MATH 2101 Linear Algebra I–System of Linear Equations II

Row Echelon form

We first consider some examples.

Example

$$
\begin{aligned}
x_1 + \quad 2x_3 &= 1 \\
x_2 + 3x_3 &= 2
\end{aligned}
$$

We can set t as a variable. Set $x_1 = 1 - 2t$, $x_2 = 2 - 3t$. The solution set is given by:

$$\{(1 - 2t, 2 - 3t, t) : t \in \mathbb{R}\}.$$

The above system of linear equations is *quite simple* since one can almost read off the solution. Our goal is to 'transform' the system of linear equations to such simple forms.

Row echelon form

We first introduce a notion involving in such form:

Definition

A matrix is said to be in a **reduced row echelon form** if the following conditions are satisfied:

- The first nonzero entry in each row is the only non-zero entry in its column.
- The rows with all zero entries are in the bottom of the matrices (if any).
- The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

The first non-zero entry in each row is called a **leading entry**. Those 1 in the leading entry is called a leading one.

Summary: The first bullet defines leading entries of a matrix and describes the column with a leading entry. The second and third bullets describe the distribution of leading entries.

Examples

Example

The matrices $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are in reduced row echelon forms.

Remark

We rephrase some main features of a matrix in reduced row echelon form:

1. The leading one is the first nonzero entry in a row.
2. The leading ones come from left to right columns when going the rows from top to bottom.
3. The leading ones are in different rows.
4. For a column with a leading one, the leading entry is the only non-zero entry in the column.
5. Rows without a leading entry is the row with all entries to be zero.

Examples

Example

(Non-examples)

- $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is not in a reduced row echelon form since the row of all zeros is in the second one, but the third row is not of all zeros.
- $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not a reduced row echelon form since the first non-zero entries in the first and second rows are in the same column.
- $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not a reduced row echelon form since the leading entries do not come from left to right when going from the first row to the second row.

2×2 matrices

Example

2×2 matrices in reduced roe echelon form:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example

The following matrices are not in reduced row echelon forms:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Exercise

Write down all the reduced row echelon forms for 3×2 matrices.

Reduced row echelon form for an invertible matrix

Theorem

Let A be an $n \times n$ matrix in a reduced row echelon form. Then the following statements are equivalent:

1. *A has n leading ones;*
2. *A is invertible;*
3. *$A = I_n$.*

Proof.

(3)$\Rightarrow$(1) and (3)$\Leftarrow$(2) are clear. For $(2) \Rightarrow (1)$, $\det(A) \neq 0$ and so A has no zero rows and so there are n leading ones.

It suffices to show that $(1) \Rightarrow (3)$.

Suppose (1) holds. Since leading ones appear in different rows, the n leading entries have to be in each of the n rows. Since leading entries come from left to right, it must have to be in the diagonals. Moreover, in each column, the leading entry is the only non-zero entry. We must then have that $A = I_n$. This shows (3). $\square$

Jordan-Gaussian eliminations

- The Jordan-Gaussian elimination is the process that transforms a $n \times m$ matrix to a reduced row echelon form by a sequence of elementary row operations.
- More precisely, given a system of linear equations $Ax = b$, we represent the system of linear equations by the augmented matrix $(A|b)$. If we carry out the elementary row operation on $(A|b)$, we obtain $E(A|b) = (EA|Eb)$ for some elementary matrix E. Then the corresponding system of linear equation is $EAx = Eb$. Since E is invertible, $Ax = b$ and $EAx = Eb$ has the same solution set.

Exercise

Show that $E(A|b) = (EA|Eb)$. (Here $E(A|b)$, EA and Eb are the matrix multiplications.)

Jordan-Gaussian elimnations

Suppose we have to solve the system of linear equations:

$$\begin{aligned} x_3 &= 2 \\ 2x_1 + 2x_2 + 4x_3 &= 6 \\ x_1 + \quad x_3 &= 1 \end{aligned}$$

We write the augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 2 & 2 & 4 & 6 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

Jordan-Gaussian elimination

(1) Find the leftmost non-zero column. By interchanging rows (Type I), make the first row to contain a non-zero entry in that column e.g.

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 2 & 2 & 4 & 6 \\ 1 & 0 & 1 & 1 \end{array}\right) \overset{2\leftrightarrow 1}{\to} \left(\begin{array}{ccc|c} 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

(2) Change that entry to 1 by a scalar multiplication on the first row (Type II) e.g.

$$\left(\begin{array}{ccc|c} 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{array}\right) \overset{1\times(\frac{1}{2})}{\to} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

(3) Add the scalar multiple of the first row to other rows to make the other entries in that column to be zero (Type III).

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{array}\right) \overset{1\times(-1)+3}{\to} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{array}\right)$$

Jordan-Gaussian eliminations

(4) Repeat a similar process for the second row until all the entries in remaining rows are zero.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{array}\right) \overset{2\leftrightarrow 3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \overset{2\times(-1)}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

(5) Start from the last row with non-zero entries, we add a scalar multiple to the row above to creates zero above that 1.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \overset{3\times(-1)+2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right) \overset{3\times(-2)+1}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

(6) Repeat the process for preceding rows.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right) \overset{2\times(-1)+1}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

(7) The solution is $x_1 = -1, x_2 = 0, x_3 = 2$.

Leading variables and free variables

Definition

Given a system of linear equations $Ax = b$, we obtain an augmented matrix $(A|b)$. By the Gaussian elimination, we obtain a reduced row echelon form:

$$(A'|c').$$

A variable corresponding to a leading entry (of the reduced row echelon form) is called a **leading variable**. A variable which is not a leading variable is called a **free variable**. The solution (if any) for $Ax = b$ is given by setting free variables to be some arbitrary variables and then solve for the leading variables.

Example

For example, if the reduced row echelon form takes:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right).$$

Then, x_2, x_4 are leading variables and x_1, x_3, x_5 are free variables. Set $x_1 = s, x_3 = t, x_5 = u$. We have $x_2 = 2 - 2t - 3u$ and $x_4 = 3 - 2u$.

More example on Jordan-Gaussian eliminations

Example

Solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= -1 \\ 2x_1 + 4x_2 - x_3 + 8x_4 &= -3 \\ x_2 + \ \ x_4 &= 2 \end{aligned}$$

The augmented matrix takes the form:

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & -1 \\ 2 & 4 & -1 & 8 & -3 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

Example (con'd)

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & -1 \\ 2 & 4 & -1 & 8 & -3 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right) \overset{1\times(-2)+2}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

$$\overset{2\leftrightarrow 3}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & -1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$\overset{3\times(-1)+1}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & -2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right) \overset{2\times(-2)+1}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -6 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

Hence, x_4 is a free variable and we set $x_4 = t$. Then, the solution set is

$$\{(-2t-6, -t+2, -1) : t \in \mathbb{R}\}.$$

Another viewpoint on Jordan-Gaussian eliminations

To summarize again, the Jordan-Gaussian elimination is a process of taking a matrix to a reduced row echelon form. Another way to see the process is to reduce the redundacy of the sytem as well as the complexity of the system. For example, one may have some trivial 'redundant' equations like:

$$x_1 + x_2 = 1 \tag{1}$$
$$2x_1 + 2x_2 = 2 \tag{2}$$
$$3x_1 + 3x_2 = 3 \tag{3}$$

The second and third equations are just essentially differed from the first one by a scalar.The elimination process reduces three equations into solving the first one. Of course, the elimination process can handle much more complicated situation than the above one. We now have a closer look on some special situations: systems of no solutions, systems of infinite solutions.

System of linear equations that has no solutions

Example

The equation $0x_1 + 0x_2 = 1$ has no solution!

Example

The system of linear equations:

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ 4x_1 + 2x_2 &= 3 \end{aligned}$$

also has no solution!

System of linear equations that has no solutions

Definition

A system of linear equations is said to be **consistent** if the solution set to that system is non-empty. Otherwise, we say that the system of lienar equations is **inconsistent**.

Definition

A matrix B is said to be a **reduced row echelon form of a matrix** A if B is obtained by the Jordan-Gaussian elimination from A.

Theorem

A system of linear equations $Ax = b$ is inconsistent if and only if the reduced row echelon form of the augmented matrix $(A|b)$ has a row which has only one non-zero entry in the last column.

Example

If the reduced row echelon form of $(A|b)$ takes the form

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

then the system $Ax = b$ is inconsistent. (Actually, if one writes the system of linear equations for the reduced row echelon form, it takes

$$\begin{aligned} x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}$$

More example

Example

Solve the following system of linear equations:

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 2x_1 - 3x_2 + x_3 &= 0 \\ 3x_1 - 4x_2 - 2x_3 &= -6 \\ x_1 + \quad 5x_3 &= 4 \end{aligned}$$

We first obtain an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 2 & -3 & 1 & 0 \\ 3 & -4 & -2 & -6 \\ 1 & 0 & 5 & 4 \end{array}\right)$$

Example (con'd)

This gives that

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 2 & -3 & 1 & 0 \\ 3 & -4 & -2 & -6 \\ 1 & 0 & 5 & 4 \end{array}\right) \xrightarrow{1\times(-2)+2} \left(\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 1 & 3 & 4 \\ 3 & -4 & -2 & -3 \\ 1 & 0 & 5 & 4 \end{array}\right)$$

$$\xrightarrow{1\times(-3)+3} \left(\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 5 & 4 \end{array}\right) \xrightarrow{1\times(-1)+4} \left(\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 6 & 6 \end{array}\right)$$

$$\xrightarrow{2\times(-2)+3} \left(\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -5 \\ 0 & 2 & 6 & 6 \end{array}\right) \xrightarrow{2\times(-2)+4} \left(\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

From this, we know the reduced row echelon form has a row with only non-zero in the last column. Hence, the system is inconsistent.

Infinitely many solutions

Definition

Let A be an $n \times m$ matrix. A system of linear equations $Ax = b$ is called **homogeneous** if $b = 0$. A homogeneous solution always has a solution, that is

$$x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Such solution is called the **trivial** solution.

Infinitely many solutions

Theorem

Let $Ax = 0$ be a homogeneous system of n linear equations and m variables. If $m > n$, then there is an infinite number of solutions.

Proof.

We consider the augmented matrix of the form $(A \mid 0_{n\times 1})$. By carrying out the Jordan-Gaussian elimination, we obtain a reduced row echelon form, say $(A' \mid 0_{n\times 1})$. Since $m > n$, we must have some columns which do not contain a leading entry. In other words, the reduced row echelon form has some free variables. Those free variables constitute an infinite number of solutions. $\square$

Example

The equation $2x_1 + 3x_2 - x_3 = 0$ has the solution set:

$$\left\{ (-\frac{3}{2}s + \frac{1}{2}t, s, t)^T : s, t \in \mathbb{R} \right\}.$$

In general, we have the following:

Theorem

Let $Ax = b$ be a system of n linear equations with m variables. Suppose $m > n$.

- *The system has a solution, then there is an infinite number of solutions for the system.*
- *Moreover, let $x_p \in \mathbb{R}^m$ satisfy $Ax_p = b$. The solution set for the system $Ax = b$ is*

$$\{x_p + y \in \mathbb{R}^m : Ay = 0\}.$$

Example

Let A be an $n \times 4$ matrix. Suppose the solution set of the system of linear equations $Ax = 0$ is given by $\left\{(2t, s, 3t+2s, t-s)^T : t, s \in \mathbb{R}\right\}$. If $(0,1,0,0)^T$ is a solution to $Ax = b$, find the solution set of $Ax = b$.

Solution: The solution set is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2t \\ s \\ 3t+2s \\ t-s \end{pmatrix} : t, s \in \mathbb{R} \right\}.$$

Proof of the theorem

Theorem

Let $Ax = b$ be a system of n linear equations with m variables. Suppose $m > n$.

- *The system has a solution, then there is an infinite number of solutions for the system.*
- *Moreover, let x_p satisfy $Ax_p = b$. The solution set for the system $Ax = b$ is*

$$\{x_p + y \in \mathbb{R}^m : Ay = 0\}.$$

Proof.

The proof for the first bullet is similar to that of the previous theorem and we leave as an exercise.
We only consider the second bullet. If $x' = x_p + y$ for some $Ay = 0$, we have that $Ax' = A(x_p + y) = Ax_p + Ay = b$. Hence, $x_p + y$ is a solution for $Ax = b$. Conversely, if $Ax' = b$, then $A(x' - x_p) = b - b = 0$. Hence,

$$x' = x_p + (x' - x_p) \in \{x_p + y \in \mathbb{R}^m : Ay = 0\}.$$

$\square$

Invertibility and solution set

Theorem

Let A be an $n \times n$ matrix. Then the system $Ax = 0$ has a non-trivial solution if and only if A is not invertible.

Proof.

Suppose A is invertible. By taking the inverse, $x = 0$ is the only solution.
If A is not invertible, then the reduced row echelon form of A cannot be I_n. Thus the reduced row echelon form of $(A|0)$ has less leading entries than the number of columns. In other words, the system has free variables. Hence, the solution has non-trivial solution. $\square$

Indeed, the proof of the previous theorem also gives:

Theorem

Let A be an $n \times n$ matrix. Then the following statements are equivalent:

(a) *the system $Ax = 0$ has a non-trivial solution;*

(b) *the system $Ax = 0$ has infinitely many solutions;*

(c) *A is not invertible.*

Example

Example

Determine if the following system of linear equations has infinitely many solutions:

$$\begin{aligned} 3x_1 - 4x_2 + 5x_3 &= 0 \\ 2x_1 - x_2 + 4x_3 &= 0 \\ -x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$