Canonical Quantum Field Theory

Mathias Driesse

November 10, 2022

1 Canonical Quantization of a Scalar Field

A scalar field $\phi(x)$ that obeys the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0 \tag{1}$$

also obeys the following *equal-time commutation relations* (ETCRs):

$$[\phi(t,\underline{x}), \dot{\phi}(t,\underline{x}')] = i\delta(\underline{x} - \underline{x}') \tag{2}$$

$$[\phi(t,\underline{x}), \phi(t,\underline{x}')] = [\dot{\phi}(t,\underline{x}), \dot{\phi}(t,\underline{x}')] = 0 \tag{3}$$

A scalar field may be expanded in terms of mode operators:

$$\phi(t,\underline{x}) = \phi^+(x) + \phi^-(x) \tag{4}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \left(a(\underline{p}) e^{-ip\cdot x} + a(\underline{p})^\dagger e^{ip\cdot x} \right) \tag{5}$$

where the creation and annihilation operators in momentum space obey the commutation relations

$$[a(\underline{p}), a^\dagger(\underline{p}')] = 2\omega(\underline{p})(2\pi)^3 \delta(\underline{p} - \underline{p}') \tag{6}$$

$$[a(\underline{p}), a(\underline{p}')] = [a^\dagger(\underline{p}), a^\dagger(\underline{p}')] = 0. \tag{7}$$

To prevent infinite contributions to the energy, we require that all terms be *normal ordered* - that is, all annihilation operators stand to the right of all creation operators. This results in the convenient property

$$\boxed{\langle 0| {:}\phi(x)\phi(y){:} |0\rangle = 0.}$$

We may simplify covariant commutation relations for the Klein-Gordon field by introducing three invariant functions

$$\Delta^+(x) \equiv -i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{-ip\cdot x} \tag{8}$$

$$\Delta^-(x) \equiv \Delta^+(x)^* = i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{ip\cdot x} \tag{9}$$

$$\Delta(x) \equiv \Delta^+(x) + \Delta^-(x). \tag{10}$$

All three satisfy the Klein-Gordon equation, and we may furthermore write Δ in a manifestly covariant form:

$$\Delta(x) = -i \int \hat{d}^4p \hat{\delta}(p^2 - m^2)\theta(p_0) e^{-ip\cdot x}. \tag{11}$$

Δ also satisfies the microcausality condition

$$\boxed{[\phi(x), \phi(y)] = 0, \quad (x-y)^2 < 0.}$$

2 Interactions and Scattering in ϕ^3 Theory

The Lagrangian for this theory is given by

$$\mathcal{L} \equiv \mathcal{L}_0 + \mathcal{L}_I \tag{12}$$

$$= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}g\phi^3. \tag{13}$$

We may write the Euler-Lagrange equation as

$$(\partial^2 + m^2)\phi = -\frac{1}{2}g\phi^2. \tag{14}$$

Since this is a nonlinear partial differential equation, we would like to proceed by solving it in terms of powers of coupling g.
We define the S-matrix as the time evolution operator between two states in the far past and the far future at which the particles involved may be considered non-interacting:

$$|\Psi, \infty\rangle = S\,|\Psi, -\infty\rangle\,. \tag{15}$$

Since in general, we must consider many possible initial and final states, we consider the S-matrix elements defined as

$$S_{fi} = \langle f|S|i\rangle \tag{16}$$

Note that the S-matrix is unitary ($S^\dagger S = SS^\dagger = 1$) even when particles in the state are destroyed and/or created.
Furthermore, in the future we will work in the interaction picture of quantum mechanics - a mix of the Schrödinger and Heisenberg pictures. We split the Hamiltonian into free and non-interacting parts $H = H_0 + H_I$ and let the time evolution of the state be governed by H_I, and the time evolution of the operators by H_0:

$$\frac{\partial}{\partial t}O(t) = i[H_0, O(t)] \tag{17}$$

$$i\frac{\partial}{\partial t}\,|\psi, t\rangle = H_I\,|\psi, t\rangle\,. \tag{18}$$

Integrating Eq. 18, we obtain a recursion relation for $|\Psi, t\rangle$:

$$|\Psi, t\rangle = |\Psi, -\infty\rangle + \int_{-\infty}^{t} dt_1 \frac{\partial}{\partial t}\,|\Psi, t_1\rangle \tag{19}$$

$$= |i\rangle - i\int_{-\infty}^{t} dt_1 H_I(t_1)\,|\Psi, t_1\rangle \tag{20}$$