The Kinematics of Fluid Motion

4

ξ – material coordinates
x – spatial coordinates
$x = x(\xi, t)$
(ξ point is position at $t = 0$.

4.11. Particle paths

Kinematics is the description of motion per se. It takes no account of how the motion is brought about or of the forces involved for these are in the realm of dynamics. Consequently the results of kinematical studies apply to all types of fluid and are the ground work on which the dynamical results are constructed.

The basic mathematical idea of a fluid motion is that it can be described by a *point transformation*. At some instant we look at the fluid and remark that a certain "particle" is at a position $\boldsymbol{\xi}$ and at a later time the same particle is at position $\mathbf{x}$. Without loss of generality, we can take the first instant to be the time $t = 0$ and if the later instant is time t we say that $\mathbf{x}$ is a function of t and the initial position $\boldsymbol{\xi}$,

ξ – material coordinates

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t) \quad \text{or} \quad x_i = x_i(\xi_1, \xi_2, \xi_3, t). \tag{4.11.1}$$

Of course we have immediately violated the concepts of the kinetic theory of fluids for in this theory the particles are the molecules and these are in random motion. In fact we have replaced the molecular picture by that of a continuum whose velocity at any point is the average velocity of the molecules in a suitable neighborhood of the point. As we have noted in Chapter 1, the

definition of average needs some care in this context, but this idealization which endows the elementary portions of the fluid with a permanence denied them by molecular theory is the key to the classical treatment of fluid motion.

The initial coordinates $\boldsymbol{\xi}$ of a particle will be referred to as the *material coordinates* of the particle and, when convenient, the particle itself may be called the particle $\boldsymbol{\xi}$. The terms *convected* and *Lagrangian* coordinates are also used. The former is a sensible term since the material coordinate system is convected with the fluid; the latter is both a misnomer* and, lacking descriptive quality, is often forgotten or confused by the student. The *spatial coordinates* $\mathbf{x}$ of the particle may be referred to as its *position* or *place*. It will be assumed that the motion is continuous, single valued and that Eq. (4.11.1) can be inverted to give the initial position or material coordinates of the particle which is at any position $\mathbf{x}$ at time t; that is,

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, t) \quad \text{or} \quad \xi_i = \xi_i(x_1, x_2, x_3, t) \tag{4.11.2}$$

are also continuous and single valued. Physically this means that a continuous arc of particles does not break up during the motion or that the particles in the neighborhood of a given particle continue in its neighborhood during the motion. The single valuedness of the equations means that a particle cannot split up and occupy two places nor can two distinct particles occupy the same place. Assumptions must also be made about the continuity of derivatives. It is usual (see, for example, Serrin, Handbuch der Physik Bd. VIII/1, p. 129) to assume continuity up to the third order derivatives. Exception to these requirements may be allowed on a finite number of singular surfaces, lines or points, as for example when a fluid divides around an obstacle. It is shown in Appendix B that a necessary and sufficient condition for the inverse functions to exist is that the Jacobian

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \tag{4.11.3}$$

should not vanish.

The transformation (4.11.1) may be looked at as the parametric equation of a curve in space with t as parameter. The curve goes through the point $\boldsymbol{\xi}$, corresponding to the parameter $t = 0$, and these curves are called the *particle paths*. Any property of the fluid may be followed along the particle path. For example, we might be given the density in the neighborhood of a particle as a function $\rho(\boldsymbol{\xi}, t)$, meaning that for any prescribed particle $\boldsymbol{\xi}$ we have the density as a function of time, that is, the density that an observer riding on the particle would see. (Position itself is a "property" in this general sense so that the equations of the particle path are of this form.) This *material*

* The term Eulerian is also applied to the spatial coordinates x. Truesdell, in a footnote of customary erudition (Kinematics of vorticity, p. 30), has traced the origin of this incorrect usage.

description of the change of some property, say $\mathscr{F}(\boldsymbol{\xi}, t)$, can be changed into a *spatial* description $\mathscr{F}(\mathbf{x}, t)$ by Eq. (4.11.2),

$$\mathscr{F}(\mathbf{x}, t) = \mathscr{F}[\boldsymbol{\xi}(\mathbf{x}, t), t]. \tag{4.11.4}$$

Physically this says that the value of the property at position $\mathbf{x}$ and time t is the value appropriate to the particle which is at $\mathbf{x}$ at time t. Conversely, the material description can be derived from the spatial one by Eqs. (4.11.1)

$$\mathscr{F}(\boldsymbol{\xi}, t) = \mathscr{F}[\mathbf{x}(\boldsymbol{\xi}, t), t], \tag{4.11.5}$$

meaning that the value as seen by the particle at time t is the value at the position it occupies at that time.

Associated with these two descriptions are two derivatives with respect to time. We shall denote them by

$$\frac{\partial}{\partial t} \equiv \left(\frac{\partial}{\partial t}\right)_x \equiv \text{derivative with respect to time keeping } \mathbf{x} \text{ constant}, \tag{4.11.6}$$

and

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_\xi \equiv \text{derivative with respect to time keeping } \boldsymbol{\xi} \text{ constant}. \tag{4.11.7}$$

Thus $\partial\mathscr{F}/\partial t$ is the rate of change of $\mathscr{F}$ as observed at a fixed point $\mathbf{x}$, whereas $d\mathscr{F}/dt$ is the rate of change as observed when moving with the particle. The latter we call the *material derivative.** In particular the material derivative of the position of a particle is its velocity. Thus, putting $\mathscr{F} = x_i$, we have

$$v_i = \frac{dx_i}{dt} = \frac{\partial}{\partial t} x_i(\xi_1, \xi_2, \xi_3, t) \tag{4.11.8}$$

or

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}.$$

This allows us to establish a connection between the two derivatives, for

$$\begin{aligned} \frac{d\mathscr{F}}{dt} &= \frac{\partial}{\partial t}\mathscr{F}(\boldsymbol{\xi}, t) = \frac{\partial}{\partial t}\mathscr{F}[\mathbf{x}(\boldsymbol{\xi}, t), t] \\ &= \frac{\partial\mathscr{F}}{\partial x_i}\left(\frac{\partial x_i}{\partial t}\right)_\xi + \left(\frac{\partial\mathscr{F}}{\partial t}\right)_x \\ &= v_i\frac{\partial\mathscr{F}}{\partial x_i} + \frac{\partial\mathscr{F}}{\partial t}. \end{aligned} \tag{4.11.9}$$

* It is also called the convected or convective derivative and often denoted by D/Dt.

It is sometimes convenient to write this as

$$\frac{d\mathscr{F}}{dt} = \frac{\partial\mathscr{F}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathscr{F}. \tag{4.11.10}$$

Exercise 4.11.1. It is not always necessary to use the initial position as material coordinate. Consider the equations for the particle paths in Gerstner waves

$$\begin{aligned} x_1 &= a + (e^{-bk}/k)\sin k(a + ct), \\ x_2 &= -b - (e^{-bk}/k)\cos k(a + ct), \\ x_3 &= \text{constant}. \end{aligned}$$

Relate the constants a and b to the initial position and show that the particle paths are circles. Find the velocity vector and show that $d\,|\mathbf{v}|\,/dt = 0$.

Exercise 4.11.2. Show that the Jacobian Eq. (4.11.3) is 1 for the Gerstner wave.

Exercise 4.11.3. Show that $f(\mathbf{x}, t) = 0$ is a surface of the same material particles if and only if

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} v_i = 0.$$

Exercise 4.11.4. If $f(\mathbf{x}, t)$ is not a material surface but moves with a speed $\mathbf{u}$ different from the stream speed $\mathbf{v}$, show that

$$(\mathbf{v} - \mathbf{u})\cdot\mathbf{n} = \frac{df/dt}{|\nabla f|}$$

where $\mathbf{n}$ is the normal to the surface.

4.12. Streamlines

From the material description $\mathbf{x}(\boldsymbol{\xi}, t)$ of the flow we have derived a vector field

$$\mathbf{v} = d\mathbf{x}/dt = \mathbf{v}(x_1, x_2, x_3, t). \tag{4.12.1}$$

The flow is called *steady* if the velocity components are independent of time. The trajectories of the velocity field are called *streamlines*; they are the solutions of the three simultaneous equations

$$\frac{d\mathbf{x}}{ds} = \mathbf{v} \quad \text{or} \quad \frac{dx_i}{ds} = v_i(x_1, x_2, x_3, t) \tag{4.12.2}$$

where s is a parameter along the streamline. This parameter s is not to be confused with the time, for in Eq. (4.12.2) t is held fixed while the equations are integrated, and the resulting curves are the streamlines *at the instant t*. These may vary from instant to instant and in general will not coincide with the particle paths.

To obtain the particle paths from the velocity field we have to follow the motion of each particle. This means we have to solve the differential equations

$$\frac{dx_i}{dt} = v_i(x_1, x_2, x_3, t)$$

subject to $x_i = \xi_i$ at $t = 0$.

If the functions v_i do not depend on t, then the parameter s along the streamlines may be taken to be t and clearly the streamlines and particle paths will coincide. Exercise 4.12.1 shows that streamlines and particle paths may coincide for an unsteady motion.

The acceleration or rate of change of velocity is defined as

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{4.12.3}$$

Notice that in steady flow this does not vanish but reduces to

$$\mathbf{a} = (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{4.12.4}$$

The higher rates of change are sometimes used and defined by repeated material differentiation; thus the $(n-1)^{\text{th}}$ acceleration* is

$$\mathbf{v}^{(n)} = d\mathbf{v}^{(n-1)}/dt. \tag{4.12.5}$$

If C is a closed curve in region of flow, the streamlines through every point of C generate a surface known as a *stream tube*. Let S be a surface with C as bounding curve, then

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dS$$

is known as the *strength of the stream tube* at its cross-section S.

Exercise 4.12.1. Show that the streamlines and particle paths coincide for the flow $v_i = x_i/(1+t)$.

* This is the notation of Rivlin and Ericksen, "Stress deformation relations for isotropic materials," J. Rat. Mech. and Anal. **4** (1955), p. 329.

Exercise 4.12.2. Show that if $v_i/|\mathbf{v}|$ is independent of t, then streamlines and particle paths will coincide.

Exercise 4.12.3. Find the streamlines and particle paths for

$$v_i = \frac{x_i}{1 + a_i t}$$

where a_i are positive constants. (There is no summation on i here.) Describe the paths and streamlines if $a_1 = 2a_2 = 1$, $a_3 = 0$.

4.13. Streaklines

The name streakline is applied to the curve traced out by a plume of smoke or dye which is continuously injected at a fixed point but does not diffuse. Thus at time t the streakline through a fixed point $\mathbf{y}$ is a curve going from $\mathbf{y}$ to $\mathbf{x}(\mathbf{y}, t)$, the position reached by the particle which was at $\mathbf{y}$ at time $t = 0$. A particle is on the streakline if it passed the fixed point $\mathbf{y}$ at some time between 0 and t. If this time were s, then the material coordinates of the particle would be given by Eq. (4.11.2) $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{y}, s)$. However, at time t this particle is at $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$ so that the equation of the streakline at time t is given by

$$\mathbf{x} = \mathbf{x}[\boldsymbol{\xi}(\mathbf{y}, s), t], \tag{4.13.1}$$

where the parameter s along it lies in the interval $0 \leqq s \leqq t$. If we regard the motion as having been proceeding for all time, then the origin of time is arbitrary and s can take negative values $-\infty < s \leqq t$.

These concepts may be illustrated by the simple plane flow

$$v_1 = x_1/(1+t), \qquad v_2 = x_2, \qquad v_3 = 0.$$

Here the streamlines at time t are the solutions of

$$\frac{dx_1}{ds} = \frac{x_1}{1+t}, \qquad \frac{dx_2}{ds} = x_2, \qquad \frac{dx_3}{ds} = 0. \tag{4.13.2}$$

Thus keeping t constant the streamline through a is

$$x_1 = a_1 e^{s/(1+t)}, \qquad x_2 = a_2 e^s, \qquad x_3 = a_3$$

which is a curve in the plane $x_3 = a_3$

$$\frac{x_2}{a_2} = \left(\frac{x_1}{a_1}\right)^{(1+t)} \tag{4.13.3}$$

The streamlines are shown for increasing t in Fig. 4.1a.

The particle paths are solutions of

$$\frac{dx_1}{dt} = \frac{x_1}{1+t}, \qquad \frac{dx_2}{dt} = x_2, \qquad \frac{dx_3}{dt} = 0. \tag{4.13.4}$$

These are $x_1 = \xi_1(1+t)$, $x_2 = \xi_2 e^t$, $x_3 = \xi_3$ or the curves in the plane $x_3 = \xi_3$

$$x_2 = \xi_2\, e^{(x_1 - \xi_1)/\xi_1} \tag{4.13.5}$$

They are shown for several initial positions in Fig. 4.1b.

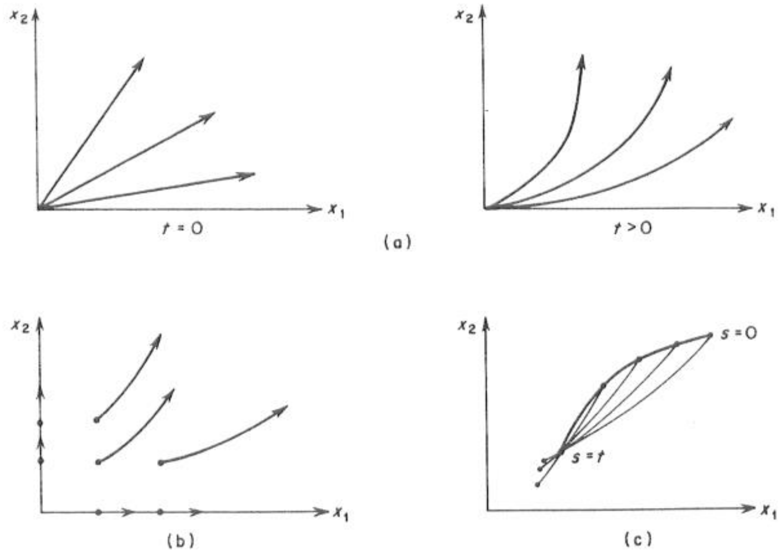

Fig. 4.1

For the inverse relations defining the particle at $\mathbf{y}$, at time s we have

$$\xi_1 = \frac{y_1}{1+s}, \qquad \xi_2 = y_2 e^{-s}, \qquad \xi_3 = y_3. \tag{4.13.6}$$

Hence, the streakline is given by

$$x_1 = y_1 \frac{1+t}{1+s}, \qquad x_2 = y_2 e^{t-s}, \qquad x_3 = y_3. \tag{4.13.7}$$

This, with some of the particle paths that contribute to it, is shown in Fig. 4.1c. (For other examples of streamlines, streaklines and particle paths see Truesdell and Toupin, Handbuch der Physik III/1, Berlin, Springer-Verlag, 1960, pages 331–336, where further references will be found.)

4.21. Dilatation

We have noticed in Section 3.16 that if the coordinate system is changed from coordinates $\boldsymbol{\xi}$ to coordinates $\mathbf{x}$, then the element of volume changes by the formula

$$dV = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}\, d\xi_1\, d\xi_2\, d\xi_3 = J\, dV_0. \tag{4.21.1}$$

If we think of $\boldsymbol{\xi}$ as the material coordinates, they are Cartesian coordinates at $t = 0$, so that $d\xi_1\, d\xi_2\, d\xi_3$ is the volume dV_0 of an elementary parallelepiped. Consider this elementary parallelepiped about a given point $\boldsymbol{\xi}$ at the initial instant. By the motion this parallelepiped is moved and distorted but because the motion is continuous it cannot break up and so at time t is some neighborhood of the point $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$. By Eq. (4.21.1), its volume is $dV = J\, dV_0$ and hence

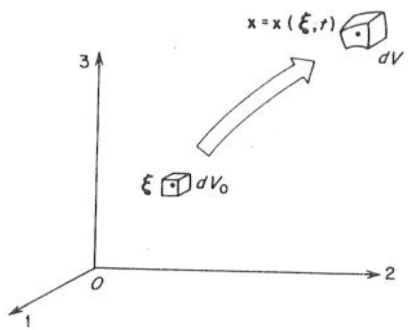

Fig. 4.2

$$J = \frac{dV}{dV_0} = \text{ratio of an elementary material volume to its initial volume.} \tag{4.21.2}$$

It is called the *dilatation* or *expansion*. The assumption that the Eq. (4.11.1) can be inverted to give Eq. (4.11.2), and vice versa, is equivalent to requiring that neither J nor J^{-1} vanish. Thus,

$$0 < J < \infty. \tag{4.21.3}$$

We can now ask how the dilatation changes as we follow the motion. To answer this we calculate the material derivative dJ/dt. However,

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_1}{\partial \xi_3} \\ \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_3} \\ \dfrac{\partial x_3}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_3} \end{vmatrix}. \tag{4.21.4}$$

Now

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial \xi_j}\right) = \frac{\partial}{\partial \xi_j}\frac{dx_i}{dt} = \frac{\partial v_i}{\partial \xi_j} \tag{4.21.4}$$

for d/dt is differentiation with $\boldsymbol{\xi}$ constant so that the order can be interchanged. Now if we regard v_i as a function of x_1, x_2, and x_3,

$$\frac{\partial v_i}{\partial \xi_j} = \frac{\partial v_i}{\partial x_1}\frac{\partial x_1}{\partial \xi_j} + \frac{\partial v_i}{\partial x_2}\frac{\partial x_2}{\partial \xi_j} + \frac{\partial v_i}{\partial x_3}\frac{\partial x_3}{\partial \xi_j} = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial \xi_j}. \tag{4.21.5}$$

In Appendix A it is shown that the derivative of the determinant is the sum of three terms in each of which only one row is differentiated. Thus for dJ/dt we should have the sum of three determinants of which the first would be

$$\begin{vmatrix} \dfrac{\partial v_1}{\partial \xi_1} & \dfrac{\partial v_1}{\partial \xi_2} & \dfrac{\partial v_1}{\partial \xi_3} \\ \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_3} \\ \dfrac{\partial x_3}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_3} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial v_1}{\partial x_k}\dfrac{\partial x_k}{\partial \xi_1} & \dfrac{\partial v_1}{\partial x_k}\dfrac{\partial x_k}{\partial \xi_2} & \dfrac{\partial v_1}{\partial x_k}\dfrac{\partial x_k}{\partial \xi_3} \\ \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_3} \\ \dfrac{\partial x_3}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_3} \end{vmatrix}.$$

In expanding this determinant by the first row we see that only the first term ($k = 1$) of the elements in the first row survive, for with $k = 2$ or 3 the coefficient of $\partial v_1/\partial x_k$ is a determinant with two rows the same and so vanishes. The value of this determinant is thus $(\partial v_1/\partial x_1)J$. Considering also the other two terms in the differentiation we see that

$$\frac{dJ}{dt} = \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)J$$

or

$$d(\ln J)/dt = \operatorname{div} \mathbf{v}. \tag{4.21.5}$$

We thus have an important physical meaning for the divergence of the velocity field. It is the relative rate of change of the dilatation following a particle path. It is evident, that for an incompressible fluid motion,

$$\nabla \cdot \mathbf{v} = 0. \tag{4.21.6}$$

Exercise 4.21.1. Calculate J for (a) Gerstner waves and (b) the flow of Ex. 4.12.3 and confirm Eq. (4.21.5).

Exercise 4.21.2. Show that in steady flow $d^2J/dt^2 = J\nabla \cdot \{(\nabla \cdot \mathbf{v})\mathbf{v}\}$ and find a similar expression for higher derivatives.

4.22. Reynolds' transport theorem

An important kinematical theorem can be derived from the so-called Euler expansion formula, Eq. (4.21.5). It is due to Reynolds and concerns the rate of change not of an infinitesimal element of volume but of any volume

integral. Let $\mathscr{F}(\mathbf{x}, t)$ be any function and $V(t)$ be a closed volume moving with the fluid, that is, consisting of the same fluid particles. Then

$$F(t) = \iiint_{V(t)} \mathscr{F}(\mathbf{x}, t)\, dV \tag{4.22.1}$$

is a function of t that can be calculated. We are interested in its material derivative dF/dt. Now the integral is over the varying volume $V(t)$ so we cannot take the differentiation through the integral sign. If, however, the integration were with respect to a volume in $\boldsymbol{\xi}$-space it would be possible to interchange differentiation and integration since d/dt is differentiation with respect to time keeping $\boldsymbol{\xi}$ constant. However, the transformation $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$, $dV = J\, dV_0$ allows us to do just this, for $V(t)$ has been defined as a moving material volume and so has come from some fixed volume V_0 at time $t = 0$. Thus

$$\begin{aligned} \frac{d}{dt}\iiint_{V(t)} \mathscr{F}(\mathbf{x}, t)\, dV &= \frac{d}{dt}\iiint_{V_0} \mathscr{F}[\mathbf{x}(\boldsymbol{\xi}, t), t] J\, dV_0 \\ &= \iiint_{V_0} \left(\frac{d\mathscr{F}}{dt} J + \mathscr{F}\frac{dJ}{dt}\right) dV_0 \\ &= \iiint_{V_0} \left\{\frac{d\mathscr{F}}{dt} + \mathscr{F}(\nabla \cdot \mathbf{v})\right\} J\, dV_0 \\ &= \iiint_{V(t)} \left\{\frac{d\mathscr{F}}{dt} + \mathscr{F}(\nabla \cdot \mathbf{v})\right\} dV. \end{aligned} \tag{4.22.2}$$

Since $d/dt = (\partial/\partial t) + \mathbf{v} \cdot \nabla$ we can throw this formula into a number of different shapes. Substituting for the material derivative and collecting the gradient terms gives

$$\frac{d}{dt}\iiint_{V(t)} \mathscr{F}(\mathbf{x}, t)\, dV = \iiint_{V(t)} \left\{\frac{\partial \mathscr{F}}{\partial t} + \nabla \cdot (\mathscr{F}\mathbf{v})\right\} dV \tag{4.22.3}$$

Now applying Green's theorem to the second integral we have

$$\frac{d}{dt}\iiint_{V(t)} \mathscr{F}(\mathbf{x}, t)\, dV = \iiint_{V(t)} \frac{\partial \mathscr{F}}{\partial t}\, dV + \iint_{S(t)} \mathscr{F}\mathbf{v} \cdot \mathbf{n}\, dS, \tag{4.22.4}$$

where $S(t)$ is the surface of $V(t)$. This admits of an immediate physical picture for it says that the rate of change of the integral of $\mathscr{F}$ within the moving volume is the integral of the rate of change at a point plus the net flow of $\mathscr{F}$ over the surface. $\mathscr{F}$ can be any scalar or tensor component, so that this is a kinematical result of wide application.

Exercise 4.22.1. Show that if $\nabla \cdot \mathbf{v} = 0$ the motion proceeds without change of volume. (Such a motion is called *isochoric*. An incompressible fluid cannot but be in isochoric motion, but isochoric motions of a compressible fluid are also possible.)

Exercise 4.22.2. Show that in isochoric motion the strength of any stream tube is constant.

Exercise 4.22.3. Show that the moment

$$\iiint_V \{\mathbf{v} \cdot \mathbf{x}^{(n)}\}\, dV = \iint_{S_2} (\mathbf{v} \cdot \mathbf{n})\mathbf{x}_2^{(n+1)}\, dS - \iint_{S_1} (\mathbf{v} \cdot \mathbf{n})\mathbf{x}_1^{(n+1)}\, dS.$$

V is the volume between any two sections S_1 and S_2 of a stream tube and the motion is isochoric. (See Ex. 3.23.4 for notation and hint.) What is the value of the moment if the stream tube should close on itself?

Exercise 4.22.4. V is composed of two volumes V_1 and V_2 divided by an internal surface Σ and S is the external surface of V. V is a material volume but as Σ moves with arbitrary velocity $\mathbf{u}$ and across it $\mathscr{F}$ suffers a discontinuity, $\mathscr{F}_1$ and $\mathscr{F}_2$ being its values on either side. If $\boldsymbol{\nu}$ is the normal to Σ in the direction from V_1 to V_2 show that Eq. (4.22.4) may be generalised to

$$\frac{d}{dt}\iiint_V \mathscr{F}\, dV = \iiint_V \frac{\partial \mathscr{F}}{\partial t}\, dV + \iint_S \mathscr{F}\mathbf{v} \cdot \mathbf{n}\, dS + \iint_\Sigma (\mathscr{F}_1 - \mathscr{F}_2)\mathbf{u} \cdot \boldsymbol{\nu}\, dS.$$

(Truesdell and Toupin)

Exercise 4.22.5. The surface element dS with normal $\mathbf{n}$ corresponding to two displacements $d\mathbf{x}$ and $d\mathbf{y}$ is given by $\mathbf{n}\, dS = d\mathbf{x} \wedge d\mathbf{y}$. By making these displacements correspond to differences in material coordinates $d\boldsymbol{\xi}$ and $d\boldsymbol{\eta}$ that define an area element $\boldsymbol{\nu}\, d\sigma$, show that

$$n_i \frac{\partial x_i}{\partial \xi_p}\, dS = J\nu_p\, d\sigma.$$

By differentiating this materially with respect to time, show that

$$\frac{d}{dt}(n_i\, dS) = \frac{\partial v_j}{\partial x_j} n_i\, dS - \frac{\partial v_j}{\partial x_i} n_j\, dS.$$

Exercise 4.22.6. Obtain a transport theorem for surface integrals in the form

$$\frac{d}{dt}\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = \iint_S \left[\frac{d\mathbf{a}}{dt} + \mathbf{a}(\nabla \cdot \mathbf{v}) - (\mathbf{a} \cdot \nabla)\mathbf{v}\right] \cdot \mathbf{n}\, dS,$$

and reconcile this with Ex. 3.32.5.

Exercise 4.22.7. Show that if

$$\frac{\partial \mathbf{a}}{\partial t} + \mathbf{v}(\nabla \cdot \mathbf{a}) + \nabla \wedge (\mathbf{a} \wedge \mathbf{v}) = 0,$$

the strength of any vector tube in the field $\mathbf{a}$ at a material cross-section remains constant.

4.3. Conservation of mass and the equation of continuity

Although the idea of mass is not a kinematical one it is convenient to introduce it here and to obtain the continuity equation. Let $\rho(\mathbf{x}, t)$ be the mass per unit volume of a homogeneous fluid at position $\mathbf{x}$ and time t. Then the mass of any finite volume V is

$$m = \iiint_V \rho(\mathbf{x}, t)\, dV. \tag{4.3.1}$$

If V is a material volume, that is, if it is composed of the same particles, and there are no sources or sinks in the medium we take it as a principle that the mass does not change. By inserting $\mathscr{F} = \rho$ in Eq. (4.22.2) we have

$$\frac{dm}{dt} = \iiint_V \left\{\frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v})\right\} dV = 0. \tag{4.3.2}$$

Now this is true for an arbitrary volume and hence the integrand itself must vanish everywhere. Suppose it did not vanish at some point P, but were positive there. Then since the integrand is continuous it would have to be positive for some neighborhood of P and we might take V to be entirely within this neighborhood, and for this V the integral would not vanish. It follows that

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{4.3.3}$$

which is the equation of continuity.

Combining the equation of continuity with Reynolds' transport theorem for a function $\mathscr{F} = \rho F$ we have

$$\begin{aligned} \frac{d}{dt}\iiint_{V(t)} \rho F\, dV &= \iiint_{V(t)} \left\{\frac{d}{dt}(\rho F) + \rho F(\nabla \cdot \mathbf{v})\right\} dV \\ &= \iiint_{V(t)} \left\{\rho \frac{dF}{dt} + F\left(\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v}\right)\right\} dV \\ &= \iiint_{V(t)} \rho \frac{dF}{dt}\, dV \end{aligned} \tag{4.3.4}$$

since the second term vanishes by (4.3.3).

A fluid for which the density ρ is constant is called *incompressible*. In this case the equation of continuity becomes

$$\nabla \cdot \mathbf{v} = 0 \tag{4.3.5}$$

and the motion is isochoric or the velocity field solenoidal.

Exercise 4.3.1. Show that if $\rho_0(\boldsymbol{\xi})$ is the distribution of density of the fluid at time $t = 0$ and $\nabla(\nabla \cdot \mathbf{v}) = 0$, then the distribution at time t is

$$\rho(\mathbf{x}, t) = \rho_0[\boldsymbol{\xi}(\mathbf{x}, t)] \exp -\int_0^t (\nabla \cdot \mathbf{v})\, dt.$$

Exercise 4.3.2. Show that, for the motion of Ex. 4.12.3,

$$\rho x_1 x_2 x_3 = \rho_0 \xi_1 \xi_2 \xi_3.$$

4.41. Deformation and rate of strain

Consider two nearby points P and Q with material coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + d\boldsymbol{\xi}$. At time t they are to be found at $\mathbf{x}(\boldsymbol{\xi}, t)$ and $\mathbf{x}(\boldsymbol{\xi} + d\boldsymbol{\xi}, t)$. Now

$$x_i(\boldsymbol{\xi} + d\boldsymbol{\xi}, t) = x_i(\boldsymbol{\xi}, t) + \frac{\partial x_i}{\partial \xi_j}\, d\xi_j + 0(d^2), \tag{4.41.1}$$

where $0(d^2)$ represents terms of order $d\boldsymbol{\xi}^2$ and higher which will be neglected from this point onwards. Thus the small displacement vector $d\boldsymbol{\xi}$ has now become

$$d\mathbf{x} = \mathbf{x}(\boldsymbol{\xi} + d\boldsymbol{\xi}, t) - \mathbf{x}(\boldsymbol{\xi}, t)$$

where

$$dx_i = \frac{\partial x_i}{\partial \xi_j}\, d\xi_j. \tag{4.41.2}$$

It is clear from the quotient rule (since $d\boldsymbol{\xi}$ is arbitrary) that the nine quantities $\partial x_i/\partial \xi_j$ are the components of a tensor. It may be called the displacement gradient tensor and is basic to the theory of elasticity. For fluid motion, its material derivative is of more direct application and we will concentrate on this.

If $\mathbf{v} = d\mathbf{x}/dt$ is the velocity, the relative velocity of two particles $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + d\boldsymbol{\xi}$ has components

$$dv_i = \frac{\partial v_i}{\partial \xi_k}\, d\xi_k = \frac{d}{dt}\left(\frac{\partial x_i}{\partial \xi_j}\right) d\xi_j. \tag{4.41.3}$$

However, by inverting the relation of Eq. (4.41.2), we have

$$dv_i = \frac{\partial v_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j}\, dx_j = \frac{\partial v_i}{\partial x_j}\, dx_j \tag{4.41.4}$$

expressing the relative velocity in terms of current relative position. Again it is evident that the $(\partial v_i/\partial x_j)$ are components of a tensor, the *velocity gradient tensor*, for which we need to obtain a sound physical feeling.

We first observe that if the motion is a rigid body translation with velocity $\mathbf{u}$,

$$\mathbf{x} = \boldsymbol{\xi} + \mathbf{u}t \tag{4.41.5}$$

and the velocity gradient tensor vanishes identically. Secondly, the velocity gradient tensor can be written as the sum of symmetric and antisymmetric parts,

$$\begin{aligned}\frac{\partial v_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)\\ &= e_{ij} + \Omega_{ij}.\end{aligned} \tag{4.41.5}$$

Now we have seen (Section 2.45) that a relative velocity dv_i related to the relative position dx_j by an antisymmetric tensor Ω_{ij}: therefore $dv_i = \Omega_{ij}\, dx_j$, represents a rigid body rotation with angular velocity $\boldsymbol{\omega} = -\text{vec}\, \boldsymbol{\Omega}$. In this case

$$\omega_i = -\frac{1}{2}\epsilon_{ijk}\Omega_{jk} = \frac{1}{2}\epsilon_{ijk}\frac{\partial v_k}{\partial x_j} \tag{4.41.6}$$

or

$$\boldsymbol{\omega} = \tfrac{1}{2}\,\text{curl}\, \mathbf{v}.$$

(Cf. Ex. 3.24.8.) Thus the antisymmetric part of the velocity gradient tensor corresponds to a rigid body rotation, and, if the motion is a rigid one (composed of a translation plus a rotation), the symmetric part of the velocity gradient tensor will vanish. For this reason the tensor e_{ij} is called the *deformation* or *rate of strain* tensor and its vanishing is necessary and sufficient for the motion to be without deformation, that is, rigid.

Exercise 4.41.1. If $e_{ij} \equiv 0$ show that

$$v_i = \epsilon_{ijk}\omega_j x_k + u_i$$

where ω_i and u_i are constants.

4.42. Physical interpretation of the deformation tensor

To interpret the tensor e_{ij} we shall see how a small element is changing during the motion. The length of the line segment from P to Q is ds, where

$$ds^2 = dx_i\, dx_i = \frac{\partial x_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k}\, d\xi_j\, d\xi_k. \tag{4.42.1}$$

Now P and Q are the material particles $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + d\boldsymbol{\xi}$ so that $d\xi_j$ and $d\xi_k$ do not change during the motion. Thus

$$\frac{d}{dt}(ds^2) = \left(\frac{\partial v_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k} + \frac{\partial x_i}{\partial \xi_j}\frac{\partial v_i}{\partial \xi_k}\right) d\xi_j\, d\xi_k = 2\frac{\partial v_i}{\partial \xi_j}\frac{\partial x_i}{\partial \xi_k} d\xi_j\, d\xi_k$$

by symmetry. However,

$$\frac{\partial v_i}{\partial \xi_j} d\xi_j = \frac{\partial v_i}{\partial x_j} dx_j \quad \text{and} \quad \frac{\partial x_i}{\partial \xi_k} d\xi_k = dx_i$$

Thus

$$\frac{1}{2}\frac{d}{dt}(ds^2) = (ds)\frac{d}{dt}(ds) = \frac{\partial v_i}{\partial x_j} dx_j\, dx_i = e_{ij}\, dx_i\, dx_j \qquad (4.42.2)$$

by symmetry, or

$$\frac{1}{(ds)}\frac{d}{dt}(ds) = e_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds}. \qquad (4.42.3)$$

Now dx_i/ds is the i^{th} component of a unit vector in the direction of the segment PQ, so that this equation says that the rate of change of the length of the segment as a fraction of its length is related to its direction through the deformation tensor.

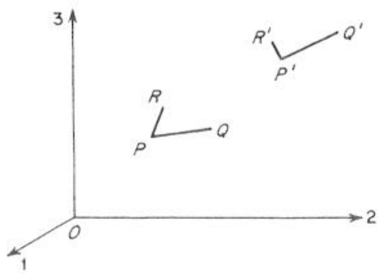

Fig. 4.3

In particular, if PQ is parallel to the coordinate axis $O1$ we have $d\mathbf{x}/ds = \mathbf{e}_{(1)}$ and

$$\frac{1}{dx_1}\frac{d}{dt}(dx_1) = e_{11}. \qquad (4.42.4)$$

Thus e_{11} is the rate of longitudinal strain of an element parallel to the $O1$ axis. Similar interpretations apply to e_{22} and e_{33}.

Again consider two segments PQ and PR, where R is the particle $\boldsymbol{\xi} + d\boldsymbol{\xi}'$. If θ is the angle between them and ds' is the length of PR,

$$ds\, ds' \cos\theta = dx_i\, dx_i'.$$

Differentiating with respect to time we have

$$\frac{d}{dt}[ds\, ds' \cos\theta] = dv_i\, dx_i' + dx_i\, dv_i'$$

$$= \frac{\partial v_i}{\partial x_j} dx_j\, dx_i' + dx_i \frac{\partial v_i}{\partial x_j} dx_j'$$

since $dv_i' = (\partial v_i/\partial x_j)\, dx_j'$. The i and j are dummy suffixes so we may interchange them in the first term on the right, then performing the differentiation we have

$$\cos\theta\left\{\frac{1}{(ds)}\frac{d}{dt}(ds) + \frac{1}{(ds')}\frac{d}{dt}(ds')\right\} - \sin\theta\frac{d\theta}{dt}$$

$$= \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\frac{dx_j}{ds}\frac{dx_i'}{ds'} = 2e_{ij}\frac{dx_i'}{ds'}\frac{dx_j}{ds}$$

Now suppose that dx' is parallel to the axis $O1$ and dx to the axis $O2$, so that $(dx_i'/ds') = \delta_{i1}$ and $(dx_j/ds) = \delta_{j2}$ and $\theta_{12} = \pi/2$. Then

$$-\frac{d\theta_{12}}{dt} = 2e_{12}. \qquad (4.21.5)$$

Thus e_{12} is to be interpreted as one-half the rate of decrease of the angle between two segments originally parallel to the $O1$ and $O2$ axes respectively. Similar interpretations are appropriate to e_{23} and e_{31}.

The fact that the deformation tensor is linear in the velocity field has an important consequence. Since we may superimpose two velocity fields to form a third, it follows that the deformation tensor of this is the sum of the deformation tensors of the fields from which it is composed. Thus a flow with $v_1 = \lambda_1 x_1$, $v_2 = v_3 = 0$ would have only one nonvanishing component of the rate of strain tensor, $e_{11} = \lambda_1$. This represents a *pure stretching* in the $O1$ direction with no deformation of an element perpendicular. Again, if $v_i = \lambda_i x_i$ (no summation on i), we have a deformation which is the superposition of three stretchings parallel to the three axes. However, if $v_1 = f(x_2)$, $v_2 = v_3 = 0$ so that the only nonzero component of the deformation tensor is $e_{12} = \frac{1}{2}f'(x_2)$, the motion is one of *pure shear* in which elements parallel to the coordinate axes are not stretched at all. Note however that in pure stretching an element not parallel or perpendicular to the direction of stretching will suffer rotation. Likewise in pure shear an element not normal to or in the plane of shear will suffer stretching.

Exercise 4.42.1. Follow through the ideas of this section for the plane stagnation flow $v_1 = x_1$, $v_2 = -x_2$, $v_3 = 0$. Show that if θ is the angle between an infinitesimal material segment and the axis $O1$, then the rate of change of log tan θ is constant along a particle path.

Exercise 4.42.2. Find an expression for the rate of change of the angle between a material line segment and a fixed direction and analyse it.

4.43. Principal axes of deformation

The quadratic form (4.42.3) may be written

$$\frac{d}{dt}\ln(ds) = e_{ij}l_il_j \tag{4.43.1}$$

where $\mathbf{l}$ is a unit vector in the direction PQ. From our knowledge of symmetric second order tensors we know that there are three mutually perpendicular directions along which this expression has stationary values (see Appendix A.12). Moreover we know from Section 2.5 and Appendix A.11 that we can find a rotation of coordinates to a frame $O\bar{1}\bar{2}\bar{3}$ such that the component $\bar{e}_{ij}$ in this frame of reference are zero if $i \neq j$. If $d_{(1)}, d_{(2)}, d_{(3)}$ are the values of $\bar{e}_{11}$, $\bar{e}_{22}$, and $\bar{e}_{33}$, they are roots of the cubic

$$\det(e_{ij} - d\delta_{ij}) = \Psi - d\Phi + d^2\Theta - d^3 = 0 \tag{4.43.2}$$

(cf. Eq. 2.5.4). The three directions $O\bar{1}$, $O\bar{2}$, $O\bar{3}$ are called the *principal axes of stretching* or rate of strain and $d_{(1)}, d_{(2)}, d_{(3)}$ the *principal rates of strain.* The three scalars Θ, Φ, Ψ are the *invariants of the deformation tensor* and the first of them we have already encountered as the dilatation. In fact

$$\Theta = e_{11} + e_{22} + e_{33} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

$$\Phi = e_{22}e_{33} - e_{23}e_{32} + e_{33}e_{11} - e_{31}e_{13} + e_{11}e_{22} - e_{12}e_{21}$$

$$\Psi = \det e_{ij} = \epsilon_{ijk}e_{1i}e_{2j}e_{3k}.$$

We know that Θ is to be interpreted as the fractional rate of change of an infinitesimal volume. Dishington (Physics of Fluids, 3, 1960, p. 482) has given a physical interpretation to the other invariants. He finds that

$$\lim_{V\to 0}\frac{1}{V}\frac{d^2V}{dt^2} = (\nabla\cdot\mathbf{a})_0 + 2\Phi$$

where $(\nabla\cdot\mathbf{a})_0$ is the divergence of the acceleration as measured by an observer moving and rotating with the element.* Thus 2Φ is the part contributed to the fractional acceleration of volume by the steady velocity of the surface. In a similar way 6Ψ is found to be the contribution to the limit of $(d^3V/dt^3)/V$.

A picture of the deformation may be formed by considering what happens to a small sphere of radius dr during a short interval of time ds. Suppose that we have chosen the coordinate system so that the axes are parallel to the principal axes of stretching at the point $\mathbf{x}$. The particles on a sphere of

* This interpretation is also implicit in the formulae given by Truesdell for $\nabla\cdot\mathbf{a}$ (Kinematics of Vorticity, p. 79).

radius dr and center $\mathbf{x}$ are $\mathbf{x} + \mathbf{l}\,dr$, where $\mathbf{l}$ is a unit vector and they have material coordinates $\boldsymbol{\xi} + d\boldsymbol{\xi}$ where

$$l_i\,dr = \left(\frac{\partial x_i}{\partial \xi_j}\right)_t d\xi_j. \tag{4.43.3}$$

In the interval from t to $t + dt$ the center moves from $\mathbf{x}(\boldsymbol{\xi}, t)$ to $\mathbf{x}(\boldsymbol{\xi}, t + dt)$. If dy is the position of a particle that was on the surface of the sphere relative to the new position of the center,

$$\begin{aligned} dy_i &= x_i(\boldsymbol{\xi} + d\boldsymbol{\xi}, t + dt) - x_i(\boldsymbol{\xi}, t + dt) \\ &= \left(\frac{\partial x_i}{\partial \xi_k}\right)_{t+dt} d\xi_k \end{aligned} \tag{4.43.4}$$

However, here $\partial x_i/\partial\xi_k$ is evaluated at $\boldsymbol{\xi}$ and $t + dt$, that is,

$$\begin{aligned} \left(\frac{\partial x_i}{\partial \xi_k}\right)_{t+dt} &= \left(\frac{\partial x_i}{\partial \xi_k}\right)_t + dt\,\frac{d}{dt}\left(\frac{\partial x_i}{\partial \xi_k}\right) \\ &= \left(\frac{\partial x_i}{\partial \xi_k}\right)_t + \left(\frac{\partial v_i}{\partial \xi_k}\right) dt. \end{aligned}$$

Substituting this value back into Eq. (4.43.4) and using the relation (4.43.3) in the form

$$d\xi_k = \left(\frac{\partial \xi_k}{\partial x_i}\right)_t l_i\,dr,$$

we have

$$\begin{aligned} dy_i &= l_i\,dr + \frac{\partial v_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j} l_j\,dr\,dt \\ &= \left(\delta_{ij} + \frac{\partial v_i}{\partial x_j}dt\right) l_j\,dr = A_{ij}l_j\,dr \end{aligned} \tag{4.43.5}$$

Now since the coordinate axes were chosen parallel to the principal axes of deformation $e_{ij} = 0$ for $i \neq j$,

$$A_{ii} = 1 + e_{ii}\,dt, \qquad i = j \qquad A_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = \Omega_{ij}, \quad i \neq j.$$

The off-diagonal terms thus represent the rigid body rotation as before and the remaining terms are purely diagonal giving, in the absence of rotation,

$$dy_i = (1 + e_{ii}\,dt)l_i\,dr \qquad \text{(no summation).}$$

Since $\mathbf{l}$ is a unit vector,

$$1 = l_il_i = \frac{dy_i\,dy_i}{(1 + e_{ii}\,dt)^2\,dr^2},$$

and this is an infinitesimal ellipsoid whose axes are coincident with the principal axes of stretching and of lengths $(1 + e_{ii}\,dt)\,dr$, $i = 1, 2, 3$. Thus in the complete deformation a small sphere is distorted into an ellipsoid and rotated, as shown in Fig. 4.4.

This insight into the character of deformation is expressed by the so-called *Cauchy-Stokes decomposition theorem*, which Truesdell formulates as follows: *an arbitrary instantaneous state of motion may be resolved at each*

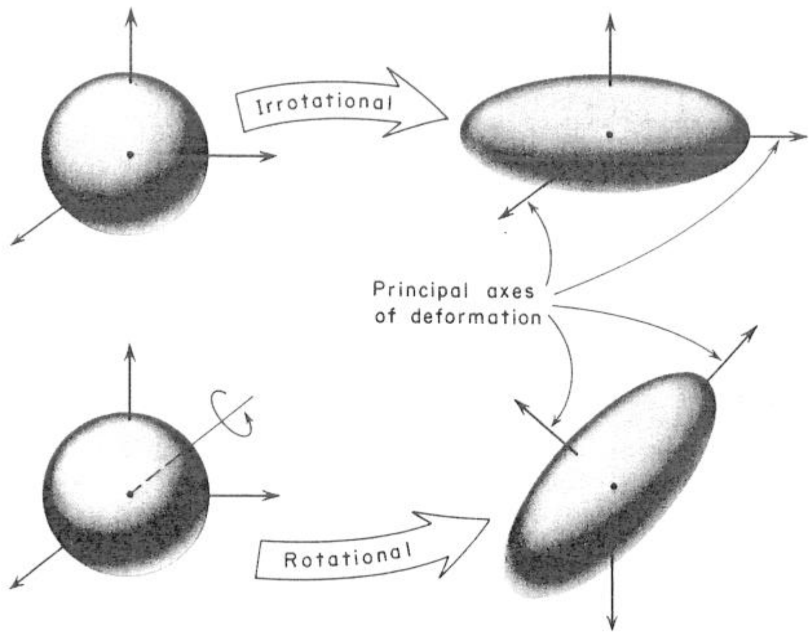

Fig. 4.4

point into a uniform translation, a dilatation along three mutually perpendicular axes, and a rigid rotation of these axes.

Exercise 4.43.1. Show that $e_{ij}e_{ij} = \Theta^2 - 2\Phi$ and deduce that $\Phi \leqq 0$ for an isochoric motion.

Exercise 4.43.2. The stretching is called spherical if all the principal rates of strain are equal. Show that in this case

$$\Psi^2 = (\tfrac{1}{3}\Phi)^3 = (\tfrac{1}{3}\Theta)^6.$$

Exercise 4.43.3. If $v_1 = f(r)(x_2/r)$, $v_2 = -f(r)(x_1/r)$, $v_3 = 0$, $r^2 = x_1^2 + x_2^2$, the motion is a steady one with circular streamlines. Show that the deformation tensor is

$$e_{11} = -e_{22} = F \sin 2\theta \qquad e_{12} = -F \cos 2\theta$$

where $\tan\theta = x_2/x_1$ and $F = \frac{1}{2}\{f'(r) - f(r)/r\}$. Show that the principal rates of strain are equal and opposite and find the principal axes.

4.5. Vorticity, vortex lines, and tubes

We have seen that the antisymmetric part of the rate of strain tensor represents the local rotation, in fact $\operatorname{vec} \mathbf{\Omega} = \frac{1}{2} \operatorname{curl} \mathbf{v}$. The curl of the velocity is known as the *vorticity*,

$$\mathbf{w} = \nabla \wedge \mathbf{v}. \tag{4.5.1}$$

For an irrotational flow the vorticity vanishes everywhere. The trajectories of the vortex field are called *vortex lines* and the surface generated by the vortex lines through a closed curve C is a *vortex tube*. Since the strength of a vector tube of the field $\mathbf{a}$ at its section S has been defined as

$$\iint_S \mathbf{a} \cdot \mathbf{n}\, dS$$

we see that the strength of a vortex tube equals the circulation round the closed curve C which bounds the cross-section S, for

$$\iint_S (\nabla \wedge \mathbf{v}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{v} \cdot \mathbf{t}\, ds \tag{4.5.2}$$

by Stokes' theorem.

The kinematics of vorticity has been elegantly and extensively treated by Truesdell in a monograph of that title (Indiana University Press, 1954) and in the appropriate sections of Bd. III/1 of the Handbuch der Physik. Here we have no intention of covering or even attempting to survey the full scope of the theory; it will be sufficient if the reader's buoyancy is increased enough for him to enjoy swimming in these waters. Truesdell collects four interpretations of the vorticity some of which we have already encountered. First, since the curl can be defined in terms of the circulation around an infinitesimal curve, the component of vorticity in a given direction is the circulation around a small circuit in a plane normal to that direction. Second, for a rigid body rotation the angular velocity is $\frac{1}{2}$ curl $\mathbf{v}$. Now the principal axes are unchanged by the deformation part of the rate of strain tensor and hence $\frac{1}{2}\mathbf{w}$ is the angular velocity of the principal axes at a point with respect to a fixed coordinate system. Third, it may be shown that $\frac{1}{2}w_1$ is the mean value of the angular velocity of two line segments through the point parallel to the $O2$ and $O3$ axes. The fourth interpretation is related to the last and identifies $\frac{1}{2}w_1$ as the mean value of all the rates of rotation about an axis parallel to $O1$ of line segments in a plane normal to $O1$. The last two interpretations can of course be generalized to relate $\frac{1}{2}\mathbf{w} \cdot \mathbf{n}$ to the mean rates of rotation of segments in a plane with normal $\mathbf{n}$.

We observe that the strength of a vortex tube at any cross-section is the same, for $\mathbf{w}$ is a solenoidal vector. One characterisation of the solenoidal field is the vanishing of $\iint \mathbf{w}\cdot\mathbf{n}\,dS$ over any closed surface. Take this closed surface to be a stream tube with sections S_1 and S_2 whose boundary curves are C_1 and C_2. If $\mathbf{n}$ is the outward normal, it will be the positive normal (in the sense of being right-handed for a given circuit of C) for one of S_1 and S_2 and negative for the other. Since $\mathbf{w}\cdot\mathbf{n}$ vanishes identically over the surface of the vortex tube,

$$\iint_{S_1} \mathbf{w}\cdot\mathbf{n}\,dS = \iint_{S_2} \mathbf{w}\cdot(-\mathbf{n})\,dS \tag{4.5.3}$$

and the strength is constant.

Taking the curl of the formula (Ex. 4.5.1) for the acceleration

$$\begin{aligned}\nabla\wedge\mathbf{a} &= \partial\mathbf{w}/\partial t + \nabla\wedge(\mathbf{w}\wedge\mathbf{v}) \\ &= d\mathbf{w}/dt - (\mathbf{w}\cdot\nabla)\mathbf{v} + \mathbf{w}(\nabla\cdot\mathbf{v}),\end{aligned} \tag{4.5.4}$$

where we have made use of Ex. 3.24.4. Thus, using the continuity equation,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\mathbf{w}}{\rho}\right) &= \frac{1}{\rho}\frac{d\mathbf{w}}{dt} - \frac{\mathbf{w}}{\rho^2}\frac{d\rho}{dt} \\ &= \frac{1}{\rho}\{\nabla\wedge\mathbf{a} + (\mathbf{w}\cdot\nabla)\mathbf{v} - \mathbf{w}(\nabla\cdot\mathbf{v})\} + \frac{\mathbf{w}}{\rho}(\nabla\cdot\mathbf{v}) \\ &= \left(\frac{\mathbf{w}}{\rho}\cdot\nabla\right)\mathbf{v} + \frac{1}{\rho}\nabla\wedge\mathbf{a}.\end{aligned} \tag{4.5.5}$$

If the acceleration is irrotational this equation may be solved. Setting $w_j = \rho c_i(\partial x_j/\partial\xi_i)$, where the c_i are components of a new vector $\mathbf{c}$, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{w_j}{\rho}\right) &= \frac{dc_i}{dt}\frac{\partial x_j}{\partial\xi_i} + c_i\frac{\partial v_j}{\partial\xi_i} \\ &= c_i\frac{\partial x_k}{\partial\xi_i}\frac{\partial v_j}{\partial x_k} = c_i\frac{\partial v_j}{\partial\xi_i}.\end{aligned}$$

Since J, the determinant of the coefficients of the dc_i/dt, is not zero, this gives

$$\frac{dc_i}{dt} = 0$$

or $\mathbf{c} = \mathbf{c}(\xi_1, \xi_2, \xi_3)$, and

$$\mathbf{w} = \rho\left[c_i(\boldsymbol{\xi})\frac{\partial}{\partial\xi_i}\right]\mathbf{x}. \tag{4.5.6}$$

If $\mathbf{w}_0$ and ρ_0 are the initial values of $\mathbf{w}$ and ρ of a particle,

$$(\mathbf{w}_0)_i = \rho_0 c_i(\boldsymbol{\xi})$$

and hence

$$\frac{\mathbf{w}}{\rho} = \left[\frac{(\mathbf{w}_0)_i}{\rho_0}\frac{\partial}{\partial\xi_i}\right]\mathbf{x}. \tag{4.5.7}$$

If initially an element $d\boldsymbol{\xi}$ is in a vortex line, $d\boldsymbol{\xi} = \mathbf{w}_0\,d\sigma$. However, under the motion, this element becomes

$$d\mathbf{x} = d\xi_i\frac{\partial\mathbf{x}}{\partial\xi_i} = (\mathbf{w}_0)_i\frac{\partial\mathbf{x}}{\partial\xi_i}\,d\sigma = \frac{\rho_0}{\rho}\mathbf{w}\,d\sigma;$$

in other words a material element tangent to a vortex line remains tangent to it. It follows that, *if* $\nabla\wedge\mathbf{a} = 0$ *then vortex lines are material lines.* It can be shown that this is also true under the broader condition $\mathbf{w}\wedge(\nabla\wedge\mathbf{a}) = 0$.

We shall return to this subject in Chapter 6, but must now pass on to some dynamical considerations.

Exercise 4.5.1. Show that the acceleration $\mathbf{a}$ is given by

$$\mathbf{a} = \partial\mathbf{v}/\partial t + \nabla(\tfrac{1}{2}|\mathbf{v}|^2) + \mathbf{w}\wedge\mathbf{v}.$$

Exercise 4.5.2. Establish the third interpretation given above.

Exercise 4.5.3. Show that the abnormality of the velocity field is the ratio of the component of vorticity in the direction of motion to the speed. (Abnormality is defined in Section 3.45) (Truesdell).

Exercise 4.5.4. Show that $v_{i,j}v_{j,i} = e_{ij}e_{ij} - \frac{1}{2}w_iw_i$ and hence that

$$\nabla\cdot\mathbf{a} = d\Theta/dt + \Theta^2 - 2\Phi - \tfrac{1}{2}|\mathbf{w}|^2 \qquad \text{(Truesdell)}$$

Exercise 4.5.5. Show that the strength of a vortex tube remains constant if

$$\frac{\partial\mathbf{w}}{\partial t} + \nabla\wedge(\mathbf{w}\wedge\mathbf{v}) = 0.$$

(Cf. Ex. 4.22.7.)

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The basic material on kinematics goes back to the seventeenth century. Full references can be found in the works of Truesdell and a valuable survey has been given by him in his introduction to volume (2) *12* of L. Euleri Opera Omnia (Lausannae MCMLIV), "Rational Fluid Mechanics," 1687–1765.

4.1. The most extensive exposition is to be found in chapter B of

Truesdell, C. A., and R. Toupin, The classical field theories, in Handbuch der Physik III/1, Ed. S. Flugge. 1960.

The "peculiar and characteristic glory of three-dimensional kinematics" is the subject of an earlier treatise

Truesdell, C. A., The kinematics of vorticity. Bloomington, Ind: Indiana University Press, 1954.

The basic material is given in

Serrin, J., Mathematical principles of classical fluid mechanics, in Handbuch der Physik VIII/1, Ed. S. Flugge, 1959.

We have followed his exposition in treating the vortex line in the last section.

5 *Stress in Fluids*

The object of this chapter is to show the tensorial character of stress and exhibit some of the ways in which it may be related to strain. We have already referred to stress as being a force per unit area and so having two directions (those of the force and the normal to the area) associated with it. This gives reason to suspect that stress can be represented by a tensor, but to establish this we follow a very elegant line of reasoning laid down by Cauchy in 1823. On the principle of the conservation of momentum we can then establish certain properties of the stress tensor. The relation between the stress tensor and the deformation tensor is known as the *mechanical constitutive equation* for the material and the remainder of the chapter will treat some elementary examples of these defining relations.

5.11. Cauchy's stress principle and the conservation of momentum

The forces acting on an element of a continuous medium may be of two kinds. *External or body forces*, such as gravitation or electromagnetic forces, can be regarded as reaching into the medium and acting throughout the volume. *Internal or contact forces* are to be regarded as acting on an element of volume through its bounding surface. If the element of volume has an external bounding surface, the forces there may be specified, as, for example, when a constant pressure is applied over a free surface. If the element is internal, the resultant force is that exerted by the material outside the surface upon that inside. Let **n** be the unit outward normal at a point of the surface S