5 Vector Calculus

5.1 Length of Curves

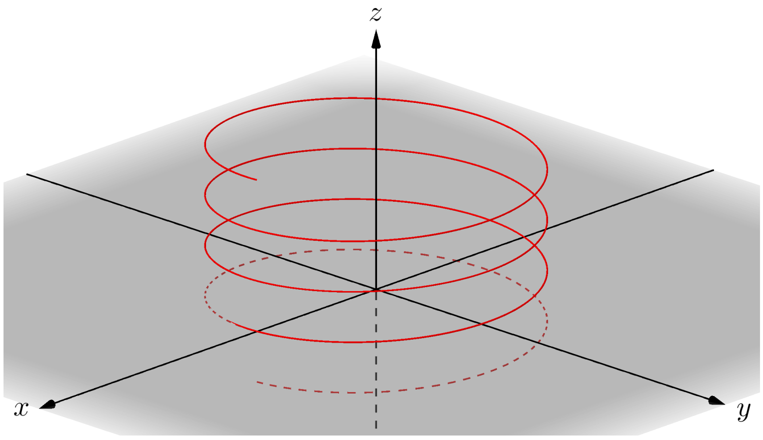
Definition 5.1. A *path* in $\mathbb{R}^n$ is a continuous function $\gamma : I \to \mathbb{R}^n$ where I is an interval in $\mathbb{R}$. The image set $\gamma(I)$ is called a *curve*.

If $I = [a, b]$ is a closed and bounded interval, we call $\gamma(a)$ and $\gamma(b)$ the *endpoints* of γ.

Example 5.1. Define $\gamma : \mathbb{R} \to \mathbb{R}^3$ by

$$\gamma(t) = (a \cos t, a \sin t, bt)$$

where a and b are some positive constants. The image of γ is called a *circular helix*.



Let $\gamma(t)$ be a path which is differentiable. We use $\gamma'(t)$ to denote the vector containing all derivatives of the component functions of $\gamma(t)$.

Definition 5.2. Let $\gamma : I \to \mathbb{R}^n$ be a differentiable path. Then $\mathbf{v}(t) = \gamma'(t)$ is called the *velocity vector* of the path, and its length $\|\mathbf{v}(t)\|$ is called the *speed* of the path.

It can be shown that $\mathbf{v}(t)$ is parallel to the tangent vector to $\gamma(t)$.

Example 5.2. Consider the path $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\gamma(t) = (a \cos t, a \sin t, bt).$$

Then we have

$$\mathbf{v}(t) = (-a \sin t, a \cos t, b)$$

and hence $\|\mathbf{v}(t)\| = \sqrt{a^2 + b^2}$.

Proposition 5.1. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a differentiable path of class C^1, where $a, b \in \mathbb{R}$. Then the length of γ is given by

$$\int_a^b \|\gamma'(t)\| \, dt.$$

If a path is only piecewise C^1, which means it can be partitioned into a finite number of C^1 paths, then we can find its length by adding up the lengths of the pieces.

Example 5.3. The length of the path $\gamma : [0, 4\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$ is

$$\int_0^{4\pi} \|(-\sin t, \cos t)\| \, dt = 4\pi.$$

5.2 Differential Operators

Definition 5.3. A *vector field* on $\mathbb{R}^n$ is a function $\mathbf{F} : X \to \mathbb{R}^n$ where $X \subset \mathbb{R}^n$.

We can visualize a vector field by drawing a vector $\mathbf{F}(\mathbf{x})$ at each point $\mathbf{x} \in X$ whose tail is at $\mathbf{x}$.

Definition 5.4. A *scalar field* is a function $f : X \to \mathbb{R}$ where $X \subset \mathbb{R}^n$.

Definition 5.5. A *gradient field* or a *conservative vector field* on $\mathbb{R}^n$ is a vector field $\mathbf{F} : X \to \mathbb{R}^n$ which is the gradient of some function $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$.

Example 5.4. Let c be a nonzero constant. The vector field $\mathbf{F} : \mathbb{R}^3 - \{\mathbf{0}\} \to \mathbb{R}^3$ defined by

$$\mathbf{F}(\mathbf{v}) = \frac{c}{\|\mathbf{v}\|^3}\mathbf{v}$$

is a gradient field. In fact, we have $\nabla f = \mathbf{F}$ where $f : \mathbb{R}^3 - \{\mathbf{0}\} \to \mathbb{R}$ is defined by

$$f(\mathbf{v}) = -\frac{c}{\|\mathbf{v}\|}.$$

Definition 5.6. The *del operator* in $\mathbb{R}^n$ is defined by

$$\nabla = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n.$$

It maps a scalar field to a vector field.

For example, given any differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

This is how we define the gradient of a scalar-valued function.

Definition 5.7. Let $\mathbf{F} : X \to \mathbb{R}^n$ be a differentiable vector field where $X \subset \mathbb{R}^n$. The *divergence* of $\mathbf{F}$ is defined by

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n},$$

where $F_1, F_2, \ldots, F_n$ are the component functions of $\mathbf{F}$. The divergence maps a vector field to a scalar field.

Example 5.5. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (x^3, xyz, 2y + z + 1).$$

Then $\operatorname{div}\mathbf{F} = 3x^2 + xz + 1$.

Definition 5.8. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a differentiable vector field where $X \subset \mathbb{R}^3$. The *curl* of $\mathbf{F}$ is defined by

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k},$$

where F_1, F_2, F_3 are the component functions of $\mathbf{F}$. The curl maps a vector field to a vector field.

Note that the curl only acts on vector fields in $\mathbb{R}^3$ (or $\mathbb{R}^2$ by regarding them as vector fields in $\mathbb{R}^3$).

Example 5.6. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (x^3, xyz, 2y + z + 1).$$

Then $\operatorname{curl}\mathbf{F}$ is the function mapping (x, y, z) to $(2 - xy, 0, yz)$.

Proposition 5.2. Let $f : X \to \mathbb{R}$ be a scalar field of class C^2 where $X \subset \mathbb{R}^3$. Then

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Proposition 5.3. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a vector field of class C^2 where $X \subset \mathbb{R}^3$. Then

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

5.3 Line Integrals

Definition 5.9. Let $\gamma : [a, b] \to X$ be a path of class C^1 where $X \subset \mathbb{R}^n$. Let $f : X \to \mathbb{R}$ be a continuous function. The *scalar line integral* of f along γ is defined by

$$\int_\gamma f\, ds = \int_a^b f(\gamma(t))\, \|\gamma'(t)\|\, dt.$$

If γ is only piecewise C^1 or f is piecewise continuous, we can define the integral by adding up the contributions from different pieces.

Example 5.7. Define $\gamma : [0, 1] \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$\gamma(t) = (2t, 3t, 6t) \quad \text{and} \quad f(x, y, z) = x^2 + yz.$$

Then

$$\int_\gamma f\, ds = \int_0^1 154t^2\, dt = \frac{154}{3}.$$

Example 5.8. Consider a wire in $\mathbb{R}^2$ represented by the path $\gamma(t) = (t, |t|)$ with $-1 \leqslant t \leqslant 1$. Suppose the density function of the wire is $f(x, y) = x^2 y$. Then the mass of the wire is

$$\int_\gamma f\, ds = \int_{-1}^0 -\sqrt{2}t^3\, dt + \int_0^1 \sqrt{2}t^3\, dt = \frac{\sqrt{2}}{2}.$$

Definition 5.10. Let $\gamma : [a, b] \to X$ be a path of class C^1 where $X \subset \mathbb{R}^n$. Let $\mathbf{F} : X \to \mathbb{R}^n$ be a continuous function. The *vector line integral* of $\mathbf{F}$ along γ is defined by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt.$$

If γ is only piecewise C^1 or $\mathbf{F}$ is piecewise continuous, we can define the integral by adding up the contributions from different pieces. If the component functions of $\mathbf{F}$ are $F_1, F_2, \ldots, F_n$ respectively, we can also write

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\gamma \left(F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n \right).$$

Example 5.9. Define $\gamma : [0,6] \to \mathbb{R}^3$ and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\gamma(t) = (2t, t^2 + t, t + 1) \quad \text{and} \quad \mathbf{F}(x,y,z) = (y, z, x).$$

Then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_0^6 (4t^2 + 7t + 1)\, dt = 420.$$

Example 5.10. Define $\gamma : [0,1] \to \mathbb{R}^2$ by $\gamma(t) = (t^2, 2t^4)$. Then

$$\int_\gamma (2xy)\, dx + (7xy^2)\, dy = \int_0^1 (8t^7 + 224t^{13})\, dt = 17.$$

Definition 5.11. Let C be a curve in $\mathbb{R}^n$. A *parametrization* of C is a path $\gamma : [a,b] \to \mathbb{R}^n$ of class C^1 such that the image of γ is C and γ is injective except possibly at finitely many points.

We can extend the definition by using a piecewise C^1 path. A path $\gamma : [a,b] \to \mathbb{R}^n$ is *closed* if $\gamma(a) = \gamma(b)$, and is *simple* if γ is injective, except possibly that $\gamma(a) = \gamma(b)$. A curve is closed or simple if the corresponding parametrization is closed or simple.

Definition 5.12. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be paths of class C^1. We say that γ_2 is a *reparametrization* of γ_1 if there exists a bijective function $\phi : [c,d] \to [a,b]$ of class C^1 such that $\gamma_2 = \gamma_1 \circ \phi$ and the inverse $\phi^{-1} : [a,b] \to [c,d]$ is of class C^1.

In addition, we say that γ_2 is *orientation-preserving* if $\phi(c) = a$ and $\phi(d) = b$; and say that γ_2 is *orientation-reversing* if $\phi(c) = b$ and $\phi(d) = a$.

Given a path $\gamma : [a,b] \to \mathbb{R}^n$, we usually use $-\gamma$ to denote the path defined by $(-\gamma)(t) = \gamma(a + b - t)$ where $t \in [a,b]$. This gives a reparametrization of γ which is orientation-reversing.

Example 5.11. A parametrization of the unit circle in $\mathbb{R}^2$ is given by $\gamma_1 : [0, 2\pi] \to \mathbb{R}^2$ where $\gamma_1(t) = (\cos t, \sin t)$.

A reparametrization of γ_1 is given by $\gamma_2 : [0, \pi] \to \mathbb{R}^2$ where $\gamma_2(t) = (\cos 2t, -\sin 2t)$. This reparametrization is orientation-reversing. The two paths are closed and simple.

Proposition 5.4. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be paths of class C^1 where γ_2 is a reparametrization of γ_1. Let $f : X \to \mathbb{R}$ be a continuous function where $X \subset \mathbb{R}^n$ contains the image of γ_1. Then

$$\int_{\gamma_1} f\, ds = \int_{\gamma_2} f\, ds.$$

Proposition 5.5. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be paths of class C^1 where $\gamma_2 = \gamma_1 \circ \phi$ is a reparametrization of γ_1. Let $\mathbf{F} : X \to \mathbb{R}^n$ be a continuous function where $X \subset \mathbb{R}^n$ contains the image of γ_1. Then the following hold.

(a) $\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s}$ if ϕ is orientation-preserving

(b) $\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = -\int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s}$ if ϕ is orientation-reversing

In view of these results, we can define $\int_C f\, ds$ for a curve C by choosing a parametrization γ of C and then define

$$\int_C f\, ds = \int_\gamma f\, ds.$$

Similarly, if the orientation of C is given, then $\int_C \mathbf{F} \cdot d\mathbf{s}$ is well-defined. If C is a closed curve or can be decomposed as a finite number of closed curves, we sometimes use the notations

$$\oint_C f\, ds \quad \text{and} \quad \oint_C \mathbf{F} \cdot d\mathbf{s}$$

to mean $\int_C f\, ds$ and $\int_C \mathbf{F} \cdot d\mathbf{s}$ respectively.

Example 5.12. Let C be the segment in $\mathbb{R}^2$ joining $(0,0)$ and $(2,2)$. Consider two parametrizations $\gamma_1 : [0,2] \to \mathbb{R}^2$ and $\gamma_2 : [0,1] \to \mathbb{R}^2$ defined by

$$\gamma_1(t) = (t,t) \quad \text{and} \quad \gamma_2(t) = (2-2t, 2-2t).$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = x + 2y \quad \text{and} \quad \mathbf{F}(x,y) = (x+y, x-y).$$

Then

$$\int_{\gamma_1} f\, ds = \int_0^2 3\sqrt{2}t\, dt = 6\sqrt{2},$$
$$\int_{\gamma_2} f\, ds = \int_0^1 12\sqrt{2}(1-t)\, dt = 6\sqrt{2}$$

and

$$\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 2t\, dt = 4,$$
$$\int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 -8(1-t)\, dt = -4.$$

Definition 5.13. Let $C = \partial D$ be the boundary of a closed and bounded region D in $\mathbb{R}^2$. Suppose C is the union of finitely many simple closed curves. We say that C is *positively oriented* if D always lies on the left when one transverses C. Otherwise, we say that C is *negatively oriented*.

Theorem 5.1. (Green's theorem) Let $C = \partial D$ be the boundary of a closed and bounded region D in $\mathbb{R}^2$. Suppose C is the union of finitely many simple closed curves of class C^1 and C is positively oriented. Let $\mathbf{F} : X \to \mathbb{R}^2$ be a vector field of class C^1 where $X \subset \mathbb{R}^2$ contains D. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \oint_C F_1\, dx + F_2\, dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

where F_1 and F_2 are the component functions of $\mathbf{F}$.

If we regard $\mathbf{F}$ as a vector field in $\mathbb{R}^3$ with k-component zero, then we can rewrite the result as

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Example 5.13. Consider the unit disc $D = \overline{B}(\mathbf{0}, 1)$ in $\mathbb{R}^2$ and orient $C = \partial D$ in the anticlockwise direction. Define $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (x + y, 2x - y)$. On the one hand, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (-\sin^2 t - 2\sin t \cos t + 2\cos^2 t)\, dt = \pi.$$

On the other hand, we have

$$\iint_D \left(\frac{\partial(2x - y)}{\partial x} - \frac{\partial(x + y)}{\partial y} \right) dx\, dy = \iint_D 1\, dA = \pi.$$

Example 5.14. Consider the square $D = [0, 1] \times [0, 1]$ and orient $C = \partial D$ in the anticlockwise direction. Define $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (x^2 + xy, x^2 y + 2y + 1)$. By Green's theorem, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial(x^2 y + 2y + 1)}{\partial x} - \frac{\partial(x^2 + xy)}{\partial y} \right) dx\, dy = \iint_D (2xy - x)\, dx\, dy = 0.$$

Theorem 5.2. (Divergence theorem in $\mathbb{R}^2$) Let $C = \partial D$ be the boundary of a closed and bounded region D in $\mathbb{R}^2$. Suppose C is the union of finitely many simple closed curves of class C^1. Let $\mathbf{F} : X \to \mathbb{R}^2$ be a vector field of class C^1 where $X \subset \mathbb{R}^2$ contains D. If $\mathbf{n}$ is the outward unit normal vector to D, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA.$$

Here, a normal vector to the region D means a vector which is perpendicular to the tangent vector to C.

Example 5.15. Consider the unit disc $D = \overline{B}(\mathbf{0}, 1)$ in $\mathbb{R}^2$ and let $C = \partial D$. Define $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (x, y)$. On the one hand, we have

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} 1\, ds = 2\pi.$$

On the other hand, we have

$$\iint_D \nabla \cdot \mathbf{F}\, dA = \iint_D 2\, dA = 2\pi.$$

5.4 Surface Integrals

Definition 5.14. Let $\Phi : D \to \mathbb{R}^3$ be a continuous function where $D \subset \mathbb{R}^2$ is open and connected, possibly together with some or all of its boundary points. If Φ is injective, except possibly along ∂D, then we say that Φ is a *parametrized surface*, and we say that $\Phi(D)$ is the *surface* parametrized by Φ.

A subset D is *connected* if every two points in D can be joined by a path in D.

Example 5.16. Define $\Phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

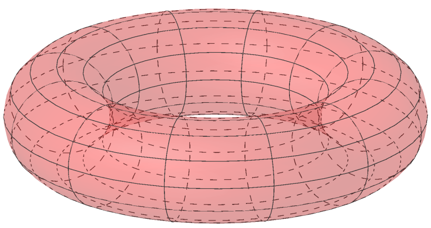
$$\Phi(s, t) = (\sin s \cos t, \sin s \sin t, \cos s).$$

Then Φ defines a surface in $\mathbb{R}^3$ which is a sphere.

Example 5.17. Define $\Phi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

$$\Phi(s, t) = ((3 + \cos s) \cos t, (3 + \cos s) \sin t, \sin s).$$

Then Φ defines a surface in $\mathbb{R}^3$ which is called a *torus*.



Definition 5.15. Let $\Phi : D \to \mathbb{R}^3$ be a parametrized surface where $D \subset \mathbb{R}^2$. Let $(a, b) \in D$. The *s-coordinate curve* at $t = b$ is the image of the function $f : I_1 \to \mathbb{R}^3$ defined by

$$f(s) = \Phi(s, b)$$

where $I_1 \subset \mathbb{R}$ consists of all values of s such that $(s, b) \in D$.

Similarly, the *t-coordinate curve* at $s = a$ is the image of the function $g : I_2 \to \mathbb{R}^3$ defined by

$$g(t) = \Phi(a, t)$$

where $I_2 \subset \mathbb{R}$ consists of all values of t such that $(a, t) \in D$.

We use the notations $\mathbf{T}_s = \dfrac{\partial \Phi}{\partial s}(s, b)$ and $\mathbf{T}_t = \dfrac{\partial \Phi}{\partial t}(a, t)$ to denote the *tangent vectors* to the coordinate curves.

Definition 5.16. Let $\Phi : D \to \mathbb{R}^3$ be a parametrized surface of class C^1 where $D \subset \mathbb{R}^2$. Let $(a, b) \in D$. The *standard normal vector* (with respect to the parametrization Φ) is defined by

$$\mathbf{N}(a, b) = \mathbf{T}_s(a, b) \times \mathbf{T}_t(a, b).$$

We say that the surface $S = \Phi(D)$ is *smooth* at the point $\Phi(a, b)$ if $\mathbf{N}(a, b) \neq \mathbf{0}$. If S is smooth at every point, then it is said to be *smooth*.

We can extend the definition of parametrized surfaces to piecewise parametrized surfaces. Loosely speaking, this simply means the union of a finite number of disjoint surfaces, possibly except the boundary. Similarly, we can define a piecewise smooth surface.

Example 5.18. Consider the parametrization $\Phi : D \to \mathbb{R}^3$ of the sphere defined by

$$\Phi(s, t) = (\sin s \cos t, \sin s \sin t, \cos s)$$

where $D = [0, \pi] \times [0, 2\pi]$. Then

$$\begin{aligned}
\mathbf{T}_s(s, t) &= (\cos s \cos t, \cos s \sin t, -\sin s), \\
\mathbf{T}_t(s, t) &= (-\sin s \sin t, \sin s \cos t, 0), \\
\mathbf{N}(s, t) &= (\sin^2 s \cos t, \sin^2 s \sin t, \sin s \cos s).
\end{aligned}$$

Example 5.19. Consider the parametrization $\Phi : D \to \mathbb{R}^3$ of the torus defined by

$$\Phi(s, t) = ((3 + \cos s) \cos t, (3 + \cos s) \sin t, \sin s).$$

where $D = [0, 2\pi] \times [0, 2\pi]$. Then

$$\begin{aligned}
\mathbf{T}_s(s, t) &= (-\sin s \cos t, -\sin s \sin t, \cos s), \\
\mathbf{T}_t(s, t) &= (-(3 + \cos s) \sin t, (3 + \cos s) \cos t, 0), \\
\mathbf{N}(s, t) &= (-(3 + \cos s) \cos s \cos t, -(3 + \cos s) \cos s \sin t, -(3 + \cos s) \sin s).
\end{aligned}$$

Proposition 5.6. Let $\Phi : D \to \mathbb{R}^3$ be a smooth parametrized surface of class C^1 where $D \subset \mathbb{R}^2$. Then the surface area of $S = \Phi(D)$ is

$$\iint_D \|\mathbf{N}(s,t)\|\ dA.$$

Example 5.20. Consider the parametrization $\Phi : D \to \mathbb{R}^3$ of the sphere defined by

$$\Phi(s,t) = (\sin s \cos t, \sin s \sin t, \cos s)$$

where $D = [0,\pi] \times [0, 2\pi]$. Then the surface area of this sphere is

$$\int_0^{2\pi} \int_0^{\pi} \sin s \, ds \, dt = 4\pi.$$

Example 5.21. A doughnut can be described by the surface

$$\Phi(s,t) = ((3 + \cos s)\cos t, (3 + \cos s)\sin t, \sin s)$$

where $0 \leqslant s \leqslant 2\pi$ and $0 \leqslant t \leqslant 2\pi$. Then the surface area of this doughnut is

$$\int_0^{2\pi} \int_0^{2\pi} (3 + \cos s) \, ds \, dt = 12\pi^2.$$

Definition 5.17. Let $\Phi : D \to \mathbb{R}^3$ be a smooth parametrized surface where $D \subset \mathbb{R}^2$. Let $f : S \to \mathbb{R}$ be a continuous function where $S = \Phi(D)$. The *scalar surface integral* of f along Φ is defined by

$$\iint_\Phi f \, dS = \iint_D f(\Phi(s,t)) \, \|\mathbf{N}(s,t)\| \ ds \, dt.$$

If S is only piecewise smooth or f is piecewise continuous, we can define the integral by adding up the contributions from different pieces.

Example 5.22. Define $\Phi : D \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$\Phi(s,t) = ((3 + \cos s)\cos t, (3 + \cos s)\sin t, \sin s) \quad \text{and} \quad f(x,y,z) = z + 1$$

where $D = [0, 2\pi] \times [0, 2\pi]$. Then

$$\iint_\Phi f \, dS = \int_0^{2\pi} \int_0^{2\pi} (\sin s + 1)(3 + \cos s) \, ds \, dt = 12\pi^2.$$

Definition 5.18. Let $\Phi : D \to \mathbb{R}^3$ be a smooth parametrized surface where $D \subset \mathbb{R}^2$. Let $\mathbf{F} : S \to \mathbb{R}^3$ be a continuous function where $S = \Phi(D)$. The *vector surface integral* of $\mathbf{F}$ along Φ is defined by

$$\iint_\Phi \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\Phi(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt.$$

If S is only piecewise smooth or $\mathbf{F}$ is piecewise continuous, we can define the integral by adding up the contributions from different pieces.

Example 5.23. Define $\Phi : D \to \mathbb{R}^3$ and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Phi(s,t) = (s, t, 4 - s^2 - t^2) \quad \text{and} \quad \mathbf{F}(x,y,z) = (x, y, z - 2y)$$

where $D = B(\mathbf{0}, 2)$ is a subset of $\mathbb{R}^2$. Then

$$\iint_\Phi \mathbf{F} \cdot d\mathbf{S} = \iint_D (s^2 + t^2 - 2t + 4)\, dA = 24\pi.$$

Definition 5.19. Let $\Phi_1 : D_1 \to \mathbb{R}^3$ and $\Phi_2 : D_2 \to \mathbb{R}^3$ be parametrized surfaces where $D_1, D_2 \subset \mathbb{R}^2$. We say that Φ_2 is a *reparametrization* of Φ_1 if there exists a bijective function $\phi : D_2 \to D_1$ such that $\Phi_2 = \Phi_1 \circ \phi$. If Φ_1 and Φ_2 are smooth and both ϕ and ϕ^{-1} are of class C^1, we say that Φ_2 is a *smooth reparametrization* of Φ_1.

Example 5.24. Define $\Phi_1 : \{(s,t) : 1 \leqslant s^2 + t^2 \leqslant 4\} \to \mathbb{R}^3$ by

$$\Phi_1(s,t) = (s, t, 4 - s^2 - t^2).$$

A smooth reparametrization of Φ_1 is given by $\Phi_2 : [1,2] \times [0, 2\pi) \to \mathbb{R}^3$ where

$$\Phi_2(s,t) = (s\cos t, s\sin t, 4 - s^2).$$

Definition 5.20. Let $\Phi_1 : D_1 \to \mathbb{R}^3$ and $\Phi_2 : D_2 \to \mathbb{R}^3$ be parametrized surfaces where $D_1, D_2 \subset \mathbb{R}^2$. Suppose Φ_2 is a reparametrization of Φ_1 where $\Phi_2 = \Phi_1 \circ \phi$. We say that the reparametrization is *orientation-preserving* if the Jacobian of ϕ is always positive, and *orientation-reversing* if the Jacobian is always negative.

Proposition 5.7. Let $\Phi_1 : D_1 \to \mathbb{R}^3$ and $\Phi_2 : D_2 \to \mathbb{R}^3$ be smooth parametrized surfaces where $D_1, D_2 \subset \mathbb{R}^2$. Suppose Φ_2 is a smooth reparametrization of Φ_1. Let $f : X \to \mathbb{R}$ be a continuous function where $X \subset \mathbb{R}^3$ contains the image of Φ_1. Then

$$\iint_{\Phi_1} f\, dS = \iint_{\Phi_2} f\, dS.$$

Proposition 5.8. Let $\Phi_1 : D_1 \to \mathbb{R}^3$ and $\Phi_2 : D_2 \to \mathbb{R}^3$ be smooth parametrized surfaces where $D_1, D_2 \subset \mathbb{R}^2$. Suppose Φ_2 is a smooth reparametrization of Φ_1. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a continuous function where $X \subset \mathbb{R}^3$ contains the image of Φ_1. Then the following hold.

(a) $\iint_{\Phi_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Phi_2} \mathbf{F} \cdot d\mathbf{S}$ if Φ_2 is orientation-preserving

(b) $\iint_{\Phi_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\Phi_2} \mathbf{F} \cdot d\mathbf{S}$ if Φ_2 is orientation-reversing

In view of these results, we can define $\iint_S f\, dS$ for a smooth surface S by choosing a parametrization Φ of S and then define

$$\iint_S f\, dS = \iint_\Phi f\, dS.$$

Similarly, if the orientation of S is given, then $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is well-defined. A surface is called a *closed surface* if it is compact and has no boundary. (In this context, the boundary ∂S of a surface S is different from Definition 2.11.) If S is a closed surface or can be decomposed as a finite number of closed surfaces, we sometimes use the notations

$$\oiint_S f\, dS \quad \text{and} \quad \oiint_S \mathbf{F} \cdot d\mathbf{S}$$

to mean $\iint_S f\, dS$ and $\iint_S \mathbf{F} \cdot d\mathbf{S}$ respectively.

Example 5.25. Let S be the surface of the cylinder with radius 3 and height 15, whose axis of symmetry is the z-axis and whose circular faces are located at $z = 0$ and $z = 15$ respectively. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = z$. Then

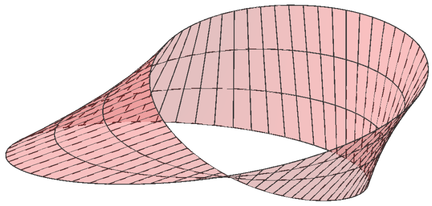
$$\iint_S f\, dS = \int_0^{2\pi} \int_0^{15} 3t\, dt\, ds + \int_0^3 \int_0^{2\pi} 15s\, dt\, ds = 810\pi.$$

Definition 5.21. A smooth and connected surface S is *orientable* if we can define a single unit normal vector at each point of S so that the normal vectors vary continuously. Otherwise, S is *nonorientable*.

Example 5.26. Define $\Phi : [0, 2\pi] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}^3$ by

$$\Phi(s, t) = \left(\left(1 + t\cos\frac{s}{2}\right)\cos s, \left(1 + t\cos\frac{s}{2}\right)\sin s, t\sin\frac{s}{2}\right).$$

Then Φ defines a surface in $\mathbb{R}^3$ which is called a *Möbius strip*. This surface is nonorientable.



Definition 5.22. Let S be a bounded, piecewise smooth, orientable surface in $\mathbb{R}^3$. Suppose the boundary ∂S of S consists of finitely many piecewise C^1, simple, closed curves. We say that ∂S is *oriented consistently* if the orientation of each closed curve in ∂S is chosen such that the right-hand rule is satisfied (which means if we use the fingers of the right hand to curl in the direction of the curve, then the thumb will point in the direction of the normal to S).

Theorem 5.3. (Stokes' theorem) Let S be a bounded, piecewise smooth, orientable surface in $\mathbb{R}^3$. Suppose the boundary ∂S of S consists of finitely many piecewise C^1, simple, closed curves which are oriented consistently with S. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a vector field of class C^1, where $X \subset \mathbb{R}^3$ contains S. Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Green's theorem is a special case of Stokes' theorem in $\mathbb{R}^2$. On the other hand, Stokes' theorem has a generalization in $\mathbb{R}^n$ using differential forms. The case $n = 1$ corresponds to the fundamental theorem of calculus. For this reason, Stokes' theorem and Green's theorem (together with Gauss's theorem) are sometimes known as the fundamental theorems of multivariable calculus.

Example 5.27. Consider the hemisphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z \geqslant 0\}$ in $\mathbb{R}^3$. Orient the surface using the outward normals. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (x + y, -x + y, 0).$$

On the one hand, we have

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = -\int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2 \sin s \cos s \, ds \, dt = -2\pi.$$

On the other hand, we have

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} -1 \, dt = -2\pi.$$

Example 5.28. Consider the surface S defined by $z = e^{-(x^2+y^2)}$ where $x^2 + y^2 \leqslant 1$. Orient the surface using normals pointing upwards. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (e^{y+z} - 2y, xe^{y+z} + y, e^{x+y}).$$

Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 2\pi.$$

Theorem 5.4. (Gauss's theorem/Divergence theorem) Let D be a bounded solid region in $\mathbb{R}^3$. Suppose the boundary ∂D of D consists of finitely many piecewise smooth, closed orientable surfaces which are oriented by normals pointing away from D. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a vector field of class C^1, where $X \subset \mathbb{R}^3$ contains D. Then

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

Example 5.29. Consider the solid $D = \{(x, y, z) : 0 \leqslant z \leqslant 9 - x^2 - y^2\}$ in $\mathbb{R}^3$. Orient the surface using the outward normals. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (x, y, z).$$

On the one hand, we have

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \int_{-3}^{3} \int_{-\sqrt{9-t^2}}^{\sqrt{9-t^2}} (s^2 + t^2 + 9)\, ds\, dt + 0 = \frac{243}{2}\pi.$$

On the other hand, we have

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV = \frac{243}{2}\pi.$$

Example 5.30. Consider the surface S defined by $z = (1 - x^2 - y^2)e^{1-x^2-3y^2}$ where $x^2 + y^2 \leqslant 1$. Orient the surface using normals pointing upwards. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (e^y \cos z, \sqrt{x^3 + 1} \sin z, x^2 + y^2 + 3).$$

Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{7}{2}\pi.$$

Links

Theorems

- 5.1: Green's theorem
- 5.2: Divergence theorem in $\mathbb{R}^2$
- 5.3: Stokes' theorem
- 5.4: Gauss's theorem

Propositions

- 5.1: Length of a differentiable path
- 5.2: Curl of a gradient field
- 5.3: Divergence of the curl
- 5.4: Effect of reparametrization on the scalar line integral
- 5.5: Effect of reparametrization on the vector line integral
- 5.6: Surface area of a smooth parametrized surface
- 5.7: Effect of reparametrization on the scalar surface integral
- 5.8: Effect of reparametrization on the vector surface integral

Terminologies and Notations

- $-\gamma$
- circular helix
- closed path
- closed surface
- connected
- conservative vector field
- coordinate curves
- curl $\nabla \times \mathbf{F}, \operatorname{curl} \mathbf{F}$
- curve
- del operator ∇f
- divergence $\nabla \cdot \mathbf{F}, \operatorname{div} \mathbf{F}$
- endpoint
- gradient field
- Möbius strip

- negatively oriented
- nonorientable surface
- orientable surface
- orientation-preserving (curve)
- orientation-preserving (surface)
- orientation-reversing (curve)
- orientation-reversing (surface)
- oriented consistently
- parametrization
- parametrized surface
- path
- positively oriented
- reparametrization (curve)
- reparametrization (surface)
- scalar field
- scalar line integral $\int_\gamma f\, ds$
- scalar surface integral $\iint_\Phi f\, dS$
- simple path
- smooth reparametrization
- smooth surface
- speed
- standard normal vector $\mathbf{N}$
- surface
- tangent vectors $\mathbf{T}_s$, $\mathbf{T}_t$
- torus
- vector field
- vector line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{s}$
- vector surface integral $\iint_\Phi \mathbf{F} \cdot d\mathbf{S}$
- velocity vector $\mathbf{v}(t)$