Warm-up Questions

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1. Consider a Poisson arrival process with rate λ. Whenever an arrival occurs, it is removed with a probability p and kept with a probability $1-p$.
   - Prove that the removed arrivals form a Poisson arrival process. Find its rate.
   - What is the rate of the arrivals that are not removed?
2. For a Poisson arrival process with rate λ, what is the probability that exactly one arrival occurs in an interval of length t? What is the arrival rate for which this probability is maximized?
3. Let $X_1, X_2, X_3 \cdots$ be a sequence of i.i.d. Uniform(0,1) random variables. Define the sequence Y_n as

$$Y_n = \min\{X_1, X_2, \cdots, X_n\}. \tag{1}$$

   Prove the following convergence results independently (i.e, do not conclude the weaker convergence modes from the stronger ones).

   (a) The distribution of Y_n converges to the unit step function at 0.

   (b) Y_n converges in the mean square sense to 0.

   (c) Y_n converges in probability to 0.

4. Let X be a continuous random variable, with PDF

$$f_X(x) = \begin{cases} 0; & x \leq 0 \\ 0.5; & 0 < x \leq 1 \\ ce^{-x}; & x > 1. \end{cases} \tag{2}$$

   (a) What is the value of c?

   (b) What is the conditional expectation of X, given $X < 1$?

   (c) What is the conditional expectation of X, given $X \geq 1$?

   (d) What is the expectation of X?

5. Bob and Eve play chess every day and Bob wins a fraction p of these games. Assume that each game of chess is independent. Let X_n denote the number of games played until Bob has a set of n consecutive victories.

   (a) What is $\mathbb{E}[X_1]$?

   (b) Use the conditional expectation theorem to show that

$$\mathbb{E}[X_n] = \frac{1}{p}\left(1 + \mathbb{E}[X_{n-1}]\right). \tag{3}$$

   (c) Use (a) and (b) to show that

$$\mathbb{E}[X_n] = \sum_{i=1}^{n} \frac{1}{p^i}. \tag{4}$$

6. Let $X_1, X_2, \ldots, X_n$ are random variables with mean $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$, respectively. Show that:

$$\mathbb{E}\left[\sum_i^n X_i\right] = \sum_i^n \mathbb{E}\left[X_i\right] \tag{5}$$

   You may assume that the random variables have joint distribution but do not assume that they are independent.

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(1) We consider Poisson arrival process $K(t)$ with rate λ

The process is split into 2 :

(i) $K_1(t)$: arrival removed with probability p

(ii) $K_2(t)$: " not " " " $1-p$

$\because$ The process of splitting of $K(t)$ has only 2 possible outcomes & probability of each outcome remains same throughout the process

$\therefore$ It's a Bernoulli process

- To prove: The 2 possible outcomes are Poisson

Proof: $K_1(t)$ occurs with probability p

$K_2(t)$ " " " $1-p$

Let total number of arrivals be K for time t

& " " " removed $= L$

Then, " " " not " $= K-L$

Joint PMF for $K_1(t)$ & $K_2(t)$:

$$P(K_1(t) = L, \quad K_2(t) = K-L \mid K(t) = K)$$

$$= \frac{K!}{L!\,(K-L)!}\, p^L \,(1-p)^{K-L}$$

Process of arrivals &

" " splitting of arrivals ~~is an~~ are independent processes

$$\therefore\ P(k_1(t) = L,\ k_2(t) = k-L)$$

$$= P(k_1(t) = L,\ k_2(t) = k-L \mid k(t) = k) \times P(k(t) = k)$$

$$\Rightarrow\ P(k_1(t) = L,\ k_2(t) = k-L)$$

$$= \frac{k!}{L!\,(k-L)!}\, p^L (1-p)^{k-L}\, \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad - ⓧ$$

(where, $P(k(t) = k) = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!}$

as it is a Poisson process with (parameter) rate $= \lambda$)

Multiplying & dividing eq^n ⓧ by $e^{\lambda t p}$, we rearrange & then, we get

$$P(k_1(t) = \cancel{k}L,\ k_2(t) = k-L)$$

$$= \frac{(p\lambda t)^L}{L!} e^{-\lambda p t}\ \frac{((1-p)\lambda t)^{k-L} e^{-\lambda(1-p)t}}{(k-L)!}$$

$$P(k_1(t) = L) = \sum_{k-L=1}^{\infty} P(k_1(t) = L,\ k_2(t) = k-L)$$

$$= \frac{(p\lambda t)^L}{L!} e^{-\lambda p t}\, e^{-\lambda(1-p)t}\ \frac{\sum((1-p)\lambda t)^{k-L}}{(k-L)!}$$

$$= \frac{(p\lambda t)^L e^{-\lambda p t}}{L!} \; e^{-\lambda(1-p)t} \; e^{\lambda(1-p)t}$$

$$= \frac{(p\lambda t)^L e^{-\lambda p t}}{L!}$$

Thus, proving that $N_1(t)$ is a Poisson process with rate λp

- The rate of arrivals that are not removed is $\lambda(1-p)$

(2) $P(N(t)=1) = e^{-\lambda t}\lambda t$ — (X)

To maximize this probability, we differentiate with respect to rate λ & equate to 0

$$\cancel{t}(-t)e^{-\lambda t}\lambda t - \cancel{\lambda}t\, e^{-\lambda t} = 0$$

or $-t(-1+\lambda t)e^{-\lambda t}$

Then $\lambda t - 1 = 0$

or $\lambda t = 1$

or $\lambda = \frac{1}{t}$

is the arrival rate for which Probability (X) is maximized

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(3) $Y_n = \min(X_1, X_2, \dots, X_n)$

where X_i's are sequence of independent & identically distributed uniform (0, 1) random variables

(a) ~~CDF~~ CDF of X_n $\forall x \in \mathbb{R}$ is given by

$$F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ \frac{x-0}{1-0} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

or

$$F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Range of $Y_n = [0, 1]$

for $0 \leq y \leq 1$

CDF of Y_n:

$$F_{Y_n}(y) = P(Y_n \leq y)$$

$$= 1 - P(Y_n > y)$$

$$= 1 - P(X_1 > y, X_2 > y, \dots, X_n > y)$$

$$= 1 - [P(X_1 > y) P(X_2 > y) \dots P(X_n > y)]$$

(due to independence of X_i's)

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$$= 1 - [(1 - F_{X_1}(y))(1 - F_{X_2}(y)) \cdots (1 - F_{X_n}(y))]$$

$$= 1 - (1-y)^n$$

~~for $y \in [0,1]$~~

at $n \to \infty$, $\lim_{n \to \infty} 1 - (1-y)^n$

$$= 0 \qquad y \le 0$$

or

$$1 \qquad y > 0$$

, i.e.,

$$\lim_{n \to \infty} F_{Y_n}(y) = \begin{cases} 0 & y \le 0 \\ 1 & y > 0 \end{cases}$$

(b) PDF of $Y_n(y)$ is given by

$$f_{Y_n}(y) = \frac{dF_{Y_n}(y)}{dy}$$

$$\Rightarrow f_{Y_n}(y) = \begin{cases} n(1-y)^{n-1} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$E|Y_n|^2 = \int_0^1 n y^2 (1-y)^{n-1} \, dy$$

$$\le \int_0^1 n y (1-y)^{n-1} \qquad (\because 2 \ge 1)$$

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$$= \left[-y(1-y)^n \right]_0^1 + \int_0^1 (1-y)^n \, dy$$

$$= \frac{1}{n+1}$$

at $n \to \infty$, $\lim_{n \to \infty} (E|Y_n|^2) = 0$

(c) Take arbitrary $\varepsilon > 0$

Then,

$$P(Y_n \geq \varepsilon) = 1 - P(Y_n < \varepsilon)$$

$$= 1 - P(Y_n \leq \varepsilon)$$

($\because Y_n$ is a continuous rv)

$$= 1 - F_{Y_n}(\varepsilon)$$

$$= (1-\varepsilon)^n$$

& at $n \to \infty$, $(1-\varepsilon)^n = 0$

$\forall \varepsilon \in (0, 1]$

(4) X is continuous rv with PDF

$$f_X(x) = \begin{cases} 0 & x \leq 0 \\ 0.5 & 0 \leq x \leq 1 \\ ce^{-x} & x > 1 \end{cases}$$

(a) $\because f_X(x)$ is PDF, $\therefore \int_{-\infty}^{\infty} f(x)\, dx = 1$

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$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 0.5\,dx + \int_{1}^{\infty} ce^{-x} = 1$$

$$\text{or } 0 + 0.5x + \left(-ce^{-x}\right)\Big|_{1}^{\infty} = 1$$

$$\text{or } 0.5x + \left(-ce^{-\infty}\right) - \left(-\frac{c}{e}\right) = 1$$

$$\Rightarrow 0.5x + \frac{c}{e} = 1$$

$$\text{or } c = \frac{e}{2}$$

ⓑ

$$E[X \mid X < 1] = \frac{\int_{0}^{1} x\,f_X(x)\,dx}{\int_{0}^{1} f_X(x)\,dx}$$

$$= \frac{\int_{0}^{1} 0.5x\,dx}{\int_{0}^{1} 0.5\,dx}$$

$$= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

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(c) $E[X | X > 1] = \dfrac{\int_1^\infty x f_X(x)\, dx}{\int_1^\infty f_X(x)\, dx}$

$$= \frac{\int_1^\infty x \frac{e}{2} e^{-x}\, dx}{\int_1^\infty \frac{e}{2} e^{-x}\, dx}$$

$$= \frac{-(x+1)e^{-x} \Big|_1^\infty}{-e^{-x} \Big|_1^\infty}$$

$$\approx \frac{0.74}{0.37} = 2$$

(d) $E[X | -\infty < X < \infty] = \dfrac{\int_{-\infty}^{\infty} x f_X(x) dx}{\int_1^\infty f_X(x)\, dx}$

$$= E[X | X < 1] + E[X | X > 1]$$

as X is continuous RV

$$= \frac{1}{2} + 2 = 2.5$$

(5) No. of games played until Bob ones 1 follows geometric distribution & has probability of success $= p$
Then, we know ~~one~~ expected value of same $= \frac{1}{p}$

Hence,

(a) $E[X_1] = \frac{1}{p}$

(b) let Bob win $n-1$ consecutive games
Then, there are 2 cases
(i) Bob wins next game
(ii) Bob loses next game

In case (i), we get desired result & Bob has played $E[X_{n-1}]+1$ games

In case (ii), Bob has to start from scratch & play $E[X_{n-1}]+1+E[X_n]$ games

Probability for case (i) to occur
$= p(E[X_{n-1}]+1)$
" for " (ii) " occur
$= (1-p)(E[X_{n-1}]+1 + E[X_n])$

Then, $E[X_n] = p(E[X_{n-1}]+1) + (1-p)(E[X_{n-1}]+1 + E[X_n]$

$\Rightarrow E[X_n] = \frac{1}{p}(1+E[X_{n-1}])$

QED

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(c) Putting $E[X_1] = \frac{1}{p}$ in equation

$$E[X_n] = \frac{1}{p}(1 + E[X_{n-1}])$$, we get

$$E[X_n] = \sum_{i=1}^{n} \frac{1}{p^i}$$

(b) We know, expectation is linear for summation over any kind of random variables, if their means are well-defined.

Let $X_1, X_2, \dots, X_n$ be n random variables
Taking their summation

$$\sum_{i=1}^{n} X_i$$

Then by linearity of Expectation, we get

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$