + 2e^3 \\ e^2 - e^3 & -e^2 + 2e^3 \end{pmatrix}.$$