Fys3120 Oblig 11

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May 1, 2017

1a

- Problem 1A non-relativistic particle, with electric charge q and mass m moves in a magnetic dipole field, given by the vector potential

$$\vec{A} = \frac{\mu_0}{4\pi r^3}(\vec{\mu} \times \vec{r}) \tag{1}$$

  Show that the Lagrangian is

$$L = \frac{1}{2}m\vec{v}^2 + \frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{l}, \tag{2}$$

  Where $\vec{l} = m\vec{r} \times \vec{v}$

We begin with defining the Lagrangian:

$$L = T - V$$

From equation 2 it is obvious that $T = 1/2mv^2$, which means, we need to prove that the potential is:

$$V = \frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{l}$$

In order to prove this, we use the fact that $-q\vec{v} \cdot \vec{A} = V$:

$$\begin{aligned}
V &= q\vec{v} \cdot \vec{A} \\
&= \frac{q\mu_0}{4\pi r^3}\vec{v}(\vec{\mu} \times \vec{r}) \\
&= \frac{q\mu_0}{4\pi m r^3}\vec{p}(\vec{\mu} \times \vec{r}) \\
&= -\frac{q\mu_0}{4\pi m r^3}\vec{\mu}(\vec{r} \times \vec{p}) \\
&= \underline{\underline{-\frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{l}}}
\end{aligned} \tag{3}$$

1b

- Particle moves in the xy-plane and ϕ is the angle between the x-axis and the particle's position vector $\vec{r}$. The magnetic dipole momentum, $\vec{\mu}$, has a direction along the z-axis. Express the Lagrangian as :

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \lambda\frac{\dot{\phi}}{r} \tag{4}$$

  where

$$\lambda = q\mu_0|\vec{\mu}|/4\pi \tag{5}$$

To express the Lagrangian in equation 4, we start by expressing the kinetic energy in polar coordinates:

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = 0$$

Finding velocity to express T:

$$\vec{v} = (\dot{x},\ \dot{y},\ 0)$$
$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$$
$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$
$$\dot{z} = 0$$
$$\implies |\vec{v}| = \sqrt{\dot{r}^2 + r^2\dot{\phi}^2}$$
$$\implies T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

We now have the kinetic term of equation 4, but we have yet to express the potential which should be: $V = \dot{\phi}\lambda/r$. To derive this, we need to substitute λ from equation 5 into the potential term of our Lagrangian found in both eq. 2 and 3.

First trick is to express the dipole momentum vector, using equation 5:

$$\vec{\mu} = |\vec{\mu}|\vec{k} \tag{6}$$

Before substituting λ into the potential, we split the magnetic dipole vector as we did in 6:

$$\begin{aligned}
-V &= \frac{q\mu_0}{4\pi m r^3}\vec{\mu}\cdot\vec{l} \\
&= \frac{q\mu_0}{4\pi m r^3}|\vec{\mu}|\vec{k}\cdot\vec{l} \\
&= \frac{\lambda}{mr^3}\vec{k}\cdot\vec{l} \\
&= |\vec{l}|\frac{\lambda}{mr^3} = |\vec{r}\times\vec{v}|\frac{m\lambda}{mr^3} = |\vec{r}\times\vec{v}|\frac{\lambda}{r^3}
\end{aligned} \tag{7}$$

This crossproduct requires us to look back when we derived the kinetic term:

$$\begin{aligned}\vec{r} &= (x,\ y,\ z)\\ &= (r\cos\phi,\ r\sin\phi,\ 0) \quad (8)\\ \vec{v} &= (\dot{x},\ \dot{y},\ 0)\\ &= (\dot{r}\cos\phi - r\dot{\phi}\sin\phi,\ \dot{r}\sin\phi + r\dot{\phi}\cos\phi,\ 0) \quad (9)\\ \implies & \quad (10)\end{aligned}$$

$$\vec{r}\times\vec{v} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & 0\\ \dot{x} & \dot{y} & 0\end{vmatrix} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & 0\\ \dot{x} & \dot{y} & 0\end{vmatrix} \quad (11)$$

$$\begin{aligned}&= (x\dot{y} - y\dot{x})\vec{k}\\ &= r\cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) - r\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) \quad (12)\\ &= r^2\dot{\phi}\cos^2\phi + r^2\dot{\phi}\sin^2\phi \quad (13)\\ &= r^2\dot{\phi} \quad (14)\end{aligned}$$

Now that we have the crossproduct, we can insert it into our potential, back in equation 7

$$\begin{aligned}-V =& |\vec{r}\times\vec{v}|\frac{\lambda}{r^3}\\ =& r^2\dot{\phi}\frac{\lambda}{r^3}\\ =& \dot{\phi}\frac{\lambda}{r}\end{aligned}$$

If we look back at our kinetic term, we saw that $T = \frac{1}{2}m(\dot{r}^2 + r^2\phi^2)$, with this we can now express the Lagrangian on the form we set out to:

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \lambda\frac{\dot{\phi}}{r} \quad (15)$$

1c

- Write Lagrange's equation for the coordinate r, expressed in terms of$\dot{r}$, $\ddot{r}$ and p_ϕ, and use the equation to find $\dot{r}^2$ as a function of r and p_ϕ. Compare the expression with that of the particle's kinetic energy.

From the Lagrangian in equation 15, we derive Lagrange's equation for ϕ. Notice that the second term of the Lagrange equation is zero, which gives us the constant conjugate momentum:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \phi} = 0 \tag{16}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(mr^2\dot{\phi} + \frac{\lambda}{r}\right) \tag{17}$$

$$\implies p_\phi = mr^2\dot{\phi} + \frac{\lambda}{r} \tag{18}$$

$$\implies \dot{\phi} = \frac{p_\phi r - \lambda}{mr^3} \tag{19}$$

We can use equation 19 to substitute away $\dot{\phi}$ out of Lagrange's equation of r:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{dt}m\dot{r} = m\ddot{r} \tag{20}$$

$$\begin{aligned}
\frac{\partial L}{\partial r} &= mr\dot{\phi}^2 - \lambda\frac{\dot{\phi}}{r^2} \qquad \text{let's substitute } \dot{\phi} \\
&= mr\left(\frac{p_\phi r - \lambda}{mr^3}\right)^2 - \frac{p_\phi r\lambda - \lambda^2}{mr^5} \\
&= m\left(\frac{p_\phi^2 r^2 - 2p_\phi r\lambda + \lambda^2 - (p_\phi r\lambda - \lambda^2)}{m^2 r^5}\right) \\
&= m\left(\frac{p_\phi^2 r^2 - 3p_\phi r\lambda}{m^2 r^5}\right)
\end{aligned}$$

Now let's set up Lagrange's equation since we have both derivative terms figured out:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} =& m\left(\ddot{r} - \frac{p_\phi^2 r^2 + 3p_\phi r\lambda}{m^2 r^5}\right) && (21)\\
\implies \ddot{r} =& \frac{p_\phi^2 r^2 - 3p_\phi r\lambda}{m^2 r^5} && (22)\\
\implies \dot{r} =& \frac{p_\phi^2}{m^2}\int \frac{1}{r^3}dt - 3\lambda\frac{p_\phi}{m^2}\int \frac{1}{r^4}dt && (23)
\end{aligned}
$$

Since our goal is to express $\dot{r}^2$ we need to integrate $\ddot{r}$, but first we gotta substitute dt out of the equation.

$$
dt = \frac{dr}{dr}dt = \frac{dt}{dr}dr = \frac{1}{\dot{r}}dr
$$

Let's substitute this into equation 23 to solve it:

$$
\begin{aligned}
\dot{r} &= \frac{p_\phi^2}{m^2}\int \frac{1}{r^3}dt - 3\lambda\frac{p_\phi}{m^2}\int \frac{1}{r^4}dt\\
\dot{r} &= \frac{p_\phi^2}{m^2}\int \frac{1}{\dot{r}r^3}dr - 3\lambda\frac{p_\phi}{m^2}\int \frac{1}{\dot{r}r^4}dr && (24)
\end{aligned}
$$

Let's do a cool trick and multiply with $\dot{r}$ on both sides of equation 24:

$$
\begin{aligned}
\dot{r}^2 =& \frac{p_\phi^2}{m^2}\int \frac{1}{r^3}dr - 3\lambda\frac{p_\phi}{m^2}\int \frac{1}{r^4}dr\\
\\
=& -\frac{p_\phi^2}{2m^2}\frac{1}{r^2} + \lambda\frac{p_\phi}{m^2}\frac{1}{r^3} && (25)\\
=& \underline{\underline{\frac{p_\phi}{2m^2 r^2}\left(\frac{2\lambda}{r} - p_\phi\right)}} && (26)
\end{aligned}
$$

Comparing this mess with the kinetic energy is completely lost on me. Perhaps I made a miscalculation.

$$
T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \qquad (27)
$$

2 Relatively cool circuit