MATH 2101 Linear Algebra I–Determinants and inverses

2×2 determinant

Determinant of a matrix is to compute a scalar value from a matrix. It has many applications.

Definition

(2×2 matrix) The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$.

Example

$\det \begin{pmatrix} 3 & 4 \\ 2 & 9 \end{pmatrix} = 27 - 8 = 19$

Remark

The area formed by the square $(0,0)^T, (1,0)^T, (0,1)^T, (1,1)^T$ is 1. Let A be a 2×2 matrix. The area of the parallelogram formed by

$$A(0,0)^T, A(1,0)^T, A(0,1)^T, A(1,1)^T$$

is given by $|\det(A)|$.

Definition for general cases

Definition

($n \times n$ matrix) The *determinant* of an $n \times n$ matrix A is defined inductively as follows. We first define a minor $\widetilde{A}_{ij}$ to be the matrix obtained by deleting the i-th row and j-th column. Fixing certain j, we then define:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(\widetilde{A}_{ij})$$

or fixing certain i, define

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\widetilde{A}_{ij}).$$

The above definition is *independent of a choice of a row or a column*.

Example

Example

Let $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 9 & 1 \\ 0 & -1 & -5 \end{pmatrix}$. The minors for the second column is:

$$\widetilde{A}_{12} = \begin{pmatrix} 2 & 1 \\ 0 & -5 \end{pmatrix}, \quad \widetilde{A}_{22} = \begin{pmatrix} 3 & 1 \\ 0 & -5 \end{pmatrix}, \quad \widetilde{A}_{32} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} \det(A) &= (-1)^{1+2}(4)\det\begin{pmatrix} 2 & 1 \\ 0 & -5 \end{pmatrix} + (-1)^{2+2}(9)\det\begin{pmatrix} 3 & 1 \\ 0 & -5 \end{pmatrix} + (-1)^{2+3}(-1)\det\left(\begin{matrix} 3 \\ 2 \end{matrix}\right. \\ &= (-1)(4)(-10) + (9)(-15) - (1)1 \end{aligned}$$

Determinant for identity matrices

Theorem

$$\det(I_n) = 1$$

$$\det(0_{n\times n}) = 0.$$

Proof.

Inductively, we have

$$\det(I_n) = (-1)^{1+1}(1)\det(I_{n-1}) = 1$$

We leave $\det(0_{n\times n}) = 0$ for an exercise.

$\square$

Exercise on determinants

Exercise

Find

$$\det \begin{pmatrix} 3 & 4 & 0 & 0 \\ 2 & 9 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & -5 \end{pmatrix}$$

Exercise

Let A be a 3×3 matrix. Prove that there exists $t \in \mathbb{R}$ such that $\det(A - tI_3) = 0$. (Hint: Consider $\det(A - xI_3)$ as a polynomial in x and solve for x.)

Property 1: Switching rows

We shall not show that the definition is independent of a choice of a row or a column, but we shall use this fact to show the followings:

Theorem

(Property 1: switching rows) Let A be an $n \times n$ matrix. If B is obtained from A by switching two rows, then

$$\det(A) = -\det(B).$$

Proof.

We first consider that the two rows are consecutive i.e. B is obtained from A by interchanging the i-th and $(i+1)$-th rows. Now, we compute $\det(B)$ by using the $i+1$-th row and so we have:

$$\det(B) = \sum_{j=1}^{n} (-1)^{i+1+j} \widetilde{B}_{i+1,j}.$$

But, we have that $\widetilde{B}_{i+1,j} = \widetilde{A}_{i,j}$. On the other hand, by computing $\det(A)$ by using the i-th row,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \widetilde{A}_{ij}.$$

The general case follows from switching two consecutive rows multiple times. □

Example

Theorem

(Property 1: swtiching rows) Let A be an $n \times n$ matrix. If B is obtained from A by switching two rows, then

$$\det(A) = -\det(B).$$

Example

$$\det \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} = -\det \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}.$$

Property II: Two identical rows

Corollary

(Property 2: two identical rows) Let A be an $n \times n$ matrix. If A has two identical rows, then $\det(A) = 0$.

Proof.

By using the previous theorem, we have $\det(A) = -\det(A)$ by switching the two identical rows. Hence $2\det(A) = 0$ and so $\det(A) = 0$. $\square$

Example

$$\det \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix} = 0.$$

Property: adding a scalar multiple on a row vector

Theorem

(Adding a scalar multiple on a row vector) Let $v_1, \ldots, v_n$ be row vectors in $\mathbb{R}^n$ and let u be another row vector in $\mathbb{R}^n$. Let $k \in \mathbb{R}$. Then

$$\det\begin{pmatrix} v_1 \\ \vdots \\ v_{r-1} \\ v_r + ku \\ v_{r+1} \\ \vdots \\ v_n \end{pmatrix} = \det\begin{pmatrix} v_1 \\ \vdots \\ v_{r-1} \\ v_r \\ v_{r+1} \\ \vdots \\ v_n \end{pmatrix} + k\det\begin{pmatrix} v_1 \\ \vdots \\ v_{r-1} \\ u \\ v_{r+1} \\ \vdots \\ v_n \end{pmatrix}$$

Proof.

The main idea of the proof is to choose the r-th row for computing the determinant. The details are left as an exercise. □

Example

Example

Let $v_1 = \begin{pmatrix}1 & 2 & 3\end{pmatrix}$, $v_2 = \begin{pmatrix}1 & 0 & -1\end{pmatrix}$, $v_3 = \begin{pmatrix}0 & 2 & 1\end{pmatrix}$. Let $u = \begin{pmatrix}1 & 4 & -3\end{pmatrix}$.
Then

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 1+2(1) & 0+2(4) & -1+2(-3) \\ 0 & 2 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix} + 2\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & -3 \\ 0 & 2 & 1 \end{pmatrix}$$

Example

$$\det \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \det \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}$$

Exercise

Find the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 2 & 1 & 0 \end{pmatrix}$$

Exercise

Show that

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} = 0.$$

Transpose and determinants

Theorem

(Taking transpose) Let A be an $n \times n$ matrix. Then $\det(A^T) = \det(A)$.

Proof.

We shall prove by an induction. When $n = 1$, it is clear. Write $A = (a_{ij})$. We first pick the first row for computing $\det(A)$. Then we have:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(\widetilde{A}_{1j})$$

Write $A^T = (b_{ij})$. We now pick the first column for computing $\det(A^T)$ and we have:

$$\det(A^T) = \sum_{i=1}^{n} (-1)^{i+1} b_{i1} \det(\widetilde{A^T}_{i1}).$$

Note that $A^T_{1x} = \widetilde{A^T}_{x1}$ and $a_{1x} = b_{x1}$. Inductively, we have $\det(A^T_{1x}) = \det(A_{1x})$. Combining the formulas, we see that two expressions coincide. □

From rows to columns

One can use the previous result to switch rows and columns to obtain:

Theorem

Let A be an $n \times n$ matrix. Then

1. *If two columns of A are switched to obtain B, then* $\det(B) = -\det(A)$.
2. *If two columns of A are identical, then* $\det(A) = 0$.
3. *Let $v_1, \ldots, v_n$ be column vectors in $\mathbb{R}^n$ and let u be another column vector in $\mathbb{R}^n$. Let $k \in \mathbb{R}$. Then*

$$\begin{aligned} &\det\begin{pmatrix} v_1 & \ldots & v_{r-1} & v_r + ku & v_{r+1} & \ldots & v_r \end{pmatrix} \\ =&\det\begin{pmatrix} v_1 & \ldots & v_{r-1} & v_r & v_{r+1} & \ldots & v_n \end{pmatrix} \\ &+ k\det\begin{pmatrix} v_1 & \ldots & v_{r-1} & u & v_{r+1} & \ldots & v_n \end{pmatrix} \end{aligned}$$

Determinant for scalar multiplications

Theorem

(Scalar multiplication) For an $n \times n$ matrix A and a scalar c, $\det(cA) = c^n \det(A)$.

Proof.

We shall prove by an induction on n. When $n = 1$, the statement is trivial. We now consider $n \geq 2$. Then, by definitions,

$$\det(cA) = \sum_{i=1}^{n} (-1)^{i+j} (ca_{ij}) \det(\widetilde{cA}_{ij}).$$

By inductive hypothesis, we have that:

$$\det(\widetilde{cA}_{ij}) = c^{n-1} \det(\widetilde{A}_{ij}).$$

Hence, $\det(cA) = \sum_{i=1}^{n} (-1)^{i+j} c^n \left(a_{ij} \det(\widetilde{A}_{ij}) \right)$.

□

Multiplicative property

Theorem

(Multiplicative property) Let A, B be two $n \times n$-matrices. Then $\det(AB) = \det(A) \cdot \det(B)$.

Exercise

An $n \times n$ matrix A is said to be *orthogonal* if $AA^T = I_n$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

How about determinant on matrix additons?

There is no formula between determinant and matrix additions! In general,

$$\det(A+B) \neq \det(A) + \det(B).$$

Example

$$\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \neq \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Definition of an inverse

Definition

An $n \times n$ matrix A is said to be *invertible* if there exists a matrix B such that $AB = BA = I_n$. We say call B to be the *inverse* of A. We shall write the inverse of A to be A^{-1}.

Remark

The inverse of an invertible matrix A is unique. If C is another inverse of A, then $C(AB) = CI_n = C$ and $(CA)B = I_nB = B$. By associativity, $C = C(AB) = (CA)B = B$.

Determinant of A^{-1}

Theorem

Let A be an invertible $n \times n$-matrix. Then $\det(A) \neq 0$. *Moreover,* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof.

$$AA^{-1} = I_n \overset{\text{taking det}}{\Longrightarrow} \det(AA^{-1}) = \det(I_n) \overset{\mathrm{Det}(I_n)=1}{\Longrightarrow} \det(AA^{-1}) = 1$$

$$\overset{\text{use } \det(AB) = \det(A)\det(B)}{\Longrightarrow} \det(A)\det(A^{-1}) = 1$$

Hence, $\det(A^{-1}) = \frac{1}{\det(A)}$. □

Proof

Theorem

Let A be an $n \times n$ matrix. Define the adjugate of A as:

$$\text{adj}(A) = \begin{pmatrix} (-1)^{1+1}\det\widetilde{A}_{11} & \dots & (-1)^{1+n}\det\widetilde{A}_{1n} \\ \vdots & & \vdots \\ (-1)^{n+1}\det\widetilde{A}_{n1} & \dots & (-1)^{n+n}\det\widetilde{A}_{nn} \end{pmatrix}^T$$

If $\det(A) \neq 0$*, then* $\frac{1}{\det A} \cdot \text{adj}(A)$ *is the inverse of* A*.*

Proof.

One needs to use the following two formulas: when multiplying the i-th row of A with i-th column in $\text{adj}(A)$,

$$\sum_{x=1}^{n} (-1)^{i+x} a_{ix} \det\widetilde{A}_{ix} = \det(A)$$

when multiplying the i-th row with i'-th column in $\text{adj}(A)$ ($i' \neq i$),

$$\sum_{x=1}^{n} (-1)^{i+x} a_{ix} \det\widetilde{A}_{i'x} = 0$$

(Why?) $\square$

Detecting invertibility of a matrix by computing determinant

It is *not flesible* to check the invertibility of a matrix from definitions. The determinant provides a convenient way to do so:

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$.

A formula for computing A^{-1}

Theorem

Let A be an $n \times n$ matrix. Define the adjugate of A as:

$$\mathrm{adj}(A) = \begin{pmatrix} (-1)^{1+1}\det\widetilde{A}_{11} & \dots & (-1)^{1+n}\det\widetilde{A}_{1n} \\ \vdots & & \vdots \\ (-1)^{n+1}\det\widetilde{A}_{n1} & \dots & (-1)^{n+n}\det\widetilde{A}_{nn} \end{pmatrix}^T,$$

where $\widetilde{A}_{ij}$ is the (i,j)-minor matrix of A. If $\det(A) \neq 0$, then

$$\frac{1}{\det A} \cdot \mathrm{adj}(A)$$

is the inverse of A.

Example

Example

Determine if the following matrices are invertible. If it is invertible, find the inverse.

- $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}$
- $C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$.

Solution:

- $\det(A) = \det(B) = 0$ (why? explain it!) and so A and B are not invertible.
- $\det(C) = 1 \neq 0$ and so C is invertible. Then

$$\operatorname{adj}(C) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}^T, C^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

Inverse with other operations

Theorem

Let A, B be an invertible $n \times n$ matrix. Let $c \in \mathbb{R}$. Then

1. $(AB)^{-1} = B^{-1}A^{-1}$;
2. $(cA)^{-1} = c^{-1}A^{-1}$;
3. $(A^T)^{-1} = (A^{-1})^T$;
4. $(A^{-1})^{-1} = A$.

Proof.

We only check for the left inverses.

1. $(B^{-1}A^{-1})(AB) = B^{-1}I_nB = B^{-1}B = I_n$
2. $(c^{-1}A^{-1})(cA) = c^{-1}cA^{-1}A = I_n$
3. $(A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n$
4. $AA^{-1} = I_n$

Applications of determinants and inverses

Determinants and matrix inversions are useful in solving system of linear equations e.g. Carmer's rule. We will soon talk about this in next section. To give you an idea, let A be an $n \times n$ matrix and b be a column vector in $\mathbb{R}^n$. If we want to *solve* a column vector $x \in \mathbb{R}^n$ such that $Ax = b$ and A is invertible, then the *solution* to the equation is:

$$x = A^{-1}b$$

since $A(A^{-1}b) = I_n b = b$.