4 Integration in Several Variables

4.1 Multiple Integration

Definition 4.1. Let $f : R \to \mathbb{R}$ be a function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. Consider a partition of R into the rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i, j = 1, 2, \ldots, n$ where

$$a = x_0 < x_1 < \cdots < x_n = b,$$
$$c = y_0 < y_1 < \cdots < y_n = d.$$

For $i, j = 1, 2, \ldots, n$, let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$, and choose an arbitrary point $\mathbf{x}_{ij} \in R_{ij}$. Then the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(\mathbf{x}_{ij}) \Delta x_i \Delta y_j$$

is called a *Riemann sum* of f on R.

Definition 4.2. Let $f : R \to \mathbb{R}$ be a function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. We say that f is *integrable* over R if the limit of the Riemann sum

$$\lim_{\Delta x, \Delta y \to 0} \sum_{i=1}^{n} \sum_{j=1}^{n} f(\mathbf{x}_{ij}) \Delta x_i \Delta y_j$$

exists, where the limit is taken over all partitions of R into rectangles such that $\Delta x = \max\limits_{1 \leqslant i \leqslant n} \Delta x_i$ and $\Delta y = \max\limits_{1 \leqslant j \leqslant n} \Delta y_j$ approach 0. In that case, we denote this limit by

$$\iint_R f \, dA$$

and call it the *double integral* of f over R.

These definitions can be easily extended to higher dimensional cases. For a *triple integral*, we can consider the integrability over a box $B = [a,b] \times [c,d] \times [q,r]$ and use the notation

$$\iiint_B f \, dV.$$

Proposition 4.1. Let $f : R \to \mathbb{R}$ be a function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. Suppose $f(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in R$. Then the volume under the graph of f over R is given by

$$\iint_R f \, dA,$$

if the integral exists.

Proposition 4.2. Let $f : R \to \mathbb{R}$ be a bounded function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. Suppose the set of discontinuities of f on R has zero area. Then $\iint_R f\,dA$ exists.

A function f is said to be *bounded* if there exists $M > 0$ such that $|f(\mathbf{x})| \leqslant M$ for all $\mathbf{x}$ in the domain of f. As a corollary of this proposition, a continuous function defined on R must be integrable. In $\mathbb{R}^3$, the condition that the set of discontinuities has zero area is replaced by having zero volume.

Proposition 4.3. Let $f, g : R \to \mathbb{R}$ be integrable functions where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. Then the following hold.

(a) $f \pm g$ is integrable over R and $\iint_R (f \pm g)\,dA = \iint_R f\,dA \pm \iint_R g\,dA$

(b) cf is integrable over R and $\iint_R (cf)\,dA = c\iint_R f\,dA$ for any $c \in \mathbb{R}$

(c) if $f(\mathbf{x}) \leqslant g(\mathbf{x})$ for all $\mathbf{x} \in R$, then $\iint_R f\,dA \leqslant \iint_R g\,dA$

(d) $|f|$ is integrable over R and $\left|\iint_R f\,dA\right| \leqslant \iint_R |f|\,dA$

Analogous results hold for higher dimensional cases.

Definition 4.3. Let $f : R \to \mathbb{R}$ be a function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. We define the *iterated integrals* by

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

and

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Theorem 4.1. (Fubini's theorem) Let $f : R \to \mathbb{R}$ be a bounded function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. Suppose the set of discontinuities of f on R has zero area, and every line parallel to the coordinate axes meets this set in at most finitely many points. Then

$$\iint_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Analogous definitions and results hold for higher dimensional cases.

Example 4.1. We have $\int_{-3}^{3}\int_{-1}^{1}\int_{0}^{2}(3y^2+2xy+2xz)\,dz\,dy\,dx = 24$.

Example 4.2. The volume under the graph of $z = \sin x \cos y$ over the rectangle $[0,\pi]\times\left[0,\frac{\pi}{2}\right]$ is

$$\int_0^{\pi}\int_0^{\frac{\pi}{2}} \sin x \cos y\,dy\,dx = 2.$$

Definition 4.4. Let D be a subset of $\mathbb{R}^2$. We say that D is an *elementary region* if it is of one of the following types.

- (Type 1) $D = \{(x,y) : a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant h(x)\}$ where g and h are continuous
- (Type 2) $D = \{(x,y) : c \leqslant y \leqslant d,\ g(y) \leqslant x \leqslant h(y)\}$ where g and h are continuous
- (Type 3) D is of both type 1 and type 2

Definition 4.5. Let $f : D \to \mathbb{R}$ be a continuous function where D is an elementary region in $\mathbb{R}^2$. Let R be a rectangle containing D and define $\widetilde{f} : R \to \mathbb{R}$ by

$$\widetilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y)\in D,\\ 0 & \text{if } (x,y)\notin D.\end{cases}$$

Then we define

$$\iint_D f\,dA = \iint_R \widetilde{f}\,dA.$$

Proposition 4.4. Let $f : D \to \mathbb{R}$ be a continuous function where D is an elementary region in $\mathbb{R}^2$. Suppose $f(\mathbf{x}) \geqslant 0$ for all $\mathbf{x}\in D$. Then the volume under the graph of f over D is given by

$$\iint_D f\,dA.$$

Proposition 4.5. Let D be an elementary region in $\mathbb{R}^2$. Then the area of D is

$$\iint_D 1\,dA.$$

The same definition and results apply if D is a more general region in $\mathbb{R}^2$, given that the integrals are well-defined.

Theorem 4.2. (Fubini's theorem) Let $f : D \to \mathbb{R}$ be a continuous function where D is an elementary region in $\mathbb{R}^2$. Then the following hold.

(a) If $D = \{(x, y) : a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant h(x)\}$ where g and h are continuous, then

$$\iint_D f\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

(b) If $D = \{(x, y) : c \leqslant y \leqslant d,\ g(y) \leqslant x \leqslant h(y)\}$ where g and h are continuous, then

$$\iint_D f\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

Example 4.3. Let D be the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Then $\displaystyle\iint_D x^2 y\, dA = \int_0^1 \int_0^{1-x} x^2 y\, dy\, dx = \frac{1}{60}$.

Example 4.4. Let D be the bounded region in $\mathbb{R}^2$ enclosed by the straight line $y = x$ and the parabola $y = -2x^2 - 2x + 2$. Then

$$\iint_D x\, dA = \int_{-2}^{\frac{1}{2}} \int_x^{-2x^2-2x+2} x\, dy\, dx = -\frac{125}{32}.$$

Example 4.5. Let $a, b, c > 0$ be real numbers. The volume of the tetrahedron in $\mathbb{R}^3$ with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx = \frac{1}{6}abc.$$

Example 4.6. We have $\displaystyle\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = e - 1.$

Proposition 4.6. Let $f : D \to \mathbb{R}$ be a continuous function where $D \subset \mathbb{R}^2$ is bounded. If $D = D_1 \cup D_2$ where $D_1 \cap D_2$ has zero area, then

$$\iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$$

if the integrals exist.

The definition of elementary regions can be extended to $\mathbb{R}^3$ and even $\mathbb{R}^n$. For example, a subset D of $\mathbb{R}^3$ satisfying

$$D = \{(x, y, z) : a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant h(x),\ \varphi(x, y) \leqslant z \leqslant \psi(x, y)\}$$

for some continuous functions g, h, φ and ψ is called an elementary region. In that case, for any continuous function $f : D \to \mathbb{R}$, we have

$$\iiint_D f\,dV = \int_a^b \int_{g(x)}^{h(x)} \int_{\varphi(x,y)}^{\psi(x,y)} f\,dz\,dy\,dx.$$

This gives the 4-dimensional volume under the graph of f over D. Also, the volume of D is

$$\iiint_D 1\,dV.$$

Example 4.7. The volume of a solid hemisphere of radius r is

$$\int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} 1\,dz\,dy\,dx = \frac{2}{3}\pi r^3.$$

Example 4.8. Let D be the bounded region in $\mathbb{R}^3$ enclosed by the elliptic paraboloid $y = x^2 + 4z^2$ and the plane $y = 4$. Then

$$\iiint_D \frac{1}{\sqrt{1-z^2}}\,dV = \int_{-1}^{1} \int_{-2\sqrt{1-z^2}}^{2\sqrt{1-z^2}} \int_{x^2+4z^2}^{4} \frac{1}{\sqrt{1-z^2}}\,dy\,dx\,dz = \frac{128}{9}.$$

4.2 Change of Variables

Definition 4.6. Let $x, y : \mathbb{R}^2 \to \mathbb{R}$ be functions of class C^1 in the variables u and v. We define the *Jacobian* by

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = x_u y_v - x_v y_u.$$

Theorem 4.3. Let $\mathbf{g}(u, v) = (x, y)$ be an injective function of class C^1 that maps from the uv-plane to the xy-plane. Let D and D' be elementary regions in the xy-plane and the uv-plane respectively such that $\mathbf{g}(D') = D$. For any integrable function $f : D \to \mathbb{R}$, we have

$$\iint_D f(x, y)\,dx\,dy = \iint_{D'} (f \circ \mathbf{g})(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv.$$

In other words, if we want to make a change of variables from x, y to u, v, we need to change the region as well as multiplying the integrand by the Jacobian.

Example 4.9. Let D be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,-1)$ and $(2,1)$. Then

$$\iint_D (x+y)^2(x-2y)^2\,dx\,dy = \iint_{D'} \frac{1}{3}u^2v^2\,du\,dv = \frac{27}{20},$$

where D' is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(3,0)$ and $(0,3)$.

Example 4.10. Let D be the bounded region in the first quadrant of $\mathbb{R}^2$ enclosed by the hyperbolas $xy = 1$, $xy = 4$ and the straight lines $y = x$, $y = x + 2$. Then

$$\iint_D (x^2-y^2)e^{xy}\,dx\,dy = \iint_{D'} -ve^u\,du\,dv = 2(e-e^4),$$

where D' is the rectangle $[1,4] \times [0,2]$.

Example 4.11. Let D be the disc in $\mathbb{R}^2$ with centre $\mathbf{0}$ and radius a. Then

$$\iint_D e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \iint_{D'} re^{-\frac{r^2}{2}}\,dr\,d\theta = 2\pi\left(1-e^{-\frac{a^2}{2}}\right),$$

where D' is the rectangle $[0,a] \times [0,2\pi]$.

In general, the conversion between the area elements of the Cartesian coordinates and the polar coordinates is

$$dA = dx\,dy = r\,dr\,d\theta.$$

One easily extends the change of variable formula for double integrals to multiple integrals. For example, in $\mathbb{R}^3$, the Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

and the change of variable formula becomes

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D'} (f \circ \mathbf{g})(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,du\,dv\,dw.$$

The conversion between the volume elements of the Cartesian coordinates and the cylindrical coordinates is

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz,$$

while the corresponding formula for spherical coordinates is

$$dV = dx\,dy\,dz = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Example 4.12. The volume of a cone with radius a and height h is

$$\iiint_{\text{cone}} 1\, dV = \int_0^{2\pi} \int_0^a \int_0^{h\left(1-\frac{r}{a}\right)} r\, dz\, dr\, d\theta = \frac{\pi}{3} a^2 h.$$

Example 4.13. A lemon shape is the region that lies inside both of the spheres $x^2 + y^2 + (z-1)^2 = 5$ and $x^2 + y^2 + (z+1)^2 = 5$ in $\mathbb{R}^3$. The volume of this lemon shape is

$$\iiint_{\text{lemon}} 1\, dV = \int_0^{2\pi} \int_0^2 \int_{1-\sqrt{5-r^2}}^{-1+\sqrt{5-r^2}} r\, dz\, dr\, d\theta = \frac{20\sqrt{5} - 28}{3}\pi.$$

Links

Theorems

- 4.1: Fubini's theorem on rectangular regions
- 4.2: Fubini's theorem on elementary regions
- 4.3: Change of variable formula for double integrals

Propositions

- 4.1: Volume under the graph over rectangular regions
- 4.2: A sufficient condition for integrability
- 4.3: Algebraic properties of double integrals
- 4.4: Volume under the graph over elementary regions
- 4.5: Area of an elementary region
- 4.6: Double integral over a union of regions

Terminologies and Notations

- bounded function
- double integral $\iint_R f \, dA$
- elementary region
- integrable
- iterated integral $\int_a^b \int_c^d f \, dy \, dx$
- Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$
- Riemann sum
- triple integral $\iiint_B f \, dV$