CS240: Homework

Sahit Mandala

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Problem 1

Let some real numbers $a, b \in \mathbb{R}$ be given. First, we will prove (i) iff (ii).

So first suppose $a < b$. Then we can add a to both sides, preserving the inequality, and see that $a + a < a + b$, or equivalently, $2 * a < a + b$. We can multiply both sides by $\frac{1}{2}$ and see that: $\frac{1}{2} * 2 * a < \frac{1}{2}(a+b) \rightarrow a < (a+b)/2$. But $(a+b)/2$ is merely the definition of the average of a and b. Whence, we have show that the average of a and b is in fact greater than a.

Now suppose that the average of a and b is greater than a. That is, $a < (a+b)/2$. We can multiple both sides by 2 and subtract a, preserving the inequality, and see that: $(2*a) - a < 2*((a+b)/2) - a \rightarrow a < a + b - a = b$. So $a < b$, as expected.

Overall, we have shown that $a < b \iff a < (a+b)/2$.

Now we shall prove (ii) iff (iii). Suppose again that the average of a and b is greater than a. That is, $a < (a+b)/2$. We can multiple both sides by 2 and subtract a, preserving the inequality, and see that: $(2*a) - a < 2*((a+b)/2) - a \rightarrow a < a + b - a = b$. So $a < b$. Next, we can add b to both sides: $a + b < 2 * b$. Finally, we can divide both sides by 2: $\frac{1}{2}(a+b) < 2b/2 = b$. So the average of a and b is less than b itself, as expected.

Next, assume that the average of a and b is less than b itself. That is, $\frac{1}{2}(a+b) < b$. We can multiply both sides by 2 and subtract b, preserving the inequality, and see that: $2 * \frac{1}{2}(a+b) - b < 2 * b - b \rightarrow a + b - b = a < b$. By our proof above (namely that (i) iff (ii)), we know that if $a < b$, then $a < (a+b)/2$. Hence, we have shown that if $\frac{1}{2}(a+b) < b$, then $a < (a+b)/2$. So overall, (ii) iff (iii)

Because (i) iff (ii) and (ii) iff (iii), we see that (i),(ii),(iii) are all logically equivalent.

Problem 2

Let some $n \in \mathbb{N}$ be given. Then either $n = 3k$, $n = 3k + 1$, or $n = 3k + 2$ for some $k \in \mathbb{N}$.

Case 1: $n = 3k$

Then $n^3 - n = (3k)^3 - 3k = 27k^3 - 3 = 3(9k^3 - 3)$. Clearly, this shows that $n^3 - n$ is divisible by 3 in this case.

Case 2: $n = 3k + 1$

Then $n^3 - n = (3k+1)^3 - (3k+1) = (27k^3 + 27k^2 + 9k + 1) - (3k+1) = (27k^3 + 27k^2 + 6k) = 3(9k^3 + 9k^2 + 2k)$. Clearly, this shows that $n^3 - n$ is divisible by 3 in this case.

Case 3: $n = 3k + 2$

$n^3 - n = (3k+2)^3 - (3k+2) = (27k^3 + 54k^2 + 36k + 8) - (3k+2) = (27k^3 + 54k^2 + 33k + 6) = 3(9k^3 + 18k^2 + 11k + 2)$. Clearly, this shows that $n^3 - n$ is divisible by 3 in this case.

Because every natural number is either $0, 1, 2$ mod 3, we have exhausted all cases on n, so we have shown that $3 | n^3 - n, \forall n \in \mathbb{N}$.

Problem 3

To prove $3|4^n - 1, \forall n \in \mathbb{N}$, we shall use induction. First, consider $n = 0$. Clearly, $4^0 - 1 = 1 - 1 = 0$ is divisible by 3.

Next, we shall assume the inductive hypothesis on k that $3|4^k - 1$.

We want to now show that our proposition is true on k+1 (that is, $3|(4^{k+1} - 1)$. $4^{k+1} - 1 = 4^{k+1} - 4 + 3 = 4(4^k - 1) + 3$. By our inductive hypothesis, $4^k - 1 = 3m$ for some $m \in \mathbb{N}$. So $4(4^k - 1) + 3 = 4(3m) + 3 = 3(4m) + 3 = 3(4m + 1)$, which is clearly divisible by 3. Hence, we have shown that $4^{k+1} - 1$ is divisible by 3, from which it follows that $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

Problem 4

We shall prove that $\forall n \in \mathbb{N}^+, \sum_{i=1}^{n}(2n - 1) = n^2$

Base case: $n = 1$, $\sum_{i=1}^{1}(2n - 1) = 2 - 1 = 1 = n^2$

Inductive step: Assume $n = k$, so $\sum_{i=1}^{k}(2n - 1) = k^2$

We want to show the propostion for k+1: $\sum_{i=1}^{k+1}(2n - 1) = \sum_{i=1}^{k}(2n - 1) + 2(k + 1) - 1 = \sum_{i=1}^{k}(2n - 1) + 2k + 1$. By the inductive assumption, $\sum_{i=1}^{k}(2n - 1) + 2k + 1 = k^2 + 2k + 1 = (k + 1)^2$. So $\sum_{i=1}^{k+1}(2n - 1) = (k + 1)^2$.

Overall, we have shown that $\forall n \in \mathbb{N}^+, \sum_{i=1}^{n}(2n - 1) = n^2$

Problem 5

The flaw is that the inductive step's statement is incorrect . Let some $a \in \mathbb{R}^+$ be given. Setting $n = 0$, we see that $a^n = a^0 = 1$. Then the inductive hypothesis (in this flawed proof) states that $a^k = 1, \forall k \leq n$, from which the proof follows that $a^{k+1} = a^k * a^k / a^{k-1} = 1 * 1/1 = 1$, claiming $a^{k-1} = a^k = 1$ by the inductive hypothesis. But in fact, when $k = 0$, $a^{k-1} = a^{0-1} = a^{-1} \neq 1$ for any arbirary a. The proof, upon showing the base case at $n = 0$, uses the incorrect strong inductive hypothesis $a^n = 1, \forall k \leq n$; in fact, k must be strictly greater than or equal to 0, so $a^n = 1, \forall k$ s.t. $0 \leq k \leq n$ is the correct strong inductive hypothesis. From this, the assumption that $a^{k-1} = 1$ fails when $k = 0$ (e.g. when inductively proving P(1) from P(0)), from which all subsequent cases $(P(n), n > 1)$ fail.