Algorithm for calculating and updating Moore-Penrose inverse.

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Abstract

The paper presents an algorithm for calculating a Moore-Penrose inverse. The algorithm is similar in spirit to the well known Gram-Schmidt process. It first calculates the More-Penrose inverse for a matrix V having two columns then it gives an efficient recipe for the updating of the Moore-Penrose inverse in response to appending a new column to V. It is assumed that the columns of V are linearly independent.

Keywords: Moore-Penrose inverse, Gram-Schmidt process, linear space, algorithm.

1 Introduction

The Moore-Penrose inverse has a diverse range of important applications in neural networks [**5**], cryptology [**6**], control theory [**8**], image processing [**7**] to mention just a few. It is closely related to various matrix factorizations such as, for example, the QR-factorization and the Singular Value Decomposition. The technique developed in this publication is associated with a modified Gram-Schmidt orthogonalization procedure (see [**1**] and [**4**]). The paper presents an algorithm for calculating the Moore-Penrose inverse

$$(V^T V)^{-1} V^T$$

The algorithm is not only rooted in a modified Gram-Schmidt type of orthogonalization but its organization is similar in spirit to that of the well known Gram-Schmidt process. It first calculates the More-Penrose inverse for a matrix V having two columns then it gives an efficient recipe for the updating of the Moore-Penrose inverse in response to appending a new column to V. The newly proposed update scheme requires the number of arithmetic operations that is proportional to the number of rows in V. From this point of view it is more efficient than methods using Givens, Householder transformations or other rotations (see [**2**]). The algorithm is designed with an intention to be used in applications where the size of the column is much greater than the number of columns in V.

2 Moore-Penrose inverse for a matrix with two columns

Consider a matrix

$$V = (V_1 \; V_2)$$

with two linearly independent columns V_1 and V_2. The goal is to calculate the Moore-Penrose inverse

$$MP_2 = (V^T V)^{-1} V^T$$

where V^T denotes the transpose of V and $(V^T V)^{-1}$ is the inverse of $(V^T V)$.

For this purpose let us use an approach based on a modified version of Gram-Schmidt orthogonalization algorithm. This approach is well known, for details see [**1**], [**2**], [**4**] and for further references see chapter 7 in [**3**]. In accordance with the modified version of Gram-Schmidt algorithm (see [**4**])

$$\begin{aligned} V_1 &= \eta_1 \\ V_2 &= \eta_2 + \frac{\langle \eta_1, V_2 \rangle}{\mid \eta_1 \mid^2} \eta_1 \end{aligned} \tag{1}$$

where $\langle \eta_1, V_1 \rangle$ denotes the dot product between η_1 and V_1, while $\mid \eta_1 \mid^2 = \langle \eta_1, \eta_1 \rangle$. One can see that

$$\langle \eta_1, \eta_2 \rangle = 0$$

and

$$\mid \eta_2 \mid^2 = \mid V_2 \mid^2 - \frac{\langle \eta_1, V_2 \rangle^2}{\mid \eta_1 \mid^2}.$$

Let us write (1) in a matrix form,

$$(V_1 \; V_2) = (\eta_1 \; \eta_2) \begin{pmatrix} 1 & \frac{\langle \eta_1, V_2 \rangle}{|\eta_1|^2} \\ 0 & 1 \end{pmatrix}$$

Then the Moore-Penrose inverse MP_2 can be calculated as follows.

$$(V^T V)^{-1} V^T = \left(\begin{pmatrix} 1 & \frac{\langle \eta_1, V_2 \rangle}{|\eta_1|^2} \\ 0 & 1 \end{pmatrix}^T (\eta_1 \; \eta_2)^T (\eta_1 \; \eta_2) \begin{pmatrix} 1 & \frac{\langle \eta_1, V_2 \rangle}{|\eta_1|^2} \\ 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & \frac{\langle \eta_1, V_2 \rangle}{|\eta_1|^2} \\ 0 & 1 \end{pmatrix}^T (\eta_1 \; \eta_2)^T \tag{2}$$

It follows from (2) and

$$(\eta_1 \; \eta_2)^T (\eta_1 \; \eta_2) = \begin{pmatrix} \mid \eta_1 \mid^2 & 0 \\ 0 & \mid \eta_2 \mid^2 \end{pmatrix}$$

that

$$MP_2 = (V^T V)^{-1} V^T = \begin{pmatrix} 1 & \frac{\langle \eta_1, V_2 \rangle}{|\eta_1|^2} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{|\eta_1|^2} & 0 \\ 0 & \frac{1}{|\eta_2|^2} \end{pmatrix} (\eta_1 \; \eta_2)^T$$

Since

$$\begin{pmatrix} 1 & \frac{\langle \eta_1, V_2 \rangle}{|\eta_1|^2} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\frac{\langle \eta_1, V_2 \rangle}{|\eta_1|^2} \\ 0 & 1 \end{pmatrix}$$

let us introduce

$$W_2 = \begin{pmatrix} \frac{1}{|\eta_1|^2} & -\frac{\langle \eta_1, V_2 \rangle}{|\eta_1|^2 |\eta_2|^2} \\ 0 & \frac{1}{|\eta_2|^2} \end{pmatrix} \tag{3}$$

Then

$$MP_2 = W_2(\eta_1\ \eta_2)^T \tag{4}$$

and we obtain the following formula

$$MP_2 = \begin{pmatrix} \frac{1}{|\eta_1|^2} & -\frac{\langle \eta_1, V_2\rangle}{|\eta_1|^2|\eta_2|^2} \\ 0 & \frac{1}{|\eta_2|^2} \end{pmatrix} (\eta_1\ \eta_2)^T = \begin{pmatrix} \frac{1}{|\eta_1|^2}\eta_1^T - \frac{\langle \eta_1, V_2\rangle}{|\eta_1|^2|\eta_2|^2}\eta_2^T \\ \frac{1}{|\eta_2|^2}\eta_2^T \end{pmatrix} \tag{5}$$

3 Updating Moore-Penrose inverse

Given Mure-Penrose inverse MP_n for linearly independent vectors $V_1,\ V_2,\ \ldots V_n$ the goal of this section is to calculate More-Penrose inverse MP_{n+1} after adding V_{n+1} column to the original set of vectors. It is assumed that the updated set of vectors $V_1,\ V_2,\ \ldots V_n,\ V_{n+1}$ is linearly independent. Section 3.2 in [**2**] considers similar problem for QR decomposition but it describes the technique employing Givens rotations which is different from the machinery developed in this section.

Application of the modified Gram-Schmidt algorithm to $V_1,\ V_2,\ \ldots V_n$ yields

$$\begin{aligned} \eta_1 &= V_1 \\ \eta_2 &= V_2 - \frac{\langle \eta_1, V_2\rangle}{\mid \eta_1 \mid^2}\eta_1 \\ &\vdots \\ \eta_n &= V_n - \sum_{j=1}^{n-1} \frac{\langle \eta_j, V_n\rangle}{\mid \eta_j \mid^2}\eta_j \\ \eta_{n+1} &= V_{n+1} - \sum_{j=1}^{n} \frac{\langle \eta_j, V_{n+1}\rangle}{\mid \eta_j \mid^2}\eta_j \end{aligned} \tag{6}$$

Theorem 3.1. *Let $V_1,\ V_2,\ \ldots\ V_{n+1}$ be linearly independent and MP_n be the Moore-Penrose inverse for $(V_1,\ V_2,\ \ldots\ V_n)$. Then*

$$MP_{n+1} = \begin{pmatrix} MP_n - \frac{1}{|\eta_{n+1}|^2}(MP_n V_{n+1})\eta_{n+1}^T \\ \frac{1}{|\eta_{n+1}|^2}\eta_{n+1}^T \end{pmatrix}$$

is the Moore-Penrose inverse for $(V_1,\ V_2,\ \ldots\ V_{n+1})$.

Proof. Modified Gram-Schmidt algorithm (6) yields that

$$\langle \eta_{n+1}, V_j\rangle = 0 \quad \text{for} \quad 1 \le j \le n$$

Hence, for $1 \le j \le n$

$$MP_{n+1}V_j = \begin{pmatrix} MP_n V_j \\ 0 \end{pmatrix} = e_j$$

where e_j has all its entries equal to 0 but its jth entry is 1.

It remains to show that $MP_{n+1}V_{n+1} = e_{n+1}$. Indeed, it follows from

$$\langle \eta_{n+1}, V_{n+1} \rangle = \mid V_{n+1} \mid^2 - \sum_{j=1}^{n} \frac{\langle \eta_j, V_{n+1} \rangle^2}{\mid \eta_j \mid^2} = \mid \eta_{n+1} \mid^2$$

that

$$MP_{n+1}V_{n+1} = \begin{pmatrix} MP_n V_{n+1} - \frac{1}{|\eta_{n+1}|^2}(MP_n V_{n+1})\langle \eta_{n+1}, V_{n+1} \rangle \\ \frac{\langle \eta_{n+1}, V_{n+1} \rangle}{|\eta_{n+1}|^2} \end{pmatrix} = e_{n+1}$$

Q.E.D. □

Example 3.1. Calculate Moore-Penrose inverse MP_3 for

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The modified Gram-Schmidt process yields

$$\eta_1 = V_1, \quad \eta_2 = V_2 - \frac{\langle \eta_1, V_2 \rangle}{\mid \eta_1 \mid^2}\eta_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 1 \\ \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

It follows from (3), (4) that

$$W_2 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{8} \\ 0 & \frac{3}{8} \end{pmatrix}$$

and

$$MP_2 = \begin{pmatrix} \frac{1}{3}(1 \;\; 0 \;\; 1 \;\; 1 \;\; 0) - \frac{1}{8}(-\frac{1}{3} \;\; 1 \;\; \frac{2}{3} \;\; -\frac{1}{3} \;\; 1) \\ \frac{3}{8}(-\frac{1}{3} \;\; 1 \;\; \frac{2}{3} \;\; -\frac{1}{3} \;\; 1) \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix}$$

It follows from (6) that

$$\eta_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -\frac{1}{3} \\ 1 \\ \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Making use of theorem 3.1 yields

$$MP_3 = \begin{pmatrix} \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ \frac{1}{2} \quad \frac{1}{2} \quad 0 \quad -\frac{1}{2} \quad -\frac{1}{2} \end{pmatrix}$$

Hence,

$$MP_3 = \begin{pmatrix} \frac{1}{8} & -\frac{3}{8} & \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ -\frac{3}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{5}{8} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

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