Canonical Quantum Field Theory

Mathias Driesse

November 14, 2022

1 Canonical Quantization of a Scalar Field

A scalar field $\phi(x)$ that obeys the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0 \tag{1}$$

also obeys the following *equal-time commutation relations* (ETCRs):

$$[\phi(t,\underline{x}), \pi(t,\underline{x}')] = [\phi(t,\underline{x}), \dot{\phi}(t,\underline{x}')] = i\delta(\underline{x} - \underline{x}') \tag{2}$$

$$[\phi(t,\underline{x}), \phi(t,\underline{x}')] = [\pi(t,\underline{x}), \pi(t,\underline{x}')] = [\dot{\phi}(t,\underline{x}), \dot{\phi}(t,\underline{x}')] = 0 \tag{3}$$

where the conjugate momentum $\pi(x)$ is the usual $\frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$. A scalar field may be expanded in terms of mode operators:

$$\phi(t,\underline{x}) = \phi^+(x) + \phi^-(x) \tag{4}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \left(a(\underline{p}) e^{-ip\cdot x} + a(\underline{p})^\dagger e^{ip\cdot x} \right) \tag{5}$$

where the creation and annihilation operators in momentum space obey the commutation relations

$$[a(\underline{p}), a^\dagger(\underline{p}')] = 2\omega(\underline{p})(2\pi)^3 \delta(\underline{p} - \underline{p}') \tag{6}$$

$$[a(\underline{p}), a(\underline{p}')] = [a^\dagger(\underline{p}), a^\dagger(\underline{p}')] = 0. \tag{7}$$

To prevent infinite contributions to the energy, we require that all terms be *normal ordered* - that is, all annihilation operators stand to the right of all creation operators. This results in the convenient property

$$\boxed{\langle 0| {:}\phi(x)\phi(y){:} |0\rangle = 0.}$$

We may simplify covariant commutation relations for the Klein-Gordon field by introducing three invariant functions

$$\Delta^+(x) \equiv -i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{-ip\cdot x} \tag{8}$$

$$\Delta^-(x) \equiv \Delta^+(x)^* = i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{ip\cdot x} \tag{9}$$

$$\Delta(x) \equiv \Delta^+(x) + \Delta^-(x). \tag{10}$$

All three satisfy the Klein-Gordon equation, and we may furthermore write Δ in a manifestly covariant form:

$$\Delta(x) = -i \int \hat{d}^4p \hat{\delta}(p^2 - m^2)\theta(p_0) e^{-ip\cdot x}. \tag{11}$$

Δ also satisfies the microcausality condition

$$\boxed{[\phi(x), \phi(y)] = 0, \quad (x-y)^2 < 0.}$$

2 Interactions and Scattering in ϕ^3 Theory

The Lagrangian for this theory is given by

$$\begin{aligned}\mathcal{L} &\equiv \mathcal{L}_0 + \mathcal{L}_I \quad (12)\\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}g\phi^3. \quad (13)\end{aligned}$$

We may write the Euler-Lagrange equation as

$$(\partial^2 + m^2)\phi = -\frac{1}{2}g\phi^2. \quad (14)$$

Since this is a nonlinear partial differential equation, we would like to proceed by solving it in terms of powers of coupling g.
We define the S-matrix as the time evolution operator between two states in the far past and the far future at which the particles involved may be considered non-interacting:

$$|\Psi,\infty\rangle = S\,|\Psi,-\infty\rangle\,. \quad (15)$$

Since in general, we must consider many possible initial and final states, we consider the S-matrix elements defined as

$$S_{fi} = \langle f|S|i\rangle \quad (16)$$

Note that the S-matrix is unitary ($S^\dagger S = SS^\dagger = 1$) even when particles in the state are destroyed and/or created.
Furthermore, in the future we will work in the interaction picture of quantum mechanics - a mix of the Schrödinger and Heisenberg pictures. We split the Hamiltonian into free and non-interacting parts $H = H_0 + H_I$ and let the time evolution of the state be governed by H_I, and the time evolution of the operators by H_0:

$$\begin{aligned}\frac{\partial}{\partial t}O(t) &= i[H_0, O(t)] \quad (17)\\ i\frac{\partial}{\partial t}\,|\psi,t\rangle &= H_I\,|\psi,t\rangle\,. \quad (18)\end{aligned}$$

Integrating Eq. 18, we obtain a recursion relation for $|\Psi,t\rangle$:

$$\begin{aligned}|\Psi,t\rangle &= |\Psi,-\infty\rangle + \int_{-\infty}^{t} dt_1 \frac{\partial}{\partial t}\,|\Psi,t_1\rangle \quad (19)\\ &= |i\rangle - i\int_{-\infty}^{t} dt_1 H_I(t_1)\,|\Psi,t_1\rangle\,. \quad (20)\end{aligned}$$

By substituting the recursion relation into itself, we write the Dyson series explicitly as

$$S = 1 + \sum_{n=1}^{\infty}(-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_I\left(t_1\right) H_I\left(t_2\right) \cdots H_I\left(t_n\right). \quad (21)$$

The introduction of the time ordered product of two operators

$$T(A(t_1)B(t_2)) = \theta(t_1 - t_2)A(t_1)B(t_2) + \theta(t_2 - t_1)B(t_2)A(t_1) \quad (22)$$

and the use of the Hamiltonian density instead of the Hamiltonain yields the manifestly covariant expression

$$\boxed{S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1 \int d^4x_2 \cdots \int d^4x_n T\left(\mathcal{H}_I\left(x_1\right)\mathcal{H}_I\left(x_2\right)\cdots\mathcal{H}_I\left(x_n\right)\right).}$$

Since our calculations involve time-ordered products through *Wick's Theorem*, which we will see shortly, it is convenient to define the Feynman propagator

$$\Delta_F(x-x') \equiv \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x). \tag{23}$$

Thus, making use of our knowledge of the functions $\Delta^\pm$, we find that

$$\boxed{\quad i\Delta_F(x-x') = \wick{\c{\phi}(x)\c{\phi}(x')} = \langle 0|T\left(\phi(x)\phi(x')\right)|0\rangle\,, \quad}$$

where the bracket on top denotes a *Wick contraction*. How can we relate the time-ordered product in general, normal ordering and Wick contractions? The answer is *Wick's Theorem*. For unequal times

$$\begin{aligned} T(ABCD\cdots WXYZ) = {}&{:}ABCD\cdots WXYZ{:} \\ &+ {:}\wick{\c{A}\c{B}}C\cdots YZ{:} + {:}\wick{\c{A}B\c{C}}\cdots YZ{:} + \cdots + {:}ABC\cdots \wick{\c{Y}\c{Z}}{:} \\ &+ {:}\wick{\c1{A}\c1{B}\c2{C}\c2{D}}\cdots WXYZ{:} + \cdots + {:}ABCD\cdots \wick{\c1{W}\c1{X}\c2{Y}\c2{Z}}{:} \\ &+ \cdots\,, \end{aligned} \tag{24}$$

which simplifies to, for example, in the case of a single time ordering to

$$T(A(x)B(x')) = {:}A(x)B(x'){:} + \langle 0|T(A(x)B(x'))|0\rangle\,. \tag{25}$$

When performing the Dyson expansion, we will need to evaluate the Feynman propagator many times. Therefore, it is usefeful to rewrite it in terms of a single integral

$$\Delta_F(x) = \int \hat{d}^4p \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon} \tag{26}$$

where $\varepsilon \equiv 2\eta\omega(\underline{p})$ is a small number that takes care of the contours in the complex plane that we have to take due to the poles at $p = \pm m$. In fact, Δ_F is a Green's function for the Klein-Gordon equation, corresponding to Dirichlet boundary conditions (initial and final conditions on $\phi(x)$), and is thus appropriate to our quantum paradigm of initial and final states.
As an example, let us expand the S-matrix in position space. In ϕ^3 theory, we have $\mathcal{H}_I(x) = \frac{g}{3!}\,{:}\phi^3(x){:}$, which we expand in terms of ϕ^+ (which destroys a particle at x) and ϕ^- (which creates a particle at x).

1. For $n = 0$, nothing happens.

2. For $n = 1$,

   $$S^{(1)} = \frac{-ig}{3!}\int d^4x\, T({:}\phi^3(x){:}). \tag{27}$$

   Since there are no *unequal times*, there are no terms with contractions. The terms without contractions cannot conserve momentum, so they vanish.

3. For $n = 2$,

   $$S^{(2)} = -\frac{g^2}{2!(3!)^2}\int d^4x d^4y\, T\left({:}\phi^3(x){:}{:}\phi^3(y){:}\right), \tag{28}$$

   which we may split up into terms containing zero, one, two and three contractions.

   (a) $S_0^{(2)}$ only contains one term, and it is proportional to ${:}\phi^3(x)\phi^3(y){:}$. The interactions at x and y are unrelated, so they factorize and vanish due to momentum conservation as in the $n = 1$ case.

   (b) $S_1^{(2)}$ contains terms with one contraction, i.e. proportional to $\wick{\c{\phi}(x)\c{\phi}(y)}\ {:}\phi^2(x)\phi^2(y){:}$. Expanding it out in terms of ϕ^+ and ϕ^-, there are three terms that survive momentum conservation:

   $$\begin{aligned} &\phi^-(y)\phi^-(y)\wick{\c{\phi}(x)\c{\phi}(y)}\phi^+(x)\phi^+(x), \\ &\phi^-(x)\phi^-(y)\wick{\c{\phi}(x)\c{\phi}(y)}\phi^+(x)\phi^+(y), \\ &\phi^-(y)\phi^-(x)\wick{\c{\phi}(x)\c{\phi}(y)}\phi^+(x)\phi^+(y). \end{aligned} \tag{29}$$

There is also a combinatorial 3^2 factor, since there are 3 ways of contracting $\phi(x)$ and 3 ways of contracting $\phi(y)$.

(c) $S_2^{(2)}$ is given by the term $\overbracket{\phi(x)\phi(y)}\overbracket{\phi(x)\phi(y)}$ $:\phi(x)\phi(y):$, which diverges and forms the so-called *self-energy correction*.

(d) $S_3^{(2)}$ is given by the term $\overbracket{\phi(x)\phi(y)}\overbracket{\phi(x)\phi(y)}\overbracket{\phi(x)\phi(y)}$, which also diverges, but as it is disconnected it makes no contribution to the scattering process.

This is a lot of work for calculating the S-matrix! Fortunately, we are more interested in calculating the matrix elements $S_{fi} = \langle f|S|i\rangle$ than S itself. Since we label free particle initial and final states by their definite on-shell momentum $\left|\underline{p}\right\rangle = a^\dagger(\underline{p})\left|0\right\rangle$, we should work in momentum space. Then

$$\begin{aligned}\phi^+(x)\left|\underline{p}\right\rangle &= \int \frac{\hat{d}^3\underline{p}}{2E'}e^{-ip'\cdot x}a(\underline{p}')a^\dagger(\underline{p})\left|0\right\rangle \\ &= \int d^3\underline{p}'e^{-ip'\cdot x}\delta^3(\underline{p}-\underline{p}')\left|0\right\rangle \\ &= e^{-ip\cdot x}\left|0\right\rangle\end{aligned} \tag{30}$$

while

$$\begin{aligned}\left\langle\underline{p}\right|\phi^-(x) &= \left\langle 0\right|a(\underline{p})\int \frac{\hat{d}^3\underline{p}}{2E'}e^{ip'\cdot x}a^\dagger(\underline{p}) \\ &= \left\langle 0\right|\int d^3\underline{p}'e^{ip'\cdot x}\delta^3(\underline{p}-\underline{p}') \\ &= e^{ip\cdot x}\left\langle 0\right|.\end{aligned} \tag{31}$$

As an example, let us perform a momentum-space calculation of the 2→2 process through the s-channel. We may do this by simply inserting the first line of Eq. 29:

$$\begin{aligned}\langle f|S_{1,s}^{(2)}|i\rangle &= -\frac{3^2g^2}{(3!)^2}\int d^4x\int d^4y\,\langle q;q'|\phi^-(y)\phi^-(y)\overbracket{\phi(x)\phi(y)}\phi^+(x)\phi^+(x)|p;p'\rangle \\ &= -\frac{g^2}{4}\int d^4x\int d^4y\,4e^{i(q+q')\cdot x}\,i\int \hat{d}^4k\frac{e^{-ik\cdot(y-x)}}{k^2-m^2+i\varepsilon}e^{-i(p+p')\cdot x} \\ &= -ig^2\int \hat{d}^4k\,\hat{\delta}^4\left(q+q'-k\right)\hat{\delta}^4\left(k-p-p'\right)\frac{1}{k^2-m^2+i\varepsilon} \\ &= \hat{\delta}^4\left(q+q'-p-p'\right)\frac{-ig^2}{(p+p')^2-m^2},\end{aligned} \tag{32}$$

where on the second line we have applied Eqs. 30, 31 and inserted our expression for the Feynman propagator, with an additional factor of 4 coming from the choice of apply $(\phi^+)^2$ on $|p;p'\rangle$. For the other two contributions we have the same, replacing the denominator by $(p-q)^2-m^2$ and $(p-q')-m^2$ respectively. It is therefore useful to write down the *Mandelstam invariants* s, t, u, defined as

$$\begin{aligned}s &= (p_1+p_2)^2 = (p_3+p_4)^2 \\ t &= (p_1-p_3)^2 = (p_2-p_4)^2 \\ u &= (p_1-p_4)^2 = (p_2-p_3)^2.\end{aligned} \tag{33}$$

Applying a similar process to the loop diagram leads to a divergent term, which we will look at in more detail through *renormalization*.
An attentive reader, having calculated all three channels, might have noticed that the process above seems fairly simple and algorithmic. Indeed, we can define the so-called *Feynman rules*, which for any theory allow us to write down the contribution from each diagram without performing any nasty integrations. For scalar ϕ^3 theory, we have:

- for each ϕ^3 vertex, a factor of $-ig$ and a delta function ensuring momentum conservation

- for each external line, a factor of 1
- for each internal line of momentum k, a factor $i/(k^2 - m^2 + i\varepsilon)$
- for each momentum k not fixed by momentum conservation, an integral $\int \hat{d}^4 k$
- a symmetry factor.

3 Complex Scalar Fields and Charge Conservation

In quantum field theory, for the S-matrix to be unitary, we only require that the Lagrangian density $\mathcal{L}$, the Hamiltonian density $\mathcal{H}$, and the action S be real. Thus, a quantum field may also be complex as long as above conditions are satisfied. They also have the important new property of charge conservation.
If we have a multiplet (several) of scalar fields $\phi_r(x)$, where $r = 1, 2, \cdots N$, then we may quantize them as we have done for a single scalar field, the only difference being an additional delta function in the nonzero commutators.
A complex scalar field $\phi(x)$ can be expressed in terms of two independent fields ϕ_1 and ϕ_2:

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)). \tag{34}$$

Then we have the Lagrangian density

$$\mathcal{L} = \left(\partial_\mu \phi^\dagger(x)\right)\left(\partial^\mu \phi(x)\right) - m^2 \phi^\dagger(x)\phi(x) \tag{35}$$

along with the conjugate momentum $\pi(x) = \frac{\partial}{\partial \mathcal{L}\dot{\phi}(x)} = \dot{\phi}^\dagger(x)$, which provides the ETCRs

$$[\phi(t, \underline{x}), \pi(t, \underline{x}')] = [\phi^\dagger(t, \underline{x}), \pi^\dagger(t, \underline{x}')] = i\delta(\underline{x} - \underline{x}'). \tag{36}$$

Each of the fields $\phi(x)$ and $\phi^\dagger(x)$ obeys the Klein-Gordon equation independently. We expand the fields in terms of mode operators

$$\phi(x) \tag{37}$$