Advance Problems

1. Consider the singlet state,

   $|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2]$, where $|\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are two orthogonal states.

   (a) Show that the quantum mechanical expectation value, $E(n,m) = \langle\psi|\vec{\sigma}.\vec{n} \otimes \vec{\sigma}.\vec{m}|\psi\rangle$, is given by $E(n,m) = -cos\phi_{n,m}$, where, $\phi_{n,m}$ is the angle between unit vectors $\hat{m}$ and $\hat{n}$. [2]

   (b) The CHSH-inequality is given by,
   $|E(n,m) - E(n,m')| + |E(n',m') + E(n',m)| \leq 2$.
   Find the angles where the inequality is maximally violated. Interpret the result. [2]

2. (a) For a two qubit state,

   $|\Phi\rangle = \frac{1}{2}|\uparrow\rangle_A \left(\frac{1}{2}|\uparrow\rangle_B + \frac{\sqrt{3}}{2}|\downarrow\rangle_B\right) + \frac{1}{2}|\downarrow\rangle_A \left(\frac{\sqrt{3}}{2}|\uparrow\rangle_B + \frac{1}{2}|\downarrow\rangle_B\right)$.
   Perform the the Schmidt decomposition of $|\Phi\rangle$. [5]

   (b) Is there a Schmidt decomposition for an arbitrary *tripartite* pure state?
   That is, if $|\psi\rangle_{ABC}$ is an arbitrary vector in $H_A \otimes H_B \otimes H_C$, can we find orthonormal bases $\{|i_A\rangle\}, \{|i_B\rangle\}, \{|i_C\rangle\}$, such that $|\psi\rangle_{ABC} = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle \otimes |i_C\rangle$?
   Explain your answer. [3]

3. The GHZ state is given by,
   $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.
   Suppose that Alice takes the first qubit and Bob takes qubits 2 and 3. Show that this scenario can be used for dense coding and give the dense coding protocol explicitly. [6]

4. (a) Using the Peres Horodecki partial transpose criterion, examine if the state,
   $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$,
   is entangled or not. [3]

(b) How can we say that the set of separable density matrices form a convex set. Using this property comment on the validity of Peres Horodecki partial transpose criterion to determine entanglement. [2]

5. Answer any one of the two sub-parts in this equation.

(a) Consider a bipartite continuous variable state formed from ground and excited states of the harmonic oscillators:

$$\psi(x_a, x_b) = \sqrt{\frac{2}{\pi}}(\alpha x_a + \beta x_b)e^{-(x_a^2+x_b^2)/2}\,, \qquad |\alpha|^2 + |\beta|^2 = 1 \quad (1)$$

Examine the entanglement property of this state.

(b) Consider the superposition state,
$|\phi_0\rangle = c_0(|\alpha\rangle + |-\alpha\rangle)$ and $|\phi_1\rangle = c_1(|\beta\rangle + |-\beta\rangle)$,
where, $c_0, c_1 \epsilon\ \mathcal{C}$ and $|\alpha\rangle, |-\alpha\rangle), |\beta\rangle, |-\beta\rangle$, are coherent states. Now, consider the state,

$|\phi = |0\rangle \otimes |\phi_0\rangle + |1\rangle \otimes |\phi_1\rangle$, where $\{|0\rangle, |1\rangle\}$ form an orthonormal basis.

Discuss the entanglement of this state and find out the condition for maximum entanglement.

[4]

6. (a) Given the Pauli spin matrices σ_x, σ_y and σ_z, find out the eigenvalues and the eigenvectors of the matrix $e^{\vec{\sigma}\cdot\hat{n}}$, where $\hat{n}$ is the unit vector on the Bloch sphere, given by $\hat{n} = (sin\theta\ cos\phi, sin\theta\ sin\phi, cos\theta)$ and $\vec{\sigma} = \hat{i}\ \sigma_x + \hat{j}\ \sigma_y + \hat{k}\ \sigma_z$. Express the eigenvectors in terms of the computational basis $\{|0\rangle, |1\rangle\}$. [3]
(b) Show explicitly using the different rotation matrices expressed in terms of the Euler angles, how you will arrive at these eigenstates from the state $|0\rangle$ in the Bloch sphere. [3]

The three Pauli spin matrices are given by,
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

7. (a) Consider a quantum state of an electron consisting of a magnetic moment, $\vec{\mu} = \gamma\, \hbar\, \vec{\sigma}$ evolving under a magnetic field. The Hamiltonian, H under a static magnetic field B applied along the x-axis is given by,

$$H = -\frac{1}{2}\hbar\, \omega\, \sigma_x, \tag{2}$$

where, $\omega = \gamma\, B$ and γ is the gyromagnetic ratio. If initially the electron is in the state $|0\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find out the minimum time necessary to flip the spin from $|\uparrow\rangle$ to $|\downarrow\rangle$. Find also the time necessary to take the initial state, $|\uparrow\rangle$ to an arbitrary state, $\phi = \frac{|\uparrow\rangle + i|\downarrow\rangle}{\sqrt{2}}$. Why this state cannot be distinguished from the state $|\uparrow\rangle$. [4]

(b) Calculate the expectation value of $\vec{\mu}$ in an arbitrary state after a time t since the evolution has started. Please give a physical interpretation of the result. [3]

8. The Hamiltonian of an electron placed in a magnetic field is given by,

$$H = -\frac{1}{2}\hbar\, \gamma\, \vec{\sigma}.\, \vec{B} \tag{3}$$

Write down the density matrix for the electron whose generalized Bloch vector is $\vec{v}$ and hence using the density matrix approach (formulation) show that the equation of motion governing the Bloch vector of the electron is given by,

$$\frac{d\vec{v}}{dt} = \gamma\, \vec{v} \times \vec{B} \tag{4}$$

[5]

9. (a) Consider a symmetric matrix A over $\mathbf{R}$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \tag{5}$$

Let the Bell states be given by,

$$|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_1 |\downarrow\rangle_2 \pm |\downarrow\rangle_1 |\uparrow\rangle_2], \tag{6}$$

$$|\phi_\pm\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_1|\uparrow\rangle_2 \pm |\downarrow\rangle_1|\downarrow\rangle_2],, \tag{7}$$

Where, $|\uparrow\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the two orthogonal states. Show that the four Bell states form a basis. Let $\bar{A}$ denote the matrix A in the Bell basis. What is the condition on the entries a_{ij}, such that $\bar{A}$ is diagonal. [4]

(b) The three particle GHZ state is defined as,

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_1|\uparrow\rangle_2|\uparrow\rangle_3 + |\downarrow\rangle_1|\downarrow\rangle_2|\downarrow\rangle_3] = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle], \tag{8}$$

Write down the density matrix for the state $|\psi\rangle$ and verify/check the properties of density matrix here. Subsequently show that the GHZ state is a maximally mixed state. [Hint evaluate the two particle reduced density matrix]. [3]

10. (a) Write down the truth table for the Toffoli Gate (a gate for reversible classical computation, also known as the controlled-controlled-NOT gate) which acts on three bits as follows: It flips the state of the third bit iff both the first and the second bits are in the state 1. [2]

(b) Suppose one represents the 8 possible states of the three bits by column vectors as follows:

$$000 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; 001 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; 010 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; 011 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; 100 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

$$101 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; 110 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; 111 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Construct the 8 x 8 matrix operator G which corresponds to the action of Toffoli Gate on the above vectors. [5]
(c) Show that $G^\dagger G = I$, where $G^\dagger$ is the adjoint operator of G (transpose of complex conjugate of G) and I is 8 x 8 the identity operator (this proves that G is a unitary operator, and thereby a valid quantum gate; use a clever method of identifying known matrices inside the gate before multiplying). [3]

11. (a) Express the Hadamard gate H as a product of the rotation matrices R_x and R_z and $e^{i\phi}$ for some phase ϕ. [3]
(b) An arbitrary single qubit unitary operator can be written in the form

$$U = e^{i\alpha} R_{\hat{n}}$$

for some real numbers α and θ, and a real 3-D vector $\hat{n}$.
(i) Prove this fact. (ii) Find the values of α, θ and $\hat{n}$ yielding the Hadamard gate H. (iii) Find the values of α, θ and $\hat{n}$ yielding the phase gate,

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

[2+2+2]
12.Consider the 4 particle state,

$$|W\rangle = \frac{1}{2\sqrt{2}}(|1100\rangle + \sqrt{2}|1010\rangle - |1001\rangle + |0011\rangle - \sqrt{2}|0101\rangle + |0110\rangle). \quad (9)$$

Table 1: The operations of Alice and the subsequent state obtained by her.

operations by Alice	**state obtained**
$I \otimes I$	
$I \otimes \sigma_3$	
$\sigma_3 \otimes I$	
$\sigma_3 \otimes \sigma_3$	
$I \otimes \sigma_1$	
$I \otimes i\sigma_2$	
$\sigma_3 \otimes \sigma_1$	
$\sigma_3 \otimes i\sigma_2$	
$\sigma_1 \otimes I$	
$\sigma_1 \otimes \sigma_3$	
$i\sigma_2 \otimes I$	
$i\sigma_2 \otimes \sigma_3$	
$\sigma_1 \otimes \sigma_1$	
$i\sigma_1 \otimes \sigma_1$	
$\sigma_1 \otimes i\sigma_2$	
$i\sigma_2 \otimes \sigma_1$	
$\sigma_2 \otimes \sigma_2$	

1 (a) $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right]$

$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right]$

$\langle\psi| = \frac{1}{\sqrt{2}}\left[(1\ 0) \otimes (0\ 1) - (0\ 1) \otimes (1\ 0)\right]$

$E(n,m) = \langle\psi|\vec{\sigma}\cdot\vec{n} \otimes \vec{\sigma}\cdot\vec{m}|\psi\rangle$ (given)

To prove

$E(n,m) = -\cos\phi_{n,m}$, we just need to apply Bell's inequalities

We know, we've to find $E(n,m)$ of operator $\vec{\sigma}\cdot\vec{n} \otimes \vec{\sigma}\cdot\vec{m}$

We know,

$E(n,m) = (I \otimes \sigma\cdot n)(\sigma\cdot m \times I)$

By definition of Pauli matrices,

$\sigma\cdot m \times I = \begin{bmatrix} m_z & m_x - im_y \\ m_x + im_y & -m_z \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

~~$\sigma\cdot m \otimes I|\psi\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} m_z & m_x - im_y \\ m_x + im_y & -m_z \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right.$~~

$\sigma\cdot m \otimes I|\psi\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} m_z \\ m_x + im_y \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} m_x - im_y \\ -m_z \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

Similarly,

$$\langle\psi| I \otimes \sigma\cdot n = \frac{1}{\sqrt{2}}\left([1 \;\; 0] \otimes [n_x + i n_y \;\; -n_z] - [0 \;\; 1] \otimes [n_z \;\; n_x - i n_y]\right)$$

Now,

$$\langle\psi|(\vec{\sigma}\cdot\vec{n}) \otimes (\vec{\sigma}\cdot\vec{m})|\psi\rangle = -n\cdot m$$
$$= -\cos\phi_{n,m}$$

(as $\vec{n}$ & $\vec{m}$ are $\hat{n}$ & $\hat{m}$ (unit vectors))

(b) CHSH inequality says

$$|A| = |E(n,m) - E(n,m')| + |E(n',m') + E(n',m)| \leq 2$$

~~& CHSH is violated when~~ [illegible]

~~because~~ [illegible]

$$|A| = |\cos\phi_{n,m} - \cos\phi_{n,m'}| + |\cos\phi_{n',m'} + \cos\phi_{n',m}| \leq 2$$

Similarly,

$$\langle\psi| I \otimes \sigma \cdot n = \frac{1}{\sqrt{2}}\Big([1\,0] \otimes [n_x + i n_y \quad -n_z] - [0 \quad 1] \otimes [n_z \quad n_x - i n_y]\Big)$$

Now, $\langle\psi|(\vec{\sigma}\cdot\vec{n}) \otimes (\vec{\sigma}\cdot\vec{m})|\psi\rangle = -n \cdot m$
$= -\cos\phi_{n,m}$
(as $\vec{n}$ & $\vec{m}$ are $\hat{n}$ & $\hat{m}$ (unit vectors))

(b) CHSH inequality →

Let $\{A\} = |E(n,m) - E(n,m')| + |E(n',m') + E(n',m)| \leq 2$

we know $E(n,m) = -\cos\phi_{n,m}$ (& $\phi \in [0,\pi]$)

Putting this in A, we get

$$|\cos\phi_{n,m} - \cos\phi_{n,m'}| + |\cos\phi_{n',m'} + \cos\phi_{n',m}| \leq 2$$

We've to maximize LHS & Hardy's genetic algorithms can be used to provide approx value of ϕ

$$\phi_{n,m'} = \frac{3\pi}{4} \quad \& \quad \phi_{n,m} = \phi_{n',m'} = \phi_{n',m} = \frac{\pi}{4}$$

$$\cos\left(\frac{3\pi}{4}\right) = \frac{-1}{2} \quad \& \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

& putting this, in CHSH inequality, we get:

$$\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 2\sqrt{2}$$

Maximal violation of Bell Inequality $= 2\sqrt{2}$

[redacted]

② ⓐ $$|\phi\rangle = \frac{1}{2}|\uparrow\rangle_A \left(\frac{1}{2}|\uparrow\rangle_B + \frac{\sqrt{3}}{2}|\downarrow\rangle_B\right) + \frac{1}{2}|\downarrow\rangle_A \left(\frac{\sqrt{3}}{2}|\uparrow\rangle_B + \frac{1}{2}|\downarrow\rangle_B\right)$$

$$= \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}_A \left(\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}_B + \frac{\sqrt{3}}{2}\begin{pmatrix}0\\1\end{pmatrix}_B\right) + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}_A \left(\frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\end{pmatrix}_B + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}_B\right)$$

$$= \frac{1}{4}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \frac{\sqrt{3}}{4}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{\sqrt{3}}{4}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\rho = \frac{1}{8}(|00\rangle\langle 00|) + \frac{3}{8}(|01\rangle\langle 01| + \frac{3}{8}(|10\rangle\langle 10| + \frac{1}{8}|11\rangle\langle 11|$$

$$\rho^{T_A} = \frac{1}{4}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$= \frac{1}{4}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\rho^{T_B} = \frac{1}{4}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right.$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ $\frac{1}{4}$ eigenvalues are

eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

More than 1 non-zero eigenvalues,

$\therefore$ entangled state

(b) ~~For any tripartite state~~ $\cancel{|\psi\rangle_{ABC} \in H_A \otimes H_B \otimes H_C}$
~~of a tripartite system~~

(b) For any ~~tripartite~~ state $|\psi\rangle_{ABC} \in H_A \otimes H_B \otimes H_C$ of a tripartite system, if for say dim H_A is smallest among Hilbert Space of all 3 particles, then on performing 'partial inner product' of basis $|u_i\rangle_A$ with the state $|\psi\rangle_{ABC}$, i.e., ${}_A\langle u_i|\psi\rangle_{ABC} = |\psi_1\rangle_{BC}$, if we get Schmidt number 1, then Schmidt decomposition for the tripartite system $H_A \otimes H_B \otimes H_C$ exists.

As we know if Schmidt rank (no. of non-zero eigenvalues of reduced density matrix / partial trace of a bipartite system and is equal to no of terms in Schmidt decomposition of bipartite state) is 1, then it's separable & if it's greater than 1, it's entangled. For tripartite state, we get reduced density matrix for one particle by taking partial trace over other two.

$$|\psi\rangle_{ABC} = \sum \lambda_i |i\rangle_A |i\rangle_B |i\rangle_C$$ is Schmidt decomposition

3 types of tripartite states (in general) $\rightarrow$

(i) Fully separable, no quantum entanglement
(ii) Biseparable, bipartite entanglement
(iii) Full inseparable, tripartite entanglement $\rightarrow$ GHZ & W states

③ Case 1 → If Alice does nothing to her qubit, then sent qubit has state

$$|\psi\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

$$= \cancel{|W_{00}\rangle} \quad |W_{00}\rangle$$

Case 2 → If Alice applies X-gate, then

$$X \otimes I \otimes I |\psi\rangle = \frac{|100\rangle + |011\rangle}{\sqrt{2}} = |W_{01}\rangle$$

Case 3 → If Alice applies Y gate,

$$Y \otimes I \otimes I |\psi\rangle = \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

$$= |W_{10}\rangle$$

Case 4 → If Alice applies Z gate,

$$Z \otimes I \otimes I |\psi\rangle = \frac{-|100\rangle + |011\rangle}{\sqrt{2}}$$

$$= |W_{11}\rangle$$

Now that these states are orthogonal,

$$\langle W_{00} | W_{01} \rangle = \frac{1}{2} \left[(\langle 000| + \langle 111|)(|100\rangle + |011\rangle) \right] = 0$$

This means that after Alice sends 1st qubit to Bob, he can distinguish by making a 3 qubits projective measurement.

Bob can recover classical bits 00, 01, 10, 11 when we use qubits obtained in specified 4 cases.

4 (a) Density operator is

$$\rho = |\psi\rangle\langle\psi|$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)\frac{1}{2}(\langle 00| + \langle 01| + \langle 10| - \langle 11|)$$

$$= \frac{1}{4}\big(|00\rangle\langle 00| + |00\rangle\langle 01| + |00\rangle\langle 10| - |00\rangle\langle 11|$$

$$+ |01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| - |01\rangle\langle 11|$$

$$+ |10\rangle\langle 00| + |10\rangle\langle 01| + |10\rangle\langle 10| - |10\rangle\langle 11|$$

$$- |11\rangle\langle 00| - |11\rangle\langle 01| - |11\rangle\langle 10| + |11\rangle\langle 11|\big)$$

In the $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ basis,

representation of density matrix is

$$\rho = \begin{pmatrix} 1/4 & 1/4 & 1/4 & -1/4 \\ 1/4 & 1/4 & 1/4 & -1/4 \\ 1/4 & 1/4 & 1/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 1/4 \end{pmatrix}$$

Partial transpose is found by swapping B qubits in each term.

$$\rho^{T_B} = \frac{1}{4}\big(|00\rangle\langle 00| + |01\rangle\langle 00| + |00\rangle\langle 10| - |01\rangle\langle 10| + |00\rangle\langle 01| + |01\rangle\langle 01| + |00\rangle\langle 11| - |01\rangle\langle 11| + |10\rangle\langle 00| + |11\rangle\langle 00| + |10\rangle\langle 10| - |11\rangle\langle 10| - |10\rangle\langle 01| - |11\rangle\langle 01| - |10\rangle\langle 11| + |11\rangle\langle 11| \big)$$

Matrix representation of ~~ρ^{T_B}~~ ρ^{T_B} is

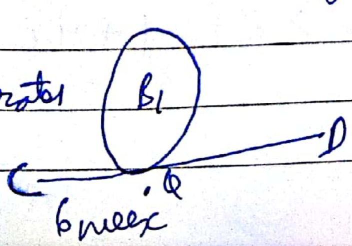
$$= \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 1/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 1/4 \end{bmatrix}$$

eigenvalues of ρ^{T_B} are $\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right\}$

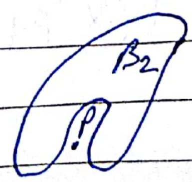
$\because$ all eigenvalues are +ve or 0, $\therefore$ $|\psi\rangle$ is not entangled

(b) Separable states form a convex state. Which means that given two separable states A & B, any convex sum $\lambda A + (1-\lambda)B$ $(1 \geq \lambda \geq 0)$ is again separable. Geometrically, it means ~~separable~~ set of separable states have no trough. For ~~any~~ point P outside B_2, a straight line $\nexists$ that separates P from B_2

Line CD separates P from B_1



Convex



Not convex

For dimensions ≤ 6, PPT is necessary & sufficient. If +ve partial transpose exists of density matrix, then state is separable.

For dimensions >6, PPT criterion is only necessary as in higher dimensions, $\exists$ entangled states with PPT.

5) (a) $\psi(x_a, x_b) = \sqrt{\frac{2}{\pi}}(\alpha x_a + \beta x_b) e^{-(x_a^2+x_b^2)/2}$

$$|\alpha^2| + |\beta^2| = 1$$

Density matrix of ψ =

$$\rho = |\psi\rangle\langle\psi| = |\alpha|^2 |1,0\rangle\langle 1,0| + |\beta|^2 |0,1\rangle\langle 0,1| + (\alpha^\dagger \beta |0,1\rangle\langle 1,0| + H.c.)$$

where $|1,0\rangle = \sqrt{\frac{2}{\pi}} x_a e^{-\frac{x_a^2+x_b^2}{2}}$

Taking partial transpose of 2nd subsystem, we obtain reduced density matrix

$$\rho^T = |\alpha|^2 |1,0\rangle\langle 1,0| + |\beta|^2 |0,1\rangle\langle 0,1| + (\alpha^\dagger \beta |0,0\rangle\langle 1,1| + H.c.)$$

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Its 4 eigenvalues are $|\alpha|^2$, $|\beta|^2$ & $\pm|\alpha||\beta|$

$\exists$ -ve eigenvalue of partial transpose $\therefore$ state is entangled.

7 (a) e^-'s state as a function of time $|\psi(t)\rangle = \begin{bmatrix} \cos(\gamma\beta t/2) \\ i\sin(\gamma\beta t/2) \end{bmatrix}$

To flip spin from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\sin \frac{\gamma\beta t}{2} = 1$$

$$t = \pi/\gamma\beta$$

For arbitrary state,

$$\phi = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$

$$|\psi(t)\rangle = \begin{pmatrix} e^{-i\gamma\beta t/2} \\ e^{i\gamma\beta t/2} \end{pmatrix}$$

@ $\frac{\gamma\beta t}{2} = 0$ or $360^\circ = 2\pi$

we get required

arbitrary state

@ $t = \frac{\pi}{\gamma\beta}$

This state can't be distinguished from initial state as with time evolution, only phase changes.

(7) (a) Electron's state as a function of time =

$$|\psi(t)\rangle = \begin{bmatrix} \cos\left(\frac{\gamma B t}{2}\right) \\ i \sin\left(\frac{\gamma B T}{2}\right) \end{bmatrix}$$

To flip spin from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\sin \frac{\gamma B t}{2} = 1$$

$$t = \frac{\pi}{\gamma B}$$

For arbitrary state,

$$\phi = \frac{|\uparrow\rangle + |i\downarrow\rangle}{\sqrt{2}}$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\gamma B t}{2}} \\ e^{i\gamma B t/2} \end{pmatrix}$$

at $\frac{\gamma B t}{2} = 0$ or $360^\circ = 2\pi$

, we get required arbitrary state

$$t = \frac{2\pi}{2\gamma B} = \frac{\pi}{\gamma B}$$

It can't be distinguished from initial state as with time evolution, only phase (angle) changes.

(b) Expectation value of $\mu = \langle \psi(0) | \mu | \psi(0) \rangle$

$$= \langle \uparrow | \mu | \uparrow \rangle$$

(8) Generalized Bloch vector $= \vec{v}$

$$= v_x, v_y, v_z$$

$$= \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta$$

$$\rho = \frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})$$

$$\cancel{H = \gamma \vec{B}}$$

$$\langle \vec{v} \rangle = \langle v_x \rangle \vec{x} + \langle v_y \rangle \vec{y} + \langle v_z \rangle \vec{z}$$

which $\Rightarrow$ that instantaneous value of $\langle \vec{v} \rangle$

$$\frac{d\langle \vec{v} \rangle}{dt} = \gamma \vec{v} \times \vec{B}$$

(9) (a) Bell basis is given by

$$\psi_{00} = (|00\rangle + |11\rangle)/\sqrt{2}$$
$$\psi_{01} = (|00\rangle - |11\rangle)/\sqrt{2}$$
$$\psi_{10} = (|10\rangle + |01\rangle)/\sqrt{2}$$
$$\psi_{11} = (|01\rangle - |10\rangle)/\sqrt{2}$$

For 2 e^- states,

$\langle 00|00\rangle = 1$

$\langle 00|11\rangle = 0$

$\langle 00|10\rangle = 0$

$\langle 00|01\rangle = 0$

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$\langle 11 | 11 \rangle = 1$ $\langle 11 | 01 \rangle = 0$

$\langle 11 | 10 \rangle = 0$ $\langle 10 | 10 \rangle = 1$

$\langle 10 | 01 \rangle = 0$ $\langle 01 | 01 \rangle = 1$

$\Rightarrow \langle i | j \rangle = \delta_{ij}$ & thus Bell states form orthonormal basis

(b) GHZ state $= \dfrac{|000\rangle + |111\rangle}{\sqrt{2}}$

$$\rho = \frac{1}{2}(|000\rangle + |111\rangle)(\langle 000| + \langle 111|)$$

$$= \frac{1}{2}(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|)$$

GHZ state isn't separable into a product. ~~Or any~~ It's tripartite entangled state.

~~To~~ On measuring particle A of GHZ state (in the 0/1 b

, we get the state =

$$\frac{1}{2}\left(|0\rangle\langle 0| \otimes |0,0\rangle\langle 0,0| + |1\rangle\langle 1| \otimes |1,1\rangle\langle 1,1|\right)$$

(10) (a) Truth table for Toffoli Gate :

Input	Output
a b c	a′ b′ c′
$\|000\rangle$	$\|000\rangle$
$\|001\rangle$	$\|001\rangle$
$\|010\rangle$	$\|010\rangle$
$\|011\rangle$	$\|011\rangle$
$\|100\rangle$	$\|100\rangle$
$\|101\rangle$	$\|101\rangle$
$\|110\rangle$	$\|111\rangle$
$\|111\rangle$	$\|110\rangle$

(b) Matrix operator G = Matrix representation of Toffoli Gate

$$= |000\rangle\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| + |011\rangle\langle 011| + |100\rangle\langle 100| + |101\rangle\langle 101| + |110\rangle\langle 111| + |111\rangle\langle 110|$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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(c) Clearly $G^{T}G = I$ (Using Mathematica)

(11)(a) Matrix of Hadamard ~~gate~~ gate in computational basis ($|\uparrow\rangle$ & $|\downarrow\rangle$) is given by

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{X+Z}{\sqrt{2}}$$

Also $e^{ixA} = \cos x\, I + \cancel{i\sin x\, I}\ i\sin x\, A$

$$e^{i\pi X/4} = (I+iX)/\sqrt{2} \ \&$$

$$e^{i\pi Z/4} = (I+iZ)/\sqrt{2}$$ are R_x, R_z rotations of $\angle\pi/4$.

$$\frac{(1+iX)}{\sqrt{2}}\frac{(1+iZ)}{\sqrt{2}}\frac{(1+iX)}{\sqrt{2}} = \frac{1}{2\sqrt{2}}(I+i(X+Z)-XZ)(I+iX)$$

$$= \frac{1}{2\sqrt{2}}(I+i(X+Z)-XZ+iX-I-ZX-iXZX)$$

$$= \frac{1}{\sqrt{2}}(X+Z) = iH$$

$$(XZ = -ZX \ \& \ XZX = -Z)$$

Thus, $H = e^{-i\pi/2}\, e^{i\pi X/4}\, e^{i\pi Z/4}\, e^{i\pi X/4}$

let $\hat{m}$ be a unit vector & $\hat{m} \in R^3$.

$$(\hat{m}\cdot\vec{\sigma})^2 = (m_x X + m_y Y + m_z Z)^2$$
$$= (m_x^2 + m_y^2 + m_z^2)I + m_x m_y (XY+YX) + m_y m_z (YZ+ZY) + m_z m_x (ZX+XZ)$$

$= I \quad (\because XY = -YX \;\&\; YZ + ZY = ZX + XZ)$

(b) $U = e^{i\alpha} R_{\hat{n}}(\theta) \rightarrow$ Proof after (ii) & (iii)

(ii) ~~Given~~ $\Rightarrow$ $\cancel{H = (X+Z)/\sqrt{2}}$

$H = (X+Z)/\sqrt{2}$

$\because U = e^{i\alpha} R_{\hat{n}}(\theta)$

$\therefore \alpha = -\pi/2, \quad \theta = \pi/2 \;\&\; \cancel{\hat{n} = (0,0,1)}$

$\hat{n} = \frac{1}{\sqrt{2}}(1,0,1)$

giving

$$-i\left\{\cos \pi/2 + i \sin \pi/2 \left(\frac{X+Z}{\sqrt{2}}\right)\right\}$$

$$= \frac{X+Z}{\sqrt{2}} = H$$

(iii) $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

As above, choosing $\alpha = \pi/4, \; \theta = -\pi/4$ &

$\hat{n} = (0,0,1)$

gives

$$S = e^{i\pi/4} e^{-i\pi Z/4}$$

(i) Let U be a 2×2 Unitary matrix

det U is a pure phase

$|\det U|^2 = \det U \det U^\dagger = 1$

$\therefore \exists\ \alpha \in \mathbb{R}$ s.t.

$$\det U = e^{i2\alpha}$$

Hence $U = e^{i\alpha} W$ (with W being Unitary & $\det W = 1$)

$\because$ W is unitary, it's normal & can be diagonalized in an orthonormal basis with eigenvalues $\lambda_\pm$.

Unitarity $\Rightarrow$ both eigenvalues are pure phases

$\because \det W = \lambda_+ \lambda_- = 1$,

$\exists\ \theta \in \mathbb{R}$ s.t.

$$\lambda_\pm = e^{\pm i\theta}$$

$$\Rightarrow \operatorname{tr} W = 2\cos\theta$$

Being 2×2 matrix, W can be decomposed in a unique way in Pauli basis,

$$W = \cos\theta\, I + \vec{u}\cdot\vec{\sigma} + \vec{w}\cdot\vec{\sigma}$$
$$W^\dagger = \cos\theta\, I + \vec{u}\cdot\vec{\sigma} - \vec{w}\cdot\vec{\sigma}$$

By Unitarity, we get

$$I = WW^\dagger$$
$$= (\cos^2\theta + |\vec{u}|^2 + |\vec{v}|^2) I + (2\cos\theta\, \vec{u} + \vec{u}\wedge\vec{v})\cdot\vec{\sigma}$$

giving, $\cos^2\theta + |\vec{u}|^2 + |v|^2 = 1$ & $2\cos\theta\,\vec{u} = -\vec{u}\wedge\vec{v}$

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$\Rightarrow 2\cos\theta\, |\vec{u}|^2 = 0$ so either

$2\cos\theta = 0$ or $\vec{u} = 0$

If $\vec{u} \neq 0$, then $\vec{u}$ & $\vec{v}$ are collinear &

by L.H.S. $|\vec{u}|^2 + |v|^2 = 1$

s.t. $\exists\, \phi \in R$ & unit vector $\hat{n}$

s.t. $\vec{u} + i\vec{v} = e^{i\phi}\hat{n}$.

$\because \det \hat{n}\vec{\sigma} = -1 \Rightarrow \phi = \pi$

& Given (in ques) holds with $\theta = \pi/2$

If $\vec{u} = 0$, then $|\vec{v}|^2 = \sin^2\theta$ & $\therefore \exists\, \hat{n}$ s.t.

$W = \cos\theta\, I + i\sin\theta\, \hat{n}\cdot\vec{\sigma} = R_{\hat{n}}(\theta)$

(12) $I \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$I \otimes \sigma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$\sigma_3 \otimes I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$I \otimes \sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$I \otimes i\sigma_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

$\sigma_3 \otimes \sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

$$\sigma_3 \otimes i\sigma_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$i\sigma_2 \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$i\sigma_2 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \times \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$i\sigma_1 \otimes \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes i\sigma_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$i\sigma_2 \otimes \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$|W\rangle = \frac{1}{2\sqrt{2}} \left(|1100\rangle + \sqrt{2}\,|1010\rangle - |1001\rangle + |0011\rangle - \sqrt{2}\,|0101\rangle + |0110\rangle \right)$$

$$\sigma_i^1 \otimes \sigma_j^3 |W\rangle \longrightarrow |W\rangle'$$

$$I \otimes I |\omega\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2\sqrt{2} \\ 1/2 \\ -1/2\sqrt{2} \\ 1/2\sqrt{2} \\ -1/2 \\ 1/2\sqrt{2} \end{bmatrix}$$

Similarly, they all can be solved.