MATH 2101 Linear Algebra I–Vector Spaces III (Linear independence and basis)

Some ideas on linear independence

We have discussed the concept of a spanning set. In general, one may get a 'large' spanning set for a vector space V. For example, one may simply take $S = V$ (and so S is an infinite set), but one can ask if it is possible to find a smaller subset of V to span V. This will lead to a question on whether a vector is in a linear combination of some others. For example, suppose a set $\{v_1, v_2, v_1 + v_2\}$ spans a vector space V. Then, $v_1 + v_2$ is a linear combination of v_1 and v_2, and so $\{v_1, v_2\}$ also spans V.

We shall introduce a concept of linearly independence to systematically address this.

Definition of linear dependence

Definition

A subset S of a vector space v is called *linearly dependent* if there exists a finite number of distinct vectors $v_1, \dots v_n$ in S and $a_1, \dots, a_n$ in $\mathbb{R}$ not all zero such that

$$a_1 v_1 + \ldots + a_n v_n = 0.$$

Roughly speaking, the linear independence on a set S of vectors in V is to *ask if one can do the addition and the scalar multiplication on some vectors in S to obtain some other vectors in S*.

Example

Let $u_1 = (1, 0, -1)$, $u_2 = (0, 1, -1)$ and $u_3 = (1, -1, 0)$. Then u_1, u_2, u_3 are linearly dependent since $u_1 + (-1)u_2 + (-1)u_3 = 0$.

Example

Let u, v be two non-zero vectors in $\mathbb{R}^n$. Then u and v are linearly dependent if and only if $u = av$ for some $a \in \mathbb{R}$.

Definition of linear independence

Definition

A subset S of a vector space V is said to be *linearly independent* if S is not linearly dependent.

We have the following logical reformulation of linearly independence:

Theorem

Let $S = \{v_1, \ldots, v_r\}$ be a finite subset in V. Then S is said to be linearly independent if the only $a_1, \ldots, a_r \in \mathbb{R}$ satisfying

$$a_1 v_1 + \ldots + a_r v_r = 0$$

are $a_1 = \ldots = a_r = 0$.

The above reformulation is more convenient if one thinks to *solve some system of linear equations involving $a_1, \ldots, a_r$.*

Example

Example

Show that the vectors $(1,0,0,2), (0,1,0,1), (0,0,1,0)(0,0,1,1)$ form a set of linearly independent vectors.
Solution: We have to solve x_1, x_2, x_3, x_4 such that

$$x_1(1,0,0,2) + x_2(0,1,0,1) + x_3(0,0,1,0) + x_4(0,0,1,1) = (0,0,0,0).$$

The corresponding augmented matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right)$$

The Gaussian eliminations (here I skip the details) take to the following reduced row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

and so $x_1 = x_2 = x_3 = x_4 = 0$ is the only solution.

Example

Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. (Recall that we have defined the addition and scalar multiplication before.) Let $f_1(x) = x$ and $f_2(x) = x^2$. Show that $\{f_1, f_2\}$ is a linearly independent set.
Sol: Let $a_1, a_2 \in \mathbb{R}$ such that $a_1 f_1 + a_2 f_2 = 0$. (What is 0 in that vector space?)

- If we substitute $x = 1$, we then have $a_1 + a_2 = 0$.
- If we substitute $x = 2$, we then have $2a_1 + 4a_2 = 0$.

Solving the equations, we have $a_1 = a_2 = 0$. Hence, $\{f_1, f_2\}$ is a linearly independent set.

Relations of linear dependence among subsets

Theorem

Let $S_1 \subset S_2 \subset \mathbb{R}^n$.

- *If S_2 is linearly independent, then S_1 is also linearly independent.*
- *If S_1 is linearly dependent, then S_2 is also linearly dependent.*

Basis

Definition

A **basis** for a vector space V is a *linearly independent* subset of V which also *spans* for V.

One may think that a basis is a smallest spanning set for V. This is because if S is a basis for V, then $\text{span}(S') \neq V$ for any proper subset S' in S.

Examples

Example

Let $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, …, $\mathbf{e}_n = (0, \ldots, 0, 1)$. The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V, which will be called the *standard basis* for $\mathbb{R}^n$.

Examples

Example

Let $v_1 = (1, 2)$ and $v_2 = (2, 3)$.

- We first show that $\{v_1, v_2\}$ is linearly independent: For any $a_1, a_2 \in \mathbb{R}$ with
$$a_1v_1 + a_2v_2 = (0, 0),$$
we have
$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
and so
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
- Show $\{v_1, v_2\}$ spans $\mathbb{R}^2$: Exercise.

Examples

Example

Let $S = \{(1, 2, 0), (2, 3, 0)\}$. Note that S is a linearly independent set. Then S is a basis for $\text{span}(S)$.

Example

Let $S = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\}$. Then, by solving the system of linear equations, we have:

$$\{(-s - t - u, s, t, u) : s, t, u \in \mathbb{R}\}.$$

Then a basis for S is

$$\{(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)\}.$$

Unique linear combination from a basis

One important property for a basis is the following, which will be useful to convert things to matrices later:

Theorem

Let V be a vector space and let $\beta = \{v_1, \ldots, v_n\}$ be a subset of V. Then β is a basis for V if and only if for each vector $u \in V$, u can be **uniquely** *written as a linear combination of vectors in β i.e.*

$$u = a_1 v_1 + \ldots a_n v_n$$

for some unique scalars $a_1, \ldots, a_n$ in $\mathbb{R}$.

Proof of the theorem

Theorem

Let V be a vector space and let $\beta = \{v_1, \ldots, v_n\}$ be a subset of V. Then β is a basis for V if and only if for each vector $u \in V$, u can be **uniquely** *written as a linear combination of vectors in β i.e. $u = a_1v_1 + \ldots a_nv_n$ for some unique scalars $a_1, \ldots, a_n$ in $\mathbb{R}$.*

Proof.

By the definition of a spanning set, there exist integers $a_1, \ldots, a_n$ in $\mathbb{R}$ such that

$$u = a_1v_1 + \ldots + a_nv_n.$$

It remains to show the uniqueness. Suppose $u = b_1v_1 + \ldots + b_nv_n$ for another linear combination of vectors in β. Then,

$$a_1v_1 + \ldots + a_nv_n = b_1v_1 + \ldots + b_nv_n$$

and so

$$(a_1 - b_1)v_1 + \ldots + (a_n - b_n)v_n = 0.$$

Since $v_1, \ldots, v_n$ are linearly independent, the only solution is

$$a_1 - b_1 = a_2 - b_2 = \ldots = a_n - b_n = 0.$$

Hence, $a_1 = b_1, \ldots, a_n = b_n$. This shows the uniqueness. $\square$

Number of vectors in a basis

Theorem

Let β be a basis for a vector space V. Let n be the number of vectors in β. Let β' be a set of linearly independent vectors in V. Let m be the number of vectors in β'. Then $m \leq n$.

Proof.

Let $\beta = \{v_1, \ldots, v_n\}$. Let $\beta' = \{u_1, \ldots, u_m\}$. Assume $m > n$ to arrive a contradiction. Since β is a basis for V, β particularly spans V. Hence, we can write as:

$$u_j = a_{1j}v_1 + \ldots + a_{nj}v_n$$

Now, we consider the solution set $(x_1, \ldots, x_m)$ of the equation:

$$x_1u_1 + \ldots + x_mu_m = 0$$

We can rewrite as:

$$\sum_{i=1}^{n}(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{im}x_m)v_i = 0$$

Since $v_1, \ldots, v_n$ are linearly independent, we obtain the following system of linear equations:

$$a_{11}x_1 + \ldots + a_{1m}x_m = 0$$

$$\vdots$$

$$a_{n1}x_1 + \ldots + a_{nm}x_m = 0$$

Since $m > n$, the solution set for $(x_1, \ldots, x_m)$ is infinite. In particular, it contains a non-trivial solution. This contradicts that $u_1, \ldots, u_m$ are linearly independent. □

Corollary

Let β_1 and β_2 be any two bases for a vector space V. Then β_1 and β_2 contain the same number of vectors.

Definition

The number of vectors in a basis for a vector space V is called the *dimension*. The above discussion implies that the definition of the dimension is independent of a choice of a basis.

Example

The dimension of $\mathbb{R}^n$ is n since we have the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

Example

The dimension of the solution set for the equation

$$x_1 + \ldots + x_m = 0$$

is $m - 1$.

Subspaces and dimensions

Corollary

Let W be a vector subspace of a vector space V. Then

$$\dim(W) \leq \dim(V).$$