Path Integral Quantum Field Theory

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1 Gaussian Integrals

The General Gaussian integral, for a complex, symmetric $n \times n$ matrix A such that $\operatorname{Re} A \geq 0$ and the eigenvalues a_i of A are nonzero, is given by

$$\boxed{Z_A(b) = \int d^n x \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j + \sum_{i=1}^{n} b_i x_i\right) = (2\pi)^{n/2} (\det A)^{-1/2} \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} b_i A_{ij}^{-1} b_j\right).} \tag{1.1}$$

Let μ be a measure in $\mathbb{R}^n$; we define the expectation value

$$\begin{aligned} \langle F \rangle_\mu &= \int d\mu(x) F(x) \\ &= \int d^n x\, \Omega(x) F(x). \end{aligned} \tag{1.2}$$

The measure is normalized so that

$$\int d\mu(x) = 1. \tag{1.3}$$

We define the generating function

$$Z_\mu(b) = \left\langle e^{(b,x)} \right\rangle_\mu = \int d\mu(x) \exp\left(\sum_{i=1}^{n} b_i x_i\right), \tag{1.4}$$

which is a function of the n-dimensional vector b and the measure μ. The integrand can be expanded

$$\exp\left(\sum_{i=1}^{n} b_i x_i\right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{i_1 \cdots i_\ell = 1}^{n} b_{i_1} \cdots b_{i_\ell} x_{i_1} \cdots x_{i_\ell}. \tag{1.5}$$

Therefore, substituting the definition of the correlator

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu = \int d\mu(x) x_{i_1} \cdots x_{i_\ell} \tag{1.6}$$

we obtain

$$Z_\mu(b) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{i_1 \cdots i_\ell = 1}^{n} b_{i_1} \cdots b_{i_\ell} \langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu. \tag{1.7}$$

Furthermore, it is useful to notice that

$$\frac{\partial}{\partial b_k} Z_\mu(b) = \int d\mu(x) x_k \exp\left(\sum_{i=1}^{n} b_i x_i\right), \tag{1.8}$$

which allows the correlaters to be written as

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} Z_\mu(b) \Big|_{b=0} . \tag{1.9}$$

Let us consder the Gaussian measure

$$d\mu_0(x) = d^n x\, \Omega_0(x) = d^n x\, \mathcal{N}_0 \exp\left(-\frac{1}{2} \sum_{i,j=0}^{n} x_i A_{ij} x_j \right) , \tag{1.10}$$

where the normalization $\mathcal{N}_0$ is fixed by the normalization of the measure:

$$\mathcal{N}_0 = (2\pi)^{-n/2} (\det A)^{1/2}. \tag{1.11}$$

The generating function in this case can be readily computed using the boxed equation above:

$$Z(b) = \frac{Z_A(b)}{Z_A(0)} = \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i A^{-1}_{ij} b_j \right) . \tag{1.12}$$

In the future, we will drop any subscript A when referring to a Gaussian measure for simplicity. And therefore, defining $\Delta_{ij} = A^{-1}_{ij}$,

$$\boxed{\langle x_{i_1} \cdots x_{i_\ell} \rangle_0 = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} .} \tag{1.13}$$

1.1 Wick's Theorem

Let us start with a couple explicit examples, which we will compute in full detail in order to get familiar with the algebraic manipulations.

One-point function: Let k be an integer between 1 and n, we have

$$\begin{aligned}
\langle x_k \rangle_0 &= \frac{\partial}{\partial b_k} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= \left(\frac{1}{2} \sum_{j=1}^{n} \Delta_{kj} b_j + \frac{1}{2} \sum_{i=1}^{n} b_i \Delta_{ik} \right) \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= \left(\sum_{j=1}^{n} \Delta_{kj} b_j \right) \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= 0,
\end{aligned} \tag{1.14}$$

where the last equality comes from the fact that the expression is linear in b, and we need to set $b = 0$.

Two-point function: We now consider a pair of indices k, l, and compute

$$\begin{aligned}
\langle x_k x_l \rangle_0 &= \frac{\partial}{\partial b_l} \frac{\partial}{\partial b_k} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= \left[\Delta_{kl} + \left(\sum_{j=1}^{n} \Delta_{kj} b_j \right) \left(\sum_{m=1}^{n} \Delta_{lm} b_m \right) \right] \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= \Delta_{kl}
\end{aligned} \tag{1.15}$$

Thus, the generating function $Z(b)$ provides a systematic way to compute all correlators for a multi-dimensional Gaussian distribution. Having understood the general rule for such a process, we can generate a recipe to compute $\langle x_{i_1} \cdots x_{i_\ell} \rangle_0$, known as *Wick's Theorem*:

- Write down each $x_{i_1} \cdots x_{i_\ell}$ and organize them pairwise (i_p, i_q). Note that if ℓ must be even for the correlator to be non-zero.
- There are $(\ell - 1) \times (\ell - 3) \times \cdots \times 3 \times 1$ ways of doing this. Sum over all of these possible pairings.
- To each pair (i_p, i_q) associate a factor $\Delta_{i_p i_q}$.

Let us revisit our result for the **two-point function**: for $\langle x_{i_1} x_{i_2} \rangle_0$, there is only one possible pairing (i_1, i_2). Therefore

$$\langle x_{i_1} x_{i_2} \rangle_0 = \Delta_{i_1 i_2}. \tag{1.16}$$

Four-point function: For $\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_0$, there are three different pairings

$$P = \left\{ \left\{ (i_1, i_2), (i_3, i_4) \right\}, \left\{ (i_1, i_3), (i_2, i_4) \right\}, \left\{ (i_1, i_4), (i_2, i_3) \right\} \right\} \tag{1.17}$$

Wick's theorem then yields

$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_0 = \Delta_{i_1 i_2} \Delta_{i_3 i_4} + \Delta_{i_1 i_3} \Delta_{i_2 i_4} + \Delta_{i_1 i_4} \Delta_{i_2 i_3}. \tag{1.18}$$

We can also represent these pairings, *Wick contractions*, as should be familiar from canonical quantum field theory:

$$\langle x_i x_j \rangle_0 = \Delta_{ij} = \contraction{}{x}{_i}{x} x_i x_j \tag{1.19}$$

1.2 Perturbed Gaussian Measure

Let us now consider a more complicated measure,

$$\Omega(x) = \frac{1}{Z(\lambda)} e^{-S(x,\lambda)}, \tag{1.20}$$

where the normalization is given as usual by

$$Z(\lambda) = \int d^n x e^{-S(x,\lambda)} \tag{1.21}$$

and

$$\begin{aligned} S(x, \lambda) &= \frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j + \lambda V(x) \\ &= S_0(x) + \lambda V(x). \end{aligned} \tag{1.22}$$

We call $V(\lambda)$ the potential term, foreshadowing the physics we will be doing using the perturbed Gaussian measure. Furthermore

$$\begin{aligned} Z(\lambda) &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int d^n x \, V^k(x) \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j \right) \\ &= Z(0) \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left\langle V^k(x) \right\rangle_0, \end{aligned} \tag{1.23}$$

where in the first line we have expanded the exponential term containing $e^V(x)$ and in the second line used

$$\langle F(x) \rangle_0 = \frac{1}{Z(0)} \int d^n x \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j \right) F(x), \tag{1.24}$$

and $1/Z(0)$ referring to the normalization constant $\mathcal{N}_0$ from Eq. 1.10. Recalling Eq. 1.13, given that $F(x)$ admits a Taylor expansion

$$\begin{aligned} F(x) &= \sum_{\ell=0}^{\infty} \sum_{i_1 \cdots i_\ell = 1}^{n} F_{i_1 \cdots i_\ell} x_{i_1} \cdots x_{i_\ell} \\ &= F_0 + F_1 x_1 + \cdots + F_n x_n \\ &\quad + F_{11} x_1^2 + F_{12} x_1 x_2 + \cdots + F_{nn} x_n^2 \\ &\quad + \cdots \end{aligned} \tag{1.25}$$

where $F_{i_1 \cdots i_\ell}$ refer to the constant expansion coefficients, we can define the expectation value of F for a generic measure μ as

$$\langle F(x) \rangle_\mu = \sum_{\ell=0}^{\infty} \sum_{i_1 \cdots i_\ell = 1}^{n} F_{i_1 \cdots i_\ell} \langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu ; \tag{1.26}$$

for the case of a Gaussian measure we obtain

$$\begin{aligned} \langle F(x) \rangle_0 &= \sum_{\ell=0}^{\infty} \sum_{i_1 \cdots i_\ell = 1}^{n} F_{i_1 \cdots i_\ell} \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\ &= F\left(\frac{\partial}{\partial b} \right) \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} . \end{aligned} \tag{1.27}$$

Thus, returning to our potential term,

$$\begin{aligned} \frac{Z(\lambda)}{Z(0)} &= \left\langle e^{-\lambda V(x)} \right\rangle_0 \\ &= \exp\left[-\lambda V\left(\frac{\partial}{\partial b} \right) \right] \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} . \end{aligned} \tag{1.28}$$

1.3 Perturbed Gaussian Correlators

The perturbative treatment discussed above can be extended to compute moments of the distribution:

$$\langle x_{i_1} \cdots x_{i_l} \rangle = \frac{\int d^n x \, e^{-S(x,\lambda)} x_{i_1} \cdots x_{i_l}}{\int d^n x \, e^{-S(x,\lambda)}} = \frac{1}{Z(\lambda)} \int d^n x \, e^{-S(x,\lambda)} x_{i_1} \cdots x_{i_l} \tag{1.29}$$

which is often referred to as an ℓ-point correlator/function. Let us start our discussion with a simple example.
Two-point function: we need to evaluate

$$\begin{aligned} \int d^n x \, e^{-S(x,\lambda)} x_{i_1} x_{i_2} &= \int d^n x \, e^{-S_0(x,\lambda)} \left[\sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \lambda^k V^k(x) \right] x_{i_1} x_{i_2} \\ &= Z(0) \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \left\langle V^k(x) x_{i_1} x_{i_2} \right\rangle , \end{aligned} \tag{1.30}$$

where in the second line we have expressed the initial correlator in terms of correlators computed in teh Gaussian theory, denoted by $\langle \cdots \rangle_0$. The Gaussian correlators can be computed using Wick's theorem as before. We shall consider again a quartic potential

$$V(x) = \frac{1}{4!} \sum_{i=1}^{n} x_i^4 , \tag{1.31}$$

and compute all the terms in Eq. 1.30 order by order in λ up to order λ^2.

- $\mathcal{O}(\lambda^0)$: For $k=0$ we simply get the two-point Gaussion correlator

$$\langle x_{i_1} x_{i_2} \rangle = \Delta_{i_1 i_2}. \tag{1.32}$$

- $\mathcal{O}(\lambda^1)$: At first order in λ we have one insertion of V:

$$\langle V(x) x_{i_1} x_{i_2} \rangle_0 = \frac{1}{4!} \sum_{i=1}^{n} \left\langle x_i^4 x_{i_1} x_{i_2} \right\rangle_0 . \tag{1.33}$$

This Gaussian expectation value involving six factors of x can be evaluated using Wick's theorem. There are two types of contractions.

1. x_1 is contracted with x_2, and the four x_i are contracted amongst themselves:

$$x_{i_1} x_{i_2} (x_i x_i x_i x_i + x_i x_i x_i x_i + x_i x_i x_i x_i) = \Delta_{i_1 i_2} \left\langle x_i^4 \right\rangle_0 \tag{1.34}$$

2. x_1 and x_2 are contracted with some of the x_i; there are 12 such contractions:

$$x_{i_1} x_{i_2} x_i x_i x_i x_i + \cdots = \Delta_{i i_1} \Delta_{i i_2} \Delta_{ii} \times 4 \times 3. \tag{1.35}$$

Collecting all the terms yields

$$\frac{1}{4!} \sum_{i=1}^{n} \left\langle x_i^4 x_{i_1} x_{i_2} \right\rangle_0 = \Delta_{i_1 i_2} \frac{1}{4!} \sum_{i=1}^{n} \left\langle x_i^4 \right\rangle_0 + \frac{1}{4!} \times 4 \times 3 \sum_{i=1}^{n} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ii}. \tag{1.36}$$

- $\mathcal{O}(\lambda^2)$: At this order we need to evaluate

$$\frac{1}{2!} \left\langle V(x)^2 x_{i_1} x_{i_2} \right\rangle_0 = \frac{1}{2!} \frac{1}{(4!)^2} \sum_{i,j=1}^{n} \left\langle x_i^4 x_j^4 x_{i_1} x_{i_2} \right\rangle_0 . \tag{1.37}$$

There are five different types of contractions, each of them coming with a given multiplicity. We encourage the interested reader to compute those contributions, and check carefullly that the correct multiplicities are recovered. Collecting all contributions yields

$$\begin{aligned}
\int d^n x e^{-S(x,\lambda)} x_{i_1} x_{i_2} = Z(0) \Bigg[\Delta_{i_1 i_2} - \lambda \left(\Delta_{i_1 i_2} \frac{1}{4!} \sum_{i=1}^{n} \left\langle x_i^4 \right\rangle_0 + \frac{1}{2} \sum_{i=1}^{n} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ii} \right) \\
+ \lambda^2 \Bigg(\frac{1}{2!} \Delta_{i_1 i_2} \frac{1}{(4!)^2} \sum_{i,j=1}^{n} \left\langle x_i^4 x_j^4 \right\rangle_0 + \frac{1}{2!} \sum_{i=1}^{n} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ii} \frac{1}{4!} \sum_{j=0}^{n} \left\langle x_j^4 \right\rangle_0 \\
+ \frac{1}{4} \sum_{i,j=1}^{n} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ij}^2 \Delta_{jj} + \frac{1}{6} \sum_{i,j=1}^{n} \Delta_{i i_1} \Delta_{j i_2} \Delta_{ij}^3 \\
+ \frac{1}{4} \sum_{i,j=1}^{n} \Delta_{i i_1} \Delta_{j i_2} \Delta_{ij} \Delta_{ii} \Delta_{jj} \Bigg) \Bigg].
\end{aligned} \tag{1.38}$$

A term that factorises as the product of a subdiagram with external lines and a subdiagram that is made of loops only is called a *vacuum contribution*. For instance, $\Delta_{i_1 i_2} \frac{1}{4!} \sum_{i=1}^{n} \left\langle x_i^4 \right\rangle_0$ is a vacuum contribution because $\left\langle x_i^4 \right\rangle_0$ is one.

Finally, we need to divide this expression by $Z(\lambda)$ in order to obtain the two-point correlator as defined in Eq. 1.29. As a result, we obtain a factor $Z(0)/Z(\lambda)$ multiplying the expression inside the square bracket in Eq. 1.38. The ratio $Z(0)/Z(\lambda)$ may be computed using Eq. 1.28 and expanding as $1/(1-x) = 1 + x + x^2 + \cdots$ and sorting the terms by powers of λ to obtain

$$Z(0)/Z(\lambda) = 1 + \frac{1}{4!} \sum_{i=1}^{n} \left\langle x_i^4 \right\rangle_0 - \frac{1}{2!} \frac{1}{(4!)^2} \lambda^2 \sum_{i,j=1}^{n} \left\langle x_i^4 x_j^4 \right\rangle_0 + \frac{1}{(4!)^2} \lambda^2 \left(\sum_{i=1}^{n} \left\langle x_i^4 \right\rangle_0 \right)^2 + \mathcal{O}(\lambda^3) \tag{1.39}$$

This cancels all vacuum contributions.

1.4 Generating Functions for the Perturbed Gaussian Measure

We can now generalise the idea of a generating function to the case of a non-Gaussian measure. Introducing

$$Z(b,\lambda) = \int d^n x \exp[-S(x,\lambda) + b_i x_i], \tag{1.40}$$

and remembering that we can also write

$$\left\langle e^{b_i x_i} \right\rangle = Z(b,\lambda)/Z(\lambda), \tag{1.41}$$

the correlators in the perturbed measure are obtained by differentiation

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle = \frac{1}{Z(\lambda)} \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} Z(b,\lambda) \bigg|_{b=0} . \tag{1.42}$$

The logarithm of $Z(b,\lambda)$ is usually denoted $W(b,\lambda)$,

$$Z(b,\lambda) = e^{W(b,\lambda)}; \tag{1.43}$$

$W(b,\lambda)$ is the generator of the connected ℓ-point correlators $W_{i_1\cdots i_\ell}$, i.e. the correlators that can be represented as a single diagram, with ℓ open ends:

$$W_{i_1\cdots i_\ell} = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} W(b,\lambda) \bigg|_{b=0} . \tag{1.44}$$

In statistics, the $W_{i_1\cdots i_\ell}$ are called the *cumulants* of the probability distribution $e^{-S(x,\lambda)}$.