CS240: Homework

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Problem 1

a)

$A = \{2, 4, 6, 8\}$
We can construct the power set by considering every possible subset of A.
$\wp(A) = \{\{\}, \{2\}, \{4\}, \{6\}, \{8\}, \{2, 4\}, \{2, 6\}, \{2, 8\}, \{4, 6\}, \{4, 8\}, \{6, 8\}$
$\{2, 4, 6\}, \{2, 4, 8\}, \{2, 6, 8\}, \{4, 6, 8\}, \{2, 4, 6, 8\}\}$

b)

Recall that $|\wp(A)| = 2^{|A|} = 2^4 = 16$.
Simply counting, we see that this set has 16 unique elements, as expected.

Problem 2

It is easy to see that S is not finite. Hence, we shall construct a pairing (i.e. a bijection) between the natural numbers and S. Consider the function $f : \mathbb{N} \to S$ defined by $f(n) = 2^n$ (noting that the image of this function is all powers of 2, i.e. S). I claim that this function is a bijection.

First, we show that the function is injective. Suppose $f(i) = f(j)$ for some $i, j \in \mathbb{N}$. Then $2^i = 2^j$ by our definition of $f(x)$. But this implies that $i = j$, since the bases ($\neq 0, 1, -1$) are the same which implies the exponents are equal as well. This shows that $f(x)$ is injective.

To show surjectivity, let some value $y \in S$ be given. By the definition of S, we can choose $i \in \mathbb{N}$ s.t. $y = 2^i$ (since S is the set of all nonnegative powers of 2). So clearly there exists an element $x \in \mathbb{N}$ s.t. $f(x) = y$ (namely $x = i$). Hence, we have shown surjectivity.

So we have shown that f is a bijection from $\mathbb{N}$ to S, from which it follows that S is countable.

Problem 3

First we prove a quick lemma that, for any finite sets A and B, if $B \subseteq A$, then $|A - B| = |A| - |B|$. If an element $x \in A$ and $x \notin B$, then $x \in A - B$. However, if an element $x \in A$ and $x \in B$, then $x \notin A - B$. So the number of elements in $|A - B|$ are the number of elements in $|A|$ minus the elements in both A and B, or specifically, $A \cap B$. But because $B \subseteq A$, we see that $B = A \cap B$ (since all elements in B are also in A). So $|A - B| = |A| - |A \cap B| = |A| - |B|$.

Let A and B be arbitrary finite sets. Consider $A \cup B$.

Suppose that $A \cap B = \emptyset$. If S and T are finite disjoint sets, then $|S \cup T| = |S| + |T|$ by the provided lemma.

Now suppose that $A \cap B \neq \emptyset$. Consider the set $B - (A \cap B)$. Notice that $(B - (A \cap B)) \cap A = \emptyset$ (if $(B - (A \cap B)) \cap A \neq \emptyset$, then there exists some element $x \in (B - (A \cap B)) \cap A$, such that $x \in (B - (A \cap B)) \subseteq B$ and $x \in A$; but then $x \in A \cap B$, which contradicts the fact that $x \in (B - (A \cap B))$). So then, by the provided lemma, $|A \cup (B - (A \cap B))| = |A| + |(B - (A \cap B))|$.

Notice that $A \cup (B - (A \cap B)) \subseteq A \cup B$, given that $\forall x \in A \cup (B - (A \cap B))$, either $x \in A$ or $x \in B - (A \cap B) \subseteq B$; so $x \in A \cup B$. Let some element $x \in A \cup B$ be given. Then either $x \in A$ or $x \in B$. If $x \in A$, then clearly $x \in A \cup (B - (A \cap B))$. If $x \notin A$ but $x \in B$, then $x \in B - (A \cap B)$ since x cannot be in $A \cap B \subseteq A$. The point is, $A \cup (B - (A \cap B)) = A \cup B$

Furthermore, $(A \cap B) \subseteq B$, so we can apply our lemma above to see that: $|B - (A \cap B)| = |B| - |A \cap B|$. We can put this all together into our equation from eariler, $|A \cup (B - (A \cap B))| = |A| + |(B - (A \cap B))|$, which gives us:

$|A \cup (B - (A \cap B))| = |A \cup B| = |A| + |B - (A \cap B)| = |A| + |B| - |A \cap B|$. Hence, we have shown that $|A \cup B| = |A| + |B| - |A \cap B|$ when $A \cap B \neq \emptyset$.

Overall, we have shown $|A \cup B| = |A| + |B| - |A \cap B|$ for any arbitrary finite sets.

Problem 4

We first note that for any arbitrary set A, $A - \{\} = A$, because $A - \{\} \subseteq A$ by definition of compliment (that is, $A - \{\} = \{x \in A | x \notin \{\}\}$) and $\forall x \in A, x \notin \{\}$, so by the definition of compliment, $x \in A - \{\}$, implying that $A \subseteq A - \{\}$. Hence, $A - \{\} = A$.

Counterexample. Consider the sets $A = \{1, 2\}, B = \{2, 3\}, C = \{\}$. Then $(A \cup B) - C = (A \cup B) = \{1, 2, 3\}$, using our lemma above. Furthermore, $A - C = A - \{\} = A$ and $B - C = A$ using the same property. So $(A - C) \cap (B - C) = A \cap B = \{1, 2\} \cap \{2, 3\} = \{2\}$. But $(A \cup B) - C = \{2\} \neq \{1, 2, 3\} = (A - C) \cap (B - C) = A \cap B$. Hence, the proposition is false.

Problem 5

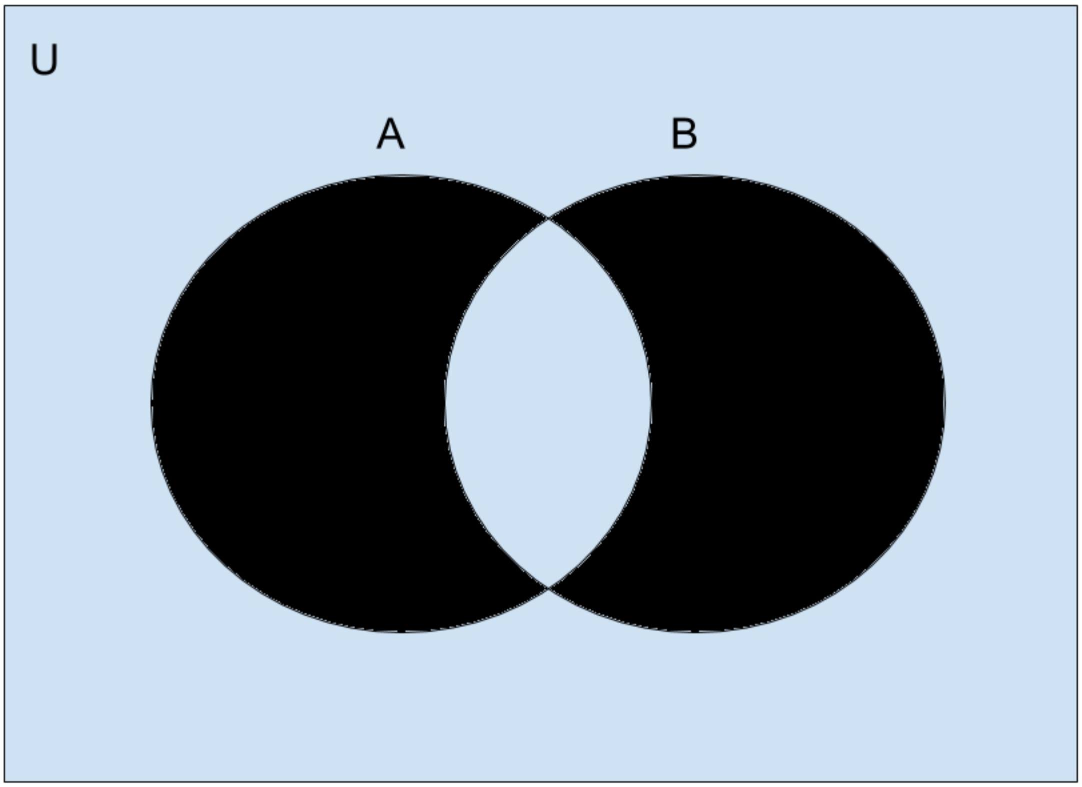
a)

First, consider the elements in {1,3,4,5,7} not in {1,2,3,6,7}: $1, 3, 7 \in 1, 2, 3, 6, 7, 4, 5 \notin 1, 2, 3, 6, 7$

Now we consider elements in {1,2,3,6,7} not in {1,3,4,5,7}: $1, 3, 7 \in 1, 3, 4, 5, 7, 2, 6 \notin 1, 3, 4, 5, 7$

So the symmetric difference must be: $\{2, 4, 5, 6\}$

b)



c)

Let A,B be arbitrary sets. We shall show that $A \oplus B = (B - A) \cup (A - B)$ by mutual inclusion.

First, let some element $x \in A \oplus B$ be given. Then either $x \in A, x \notin B$ or $x \in B, x \notin A$ (note both propositions cannot be simultaneously true, since x can't be and not be in A,B at the same time).

Case 1: Suppose that $x \in A, x \notin B$. Then $x \in A - B$ by definition of compliment. So then $x \in (B - A) \cup (A - B)$, by definition of union (since $(A - B) \subseteq (B - A) \cup (A - B)$)

Case 2: Suppose that $x \in B, x \notin A$. Then $x \in B - A$ by definition of compliment. So then $x \in (B - A) \cup (A - B)$, by definition of union (since $(B - A) \subseteq (B - A) \cup (A - B)$)

So, in either case, $x \in (B - A) \cup (A - B)$, for any $x \in A \oplus B$. So $A \oplus B \subseteq (B - A) \cup (A - B)$

Now let some element $x \in (B - A) \cup (A - B)$ be given. Then, by definition, either $x \in (B - A)$ or $x \in (A - B)$.

Case 1: Suppose that $x \in (B - A)$. Then $x \in B, x \notin A$ by definition of compliment. But this implies that $x \in A \oplus B$, since x is in either A or B, but not in both A and B.

Case 2: Suppose that $x \in (A - B)$. Then $x \in A, x \notin B$ by definition of compliment. But this implies that $x \in A \oplus B$, since x is in either A or B, but not in both A and B.

So $x \in A \oplus B$, for all $x \in (B - A) \cup (A - B)$. So $A \oplus B \subseteq B - A) \cup (A - B)$.

Overall, this implies that $A \oplus B = (B - A) \cup (A - B)$