Canonical Quantum Field Theory

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1 Canonical Quantization of a Scalar Field

A scalar field $\phi(x)$ that obeys the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0 \tag{1}$$

also obeys the following *equal-time commutation relations* (ETCRs):

$$[\phi(t,\underline{x}), \dot{\phi}(t,\underline{x}')] = i\delta(\underline{x} - \underline{x}') \tag{2}$$
$$[\phi(t,\underline{x}), \phi(t,\underline{x}')] = [\dot{\phi}(t,\underline{x}), \dot{\phi}(t,\underline{x}')] = 0 \tag{3}$$

A scalar field may be expanded in terms of mode operators:

$$\phi(t,\underline{x}) = \phi^+(x) + \phi^-(x) \tag{4}$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(\underline{p})}\left(a(\underline{p})e^{-ip\cdot x} + a(\underline{p})^\dagger e^{ip\cdot x}\right) \tag{5}$$

where the creation and annihilation operators in momentum space obey the commutation relations

$$[a(\underline{p}), a^\dagger(\underline{p}')] = 2\omega(\underline{p})(2\pi)^3\delta(\underline{p} - \underline{p}') \tag{6}$$
$$[a(\underline{p}), a(\underline{p}')] = [a^\dagger(\underline{p}), a^\dagger(\underline{p}')] = 0. \tag{7}$$

To prevent infinite contributions to the energy, we require that all terms be *normal ordered* - that is, all annihilation operators stand to the right of all creation operators. This results in the convenient property

$$\boxed{\langle 0|{:}\phi(x)\phi(y){:}|0\rangle = 0.}$$

We may simplify covariant commutation relations for the Klein-Gordon field by introducing three invariant functions

$$\Delta^+(x) \equiv -i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(\underline{p})}e^{-ip\cdot x} \tag{8}$$
$$\Delta^-(x) \equiv \Delta^+(x)^* = i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(\underline{p})}e^{ip\cdot x} \tag{9}$$
$$\Delta(x) \equiv \Delta^+(x) + \Delta^-(x). \tag{10}$$

All three satisfy the Klein-Gordon equation, and we may furthermore write Δ in a manifestly covariant form:

$$\Delta(x) = -i\int \hat{d}^4p\,\hat{\delta}(p^2 - m^2)\theta(p_0)e^{-ip\cdot x}. \tag{11}$$

Δ also satisfies the microcausality condition

$$\boxed{[\phi(x), \phi(y)] = 0, \quad (x-y)^2 < 0.}$$

2 Interactions and Scattering in ϕ^3 Theory

The Lagrangian for this theory