

CS240: Homework 8

Sahit Mandala

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Problem 1

Part a)

Let $R = \{((a, b), (c, d)) \in (\mathbb{Z}^+ \times \mathbb{Z}^+) \times (\mathbb{Z}^+ \times \mathbb{Z}^+) \mid ad = bc\}$. We want to show this relation is an equivalence relation. So let some $(a, b), (c, d), (e, f) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ be given. We will show the following 3 properties:

Symmetric: Suppose $(a, b)R(c, d)$. It follows then that $ad = bc$. But clearly, it follows that $cb = bc = ad$, so $(c, d)R(a, b)$. Hence, we have shown R is symmetric.

Reflexive: Clearly, $ab = ab$, so $(a, b)R(a, b)$.

Transitive: Suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Then we know that $ad = bc$ and $cf = de$. So $c = \frac{de}{f}$ (noting these are all strictly positive integer; furthermore, since $cf = de$, we know f divides de). Substitution reveals that $ad = b\frac{de}{f}$. Dividing both sides by d and reduction, we see that $a = b\frac{e}{f} \rightarrow af = be$. That is, $(a, b)R(e, f)$. Hence, we have shown transitivity.

Overall, we have shown that R is an equivalence relation.

Part b)

i) Suppose that $R_1 = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 3), (3, 1), (1, 1), (3, 3), (2, 2)\}$, which are over the set $\{1, 2, 3\}$. Notice that both are equivalence relations. As a proof for R_1 (R_2 is virtually identical), we see that $1R_11, 2R_12, 3R_13$, which are all distinct elements in the pairs, so R_1 is reflexive. Similarly, we see that $1R_12$ and $2R_11$, so R_1 is reflexive (the others are trivially reflexive). To show transitivity, we exhaustively consider all relevant pairs fitting the form aRb and bRc : $1R_11, 1R_11 \rightarrow 1R_11, 1R_12, 2R_11 \rightarrow 1R_11, 1R_12, 2R_11 \rightarrow 1R_11, 2R_11, 1R_12 \rightarrow 2R_12$. This shows transitivity for R_1 and thus that R_1 is an equivalence relation overall. A very similar proof can be used to show R_2 is an equivalence relation.

Now consider $R = R_1 \cup R_2 = \{(1, 2), (2, 1), (1, 1), (2, 2), (1, 3), (3, 1), (1, 1), (3, 3)\}$. We see that $2R_11$ and $1R_23$. However, we see that $(2, 3) \notin R$, which means $2 \not R 3$, so R does not satisfy transitivity. Hence, we have shown a counterexample to the proposition that $R_1 \cup R_2$ is also an equivalence relation.

ii) Consider some relations $R_1 \cap R_2$ over a set X with $R_1 \cap R_2$. Let some $a, b, c \in X$ be given. We will show the properties of an equivalence relation:

Symmetry: Because aR_1a and aR_2a since both are equivalence relations, $(a, a) \in R_1$ and $(a, a) \in R_2$. So $(a, a) \in R_1 \cap R_2$. Hence, we have shown that $R_1 \cap R_2$ is reflexive.

Reflexive: Suppose that $a(R_1 \cap R_2)b$. That is, $(a, b) \in R_1 \cap R_2$. So $(a, b) \in R_1$ and $(a, b) \in R_2$. So then $(b, a) \in R_1$ and $(b, a) \in R_2$. So $(b, a) \in R_1 \cap R_2$, which shows reflexivity.

Transitive: Suppose that $a(R_1 \cap R_2)b$ and $b(R_1 \cap R_2)c$. Then we know that $(a, b) \in R_1, (a, b) \in R_2, (b, c) \in R_1, (b, c) \in R_2$, by definition of relation and intersection. So, because R_1, R_2 satisfy transitivity as equivalence relations, then $(a, c) \in R_1, (a, c) \in R_2$. So $(a, c) \in R_1 \cap R_2$, which proves transitivity.

Hence, this overall shows that the intersection of 2 equivalence relations is also an equivalence relation.

Part c)

i)

Let R be as defined from the set X of function on $\mathbb{Z}^+ \rightarrow \mathbb{Z}^+$. Let some $f, g, h \in X$ be given. We will show R is an equivalence relation.

Symmetry: Notice that $\forall x, \frac{f(x)}{f(x)} = 1$, so clearly, $0 \leq \frac{f(x)}{f(x)} \leq 2$. So $f = \theta(f)$, showing reflexivity.

Reflexivity: Suppose that fRg . That is, $f = \theta(g)$. Then we can choose c, d, N s.t. $\forall n \geq N, c \leq \frac{f(n)}{g(n)} \leq d$ as given by the definition of big-Theta. Then, multiplying our through by $\frac{g(n)}{f(n)}$, we see that $c \frac{g(n)}{f(n)} \leq \frac{g(n)}{f(n)} * \frac{f(n)}{g(n)} = 1 \leq d \frac{g(n)}{f(n)}$. This implies that $c \frac{g(n)}{f(n)} \leq 1$ and $1 \leq d \frac{g(n)}{f(n)}$. So then $\frac{g(n)}{f(n)} \leq \frac{1}{c}$ and $\frac{1}{d} \leq \frac{g(n)}{f(n)}$. Hence, $\forall n \geq N, \frac{1}{d} \leq \frac{g(n)}{f(n)} \leq \frac{1}{c}$. So we have shown that $g = \theta(f)$. Whence, we have shown gRf and thus that R is reflexive.

Transitivity: Suppose that fRg and gRh . That is, $f = \theta(g)$ and $g = \theta(h)$. Then we can choose c, d, N_1 s.t. $\forall n \geq N_1, c \leq \frac{f(n)}{g(n)} \leq d$ and e, f, N_2 s.t. $\forall n \geq N_2, e \leq \frac{g(n)}{h(n)} \leq f$ as given by the definition of big-Theta. Now construct $N = \max(N_1, N_2)$. So for $\forall n \geq N$, both inequality relations are still satisfied. Furthermore, notice that $c \leq \frac{f(n)}{g(n)}$ and $e \leq \frac{g(n)}{h(n)}$, so $ce \leq \frac{f(n)}{g(n)} * \frac{g(n)}{h(n)} = \frac{f(n)}{h(n)}$. Similarly, since $\frac{f(n)}{g(n)} \leq d$ and $\frac{g(n)}{h(n)} \leq f$, so $\frac{f(n)}{g(n)} * \frac{g(n)}{h(n)} = \frac{f(n)}{h(n)} \leq d * f$. So $\forall n \geq N, ce \leq \frac{f(n)}{h(n)} \leq df$, which implies that $f = \theta(h)$. That is, fRh , which shows transitivity. Hence, overall, it shows that R is a equivalence relation.

ii)

This equivalence class is formed by all polynomials strictly of degree 2 (that is, a nonzero coefficient on their 2nd degree term). This is because, for any polynomial of the form $an^2 + bn + c$, the ratio $\lim_{n \rightarrow \infty} \frac{an^2 + bn + c}{n^2} = \lim_{n \rightarrow \infty} \frac{an^2}{n^2} + \frac{bn}{n^2} + \frac{c}{n^2} = \lim_{n \rightarrow \infty} a + \frac{b}{n} + \frac{c}{n^2} = a$, so $an^2 + bn + c = \theta(n^2)$.

Problem 2

Part a)

i)

Let A, R be as defined. We want to show that R is an order relation on A . So let some $a, b, c \in A$ be given.

Suppose that aRb and bRa . That is, $a \subseteq b$ and $b \subseteq a$. This implies that $a = b$ since both are mutually inclusive. Hence, we have shown that R is antisymmetric. Next, suppose that aRb and bRc . This implies that $a \subseteq b$ and $b \subseteq c$. But this clearly implies that $a \subseteq c$ (since all elements of a are in b and all elements of b are in c). So aRc ; hence, we have shown transitivity. Overall, we have shown that R forms an order relation on A .

ii)

R is not a total order relation on A . Consider $\{2, 4\}, \{3, 4\} \in A$. Notice that $\{2, 4\} \neq \{3, 4\} \in$ but $\{2, 4\} \not\subseteq \{3, 4\}$ and $\{2, 4\} \not\supseteq \{3, 4\}$.

iii)

R is not a strict order relation since we can choose an element in A , say $\{2, 4\} \in A$, as see that $\{2, 4\} \subseteq \{2, 4\}$, so $\{2, 4\} R \{2, 4\}$; this retributes the definition of antireflexive and thus the definition of strict order relation. (In fact, $X \subseteq X$ generally holds for any arbitrary sets, so R is in fact reflexive).

Part b)

i)

The maximal sets are $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2\}$, noting that there are no other sets $x \in A$ satisfying $\{1, 3, 4\} \subseteq x$, $\{2, 3, 4\} \subseteq x$, $\{1, 2\} \subseteq x$ (except themselves).

ii)

The minimal sets are just \emptyset , since the only element that is a subset of the empty set is itself.

iii)

There are no greatest elements since every set has a contradiction on the definition. For instance:

$\{2, 3, 4\} \not\subseteq \emptyset$, $1, 2, 4, 1, 2, 1, 4, 2, 4, 3, 4, 1, 3, 4$ and $\{1, 3, 4\} \not\subseteq \{2, 3, 4\}$

vi)

The empty set \emptyset is a least element since, for any set $x \in A$ (or in general), $\emptyset \subseteq x$, satisfying the definition of minimal.

Problem 3

Part a)

Consider the relation on strict equality “ $=$ ”. That is, $(a, b) \in R$ if $a = b$. Then $R = \{(a, a), (b, b), (c, c), (d, d), (e, e)\}$. Trivially, the relation satisfies reflexivity and symmetry (every pair is its own symmetric counter part). Transitivity is always preserved since $xRx, xRx \rightarrow xRx$ trivially (for all $x \in A$). So this relation is an equivalence relation. To show it is an order relation, we need to show that it is antisymmetric (we already showed transitivity). But, for any $a, b \in A$, we know that when $a \neq b$, it follows that $aRb \wedge bRa$ is always false, since none of our pairs contain 2 distinct elements so $(a, b) \notin R$. That is, $a \neq b \rightarrow \neg(aRb \wedge bRa)$. By the contrapositive, this shows that $aRb \wedge bRa \rightarrow a = b$, which shows antisymmetry. So R is a order relation as well.

The equivalence classes on R are just the singleton sets of each element, so $[a] = \{a\}$, $[b] = \{b\}$, $[c] = \{c\}$, $[d] = \{d\}$, $[e] = \{e\}$.

Part b)

We know that our relation contains (a, b) , (a, c) , and (d, e) . This implies that (a, a) , (b, b) , (c, c) , (d, d) , (e, e) are also in our relation by reflexivity. By symmetry, we also know that (b, a) , (c, a) , (e, d) are also in our relation (their reflexes are already in the relation). We also need to provide transitive closure, which means we need to include (ignoring those we already have): $(b, a), (a, c) \rightarrow (b, c)$ and $(c, a), (a, b) \rightarrow (c, b)$. We note that these are symmetric counterparts.

Putting these together, our relation $R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (d, e), (b, a), (c, a), (e, d), (b, c), (c, b)\}$ contains the minimum number of necessary elements by construction.

Part c)

i) Consider the subsets $\{a\}, \{a, b\}$. Notice that $|\{a\}| = 1 \leq 2 = |\{a, b\}|$, so $\{a\} R \{a, b\}$. However, $|\{a, b\}| = 2 \not\leq 1 = |\{a\}|$, so $(\{b, a\}, \{a\}) \notin R$. This violates symmetry, so R is not an equivalence relation.

ii)

R is not an order relation. Consider the subsets $\{a, c\}, \{a, b\}$. Notice that $|\{a, c\}| = 2 = |\{a, b\}|$, so $|\{a, c\}| \leq |\{a, b\}|$ and $|\{a, c\}| \geq |\{a, b\}|$. So $\{a, c\}R\{a, b\}$ and $\{a, b\}R\{a, c\} \in R$. However, we see that $\{a, c\} \neq \{a, b\}$, which violates the antisymmetric property. So R is not an order relation either.

Problem 4

Part a)

Not a bijection. Notice that $f(-1) = -3(-1)^2 + 7 = -3 + 7 = 4$ and $f(1) = -3(1)^2 + 7 = -3 + 7 = 4$. That is, $f(1) = f(-1)$ contradicting the injective property of bijections.

Part b)

Yes. We utilize the fact that $g(x) = \sqrt[5]{x}$ is a well defined function defined on all of \mathbb{R} (which technically follows from the fact that x^5 is invertible on all \mathbb{R}). Suppose that for some x, y that $x^5 + 1 = y^5 + 1$. So then $x^5 = y^5$, and by applying the 5th root, we see that $\sqrt[5]{x^5} = x = y = \sqrt[5]{y^5}$, which implies injectivity. Furthermore, for arbitrary $x \in \mathbb{R}$, we can construct $\sqrt[5]{x-1}$ and see that $f(\sqrt[5]{x-1}) = (\sqrt[5]{x-1})^5 + 1 = x - 1 + 1 = x$, so the function is also surjectively. Hence, it is bijective.

Part c)

Not a bijection. Consider $1 \in \mathbb{R}$. Notice that if $f(x) = \frac{x+1}{x+2} = 1$ for some x , then $x+1 = 1(x+2)$ which then implies that $x+1-x = 1 = x+2-x = 2$, so $1 = 2$. Hence, there is no $x \in \mathbb{R}$ satisfying $f(x) = \frac{x+1}{x+2} = 1$ contradicting the surjective property of bijections.