# CS / ISyE 730 - Spring 2016 - Homework 3

(assigned 2/7/16, due in class 2/17/16)

1. The following technical result came up as a final step in our proof that $\mathcal{F}(x^*) = T_\Omega(x^*)$ when LICQ holds at $x^*$. Prove rigorously that if we have two sequences of vectors $\{w^k\}$ and $\{v^k\}$ and a sequence of matrices $M^k$ such that
$$w^k = M^k v^k + o(1), \quad , M^k \to M,$$
where $M$ is a nonsingular matrix, and $v^k = o(\|w^k\|)$, then $w^k \to 0$. (Hint: show by contraction that there cannot be a subsequence $\mathcal{S}$ such that $\{w^k\}_{k\in\mathcal{S}}$ is bounded away from zero.)

2. The MFCQ conditions for the case in which both equality and inequality constraints are present are that:

   - The vectors $\{\nabla c_i(z) \, : \, i \in \mathcal{E}\}$ are linearly independent, where $\mathcal{E}$ is the set of equality constraints;
   - There is a vector $t$ such that $\nabla c_i(z)^T t = 0$ for all $i \in \mathcal{E}$ and $\nabla c_i(z)^T t > 0$ for all $i \in \mathcal{A} \cap \mathcal{I}$, where $\mathcal{A}$ is the set of active constraints and $\mathcal{I}$ is the set of inequality constraints.

   Show that the set of Lagrange multipliers that satisfy KKT conditions is bounded when these conditions hold. Recall that, among other conditions, the KKT conditions require existence of Lagrange multipliers $\lambda_i$ such that
$$\nabla f(z) - \sum_{i\in\mathcal{A}} \lambda_i \nabla c_i(z),$$
where $\lambda_i \geq 0$ for $i \in \mathcal{A} \cap \mathcal{I}$.

3. Consider the following problem (called a "Mathematical Program with Equilibrium Constraints" or MPEC):
$$\min \, f(x) \;\text{ s.t. }\; g(x) \geq 0, \; h(x) \geq 0, \; g(x)^T h(x) = 0,$$

where $g : \mathbf{R}^n \to \mathbf{R}^m$ and $h : \mathbf{R}^n \to \mathbf{R}^m$ are smooth functions. Show that LICQ is not satisfied at any feasible point for this problem.

4. Suppose that we have a nonlinear program with equality constraints:

$$\min f(x) \quad \text{s.t. } c_i(x) = 0,\ i = 1, 2, \dots, m.$$

Suppose that $x^*$ is a solution at which the LICQ condition is satisfied, so that the KKT conditions hold at this point, with optimal Lagrange multipliers $\lambda_i^*$, $i = 1, 2, \dots, m$. Now consider the equivalent reformulated problem in which each equality constraint is replaced by two inequalities:

$$\min f(x) \quad \text{s.t. } c_i(x) \geq 0,\ -c_i(x) \geq 0,\ i = 1, 2, \dots, m.$$

(a) Are the LICQ conditions satisfied at $x^*$, for the reformulated problem?

(b) Do the KKT conditions hold at $x^*$ for this reformulated problem? If so, are the Lagrange multipliers unique?

(c) Do the MFCQ conditions hold for the reformulated problem?