Homework 2

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Problem 1

Let $C = \{x|Dx \leq d, Gx = g\}$ and $x_0 \in C$ be given with active set A. Consider the tangent cone $T_C(x_0)$ and define $S = \{z|D_A z \leq 0, Gz = 0\}$.

To show $T_C(x_0) = S$, first let some $v \in T_C(x_0)$ be given. That is, we can choose some sequence v_n s.t. for some $\alpha_n > 0 \to 0$, the sequence $x_0 + \alpha_n v_n \in C$ and $v_n \to v$. Now let some $i \in A$ be given. Then we know that $D_i x_0 = d_i$. Notice that $D_i(x_0 + \alpha_n v_n) = D_i x_0 + \alpha_n D_i v_n = d_i + \alpha_n D_i v_n \leq d_i$, so $\alpha_n D_i v_n \leq 0$, which can only be true for arbitrary n if $D_i v_n \leq 0$ since $\alpha_n > 0$. Hence, in the limit, we see that $D_i v_n \leq 0 \to D_i v \leq 0$. And because $i \in A$ was arbitary, we see that $D_A v \leq 0$. Next, notice that $Gx_0 = g$ and that $G(x_0 + \alpha_n v_n) = Gx_0 + \alpha_n G v_n = g$. So $\alpha_n G v_n = 0$, which implies that $Gv_n = 0$ for arbitrary n since $\alpha_n > 0$. Hence, in the limit, $Gv_n = 0 \to Gv = 0$. Overall, we have shown that $v \in S$ and thus $T_C(x_0) \subseteq S$.

Now let some element $z \in S$ be given. So $D_A z \leq 0, Gz = 0$. Consider the following 2 cases:

Case 1: $D = D_A$

Here, we then have that $D_A z = Dz \leq 0$. So then $D(x_0 + z) = Dx_0 + Dz$. Since $D_A = D$, we know that $Dx_0 = d$ and thus $Dx_0 + Dz = d + Dz \leq d$. We see that setting $z_n = z$ and $\alpha_n = \frac{1}{n}$ suffices as sequences s.t. $x_0 + \alpha_n z_n \in C$ since $D(x_0 + \frac{1}{n} z) = Dx_0 + \frac{1}{n} Dz = d + \frac{1}{n} Dz \leq d$ since $Dz \leq 0 \to \frac{1}{n} Dz \leq 0$; furthermore, $G(x_0 + \frac{1}{n} z) = Gx_0 + \frac{1}{n} Gz = g + \frac{1}{n} 0 = g$, showing that $x_0 + \alpha_n z_n \in C$. Since $z_n = z \to z$ trivially, this shows that $z \in T_C(x_0)$.

Case 2: $D \neq D_A$

In this case, there exists some nonempty set B s.t. $\forall i \in B$, $D_i x_0 \neq d_i$. But because $Dx_0 \leq d$, this implies that $D_i x_0 < d_i$, so $d_i - D_i x_0 > 0$. So let us set $r = min_{i \in B}(d_i - D_i x_0)$ and $s = max_{i \in B}(|D_i z|)$, which clearly satisfies $r, s > 0$, so $\frac{r}{s} > 0$. To show that $z \in T_C(x_0)$, it suffices to show that $\frac{r}{s} z \in T_C(x_0)$ since $T_C(x_0)$ is a cone. Consider the sequences $\alpha_n = \frac{1}{n}$ and $v_n = \frac{r}{s} z$. Notice that, for $i \in A$, $D_i z \leq 0$ so $D_i(\frac{r}{s} z) = \frac{r}{s} Dz \leq 0$; hence, $D_i(x_0 + \alpha_n v_n) = D_i x_0 + \frac{1}{n}\frac{r}{s} D_i z = d_i + \frac{1}{n}\frac{r}{s} D_i z \leq d_i + \frac{1}{n}\frac{r}{s}(0) = d_i$. So $D_i(x_0 + \alpha_n v_n) \leq d_i$ when $i \in A$.

For $i \in B$, consider $D_i(x_0 + \alpha_n v_n) = D_i x_0 + \frac{1}{n}\frac{r}{s} D_i z$. Note that since $s \geq |D_i x_0| \geq 0$, it follows that $\frac{1}{s} D_i z \leq 1$; hence, we see that $D_i x_0 + \frac{1}{n}\frac{r}{s} D_i z \leq D_i x_0 + \frac{1}{n} r$. Furthermore, we see that $0 < r \leq d_i - D_i x_0$, so $\frac{1}{n} r \leq d_i - D_i x_0$, so $D_i x_0 + \frac{1}{n} r \leq D_i x_0 + d_i - D_i x_0 = d_i$. Chaining these inequalities together, we see that $D_i(x_0 + \alpha_n v_n) \leq d_i$ for $i \in B$.

Overall, this implies that $D(x_0 + \alpha_n v_n) \leq d$. Furthermore, we see that $G(x_0 + \alpha_n v_n) = Gx_0 + \frac{1}{n}\frac{r}{s} Gz = g + \frac{1}{n}\frac{r}{s} 0 = g$, since $Gx_0 = g, Gz = 0$. Overall, this shows that $(x_0 + \alpha_n v_n)$ and thus $v_n \to \frac{r}{s} z \in T_C(x_0)$. So z must also be in $T_C(x_0)$, so $S \subseteq T_C(x_0)$.

Problem 2

Let some A, x, b, β be given. For the sake of convenience, we shall indicate the 2 predicates of II by (IIa) ($\exists y \geq 0$ s.t. $A^T y = b$ and $c^T y \leq \beta$) and (IIb) ($\exists y \geq 0$ s.t. $A^T y = 0, c^T y < 0$).

Suppose that (I) is true. That is, we can choose some x^* s.t. $b^T x^* > \beta, Ax^* \leq c$. We want to show that (II) is false, which we shall do by showing each predicate is false. Consider the LP: max $b^T x$ s.t. $Ax \leq c$ (x free). The primal of this problem is min $c^T u$ s.t. $A^T u = b, u \geq 0$. Because $Ax^* \leq c$, we see that the original (dual) problem is feasible at x^*. Furthermore, this implies that the corresponding dual objective $b^T x^* > \beta$, and by weak duality, we know that $\forall u \geq 0$ s.t. $A^T u = b$, the objective $c^T u \geq b^T c > \beta$. Hence, $c^T u \nleq \beta$ for $A^T u = b, u \geq 0$, which implies the first predicate (IIa) is false. Now consider IIb and the related primal LP: min $c^T y$ s.t. $A^T y = 0, y \geq 0$. The dual of this problem is max 0 s.t. $Ax \leq c$,x free. Notice that because $x^*$is feasible in I, we know that $Ax^* \leq c$ and thus x^* is feasible in the dual problem aformentioned. Hence, we we know that the objective at x^* in this dual problem is 0 and thus, by weak duality, $\forall y \geq 0$ s.t. $A^T y = 0$, we see that $c^T y \geq 0$. But this means that $\forall y \geq 0$ s.t. $A^T y = 0$, $c^T y \nless 0$. Hence, we have shown that IIb is false, overall showing that II is false.

Now suppose that I is false. We would like to show that exactly one of IIa,IIb is true. If IIa is true, we are done, so suppose that IIa is false. That is, we cannot choose $y \geq 0$ s.t. $A^T y = b$ and $c^T y \leq \beta$. We want to show that IIb is true.

Consider the LP: min$c^T y$ s.t. $A^T y = 0, y \geq 0$. This dual of this LP is: max 0 s.t. $Ax \leq c$,x free. Clearly, this primal LP is feasible with $y = 0$. To show that it is unbounded, we would like to show that the dual is infeasible. So suppose the dual had some feasible point x^*. Then $Ax^* \leq c$. Because I is false, we know then that $b^T x^* \leq \beta$.

Consider the LP: min $c^T y$ s.t. $A^T y = b, y \geq 0$. The dual of this LP is: max $b^T x$ s.t. $Ax \leq c$,x free. Notice here that the dual LP is feasible at x^*. Furthermore, we know that $\forall x$ s.t. $Ax \leq c$, $b^T x \leq \beta$ since I is false; hence, the dual LP is feasible but bounded above. Hence, the primal LP must be feasible but also not unbounded. But we know then, since IIa is false, that $\forall y$ s.t. $A^T y = b, y \geq 0$, $c^T y > \beta$. Because both problems are feasible, by strong duality, there must exist some (x_1, y_1) s.t. $b^T x_1 = c^T y_1$. But $b^T x_1 \leq \beta$ and $c^T y_1 > \beta$, a contradiction. Hence, our assumption was false and there is no such feasible point x^*. Hence the original dual (max 0 s.t. $Ax \leq c$,x free) is infeasible and thus the primal (min$c^T y$ s.t. $A^T y = 0, y \geq 0$) must be unbounded (in the $-\infty$ direction). That is, we can choose some y s.t. $c^T y < 0$ with $A^T y = 0, y \geq 0$. That is, we can construct a y satisfying IIb.

Problem 3

A)

For the point 0, the tangent cone would be any points $\underline{u}$ such that $\exists$ sequences $u_k \to u, \alpha_k \downarrow 0$ s.t. $x + \alpha_k u_k = \alpha_k u_k \in \Omega$. Notice that this set $\Omega \subseteq \mathbb{R}$. In this space, the tangent cone would be a subset of this space.

First, notice that for $u_k = 0, \alpha_k \downarrow 0$, every k satisfies $\alpha_k u_k = 0 \in \Omega$, so clearly $u_k \to 0 \in T_\Omega(0)$. So let some point $x \in \mathbb{R}, x \neq 0$ be given and set $u_k = x$. If x is positive, we can then choose $\alpha_k := \frac{1}{x}\frac{1}{k\pi} > 0$. Clearly, as $k \to \infty$, $\alpha_k \to 0$. Notice that for arbitrary $k \in \mathbb{Z}^+$, $\alpha_k u_k = \frac{1}{x}\frac{1}{k\pi} * x = \frac{1}{k\pi} \in \Omega$. Hence, it follows that $u_k \to x \in T_\Omega(0)$. For negative x, we can choose $\alpha_k := \frac{1}{x}\frac{1}{(-k)\pi} > 0$ and we see that the same argument holds since for $k \in \mathbb{Z}^+, \alpha_k u_k = \frac{1}{x}\frac{1}{-k\pi} * x = \frac{1}{-k\pi} \in \Omega$, implying that $x \in T_\Omega(0)$. But this implies that $\mathbb{R} \subseteq \Omega$, so $\mathbb{R} = \Omega$. In this case, the normal cone (that is, the set of points which make a negative inner product with $T_\Omega(0)$) would just be y s.t. $y * x \leq 0$ for all $x \in \mathbb{R}$. But the only choice would be a y s.t. y is nonnegative (e.g. $y * -1 \leq 0$) and nonpositive (e.g. $y * 1 \leq 0$), which is only satisfied by 0. So the normal cone is just: $\{0\}$

B)

Let some $x \in \Omega$ be given s.t. $x \neq 0$. Then we can choose $j \in \mathbb{Z}, j \neq 0$ s.t. $x = \frac{1}{j\pi}$. I claim that the tangent cone $T_\Omega(x) = \{0\}$. First, notice that for $u_n = 0, \alpha_n \downarrow 0$, every n satisfies $x + \alpha_n u_n = x + 0 = x \in \Omega$, so clearly $u_n \to 0 \in T_\Omega(x)$. Now let some $u \in \mathbb{R}, u \neq 0$ be given. Suppose $u \in T_\Omega(x)$, so that we may choose a $u_n \to u, \alpha_n \downarrow 0$ s.t. $x + \alpha_n u_n \in \Omega$ for all n.

Case 1: As edge cases, consider $j = 1$.

Then set $\epsilon = \frac{1}{2\pi}$. Now choose some k large enough s.t. $|u_k - u| < |u|$ (making $u_k \neq 0$ since u is nonzero) and $|(x + \alpha_k u_k) - x| < \frac{1}{2\pi}$. So $\frac{1}{2\pi} = \frac{1}{\pi} - \frac{1}{2\pi} = x - \frac{1}{2\pi} < (x + \alpha_k u_k) < x + \frac{1}{2\pi} = \frac{1}{\pi} + \frac{1}{2\pi} = \frac{3}{2\pi}$. But we know that $(x + \alpha_k u_k) \in \Omega$. But the only $x + \alpha_k u_k \in \Omega$ s.t. $\frac{1}{2\pi} < (x + \alpha_k u_k) < \frac{3}{2\pi}$ is $x + \alpha_k u_k = \frac{1}{\pi}$. But this implies that either $\alpha_k = 0$ or $u_k = 0$, a contradiction.

Case 2: Now consider $j = -1$.

Then set $\epsilon = \frac{1}{2\pi}$. Now choose some k large enough s.t. $|u_k - u| < |u|$ (making $u_k \neq 0$ since u is nonzero) and $|(x+\alpha_k u_k)-x| = |(x+\alpha_k u_k) - \frac{1}{\pi}| < \frac{1}{2\pi}$. So $\frac{-3}{2\pi} = -\frac{1}{\pi} - \frac{1}{2\pi} < (x+\alpha_k u_k) < -\frac{1}{\pi} + \frac{1}{2\pi} = -\frac{1}{2\pi}$. But we know that $(x+\alpha_k u_k) \in \Omega$. But the only $x+\alpha_k u_k \in \Omega$ s.t. $\frac{-3}{2\pi} < (x+\alpha_k u_k) < -\frac{1}{2\pi}$ is $x+\alpha_k u_k = -\frac{1}{\pi}$. But this implies that either $\alpha_k = 0$ or $u_k = 0$, a contradiction.

Case 3: If j is positive (excluding the case $j = 1$)

Set $\epsilon = \frac{1}{j\pi} - \frac{1}{(j+1)\pi} > 0$ Because $u_n \to u, \alpha_n \downarrow 0$, we know $x+\alpha_n u_n \to x$ so can choose some k large enough s.t. $|u_k - u| < |u|$ (making $u_k \neq 0$ since u is nonzero) and $|(x+\alpha_k u_k)-x| < \frac{1}{j\pi} - \frac{1}{(j+1)\pi}$. That is, $x+-\frac{1}{j\pi} + \frac{1}{(j+1)\pi} < (x+\alpha_k u_k) < x+\frac{1}{j\pi} - \frac{1}{(j+1)\pi}$. We can also choose We know that $x+\alpha_k u_k \in \Omega$. Notice that: $x+-\frac{1}{j\pi} + \frac{1}{(j+1)\pi} = \frac{1}{j\pi} - \frac{1}{j\pi} + \frac{1}{(j+1)\pi} = \frac{1}{(j+1)\pi} < (x+\alpha_k u_k) < x + \frac{1}{j\pi} - \frac{1}{(j+1)\pi} = \frac{2}{j\pi} - \frac{1}{(j+1)\pi} = \frac{j+2}{j(j+1)\pi}$. Notice however that $\frac{j+2}{j(j+1)\pi} \leq \frac{1}{j-1\pi}$ since $(j+2)(j-1)\pi = \pi(j^2+j-2) \leq \pi(j^2+j) = j(j+1)\pi$, $\frac{1}{(j+1)\pi} < (x+\alpha_k u_k) < \frac{1}{j-1\pi}$. However, this is impossible since the the only element $\omega \in \Omega$ s.t. $\frac{1}{(j+1)\pi} < \omega < \frac{1}{j-1\pi}$ is $\omega = \frac{1}{j\pi}$ (for any other $k \in \mathbb{Z}, k \neq 0$, either $k \leq j-1$ which makes $\frac{1}{k\pi} \geq \frac{1}{(j-1)\pi}$ or $k \geq j+1$ which makes $\frac{1}{k\pi} \leq \frac{1}{(j+1)\pi}$). But this implies $\alpha_k = 0$ or $u_k = 0$, a contradiction.

Case 4: If j is negative (excluding the case $j = -1$)

Set $\epsilon = \frac{1}{(j-1)\pi} - \frac{1}{j\pi} > 0$ Because $u_n \to u, \alpha_n \downarrow 0$, we know $x+\alpha_n u_n \to x$ so can choose some k large enough s.t. $|u_k - u| < |u|$ (making $u_k \neq 0$ since u is nonzero) and $|(x+\alpha_k u_k)-x| < \frac{1}{(j-1)\pi} - \frac{1}{j\pi}$. That is, $x+\frac{1}{j\pi} - \frac{1}{(j-1)\pi} < (x+\alpha_k u_k) < x+\frac{1}{(j-1)\pi} - \frac{1}{j\pi}$. We can also choose We know that $x+\alpha_k u_k \in \Omega$ for $k \geq K$.

Notice that: $\frac{2}{j\pi} - \frac{1}{(j-1)\pi} = \frac{j-2}{j(j-1)\pi} < (x+\alpha_k u_k) < \frac{1}{(j-1)\pi}$. Notice however that $\frac{1}{(j+1)\pi} \leq \frac{j-2}{j(j-1)\pi}$ since $j(j-1)\pi = \pi(j^2-j) \leq \pi(j^2-j-2) = (j-2)(j+1)\pi$, s0 $\frac{1}{(j+1)\pi} < (x+\alpha_k u_k) < \frac{1}{(j-1)\pi}$. However, this is impossible since the the only element $\omega \in \Omega$ s.t. $\frac{1}{(j+1)\pi} < \omega < \frac{1}{j-1\pi}$ is $\omega = \frac{1}{j\pi}$. But this implies $\alpha_k = 0$ or $u_k = 0$, a contradiction.

Hence, our assumption was false and we have shown that $u \notin T_\Omega(x)$ by creating an open interval around x in which no elements of $x+\alpha_n u_n$ exist. This shows that the tangent cone is in fact only $T_\Omega(x) = \{0\}$. The normal cone is $\mathbb{R}$ since $\forall x \in \mathbb{R}$, $x * 0 \leq 0$.

C)

We assume that f is smooth implies that it is differentiable. For 0 to be a local solution, one necessary condition is denoted by theorem 12.3. That is:

$\nabla f(0)^T d \geq 0$ for all $d \in T_\Omega(0)$.

From our above results, we saw that $T_\Omega(0) = \mathbb{R}$. This implies that $\nabla f(0)^T d = \nabla f(0) * (-1) \geq 0$ and $\nabla f(0)^T d = \nabla f(0) * (1) \geq 0$. So $\nabla f(0) = 0$ is neccesary by theorem 12.3

D)

For $x \neq 0$ to be a local solution, it must satisfy theorem 12.3: $\nabla f(x)^T d \geq 0$ for all $d \in T_\Omega(x)$. For these x, recall that $T_\Omega(x) = \{0\}$, so for any $d \in T_\Omega(x)$, $\nabla f(x)^T d = \nabla f(x) * 0 = 0 \geq 0$. So the gradient at any of these points are not restricted by theorem 12.3. In fact, regardless of f, we can construct an open ball $Ball_\epsilon(x)$ around each x using our choices for ϵ from (B) s.t. $\{x\} = Ball_\epsilon(x) \cap \Omega$. Then, within this neighborhood, $f(x) \leq f(y), \forall y \in Ball_\epsilon(x) \cap \Omega = \{x\}$ so x is a local minimum. So every $x \neq 0$ is naturally a local minimum. However, for $x = 0$ to not be a local solution, it suffices for $\nabla f(x) \neq 0$ from our result in C.