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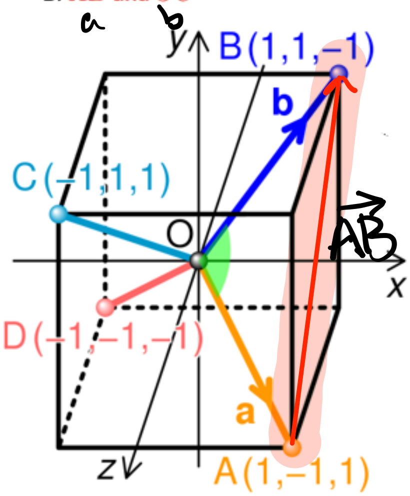
1. (0.5 points) Given the points A, B, C, and D as shown in the following figure, please compute the angle between two vectors

A. $\overset{a}{\overrightarrow{OC}}$ and $\overset{b}{\overrightarrow{OD}}$

B. $\underset{a}{\overrightarrow{AB}}$ and $\overset{b}{\overrightarrow{OC}}$



A) $\theta = \cos^{-1}\left(\frac{a \cdot b}{\|a\|\,\|b\|}\right) = \cos^{-1} \frac{\begin{pmatrix}1\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix}}{\sqrt{3}\cdot\sqrt{3}}$

$\theta = \cos^{-1} \frac{1\cdot 1 + -1\cdot 1 + 1\cdot -1}{3} = \cos^{-1}\left(\frac{-1}{3}\right)$ $\boxed{\approx 109.5^\circ}$

B) $\theta = \cos^{-1}\left(\frac{a \cdot b}{\|a\|\,\|b\|}\right) = \cos^{-1} \frac{\begin{bmatrix}2\\0\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\1\\1\end{bmatrix}}{\sqrt{4}\cdot\sqrt{3}}$

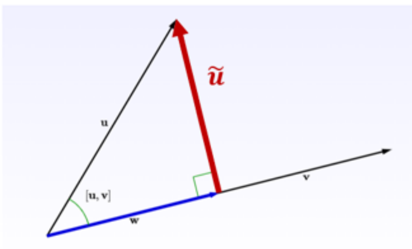
add – – – – – $\vec{AB}$

$A + B$: $\begin{bmatrix}1\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix} = \cos^{-1}\left(\frac{-2+1+1}{2\sqrt{3}}\right)$

$= \cos^{-1}(0) = \boxed{\frac{\pi}{2}}$

or 90°

2. (0.5 points) Suppose $u = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $v = \begin{bmatrix}1\\-1\\2\end{bmatrix}$, please compute the vector $\tilde{u}$ that is orthogonal to the vector v as illustrated in the following figure.



$w = \text{proj}_v(u) = \frac{u^T v}{v^T v} v = \frac{[1,2,1]\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix}}{[1,-1,2]\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix}}$

$= \frac{1 + -2 + 2}{1+1+4} = \frac{1}{6}$

$\tilde{u} = u - w = \begin{bmatrix}1\\2\\1\end{bmatrix} - \frac{1}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \boxed{\begin{bmatrix}5/6\\13/6\\-11/6\end{bmatrix}}$

3. (0.5 points) Please apply the Gram-Schmidt process to orthonormalize the set of vectors

A. $v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$$\frac{v_2^T u_1}{u_1^T u_1} u_1$$

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = e_1 \qquad u_2 = v_2 - \text{proj}_{u_1}(v_2)$$

$$= \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{[1,1,0]\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}{[1,1,0]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3/2 \\ -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix} = e_2$$

$$u_3 = v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3) = v_3 - \frac{v_3^T u_1}{u_1^T u_1} u_1 - \frac{v_3^T u_2}{u_2^T u_2} u_2$$

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{[0,-1,1]\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}{2}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{[0,-1,1]\begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix}}{1+\frac{1}{4}+\frac{1}{4}}\begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2/3 \\ -2/6 \\ 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 8/3 \\ 1/3 \\ -3/2 \end{bmatrix} = e_3 \qquad \left[\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 8/3 \\ 1/3 \\ -3/2 \end{bmatrix} \right]$$

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1. (0.5 points) Please calculate the cross product between two vectors $u = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$. Hint: please use the formula based on matrix-vector product that is given on Slide 103 of "Topic 1.1: Vectors".

2. (0.5 points) Please perform matrix-matrix multiplication to compute both AB and BA, where $A = \begin{bmatrix} 3 & 1 & 7 \\ 2 & 4 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 1 & 1 \\ -3 & -2 \end{bmatrix}$

3. (0.5 points) Consider $Q = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}$, please answer the following questions,

A. Is Q orthogonal? If Yes, please prove it. If No, why?
B. What is Q^{-1} ? Note that Q^{-1} is matrix inverse of Q. Please write down the matrix and explain how you get that matrix.

1. $u \times v = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} 1\cdot1 - (-1)\cdot(-3) \\ -1\cdot3 - 2\cdot1 \\ 2\cdot-3 - 1\cdot3 \end{bmatrix} = \begin{bmatrix} 1-3 \\ -3-2 \\ -6-3 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ -9 \end{bmatrix}$

2. matrix by matrix multiplication is <u>not</u> commutative

$AB = \begin{bmatrix} 3 & 1 & 7 \\ 2 & 4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -1 & -3 \\ 1 & 1 \\ -3 & -2 \end{bmatrix}$

$= \begin{bmatrix} 3\cdot-1 + 1\cdot1 + 7\cdot-3 & 3\cdot-3 + 1\cdot1 + 7\cdot-2 \\ 2\cdot-1 + 4\cdot1 + -5\cdot-3 & 2\cdot-3 + 4\cdot1 + -5\cdot-2 \end{bmatrix}$

$= \begin{bmatrix} -23 & 22 \\ 17 & 8 \end{bmatrix}$; $BA = \begin{bmatrix} -1 & -3 \\ 1 & 1 \\ -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 7 \\ 2 & 4 & -5 \end{bmatrix}$

$= \begin{bmatrix} -1\cdot3 + -3\cdot2 & 1\cdot3 + 1\cdot2 & -3\cdot3 + -2\cdot2 \\ -1\cdot1 + -3\cdot4 & 1\cdot1 + 1\cdot4 & -3\cdot1 + -2\cdot4 \\ -1\cdot7 + -3\cdot-5 & 1\cdot7 + 1\cdot-5 & -3\cdot7 + -2\cdot-5 \end{bmatrix} = \begin{bmatrix} -9 & 5 & 13 \\ -13 & 5 & 11 \\ 8 & 2 & -11 \end{bmatrix}$

3a)

$$Q = \begin{array}{c} \begin{matrix} \quad c_1 & \quad c_2 & \quad c_3 \end{matrix} \\ \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix} \end{array}$$

$$\vec{c}_1 \cdot \vec{c}_1 = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 = \frac{1+4+4}{9} = \frac{9}{9} = 1$$

$$\vec{c}_2 \cdot \vec{c}_2 = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{4+1+4}{9} = \frac{9}{9} = 1$$

$$\vec{c}_3 \cdot \vec{c}_3 = \left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 = \frac{4+4+1}{9} = \frac{9}{9} = 1$$

columns are unit vectors ✓

$$\vec{c}_1 \cdot \vec{c}_2 = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} + \left(-\frac{2}{3}\right) \cdot \frac{2}{3} = \frac{2+2-4}{9} = 0$$

$$\vec{c}_1 \cdot \vec{c}_3 = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right) \cdot \left(-\frac{1}{3}\right) = \frac{2-4+2}{9} = 0$$

$$\vec{c}_2 \cdot \vec{c}_3 = \frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \left(\frac{-2}{3}\right) + \frac{2}{3} \cdot \left(-\frac{1}{3}\right) = \frac{4-2-2}{9} = 0$$

Columns are mutually orthogonal

Yes, Q is orthogonal

3b)

$$Q^{-1} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix}$$

I swapped the rows with the columns.

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$$\begin{aligned} x &+3y &+2z &= 2 \\ 2x &+7y &+7z &= -1 \\ 2x &+5y &+2z &= 7 \end{aligned}$$

Naïve Gaussian Elimination Method

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 2 & 7 & 7 & -1 \\ 2 & 5 & 2 & 7 \end{array}\right]$$

1. Make first column $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Row 2 $\rightarrow$ Row 2 - 2(Row 1)

$$[2\ 7\ 7 \mid -1] - (2) \cdot [1\ 3\ 2 \mid 2] = [0\ 1\ 3 \mid -5]$$

$R_3 \rightarrow R_3 - 2R_1$

$$[2\ 5\ 2 \mid 7] - (2) \cdot [1\ 3\ 2 \mid 2] = [0\ -1\ -2 \mid 3]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -1 & -2 & 3 \end{array}\right]$$

2. eliminate row 2 col 2 entry

$R_3 \rightarrow R_3 + R_2$: $[0\ -1\ -2 \mid 3] + [0\ 1\ 3 \mid -5] = [0\ 0\ 1 \mid -2]$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

3. Lay out equations and solve.

$x + 3y + 2z = 2 \rightarrow x + 3 + -4 = 2 \rightarrow x = 3$

$y + 3z = -5 \rightarrow y = 1$

$z = -2$