# Mathematics Interview Preparation

## 1 Linear Algebra

### 1.1 Systems and Matrices

1. Define the **Rank** and **Nullity** of a matrix. State the **Dimension Formula** (Rank-Nullity Theorem) for a matrix $A \in M_{n\times m}$.
2. What are the conditions for a system $Ax = b$ to be **inconsistent**? Relate this to the RREF of the augmented matrix $(A|b)$.
3. List at least 5 equivalent conditions for an $n \times n$ matrix $A$ to be **Invertible** (The Invertible Matrix Theorem).
4. Explain **Cramer's Rule**. When is it applicable?

### 1.2 Vector Spaces and Linear Maps

1. Define a **Subspace**. Why is the union of two subspaces generally not a subspace?
2. Define **Linear Independence**. How do you test if a set of vectors is linearly independent using a homogeneous system?
3. Let $T : V \to W$ be a linear transformation. Define the **Kernel** (Null Space) and **Image** (Range).
4. What does it mean for a linear map to be an **Isomorphism**? What does this imply about the dimensions of $V$ and $W$?
5. Explain the relationship between the matrix of a transformation $[T]_{\mathcal{B}}^{\mathcal{C}}$ and a change of basis matrix $Q$. How does $Q$ relate $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{B}'}$?

   *Quick note:* The change-of-basis matrix $Q$ converts coordinates between bases: $[v]_{\mathcal{B}} = Q[v]_{\mathcal{B}'}$, and $[v]_{\mathcal{B}'} = Q^{-1}[v]_{\mathcal{B}}$. Matrices of the same transformation in different bases are related by
   $$[T]_{\mathcal{B}'} = Q^{-1}[T]_{\mathcal{B}}Q.$$
6. Define the **Dual Space** $V^*$. If $V$ is finite-dimensional, is $V \cong V^*$?

   Quick note: The dual space $V^*$ is the set of all linear functionals $f : V \to \mathbb{F}$, forming a vector space itself. If $\dim(V) = n$, then $\dim(V^*) = n$, so $V$ and $V^*$ are isomorphic as vector spaces. However, this isomorphism is *not canonical*—it depends on the choice of basis (or an extra structure such as an inner product). The dual basis $\{e^1, \ldots, e^n\}$ is defined by $e^i(e_j) = \delta_{ij}$.

### 1.3 Inner Products and Orthogonality

1. State the properties of an **Inner Product**. What is the relationship between the inner product and the norm?

2. State the **Cauchy-Schwarz Inequality**. When does equality hold?

3. Describe the **Gram-Schmidt Process**. What is its geometric interpretation?

   *Quick note:* The **Gram-Schmidt process** takes a linearly independent set of vectors $\{v_1, v_2, \ldots, v_n\}$ in an inner product space and converts it into an **orthonormal set** $\{q_1, q_2, \ldots, q_n\}$ that spans the same subspace. Each new vector is made orthogonal to the previous ones by subtracting its projection components:

   $$q_1 = \frac{v_1}{\|v_1\|}, \quad q_k = \frac{v_k - \sum_{j=1}^{k-1} \langle v_k, q_j \rangle q_j}{\|v_k - \sum_{j=1}^{k-1} \langle v_k, q_j \rangle q_j\|}.$$

   **Geometric interpretation:** Gram–Schmidt can be seen as the process of constructing a sequence of perpendicular directions from arbitrary basis vectors — like turning skewed axes into orthogonal ones. It provides an orthonormal basis suitable for projections and for QR decomposition ($A = QR$).

4. Define the **Orthogonal Complement** $S^\perp$. What is the dimension of $W^\perp$ if $W$ is a subspace of a finite-dimensional space $V$?

5. Explain the **Least Squares Problem**. Deriving from the geometry of projections, why is the solution given by the normal equation $A^T Ax = A^T y$?

   *Quick note:* The least squares solution $x_{\text{LS}}$ minimizes $\|Ax - y\|^2$ by projecting $y$ onto the column space of $A$. The residual $r = y - Ax_{\text{LS}}$ is orthogonal to $\text{Col}(A)$, giving the condition $A^T r = 0$, which leads to the **normal equation:**

   $$A^T A x_{\text{LS}} = A^T y.$$

   When the columns of $A$ are linearly independent, $A^T A$ is invertible and the unique solution is

   $$x_{\text{LS}} = (A^T A)^{-1} A^T y.$$

6. Under what condition is the matrix $A^T A$ invertible?

7. What is a **Positive Definite Matrix**? Give three equivalent characterizations (e.g., eigenvalues, pivots, energy).

   *Quick note:* A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is **positive definite** if

   $$x^T Ax > 0 \quad \text{for all } x \neq 0.$$

   This means it defines a strictly positive quadratic form. Several equivalent characterizations exist:

   - **Energy test:** $x^T Ax > 0$ for all nonzero $x$ (definition).
   - **Eigenvalue test:** All eigenvalues of $A$ are positive.
   - **Pivot (principal minors) test:** All leading principal minors $\det(A_k)$ are positive.

### 1.4 Eigenvalues and Diagonalization

1. Define **Algebraic Multiplicity** and **Geometric Multiplicity**. What is the inequality relationship between them?

   *Quick note:* Algebraic multiplicity $m_a(\lambda)$ = number of times $\lambda$ is a root of the characteristic polynomial $\det(A-\lambda I) = 0$. Geometric multiplicity $m_g(\lambda) = \dim(\mathrm{Ker}(A-\lambda I))$. Inequality: $1 \le m_g(\lambda) \le m_a(\lambda)$, with equality $\Rightarrow$ the eigenvalue's eigenspace is "complete."

2. What is the precise condition for a matrix $A$ to be **Diagonalizable**?

   *Quick note:* A matrix $A$ is diagonalizable $\iff$ it has $n$ linearly independent eigenvectors $\iff$ $m_g(\lambda) = m_a(\lambda)$ for each eigenvalue. If $A$ has $n$ distinct eigenvalues $\Rightarrow$ automatically diagonalizable.

3. If a matrix $A$ is symmetric ($A^T = A$), what can you say about its eigenvalues and eigenvectors?

   *Quick note:* Real symmetric $\Rightarrow$ all eigenvalues are real, eigenvectors for distinct eigenvalues are orthogonal, and $A$ is orthogonally diagonalizable:

   $$A = QDQ^T, \quad Q^TQ = I.$$

   (Spectral Theorem.)

4. Explain how to compute the matrix exponential $e^A$ using diagonalization.

   *Quick note:* If $A = PDP^{-1}$ with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, then

   $$e^A = Pe^DP^{-1}, \quad \text{where } e^D = \mathrm{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n}).$$

   This works since $A^k = PD^kP^{-1}$ and exponentials preserve similarity transformations.

## 2 Mathematical Analysis

### 2.1 Real Numbers and Topology

1. Define the **Supremum** (LUB). How does the **Completeness Axiom** distinguish $\mathbb{R}$ from $\mathbb{Q}$?

   *Quick note:* The **supremum** $\sup S$ of a nonempty, bounded-above set $S \subseteq \mathbb{R}$ is the smallest real number $u$ with $s \le u$ for all $s \in S$. The **Completeness Axiom** states every nonempty, bounded-above subset of $\mathbb{R}$ has a supremum in $\mathbb{R}$. $\mathbb{R}$ is **complete**; $\mathbb{Q}$ is not — e.g. $\{q \in \mathbb{Q} : q^2 < 2\}$ has no supremum in $\mathbb{Q}$.

2. State the **Archimedean Property** and the **Density of** $\mathbb{Q}$.

3. What is a **Cauchy Sequence**? State the Cauchy Criterion for convergence in $\mathbb{R}$.

4. State the **Bolzano-Weierstrass Theorem**.

   *Quick note:* Every bounded sequence in $\mathbb{R}$ has a **convergent subsequence**. This expresses the compactness of closed, bounded intervals in $\mathbb{R}$: bounded $\Rightarrow$ at least one limit (accumulation) point exists.

5. Define **Limit Superior** ($\limsup$) and **Limit Inferior** ($\liminf$). Under what condition does a sequence converge?

### 2.2 Series and Convergence

1. Distinguish between **Absolute Convergence** and **Conditional Convergence**.
2. State the **Ratio Test** and **Root Test**. When are these tests inconclusive?
3. Does $\sum a_n$ converging imply $\sum a_n^2$ converges?
4. If $a_n \to 0$, does $\sum a_n$ necessarily converge? Give a counter-example.

### 2.3 Continuity and Differentiation

1. Define **Continuity** using the $\epsilon - \delta$ definition.
2. Define **Uniform Continuity**. How does the definition differ from pointwise continuity?
3. True or False: If $f$ is continuous on a bounded interval $(a, b)$, it is uniformly continuous. (Hint: Consider $f(x) = 1/x$).
4. State the **Intermediate Value Theorem** and the **Extreme Value Theorem**. What topological properties of the domain are required?

   *Quick note:* **Intermediate Value Theorem (IVT):** If $f$ is continuous on $[a, b]$ and $y$ lies between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ such that $f(c) = y$. Requires the domain to be **connected (interval)**.

   **Extreme Value Theorem (EVT):** If $f$ is continuous on a closed, bounded interval $[a, b]$, then $\exists c, d \in [a, b]$ such that

   $$f(c) = \max_{x \in [a,b]} f(x), \qquad f(d) = \min_{x \in [a,b]} f(x).$$

   Requires the domain to be **compact (closed and bounded)**.
5. Does differentiability at a point imply continuity at that point? Does continuity imply differentiability? (Provide the standard counter-example for the latter).
6. State the **Mean Value Theorem**.

   *Quick note:* If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then

   $$\exists c \in (a, b) \text{ such that } f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   Requires $f$ to be **continuous on** $[a, b]$ and **differentiable on** $(a, b)$.
7. Can a derivative $f'$ exist everywhere but be discontinuous? (Hint: $x^2 \sin(1/x)$).

### 2.4 Sequences of Functions and Integration

1. Define **Pointwise Convergence** vs. **Uniform Convergence** of a sequence of functions $(f_n)$.
2. Why is Uniform Convergence important for swapping limits with integrals or derivatives?

   *Quick note:* Uniform convergence allows limit operations to pass through continuous processes like integration and differentiation. If $f_n \to f$ **uniformly** and each $f_n$ is integrable on $[a, b]$, then

   $$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx.$$

   Uniform convergence also ensures $f$ is continuous if each $f_n$ is continuous. Pointwise convergence alone **does not** guarantee these properties.

3. State the **Weierstrass M-Test**.
4. Define the **Riemann Integral** using partitions and Darboux sums ($U(f, P)$ and $L(f, P)$).
5. State both parts of the **Fundamental Theorem of Calculus**.

   *Quick note:* **FTC Part I (Derivative of Integral):** If $f$ is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

   **FTC Part II (Integral of Derivative):** If $F$ is differentiable on $[a, b]$ with continuous derivative $F'$, then

   $$\int_a^b F'(x)\,dx = F(b) - F(a).$$

# 3 Multivariable Calculus

## 3.1 Differentiation in $\mathbb{R}^n$

1. Define the **Gradient** $\nabla f$. What is its geometric relationship to level surfaces?
2. True or False: If all partial derivatives exist at a point, the function is differentiable at that point.

   *Quick note:* **False.** Existence of all partials does not guarantee differentiability — they must fit together to form a linear approximation. Example: $f(x, y) = \frac{x^2 y}{x^2+y^2}$ at $(0, 0)$ has partials 0, but $f$ isn't differentiable.
3. State the **Inverse Function Theorem**. What does the Jacobian determinant tell you about local invertibility?

   *Quick note:* If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable near $a$ and $\det(Df(a)) \neq 0$, then there exist neighborhoods $V$ of $a$ and $W$ of $f(a)$ such that $f : V \to W$ is a bijection with a continuously differentiable inverse $f^{-1}$.

   $$D(f^{-1})(f(a)) = [Df(a)]^{-1}.$$
4. State **Clairaut's Theorem** regarding mixed partial derivatives.

   *Quick note:* If $f_{xy}$ and $f_{yx}$ exist and are **continuous** near a point $(a, b)$, then

   $$f_{xy}(a, b) = f_{yx}(a, b).$$
5. Explain the method of **Lagrange Multipliers**. Why do we solve $\nabla f = \lambda \nabla g$?
6. How do you classify critical points using the **Hessian Matrix** (Second Derivative Test)? Relate this to positive/negative definiteness.

## 3.2 Integration and Vector Calculus

1. State **Fubini's Theorem**. When can you swap the order of integration?

   Quick note: If $f(x, y)$ is continuous (or absolutely integrable) on a rectangular region, then

   $$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy.$$

   Order of integration may be swapped when $f$ is continuous or $|f|$ is integrable (Fubini/Tonelli).

2. Explain the **Change of Variables Formula** for multiple integrals. What is the role of the Jacobian determinant?

   Quick note: For a transformation $T(u,v) = (x,y)$,

   $$\int_V f(x,y)\,dA = \int_U f(T(u,v))\,\big|\det J_T(u,v)\big|\,du\,dv.$$

   The Jacobian determinant $\big|\det J_T\big|$ gives the local area or volume scaling factor caused by the coordinate change. Examples: polar ($dA = r\,dr\,d\theta$), spherical ($dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$).

3. Define the **Curl** and **Divergence** of a vector field.

   Quick note:

   $$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}, \quad \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

   $\nabla \cdot \mathbf{F}$ (divergence) is a scalar measuring outflow or source strength. $\nabla \times \mathbf{F}$ (curl) is a vector measuring local rotation.

4. State **Green's Theorem**.

   Quick note:

   $$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

   Relates a line integral around a closed curve to a double integral over the enclosed region (circulation = curl form in $\mathbb{R}^2$).

5. State the **Divergence Theorem** (Gauss's Theorem).

   Quick note:

   $$\oint_{\partial V} \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_V (\nabla \cdot \mathbf{F})\,dV.$$

6. State **Stokes' Theorem**.

   Quick note:

   $$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS.$$

7. What is a **Conservative Vector Field**? How is it related to path independence of line integrals?

   Quick note: A vector field $\mathbf{F}$ is conservative if there exists a scalar potential $\phi$ such that $\mathbf{F} = \nabla\phi$. Then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent and $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve $C$. In simply connected regions, $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$.

# 4 Topology and Metric Spaces

1. Define a **Metric Space**. What are the three properties of a distance function?

2. Define **Open** and **Closed** sets. Can a set be both? Can a set be neither?

3. What is the definition of **Compactness** (using open covers)?

   *Quick note:* A set $K$ in a topological space is **compact** if every open cover of $K$ has a finite subcover.

4. State the **Heine-Borel Theorem**. In which spaces does it apply?

   *Quick note:* In $\mathbb{R}^n$, a set is **compact** if and only if it is **closed and bounded**. This equivalence (the Heine-Borel property) holds only in finite-dimensional Euclidean spaces. In infinite-dimensional spaces, closed and bounded sets need not be compact (for example, the unit ball in $\ell^2$).

5. Define **Connectedness**.

   *Rigorous definition:* A topological space $X$ is said to be **connected** if there do not exist two nonempty disjoint open sets $U, V \subseteq X$ such that

   $$X = U \cup V.$$

   Equivalently, $X$ is connected if the only subsets of $X$ that are both open and closed (clopen sets) are $\varnothing$ and $X$ itself. If such a decomposition $X = U \cup V$ exists with both $U$ and $V$ nonempty and open (or equivalently, both closed), then $X$ is said to be **disconnected**.

## 5 The "Trap" Section: Common Counter-Examples

1. **Trap:** If a sequence of continuous functions $f_n$ converges pointwise to $f$ on $[0, 1]$, is $f$ continuous?

   *Counterexample:* Let

   $$f_n(x) = x^n \quad \text{on } [0, 1].$$

   Each $f_n$ is continuous, but $f_n \to f$ pointwise where

   $$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

   The limit function $f$ is *not* continuous at $x = 1$. Pointwise convergence of continuous functions does not, in general, preserve continuity.

2. **Trap:** Is the union of an infinite number of closed sets always closed?

   *Counterexample:* Define

   $$F_n = \left[\frac{1}{n}, 1\right].$$

   Each $F_n$ is closed, but

   $$\bigcup_{n=1}^{\infty} F_n = (0, 1].$$

3. **Trap:** If a function has a local minimum at $a$, is the Hessian matrix $H_f(a)$ always positive definite?

   *Counterexample:* Let

   $$f(x, y) = x^4 + y^4.$$

   At $(0, 0)$, $f$ has a local (and global) minimum, yet

   $$H_f(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

   which is *positive semidefinite*, not positive definite.