Canonical Quantum Field Theory

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1 Canonical Quantization of a Scalar Field

A scalar field $\phi(x)$ that obeys the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0 \tag{1}$$

also obeys the following *equal-time commutation relations* (ETCRs):

$$[\phi(t,\underline{x}), \pi(t,\underline{x}')] = [\phi(t,\underline{x}), \dot{\phi}(t,\underline{x}')] = i\delta(\underline{x} - \underline{x}') \tag{2}$$

$$[\phi(t,\underline{x}), \phi(t,\underline{x}')] = [\pi(t,\underline{x}), \pi(t,\underline{x}')] = [\dot{\phi}(t,\underline{x}), \dot{\phi}(t,\underline{x}')] = 0 \tag{3}$$

where the conjugate momentum $\pi(x)$ is the usual $\frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$. A scalar field may be expanded in terms of mode operators:

$$\phi(t,\underline{x}) = \phi^+(x) + \phi^-(x) \tag{4}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \left(a(\underline{p}) e^{-ip\cdot x} + a(\underline{p})^\dagger e^{ip\cdot x} \right) \tag{5}$$

where the creation and annihilation operators in momentum space obey the commutation relations

$$[a(\underline{p}), a^\dagger(\underline{p}')] = 2\omega(\underline{p})(2\pi)^3 \delta(\underline{p} - \underline{p}') \tag{6}$$

$$[a(\underline{p}), a(\underline{p}')] = [a^\dagger(\underline{p}), a^\dagger(\underline{p}')] = 0. \tag{7}$$

To prevent infinite contributions to the energy, we require that all terms be *normal ordered* - that is, all annihilation operators stand to the right of all creation operators. This results in the convenient property

$$\boxed{\langle 0| {:}\phi(x)\phi(y){:} |0\rangle = 0.}$$

We may simplify covariant commutation relations for the Klein-Gordon field by introducing three invariant functions

$$\Delta^+(x) \equiv -i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{-ip\cdot x} \tag{8}$$

$$\Delta^-(x) \equiv \Delta^+(x)^* = i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{ip\cdot x} \tag{9}$$

$$\Delta(x) \equiv \Delta^+(x) + \Delta^-(x). \tag{10}$$

All three satisfy the Klein-Gordon equation, and we may furthermore write Δ in a manifestly covariant form:

$$\Delta(x) = -i \int \hat{d}^4p \hat{\delta}(p^2 - m^2)\theta(p_0) e^{-ip\cdot x}. \tag{11}$$

Δ also satisfies the microcausality condition

$$\boxed{[\phi(x), \phi(y)] = 0, \quad (x-y)^2 < 0.}$$

2 Interactions and Scattering in ϕ^3 Theory

The Lagrangian for this theory is given by

$$\begin{aligned}\mathcal{L} &\equiv \mathcal{L}_0 + \mathcal{L}_I \qquad (12)\\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}g\phi^3. \qquad (13)\end{aligned}$$

We may write the Euler-Lagrange equation as

$$(\partial^2 + m^2)\phi = -\frac{1}{2}g\phi^2. \qquad (14)$$

Since this is a nonlinear partial differential equation, we would like to proceed by solving it in terms of powers of coupling g.
We define the S-matrix as the time evolution operator between two states in the far past and the far future at which the particles involved may be considered non-interacting:

$$|\Psi, \infty\rangle = S\,|\Psi, -\infty\rangle\,. \qquad (15)$$

Since in general, we must consider many possible initial and final states, we consider the S-matrix elements defined as

$$S_{fi} = \langle f|S|i\rangle \qquad (16)$$

Note that the S-matrix is unitary ($S^\dagger S = SS^\dagger = 1$) even when particles in the state are destroyed and/or created.
Furthermore, in the future we will work in the interaction picture of quantum mechanics - a mix of the Schrödinger and Heisenberg pictures. We split the Hamiltonian into free and non-interacting parts $H = H_0 + H_I$ and let the time evolution of the state be governed by H_I, and the time evolution of the operators by H_0:

$$\begin{aligned}\frac{\partial}{\partial t}O(t) &= i[H_0, O(t)] \qquad (17)\\ i\frac{\partial}{\partial t}\,|\psi, t\rangle &= H_I\,|\psi, t\rangle\,. \qquad (18)\end{aligned}$$

Integrating Eq. 18, we obtain a recursion relation for $|\Psi, t\rangle$:

$$\begin{aligned}|\Psi, t\rangle &= |\Psi, -\infty\rangle + \int_{-\infty}^{t} dt_1 \frac{\partial}{\partial t}\,|\Psi, t_1\rangle \qquad (19)\\ &= |i\rangle - i\int_{-\infty}^{t} dt_1 H_I(t_1)\,|\Psi, t_1\rangle\,. \qquad (20)\end{aligned}$$

By substituting the recursion relation into itself, we write the Dyson series explicitly as

$$S = 1 + \sum_{n=1}^{\infty}(-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_I\left(t_1\right) H_I\left(t_2\right) \cdots H_I\left(t_n\right). \qquad (21)$$

The introduction of the time ordered product of two operators

$$T(A(t_1)B(t_2)) = \theta(t_1 - t_2)A(t_1)B(t_2) + \theta(t_2 - t_1)B(t_2)A(t_1) \qquad (22)$$

and the use of the Hamiltonian density instead of the Hamiltonain yields the manifestly covariant expression

$$\boxed{S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \int d^4x_2 \cdots \int d^4x_n T\left(\mathcal{H}_I\left(x_1\right)\mathcal{H}_I\left(x_2\right)\cdots\mathcal{H}_I\left(x_n\right)\right).}$$

Since our calculations involve time-ordered products through *Wick's Theorem*, which we will see shortly, it is convenient to define the Feynman propagator

$$\Delta_F(x - x') \equiv \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x). \tag{23}$$

Thus, making use of our knowledge of the functions $\Delta^{\pm}$, we find that

$$\boxed{i\Delta_F(x - x') = \overbracket{\phi(x)\phi(x')} = \langle 0|T\left(\phi(x)\phi(x')\right)|0\rangle\,,}$$

where the bracket on top denotes a *Wick contraction*. How can we relate the time-ordered product in general, normal ordering and Wick contractions? The answer is *Wick's Theorem*. For unequal times

$$\begin{aligned} T(ABCD\cdots WXYZ) = {}&{:}ABCD\cdots WXYZ{:} \\ &+ {:}\overbracket{AB}C\cdots YZ{:} + {:}\overbracket{ABC}\cdots YZ{:} + \cdots + {:}ABC\cdots \overbracket{YZ}{:} \\ &+ {:}\overbracket{AB}\,\overbracket{CD}\cdots WXYZ{:} + \cdots + {:}ABCD\cdots \overbracket{WX}\,\overbracket{YZ}{:} \\ &+ \cdots\,, \end{aligned} \tag{24}$$

which simplifies to, for example, in the case of a single time ordering to

$$T(A(x)B(x')) = {:}A(x)B(x'){:} + \langle 0|T(A(x)B(x'))|0\rangle\,. \tag{25}$$

When performing the Dyson expansion, we will need to evaluate the Feynman propagator many times. Therefore, it is usefeful to rewrite it in terms of a single integral

$$\Delta_F(x) = \int \hat{d}^4p \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon} \tag{26}$$

where $\varepsilon \equiv 2\eta\omega(\underline{p})$ is a small number that takes care of the contours in the complex plane that we have to take due to the poles at $p = \pm m$. In fact, Δ_F is a Green's function for the Klein-Gordon equation, corresponding to Dirichlet boundary conditions (initial and final conditions on $\phi(x)$), and is thus appropriate to our quantum paradigm of initial and final states.
As an example, let us expand the S-matrix in position space. In ϕ^3 theory, we have $\mathcal{H}_I(x) = \frac{g}{3!}\,{:}\phi^3(x){:}$, which we expand in terms of ϕ^+ (which destroys a particle at x) and ϕ^- (which creates a particle at x).

1. For $n = 0$, nothing happens.

2. For $n = 1$,

$$S^{(1)} = \frac{-ig}{3!}\int d^4x\, T({:}\phi^3(x){:}). \tag{27}$$

   Since there are no *unequal times*, there are no terms with contractions. The terms without contractions cannot conserve momentum, so they vanish.

3. For $n = 2$,

$$S^{(2)} = -\frac{g^2}{2!(3!)^2}\int d^4x d^4y\, T\left({:}\phi^3(x){:}{:}\phi^3(y){:}\right), \tag{28}$$

   which we may split up into terms containing zero, one, two and three contractions.

   (a) $S_0^{(2)}$ only contains one term, and it is proportional to ${:}\phi^3(x)\phi^3(y){:}$. The interactions at x and y are unrelated, so they factorize and vanish due to momentum conservation as in the $n = 1$ case.

   (b) $S_1^{(2)}$ contains terms with one contraction, i.e. proportional to $\overbracket{\phi(x)\phi(y)}\ {:}\phi^2(x)\phi^2(y){:}$. Expanding it out in terms of ϕ^+ and ϕ^-, there are three terms that survive momentum conservation:

$$\begin{aligned} &\phi^-(y)\phi^-(y)\overbracket{\phi(x)\phi(y)}\phi^+(x)\phi^+(x), \\ &\phi^-(x)\phi^-(y)\overbracket{\phi(x)\phi(y)}\phi^+(x)\phi^+(y), \\ &\phi^-(y)\phi^-(x)\overbracket{\phi(x)\phi(y)}\phi^+(x)\phi^+(y). \end{aligned} \tag{29}$$

There is also a combinatorial 3^2 factor, since there are 3 ways of contracting $\phi(x)$ and 3 ways of contracting $\phi(y)$.

(c) $S_2^{(2)}$ is given by the term $\overline{\phi(x)\phi(y)}\,\overline{\phi(x)\phi(y)}$ $:\phi(x)\phi(y):$, which diverges and forms the so-called *self-energy correction*.

(d) $S_3^{(2)}$ is given by the term $\overline{\phi(x)\phi(y)}\,\overline{\phi(x)\phi(y)}\,\overline{\phi(x)\phi(y)}$, which also diverges, but as it is disconnected it makes no contribution to the scattering process.

This is a lot of work for calculating the S-matrix! Fortunately, we are more interested in calculating the matrix elements $S_{fi} = \langle f|S|i\rangle$ than S itself. Since we label free particle initial and final states by their definite on-shell momentum $|\underline{p}\rangle = a^\dagger(\underline{p})\,|0\rangle$, we should work in momentum space. Then

$$\begin{aligned}
\phi^+(x)\,|\underline{p}\rangle &= \int \frac{\hat{d}^3\underline{p}}{2E'} e^{-ip'\cdot x} a(\underline{p}')a^\dagger(\underline{p})\,|0\rangle \\
&= \int d^3\underline{p}' e^{-ip'\cdot x}\delta^3(\underline{p}-\underline{p}')\,|0\rangle \\
&= e^{-ip\cdot x}\,|0\rangle
\end{aligned} \tag{30}$$

while

$$\begin{aligned}
\langle\underline{p}|\,\phi^-(x) &= \langle 0|\,a(\underline{p}) \int \frac{\hat{d}^3\underline{p}}{2E'} e^{ip'\cdot x} a^\dagger(\underline{p}) \\
&= \langle 0| \int d^3\underline{p}' e^{ip'\cdot x}\delta^3(\underline{p}-\underline{p}') \\
&= e^{ip\cdot x}\,\langle 0|\,.
\end{aligned} \tag{31}$$

As an example, let us perform a momentum-space calculation of the 2→2 process through the s-channel. We may do this by simply inserting the first line of Eq. 29:

$$\begin{aligned}
\langle f|S_{1,s}^{(2)}|i\rangle &= -\frac{3^2g^2}{(3!)^2}\int d^4x \int d^4y\,\langle q;q'|\phi^-(y)\phi^-(y)\overline{\phi(x)\phi(y)}\phi^+(x)\phi^+(x)|p;p'\rangle \\
&= -\frac{g^2}{4}\int d^4x \int d^4y\, 4e^{i(q+q')\cdot x}\, i\int \hat{d}^4k \frac{e^{-ik\cdot(y-x)}}{k^2-m^2+i\varepsilon} e^{-i(p+p')\cdot x} \\
&= -ig^2 \int \hat{d}^4k\,\hat{\delta}^4\left(q+q'-k\right)\hat{\delta}^4\left(k-p-p'\right)\frac{1}{k^2-m^2+i\varepsilon} \\
&= \hat{\delta}^4\left(q+q'-p-p'\right)\frac{-ig^2}{(p+p')^2-m^2},
\end{aligned} \tag{32}$$

where on the second line we have applied Eqs. 30, 31 and inserted our expression for the Feynman propagator, with an additional factor of 4 coming from the choice of apply $(\phi^+)^2$ on $|p;p'\rangle$. For the other two contributions we have the same, replacing the denominator by $(p-q)^2-m^2$ and $(p-q')-m^2$ respectively. It is therefore useful to write down the *Mandelstam invariants* s, t, u, defined as

$$\begin{aligned}
s &= (p_1+p_2)^2 = (p_3+p_4)^2 \\
t &= (p_1-p_3)^2 = (p_2-p_4)^2 \\
u &= (p_1-p_4)^2 = (p_2-p_3)^2.
\end{aligned} \tag{33}$$

Applying a similar process to the loop diagram leads to a divergent term, which we will look at in more detail through *renormalization*.

2.1 Feynman rules for scalar field theory

An attentive reader, having calculated all three channels, might have noticed that the process above seems fairly simple and algorithmic. Indeed, we can define the so-called *Feynman rules*, which for any theory allow us to write down the contribution from each diagram without performing any nasty integrations. For scalar ϕ^3 theory, we have:

- for each ϕ^3 vertex, a factor of $-ig$ and a delta function ensuring momentum conservation
- for each external line, a factor of 1
- for each internal line of momentum k, a factor $i/(k^2 - m^2 + i\varepsilon)$
- for each momentum k not fixed by momentum conservation, an integral $\int \hat{d}^4k$
- a symmetry factor.

3 Complex Scalar Fields and Charge Conservation

In quantum field theory, for the S-matrix to be unitary, we only require that the Lagrangian density $\mathcal{L}$, the Hamiltonian density $\mathcal{H}$, and the action S be real. Thus, a quantum field may also be complex as long as above conditions are satisfied. They also have the important new property of charge conservation.
If we have a multiplet (several) of scalar fields $\phi_r(x)$, where $r = 1, 2, \cdots N$, then we may quantize them as we have done for a single scalar field, the only difference being an additional delta function in the nonzero commutators.
A complex scalar field $\phi(x)$ can be expressed in terms of two independent fields ϕ_1 and ϕ_2:

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)). \tag{34}$$

Then we have the Lagrangian density

$$\mathcal{L} = \left(\partial_\mu \phi^\dagger(x)\right)(\partial^\mu \phi(x)) - m^2 \phi^\dagger(x)\phi(x) \tag{35}$$

along with the conjugate momentum $\pi(x) = \frac{\partial}{\partial \mathcal{L}\dot{\phi}(x)} = \dot{\phi}^\dagger(x)$, which provides the ETCRs

$$[\phi(t, \underline{x}), \pi(t, \underline{x}')] = [\phi^\dagger(t, \underline{x}), \pi^\dagger(t, \underline{x}')] = i\delta(\underline{x} - \underline{x}'). \tag{36}$$

Each of the fields $\phi(x)$ and $\phi^\dagger(x)$ obeys the Klein-Gordon equation independently. We expand the fields in terms of mode operators

$$\phi(x) = \int \hat{d}^3p \frac{1}{2\omega(\underline{p})}\left(a(\underline{p})e^{-ip\cdot x} + b^\dagger(\underline{p})e^{ip\cdot x}\right), \tag{37}$$

noting that we can no longer assume the Hermiticity relation between the operator coefficients of the positive and negative frequency terms because $\phi(x)$ is not Hermitian. Furthermore, the commutation relations

$$[a(\underline{p}), a^\dagger(\underline{p}')] = [b(\underline{p}), b^\dagger(\underline{p}')] = 2\omega(\underline{p})\hat{\delta}^3(\underline{p} - \underline{p}') \tag{38}$$

with all others being zero reproduce the ETCRs. a and b and their Hermitian conjugates are the creation and annihilation operators for two types of particles. Furthermore, momentum is conserved, with the 4-momentum operator (not to be confused with the conjugate of the field) being

$$P^\nu = \int d^3x T^{0\nu}(x) \tag{39}$$

$$T^{\mu\nu}(x) =: \partial^\mu \phi^\dagger(x)\partial^\nu \phi(x) + \partial^\nu \phi^\dagger(x)\partial^\mu \phi(x) - \eta^{\mu\nu}\mathcal{L}(x):, \tag{40}$$

or in terms of mode operators

$$P^\nu = \int \hat{d}^3p \frac{1}{2\omega(\underline{p})}p^\nu\left(a^\dagger(\underline{p})a(\underline{p}) + b^\dagger(\underline{p})b(\underline{p})\right). \tag{41}$$

3.1 Charge conservation

Additionally, notice that the Lagrangian density is invariant under the phase transformations

$$\begin{aligned} \phi(x) &\to \phi'(x) = e^{i\alpha}\phi(x) \approx (1+i\alpha)\phi(x) \\ \phi(x) &\to \phi'(x) = e^{-i\alpha}\phi(x) \approx (1-i\alpha)\phi(x), \end{aligned} \tag{42}$$

which is known as a *global phase transformation* or *gauge transformation of the first kind*. Furthermore, since the Lagrangian density itself is invariant, according to Noether's theorem there is a conserved current j and conserved charge Q:

$$Q = \int d^3x j^0(x) \tag{43}$$

$$j^\mu(x) = -\frac{1}{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\dagger}\right) = i :\phi^\dagger(x)\left(\partial^\mu\phi(x)\right) - \left(\partial^\mu\phi^\dagger(x)\right)\phi(x): . \tag{44}$$

Thus, the existence of charge is directly tied to the existence of complex scalar fields. Furthermore, we can think of a as corresponding to particles with "charge" $+1$ while b would correspond to particles with "charge" -1. Since Q is conserved, these particles must always be created or destroyed in pairs - they are anti-particles of each other.

3.2 Feynman rules for complex scalar fields

The propagator for complex scalar fields is the same as for real scalar fields:

$$\overbrace{\phi(x)\phi^\dagger}(x') = i\Delta_F(x-x'). \tag{45}$$

The external states are given by

$$|p,+\rangle = a^\dagger(\underline{p})\,|0\rangle\,, \quad |p,-\rangle = b^\dagger(\underline{p})\,|0\rangle\,, \tag{46}$$

so then

$$\phi_+(x)\,|p,+\rangle = e^{-ip\cdot x}\,|0\rangle\,, \quad \phi_+^\dagger(x)\,|p,-\rangle = e^-ip\cdot x\,|0\rangle \tag{47}$$

$$\langle p,+|\,\phi_-^\dagger(x) = e^{ip\cdot x}\,\langle 0|\,, \quad \langle p,-|\,\phi_-(x) = e^{ip\cdot x}\,\langle 0|\,. \tag{48}$$

In terms of the interactions possible, remembering that the Lagrangian must remain real, we commonly use

$$\mathcal{L}_I^4 = -\frac{1}{4}\lambda(\phi^\dagger\phi)^2 \tag{49}$$

or if there is also a real scalar field Φ

$$\mathcal{L}_I^3 = -y\Phi(\phi^\dagger\phi). \tag{50}$$

These additional processes allow much more interesting processes such as elastic scattering, decay, and charged particle elastic scattering.

3.3 C, P, and T for scalar fields

Under **parity**, $x^\mu = (t,\underline{x}) \to \bar{x}^\mu = (t,-\underline{x})$, $p^\mu = (E,\underline{p}) \to \bar{p}^\mu = (E,-\underline{p})$. This is implemented using the unitary operator $\mathcal{P}$, thus

$$\mathcal{P}\,|0\rangle\,, \quad \mathcal{P}\,\left|\underline{p}\right\rangle = \left|-\underline{p}\right\rangle. \tag{51}$$

Since $\left|\underline{p}\right\rangle = a^\dagger(\underline{p})\,|0\rangle$, we must have

$$\mathcal{P}a^\dagger(\underline{p})\mathcal{P}^\dagger = a^\dagger(-\underline{p}), \quad \mathcal{P}a(\underline{p})\mathcal{P}^\dagger = a(-\underline{p}). \tag{52}$$

Therefore

$$\phi(x) \to \mathcal{P}\phi(x)\mathcal{P}^\dagger = \phi(\bar{x}). \tag{53}$$

Under **time reversal**, $x^\mu \to -\bar{x}^\mu$, $p^\mu \to \bar{p}^\mu$. Furthermore, initial and final states are interchanged $\langle f| |i\rangle \to \langle i| |f\rangle$, or alternatively $i \to -i$. $\mathcal{T}$ is anti-unitary: unitary, but also complex conjugates. It acts as

$$\mathcal{T}|0\rangle = |0\rangle, \quad \mathcal{T}|\underline{p}\rangle = |-\underline{p}\rangle, \quad \mathcal{T}a^\dagger(\underline{p})\mathcal{T}^\dagger = a^\dagger(-\underline{p}), \quad \mathcal{T}a(\underline{p})\mathcal{T}6\dagger = a(-\underline{p}) \tag{54}$$

but

$$\phi(x) \to \mathcal{T}\phi(x)\mathcal{T}^\dagger = \phi(-\bar{x}). \tag{55}$$

Under **charge conjugation**, particles and antiparticles are exchanged: $\mathcal{C}|\underline{p},+\rangle = |\underline{p},-\rangle$, $\mathcal{C}|\underline{p},-\rangle = |\underline{p},+\rangle$, so

$$\mathcal{C}a^\dagger(\underline{p})\mathcal{C}^\dagger = b^\dagger(\underline{p}), \quad \mathcal{C}b^\dagger(\underline{p})\mathcal{C}^\dagger = a^\dagger(\underline{p}). \tag{56}$$

Furthermore (only for complex fields, since scalar fields do not carry any charge),

$$\mathcal{C}\phi(x)\mathcal{C}^\dagger = \phi^\dagger(x), \quad \mathcal{C}\phi^\dagger(x)\mathcal{C}^\dagger = \phi(x), \tag{57}$$

and

$$\mathcal{C}j^\mu\mathcal{C}^\dagger = -j^\mu, \quad \mathcal{C}Q\mathcal{C}^\dagger = -Q. \tag{58}$$

4 The Dirac Equation

The Dirac equation, written in covariant form as

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi(x) = 0,}$$

contains two objects worth studying in detail: γ^μ and $\psi(x)$. The gamma matrices γ^μ follow the *Clifford algebra*

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{59}$$

and have two common representations, known as the Dirac basis and Weyl basis respectively:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{60}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{61}$$

where we have used

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{62}$$

Recalling our knowledge of the Lorentz group from Symmetries of Particles and Fields, Lorentz transformations may be written as

$$S(\Lambda) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \quad \sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]. \tag{63}$$

Under Lorentz transformations, the four-component object $\psi(x)$ transforms as

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x), \tag{64}$$

γ^μ transforms as

$$\Lambda^\mu_\nu\gamma^\nu = S(\Lambda)^{-1}\gamma^\mu S(\Lambda). \tag{65}$$

We also introduce the conjugate $\bar{\psi} = \psi^\dagger(x)\gamma^0$.
From this, we can construct the 16 Dirac bilinears

$$\begin{array}{ccc} \bar{\psi}\psi & \text{scalar} & 1 \\ \bar{\psi}\gamma^\mu\psi & \text{vector} & 4 \\ \bar{\psi}\sigma^{\mu\nu}\psi & \text{tensor} & 6 \\ \bar{\psi}\gamma^5\gamma^\mu\psi & \text{axial vector} & 4 \\ \bar{\psi}\gamma^5\psi & \text{pseudoscalar} & 1. \end{array} \tag{66}$$

The Lagrangian for the Dirac field is given by

$$\boxed{\mathcal{L} = \bar{\psi}(x)\left(i\partial\!\!\!/ - m\right)\psi(x)}$$

if we treat $\psi(x)$ and $\bar{\psi}(x)$ as independent fields. Varying the Lagrangian with respect to $\bar{\psi}(x)$ gives the Dirac equation, while varying the Lagrangian with respect to $\psi(x)$ gives $(i\partial_\mu\gamma^\mu + m)\,\bar{\psi}(x) = 0$. Just as with complex scalar fields, we have a conserved current and charge

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \quad Q = \int d^3x\psi^\dagger\psi. \tag{67}$$

The solutions to the Dirac equation are plane wave solutions, the positive and negative energy ones given by

$$\psi^+(x) = \exp(-ip_\mu x^\mu)\,u(\underline{p}), \quad \psi^-(x) = \exp(ip_\mu x^\mu)v(\underline{p}) \tag{68}$$

respectively, where u and v are four-component *spinors* also satisfying the Dirac equation:

$$u(\underline{p}, s) = \sqrt{E+m}\begin{pmatrix} \phi^s \\ \frac{\underline{\sigma}\cdot\underline{p}}{E+m}\phi^s \end{pmatrix} \tag{69}$$

$$v(\underline{p}, s) = \sqrt{E+m}\begin{pmatrix} \frac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi^s \\ \chi^s \end{pmatrix}, \tag{70}$$

where

$$\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{71}$$

$$\chi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{72}$$

The energy states $s = 1, 2$ are degenerate in energy; the *helicity* operator

$$\hat{h}(\underline{p}) = \frac{\underline{\Sigma}\cdot\underline{p}}{|\underline{p}|}, \quad \Sigma \equiv \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} = -\frac{1}{4}\epsilon_{ijk}[\gamma_j, \gamma_k] \tag{73}$$

commutes with the Hamiltonian and measures the spin projected along the direction of motion. In accordance with our earlier definition of $\bar{\psi}(x)$, let us list out a few properties of u, $\bar{u} \equiv u^\dagger\gamma^0$, v, and $\bar{v} \equiv v^\dagger\gamma^0$:

$$\sum_s u(p, s)\bar{u}(p, s) = p\!\!\!/ + m \equiv 2m\Lambda_+ \tag{74}$$

$$\sum_s v(p, s)\bar{v}(p, s) = p\!\!\!/ - m \equiv -2m\Lambda_- \tag{75}$$

$$\psi = \alpha u + \beta v \to \Lambda_+\psi = \alpha u,\ \Lambda_-\psi = \beta v \tag{76}$$

$$\Lambda_+^2 = \Lambda_+, \quad \Lambda_-^2 = \Lambda_-, \Lambda_+\Lambda_- = 0. \tag{77}$$