1 Vectors

1.1 Vectors in the Euclidean Space

Definition 1.1. Let n be a positive integer. A *vector* is an n-tuple $(v_1, v_2, \ldots, v_n)$ of real numbers. The set $\mathbb{R}^n$ of all such vectors is called the *Euclidean space* of dimension n.

We use a boldface letter to denote a vector. For example, we write $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. In written form, we can write $\underline{v}$ or $\vec{v}$. The entries v_j are called the *coordinates* or *entries* of $\mathbf{v}$. In linear algebra, we usually define a vector as an $n \times 1$ or a $1 \times n$ matrix. More specifically, these are called *column vectors* and *row vectors* respectively. In our context, we occasionally mix up the use of column vectors and row vectors.

Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are said to be *equal* if $m = n$ and $u_j = v_j$ for each $j = 1, 2, \ldots, n$.

Definition 1.2. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{R}^n$. Then the *vector addition* is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

Proposition 1.1. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be any vectors in $\mathbb{R}^n$. Then the following hold.

(a) (commutativity) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) (associativity) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) (identity) $\mathbf{0} + \mathbf{v} = \mathbf{v}$ where $\mathbf{0}$ is the vector $(0, 0, \ldots, 0)$

(d) (inverse) $(-\mathbf{v}) + \mathbf{v} = \mathbf{0}$ where $-\mathbf{v}$ is the vector whose every coordinate is the negative of the corresponding coordinate of $\mathbf{v}$, and $\mathbf{0}$ is the vector in (c)

In view of (b), we can write $\mathbf{u} + \mathbf{v} + \mathbf{w}$ to mean the sum of three vectors. The vector $\mathbf{0}$ in (c) is called the *zero vector* in $\mathbb{R}^n$. The vector $-\mathbf{v}$ is called the *additive inverse* of $\mathbf{v}$. Using this concept, we can define $\mathbf{u} - \mathbf{v}$ by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

In other words, the coordinates of $\mathbf{u} - \mathbf{v}$ are the differences of the corresponding coordinates of $\mathbf{u}$ and $\mathbf{v}$.

Definition 1.3. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a vector in $\mathbb{R}^n$ and let c be a real number. Then the *scalar multiplication* is defined by

$$c\mathbf{v} = (cv_1, cv_2, \ldots, cv_n).$$

Proposition 1.2. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in $\mathbb{R}^n$, and let a and b be any real numbers. Then the following hold.

(a) $1 \cdot \mathbf{v} = \mathbf{v}$

(b) $(ab)\mathbf{v} = a(b\mathbf{v})$

(c) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

(d) $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Example 1.1. Given vectors $\mathbf{u} = (1, 3, 5)$ and $\mathbf{v} = (1, 0, -1)$ in $\mathbb{R}^3$, we have

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (1 + 1, 3 + 0, 5 + (-1)) = (2, 3, 4), \\ \mathbf{u} - \mathbf{v} &= (1 - 1, 3 - 0, 5 - (-1)) = (0, 3, 6), \text{ and} \\ 3\mathbf{u} &= (3 \cdot 1, 3 \cdot 3, 3 \cdot 5) = (3, 9, 15). \end{aligned}$$

Definition 1.4. Let P be a point in the two-dimensional or three-dimensional Cartesian coordinate system. The *position vector* of P is the vector represented by an arrow drawing from the origin to P.

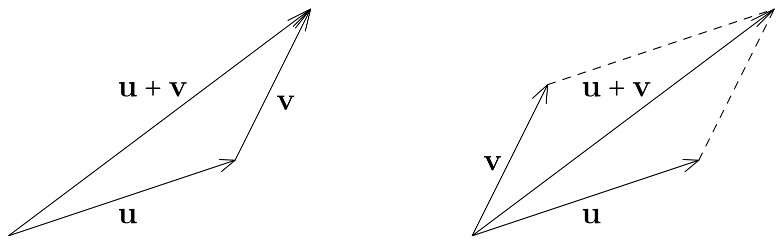
This gives us a way to visualize a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. For example, the vector $(1, 2) \in \mathbb{R}^2$ can be visualized as the position vector drawing from the origin O to the point $P(1, 2)$. We use the notation $\overrightarrow{OP}$ to denote this position vector. The point O is called the *initial point*, while P is called the *terminal point*. In general, we can use any vector of the same length and direction to represent the same vector. It is not necessary to take O to be the initial point of the arrow.

The vectors $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$ in $\mathbb{R}^2$ and $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ in $\mathbb{R}^3$ are called *standard vectors*. We can use these notations to represent a vector in another form. For example, we can write

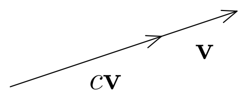
$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

In general, we can write $(v_1, v_2, \ldots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$, where $\mathbf{e}_j$ is the *standard vector* whose jth entry is 1 and all other entries are 0.

Using the geometric interpretation of vectors, we have some geometric meanings of the vector addition and scalar multiplication.



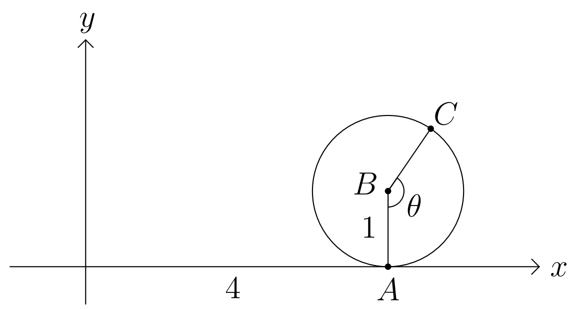
If we put the tail of $\mathbf{v}$ at the tip of $\mathbf{u}$, then the vector $\mathbf{u} + \mathbf{v}$ is represented by the arrow drawing from the tail of $\mathbf{u}$ to the tip of $\mathbf{v}$. Alternatively, if we put the tails of $\mathbf{u}$ and $\mathbf{v}$ at the same point and draw a parallelogram using $\mathbf{u}$ and $\mathbf{v}$ as two of the sides, then the vector $\mathbf{u}+\mathbf{v}$ is represented by the arrow drawing from the tail of $\mathbf{u}$ to the opposite vertex of the parallelogram.



When $c > 0$, the vector $c\mathbf{v}$ has the same direction as $\mathbf{v}$ but its length is c times that of $\mathbf{v}$. When $c = 0$, the vector $c\mathbf{v}$ is the zero vector. When $c < 0$, the vector $c\mathbf{v}$ has opposite direction as $\mathbf{v}$ and its length is $|c|$ times that of $\mathbf{v}$.

Two vectors are *parallel* if one of them is a scalar multiple of another.

Example 1.2. The figure shows a circle with centre B tangent to the x-axis. Then the position vectors of B and C are $4\mathbf{i} + \mathbf{j}$ and $(4 + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$ respectively.



1.2 Equations of Lines

Definition 1.5. A *parametric equation* in $\mathbb{R}^2$ is the representation of a set of points (x, y) given by

$$\begin{cases} x = f(t), \\ y = g(t) \end{cases}$$

for some parameter t and some functions f and g.

A parametric equation in $\mathbb{R}^3$ can be similarly defined. We can also represent this in vector form such as $\mathbf{v}(t) = (x, y) = (f(t), g(t))$.

Proposition 1.3. Let P be a point in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let $\mathbf{v}$ be the position vector of P. Let $\mathbf{u}$ be a nonzero vector. Then the equation of the straight line passing through P and parallel to $\mathbf{u}$ is

$$\mathbf{v} + t\mathbf{u}$$

where t is a real parameter.

Example 1.3. The equation of the straight line passing through the point $(1, 2)$ and perpendicular to $3\mathbf{i} + 4\mathbf{j}$ is $(1 - 4t)\mathbf{i} + (2 + 3t)\mathbf{j}$ where t is a real parameter.

Example 1.4. The equation of the straight line passing through the points $(1, -1, 1)$ and $(2, 3, 0)$ is $(1 + t)\mathbf{i} + (-1 + 4t)\mathbf{j} + (1 - t)\mathbf{k}$ where t is a real parameter.

Example 1.5. The straight lines $(1, -1, 3) + (0, 2, -1)t$ and $(-5, 0, -5) + (2, 3, 1)t$ have an intersection point, which is $(1, 9, -2)$.

Definition 1.6. Two non-parallel straight lines in $\mathbb{R}^3$ which do not intersect are called a pair of *skew lines*.

Given a pair of skew lines ℓ_1 and ℓ_2, to find the angle between them, we first find a line ℓ_3 which is parallel to ℓ_2 such that ℓ_1 and ℓ_3 intersect. Then the desired angle is the angle between ℓ_1 and ℓ_3.

1.3 Dot Product

Definition 1.7. The *dot product* of $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example 1.6. We have

$$(1, 3) \cdot (-2, 6) = (1)(-2) + (3)(6) = 16, \text{ and}$$
$$(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (-3\mathbf{i} + 4\mathbf{k}) = (2)(-3) + (1)(4) = -2.$$

Proposition 1.4. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be any vectors in $\mathbb{R}^n$, and let c be any real number. Then the following hold.

(a) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(b) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$

(c) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(d) $\mathbf{v} \cdot \mathbf{v} \geqslant 0$, and equality holds if and only if $\mathbf{v} = \mathbf{0}$

Definition 1.8. The *length* or *norm* of $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Example 1.7. We have

$$\|(3, -4)\| = \sqrt{(3)^2 + (-4)^2} = 5, \text{ and}$$
$$\|\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\| = \sqrt{(1)^2 + (-2)^2 + (2)^2} = 3.$$

Proposition 1.5. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in $\mathbb{R}^n$, and let c be any real number. Then the following hold.

(a) $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$

(b) $\|\mathbf{v}\| \geqslant 0$, and equality holds if and only if $\mathbf{v} = \mathbf{0}$

(c) (Triangle inequality) $\|\mathbf{u} + \mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$, and equality holds if and only if one of $\mathbf{u}$ and $\mathbf{v}$ is a nonnegative multiple of the other

A vector with norm 1 is called a *unit vector*. For every nonzero vector $\mathbf{v}$, the vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector having the same direction as $\mathbf{v}$. One can show that the distance between two points with position vectors $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{v} - \mathbf{u}\|$.

Proposition 1.6. Let $\mathbf{u}$ and $\mathbf{v}$ be any nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta.$$

In particular, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 1.8. Let O be the centre of a regular tetrahedron $ABCD$. Then we have $\angle AOB \approx 109^\circ$ (correct to the nearest integer).

Proposition 1.7. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in $\mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}$. Then there exist vectors $\mathbf{x}$ and $\mathbf{y}$ such that the following hold.

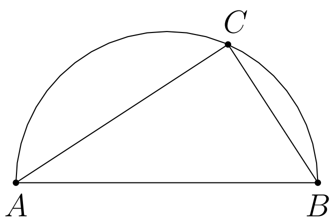
- $\mathbf{u} = \mathbf{x} + \mathbf{y}$
- $\mathbf{x}$ is parallel to $\mathbf{v}$
- $\mathbf{y}$ is perpendicular to $\mathbf{v}$

The vector $\mathbf{x}$ is called the *orthogonal projection* or *projection* of $\mathbf{u}$ onto $\mathbf{v}$.

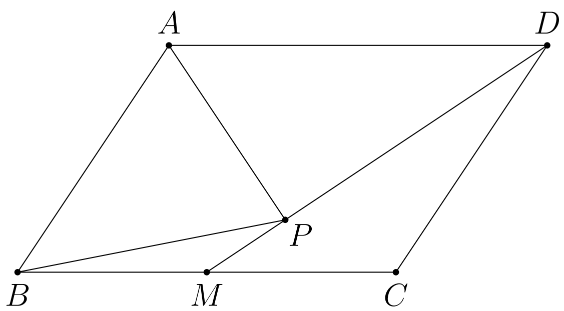
Proposition 1.8. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in $\mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}$. Then the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

Example 1.9. Let AB be the diameter of a semicircle and let C be a point on the semicircle. Then $\angle ACB = 90^\circ$.



Example 1.10. Let M be the midpoint of the side BC of a parallelogram $ABCD$. Let P be the projection of A onto MD. Then $BA = BP$.



1.4 Cross Product

Definition 1.9. The *cross product* of two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is defined to be the vector $\mathbf{u} \times \mathbf{v}$ satisfying the following properties.

- $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$
- $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$
- (right-hand rule) if we use the fingers of the right hand to curl from $\mathbf{u}$ towards $\mathbf{v}$, then the thumb will point in the direction of $\mathbf{u} \times \mathbf{v}$

The length of $\mathbf{u} \times \mathbf{v}$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. When $\mathbf{u}$ and $\mathbf{v}$ are parallel, we define $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. We can also talk about the cross product of two vectors in $\mathbb{R}^2$ by regarding them as vectors in $\mathbb{R}^3$ with the $\mathbf{k}$-component 0.

Example 1.11. We have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$.

Proposition 1.9. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be any vectors in $\mathbb{R}^3$, and let c be any real number. Then the following hold.

(a) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

(c) (anticommutativity) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

(d) $(c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v})$

Example 1.12. We have

$$\begin{aligned} (2,3,1) \times (-1,4,-2) &= (-10,3,11), \text{ and} \\ (2\mathbf{i} + 3\mathbf{j}) \times (\mathbf{i} - \mathbf{j}) &= -5\mathbf{k}. \end{aligned}$$

Proposition 1.10. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be any vectors in $\mathbb{R}^3$. Then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Proposition 1.11. Let O be the origin of the coordinate plane and let A and B be two distinct points. Let $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OB}$. Then the area of $\triangle OAB$ is

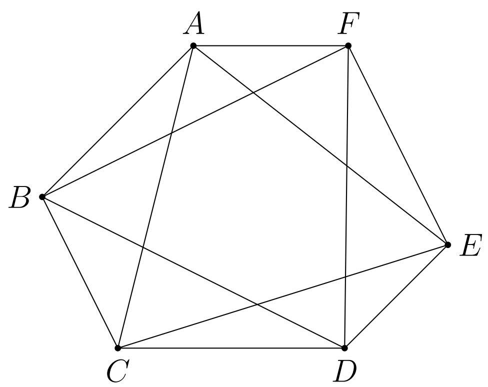
$$\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|.$$

Proposition 1.12. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be position vectors in $\mathbb{R}^3$ with the same initial point. Then the volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ is the absolute value of $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$, and we have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The quantity $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the *scalar triple product* of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Example 1.13. Let $ABCDEF$ be a convex hexagon such that $AB//DE$, $BC//EF$ and $CD//FA$. Then the areas of $\triangle ACE$ and $\triangle BDF$ are equal.



1.5 Equations of Planes

Analogous to the equation of straight lines, the parametric equation of a plane is of the form $\mathbf{u} + s\mathbf{v} + t\mathbf{w}$ where $\mathbf{v}$ and $\mathbf{w}$ are non-parallel vectors, and s and t are real parameters. However, this form of the equation of a plane is seldom used.

Definition 1.10. A *normal* to a plane is a nonzero vector which is perpendicular to every vector in the plane.

Proposition 1.13. Let (a, b, c) be the position vector of a point P in $\mathbb{R}^3$. Let $\mathbf{n} = (A, B, C)$ be a nonzero vector. Then the equation of the plane passing through P and having $\mathbf{n}$ as a normal vector is given by

$$A(x - a) + B(y - b) + C(z - c) = 0.$$

In general, for arbitrary constants A, B, C and D where A, B and C are not all zero, $Ax + By + Cz = D$ represents the equation of a plane with normal (A, B, C).

Example 1.14. The equation of the plane passing through the points $(1, 0, 1)$, $(1, 2, 2)$ and $(-1, -2, 3)$ is $3x - y + 2z = 5$.

Example 1.15. The angle between the planes $2x + 3y - 6z = 1$ and $x - 2y - 2z = 2$ is $\cos^{-1}\left(\dfrac{8}{21}\right)$.

Note that the angle between two planes is the angle between the normal vectors to the planes. A line is parallel to a plane if they have no intersection point or the line lies on the plane. A line is perpendicular to the plane if it is parallel to a normal vector to the plane.

1.6 Other Coordinate Systems

Definition 1.11. The *polar coordinates* (r, θ) of a point P in $\mathbb{R}^2$ are defined so that r is the distance from the origin O to the point P and θ is the angle in radian measured anticlockwise from the positive x-axis to the ray OP.

We usually restrict r to be a nonnegative real number and θ to be a real number such that $0 \leqslant \theta < 2\pi$. In that case, the polar coordinates of a point other than the origin are uniquely determined. For the origin, r must be 0 but θ can take any value.

Proposition 1.14. Let P be a point in $\mathbb{R}^2$ with Cartesian coordinates (x, y) and polar coordinates (r, θ). Then

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases}$$

and

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \tan\theta = \dfrac{y}{x}. \end{cases}$$

To convert from the Cartesian coordinates to the polar coordinates, note that the above formula only allows us to determine the value of θ up to a difference of π. The exact value of θ in the range $[0, 2\pi)$ can be determined based on the signs of x and y.

Example 1.16. Suppose the Cartesian coordinates of the point P are $(-2\sqrt{3}, -6)$. Then the polar coordinates of P are $\left(4\sqrt{3}, \frac{4\pi}{3}\right)$.

Example 1.17. The equation $r = 2\cos\theta$ using polar coordinates is the equation of a circle.

Definition 1.12. The *cylindrical coordinates* (r, θ, z) of a point P in $\mathbb{R}^3$ are defined so that (r, θ) are the polar coordinates of the projection of P onto the xy-plane and z is the z-coordinate of P in the Cartesian coordinates.

Proposition 1.15. Let P be a point in $\mathbb{R}^3$ with Cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z). Then

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z \end{cases}$$

and

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \tan\theta = \dfrac{y}{x}, \\ z = z. \end{cases}$$

Example 1.18. Suppose the Cartesian coordinates of the point P are $(\sqrt{2}, -\sqrt{2}, 1)$. Then the cylindrical coordinates of P are $\left(2, \frac{7\pi}{4}, 1\right)$.

Example 1.19. The equation $r = 3$ using cylindrical coordinates is the equation of a cylinder.

Example 1.20. The equation $z = 2r$ using cylindrical coordinates is the equation of a cone.

Definition 1.13. The *spherical coordinates* (ρ, φ, θ) of a point P in $\mathbb{R}^3$ are defined so that ρ is the distance from the origin O to the point P, φ is the angle in radian measured from the positive z-axis to the ray OP and θ is the θ-coordinate of P in the cylindrical coordinates.

We usually restrict ρ to be a nonnegative real number, and φ and θ be real numbers such that $0 \leqslant \varphi \leqslant \pi$ and $0 \leqslant \theta < 2\pi$. In that case, the spherical coordinates of a point other than those on the z-axis are uniquely determined. For the points on the z-axis other than the origin, φ must be 0 or π, but θ can take any value. For the origin, ρ must be 0, but φ and θ can take any values.

Proposition 1.16. Let P be a point in $\mathbb{R}^3$ with Cartesian coordinates (x, y, z) and spherical coordinates (ρ, φ, θ). Then

$$\begin{cases} x = \rho \sin\varphi \cos\theta, \\ y = \rho \sin\varphi \sin\theta, \\ z = \rho \cos\varphi \end{cases}$$

and

$$\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2}, \\ \tan\varphi = \dfrac{\sqrt{x^2 + y^2}}{z}, \\ \tan\theta = \dfrac{y}{x}. \end{cases}$$

Example 1.21. Suppose the Cartesian coordinates of the point P are $(1, 1, \sqrt{2})$. Then the spherical coordinates of P are $\left(2, \frac{\pi}{4}, \frac{\pi}{4}\right)$.

Example 1.22. The equation $\rho = 2$ using spherical coordinates is the equation of a sphere.

Example 1.23. The equation $\rho = 4\cos\varphi$ using spherical coordinates is the equation of a sphere.

Links

Propositions

- 1.1: Properties of vector addition
- 1.2: Properties of scalar multiplication
- 1.3: Equation of a straight line
- 1.4: Properties of dot product
- 1.5: Properties of norm
- 1.6: Angle between two vectors
- 1.7: Orthogonal projection
- 1.8: Formula for the orthogonal projection
- 1.9: Properties of cross product
- 1.10: Formula for the cross product
- 1.11: Area of a triangle
- 1.12: Volume of a parallelepiped
- 1.13: Equation of a plane
- 1.14: Conversion between Cartesian coordinates and polar coordinates
- 1.15: Conversion between Cartesian coordinates and cylindrical coordinates
- 1.16: Conversion between Cartesian coordinates and spherical coordinates

Terminologies and Notations

- additive inverse $-\mathbf{v}$
- column vector
- coordinate
- cross product $\mathbf{u} \times \mathbf{v}$
- cylindrical coordinates (r, θ, z)
- dot product $\mathbf{u} \cdot \mathbf{v}$
- entry
- equal (vectors)
- Euclidean space $\mathbb{R}^n$
- initial point
- length $\|\mathbf{v}\|$

- norm $\|\mathbf{v}\|$
- normal (plane)
- orthogonal projection
- parallel
- parametric equation
- polar coordinates (r, θ)
- position vector $\overrightarrow{OP}$
- projection
- row vector
- scalar multiplication $c\mathbf{v}$
- scalar triple product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
- skew lines
- spherical coordinates (ρ, φ, θ)
- standard vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{e}_j$
- terminal point
- unit vector
- vector $\mathbf{v}$
- vector addition $\mathbf{u} + \mathbf{v}$
- zero vector $\mathbf{0}$