MATH 2101 Linear Algebra I–Matrix Algebra

What to study in Linear Algebra?

1. Systems of linear equations: One consider the solution $(x_1, \ldots, x_n)$ for

$$x_1 + 2x_2 + \ldots + nx_n = 0$$

   For example, $(2, -1, 0, \ldots, 0)$ and $(0, 3, -2, 0, \ldots, 0)$ are solutions for the equation. How do you obtain more solutions form these two?

2. One central object in linear algebra is matrices.

This course will emphasis on both *conceptual* and *practical* sides of linear algebras.

Matrix

Definition

1. A $n \times m$ matrix is a rectangular array of the form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \ddots & \ddots & & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

   where a_{ij} are in $\mathbb{R}$.
2. We shall also sometimes simply write (a_{ij}) to represent an arbitrary matrix, whose (i,j)**-entry** is a_{ij}.
3. For a matrix A, we write A_{ij} to be the (i,j) entry of A.
4. The i'**-th row** of (a_{ij}) is the $1 \times m$ matrix $\begin{pmatrix} a_{i'1} & a_{i'2} & \dots & a_{i'm} \end{pmatrix}$
5. The j'**-th column** of (a_{ij}) is the $n \times 1$ matrix $\begin{pmatrix} a_{1j'} \\ a_{2j'} \\ \vdots \\ b_{nj'} \end{pmatrix}$.

Example

Let $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 7 & 9 \end{pmatrix}$. Then

- $A_{13} = 1$
- The 2nd column of A is $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$
- The 1st row of A is $\begin{pmatrix} 2 & 4 & 1 \end{pmatrix}$.

Vectors

We shall denote $\mathbb{R}^n$ to be the set of n-tuples of numbers in $\mathbb{R}$. Elements in $\mathbb{R}^n$ can be written as **column vectors**:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

and is also sometimes written as row vectors $(a_1, \ldots, a_n)$.

- We could add two vectors: e.g.

$$(2,3) + (4,-1) = (6,2)$$

- We could also do the scalar multiplication on vectors: e.g.

$$3 \cdot (3,2) = (9,6)$$

Adding two matrices

Addition of two $n \times m$ matrices:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2m}+b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \dots & a_{nm}+b_{nm} \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 2 & 4 \\ -1 & 7 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 3 & 2 \\ 1 & 3 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 6 \\ 0 & 10 & 10 \end{pmatrix}.$$

Scalar multiplication

Scalar multiplication on a $n \times m$ matrix: for a scalar $c \in \mathbb{R}$,

$$c \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1m} \\ ca_{21} & ca_{22} & \dots & ca_{2m} \\ \vdots & \vdots & & \vdots \\ ca_{n1} & ca_{n2} & \dots & ca_{nm} \end{pmatrix}$$

Matrix multiplication

Matrix multiplication: Let $A = (a_{ij})$ be a $n \times m$ matrix and let $B = (b_{ij})$ be a $m \times p$ matrix. We define the multiplication AB as:

$$(AB)_{ij} = \sum_{x=1}^{m} a_{ix} b_{xj}$$

Thus matrix multiplication is not even defined if $n \neq m$.

Example

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & 1 \\ 9 & 2 \end{pmatrix} = \begin{pmatrix} 1(4) + 2(0) + 3(9) & 1(3) + 2(1) + 3(2) \end{pmatrix} = \begin{pmatrix} 31 & 11 \end{pmatrix}$$

remark

Where does the matrix multiplication come from? (Or why define in such way?) Consider $A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$. Define a map $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T_A(v) = Av \quad (v \text{ is a column vector in } \mathbb{R}^3),$$

$T_B : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_B(w) = Bw$$

Exercise: Show that

$$(T_B \circ T_A)(v) = (BA)v$$

Property of matrix multiplication I: associativity

Theorem

(Associativity) The matrix multiplication is an associative operation, meaning that: Let A be a $n \times m$ matrix. Let B be a $m \times p$ matrix. Let C be a $p \times q$ matrix. Then

$$A(BC) = (AB)C.$$

Proof.

Write $A = (a_{ij})$, $B = (b_{jk})$ and $C = (c_{kl})$. Then

$$(AB)_{ij} = \sum_{x=1}^{m} a_{ix} b_{xj}, \quad ((AB)C)_{il} = \sum_{y=1}^{p} (AB)_{iy} C_{yl} = \sum_{y=1}^{p} \sum_{x=1}^{m} a_{ix} b_{xy} c_{yl}$$

On the other hand,

$$(BC)_{kl} = \sum_{y=1}^{p} b_{ky} c_{yl}, \quad A(BC)_{il} = \sum_{x=1}^{m} a_{ix} (BC)_{xl} = \sum_{x=1}^{m} \sum_{y=1}^{p} a_{kx} b_{xy} c_{yl}.$$

Property of matrix multiplication II: distributive

Another property is the distributive:

Theorem

For a $n \times m$ matrix A and $m \times p$ matrices B, C, we have:

$$A(B + C) = AB + AC.$$

Exercise

Verify that $A(B + C) = AB + AC$ by using definitions.

Non-commutativity

For a $n \times m$ matrix A and a $m \times p$ matrix B, we *do not* have $AB = BA$ in general.

Example

Let $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 1+ab & a \\ b & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 1 & a \\ b & 1+ab \end{pmatrix}.$$

Identity matrix

The identity matrix I_n is defined as:

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

that is

$$(I_n)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Multiplication with an identity matrix

Theorem

Let A be an $n \times m$ matrix. Then

1. $I_n A = A$*; and*
2. $A I_m = A$.

Proof.

Exercise. $\square$

Exercise

An $n \times n$ matrix A is said to be *diagonal* if for any i, j with $i \neq j$, $A_{ij} = 0$.For example, $\begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$ is diagonal but $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ is not diagonal. Let D be a $n \times n$ diagonal matrix and let E be a $m \times m$ diagonal matrix. Let A be a $n \times m$ matrix.

1. For a diagonal $n \times n$ matrix D, describe the matrix multiplication DA;
2. For a diagonal $m \times m$ matrix E, describe the matrix multiplication AE.

Zero matrices

The $n \times m$ matrix is called the **zero matrix** $0_{n\times m}$ (or sometimes simply 0) if all its entries are zero. The main properties of the zero matrix is the following:

Theorem

Let A be an $n \times m$ matrix. Then

1. $0_{p\times n}A = 0_{p\times m}$; *and*
2. $A0_{m\times q} = 0_{n\times q}$; *and*
3. $A + 0 = A$.

Proof.

Exercise. $\square$

Transpose of a matrix

Definition

The *transpose* of an $n \times m$ matrix A is the $m \times n$ matrix obtained from interchanging rows and columns i.e.

$$(A^T)_{ij} = A_{ji}$$

Example

1. Let $A = \begin{pmatrix} 3 & 5 & 0 \end{pmatrix}$. Then
$$A^T = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}.$$
2. Let $B = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$. Then $B^T = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}$.

Properties of the transpose

Theorem

1. *For $n \times m$-matrices A and B,* $(A+B)^T = A^T + B^T$.
2. *For an $n \times m$-matrix A and a $m \times p$-matrix B,* $(AB)^T = B^T A^T$.
3. *For a $n \times m$-matrix A and a scalar c,* $(cA)^T = cA^T$.

Example

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ and let $B = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix},$$

and so

$$(AB)^T = \begin{pmatrix} 1 & -8 \end{pmatrix}, \quad B^T A^T = \begin{pmatrix} -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -8 \end{pmatrix}.$$

Where does the transpose come from?

Theorem

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and* $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$*, define*

$$\langle x, y \rangle = x_1 y_1 + \ldots + x_n y_n.$$

Then, for any $n \times n$ *matrix* A*,* $\langle xA, y \rangle = \langle x, yA^T \rangle$*.*

Proof.

One can write $\langle x, y \rangle$ as:

$$\langle x, y \rangle = xy^T.$$

Hence,

$$\langle xA, y \rangle = xAy^T$$

□

Doing transpose twice

Theorem

Let A be an $n \times m$ matrix. Then $(A^T)^T = A$.

Proof.

$$((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}.$$

$\square$