CS / ISyE 730 - Spring 2016 - Homework 1

(assigned 1/21/16, due in class 2/1/16)

This assignment is a review of the basics of convexity, topology, and related issues. Definitions and observation are interspersed with questions. Draw pictures to help your understanding.

1. A set $C \subset \mathbf{R}^n$ is *convex* if

$$(1-\lambda)x + \lambda y \in C, \quad \forall x, y \in C, \;\; 0 \leq \lambda \leq 1.$$

   Show that if C_1 and C_2 are convex, then so is $C_1 \cap C_2$. Given an example (e.g. by drawing a picture) to show that $C_1 \cup C_2$ is not convex in general, for convex C_1 and C_2.

2. Show that if C_i, $i = 1, \ldots, m$ are convex sets in $\mathbf{R}^{n_i}$ then $C = C_1 \times C_2 \times \ldots \times C_m$ is convex in $\mathbf{R}^{\sum n_i}$.

3. Show the interior and closure of a convex set are convex.

4. A point $x \in \mathbf{R}^n$ is a *convex combination* of the points $\{x^1, x^2, \ldots, x^r\}$ in $\mathbf{R}^n$ if for some real numbers $\lambda_1, \lambda_2, \ldots \lambda_r$ that satisfy $\sum \lambda_i = 1$ and $\lambda_i \geq 0$, we have $x = \sum_{i=1}^r \lambda_i x^i$. Show that a set S in $\mathbf{R}^n$ is convex if and only if every convex combination of a finite number of points of S is in S.

5. Consider the matrix $A \in \mathbf{R}^{m \times n}$ and the closed convex set $C \subset \mathbf{R}^n$. Show the set $AC := \{Ax \mid x \in C\} \subset \mathbf{R}^m$ is convex. Is it closed? What if C is also compact? (Note that this result can be used to show that $C_1 + C_2$ is convex when C_1 and C_2 are, and that γC is convex for all $\gamma \in \mathbf{R}$ and C convex.)

6. The inverse image of Y under A is $A^{-1}(Y) = \{x \,|\, Ax \in Y\}$. Is this set convex if Y is convex? What if we replace the linear map by an affine map, so that the set in question is $\{x \,|\, Ax + b \in Y\}$ for some b?

7. The convex hull of a set S is the intersection of all convex sets containing S, denoted co$S$. Show that co$S$ equals the set of all convex combinations of points in S. Note also that co$S$ is compact if S is compact.

8. A set C is a *cone* if for every $\lambda > 0$ and every $x \in C$, $\lambda x \in C$. Prove that a cone is convex if and only if it is closed under addition, that is, $x, y \in C$ implies that $x + y \in C$.

9. Some functions are not defined over $\mathbf{R}^n$ but only over a subset X. For example, $-\log(x)$ is defined only for $x > 0$. Furthermore, if we define a function to be the optimal value of a linear program, for example, for different values of the right hand side vector, that function may take on the value of $-\infty$ if the (minimization) problem is unbounded, or $+\infty$ if the problem is infeasible. We thus consider *extended real-valued functions* $f : \mathbf{R}^n \to [-\infty, +\infty]$, where by convention $f(x) = +\infty$ if $x \notin X$. For such f, the *epigraph*, epi(f), is defined by $\{(x, \mu) \mid x \in X, \mu \in \mathbf{R}, f(x) \leq \mu\}$, and f is convex if epi(f) is a convex set. Prove that this definition of convexity of f is consistent with the previous definition.

10. Prove that $\{x \,|\, f(x) \leq c\}$ is convex if f is convex. Show that the converse is not true, that is, this set may be convex for all c yet f may be nonconvex? (A picture is enough.) Such functions are called *quasi-convex* in Boyd and Vandenberghe.

11. Prove the following claim about the (Euclidean) projection operator P onto a closed convex set Ω:

$$\langle y - P(y), z - P(y) \rangle \leq 0 \text{ for all } z \in \Omega.$$

12. Consider the second-order cone in $\mathbf{R}^3$:

$$\Omega := \{(x, y, z)^T \,|\, z \geq \sqrt{x^2 + y^2}\}.$$

Show that $N_\Omega(0) = -\Omega$. In particular, show by construction that any for vector $(u, v, w)^T \notin -\Omega$, there is a vector $(x, y, z)^T \in \Omega$ such that $ux + vy + wz > 0$.

13. We did an elementary proof in class that the set of positive semidefinite symmetric matrices is a pointed cone. Prove this fact using the eigenvalue factorization of symmetric matrices. That is, we can factor any symmetric matrix A as QDQ^T, where Q is orthogonal and D is a diagonal matrix, whose (real) diagonal entries are the eigenvalues of A.