MATH 2101 Linear Algebra I–Vector Spaces II (Linear combinations and vector subspaces)

Some motivating questions

Now the question is how to *produce* some vector subspaces from a vector space.

Example

If we start a *fixed* vector $v \in \mathbb{R}^n$, we can form a vector subspace of $\mathbb{R}^n$ by considering the set:

$$\{cv : c \in \mathbb{R}\}.$$

For example, let $v = (3, 4, 5)$ and then $\{(3c, 4c, 5c) : c \in \mathbb{R}\}$ is a vector subspace.

Example

How about if we add one more vector $(2, -1, 3)$? How to form a vector subspace of $\mathbb{R}^3$ on that.

Linear combinations

Definition

Let S be a non-empty subset of V. A vector v in V is called a **linear combination** of vectors in S if there exists a finite number of vectors $u_1, \ldots, u_n$ in S and scalars $a_1, \ldots, a_n$ in $\mathbb{R}$ such that

$$v = a_1u_1 + \ldots + a_nu_n$$

We also say that v is a linear combination of vectors $u_1, \ldots, u_n$.

Example

Example

Express the vector $(1, 3, 2)$ in linear combinations of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.
Solution: $(1, 3, 2) = (1, 0, 0) + 3(0, 1, 0) + 2(0, 0, 1)$.

Example

Example

Express the vector $(1,3)$ in linear combinations of $(2,3)$ and $(3,5)$.
Solution: We have to solve $x_1, x_2 \in \mathbb{R}$ for $(1,3) = x_1(2,3) + x_2(3,5)$ i.e.

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Let $A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$. Then $A^{-1} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$. We then have:

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}.$$

Determining linear combinations=solving a system of linear equations

Remark

Determining questions about linear combinations of vectors is usually done by transferring to a problem of solving system of linear equations and then solving the system.

More examples

Example

Let $u_1 = (1, 1, 0, 0)$, $u_2 = (0, 1, 1, 0)$, $u_3 = (0, 0, 1, 1)$. Show that $(1, 2, 3, 4)$ cannot be written as a linear combination of u_1, u_2 and u_3.

Solution: We have to show the system of linear equations: $x_1u_1 + x_2u_2 + x_3u_3 = (1, 2, 3, 4)$ does not have a solution. We write the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

By the Jordan-Gaussian eliminations, the row echelon form takes the form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

The last row implies there is no solution.

More examples

Example

Express $(0, -1, 4)$ as a linear combination of $(1, 2, -1)$ and $(2, 3, 2)$.
Solution: We have to solve x_1, x_2 such that

$$x_1(1, 2, -1) + x_2(2, 3, 2) = (0, -1, 4).$$

Then, we have to solve the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 3 & -1 \\ -1 & 2 & 4 \end{array}\right).$$

By the Jordan-Gaussian eliminations (exercise: fill out the details),

$$\left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

Hence, $x_1 = -2$ and $x_2 = 1$. Then

$$(0, -1, 4) = -2(1, 2, -1) + (2, 3, 2).$$

Spanning sets

We now use the concept of linear combinations to define some vector subspaces:

Definition

Let S be a non-empty subset of a vector space V. The *span* of S, denoted by $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S. For convenience, we define $\text{span}(\emptyset) = \{0\}$.

Examples of spanning sets

Example

Let $S = \{(1,0,0),(0,1,0)\}$ be a subset in $\mathbb{R}^3$. Then

$$\operatorname{span}(S) = \left\{(x,y,0) \in \mathbb{R}^3 : x,y \in \mathbb{R}\right\}.$$

Example

Let v be a non-zero vector in $\mathbb{R}^n$ and let $S = \{v\}$. Then

$$\operatorname{span}(S) = \{av : a \in \mathbb{R}\}$$

Geometrically, $\operatorname{span}(S)$ represents a line in $\mathbb{R}^n$.

Theorem

The span of any subset S of V is a vector subspace of V.

Proof.

When $S = \emptyset$, it follows from that $\text{span}(S)$ is a zero subspace.
Now assume $S \neq \emptyset$. For any $v, u \in \text{span}(S)$, there exists $a_1, \ldots, a_r \in \mathbb{R}$ and $v_1, \ldots, v_r \in S$ such that

$$v = a_1 v_1 + \ldots + a_r v_r,$$

and similarly there exists $b_1, \ldots, b_s \in \mathbb{R}$ and $u_1, \ldots, u_s \in S$ such that

$$u = b_1 u_1 + \ldots + b_s u_s.$$

Hence,

$$v + u = a_1 v + \ldots + a_r v_r + b_1 u_1 + \ldots + b_s u_s$$

and so is a linear combination of vectors in S. Hence, $v + u \in \text{span}(S)$.
For $c \in \mathbb{R}$,

$$cv = c(a_1 v_1) + \ldots + c(a_r v_r) = (ca_1) v_1 + \ldots + (ca_r) v_r.$$

Then, cv is also a linear combination of vectors in S and so is also in $\text{span}(S)$. $\square$

Exercise

Let A be an $n \times m$ matrix and let $v_1, \ldots, v_m$ be the columns of A. Show that

$$\operatorname{span}(\{v_1, \ldots, v_m\}) = \{Ax : x \in \mathbb{R}^m\} .$$

The former space is called **the column space** of A.