2 Differentiation in Several Variables

2.1 Multivariable Functions

Definition 2.1. For a function $f : X \to Y$, the set X is called the *domain* of f and Y is called the *codomain* of f. The *range* of f is the set $\{f(x) : x \in X\}$ of all images.

In this course, we study functions where the domain is a subset of some Euclidean space $\mathbb{R}^n$ and the codomain is another subset of a Euclidean space $\mathbb{R}^m$, possibly $m = n$.

Definition 2.2. Let $f : X \to Y$ and $g : Y \to Z$ be functions. The *composition* of f and g is the function $g \circ f$ from X to Z such that

$$(g \circ f)(x) = g(f(x))$$

for all $x \in X$.

Example 2.1. The functions $\mathbf{f}, \mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$\mathbf{f}(x, y) = (2x, 2y) \quad \text{and} \quad \mathbf{g}(x, y) = (x + 1, y + 1).$$

Then $(\mathbf{f} \circ \mathbf{g})(x, y) = (2x + 2, 2y + 2)$ and $(\mathbf{g} \circ \mathbf{f})(x, y) = (2x + 1, 2y + 1)$.

Definition 2.3. A function $f : X \to Y$ is *injective* or *one-to-one* if for all distinct elements $a, b \in X$, we have $f(a) \neq f(b)$.

This means the images of different elements under an injective function are different.

Definition 2.4. A function $f : X \to Y$ is *surjective* or *onto* if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

This means the range of a surjective function f is the same as the codomain.

Definition 2.5. A function $f : X \to Y$ is *bijective* or *one-to-one correspondence* if it is both injective and surjective.

The importance of a bijective function f is that it has an *inverse function*, which means a function g (or commonly denoted by f^{-1}) such that both $f \circ g$ and $g \circ f$ are identity functions.

Example 2.2. The function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{f}(x,y) = (x+y, x-y)$ is bijective. Its inverse $\mathbf{f}^{-1}$ satisfies $\mathbf{f}^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$.

Example 2.3. The function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(\mathbf{v}) = \|\mathbf{v}\|$ is not injective or surjective.

Consider a *vector-valued function* $\mathbf{f} : X \to \mathbb{R}^m$ where $X \subset \mathbb{R}^n$. As $\mathbf{f}(\mathbf{v})$ has m coordinates for any $\mathbf{v} \in X$, we can write $\mathbf{f}(\mathbf{v}) = (f_1(\mathbf{v}), f_2(\mathbf{v}), \ldots, f_m(\mathbf{v}))$ for some functions $f_1, f_2, \ldots, f_m$ where $f_j : X \to \mathbb{R}$. The functions $f_1, f_2, \ldots, f_m$ are called the *component functions* of $\mathbf{f}$. For this reason, we first focus on functions of the form $g : X \to \mathbb{R}$. Such a function is called a *scalar-valued function*.

Example 2.4. Let f_1, f_2, f_3 be the component functions of $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\mathbf{f}(x,y) = (x^2+y^2, 2xy, 1)$. Then

$$f_1(x,y) = x^2 + y^2, \quad f_2(x,y) = 2xy, \quad f_3(x,y) = 1.$$

We now restrict our discussion to a function $f : X \to \mathbb{R}$ where $X \subset \mathbb{R}^2$.

Definition 2.6. The *graph* of a function $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^2$, is the set

$$\{(x, y, f(x,y)) \in \mathbb{R}^3 : (x,y) \in X\}.$$

We can visualize the graph of a function by plotting all the points $(x, y, f(x,y))$ in the three-dimensional space. The height at (x,y) is the value of the image of (x,y) under f.

Definition 2.7. The *contour curve* at height c of a function $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^2$, is the set

$$\{(x, y, f(x,y)) \in \mathbb{R}^3 : (x,y) \in X,\ f(x,y) = c\}.$$

The *level curve* at height c of f is the set

$$\{(x,y) \in \mathbb{R}^2 : (x,y) \in X,\ f(x,y) = c\}.$$

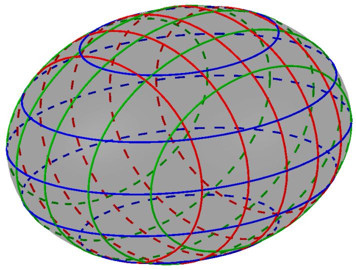
In other words, a contour curve is the intersection of the graph of f with a certain horizontal plane $z = c$, while a level curve is the projection of a contour curve onto the xy-plane. We usually only use these terminologies when the above sets really represent some curves (defined in section 5). By drawing level curves and contour curves of a function, we have more ideas on how the graph of the function looks like.

Example 2.5. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = 100 - x^2 - y^2$. Its level curves are circles.

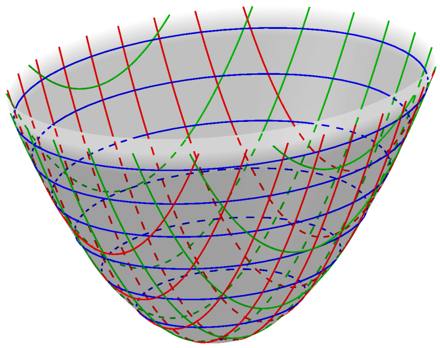
Definition 2.8. A *quadric surface* is a surface in $\mathbb{R}^3$ defined by an equation of the form

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + J = 0.$$

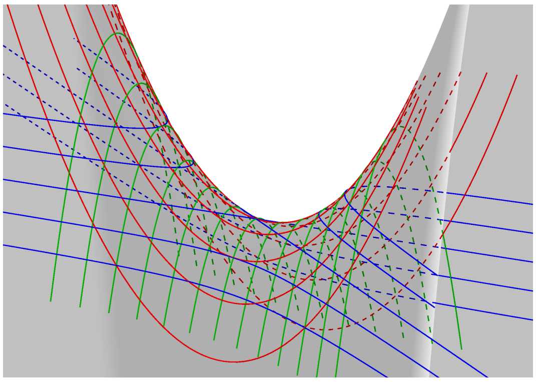
Example 2.6. The quadric surface with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is called an *ellipsoid*. In particular, when $a = b = c$, it is a sphere.



Example 2.7. The quadric surface with equation $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is called an *elliptic paraboloid*.

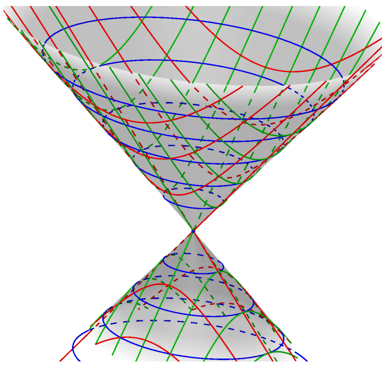


Example 2.8. The quadric surface with equation $\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ is called a *hyperbolic paraboloid*.

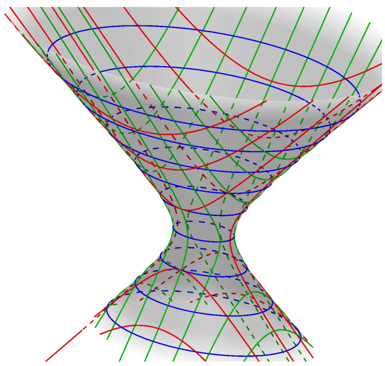


Both elliptic paraboloid and hyperbolic paraboloid are graphs of some functions.

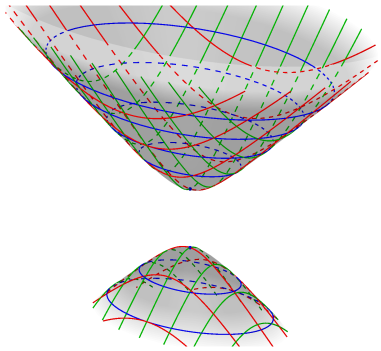
Example 2.9. The quadric surface with equation $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is called an *elliptic cone*.



Example 2.10. The quadric surface with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is called a *hyperboloid of one sheet*.



Example 2.11. The quadric surface with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ is called a *hyperboloid of two sheets*.



2.2 Limits and Continuity

Definition 2.9. For any $\mathbf{a} \in \mathbb{R}^n$ and $r > 0$, the *open ball* in $\mathbb{R}^n$ with centre $\mathbf{a}$ and radius r is the set

$$B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < r\}.$$

The *closed ball* in $\mathbb{R}^n$ with centre $\mathbf{a}$ and radius r is the set

$$\overline{B}(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| \leqslant r\}.$$

Definition 2.10. Let X be a subset of $\mathbb{R}^n$. A point $\mathbf{a} \in X$ is called an *interior point* of X if there exists $r > 0$ such that $B(\mathbf{a}, r) \subset X$. The *interior* of X is the set of all interior points of X.

Definition 2.11. Let X be a subset of $\mathbb{R}^n$. A point $\mathbf{a} \in \mathbb{R}^n$ is called a *boundary point* of X if every open ball with centre $\mathbf{a}$ contains some points in X as well as some points not in X. The *boundary* of X, denoted by ∂X, is the set of all boundary points of X.

Note that an interior point of X must belong to X, while a boundary point of X needs not belong to X.

Definition 2.12. A subset X of $\mathbb{R}^n$ is called an *open set* in $\mathbb{R}^n$ if every point in X is an interior point of X.

Definition 2.13. A subset X of $\mathbb{R}^n$ is called a *closed set* in $\mathbb{R}^n$ if the complement $\mathbb{R}^n - X$ is an open set in $\mathbb{R}^n$.

Proposition 2.1. A subset X of $\mathbb{R}^n$ is closed if and only if it contains all its boundary points.

Example 2.12. Consider the following subsets of $\mathbb{R}^2$.

- $\varnothing$ and $\mathbb{R}^2$ are both open and closed
- $\{(x,y) \in \mathbb{R}^2 : x + y < 1\}$ is open but not closed
- $\{(x,y) \in \mathbb{R}^2 : xy = 1\}$ is closed but not open
- $\{(x,y) \in \mathbb{R}^2 : 0 \leqslant x < 1,\ 0 < y \leqslant 1\}$ is neither open nor closed

Definition 2.14. Let X be a subset of $\mathbb{R}^n$. We say that $\mathbf{a} \in \mathbb{R}^n$ is a *limit point* (or *accumulation point*) of X if the following holds. For any $\delta > 0$, there exists $\mathbf{x}$ in the set $X \cap B(\mathbf{a}, \delta)$ which is different from $\mathbf{a}$.

Loosely speaking, $\mathbf{a}$ is a limit point of X if there are many points in X which are sufficiently close to $\mathbf{a}$. Note that $\mathbf{a}$ needs not belong to X. If $\mathbf{a} \notin X$, then $\mathbf{a}$ must belong to the boundary of X. On the other hand, a point $\mathbf{a} \in X$ is said to be an *isolated point* of X if it is not a limit point. Equivalently, this means there exists $\delta > 0$ such that $B(\mathbf{a}, \delta) \cap X = \{\mathbf{a}\}$.

Definition 2.15. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function where $X \subset \mathbb{R}^n$. Let $\mathbf{a}$ be a limit point of X. We say that the *limit* of $\mathbf{f}$ at $\mathbf{a}$ is $\mathbf{L} \in \mathbb{R}^m$ if the following holds. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{x} \in X$ and $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, then $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon$.

In that case, we write
$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}.$$

Under our new terminologies, the condition means that for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\mathbf{x} \in X$, $\mathbf{x} \in B(\mathbf{a}, \delta)$ and $\mathbf{x} \neq \mathbf{a}$, we have $\mathbf{f}(\mathbf{x}) \in B(\mathbf{L}, \varepsilon)$.

Proposition 2.2. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function where $X \subset \mathbb{R}^n$. Let $\mathbf{a}$ be a limit point of X. If the limit of $\mathbf{f}$ at $\mathbf{a}$ exists, then this limit is uniquely determined.

Proposition 2.3. Let $\mathbf{f}, \mathbf{g} : X \to \mathbb{R}^m$ be functions where $X \subset \mathbb{R}^n$. Let $\mathbf{a}$ be a limit point of X. Suppose

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{M}.$$

Then the following hold.

(a) $\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f} \pm \mathbf{g})(\mathbf{x}) = \mathbf{L} \pm \mathbf{M}$

(b) $\lim_{\mathbf{x}\to\mathbf{a}} (c\mathbf{f})(\mathbf{x}) = c\mathbf{L}$ for any $c \in \mathbb{R}$

(c) if $m = 1$, then $\lim_{\mathbf{x}\to\mathbf{a}} (fg)(\mathbf{x}) = LM$

(d) if $m = 1$ and $M \neq 0$, then $\lim_{\mathbf{x}\to\mathbf{a}} \left(\frac{f}{g}\right)(\mathbf{x}) = \frac{L}{M}$

Example 2.13. We have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 + xy + 2}{x + y + 1} = \frac{0 + 0 + 2}{0 + 0 + 1} = 2.$

Here are two usual ways to show that the limit of a multivariable function does not exist.

- Consider the restriction of the function on a certain curve, and show that the limit does not exist.
- Consider the restriction of the function on two different curves, and show that the limits are different.

Example 2.14. The limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x + y^3}{x^3 + y^2}$ does not exist.

Example 2.15. The limit $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2 + y^2 - 3z^2}{x^2 + y^2 + z^2}$ does not exist.

Theorem 2.1. (Sandwich theorem) Let $f, g, h : X \to \mathbb{R}$ be three functions where $X \subset \mathbb{R}^n$. Let $\mathbf{a}$ be a limit point of X. Suppose $f(\mathbf{x}) \leqslant g(\mathbf{x}) \leqslant h(\mathbf{x})$ for all $\mathbf{x} \in X$. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = L$, then $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = L$.

Example 2.16. We have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r\to 0} \frac{(r\cos\theta)^3 + (r\sin\theta)^3}{(r\cos\theta)^2 + (r\sin\theta)^2} = 0.$

Proposition 2.4. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function and let $\mathbf{f} = (f_1, f_2, \ldots, f_m)$, where $X \subset \mathbb{R}^n$. Let $\mathbf{a}$ be a limit point of X. Then

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = (L_1, L_2, \ldots, L_m)$$

if and only if

$$\lim_{\mathbf{x}\to\mathbf{a}} f_j(\mathbf{x}) = L_j \text{ for } j = 1, 2, \ldots, m.$$

Proposition 2.5. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function where $X \subset \mathbb{R}^n$. Let $\mathbf{g} : Y \to \mathbb{R}^k$ be a function where $\mathbf{f}(X) \subset Y \subset \mathbb{R}^m$. Let $\mathbf{a}$ be a limit point of X. If

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \in Y, \quad \lim_{\mathbf{y}\to\mathbf{L}} \mathbf{g}(\mathbf{y}) = \mathbf{M} \quad \text{and} \quad \mathbf{g}(\mathbf{L}) = \mathbf{M},$$

then

$$\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{M}.$$

Example 2.17. We have $\displaystyle\lim_{(x,y)\to(0,0)} (e^{xy}, \cos \sqrt[3]{x+y}) = (1, 1)$.

Definition 2.16. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function where $X \subset \mathbb{R}^n$. We say that $\mathbf{f}$ is *continuous* at $\mathbf{a} \in X$ if $\mathbf{a}$ is an isolated point of X, or $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$. Otherwise, it is said to be *discontinuous* at a.

We say that $\mathbf{f}$ is a *continuous function* if it is continuous at every point in X. Otherwise, it is called a *discontinuous function*.

We only talk about the continuity at a point in the domain. In most cases, the domain has no isolated points, so we mainly wish to see if $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$ holds or not. Loosely speaking, the graph of a continuous function has no gaps, given that the domain is nice.

Example 2.18. Define

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{x + y} & \text{if } x + y \neq 0, \\ 0 & \text{if } x = y = 0. \end{cases}$$

Then f is discontinuous at $(0, 0)$ since $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x + y}$ does not exist.

Proposition 2.6. Let $\mathbf{f}, \mathbf{g} : X \to \mathbb{R}^m$ be functions where $X \subset \mathbb{R}^n$. Suppose $\mathbf{f}$ and $\mathbf{g}$ are continuous at $\mathbf{a} \in X$. Then the following hold.

(a) $\mathbf{f} \pm \mathbf{g}$ is continuous at $\mathbf{a}$

(b) $c\mathbf{f}$ is continuous at $\mathbf{a}$ for any $c \in \mathbb{R}$

(c) if $m = 1$, then fg is continuous at $\mathbf{a}$

(d) if $m = 1$ and $g(\mathbf{a}) \neq 0$, then $\dfrac{f}{g}$ is continuous at $\mathbf{a}$

Proposition 2.7. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function and let $\mathbf{f} = (f_1, f_2, \ldots, f_m)$, where $X \subset \mathbb{R}^n$. Then $\mathbf{f}$ is continuous at $\mathbf{a} \in X$ if and only if each f_j is continuous at $\mathbf{a}$.

Proposition 2.8. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function where $X \subset \mathbb{R}^n$. Let $\mathbf{g} : Y \to \mathbb{R}^k$ be a function where $\mathbf{f}(X) \subset Y \subset \mathbb{R}^m$. If $\mathbf{f}$ and $\mathbf{g}$ are continuous, then so is $\mathbf{g} \circ \mathbf{f}$.

Example 2.19. All the functions

$$f(x,y) = \frac{(xy+1)^2}{x^2+y^2+1}, \quad g(\mathbf{x}) = \|\mathbf{x}\| \quad \text{and} \quad \mathbf{h}(x,y,z) = (xy, 2e^{yz}, 1)$$

are continuous.

2.3 Differentiation

Definition 2.17. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, and let $\mathbf{a}$ be an interior point of X. The *partial derivative* of f at $\mathbf{a}$ with respect to a variable x_j is defined as

$$\frac{\partial f}{\partial x_j}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{e}_j) - f(\mathbf{a})}{h}.$$

We also use the notations $f_{x_j}(\mathbf{a})$ and $D_{x_j} f(\mathbf{a})$.

This means we take the derivative of f with respect to x_j by regarding all other variables as constants.

Example 2.20. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = 2x^2yz - \sin(y^2+z^2) + e^{2x-z}$. Then

$$\begin{aligned} f_x(x,y,z) &= 4xyz + 2e^{2x-z}, \\ f_y(x,y,z) &= 2x^2z - 2y\cos(y^2+z^2), \text{ and} \\ f_z(x,y,z) &= 2x^2y - 2z\cos(y^2+z^2) - e^{2x-z}. \end{aligned}$$

Example 2.21. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^4 + 2y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

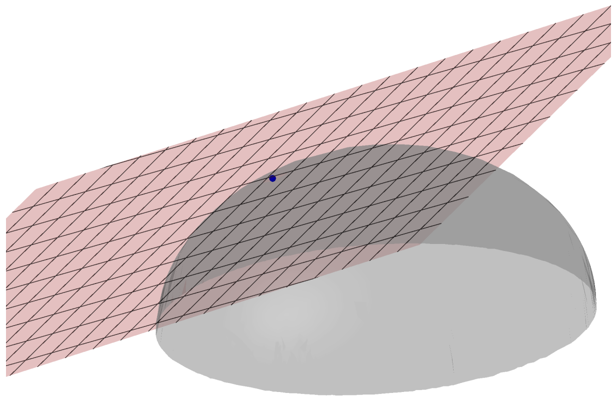
Then

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{2x(x^4 + 2x^2y^2 - 2y^3)}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

and

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{2y(-x^4 + 3x^2y + y^3)}{(x^2 + y^2)^2} & \text{if } (x,y) \neq (0,0), \\ 2 & \text{if } (x,y) = (0,0). \end{cases}$$

Definition 2.18. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^2$, ane let (a,b) be an interior point of X. The *tangent plane* at $\mathbf{x} = (a,b,f(a,b))$ to the graph S of f is the plane that contains the tangent lines at $\mathbf{x}$ to all curves on S passing through $\mathbf{x}$.



Similar to tangent lines to the graph of a one-variable function, the tangent plane at $(a,b,f(a,b))$ gives a good approximation to $f(x,y)$ when (x,y) is close to (a,b). Note that the tangent plane needs not exist.

Proposition 2.9. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^2$, ane let (a,b) be an interior point of X. Suppose the partial derivatives f_x and f_y at (a,b) exist. If the graph of f has a tangent plane at $(a,b,f(a,b))$ where $(a,b) \in X$, then the equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

More generally, we can define the tangent plane to the graph of $f : X \to \mathbb{R}$ where $X \subset \mathbb{R}^n$. If such a tangent plane exists, its equation is

$$x_{n+1} = f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + f_{x_2}(\mathbf{a})(x_2 - a_2) + \cdots + f_{x_n}(\mathbf{a})(x_n - a_n).$$

Example 2.22. Define $f : B(\mathbf{0}, \sqrt{3}) \to \mathbb{R}$ by

$$f(x,y) = \sqrt{3 - x^2 - y^2}.$$

Then the tangent plane at $(1,1,1)$ to the graph of f has equation $x + y + z = 3$.

Suppose S is a surface in $\mathbb{R}^3$ represented by $F(x,y,z) = c$ for some constant c. If S has a tangent plane at $\mathbf{a} \in \mathbb{R}^3$, then $(F_x(\mathbf{a}), F_y(\mathbf{a}), F_z(\mathbf{a}))$ is a normal vector to the tangent plane at $\mathbf{a}$.

Example 2.23. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \big||x| - |y|\big| - |x| - |y|.$$

Then the graph of f has no tangent plane at $(0,0,0)$.

Definition 2.19. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$ is open. Suppose $f_{x_j}(\mathbf{a})$ exists for $j = 1, 2, \ldots, n$ where $\mathbf{a} \in X$. Define

$$h(\mathbf{x}) = f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + f_{x_2}(\mathbf{a})(x_2 - a_2) + \cdots + f_{x_n}(\mathbf{a})(x_n - a_n).$$

We say that f is *differentiable* at $\mathbf{a}$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{f(\mathbf{x}) - h(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|} = 0.$$

We say that f is *differentiable* if it is differentiable at every point in X.

If any of $f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}), \ldots, f_{x_n}(\mathbf{a})$ does not exist, f is not differentiable at $\mathbf{a}$. It is known that the tangent plane at $\mathbf{a}$ exists if and only if f is differentiable at $\mathbf{a}$.

Example 2.24. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^2 + 2y^2$. Then f is differentiable.

Definition 2.20. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$ is open, and let $\mathbf{a}$ be an interior point of X. Suppose $f_{x_j}(\mathbf{a})$ exists for $j = 1, 2, \ldots, n$. The *gradient* of f at $\mathbf{a}$ is defined by

$$\operatorname{grad} f(\mathbf{a}) = \nabla f(\mathbf{a}) = (f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}), \ldots, f_{x_n}(\mathbf{a})).$$

Using this notation, the limit condition in the definition of differentiability can be rewritten as

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - [f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a})]}{\|\mathbf{x}-\mathbf{a}\|} = 0.$$

Definition 2.21. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$ is open, and let $\mathbf{a}$ be an interior point of X. Suppose $f_{x_j}(\mathbf{a})$ exists for $j = 1, 2, \ldots, n$. The *derivative* of f at $\mathbf{a}$ is the $1 \times n$ matrix defined by

$$Df(\mathbf{a}) = \begin{pmatrix} f_{x_1}(\mathbf{a}) & f_{x_2}(\mathbf{a}) & \cdots & f_{x_n}(\mathbf{a}) \end{pmatrix}.$$

Therefore, the scalar $\nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a})$ can also be rewritten as $Df(\mathbf{a})(\mathbf{x}-\mathbf{a})$.

Theorem 2.2. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, ane let $\mathbf{a}$ be an interior point of X. If f has continuous partial derivatives in an open set containing $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.

Proposition 2.10. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, ane let $\mathbf{a}$ be an interior point of X. If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

Example 2.25. Define $f : X \to \mathbb{R}$ by

$$f(x,y) = \tan^{-1}\frac{y}{x}$$

where $X = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. Then

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2+y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2},$$

so that f is differentiable.

Definition 2.22. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function and let $\mathbf{f} = (f_1, f_2, \ldots, f_m)$, where $X \subset \mathbb{R}^n$ is open. Suppose $\dfrac{\partial f_i}{\partial x_j}$ exists for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The *Jacobian matrix* of $\mathbf{f}$ is the $m \times n$ matrix defined by

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Example 2.26. Define $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(x, y) = (x^2 + y, \sin x \cos y).$$

Then

$$D\mathbf{f}(x, y) = \begin{pmatrix} 2x & 1 \\ \cos x \cos y & -\sin x \sin y \end{pmatrix}.$$

Definition 2.23. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function where $X \subset \mathbb{R}^n$ is open. Suppose $D\mathbf{f}(\mathbf{a})$ exists where $\mathbf{a} \in X$. Define

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

We say that $\mathbf{f}$ is *differentiable* at $\mathbf{a}$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{h}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{a}\|} = 0.$$

We say that $\mathbf{f}$ is *differentiable* if it is differentiable at every point in X.

Proposition 2.11. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function and let $\mathbf{f} = (f_1, f_2, \ldots, f_m)$, where $X \subset \mathbb{R}^n$. Let $\mathbf{a}$ be an interior point of X. Then $\mathbf{f}$ is differentiable at $\mathbf{a}$ if and only if each f_j is differentiable at $\mathbf{a}$.

Theorem 2.3. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function and let $\mathbf{f} = (f_1, f_2, \ldots, f_m)$, where $X \subset \mathbb{R}^n$. Let $\mathbf{a}$ be an interior point of X. If each f_j has continuous partial derivatives in an open set containing $\mathbf{a}$, then $\mathbf{f}$ is differentiable at $\mathbf{a}$.

Proposition 2.12. Let $\mathbf{f} : X \to \mathbb{R}^m$ be a function where $X \subset \mathbb{R}^n$, ane let $\mathbf{a}$ be an interior point of X. If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then $\mathbf{f}$ is continuous at $\mathbf{a}$.

Example 2.27. Define $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(x, y) = (xy, e^{x+y}).$$

Then

$$D\mathbf{f}(x, y) = \begin{pmatrix} y & x \\ e^{x+y} & e^{x+y} \end{pmatrix},$$

and so $\mathbf{f}$ is differentiable.

Proposition 2.13. Let $\mathbf{f}, \mathbf{g} : X \to \mathbb{R}^m$ be functions where $X \subset \mathbb{R}^n$, and let $\mathbf{a}$ be an interior point of X. Suppose $\mathbf{f}$ and $\mathbf{g}$ are differentiable at $\mathbf{a}$. Then the following hold.

(a) $\mathbf{f} + \mathbf{g}$ is differentiable at $\mathbf{a}$ and $D(\mathbf{f} + \mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) + D\mathbf{g}(\mathbf{a})$

(b) $c\mathbf{f}$ is differentiable at $\mathbf{a}$ and $D(c\mathbf{f})(\mathbf{a}) = cD\mathbf{f}(\mathbf{a})$ for any $c \in \mathbb{R}$

(c) if $m = 1$, then fg is differentiable at $\mathbf{a}$ and

$$D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a})$$

(d) if $m = 1$ and $g(\mathbf{a}) \neq 0$, then $\frac{f}{g}$ is differentiable at $\mathbf{a}$ and

$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}$$

Example 2.28. Define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x + y, \quad g(x, y) = e^{x^2+y^2}.$$

Then

$$D(fg) = gDf + fDg = \begin{pmatrix} e^{x^2+y^2} + 2x(x+y)e^{x^2+y^2} & e^{x^2+y^2} + 2y(x+y)e^{x^2+y^2} \end{pmatrix}.$$

We can investigate partial derivatives of higher order. For example, for $f : \mathbb{R}^2 \to \mathbb{R}$, we can consider $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$. In general, for a function $f : X \to \mathbb{R}$ where $X \subset \mathbb{R}^n$, we define

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

as the *second-order partial derivative* with respect to x_i and then x_j. More generally, we can define the *kth-order partial derivative*

$$\frac{\partial^k f}{\partial x_{i_k} \partial x_{i_{k-1}} \cdots \partial x_{i_1}} = f_{x_{i_1} x_{i_2} \cdots x_{i_k}} = \frac{\partial}{\partial x_{i_k}}\left(\frac{\partial}{\partial x_{i_{k-1}}}\left(\cdots\left(\frac{\partial f}{\partial x_{i_1}}\right)\right)\right).$$

Example 2.29. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = 2x^3y^4.$$

Then

$$f_x = 6x^2y^4, \quad f_y = 8x^3y^3$$

and

$$f_{xx} = 12xy^4, \quad f_{xy} = f_{yx} = 24x^2y^3, \quad f_{yy} = 24x^3y^2.$$

Definition 2.24. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$ is open. Let k be a positive integer. We say that f is of *class C^k* if all its partial derivatives of order at most k exist and are continuous.

We say that f is *smooth* or of *class C^∞* if all its partial derivatives of any order exist and are continuous.

For a vector-valued function $\mathbf{f} : X \to \mathbb{R}^m$, the same definitions apply if every component function of $\mathbf{f}$ is of class C^k or C^∞ respectively.

Theorem 2.4. Let $f : X \to \mathbb{R}$ be a function of class C^k where $X \subset \mathbb{R}^n$ is open and k is a positive integer. Suppose $(i_1, i_2, \ldots, i_k)$ is a k-tuple of positive integers between 1 and n, and $(j_1, j_2, \ldots, j_k)$ is a permutation of $(i_1, i_2, \ldots, i_k)$. Then

$$f_{x_{i_1} x_{i_2} \cdots x_{i_k}} = f_{x_{j_1} x_{j_2} \cdots x_{j_k}}.$$

Example 2.30. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = 2x^2 yz + \frac{e^{xy}}{x^2 + y^2 + 1}.$$

Then $f_{xyz}(x, y, z) = f_{zxy}(x, y, z) = 4x$.

2.4 Chain Rule and Directional Derivatives

Theorem 2.5. (Chain rule) Let $f : Y \to \mathbb{R}$ be a function where $Y \subset \mathbb{R}^n$. Let $\mathbf{g} : X \to \mathbb{R}^n$ be a function where $X \subset \mathbb{R}$ and $\mathbf{g}(X) \subset Y$, and define $\mathbf{g} = (x_1, x_2, \ldots, x_n)$. Let t_0 be an interior point of X and $\mathbf{y}_0 = \mathbf{g}(t_0)$ be an interior point of Y. Suppose $\mathbf{g}$ is differentiable at t_0 and f is differentiable at $\mathbf{y}_0$. Then $f \circ \mathbf{g} : X \to \mathbb{R}$ is differentiable at t_0 and

$$\frac{df}{dt}(t_0) = \frac{\partial f}{\partial x_1}(\mathbf{y}_0) \cdot \frac{dx_1}{dt}(t_0) + \frac{\partial f}{\partial x_2}(\mathbf{y}_0) \cdot \frac{dx_2}{dt}(t_0) + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{y}_0) \cdot \frac{dx_n}{dt}(t_0).$$

Note that $\dfrac{df}{dt}$ actually means $\dfrac{dh}{dt}$ where $h = f \circ \mathbf{g}$. The formula can be rewritten as

$$\frac{df}{dt}(t_0) = Df(\mathbf{y}_0) D\mathbf{g}(t_0).$$

Example 2.31. Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$ by

$$f(x, y) = \sin(x + y) + 2x^2 \quad \text{and} \quad \mathbf{g}(t) = (x(t), y(t)) = (2t^2, 1 - t^3).$$

Then

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = (4t - 3t^2)\cos(2t^2 + 1 - t^3) + 32t^3.$$

Example 2.32. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = f(t^2 + 1, 2t - 1).$$

Then $g'(t) = 2tf_x(t^2 + 1, 2t - 1) + 2f_y(t^2 + 1, 2t - 1)$.

Theorem 2.6. (Chain rule) Let $\mathbf{f} : Y \to \mathbb{R}^k$ be a function where $Y \subset \mathbb{R}^n$. Let $\mathbf{g} : X \to \mathbb{R}^n$ be a function where $X \subset \mathbb{R}^m$ and $\mathbf{g}(X) \subset Y$. Let $\mathbf{t}_0$ be an interior point of X and $\mathbf{y}_0 = \mathbf{g}(\mathbf{t}_0)$ be an interior point of Y. Suppose $\mathbf{g}$ is differentiable at $\mathbf{t}_0$ and $\mathbf{f}$ is differentiable at $\mathbf{y}_0$. Then $\mathbf{h} = \mathbf{f} \circ \mathbf{g} : X \to \mathbb{R}^k$ is differentiable at $\mathbf{t}_0$ and

$$D\mathbf{h}(\mathbf{t}_0) = D\mathbf{f}(\mathbf{y}_0)D\mathbf{g}(\mathbf{t}_0).$$

Example 2.33. Define $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\mathbf{f}(x, y) = (x + y, 2x^2), \quad \mathbf{g}(x, y, z) = (x - y^2 + z, 1).$$

Let $\mathbf{h} = \mathbf{f} \circ \mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^2$. Then

$$\begin{aligned} D\mathbf{h}(2, 3, 4) &= D\mathbf{f}(-3, 1)D\mathbf{g}(2, 3, 4) \\ &= \begin{pmatrix} 1 & 1 \\ -12 & 0 \end{pmatrix} \begin{pmatrix} 1 & -6 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -6 & 1 \\ -12 & 72 & -12 \end{pmatrix}. \end{aligned}$$

Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^2$ is open. If we use the Cartesian coordinates, we can find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if they exist. However, sometimes it is more convenient to use the polar coordinates. Using the substitutions $x = r\cos\theta$ and $y = r\sin\theta$, we can regard f as a function in r and θ. So we can write $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ if there is no confusion. The following result gives the relationships between different partial derivatives.

Proposition 2.14. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^2$ is open. We can regard f as a function in x and y by using the Cartesian coordinates or as a function in r and θ by using the polar coordinates. Then we have

$$\begin{cases} \dfrac{\partial f}{\partial r} = \cos\theta \cdot \dfrac{\partial f}{\partial x} + \sin\theta \cdot \dfrac{\partial f}{\partial y}, \\ \dfrac{\partial f}{\partial \theta} = -r\sin\theta \cdot \dfrac{\partial f}{\partial x} + r\cos\theta \cdot \dfrac{\partial f}{\partial y} \end{cases}$$

and

$$\begin{cases} \dfrac{\partial f}{\partial x} = \cos\theta \cdot \dfrac{\partial f}{\partial r} - \dfrac{\sin\theta}{r} \cdot \dfrac{\partial f}{\partial \theta}, \\ \dfrac{\partial f}{\partial y} = \sin\theta \cdot \dfrac{\partial f}{\partial r} + \dfrac{\cos\theta}{r} \cdot \dfrac{\partial f}{\partial \theta}. \end{cases}$$

Example 2.34. Define $f : X \to \mathbb{R}$ by

$$f(r\cos\theta, r\sin\theta) = 2r\sin^3\theta - \cos^2\theta$$

where $X = \mathbb{R}^2 - \{(0,0)\}$. Then

$$\frac{\partial f}{\partial x} = -4\sin^3\theta\cos\theta - \frac{2\sin^2\theta\cos\theta}{r} = \frac{-4xy^3 - 2xy^2}{(x^2+y^2)^2}.$$

Definition 2.25. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, and let $\mathbf{a}$ be an interior point of X. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$. The *directional derivative* of f at $\mathbf{a}$ in the direction of $\mathbf{u}$ is defined by

$$D_{\mathbf{u}} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}.$$

In other words, the directional derivative is the derivative of $g(t) = f(\mathbf{a} + t\mathbf{u})$. If $\mathbf{v}$ is not a unit vector, the directional derivative in the direction of $\mathbf{v}$ means the directional derivative in the direction of $\mathbf{u}$ where $\mathbf{u}$ is the unit vector in the same direction as $\mathbf{v}$.

Example 2.35. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = x^2 y.$$

Let $\mathbf{a} = (1,2)$ and $\mathbf{u} = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. Then $D_{\mathbf{u}} f(\mathbf{a}) = \dfrac{5}{\sqrt{2}}$.

Example 2.36. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

For any unit vector $\mathbf{u} = (a,b)$ with $a \neq 0$, we have

$$D_{\mathbf{u}} f(0,0) = \lim_{h\to 0} \frac{f(ha,hb) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^3ab^2}{h(h^2a^2 + h^4b^4)} = \frac{b^2}{a}.$$

For $\mathbf{u} = (0, \pm 1)$, we have $D_{\mathbf{u}} f(0,0) = 0$.

Proposition 2.15. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, and let $\mathbf{a}$ be an interior point of X. Suppose f is differentiable at $\mathbf{a}$. Then for any unit vector $\mathbf{u}$, the directional derivative at $\mathbf{a}$ in the direction of $\mathbf{u}$ exists and

$$D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

The converse does not hold. This means a non-differentiable function may have directional derivatives in all directions.

Proposition 2.16. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, and let $\mathbf{a}$ be an interior point of X. Suppose f is differentiable at $\mathbf{a}$. Then the directional derivative $D_{\mathbf{u}} f(\mathbf{a})$ is maximized when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{a})$, and is minimized when $\mathbf{u}$ has an opposite direction as $\nabla f(\mathbf{a})$. The optimum values are $\pm \|\nabla f(\mathbf{a})\|$.

This gives a geometric interpretation of ∇f.

Example 2.37. Suppose we are climbing a mountain whose surface is the graph of $f(x,y) = 6 - 2x^2 - 3y^2 - 3xy$ with $f(x,y) \geqslant 0$. If we are located at the point $(1,0,4)$ and we want to go upwards in the quickest way, we should move in the direction $\nabla f(1,0) = (-4,-3)$.

Links

Theorems

- 2.1: Sandwich theorem
- 2.2: A sufficient condition for $f : \mathbb{R}^n \to \mathbb{R}$ to be differentiable
- 2.3: A sufficient condition for $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable
- 2.4: Symmetry of mixed partial derivatives
- 2.5: Chain rule (special case)
- 2.6: Chain rule (general case)

Propositions

- 2.1: Criterion for a set to be closed
- 2.2: Uniqueness of the limit
- 2.3: Algebraic properties of limits
- 2.4: Limits of component functions
- 2.5: Limit of composition of functions
- 2.6: Algebraic properties of continuous functions
- 2.7: Continuity of component functions
- 2.8: Continuity of composition of functions
- 2.9: Equation of the tangent plane
- 2.10: Continuity of differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$
- 2.11: Differentiability of component functions
- 2.12: Continuity of differentiable functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
- 2.13: Algebraic properties of derivatives
- 2.14: Relations between differential operators involving polar coordinates
- 2.15: Directional derivative of a differentiable function
- 2.16: Optimizing directional derivatives

Terminologies and Notations

- accumulation point
- bijective
- boundary ∂X
- boundary point

- class C^k
- class C^∞
- closed ball $\overline{B}(\mathbf{a}, r)$
- closed set
- codomain
- component functions
- composition $g \circ f$
- continuous
- continuous function
- contour curve
- derivative Df
- differentiable $f : \mathbb{R}^n \to \mathbb{R}$
- differentiable $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$
- directional derivative $D_{\mathbf{u}} f(\mathbf{a})$
- discontinuous
- discontinuous function
- domain
- ellipsoid
- elliptic cone
- elliptic paraboloid
- gradient ∇f, $\operatorname{grad} f$
- graph
- hyperbolic paraboloid
- hyperboloid of one sheet
- hyperboloid of two sheets
- injective
- interior
- interior point
- inverse function f^{-1}
- isolated point
- Jacobian matrix $D\mathbf{f}$
- kth-order partial derivatives

- level curve
- limit $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$
- limit point
- one-to-one
- one-to-one correspondence
- onto
- open ball $B(\mathbf{a}, r)$
- open set
- partial derivative $\dfrac{\partial f}{\partial x_j}$, f_{x_j}, $D_{x_j} f$
- quadric surface
- range
- scalar-valued function
- second-order partial derivatives $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$, $f_{x_i x_j}$
- smooth function
- surjective
- tangent plane
- vector-valued function