CS730: Homework 1

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February 1, 2016

Problem 1

1)

Intersection of convex sets is convex: Proof

Let some 2 convex sets C_1, C_2 be given. Consider $C_1 \cap C_2$ and let some points $x, y \in C_1 \cap C_2$ be given. Because $x, y \in C_1$, we know then that, $\forall \lambda \in [0,1]$, $(1-\lambda)x + \lambda y \in C_1$ from convexity. And because $x, y \in C_2$, we know then that, $\forall \lambda \in [0,1]$, $(1-\lambda)x + \lambda y \in C_2$. Hence, $\forall \lambda \in [0,1], (1-\lambda)x + \lambda y \in C_1 \cap C_2$. Because this was for any arbitrary $x, y \in C_1 \cap C_2$, we see that $C_1 \cap C_2$ is also convex.

2)

Union of convex sets is convex: Counterexample

See attached

Problem 2

Let positive integer m and convex sets $C_i \in \mathbb{R}^{n_i}, i \in \{1, ..., m\}$ be given. Consider $C_1 \times ... \times C_m$ and let some $x, y \in C_1 \times ... \times C_m$. By the definition of Cartesian product, we can write $x = [x_1, ..., x_m]^T, y = [y_1, ..., y_m]^T$, for $x_i, y_i \in C_i, \forall i \in \{1, .., m\}$. Let some $\lambda \in [0,1]$ be given and consider $\lambda x + (1-\lambda)y = \lambda[x_1, ..., x_m]^T + (1-\lambda)[y_1, ..., y_m]^T$. By the properties of the vector space under Cartesian products: $\lambda[x_1, ..., x_m]^T + (1-\lambda)[y_1, ..., y_m]^T = [\lambda x_1, ..., \lambda x_m]^T + [(1-\lambda)y_1, ..., (1-\lambda)y_m]^T = [\lambda x_1 + (1-\lambda)y_1, ..., \lambda x_m + (1-\lambda)y_m]^T$

For each $i \in \{1, ..., m\}$, we know that $\lambda x_i + (1-\lambda)y_i \in C_i$ since C_i is convex. So $[\lambda x_1 + (1-\lambda)y_1, ..., \lambda x_m + (1-\lambda)y_m]^T \in C_1 \times ... \times C_m$. Because x, y, λ were arbitrary, it readily follows that $C_1 \times ... \times C_m$ is convex.

Problem 3

1)

Let some nonempty convex set S be given. First, we shall show the interior of a convex set is convex. So let some elements $x, y \in int(S)$ be given. Now let some $\lambda \in [0,1]$ be given. We want to show that $\lambda x + (1-\lambda)y \in int(S)$. Trivially, this is true for $\lambda = 0$ and $\lambda = 1$ (since $x, y \in int(S)$ by definition), so consider the case where $\lambda \in (0,1)$. Set the point $z = \lambda x + (1-\lambda)y$. Because $x, y \in S$, we see that $z \in S$ by convexity. Because $x, y \in int(S)$, we can choose some open balls of respective radius ϵ_1, ϵ_2 s.t. $Ball_{\epsilon_1}(x) \subset S$ and $Ball_{\epsilon_2}(y) \subset S$ (i.e. open balls which are contained in S) by definition of interior. So set $\epsilon = min(\epsilon_1, \epsilon_2)$. Clearly, $Ball_\epsilon(x) \subset S$ and $Ball_\epsilon(y) \subset S$ since we are just taking a ball of the same or smaller radius.

To show that $z \in int(S)$, we would like to show that we can create an open ball of radius ϵ around z contained within S. So let some element $z^* \in Ball_\epsilon(z)$ be given. Notice that $\|z^* - z\|_2 < \epsilon$ since z^* was chosen from the open ball of radius

ϵ, so it readily follows that $x + (z^* - z) \in Ball_\epsilon(x)$ and $y + (z^* - z) \in Ball_\epsilon(y)$. Because those balls contained in S, $x + (z^* - z), y + (z^* - z) \in S$ and thus by convexity: $\lambda(x + (z^* - z)) + (1 - \lambda)(y + (z^* - z)) \in S$. Substituting our definition of z and expanding: $\lambda(x + z^* - z) + (1 - \lambda)(y + z^* - z) = \lambda(x + z^* - (\lambda x + (1 - \lambda)y)) + (1 - \lambda)(y + z^* - (\lambda x + (1 - \lambda)y)) =$

$\lambda(x + z^* - \lambda x - y + \lambda y) + (1 - \lambda)(y + z^* - \lambda x - y + \lambda y) =$

$\lambda((1 - \lambda)x + z^* - y + \lambda y) + (1 - \lambda)(z^* - \lambda x + \lambda y) =$

$\lambda x - \lambda^2 x + \lambda z^* - \lambda y + \lambda^2 y + z^* - \lambda x + \lambda y - \lambda z^* + \lambda^2 x - \lambda^2 y =$

$\lambda x - \lambda x + \lambda^2 x - \lambda^2 x + \lambda z^* - \lambda z^* + \lambda y - \lambda y + \lambda^2 y - \lambda^2 y + z^* = z^*$

So, overall, this implies that $z^* \in S$. Because $z^* \in Ball_\epsilon(z)$ was arbitrary, this implies that $Ball_\epsilon(z) \subset S$, so $z = \lambda x + (1 - \lambda)y \in int(S)$. Because $\lambda \in (0, 1)$ and $x, y \in int(S)$ was also arbitrary, we see that $\forall x, y \in int(S), \forall \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in int(S)$, showing that the interior of S convex.

2)

Next, we wish to show that the closure of S is also convex. Let some $x, y \in cl(S)$ be given. Let some $\lambda \in [0, 1]$ be given. To show $\lambda x + (1 - \lambda)y \in cl(S)$, it suffices to construct a sequence of points in S s.t. $\lambda x + (1 - \lambda)y$ is the limit point. Because $x \in cl(S)$, we can construct a sequence $\{x_n\} \in S$ s.t. $x_n \in Ball_{\frac{1}{n}}(x)$ (every open ball of x with strictly positive radius contains some element in S); clearly, the limit point is x since $|x_n - x| \leq \frac{1}{n}$. Similarly, we can construct a sequence $\{y_n\} \in S$ s.t. $y_n \in Ball_{\frac{1}{n}}(y)$ (again, with limit y). So then we can construct a new sequence $z_i = \lambda x_i + (1 - \lambda)y_i \in S$, since $x_i, y_i \in S$ and S is convex. In the limit, $x_i \to x, y_i \to y$ so $z_i = \lambda x_i + (1 - \lambda)y_i \to \lambda x + (1 - \lambda)y$, so the limit point of this sequence is $\lambda x + (1 - \lambda)y$. Hence, $\lambda x + (1 - \lambda)y \in cl(S)$. Overall, because x,y were arbitrary and $\lambda \in [0, 1]$, this shows that cl(S) is also convex.

Problem 4

Let some set S be given.

Suppose S is convex. To prove the containment of convex combinations for arbitrary finite sets, we shall use induction on the number of element in S (namely, every convex combination of n elements from S is contained within S).

Base Case: $n = 1$

In this case, we have a single element, say x^1, from S and thus $\sum \lambda_i = \lambda_1 = 1$. Clearly, $\lambda_1 x^1 = x^1 \in S$

Inductive Step: Next, assume the case for $n = k$. That is, every convex combination of k elements from S is contained within S.

We want to show the case where $n = k + 1$. So let some $k + 1$ elements in S be given, say $x^1, x^2, ..., x^{k+1}$. Let some $\lambda_i \geq 0$ for $i \in \{1, 2, ..., k + 1\}$ be given s.t. $\sum_{i=1}^{k+1} \lambda_i = 1$.

Case 1: Suppose $\lambda_{k+1} = 1$. Then $\sum_{i=1}^{k+1} \lambda_i = \sum_{i=1}^{k} \lambda_i + 1 = 1 \to \sum_{i=1}^{k} \lambda_i = 0$, and because $\lambda_i \geq 0$, it readily follows that $\lambda_1, ..., \lambda_k = 0$. So the convex combination of these k+1 elements becomes: $\sum_{i=1}^{k+1} \lambda_i x^i = \sum_{i=1}^{k} \lambda_i x^i + \lambda_{k+1} x^{k+1} = \sum_{i=1}^{k} 0x^i + 1x^{k+1} = x_{k+1}$. So clearly, $\sum_{i=1}^{k+1} \lambda_i x^i = x_{k+1} \in S$.

Case 2: Now suppose that $\lambda_{k+1} \neq 1$. Note that because $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, it readily follows that $\lambda_i \leq 1$ (the sum of positive scalars must each be less than or equal to the sum).

Next, we see that $\sum_{i=1}^{k+1} \lambda_i = \sum_{i=1}^{k} \lambda_i + \lambda_{k+1} = 1 \to \sum_{i=1}^{k} \lambda_i = 1 - \lambda_{k+1}$, so $\frac{\sum_{i=1}^{k} \lambda_i}{1 - \lambda_{k+1}} = \sum_{i=1}^{k} \frac{\lambda_i}{1 - \lambda_{k+1}} = 1$.

Also notice that $\frac{\lambda_i}{1 - \lambda_{k+1}} \geq 0, \forall i \in \{1, ..., k\}$ since $\lambda_i \geq 0$ and $1 > \lambda_{k+1}$, so $0 < 1 - \lambda_{k+1}$.

So $\frac{\lambda_i}{1 - \lambda_{k+1}} \geq 0, \forall i \in \{1, ..., k\}$ can act as convex combination coefficients on k elements. If we consider convex combination formed from the first k elements $x^1, ..., x^k$, we see that $y = \sum_{i=1}^{k} \frac{\lambda_i}{1 - \lambda_{k+1}} x^i \in S$ by our inductive hypothesis.

Because S is convex and $\sum_{i=1}^{k} \frac{\lambda_i}{1 - \lambda_{k+1}} x^i, x^{k+1} \in S$, we see that for $\lambda = \lambda_{k+1}$: $(1 - \lambda_{k+1}) \sum_{i=1}^{k} \frac{\lambda_i}{1 - \lambda_{k+1}} x^i + \lambda_{k+1} x^{k+1} \in S$.

With some algebraic manipulation:

$(1-\lambda_{k+1})\sum_{i=1}^{k}\frac{\lambda_i}{1-\lambda_{k+1}}x^i+\lambda_{k+1}x^{k+1}=\sum_{i=1}^{k}(1-\lambda_{k+1})\frac{\lambda_i}{1-\lambda_{k+1}}x^i+\lambda_{k+1}x^{k+1}=\sum_{i=1}^{k}\lambda_i x^i+\lambda_{k+1}x^{k+1}=\sum_{i=1}^{k+1}\lambda_i x^i\in S.$
So this implies the convex combination $\sum_{i=1}^{k+1}\lambda_i x^i$ is in S. And because x^i,λ_i were arbitrary, we have shown convex combination of $k+1$ elements in S is contained within S.

By induction, we have shown the convex combination of finitely many elements in S is contained within S.

Problem 5

1)

Let some A, C be given as defined in the problem and consider the set AC as defined.

To show AC is convex, let some $b,c\in AC$ be given. Then we can choose $x,y\in C$ s.t. $Ax=b, Ay=c$ by definition of AC. Let some $\lambda\in[0,1]$ be given. Because C is convex, we know that $\lambda x+(1-\lambda)y\in C$. Set $z=\lambda x+(1-\lambda,)y$. Notice then that $Az=A(\lambda x+(1-\lambda)y)=\lambda Ax+(1-\lambda)Ay$ by linearity. Because $z\in C$, we know that $Az\in AC$ by definition of AC. Furthermore, this shows that $\lambda Ax+(1-\lambda)Ay\in AC$, and because x,y,λ we arbitrary, it follows that AC is convex.

2)

AC is not neccesarily closed. As a counterexample, let $A=\begin{bmatrix}0&0\\0&1\end{bmatrix}$ and $C=\{(x,\mu)|x>0,\mu\geq 0,\mu\geq\frac{1}{x}\}$ (an epigraph on strictly positive x). Notice that C is convex (because $f(x)=\frac{1}{x}$ is convex on $x>0$) and closed.

Notice that, for some point (x,μ) in C, $A*(x,\mu)=(0,\mu)$. Consider the sequence $a_n=(0,\frac{1}{n}),\forall n\geq 1$. Notice that $(n,\frac{1}{n})\in C$, so $A*(n,\frac{1}{n})=(0,\frac{1}{n})=a_n$, so $a_n\in AC$. But the limit point $a_n\to(0,0)\notin AC$ since no element in C has the form $(a,0)$. So AC is not closed in this case.

Now consider the case that C is compact. To show AC is closed, consider AC and and let some limit point x of AC be given. We want to show that $x\in AC$. To do so, first, construct a sequence $\{x_n\}\subseteq AC$ s.t. $x_n\to x$. By definition of AC, we can choose some $\{c_n\}\subseteq C$ s.t. $Ac_n=x_n$. Because C is compact, we can choose some subsequence of $\{c_n\}$ s.t. the limit point, say c, is in C. That is, for some (strictly) increasing function $f:\mathbb{N}\to\mathbb{N}$, $c_{f(n)}\to c\in C$. So $Ac_{f(n)}-Ac=A(c_{f(n)}-c)\to A*0=0$, so $Ac_{f(n)}\to Ac$. But by our choice of c_n, we know that $Ac_{f(n)}\to x$. Hence, $Ac=x\in AC$ since $c\in C$. Overall, this shows that AC is closed.

Problem 6

To show convexity under linear maps, let A,Y be given as defined in the problem and consider $A^{-1}(Y)=\{x|Ax\in Y\}$. Let some $x,y\in A^{-1}(Y)$ be given. Then we know that $Ax,Ay\in Y$. Now let some $\lambda\in[0,1]$ be given. Since Y is convex, $\lambda(Ax)+(1-\lambda)(Ay)\in Y$. So then $\lambda(Ax)+(1-\lambda)(Ay)=(A\lambda x+A(1-\lambda)y)=A(\lambda x+(1-\lambda)y)\in Y$. But this implies that $(\lambda x+(1-\lambda)y)\in A^{-1}(Y)$, and because x,y,λ were arbitrary, it readily follows that $A^{-1}(Y)$ is convex.

The set under an affine map is also convex for arbitrary b. Consider $A^{-1}(Y)=\{x|Ax+b\in Y\}$. Let some $x,y\in A^{-1}(Y)$ be given. Then we know that $Ax+b,Ay+b\in Y$. Now let some $\lambda\in[0,1]$ be given. Since Y is convex, $\lambda(Ax+b)+(1-\lambda)(Ay+b)\in Y$. So then $\lambda(Ax+b)+(1-\lambda)(Ay+b)=(A\lambda x+\lambda b+A(1-\lambda)y+(1-\lambda)b)=A(\lambda x+(1-\lambda)y)+b\in Y$. But this implies that $(\lambda x+(1-\lambda)y)\in A^{-1}(Y)$, and because x,y,λ were arbitrary, it readily follows that $A^{-1}(Y)$ is convex.

Problem 7

Let some set S be given. Consider CoS, the convex hull of S, which is defined to be the intersection of all convex sets containing S. Let $CC(S)$ be the set of all convex combinations of points in S.

To show that $CC(S) = CoS$, first we shall show that $CC(S) \subseteq CoS$. Consider some arbitrary convex set, say X, containing S. Let some element $y \in CC(S)$ be given. Then we can choose some r elements $\{x^1, ..., x^r\} \subseteq S$ and $\lambda_1, ..., \lambda_r \geq 0$ with $\sum \lambda_i = 1$ such that $y = \sum \lambda_i x^i$. Since $\{x^1, ..., x^r\} \subseteq S \subseteq X$ and X is convex, from (4), we know that every finite convex combination of elements in X is contained within X; so $y = \sum \lambda_i x^i \in X$. Because y was arbitrary in $CC(S)$, it follows that $CC(S) \subseteq X$. And because X was any arbitrary convex set containing S, it readily follows every convex set containing S also contains $CC(S)$. So $CC(S) \subseteq CoS$ since CoS is the intersection of all these sets.

To show $CoS \subseteq CC(S)$, it suffices to show that $CC(S)$ is a convex set containing S, which implies it is a set included within the intersection forming CoS . For any $x \in S$, $x \in CC(S)$ by constructing the convex combination with $\lambda_1 = 1$:$\sum \lambda_i x^i = \lambda_1 x = x$. So $S \subseteq CC(S)$. To show $CC(S)$ is convex, let some $x, y \in CC(S)$ be given. Then we can choose some r elements $\{x^1, ..., x^r\} \subseteq S$ with $\alpha_1, ..., \alpha_r \geq 0$ with $\sum \alpha_i = 1$ such that $x = \sum_{i=1}^{s} \alpha_i x^i$. Similarly, we can choose some s elements $\{y^1, ..., y^s\} \subseteq S$ with $\beta_1, ..., \beta_r \geq 0$ with $\sum \beta_i = 1$ such that $y = \sum_{i=1}^{s} \beta_i y^i$. Now let some $\lambda \in [0, 1]$ be given and consider $z = \lambda x + (1 - \lambda)y = \lambda \sum_{i=1}^{s} \alpha_i x^i + (1 - \lambda) \sum_{i=1}^{s} \beta_i y^i$. We would like to show $z \in CC(S)$.

Define $\{z^1, ..., z^{r+s}\} \subseteq S$ as $z^i = x^i, \forall i \in \{1, ..., r\}$ and $z^{r+i} = y^i, \forall i \in \{1, .., s\}$. Next, define $\{\gamma_1, ..., \gamma_{r+s}\}$ with $\gamma_i = \lambda \alpha_i, \forall i \in \{1, ..., r\}$ and $\gamma_{r+i} = (1 - \lambda)\beta_i, \forall i \in \{1, ..., s\}$. Clearly, because $\alpha_i, \beta_i, \lambda, (1 - \lambda) \geq 0$, it readily follows that $\gamma_i \geq 0, \forall i \in \{1, ..., r + s\}$.

Furthermore, $\sum_{i=1}^{r+s} \gamma_i = \sum_{i=1}^{r} \gamma_i + \sum_{i=r+1}^{r+s} \gamma_i = \sum_{i=1}^{r} \lambda \alpha_i + \sum_{i=1}^{s} (1 - \lambda)\beta_i = \lambda \sum_{i=1}^{r} \alpha_i + (1 - \lambda) \sum_{i=1}^{s} \beta_i = \lambda(1) + (1 - \lambda)(1) = 1$. Hence, $\sum_{i=1}^{r+s} \gamma_i z^i$ is a convex combination of points in S, so $\sum_{i=1}^{r+s} \gamma_i z^i \in CC(S)$. Furthermore, notice that $\sum_{i=1}^{r+s} \gamma_i z^i = \sum_{i=1}^{r} \lambda \alpha_i x^i + \sum_{i=1}^{s} (1 - \lambda)\beta_i y^i = \lambda \sum_{i=1}^{r} \alpha_i x^i + (1 - \lambda) \sum_{i=1}^{s} \beta_i y^i = z$, from before. Hence, $z \in CC(S)$. Because this was for arbitrary $x, y \in CC(S), \lambda \in [0, 1]$, it follows that $CC(S)$ is a convex set. Since $CC(S)$ is a convex set containing S, $CoS \subseteq CC(S)$.

Hence, overall, we have shown that: $CoS = CC(S)$

2)

Suppose that S is compact. To show that coS is compact, first we shall show it is bounded. Because S is compact and thus bounded, we can choose some M s.t. $\forall a \in S, \|a - 0\| = \|a - 0\| = \|a\| \leq M$. Now let some $x \in coS$ be given. Then we can choose some r elements $\{x^1, ..., x^r\} \subseteq S$ and $\lambda_1, ..., \lambda_r \geq 0$ with $\sum \lambda_i = 1$ such that $x = \sum \lambda_i x^i$. Notice then that $\|x - 0\| = \left\|\sum \lambda_i x^i\right\| = \sum |\lambda_i| \left\|x^i\right\| \leq \sum |\lambda_i| M$ by the triangle inequality and by our definition of x^i, M. We have shown that the distance from every point x in coS to 0 is bounded by $\sum |\lambda_i| M$ and thus the set itself in bounded.

Next, we will show closure. Suppose that x is a limit point of coS. That is, we can construct a sequence $\{a_n\} \in coS$ s.t. $a_n \to x$. Then $a_n = \sum_{i=1}^{r_n} \lambda_{n,i} x_n^i$ for some positive integers r_n, $x_n^i \in S$, and $\lambda_{n,i} \in [0, 1]$. In compact X, we can construct a subsequence of x_n^1 with some limit point in X, say $\{x_{n^*}^1\}$ for some $n^* \in N^* \subseteq \mathbb{N}$. Similarly, we can construct another subsequence on $\{x_{f(n)}^2\}$ s.t. it too has a limit point in X, call it $\{x_{n^{**}}^2\}$ for $n^{**} \in N^{**} \subseteq N^*$. We can repeat this process until we have some subset $N^{(r)} \subseteq \mathbb{N}$ s.t. each subsequence $\{x_{n'}^i\}$ converges for $n' \in N^{(r)}$, all i. Furthermore, we can apply a similar process to λ_n^i since $\lambda_n^i \in [0, 1]$ is a compact space, using $N^{(r)}$ as our initial set of subsequence indices. So we can construct a subset $M^{(r)} \subseteq N^{(r)} \subseteq \mathbb{N}$ s.t. $\{\lambda_n^i\}$ converges in $[0, 1]$ for all i. On the subsequence defined by $M^{(r)}$, we have $a_n = \sum_i^{r_n} \lambda_{n,i} x_n^i$, as $n \in M^{(r)} \to \infty$, we see that the terms $x_n^i \to x^i \in X, \lambda_{n,i} \to \lambda_i \in [0, 1]$ converge, particularly to elements in S, $[0, 1]$ respectively. Furthermore, $\sum_i^{r_n} \lambda_{n,i} = 1$, so in the limit, we know $\sum \lambda_i = 1$. So then the limit $\sum \lambda_{n,i} x_n^i \to \sum \lambda_i x^i = x$ is a convex combination of terms in S; hence, $\sum \lambda_i x^i = x \in coS$, showing that coS is closed.

Problem 8

Let some cone C be given. First, suppose that C is convex and let some $x, y \in C$ be given. Because C is convex, we know that $\forall \lambda \in [0, 1], (1 - \lambda)x + \lambda y \in C$. So let us set $\lambda = \frac{1}{2}$. Then $(1 - \frac{1}{2})x + \frac{1}{2}y = \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}(x + y) \in C$. And because C is a cone, it follows that $\alpha * \frac{1}{2}(x + y) \in C$ for any $\alpha > 0$. So if we choose $\alpha = 2$, we see that $2 * \frac{1}{2}(x + y) = x + y \in C$. Because x,y were arbitrary in C, we see that C is closed under addition.

Next suppose that C is closed under addition and let some $x, y \in C$ be given. We want to show that $\forall \lambda \in [0,1], (1-\lambda)x+\lambda y \in C$. So let some $\lambda \in [0,1]$ be given. If $\lambda = 0$, then $(1-0)x + 0y = x$, which we know is in C since $x \in C$; similarly, if $\lambda = 1$, then $(1-1)x + 1y = y$, which we know is in C since $y \in C$. If $\lambda \neq 0$ and $\lambda \neq 1$, then $1 > \lambda > 0$, so by the definition of a cone, $\lambda y \in C$. Furthermore, this implies that $(1-\lambda) > 0$, so $(1-\lambda)x \in C$. By the additive closure of C, we see that $(1-\lambda)x + \lambda y \in C$. Because λ, x, y were arbitrary, we see that $\forall x, y \in C, \forall \lambda \in [0,1], (1-\lambda)x + \lambda y \in C_1 \cap C_2$, so C is in fact convex.

Problem 9

Let f be given as defined and consider $epi(f)$. Suppose that $epi(f)$ is convex; we would like to show that f is convex. So let some $x_1, x_2 \in dom(f)$ be given. Now let some $\lambda \in [0,1]$ be given and $\lambda f(x_1)+(1-\lambda)f(x_2)$. We know that $(x_1, f(x_1)), (x_2, f(x_2)) \in epi(f)$ since $f(x_1) \leq f(x_1), f(x_2) \leq f(x_2)$, and because it is convex, we know that $\lambda(x_1, f(x_1)) + (1-\lambda)(x_2, f(x_2)) \in epi(f)$. But $\lambda(x_1, f(x_1)) + (1-\lambda)(x_2, f(x_2)) = (\lambda x_1, \lambda f(x_1)) + ((1-\lambda)x_2, (1-\lambda)f(x_2)) = (\lambda x_1 + (1-\lambda)x_2, \lambda f(x_1) + (1-\lambda)f(x_2)) \in epi(f)$. By definition of $epi(f)$, this implies that $\lambda x_1+(1-\lambda)x_2 \in dom(f)$ and thus $\lambda f(x_1)+(1-\lambda)f(x_2)) \geq f(\lambda x_1+(1-\lambda)x_2)$. Whence, because $x_1, x_2 \in dom(f), \lambda \in [0,1]$ were arbitrary, it follows that f itself is convex.

Suppose now that f is convex; we would like to show that $epi(f)$ is convex. So let some $(x_1, \mu_1), (x_2, \mu_2) \in epi(f)$ be given. Now let some $\lambda \in [0,1]$ be given. By the definition of $epi(f)$, we know that $f(x_1) \leq \mu_1, f(x_2) \leq \mu_2$. Now consider $\lambda(x_1, \mu_1) + (1-\lambda)(x_2, \mu_2) = (\lambda x_1 + (1-\lambda)x_2, \lambda\mu_1 + (1-\lambda)\mu_2)$ Because f is convex, we know that $\lambda x_1 + (1-\lambda)x_2 \in dom(f)$ and, furthermore, that $\lambda f(x_1) + (1-\lambda)f(x_2)) \geq f(\lambda x_1 + (1-\lambda)x_2)$. But we also know that $\lambda f(x_1) + (1-\lambda)f(x_2)) \leq \lambda\mu_1 + (1-\lambda)\mu_2$. So $\lambda\mu_1 + (1-\lambda)\mu_2 \geq \lambda f(x_1) + (1-\lambda)f(x_2)) \geq f(\lambda x_1 + (1-\lambda)x_2)$. Since $\lambda x_1 + (1-\lambda)x_2 \in dom(f)$ and $\lambda\mu_1 + (1-\lambda)\mu_2 \geq f(\lambda x_1 + (1-\lambda)x_2)$, we know that $(\lambda x_1 + (1-\lambda)x_2, \lambda\mu_1 + (1-\lambda)\mu_2) \in epi(f)$ by definition of $epi(f)$. Hence, since $(x_1, \mu_1), (x_2, \mu_2) \in epi(f), \lambda \in [0,1]$ were arbitrary, it follows that $epi(f)$ itself is convex.

Overall, we have shown that f is convex iff $epi(f)$ is convex.

Problem 10

1)

Let some function f be given and suppose that f is convex. Let some c be given and consider $S = \{x | f(x) \leq c\}$. (Note $S = \emptyset$ is convex, so we assume nonempty). Let some $x_1, x_2 \in S$ and some $\lambda \in [0,1]$ be given. By the definition of D, we know that $f(x_1) \leq c, f(x_2) \leq c$. Because f is convex, we know that $\lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2)$. But then $\lambda c + (1-\lambda)c = c \geq \lambda f(x_1) + (1-\lambda)f(x_2)$. So $c \geq \lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2)$. Hence, we see that $c \geq f(\lambda x_1 + (1-\lambda)x_2)$ and thus $\lambda x_1 + (1-\lambda)x_2 \in S$. Hence, since $x_1, x_2 \in S, \lambda \in [0,1]$, were arbitrary, it follows that S itself is convex, for any c.

2)

Converse: Counterexample $f(x) = x^3$

The convex sets are only formed on the domain (here, x) so they fail to capture the function's own behavior, particularly when the function is monotonically increasing.

Problem 11

Let $P(y)$ be given with $y \notin \Omega$ (otherwise, $\langle y - P(y), x - P(y)\rangle = \langle y - y, x - P(y)\rangle = 0$ trivially). Then, $\forall x \in \Omega$, $\|y - P(y)\| \leq \|y - x\|$, so $\|y - P(y)\|^2 = \langle y - P(y), y - P(y)\rangle \leq \|y - x\|^2 = \langle y - x, y - x\rangle$. Consider $\langle y - P(y), x - P(y)\rangle$ for some $x \in \Omega$ with $x \neq P(y)$ (otherwise, $\langle y - P(y), x - P(y)\rangle = \langle y - P(y), 0\rangle = 0$ trivially). Because $P(y), x \in \Omega$, we know that for $\lambda \in (0,1]$, $\lambda x + (1-\lambda)P(y) \in \Omega$. So then $\langle y - P(y), y - P(y)\rangle \leq \langle y - (\lambda x + (1-\lambda)P(y)), y - (\lambda x + (1-\lambda)P(y))\rangle$.

But:

$\langle y-(\lambda x+(1-\lambda)P(y)), y-(\lambda x+(1-\lambda)P(y))\rangle =$

$\langle y-\lambda x-P(y)+\lambda P(y), y-\lambda x-P(y)-\lambda P(y)\rangle = \langle y-P(y)+\lambda(P(y)-x), y-P(y)+\lambda(P(y)-x)\rangle$

$\langle y-P(y), y-P(y)+\lambda(P(y)-x)\rangle + \langle \lambda(P(y)-x), y-P(y)+\lambda(P(y)-x)\rangle =$

$\langle y-P(y), y-P(y)\rangle + \langle y-P(y), -\lambda(x-P(y))\rangle + \langle -\lambda(x-P(y)), y-P(y)\rangle + \langle -\lambda(x-P(y)), -\lambda(x-P(y))\rangle =$

$\langle y-P(y), y-P(y)\rangle + 2\lambda\langle y-P(y), -(x-P(y))\rangle + \lambda^2\langle -(x-P(y)), -(x-P(y))\rangle =$

$\langle y-P(y), y-P(y)\rangle - 2\lambda\langle y-P(y), -(x-P(y))\rangle + \lambda^2\langle (x-P(y)), (x-P(y))\rangle$

So we see that:

$\langle y-P(y), y-P(y)\rangle \leq \langle y-P(y), y-P(y)\rangle - 2\lambda\langle y-P(y), (x-P(y))\rangle + \lambda^2\langle (x-P(y)), (x-P(y))\rangle \rightarrow$

$0 \leq -2\lambda\langle y-P(y), (x-P(y))\rangle + \lambda^2\langle (x-P(y)), (x-P(y))\rangle \rightarrow$

$\langle y-P(y), (x-P(y))\rangle \leq \frac{\lambda}{2}\langle (x-P(y)), (x-P(y))\rangle$ (noting that we chose $\lambda > 0$, allowing for this division)

We know that $\langle (x-P(y)), (x-P(y))\rangle \geq 0$ since $\langle (x-P(y)), (x-P(y))\rangle = ||(x-P(y)||^2$. Because $\langle y-P(y), (x-P(y))\rangle$ is independent of λ, this inequality should hold $\forall \lambda \in (0,1]$. For any $\epsilon > 0$, we can choose $\lambda = \frac{\epsilon}{\langle (x-P(y)), (x-P(y))\rangle}$ (note the denominator is nonzero) and see that:

$\langle y-P(y), (x-P(y))\rangle \leq \frac{\lambda}{2}\langle (x-P(y)), (x-P(y))\rangle = \frac{\epsilon}{2\langle (x-P(y)), (x-P(y))\rangle}\langle (x-P(y)), (x-P(y))\rangle = \frac{\epsilon}{2}$.

So $\langle y-P(y), (x-P(y))\rangle \neq \epsilon, \forall \epsilon > 0$. Hence, $\langle y-P(y), (x-P(y))\rangle \leq 0$, as expected.

Problem 12

Now let some $(u,v,w) \notin -\Omega$ be given (note it must be nonzero). We will show that we can construct a vector $(x,y,z) \in \Omega$ s.t. $(u,v,w)^T(x,y,z) > 0$. If $(u,v,w) \in \Omega$, we can just choose (u,v,w) itself and see that $(u,v,w)^T(u,v,w) = u^2+v^2+w^2 > 0$. Now consider the case in which $(u,v,w) \notin \Omega$. In this case, we know that $w < \sqrt{u^2+v^2}$ by definition of Ω. We can also note that $-\sqrt{u^2+v^2} < w$ since $(u,v,z) \in -\Omega$ for $z \leq -\sqrt{u^2+v^2}$. We can construct the vector $(u,v,\sqrt{u^2+v^2}) \in \Omega$ since $(\sqrt{u^2+v^2})^2 \geq u^2+v^2$. Then $(u,v,w)^T(u,v,\sqrt{u^2+v^2}) = u^2+v^2+w\sqrt{u^2+v^2}$. Note that because $-\sqrt{u^2+v^2} < w$, $w\sqrt{u^2+v^2} > -\sqrt{u^2+v^2} * \sqrt{u^2+v^2} = -u^2-v^2$. So $u^2+v^2+w\sqrt{u^2+v^2} > u^2+v^2-u^2-v^2 = 0$. Hence, we have shown $(u,v,w)^T(u,v,\sqrt{u^2+v^2}) > 0$. Overall, for $(u,v,w) \notin -\Omega$, $(u,v,w) \notin N_\Omega(0)$.

Problem 13

Let some positive integer n be given and let S be the set of all n by n positive semidefinite matricies. To show S is a cone, let some $A \in S$ and some $\lambda > 0$ be given. Because A is positive semidefinite, we can decompose it via eigenvalue decomposition into QDQ^T, where Q is orthogonal and D is a real valued diagonal matrix of A's eigenvalues. Because A is positive semidefinite, D consists of positive values (on the diagonal). So consider λA. Trivially, λA is symmetric with $(\lambda A)^T = \lambda A^T = \lambda A$. Furthermore, $\lambda A = \lambda DQ^T = Q(\lambda D)Q^T$. Notice that because D is a diagonal matrix (with real values), λD is also a diagonal matrix. Furthermore, because $\lambda > 0$ and $D \geq 0$, we know that (λD) has positive entries as well. So we see that can be written as a eigendecomposition of a symmetric matrix, with (λD) as its diagonal matrix of eigenvalues. Furthermore, because (λA) is symmetric and its eigenvalues are all positive, we see that (λA) is positive semidefinite, showing that S is a cone.

To show that S is a pointed cone, it suffices to show that there is no $A \in S, A \neq 0$ s.t. $-A \in S$. To show this, let some $A \in S, A \neq 0$ be given. We can decompose A into $A = QDQ^T$ via eigendecomposition. Now consider $-A$. Again, we see that $-A = -1 * A$ is symmetric. Then $-A = -1 * A = -1(QDQ^T) = Q(-1 * D)Q^T$. We can see that $Q(-1 * D)Q^T$ forms a eigendecomposition for the symmetric matrix $-A$ with diagonal matrix $(-1 * D)$. Recall that $D \geq 0$. If $D = 0$, then $A = Q(0)Q^T = 0$, a contradiction. So D must have atleast 1 strictly positive entry on its diagonal, say $D_{i,j} > 0$ for some i,j. But this implies that $(-1 * D)_{i,j} = -1 * D_{i,j} < 0$. This implies from the eigendecomposition $-A = Q(-1 * D)Q^T$ that $-A$

has a negative eigenvalue. But this implies that $-A$ is not positive semidefinite. Because A was arbitrary, we have shown no positive semidefinite matrix A has its negative $-A$ in S. So S is a pointed cone.