Mathematical Analysis Cheat Sheet

HKPFS Math PhD Interview Prep

1 1. Real Numbers

1.1 Field Axioms

Field F with operations $+$ and $\cdot$:
Addition: (A1) Commutative, (A2) Associative, (A3) Zero exists, (A4) Additive inverse exists
Multiplication: (M1) Commutative, (M2) Associative, (M3) Unity $1 \neq 0$ exists, (M4) Multiplicative inverse exists (for $a \neq 0$)
Distributive: (D) $a(b+c) = ab + ac$
Key facts: $0 \cdot a = 0$; $(-1)(-1) = 1$; $ab = 0 \Rightarrow a = 0$ or $b = 0$

1.2 Ordered Field

Relation $\leq$ satisfying:
(a) Reflexive: $a \leq a$
(b) Antisymmetric: $a \leq b$ and $b \leq a \Rightarrow a = b$
(c) Transitive: $a \leq b$ and $b \leq c \Rightarrow a \leq c$
(d) Compatible with $+$: $a \leq b \Rightarrow a + c \leq b + c$
(e) Compatible with $\cdot$: $a \leq b, 0 \leq c \Rightarrow ac \leq bc$
Properties: $a > 0 \Leftrightarrow -a < 0$; $1 > 0$; $a > b > 0 \Rightarrow b^{-1} > a^{-1} > 0$
Absolute value: $|a| = \max\{a, -a\}$
$|a| \geq 0$; $|ab| = |a||b|$; $|a+b| \leq |a| + |b|$ (triangle inequality); $||a| - |b|| \leq |a - b|$

1.3 Completeness (LUB Property)

Supremum: For non-empty $S \subseteq \mathbb{R}$ bounded above, $\sup(S)$ exists and is the *least* upper bound
Infimum: For non-empty S bounded below, $\inf(S) = -\sup(-S)$
Archimedean Property: For any $a \in \mathbb{R}$, $\exists N \in \mathbb{N} : N > a$
Corollary: $\forall \epsilon > 0, \exists N : 1/N < \epsilon$
Density of $\mathbb{Q}$: For $a < b$, $\exists p/q \in \mathbb{Q} : a < p/q < b$
Nested Intervals: If $I_n = [a_n, b_n]$ with $I_{n+1} \subseteq I_n$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$
$\mathbb{R}$ is uncountable (Cantor's diagonal argument)

2 2. Sequences

2.1 Convergence

$(a_n) \to L$ if $\forall \epsilon > 0, \exists N : |a_n - L| < \epsilon$ for all $n \geq N$
Uniqueness: Limit is unique if it exists
Boundedness: Convergent sequences are bounded
Negation: $(a_n) \not\to L$ if $\exists \epsilon > 0$ and infinitely many n with $|a_n - L| \geq \epsilon$

2.2 Arithmetic of Limits

If $(a_n) \to L_1$ and $(b_n) \to L_2$:
- $(ca_n) \to cL_1$ - $(a_n + b_n) \to L_1 + L_2$ - $(a_n b_n) \to L_1 L_2$ - $(a_n/b_n) \to L_1/L_2$ if $b_n \neq 0, L_2 \neq 0$ - If $a_n \leq b_n$, then $L_1 \leq L_2$
Sandwich Theorem: If $a_n \leq c_n \leq b_n$ and $(a_n), (b_n) \to L$, then $(c_n) \to L$

2.3 Subsequences and Monotone Sequences

Subsequence: (a_{n_k}) where $n_1 < n_2 < \cdots$
If $(a_n) \to L$, then every subsequence $\to L$
Monotone Convergence Theorem: Bounded monotone sequence converges
- Increasing bounded above: $(a_n) \to \sup\{a_n\}$ - Decreasing bounded below: $(a_n) \to \inf\{a_n\}$

2.4 Limit Superior and Inferior

For bounded (a_n):
$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$
$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$
Always: $\liminf a_n \leq \limsup a_n$
(a_n) converges $\Leftrightarrow \liminf a_n = \limsup a_n$
Bolzano-Weierstrass: Every bounded sequence has convergent subsequence

2.5 Cauchy Sequences

(a_n) is Cauchy if $\forall \epsilon > 0, \exists N : |a_m - a_n| < \epsilon$ for all $m, n \geq N$
Cauchy Criterion: (a_n) converges $\Leftrightarrow (a_n)$ is Cauchy
Cauchy sequences are bounded

3 3. Series

3.1 Convergence of Series

Series $\sum_{i=1}^{\infty} a_i$ converges if partial sums $s_n = \sum_{i=1}^{n} a_i$ converge
If $\sum a_i$ converges, then $a_n \to 0$
Cauchy Criterion: $\sum a_i$ converges $\Leftrightarrow$
$\forall \epsilon > 0, \exists N : |a_{n+1} + \cdots + a_m| < \epsilon$ for all $m > n \geq N$

3.2 Absolute Convergence

$\sum a_i$ is **absolutely convergent** if $\sum |a_i| < \infty$
Absolute convergence $\Rightarrow$ convergence
Rearrangement: Absolutely convergent series can be rearranged without changing sum

3.3 Convergence Tests

Comparison Test: If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges
Root Test: Let $\alpha = \limsup |a_n|^{1/n}$ - If $\alpha < 1$: absolutely convergent - If $\alpha > 1$: divergent
Ratio Test: - If $\limsup |a_{n+1}/a_n| < 1$: absolutely convergent - If $\liminf |a_{n+1}/a_n| > 1$: divergent
Alternating Series Test: If $a_1 \geq a_2 \geq \cdots \geq 0$ and $a_n \to 0$, then $\sum (-1)^{n+1} a_n$ converges
Integral Test: If $f : [1, \infty) \to \mathbb{R}_{\geq 0}$ decreasing, then $\sum f(n)$ converges $\Leftrightarrow \int_1^{\infty} f(x)dx < \infty$

4 4. Limits of Functions

4.1 Definition

$\lim_{x\to\alpha} f(x) = L$ if $\forall \epsilon > 0, \exists \delta > 0 : |f(x) - L| < \epsilon$ whenever $0 < |x - \alpha| < \delta$
One-sided limits: - Right: $\lim_{x\to\alpha^+} f(x) = L$ - Left: $\lim_{x\to\alpha^-} f(x) = L$
$\lim_{x\to\alpha} f = L \Leftrightarrow \lim_{x\to\alpha^-} f = L = \lim_{x\to\alpha^+} f$

4.2 Sequential Criterion

$\lim_{x\to\alpha} f(x) = L \Leftrightarrow$ for all sequences $(a_n) \to \alpha$ (with $a_n \neq \alpha$), $f(a_n) \to L$

4.3 Arithmetic of Limits

If $\lim_{x\to\alpha} f = L_1$ and $\lim_{x\to\alpha} g = L_2$:
- $\lim(cf) = cL_1$ - $\lim(f+g) = L_1 + L_2$ - $\lim(fg) = L_1 L_2$ - $\lim(f/g) = L_1/L_2$ if $L_2 \neq 0$
Sandwich Theorem: If $f \leq h \leq g$ and $\lim f = \lim g = L$, then $\lim h = L$

4.4 Limits at Infinity

$\lim_{x\to\infty} f(x) = L$ if $\forall \epsilon > 0, \exists M : |f(x) - L| < \epsilon$ for all $x > M$
$\lim_{x\to\alpha} f(x) = \infty$ if $\forall C > 0, \exists \delta > 0 : f(x) > C$ whenever $0 < |x - \alpha| < \delta$

5 5. Continuity

5.1 Definition

$f : I \to \mathbb{R}$ is **continuous at** $\alpha \in I$ if
$\forall \epsilon > 0, \exists \delta > 0 : |f(x) - f(\alpha)| < \epsilon$ whenever $|x - \alpha| < \delta$
Equivalently: $\lim_{x\to\alpha} f(x) = f(\alpha)$
Sequential Criterion: f continuous at $\alpha \Leftrightarrow$ for all $(a_n) \to \alpha$, $f(a_n) \to f(\alpha)$

5.2 Properties

If f, g continuous at α: - $cf, f \pm g, fg$ continuous at α - f/g continuous at α if $g(\alpha) \neq 0$
Composition: If f continuous at α and g continuous at $f(\alpha)$, then $g \circ f$ continuous at α

5.3 Key Theorems

Extreme Value Theorem: If $f : [a, b] \to \mathbb{R}$ continuous, then f attains max and min on $[a, b]$
Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ continuous, then f takes every value between $f(a)$ and $f(b)$
Image of Interval: Continuous image of interval is interval
Inverse Function: If $f : I \to \mathbb{R}$ continuous and injective, then: - f strictly monotone - $f^{-1} : f(I) \to I$ is continuous

5.4 Uniform Continuity

$f : I \to \mathbb{R}$ is **uniformly continuous** if
$\forall \epsilon > 0, \exists \delta > 0 : |f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ (for all $x, y \in I$)
Key difference: δ depends only on ϵ, not on point
Uniform continuity $\Rightarrow$ continuity
Theorem: Continuous on $[a, b] \Rightarrow$ uniformly continuous
Lipschitz: $|f(x) - f(y)| \leq C|x - y|$ for all x, y ($C > 0$)
Lipschitz $\Rightarrow$ uniformly continuous

6 6. Sequences of Functions

6.1 Pointwise Convergence

$(f_n) \to f$ pointwise if $\forall x \in I, f_n(x) \to f(x)$
Pointwise limit of continuous functions may not be continuous

6.2 Uniform Convergence

$(f_n) \to f$ uniformly (written $(f_n) \Rightarrow f$) if
$\forall \epsilon > 0, \exists N : |f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in I$
Key: N depends only on ϵ, not on x
Cauchy Criterion: $(f_n) \Rightarrow f \Leftrightarrow (f_n)$ uniformly Cauchy:
$\forall \epsilon > 0, \exists N : |f_m(x) - f_n(x)| < \epsilon$ for all $m > n \geq N$ and all x

6.3 Properties

Continuity Preservation: If f_n continuous and $(f_n) \Rightarrow f$, then f continuous
Uniform convergence $\Rightarrow$ pointwise convergence (not converse!)
Weierstrass M-Test: If $|f_n(x)| \leq M_n$ for all x and $\sum M_n < \infty$, then $\sum f_n$ converges uniformly

7 7. Power Series

7.1 Radius of Convergence

Power series: $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$
Let $\beta = \limsup |a_n|^{1/n}$ and $R = 1/\beta$ (radius of convergence)
Convergence: - Absolutely for $|x - x_0| < R$ - Diverges for $|x - x_0| > R$ - Uniformly on $[x_0 - R_0, x_0 + R_0]$ for any $R_0 < R$

7.2 Differentiation and Integration

$f(x) = \sum a_n(x-x_0)^n$ with radius R
Term-by-term differentiation: $f'(x) = \sum na_n(x-x_0)^{n-1}$ has same radius R
f is infinitely differentiable on $(x_0 - R, x_0 + R)$
$f^{(n)}(x_0) = n! a_n$
Term-by-term integration: $\int_{x_0}^x f(t)dt = \sum \frac{a_n}{n+1}(x-x_0)^{n+1}$ has same radius R

7.3 Important Series

$e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ (all x)
$\sin x = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ (all x)
$\cos x = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}$ (all x)
$\frac{1}{1-x} = \sum_{n=0}^\infty x^n$ ($|x| < 1$)
$\ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}x^n}{n}$ ($|x| < 1$)

8 8. Differentiation

8.1 Definition

$f'(\alpha) = \lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha}$ if limit exists
Differentiable at α $\Rightarrow$ continuous at α

8.2 Rules

$(cf)' = cf'$; $(f \pm g)' = f' \pm g'$
Product: $(fg)' = f'g + fg'$
Quotient: $(f/g)' = \frac{f'g - fg'}{g^2}$ if $g \neq 0$
Chain Rule: $(g \circ f)'(\alpha) = g'(f(\alpha)) \cdot f'(\alpha)$
Inverse: If f differentiable, bijective, $f' \neq 0$, then
$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$

8.3 Mean Value Theorems

Rolle's Theorem: If f continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then $\exists \zeta \in (a,b) : f'(\zeta) = 0$
Mean Value Theorem: If f continuous on $[a,b]$, differentiable on (a,b), then $\exists \zeta \in (a,b)$:

$$f'(\zeta) = \frac{f(b)-f(a)}{b-a}$$

Cauchy MVT: $\exists \zeta \in (a,b)$:

$$f'(\zeta)(g(b)-g(a)) = g'(\zeta)(f(b)-f(a))$$

8.4 Applications of MVT

$f' \equiv 0$ on interval $\Rightarrow f$ constant
$f' > 0 \Rightarrow f$ strictly increasing
$f' \geq 0 \Rightarrow f$ increasing
f' bounded $\Rightarrow f$ Lipschitz

8.5 L'Hôpital's Rules

Type 0/0: If $\lim_{x\to\alpha} f(x) = 0 = \lim_{x\to\alpha} g(x)$ and $\lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$, then

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = L$$

Type ∞/∞: If $\lim_{x\to\alpha} f(x) = \pm\infty$, $\lim_{x\to\alpha} g(x) = \pm\infty$, and $\lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$, then

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = L$$

Works for $\alpha = \pm\infty$ and $L = \pm\infty$

8.6 Taylor's Theorem

If f is n times differentiable:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x)$$

Lagrange Remainder: $\exists \zeta$ between x_0 and x:

$$R_n(x) = \frac{f^{(n)}(\zeta)}{n!}(x-x_0)^n$$

Cauchy Remainder:

$$R_n(x) = \int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt$$

Taylor Series: If $\lim_{n\to\infty} R_n(x) = 0$:

$$f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

9 9. Riemann Integration

9.1 Definitions

Partition: $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$
Norm: $\|P\| = \max(x_i - x_{i-1})$
Tagged partition: Choose $t_i \in [x_{i-1}, x_i]$
Riemann sum: $S(f; \dot{P}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$
f is **Riemann integrable** if $\exists L : \forall \epsilon > 0, \exists \delta > 0$ such that $|S(f; \dot{P}) - L| < \epsilon$ whenever $\|\dot{P}\| < \delta$
Write $L = \int_a^b f(x)dx$

9.2 Darboux Sums

Upper sum: $U(f; P) = \sum \sup_{[x_{i-1}, x_i]} f \cdot (x_i - x_{i-1})$
Lower sum: $L(f; P) = \sum \inf_{[x_{i-1}, x_i]} f \cdot (x_i - x_{i-1})$
$U(f) = \inf_P U(f; P)$; $L(f) = \sup_P L(f; P)$
Always: $L(f; P) \leq L(f) \leq U(f) \leq U(f; P)$
Criterion: f Riemann integrable $\Leftrightarrow U(f) = L(f)$
$\Leftrightarrow \forall \epsilon > 0, \exists P : U(f; P) - L(f; P) < \epsilon$

9.3 Integrability

Riemann integrable functions are bounded
Monotone $\Rightarrow$ integrable
Continuous $\Rightarrow$ integrable
Lebesgue Criterion: Bounded f integrable $\Leftrightarrow$ discontinuity set has measure zero

9.4 Properties

If f, g integrable on $[a, b]$:
$\int_a^b(cf) = c \int_a^b f$
$\int_a^b(f+g) = \int_a^b f + \int_a^b g$
$f \leq g \Rightarrow \int_a^b f \leq \int_a^b g$
$|f|$ integrable and $|\int_a^b f| \leq \int_a^b |f|$
fg integrable
$\int_a^b f = \int_a^c f + \int_c^b f$ for $c \in (a, b)$

9.5 Fundamental Theorems of Calculus

FTC I: If F continuous on $[a, b]$, differentiable on (a, b), and F' integrable:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

FTC II: If f integrable on $[a, b]$, define $F(x) = \int_c^x f(t)dt$: - F is continuous on $[a, b]$ - If f continuous at x_0, then $F'(x_0) = f(x_0)$

9.6 Techniques

Integration by parts: If u, v continuous, u', v' integrable:

$$\int_a^b uv' + \int_a^b u'v = u(b)v(b) - u(a)v(a)$$

Substitution: If f continuous, u continuously differentiable:

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(t)dt$$

MVT for Integrals: If f continuous on $[a, b]$, $\exists \zeta \in [a, b]$:

$$f(\zeta) = \frac{1}{b-a} \int_a^b f(x)dx$$

9.7 Integration of Series

If $(f_n) \Rightarrow f$ uniformly on $[a, b]$ and each f_n integrable:
f is integrable and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$
For power series: can integrate term-by-term within radius of convergence

10 10. Quick Reference

10.1 Important Limits

$\lim_{n\to\infty} \frac{1}{n} = 0$
$\lim_{n\to\infty} n^{1/n} = 1$
$\lim_{n\to\infty} c^{1/n} = 1$ for $c > 0$
$\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for any c
$\lim_{x\to 0} \frac{\sin x}{x} = 1$
$\lim_{x\to\infty}(1 + 1/x)^x = e$

10.2 Common Series

Geometric: $\sum_{n=0}^\infty r^n = \frac{1}{1-r}$ if $|r| < 1$
Harmonic: $\sum_{n=1}^\infty \frac{1}{n}$ diverges
p-series: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges iff $p > 1$

10.3 Key Inequalities

Bernoulli: $(1+x)^n \geq 1 + nx$ for $x > -1, n \in \mathbb{N}$
AM-GM: $\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$
Cauchy-Schwarz: $|\sum a_i b_i| \leq \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$

10.4 Common Mistakes

1. Pointwise convergence $\not\Rightarrow$ uniform convergence. Example: $f_n(x) = x^n$ on $[0, 1]$. Then $f_n(x) \to f(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases}$. Limit f is discontinuous $\Rightarrow$ convergence not uniform.
2. Continuous $\not\Rightarrow$ differentiable. Example: $f(x) = |x|$. Continuous everywhere, not differentiable at $x = 0$. **3. Differentiable $\not\Rightarrow f'$ continuous.** Example: $f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then $f'(x) = 2x\sin(1/x) - \cos(1/x)$ for $x \neq 0$, $f'(0) = 0$. f' exists but oscillates wildly near 0 (not continuous).

4. $\sum a_n$ converges $\not\Rightarrow \sum a_n^2$ converges. Example: $a_n = \frac{(-1)^n}{\sqrt{n}}$. $\sum a_n$ converges (Alternating Series Test), but $\sum a_n^2 = \sum \frac{1}{n}$ diverges (harmonic series).
5. Ratio test inconclusive when limit $= 1$. Examples: (a) $a_n = \frac{1}{n}$: $\frac{a_{n+1}}{a_n} \to 1$, yet $\sum a_n$ diverges. (b) $a_n = \frac{1}{n^2}$: $\frac{a_{n+1}}{a_n} \to 1$, yet $\sum a_n$ converges.
$\Rightarrow$ Ratio test gives no information when the limit equals 1.