MATH 2101 Linear Algebra I–System of Linear Equations I

System of linear equations

A **system of n linear equations in m variables** is a collection of equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nm}x_m &= b_n
\end{aligned}
$$

We transform to a matrix form so that we can use techniques of matrices. Let

$$
A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{m2} & \ldots & a_{nm} \end{pmatrix}
$$

Then, we can rewrite as a matrix equation:

$$
A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{1}
$$

We also refer such matrix equation to be a system of n linear equations in m-variables. A **solution** to (1) is a column vector s in $\mathbb{R}^n$ such that $As = b$.

Method of using matrix inversions

Suppose now $m = n$ and A is invertible. Then, we multiply A^{-1} on both sides:

$$A^{-1}A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A^{-1}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A^{-1}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

i.e. $A^{-1}b$ is the solution to the system of linear equations.

Example of using matrix inversions

Example

Solve the following system of linear equations by using A^{-1}:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 2 \\ 2x_1 - 2x_2 + x_3 &= 0 \end{aligned}$$

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix}$. Then we have

$$\det(A) = 9, \quad \operatorname{adj}(A) = \begin{pmatrix} 3 & 1 & -4 \\ 0 & 3 & 6 \\ 3 & -2 & -1 \end{pmatrix}^T.$$

Hence, $A^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 0 & 3 \\ 1 & 3 & -2 \\ -4 & 6 & -1 \end{pmatrix}$. Hence, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{15}{9} \\ \frac{11}{9} \\ -\frac{8}{9} \end{pmatrix}$.

A variation of matrix inversion method: Carmer's rule

Carmer's rule is a simpler formula that lighten some computations in matrix inversion:

Theorem

Let A be an $n \times n$ invertible matrix and let b be a column vector in $\mathbb{R}^n$. Consider a system of n linear equations in n variables:

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b.$$

Then the solution to the system is given by:

$$x_{i} = \frac{\det(A_{i})}{\det(A)},$$

where A_i is the matrix obtained by replacing the i-th column by b.

Proof of Carmer's rule

Theorem

The solution to the system $Ax = b$ is given by:

$$x_i = \frac{\det(A_i)}{\det(A)},$$

where A_i is the matrix obtained by replacing the i-th column by b.

Proof.

To use the formula:

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A),$$

we reduce computing $A^{-1}b$ to computing $\mathrm{adj}(A)b$. Indeed,

$$(\mathrm{adj}(A)b)_{i1} = (-1)^{i+1}b_1\det(\widetilde{A}_{1i}) + \ldots + (-1)^{i+n}b_n\det(\widetilde{A}_{ni}) = \det(A_i),$$

where the last equality follows from computing $\det(A_i)$ by using the i-th column in the definition.

Example of using Carmer's rule

Example

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and let $b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Solve $Ax = b$.
Solution: $\det(A) = -3$. We also have:

$$\det \begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix} = -3, \det \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} = -3$$

Hence, by Carmer's rule,

$$x = \frac{1}{-3} \begin{pmatrix} -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Remark on the method of matrix inverson or Carmer's rule

Remark

The matrix inversion or Carmer's rule to solve a system of linear equation only works when the matrix is invertible. For example, if $n \neq m$ for a $n \times m$ matrix A, then we cannot solve the equation by above method.

Elementary row operations: motivation

In order to deal with general system of linear equations, we have to develop some operations on and matrices. A basic idea is like to subtract system of linear equations to get less variables in one equation e.g. Solve

$$x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 2$$

We subtract the second eqn. from the first one to get $x_2 = 1$ and then get $x_1 = 0$.

Elementary row operations

We shall carry out those operations on matrices:

Definition

Let A be an $n \times m$ matrix. Any one of the following three operations on rows (resp. columns) is called an **elementry row operation**:

1. Type (I) interchanging any two rows (resp. columns) of A;
2. Type (II) multiplying any row (resp. column) of A by a non-zero scalar;
3. Type (III) adding a scalar multiple of a row (resp. column) of A to another row of A.

Examples of elementary row operations

Example

Let $A = \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 7 & 1 & 2 \\ 9 & 2 & 0 & 4 \end{pmatrix}$.

1. (Type I) interchanging the first and third rows gives:

$$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 7 & 1 & 2 \\ 9 & 2 & 0 & 4 \end{pmatrix} \overset{1\leftrightarrow 3}{\rightarrow} \begin{pmatrix} 9 & 2 & 0 & 4 \\ 0 & 7 & 1 & 2 \\ 1 & 2 & -1 & 5 \end{pmatrix}$$

2. (Type II) multiplying the second row by $\sqrt{2}$ gives:

$$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 7 & 1 & 2 \\ 9 & 2 & 0 & 4 \end{pmatrix} \overset{2\times(\sqrt{2})}{\rightarrow} \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 7\sqrt{2} & \sqrt{2} & 2\sqrt{2} \\ 9 & 2 & 0 & 4 \end{pmatrix}$$

Examples of elementary row operations

Example

(Type III) adding the scalar multiple 2 of the first row to the third row gives:

$$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 7 & 1 & 2 \\ 9 & 2 & 0 & 4 \end{pmatrix} \overset{1\times(2)+3}{\longrightarrow} \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 7 & 1 & 2 \\ 11 & 6 & -2 & 14 \end{pmatrix}$$

Elementary matrices

We would like to convert those elementary row operations to *matrix multiplications* so that again we can use techniques of matrices. We first describe the matrices that we need.

Definition

An $n \times n$ **elementary matrix** is a matrix obtained by performing an elementary operation on I_n. An elementary matrix is called of type I, II or III according to its corresponding elementary operation.

Example

1. $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is an elementary matrix performing the interchanging the first and third row. (Type I)
2. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is an elementary matrix performing the scalar multiplication of 2 on the second row (Type II).
3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ is an elementary matrix performing the addition of the scalar multiple -2 of the first row to the third row (Type III).

Non-example of elementary matrices

Example

(Non-example) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ are not elementary.

Exercise

Write down the elementary matrices for the following elementary row operations on 4×4 matrices:

- Interchanging the first and fourth rows.
- Adding the second row to the third row.
- Scalar multiplication of 2 on the first row.

Matrix multiplications of elementary matrix= elementary row operations

Theorem

Let A be an $n \times m$ matrix.

1. *Suppose E is an elementary matrix. Then EA is the matrix obtained from A by the corresponding elementary row operation.*
2. *Conversely, if B is a matrix obtained from A by an elementary row operation, then $B = EA$ for the elementary matrix corresponding to that operation.*

Example

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4(-2) & 5(-2) & 6(-2) \end{pmatrix}.$$

(LHS: matrix multiplication EA. RHS: row operations)

Example

Let $E = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, which corresponds to the row elementary operation for adding the scalar multiple 2 of the second row to the first row. Let $A = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$. Then

$$EA = \begin{pmatrix} u_1 + 2v_1 & u_2 + 2v_2 & u_3 + 2v_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

Invertibility of elementary matrices

Theorem

Elementary matrices are invertible. Moreover, the inverse of an elementary matrix is still an elementary matrix of the same type.

Proof.

We just illustrate the idea of the proof. Finding an inverse is the same as finding an elementary operation to get back to the original matrix.

- In type I, if the elementary operation interchanges the k-th row and the l-th row, then applying the same elementary operation once more (i.e. interchanging the k-th and the l-th rows again) gives back to the original matrix.
- In type II, if the elementary operation multiplies the k-th row by a scalar a, then the elementary operation multiplying the k-th row by the scalar $\frac{1}{a}$ gives back to the original matrix.
- In type III, if the elementary operation adds the scalar multiple a of the k-th row to the l-th row, then the elementary operation for adding the scalar multiple $-a$ of the k-th row to the l-th row gives back to the original matrix.

General form of elementary matrices

We have the following general description of elementary matrices:

Remark

Elementary matrices E for each type take the following form:

- (Type I) Interchanging k-th and l-th rows: the diagonal entries $E_{ii} = 1$ for $i \neq k, l$ and $E_{ii} = 0$ for $i = k$ or l; $E_{kl} = E_{lk} = 1$; and all other entries are zero.
- (Type II) Scalar multiplication a on the k-th row: $E_{ii} = 1$ for $i \neq k$ and $E_{kk} = a$ and all other entries are zero.
- (Type III) Adding a scalar multiple a of the k-th row to the l-th row: all the diagonal entries are 1 i.e. $E_{ii} = 1$ for any i, and $E_{lk} = a$, and all other entries are zero.

Operations on elementary matrices

Exercise

Determine if the followings still give an elementary matrix:

- Addition of two elementary matrices
- Multiplication of two elementary matrices
- Transpose of an elementary matrix