# MATH 2101 Linear Algebra I–System of Linear Equations III

# Theoretical aspect: Inverse and Reduced row echelon form

In the last part of this section, we shall study more on reduced row echelon forms for a *square* matrix and related matters.

### Theorem

*Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if the reduced row echelon form of $A$ is $I_n$.*

### Proof.

We carry out the Jordan-Gaussian elimination to transfer $A$ to a reduced row echelon form $B$. Then $E_r \dots E_1 A = B$ for some elementary matrices $E_1, \dots, E_r$. Since $E_1, \dots, E_r$ are invertible, $A$ is invertible if and only if $B$ is invertible. We have shown that the latter condition is equivalent to $I_n$. $\square$

# Theoretical aspect: Invertible matrix and elmentary matrices

Corollary

*Every invertible matrix is a product of elementary matrices.*

Proof.

As shown in the previous proof,

$$E_r \dots E_1 A = I_n$$

for some elementary matrices $E_1, \dots, E_r$. Then $A = E_1^{-1} \dots E_r^{-1}$. We have shown before that the inverse of an elementary matrix is still elementary and thus we have shown the corollary.

$\square$

# Computational aspect: finding an inverse

We illustrate the idea on using elementary operations to find an inverse. Let $A$ be an invertible $n \times n$ matrix. Then we form $n \times 2n$ augmented matrix:

$$\begin{pmatrix} A & | & I_n \end{pmatrix}$$

Then, $A^{-1}\begin{pmatrix} A & | & I_n \end{pmatrix} = \begin{pmatrix} A^{-1}A & | & A^{-1} \end{pmatrix} = \begin{pmatrix} I_n & | & A^{-1} \end{pmatrix}$. Now the idea is to use elementary row operations/Jordan-Gaussian eliminations to change $\begin{pmatrix} A & | & I_n \end{pmatrix}$ to $\begin{pmatrix} I_n & | & A^{-1} \end{pmatrix}$. Then we can read $A^{-1}$ from the product of elementary matrices.

# Example

**Example**

Find the inverse of $\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$. We compute in the following way:

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{array}\right) \overset{1\times(2)+2}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 10 & 2 & 1 \end{array}\right) \overset{2\times(\frac{1}{10})}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{2}{10} & \frac{1}{10} \end{array}\right)$$

$$\overset{2\times(-3)+1}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & 0 & \frac{4}{10} & \frac{-3}{10} \\ 0 & 1 & \frac{2}{10} & \frac{1}{10} \end{array}\right)$$

Then $A^{-1} = \begin{pmatrix} \frac{4}{10} & \frac{-3}{10} \\ \frac{2}{10} & \frac{1}{10} \end{pmatrix}$.Moreover, we can write $A^{-1}$ as a product of elementary matrices:

$$A^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Example

Find the inverse of $\begin{pmatrix} 2 & 4 & 3 \\ 4 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix}$. We compute in the following way:

$$\begin{pmatrix} 2 & 4 & 3 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{1\times\frac{1}{2}} \begin{pmatrix} 1 & 2 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{1\times(-4)+2} \begin{pmatrix} 1 & 2 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -6 & -5 & -2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{2\times\frac{1}{-6}} \begin{pmatrix} 1 & 2 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{6} & \frac{1}{-6} & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{2\times(-2)+3} \begin{pmatrix} 1 & 2 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{6} & \frac{1}{-6} & 0 \\ 0 & 0 & \frac{8}{6} & \frac{-4}{6} & \frac{2}{6} & 1 \end{pmatrix}$$

$$\xrightarrow{3\times\frac{6}{8}} \begin{pmatrix} 1 & 2 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{6} & \frac{2}{6} & \frac{1}{-6} & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \xrightarrow{3\times(\frac{-5}{6})+2} \begin{pmatrix} 1 & 2 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{9}{12} & \frac{-9}{24} & -\frac{5}{8} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\xrightarrow{3\times(\frac{-3}{2})+1} \begin{pmatrix} 1 & 2 & 0 & \frac{5}{4} & \frac{-3}{8} & \frac{-9}{8} \\ 0 & 1 & 0 & \frac{9}{12} & \frac{-9}{24} & -\frac{5}{8} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \xrightarrow{2\times(-2)+1} \begin{pmatrix} 1 & 0 & 0 & \frac{-1}{4} & \frac{9}{24} & \frac{1}{8} \\ 0 & 1 & 0 & \frac{9}{12} & \frac{-9}{24} & -\frac{5}{8} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Hence, the inverse is

$$\begin{pmatrix} \frac{-1}{4} & \frac{9}{24} & \frac{1}{8} \\ \frac{9}{12} & \frac{-9}{24} & -\frac{5}{8} \\ \frac{-1}{2} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

## Exercise

Express the inverse in the previous example in terms of a product of elementary matrices.