MATH 2101 Linear Algebra I–Vector Spaces I

Vector Spaces

Our goal now is to abstractize some important properties and natures of $\mathbb{R}^n$, which involves the addition of two vectors and the scalar multiplication on a vector.

Definition of a vector space

Definition

A vector space V over $\mathbb{R}$ consists of a set on which two operations, called additions and scalar multiplications, are defined in the following ways:

1. (addition) for $x, y \in V$, a unique element $x + y$ is defined in V:
2. (scalar multiplication) for $x \in V$ and $a \in \mathbb{R}$, a unique element ax is defined in V.

such that

1. For any $x, y \in V$, $x + y = y + x$ (commutativity of addition).
2. For any $x, y, z \in V$, $(x + y) + z = x + (y + z)$. (associativity of addition)
3. There exists an element in V, denoted by 0, such that $x + 0 = 0 + x = x$. (zero element in V)
4. For each $x \in V$, there exists an element $y \in V$ such that $x + y = 0$.
5. For any $x \in V$, $1x = x$.
6. For any $a, b \in \mathbb{R}$ and $x \in V$, $a(bx) = (ab)x$. (associativity of scalar multiplications)
7. For any $a, b \in \mathbb{R}$ and $x \in V$, $(a + b)x = ax + bx$.
8. For any $a \in \mathbb{R}$ and any $x, y \in V$, $a(x + y) = ax + ay$. (Distributivity for scalar multiplications)

We write $-v$ for $(-1)v$, the scalar multiple -1 on v.

Examples

Example

The Euclidean space $\mathbb{R}^n$ is a vector space. The additon and the scalar multiplication are the usual ones: for $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$

$$c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n).$$

Example

The set of $n \times m$ matrices. The addition is the matrix addition. The scalar multiplication is also simply the one defined for matrices before.

Examples

Example

Let V be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. The addition is defined as: for $f_1, f_2 \in V$, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ for $x \in \mathbb{R}$. The scalar multiplication is defined as: for $f \in V$ and $c \in \mathbb{R}$, $(cf)(x) = c(f(x))$ for $c \in \mathbb{R}$.

Vector subspaces

Definition

A non-empty subset S of a vector space V is called a *vector subspace* of V if it is closed under matrix addition and scalar multiplication i.e. for $v_1, v_2 \in S$,

1. $v_1 + v_2 \in S$;
2. for any $c \in \mathbb{R}$ and any $v \in S$, $cv \in S$.

The way of defining is to ensure that a vector subspace is *still* a vector space under the same addition and scalar multiplication.

Examples

Example

The set $\{(x, y, 0) : x \in \mathbb{R}, y \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^3$. Also visualize the vector subspace geometrically.

Example

Let v be a non-zero vector in $\mathbb{R}^n$. The set $\{cv : c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$.

Examples

Example

The set $\{(1,2),(3,4)\}$ with only two non-zero points is not a vector subspace of $\mathbb{R}^2$. In general, a set with only finitely many **non-zero** points is not a vector subspace.

Example

The set of all diagonal $n \times n$ matrices is a vector subspace of the set of all $n \times n$ matrices.

Examples

Example

Determine if the following subsets are subspaces:

1. $S_1 = \{(1,2),(2,3)\}$;
2. $S_2 = \{(2t,3t,4t) : t \in \mathbb{R}\}$;
3. $S_3 = \mathbb{R}^3 \setminus \{(2t,3t,4t) : t \in \mathbb{R}\}$;
4. $S_4 = \{(1+2t,2t,4t) : t \in \mathbb{R}\} \cup \{(0,0,0)\}$;
5. $S_5 = \{(0,0,0,0)\}$;
6. $S_6 = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^3 + x_2^3 + x_3^2 = 0\right\}$.

Example

1. $S_1 = \{(1,2),(2,3)\}$; S_1 is not a subspace since $(0,0) = 0(1,2)$ is not in S_1.
2. $S_2 = \{(2t,3t,4t) : t \in \mathbb{R}\}$; S_2 is a subspace.
3. $S_3 = \mathbb{R}^3 \setminus \{(2t,3t,4t) : t \in \mathbb{R}\}$; S_3 is not a subspace since $(2,0,0),(0,3,4) \in S_3$ but $(2,3,4)$ is not in S_3.
4. $S_4 = \{(1+2t,2t,4t) : t \in \mathbb{R}\} \cup \{(0,0,0)\}$; S_4 is not a subspace since $(1,0,0) \in S_4$, but $(2,0,0) \notin S_4$.
5. $S_5 = \{(0,0,0,0)\}$; S_5 is a subspace, and is called **zero subspace**.
6. $S_6 = \left\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^3 + x_2^3 + x_3^2 = 0\right\}$; S_6 is not a subspace since $(-1,0,1),(0,-1,1) \in S_6$ but $(-1,-1,2) \in S_6$.

Vector subspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$

We give a geometric description of all vectors subspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$:

Example

Any vector subspace of $\mathbb{R}^2$ is one of the followings:

- $\{0\}$;
- any line passing through the origin;
- $\mathbb{R}^2$.

Example

Any vector subspace of $\mathbb{R}^3$ is one of the followings:

- $\{0\}$;
- any line passing through the origin;
- any plane passing through the origin;
- $\mathbb{R}^3$.

Example 1: Homogeneous system of linear equations and vector spaces

We first recall the following definition.

Definition

Let A be a $n \times m$ matrix and let b be a column vector in $\mathbb{R}^n$. A system of linear equations $Ax = b$ of n equations and m variables is said to be *homogeneous* if $b = 0_{n\times 1}$.

Theorem

The solution set of a homogeneous system of linear equations is a vector subspace i.e. for an $n \times m$ matrix A, $\{x \in \mathbb{R}^m : Ax = 0\}$ is a vector subspace.

Example 1: Homogeneous system of linear equations and vector spaces

Theorem

The solution set of a homogeneous system of linear equations is a vector subspace of $\mathbb{R}^m$ *i.e. for an* $n \times m$ *matrix* A, $\{x \in \mathbb{R}^m : Ax = 0\}$ *is a vector subspace of* $\mathbb{R}^m$.

Proof.

Let $S = \{x \in \mathbb{R}^m : Ax = 0\}$. For $v_1, v_2 \in S$, $Av_1 = 0$ and $Av_2 = 0$. Hence $A(v_1 + v_2) = Av_1 + Av_2 = 0 + 0 = 0$. Hence, $v_1 + v_2 \in S$. For $v \in S$ and $c \in \mathbb{R}$, $A(cv) = c(Av) = c(0) = 0$ and so $cv \in S$. □

Example 2: Multiplication with a matrix

Theorem

Let A be an $n \times m$ matrix. The set $S = \{Ax : x \in \mathbb{R}^m\}$ is a vector subspace of $\mathbb{R}^m$.

Proof.

(Addition) For $y_1, y_2 \in S$, $y_1 = Ax_1$ and $y_2 = Ax_2$ for some $x_1, x_2 \in \mathbb{R}^m$. Then $y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2) \in S$.
(Scalar multiplication) For $y' \in S$ and $c \in \mathbb{R}$, write $y' = Ax'$ for some $x' \in \mathbb{R}^m$. Then $cy' = c(Ax') = A(cx') \in S$.
Hence, S is a vector subspace. $\square$

Remark

The subspace is related to so-called the column space of a matrix, which will be discussed later.