CS240: Homework 8

Sahit Mandala

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Problem 1

Part a)

Let $R = \{((a,b),(c,d)) \in (\mathbb{Z}^+ \times \mathbb{Z}^+) \times (\mathbb{Z}^+ \times \mathbb{Z}^+) | ad = bc\}$. We want to show this relation is an equivalence relation. So let some $(a,b),(c,d),(e,f) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ be given. We will show the following 3 properties:

Symmetric: Suppose $(a,b)R(c,d)$. It follows then that $ad = bc$. But clearly, it follows that $cb = bc = ad$, so $(c,d)R(a,b)$. Hence, we have shown R is symmetric.

Reflexive: Clearly, $ab = ab$, so $(a,b)R(a,b)$.

Transitive: Suppose $(a,b)R(c,d)$ and $(c,d)R(e,f)$. Then we know that $ad = bc$ and $cf = de$. So $c = \frac{de}{f}$ (noting these are all strictly positive integer; furthermore, since $cf = de$, we know f divides de). Substitution reveals that $ad = b\frac{de}{f}$. Dividing both sides by d and reduction, we see that $a = b\frac{e}{f} \to af = be$. That is, $(a,b)R(e,f)$. Hence, we have shown transitivity.
Overall, we have shown that R is an equivalence relation.

Part b)

i) Suppose that $R_1 = \{(1,2),(2,1),(1,1),(2,2),(3,3)\}$ and $R_1 = \{(1,3),(3,1),(1,1),(3,3),(2,2)\}$, which are over the set $\{1,2,3\}$. Notice that both are equivalence relations. As a proof for R_1 (R_2 is virtually identical), we see that $1R_11$,$2R_12$, $3R_13$, which are all distinct elements in the pairs, so $R_1$is reflexive. Similarly, we see that $1R_12$ and $2R_11$, so R_1 is reflexive (the others are trivially reflexive). To show transitivity, we exhaustively consider all relevant pairs fitting the form aRb and bRc: $1R_11, 1R_11 \to 1R_11$,$1R_12, 2R_11 \to 1R_11$,$1R_12, 2R_11 \to 1R_11$,$2R_11, 1R_12 \to 2R_12$. This shows transitivity for $R_1$and thus that R_1 is an equivalence relation overall. A very similar proof can be used to show R_2 is an equivalence relation.

Now consider $R = R_1 \cup R_2 = \{(1,2),(2,1),(1,1),(2,2),(1,3),(3,1),(1,1),(3,3)\}$. We see that $2R1$ and $1R3$. However, we see that $(2,3) \notin R$, which means $2 \not R 3$, so R does not satisify transitivity. Hence, we have shown a counterexample to the proposition that $R_1 \cup R_2$ is also a equivalence relation.

ii) Consider some relations $R_1 \cap R_1$ over a set X with $R_1 \cap R_1$. Let some $a,b,c \in X$ be given. We will show the properties of an equivalence relation:

Symmetry: Because aR_1a and aR_2a since both are equivalence relations, $(a,a) \in R_1$ and $(a,a) \in R_2$. So $(a,a) \in R_1 \cap R_2$. Hence, we have shown that $R_1 \cap R_2$ is reflexive.

Reflexive: Suppose that $a(R_1 \cap R_2)b$. That is, $(a,b) \in R_1 \cap R_2$. So $(a,b) \in R_1$ and $(a,b) \in R_2$. So then $(b,a) \in R_1$ and $(b,a) \in R_2$. So $(b,a) \in R_1 \cap R_2$, which shows reflexitivity.

Transitive: Suppose that $a(R_1 \cap R_2)b$ and $b(R_1 \cap R_2)c$. Then we know that $(a, b) \in R_1$,$(a, b) \in R_2$,$(b, c) \in R_1$,$(b, c) \in R_2$, by definition of relation and intersection. So, because R_1, R_2 satisfy transitivity as equivalence relations, then $(a, c) \in R_1, (a, c) \in R_2$. So $(a, c) \in R_1 \cap R_2$, which proves transitivity.

Hence, this overall shows that the intersection of 2 equivalence relations is also an equivalence relation.

Part c)

i)

Let R be as defined from the set X of function on $\mathbb{Z}^+ \to \mathbb{Z}^+$. Let some $f, g, h \in X$ be given. We will show R is an equivalence relation.

Symmetry: Notice that $\forall x, \frac{f(x)}{f(x)} = 1$, so clearly, $0 \leq \frac{f(x)}{f(x)} \leq 2$. So $f = \theta(f)$, showing reflexivity.

Reflexivity: Suppose that fRg. That is, $f = \theta(g)$. Then we can choose c, d, N s.t. $\forall n \geq N$,$c \leq \frac{f(n)}{g(n)} \leq d$ as given by the definition of big-Theta. Then, multiplying our through by $\frac{g(n)}{f(n)}$, we see that $c\frac{g(n)}{f(n)} \leq \frac{g(n)}{f(n)} * \frac{f(n)}{g(n)} = 1 \leq d\frac{g(n)}{f(n)}$. This implies that $c\frac{g(n)}{f(n)} \leq 1$ and $1 \leq d\frac{g(n)}{f(n)}$. So then $\frac{g(n)}{f(n)} \leq \frac{1}{c}$ and $\frac{1}{d} \leq \frac{g(n)}{f(n)}$. Hence, $\forall n \geq N, \frac{1}{d} \leq \frac{g(n)}{f(n)} \leq \frac{1}{c}$. So we have shown that $g = \theta(f)$. Whence, we have shown gRf and thus that R is reflexive.

Transitivity: Suppose that fRg and gRh. That is, $f = \theta(g)$ and $g = \theta(h)$. Then we can choose c, d, N_1 s.t. $\forall n \geq N_1$,$c \leq \frac{f(n)}{g(n)} \leq d$ and e, f, N_2 s.t. $\forall n \geq N_2, e \leq \frac{g(n)}{h(n)} \leq f$ as given by the definition of big-Theta. Now construct $N = max(N_1, N_2)$. So for $\forall n \geq N$, both inequality relations are still satified. Furthermore, notice that $c \leq \frac{f(n)}{g(n)}$ and $e \leq \frac{g(n)}{h(n)}$, so $ce \leq \frac{f(n)}{g(n)} * \frac{g(n)}{h(n)} = \frac{f(n)}{h(n)}$. Similarly, since $\frac{f(n)}{g(n)} \leq d$ and $\frac{g(n)}{h(n)} \leq f$, so $\frac{f(n)}{g(n)} * \frac{g(n)}{h(n)} = \frac{f(n)}{h(n)} \leq d * f$. So $\forall n \geq N$, $ce \leq \frac{f(n)}{h(n)} \leq df$, which implies that $f = \theta(h)$. That is, fRh, which shows transitivity. Hence, overall, it shows that R is a equivalence relation.

ii)

This equivalence class is formed by all polynomials strictly of degree 2 (that is, a nonzero coefficient on their 2nd degree term). This is because, for any polynomial of the form $an^2 + bn + c$, the ratio$lim_{n\to\infty} \frac{an^2+bn+c}{n^2} = lim_{n\to\infty} \frac{an^2}{n^2} + \frac{bn}{n^2} + \frac{c}{n^2} = lim_{n\to\infty} a + \frac{b}{n} + \frac{c}{n^2} = a$, so $an^2 + bn + c = \theta(n^2)$.

Problem 2

Part a)

i)

Let A,R be as defined. We want to show that R is an order relation on A. So let some $a, b, c \in A$ be given.

Suppose that aRb and bRa. That is, $a \subseteq b$ and $b \subseteq a$. This implies that $a = b$ since both are mutually inclusive. Hence, we have shown that R is antisymmetric. Next, suppose that aRb and bRc. This implies that $a \subseteq b$ and $b \subseteq c$. But this clearly implies that $a \subseteq c$ (since all elements of a are in b and all elements of b are in c). So aRc; hence, we have shown transitivity. Overall, we have shown that R forms an order relation on A.

ii)

R is not a total order relation on A. Consider $\{2, 4\}, \{3, 4\} \in A$. Notice that $\{2, 4\}, \{3, 4\}$ but $\{2, 4\} \not\subseteq \{3, 4\}$ and $\{2, 4\} \not\supseteq \{3, 4\}$.

iii)

R is not a strict order relation since we can choose an element in A, say $\{2,4\} \in A$, as see that $\{2,4\} \subseteq \{2,4\}$, so $\{2,4\}R\{2,4\}$; this retributes the defintion of antireflexive and thus the definition of struct order relation. (In fact, $X \subseteq X$ generally holds for any arbitrary sets, so R is in fact reflexive).

Part b)

i)

The maximal sets are $\{1,3,4\}, \{2,3,4\}, \{1,2\}$, noting that there are no other sets $x \in A$ satisfying $\{1,3,4\} \subseteq x, \{2,3,4\} \subseteq x, \{1,2\} \subseteq x$ (except themselves).

ii)

The minimal sets are just $\emptyset$, since the only element that is a subset of the empty set is itself.

iii)

There are no greatest elements since every set has a contradiction on the definition. For instance:

$\{2,3,4\} \not\subseteq \emptyset, 1, 2, 4, 1, 2, 1, 4, 2, 4, 3, 4, 1, 3, 4$ and $\{1,3,4\} \not\subseteq \{2,3,4\}\}$

vi)

The empty set $\emptyset$ is a least element since, for any set $x \in A$ (or in general), $\emptyset \subseteq x$, satisfying the definition of minimal.

Problem 3

Part a)

Consider the relation on strict equality "$=$". That is, $(a,b) \in R$ if $a = b$. Then $R = \{(a,a),(b,b),(c,c),(d,d),(e,e)\}$. Trivially, the relation satisfies reflexivity and symmetry (every pair is its own symmetric counter part). Transitivity is always preserved since $xRx, xRx \to xRx$ trivially (for all $x \in A$). So this relation is an equivalence relation. To show it is an order relation, we need to show that it is antisymmetric (we already showed transitivity). But, for any $a, b \in A$, we know that when $a \neq b$, it follows that $aRb \wedge bRa$ is always false, since none of our pairs contain 2 distinct elements so $(a,b) \notin R$. That is, $a \neq b \to \neg(aRb \wedge bRa)$. By the contrapositive, this shows that $aRb \wedge bRa \to a = b$, which shows antisymmetry. So R is a order relation as well.

The equivalence classes on R are just the singleton sets of each element, so $[a] = \{a\}, [b] = \{b\}, [c] = \{c\}, [d] = \{d\}, [e] = \{e\}$.

Part b)

We know that our relation contains (a, b), (a, c), and (d, e). This implies that $(a,a),(b,b)(c,c)(d,d)(e,e)$ are also in our relation by reflexivity. By symmetry, we also know that $(b,a),(c,a),(e,d)$ are also in our relation (their reflexes are already in the relation). We also need to provide transitive closure, which means we need to include (ignoring those we already have): $(b,a),(a,c) \to (b,c)$ and $(c,a)(a,b) \to (c,b)$. We note that these are symmetric counterparts.

Putting these together, our relation R $= \{(a,a),(b,b)(c,c)(d,d)(e,e),(a,b),(a,c),(d,e),(b,a),(c,a),(e,d),(b,c),(c,b)\}$ contains the minimum number of neccesary elements by construction.

Part c)

i) Consider the subsets$\{a\}, \{a,b\}$. Notice that $|\{a\}| = 1 \leq 2 = |\{a,b\}|$, so$\{a\}R\{a,b\}$. However, $|\{a,b\}| = 2 \not\leq 1 = |\{a\}|$, so $(\{b,a\},\{a\}) \notin R$. This violates symmetry, so R is not an equivalence relation.

ii)

R is not an order relation. Consider the subsets $\{a,c\},\{a,b\}$. Notice that $|\{a,c\}| = 2 = |\{a,b\}|$, so $|\{a,c\}| \leq |\{a,b\}|$ and $|\{a,c\}| \geq |\{a,b\}|$. So $\{a,c\}R\{a,b\}$ and $\{a,b\}R\{a,c\} \in R$. However, we see that $\{a,c\} \neq \{a,b\}$, which violates the antisymmetric property. So R is not an order relation either.

Problem 4

Part a)

Not a bijection. Notice that $f(-1) = -3(-1)^2 + 7 = -3 + 7 = 4$ and $f(1) = -3(1)^2 + 7 = -3 + 7 = 4$. That is, $f(1) = f(-1)$ contradicting the injective property of bijections.

Part b)

Yes. We utilize the fact that $g(x) = \sqrt[5]{x}$ is a well defined function defined on all of $\mathbb{R}$ (which technically follows from the fact that x^5 is invertable on all $\mathbb{R}$). Suppose that for some x, y that $x^5 + 1 = y^5 + 1$. So then $x^5 = y^5$, and by applyting the 5th root, we see that $\sqrt[5]{x^5} = x = y = \sqrt[5]{y^5}$, which implies injectivity. Furthermore, for arbitrary $x \in \mathbb{R}$, we can construct $\sqrt[5]{x-1}$ and see that $f(\sqrt[5]{x-1}) = (\sqrt[5]{x-1})^5 + 1 = x - 1 + 1 = x$, so the function is also surjectively. Hence, it is bijective.

Part c)

Not a bijection. Consider $1 \in \mathbb{R}$. Notice that if $f(x) = \frac{x+1}{x+2} = 1$ for some x, then $x + 1 = 1(x+2)$ which then implies that $x + 1 - x = 1 = x + 2 - x = 2$, so $1 = 2$. Hence, there is no $x \in \mathbb{R}$ satisfying $f(x) = \frac{x+1}{x+2} = 1$contradicting the surjective property of bijections.