Multivariable Calculus

HKPFS Math PhD Interview Prep

1 1. Vectors & Geometry

1.1 Dot Product

$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$
$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$
$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$
Orthogonal: $\mathbf{u}\perp\mathbf{v} \Leftrightarrow \mathbf{u}\cdot\mathbf{v} = 0$
Projection of u onto v:
$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

1.2 Cross Product (in $\mathbb{R}^3$)

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
$= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$
$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ (area of parallelogram)
Properties: $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$, $\mathbf{u}\times\mathbf{u} = \mathbf{0}$
Scalar triple product:
$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ (volume)

1.3 Lines & Planes

Line through point P parallel to v:
$\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}$
Plane with normal $\mathbf{n} = (A, B, C)$ through (a, b, c):
$A(x-a) + B(y-b) + C(z-c) = 0$
General form: $Ax + By + Cz = D$

1.4 Coordinate Systems

Polar: $x = r\cos\theta$, $y = r\sin\theta$
$r = \sqrt{x^2+y^2}$, $\tan\theta = \frac{y}{x}$
Cylindrical: (r, θ, z) where (r, θ) is polar in xy-plane
Spherical: (ρ, ϕ, θ)

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$
$$\rho = \sqrt{x^2+y^2+z^2}, \quad \tan\phi = \frac{\sqrt{x^2+y^2}}{z}$$

2 2. Functions & Limits

2.1 Multivariable Functions

Domain: Set $X \subseteq \mathbb{R}^n$ where f is defined
Range: $\{f(\mathbf{x}) : \mathbf{x} \in X\}$
Level curve at height c: $\{(x, y) : f(x, y) = c\}$
Graph: $\{(x, y, f(x, y)) : (x, y) \in X\} \subseteq \mathbb{R}^3$

2.2 Quadric Surfaces

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Elliptic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
Hyperbolic paraboloid: $\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$
Hyperboloid (1 sheet): $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Hyperboloid (2 sheets): $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

2.3 Topology

Open ball: $B(\mathbf{a}, r) = \{\mathbf{x} : \|\mathbf{x}-\mathbf{a}\| < r\}$
Closed ball: $\overline{B}(\mathbf{a}, r) = \{\mathbf{x} : \|\mathbf{x}-\mathbf{a}\| \le r\}$
Interior point: $\mathbf{a} \in X$ is interior if $\exists r > 0 : B(\mathbf{a}, r) \subseteq X$
Boundary point: Every ball around $\mathbf{a}$ contains points in X and not in X
Open set: Every point is an interior point
Closed set: Complement is open (equivalently: contains all boundary points)
Bounded: $\exists M > 0 : \|\mathbf{x}\| \le M$ for all $\mathbf{x} \in X$
Compact: Closed and bounded

2.4 Limits & Continuity

Limit: $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if $\forall\varepsilon > 0, \exists\delta > 0 :$
$0 < \|\mathbf{x}-\mathbf{a}\| < \delta \Rightarrow \|f(\mathbf{x}) - L\| < \varepsilon$
Continuous at a: $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$
Showing limit DNE: Approach along different paths and get different limits
Sandwich theorem: If $f \le g \le h$ and $\lim f = \lim h = L$, then $\lim g = L$

3 3. Differentiation

3.1 Partial Derivatives

$\frac{\partial f}{\partial x_j}(\mathbf{a}) = \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{e}_j) - f(\mathbf{a})}{h}$
Notation: $f_{x_j}(\mathbf{a})$, $D_{x_j}f(\mathbf{a})$, $\frac{\partial f}{\partial x_j}$
Higher order: $f_{x_ix_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$
Clairaut's theorem: If $f \in C^k$, then mixed partials commute: $f_{x_ix_j} = f_{x_jx_i}$

3.2 Gradient & Derivative

Gradient: $\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$
Derivative (Jacobian): For $f : \mathbb{R}^n \to \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$:
$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$
For scalar f: $Df = (\nabla f)^T$ (row vector)

3.3 Differentiability

f is **differentiable at a** if $\nabla f(\mathbf{a})$ exists and
$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - [f(\mathbf{a}) + \nabla f(\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})]}{\|\mathbf{x}-\mathbf{a}\|} = 0$
Sufficient condition: If all partial derivatives exist and are continuous in a neighborhood of $\mathbf{a}$, then f is differentiable at $\mathbf{a}$
Class C^k: All partial derivatives up to order k exist and are continuous
Differentiable $\Rightarrow$ continuous (but not conversely)

3.4 Tangent Plane

Tangent plane to $z = f(x, y)$ at $(a, b, f(a, b))$:
$z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$
For surface $F(x, y, z) = c$: Normal vector is $\nabla F = (F_x, F_y, F_z)$

3.5 Directional Derivative

$D_{\mathbf{u}}f(\mathbf{a}) = \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{u}) - f(\mathbf{a})}{h}$
where $\mathbf{u}$ is a unit vector.
If f is differentiable: $D_{\mathbf{u}}f(\mathbf{a}) = \nabla f(\mathbf{a})\cdot\mathbf{u}$
Maximum rate of increase: Direction of ∇f, rate $= \|\nabla f\|$
Minimum rate: Direction of $-\nabla f$, rate $= -\|\nabla f\|$
∇f is perpendicular to level curves/surfaces

3.6 Chain Rule

For $h = f \circ \mathbf{g}$ where $f : \mathbb{R}^n \to \mathbb{R}^k$, $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^n$:
$Dh(\mathbf{t}) = Df(\mathbf{g}(\mathbf{t}))\cdot D\mathbf{g}(\mathbf{t})$
Special case: $z = f(x, y)$, $x = x(t)$, $y = y(t)$:
$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
Polar coordinates conversion:
$\frac{\partial f}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$
$\frac{\partial f}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}$

4 4. Optimization

4.1 Taylor's Theorem

First order: $f(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a}) + R_1(\mathbf{x}, \mathbf{a})$
where $\lim_{\mathbf{x}\to\mathbf{a}} \frac{R_1(\mathbf{x},\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|} = 0$
Second order:
$f(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T H_f(\mathbf{a})(\mathbf{x}-\mathbf{a}) + R_2$
Hessian matrix:
$H_f = \begin{pmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & \ddots & \vdots \\ f_{x_nx_1} & \cdots & f_{x_nx_n} \end{pmatrix}$
Total differential: $df(\mathbf{a}, \mathbf{h}) = Df(\mathbf{a})\mathbf{h}$
Incremental change:
$\Delta f = f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \approx df(\mathbf{a}, \mathbf{h})$

4.2 Extrema

Critical point: $\nabla f(\mathbf{a}) = \mathbf{0}$ or $\nabla f(\mathbf{a})$ DNE
Necessary condition: If $\mathbf{a}$ is interior local extremum and $\nabla f(\mathbf{a})$ exists, then $\nabla f(\mathbf{a}) = \mathbf{0}$
Saddle point: Critical point that is not a local extremum

4.3 Second Derivative Test

Let $\mathbf{a}$ be a critical point of $f \in C^2$.
- If $H_f(\mathbf{a})$ is positive definite $\Rightarrow$ local min
- If $H_f(\mathbf{a})$ is negative definite $\Rightarrow$ local max
- If $\det H_f(\mathbf{a}) \neq 0$ and $H_f(\mathbf{a})$ is indefinite $\Rightarrow$ saddle point

Sylvester's criterion: Let $d_k = \det(H_k)$ (leading principal minors)
- Positive definite: $d_k > 0$ for all $k = 1, \ldots, n$
- Negative definite: $d_k < 0$ for odd k, $d_k > 0$ for even k

For $n = 2$: Let $D = f_{xx}f_{yy} - (f_{xy})^2$
- $D > 0, f_{xx} > 0 \Rightarrow$ local min
- $D > 0, f_{xx} < 0 \Rightarrow$ local max
- $D < 0 \Rightarrow$ saddle point
- $D = 0 \Rightarrow$ inconclusive

4.4 Extreme Value Theorem

If $f : X \to \mathbb{R}$ is continuous and X is compact, then f attains global max and min on X
Strategy: Find critical points in interior, then check boundary

4.5 Lagrange Multipliers

To optimize $f(\mathbf{x})$ subject to $g(\mathbf{x}) = c$:
At extremum $\mathbf{a}$ (if $\nabla g(\mathbf{a}) \neq \mathbf{0}$):
$\exists\lambda : \nabla f(\mathbf{a}) = \lambda\nabla g(\mathbf{a})$
Multiple constraints $g_1 = c_1, \ldots, g_k = c_k$ (with $\{\nabla g_j\}$ linearly indep.):
$\nabla f = \lambda_1\nabla g_1 + \cdots + \lambda_k\nabla g_k$
Procedure:
1. Solve $\nabla f = \lambda\nabla g$ and $g = c$ simultaneously
2. Evaluate f at all solutions
3. Compare to find max/min

5 5. Integration

5.1 Double Integrals

Definition:
$\iint_R f\,dA = \lim_{\Delta x,\Delta y\to 0}\sum_{i,j} f(\mathbf{x}_{ij})\Delta x_i \Delta y_j$
Fubini's theorem (rectangle): $R=[a,b]\times[c,d]$
$\iint_R f\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$
Type 1 region: $D=\{(x,y): a\le x\le b, g(x)\le y\le h(x)\}$
$\iint_D f\,dA = \int_a^b\int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$
Type 2 region: $D=\{(x,y): c\le y\le d, g(y)\le x\le h(y)\}$
$\iint_D f\,dA = \int_c^d\int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$
Volume under graph: $V=\iint_D f(x,y)\,dA$ (if $f\ge 0$)
Area of region: $A=\iint_D 1\,dA$

5.2 Triple Integrals

$\iiint_B f\,dV = \int_a^b\int_c^d\int_p^q f(x,y,z)\,dz\,dy\,dx$
For elementary region:
$\iiint_D f\,dV = \int_a^b\int_{g(x)}^{h(x)}\int_{\varphi(x,y)}^{\psi(x,y)} f\,dz\,dy\,dx$
Volume: $V=\iiint_D 1\,dV$

5.3 Change of Variables

Jacobian: For $x=x(u,v)$, $y=y(u,v)$:
$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$
Change of variables formula:
$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$
Polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$
$dA = dx\,dy = r\,dr\,d\theta$
Cylindrical: $x=r\cos\theta$, $y=r\sin\theta$, $z=z$
$dV = r\,dr\,d\theta\,dz$
Spherical: $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$
$dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
For $\mathbb{R}^3$:
$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$

6 6. Vector Calculus

6.1 Curves & Paths

Path: $\boldsymbol{\gamma}: I\to\mathbb{R}^n$ continuous, $I\subseteq\mathbb{R}$ interval
Curve: Image $\boldsymbol{\gamma}(I)$
Velocity: $\mathbf{v}(t)=\boldsymbol{\gamma}'(t)$
Speed: $\|\mathbf{v}(t)\| = \|\boldsymbol{\gamma}'(t)\|$
Arc length: $L=\int_a^b \|\boldsymbol{\gamma}'(t)\|\,dt$
Parametrization: Injective C^1 path with image C
Reparametrization: $\boldsymbol{\gamma}_2=\boldsymbol{\gamma}_1\circ\phi$
where $\phi:[c,d]\to[a,b]$ bijective C^1
Orientation-preserving if $\phi(c)=a$, $\phi(d)=b$
Orientation-reversing if $\phi(c)=b$, $\phi(d)=a$

6.2 Differential Operators

Del operator: $\nabla = \frac{\partial}{\partial x_1}\mathbf{e}_1+\cdots+\frac{\partial}{\partial x_n}\mathbf{e}_n$
Gradient: $\nabla f = \left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)$
(scalar $\to$ vector)
Divergence:
$\nabla\cdot\mathbf{F} = \text{div }\mathbf{F} = \frac{\partial F_1}{\partial x_1}+\cdots+\frac{\partial F_n}{\partial x_n}$
(vector $\to$ scalar)
Curl (in $\mathbb{R}^3$):
$\nabla\times\mathbf{F} = \text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$
$= \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathbf{k}$
Key identities:
- $\nabla\times(\nabla f)=\mathbf{0}$ (curl of gradient is zero)
- $\nabla\cdot(\nabla\times\mathbf{F})=0$ (divergence of curl is zero)

Conservative field: $\mathbf{F}=\nabla f$ for some scalar f

6.3 Line Integrals

Scalar line integral:
$\int_{\boldsymbol{\gamma}} f\,ds = \int_a^b f(\boldsymbol{\gamma}(t))\|\boldsymbol{\gamma}'(t)\|\,dt$
Independent of orientation
Vector line integral:
$\int_{\boldsymbol{\gamma}}\mathbf{F}\cdot d\mathbf{s} = \int_a^b \mathbf{F}(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt$
Also written: $\int_{\boldsymbol{\gamma}} F_1\,dx_1+\cdots+F_n\,dx_n$
Orientation: Reversing orientation changes sign of vector line integral but not scalar
Closed curve notation: $\oint_C f\,ds$, $\oint_C \mathbf{F}\cdot d\mathbf{s}$

6.4 Surface Integrals

Parametrized surface: $\boldsymbol{\Phi}: D\subseteq\mathbb{R}^2\to\mathbb{R}^3$, D open connected
Tangent vectors: $\mathbf{T}_s=\frac{\partial\boldsymbol{\Phi}}{\partial s}$, $\mathbf{T}_t=\frac{\partial\boldsymbol{\Phi}}{\partial t}$
Normal vector: $\mathbf{N}(s,t)=\mathbf{T}_s\times\mathbf{T}_t$
Smooth surface: $\mathbf{N}\neq\mathbf{0}$ everywhere
Surface area: $A=\iint_D\|\mathbf{N}(s,t)\|\,ds\,dt$
Scalar surface integral:
$\iint_{\boldsymbol{\Phi}} f\,dS = \iint_D f(\boldsymbol{\Phi}(s,t))\|\mathbf{N}(s,t)\|\,ds\,dt$
Vector surface integral:
$\iint_{\boldsymbol{\Phi}}\mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\boldsymbol{\Phi}(s,t))\cdot\mathbf{N}(s,t)\,ds\,dt$
Orientable surface: Can define continuous unit normal everywhere
Closed surface notation: $\oint_S f\,dS$, $\oint_S \mathbf{F}\cdot d\mathbf{S}$

6.5 Fundamental Theorems

Green's theorem: Let $C=\partial D$ positively oriented, $\mathbf{F}=(F_1,F_2)$:
$\oint_C\mathbf{F}\cdot d\mathbf{s} = \oint_C F_1\,dx+F_2\,dy$
$=\iint_D\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dx\,dy$
Equivalently: $\oint_C\mathbf{F}\cdot d\mathbf{s} = \iint_D(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA$
Divergence theorem ($\mathbb{R}^2$): Let $C=\partial D$, $\mathbf{n}$ outward normal:
$\oint_C\mathbf{F}\cdot\mathbf{n}\,ds = \iint_D\nabla\cdot\mathbf{F}\,dA$
Stokes' theorem: Let S be orientable surface, ∂S oriented consistently:
$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{S} = \oint_{\partial S}\mathbf{F}\cdot d\mathbf{s}$
Gauss/Divergence theorem: Let D be solid region, ∂D oriented outward:
$\oint_{\partial D}\mathbf{F}\cdot d\mathbf{S} = \iiint_D\nabla\cdot\mathbf{F}\,dV$

6.6 Important Facts

- Positively oriented boundary: D on left when traversing C
- Right-hand rule for consistent orientation on surface
- Green's is 2D Stokes
- For conservative field $\mathbf{F}=\nabla f$:
  $\int_{\boldsymbol{\gamma}}\mathbf{F}\cdot d\mathbf{s} = f(\text{end})-f(\text{start})$ (path-independent)