Path Integral Quantum Field Theory

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1 Gaussian Integrals

The General Gaussian integral, for a complex, symmetric $n \times n$ matrix A such that $\operatorname{Re} A \geq 0$ and the eigenvalues a_i of A are nonzero, is given by

$$
Z_A(b) = \int d^n x \exp \left(-\frac{1}{2} \sum_{i,j=1}^n x_i A_{ij} x_j + \sum_{i=1}^n b_i x_i \right) = (2\pi)^{n/2} (\det A)^{-1/2} \exp \left(\frac{1}{2} \sum_{i,j=1}^n b_i A_{ij}^{-1} b_j \right).
$$

(1.1)

Let μ be a measure in $\mathbb{R}^n$; we define the expectation value

$$
\begin{aligned}
\langle F \rangle_\mu &= \int d\mu(x) F(x) \\
&= \int d^n x \, \Omega(x) F(x).
\end{aligned}
$$

(1.2)

The measure is normalized so that

$$
\int d\mu(x) = 1.
$$

(1.3)

We define the generating function

$$
Z_\mu(b) = \left\langle e^{(b,x)} \right\rangle_\mu = \int d\mu(x) \exp \left(\sum_{i=1}^n b_i x_i \right),
$$

(1.4)

which is a function of the n-dimensional vector b and the measure μ. The integrand can be expanded

$$
\exp \left(\sum_{i=1}^n b_i x_i \right) = \sum_{\ell=0}^\infty \frac{1}{\ell!} \sum_{i_1 \cdots i_\ell=1}^n b_{i_1} \cdots b_{i_\ell} x_{i_1} \cdots x_{i_\ell}.
$$

(1.5)

Therefore, substituting the definition of the correlator

$$
\langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu = \int d\mu(x) x_{i_1} \cdots x_{i_\ell}
$$

(1.6)

we obtain

$$
Z_\mu(b) = \sum_{\ell=0}^\infty \frac{1}{\ell!} \sum_{i_1 \cdots i_\ell=1}^n b_{i_1} \cdots b_{i_\ell} \langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu.
$$

(1.7)

Furthermore, it is useful to notice that

$$
\frac{\partial}{\partial b_k} Z_\mu(b) = \int d\mu(x) x_k \exp \left(\sum_{i=1}^n b_i x_i \right),
$$

(1.8)

which allows the correlaters to be written as

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} Z_\mu(b) \bigg|_{b=0}. \tag{1.9}$$

Let us consder the Gaussian measure

$$d\mu_0(x) = d^n x \, \Omega_0(x) = d^n x \, \mathcal{N}_0 \exp\left(-\frac{1}{2} \sum_{i,j=0}^{n} x_i A_{ij} x_j\right), \tag{1.10}$$

where the normalization $\mathcal{N}_0$ is fixed by the normalization of the measure:

$$\mathcal{N}_0 = (2\pi)^{-n/2} (\det A)^{1/2}. \tag{1.11}$$

The generating function in this case can be readily computed using the boxed equation above:

$$Z(b) = \frac{Z_A(b)}{Z_A(0)} = \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i A_{ij}^{-1} b_j\right). \tag{1.12}$$

In the future, we will drop any subscript A when referring to a Gaussian measure for simplicity. And therefore, defining $\Delta_{ij} = A_{ij}^{-1}$,

$$\boxed{\langle x_{i_1} \cdots x_{i_\ell} \rangle_0 = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right) \bigg|_{b=0}.} \tag{1.13}$$

1.1 Wick's Theorem

Let us start with a couple explicit examples, which we will compute in full detail in order to get familiar with the algebraic manipulations.

One-point function: Let k be an integer between 1 and n, we have

$$\begin{aligned}
\langle x_k \rangle_0 &= \frac{\partial}{\partial b_k} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right) \bigg|_{b=0} \\
&= \left(\frac{1}{2} \sum_{j=1}^{n} \Delta_{kj} b_j + \frac{1}{2} \sum_{i=1}^{n} b_i \Delta_{ik}\right) \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right) \bigg|_{b=0} \\
&= \left(\sum_{j=1}^{n} \Delta_{kj} b_j\right) \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right) \bigg|_{b=0} \\
&= 0,
\end{aligned} \tag{1.14}$$

where the last equality comes from the fact that the expression is linear in b, and we need to set $b = 0$.

Two-point function: We now consider a pair of indices k, l, and compute

$$\begin{aligned}
\langle x_k x_l \rangle_0 &= \frac{\partial}{\partial b_l} \frac{\partial}{\partial b_k} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right) \bigg|_{b=0} \\
&= \left[\Delta_{kl} + \left(\sum_{j=1}^{n} \Delta_{kj} b_j\right) \left(\sum_{m=1}^{n} \Delta_{lm} b_m\right)\right] \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right) \bigg|_{b=0} \\
&= \Delta_{kl}
\end{aligned} \tag{1.15}$$

Thus, the generating function $Z(b)$ provides a systematic way to compute all correlators for a multi-dimensional Gaussian distribution. Having understood the general rule for such a process, we can generate a recipe to compute $\langle x_{i_1} \cdots x_{i_\ell} \rangle_0$, known as *Wick's Theorem*:

- Write down each $x_{i_1} \cdots x_{i_\ell}$ and organize them pairwise (i_p, i_q). Note that if ℓ must be even for the correlator to be non-zero.

- There are $(\ell - 1) \times (\ell - 3) \times \cdots \times 3 \times 1$ ways of doing this. Sum over all of these possible pairings.

- To each pair (i_p, i_q) associate a factor $\Delta_{i_p i_q}$.

Let us revisit our result for the **two-point function**: for $\langle x_{i_1} x_{i_2} \rangle_0$, there is only one possible pairing (i_1, i_2). Therefore

$$\langle x_{i_1} x_{i_2} \rangle_0 = \Delta_{i_1 i_2}. \tag{1.16}$$

Four-point function: For $\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_0$, there are three different pairings

$$P = \{\{(i_1, i_2), (i_3, i_4)\}, \{(i_1, i_3), (i_2, i_4)\}, \{(i_1, i_4), (i_2, i_3)\}\} \tag{1.17}$$

Wick's theorem then yields

$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_0 = \Delta_{i_1 i_2} \Delta_{i_3 i_4} + \Delta_{i_1 i_3} \Delta_{i_2 i_4} + \Delta_{i_1 i_4} \Delta_{i_2 i_3}. \tag{1.18}$$

We can also represent these pairings, *Wick contractions*, as should be familiar from canonical quantum field theory:

$$\langle x_i x_j \rangle_0 = \Delta_{ij} = \overline{x_i x_j} \tag{1.19}$$

1.2 Perturbed Gaussian Measure

Let us now consider a more complicated measure,

$$\Omega(x) = \frac{1}{Z(\lambda)} e^{-S(x,\lambda)}, \tag{1.20}$$

where the normalization is given as usual by

$$Z(\lambda) = \int d^n x\, e^{-S(x,\lambda)} \tag{1.21}$$

and

$$S(x,\lambda) = \frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j + \lambda V(x) \tag{1.22}$$
$$= S_0(x) + \lambda V(x).$$

We call $V(\lambda)$ the potential term, foreshadowing the physics we will be doing using the perturbed Gaussian measure. Furthermore

$$Z(\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int d^n x\, V^k(x) \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j\right) \tag{1.23}$$
$$= Z(0) \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \langle V^k(x) \rangle_0,$$

where in the first line we have expanded the exponential term containing $e^V(x)$ and in the second line used

$$\langle F(x) \rangle_0 = \frac{1}{Z(0)} \int d^n x \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j\right) F(x), \tag{1.24}$$

and $1/Z(0)$ referring to the normalization constant $\mathcal{N}_0$ from Eq. 1.10. Recalling Eq. 1.13, given that $F(x)$ admits a Taylor expansion

$$
\begin{aligned}
F(x) &= \sum_{\ell=0}^{\infty} \sum_{i_1 \cdots i_\ell = 1}^{n} F_{i_1 \cdots i_\ell} x_{i_1} \cdots x_{i_\ell} \\
&= F_0 + F_1 x_1 + \cdots + F_n x_n \\
&\quad + F_{11} x_1^2 + F_{12} x_1 x_2 + \cdots + F_{nn} x_n^2 \\
&\quad + \cdots
\end{aligned}
\tag{1.25}
$$

where $F_{i_1 \cdots i_\ell}$ refer to the constant expansion coefficients, we can define the expectation value of F for a generic measure μ as

$$
\langle F(x) \rangle_\mu = \sum_{\ell=0}^{\infty} \sum_{i_1 \cdots i_\ell = 1}^{n} F_{i_1 \cdots i_\ell} \langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu \, ;
\tag{1.26}
$$

for the case of a Gaussian measure we obtain

$$
\begin{aligned}
\langle F(x) \rangle_0 &= \sum_{\ell=0}^{\infty} \sum_{i_1 \cdots i_\ell = 1}^{n} F_{i_1 \cdots i_\ell} \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \bigg|_{b=0} \\
&= F\left(\frac{\partial}{\partial b} \right) \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \bigg|_{b=0}.
\end{aligned}
\tag{1.27}
$$

Thus, returning to our potential term,

$$
\begin{aligned}
\frac{Z(\lambda)}{Z(0)} &= \left\langle e^{-\lambda V(x)} \right\rangle_0 \\
&= \exp\left[-\lambda V\left(\frac{\partial}{\partial b} \right) \right] \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \bigg|_{b=0}.
\end{aligned}
\tag{1.28}
$$

1.3 Perturbed Gaussian Correlators

The perturbative treatment discussed above can be extended to compute moments of the distribution:

$$
\langle x_{i_1} \cdots x_{i_l} \rangle = \frac{\int d^n x \, e^{-S(x,\lambda)} x_{i_1} \cdots x_{i_l}}{\int d^n x \, e^{-S(x,\lambda)}} = \frac{1}{Z(\lambda)} \int d^n x \, e^{-S(x,\lambda)} x_{i_1} \cdots x_{i_l}
\tag{1.29}
$$

which is often referred to as an ℓ-point correlator/function. Let us start our discussion with a simple example. **Two-point function**: we need to evaluate

$$
\begin{aligned}
\int d^n x \, e^{-S(x,\lambda)} x_{i_1} x_{i_2} &= \int d^n x \, e^{-S_0(x,\lambda)} \left[\sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \lambda^k V^k(x) \right] x_{i_1} x_{i_2} \\
&= Z(0) \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \left\langle V^k(x) x_{i_1} x_{i_2} \right\rangle,
\end{aligned}
\tag{1.30}
$$

where in the second line we have expressed the initial correlator in terms of correlators computed in teh Gaussian theory, denoted by $\langle \cdots \rangle_0$. The Gaussian correlators can be computed using Wick's theorem as before. We shall consider again a quartic potential

$$
V(x) = \frac{1}{4!} \sum_{i=1}^{n} x_i^4,
\tag{1.31}
$$

and compute all the terms in Eq. 1.30 order by order in λ up to order λ^2.

- $\mathcal{O}(\lambda^0)$: For $k = 0$ we simply get the two-point Gaussion correlator

$$\langle x_{i_1} x_{i_2} \rangle = \Delta_{i_1 i_2}. \tag{1.32}$$

- $\mathcal{O}(\lambda^1)$: At first order in λ we have one insertion of V:

$$\langle V(x) x_{i_1} x_{i_2} \rangle_0 = \frac{1}{4!} \sum_{i=1}^{n} \langle x_i^4 x_{i_1} x_{i_2} \rangle_0 . \tag{1.33}$$

This Gaussian expectation value involving six factors of x can be evaluated using Wick's theorem. There are two types of contractions.

1. x_1 is contracted with x_2, and the four x_i are contracted amongst themselves:

$$\overline{x_{i_1} x_{i_2}} (\overline{x_i x_i}\,\overline{x_i x_i} + \overline{x_i x_i x_i}\, x_i + \overline{x_i x_i x_i}\, x_i) = \Delta_{i_1 i_2} \langle x_i^4 \rangle_0 \tag{1.34}$$

2. x_1 and x_2 are contracted with some of the x_i; there are 12 such contractions:

$$\overline{x_{i_1} x_{i_2} x_i}\, x_i x_i x_i + \cdots = \Delta_{i i_1} \Delta_{i i_2} \Delta_{ii} \times 4 \times 3. \tag{1.35}$$

Collecting all the terms yields

$$\frac{1}{4!} \sum_{i=1}^{n} \langle x_i^4 x_{i_1} x_{i_2} \rangle_0 = \Delta_{i_1 i_2} \frac{1}{4!} \sum_{i=1}^{n} \langle x_i^4 \rangle_0 + \frac{1}{4!} \times 4 \times 3 \sum_{i=1}^{n} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ii}. \tag{1.36}$$

- $\mathcal{O}(\lambda^2)$: At this order we need to evaluate

$$\frac{1}{2!} \langle V(x)^2 x_{i_1} x_{i_2} \rangle_0 = \frac{1}{2!} \frac{1}{(4!)^2} \sum_{i,j=1}^{n} \langle x_i^4 x_j^4 x_{i_1} x_{i_2} \rangle_0 . \tag{1.37}$$

There are five different types of contractions, each of them coming with a given multiplicity. We encourage the interested reader to compute those contributions, and check carefullly that the correct multiplicities are recovered. Collecting all contributions yields

$$
\int d^n x\, e^{-S(x,\lambda)} x_{i_1} x_{i_2} = Z(0) \Bigg[\Delta_{i_1 i_2} - \lambda \left(\Delta_{i_1 i_2} \frac{1}{4!} \sum_{i=1}^{n} \langle x_i^4 \rangle_0 + \frac{1}{2} \sum_{i=1}^{n} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ii} \right)
$$
$$
+ \lambda^2 \left(\frac{1}{2!} \Delta_{i_1 i_2} \frac{1}{(4!)^2} \sum_{i,j=1}^{n} \langle x_i^4 x_j^4 \rangle_0 + \frac{1}{2!} \sum_{i=1}^{n} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ii} \frac{1}{4!} \sum_{j=0}^{n} \langle x_j^4 \rangle_0 \right.
$$
$$
+ \frac{1}{4} \sum_{i,j=1}^{n} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ij}^2 \Delta_{jj} + \frac{1}{6} \sum_{i,j=1}^{n} \Delta_{i i_1} \Delta_{j i_2} \Delta_{ij}^3
$$
$$
\left. + \frac{1}{4} \sum_{i,j=1}^{n} \Delta_{i i_1} \Delta_{j i_2} \Delta_{ij} \Delta_{ii} \Delta_{jj} \right) \Bigg].
\tag{1.38}
$$

A term that factorises as the product of a subdiagram with external lines and a subdiagram that is made of loops only is called a *vacuum contribution*. For instance, $\Delta_{i_1 i_2} \frac{1}{4!} \sum_{i=1}^{n} \langle x_i^4 \rangle_0$ is a vacuum contribution because $\langle x_i^4 \rangle_0$ is one.

Finally, we need to divide this expression by $Z(\lambda)$ in order to obtain the two-point correlator as defined in Eq. 1.29. As a result, we obtain a factor $Z(0)/Z(\lambda)$ multiplying the expression inside the square bracket in Eq. 1.38. The ratio $Z(0)/Z(\lambda)$ may be computed using Eq. 1.28 and expanding as $1/(1-x) = 1 + x + x^2 + \cdots$ and sorting the terms by powers of λ to obtain

$$Z(0)/Z(\lambda) = 1 + \frac{1}{4!} \sum_{i=1}^{n} \langle x_i^4 \rangle_0 - \frac{1}{2!} \frac{1}{(4!)^2} \lambda^2 \sum_{i,j=1}^{n} \langle x_i^4 x_j^4 \rangle_0 + \frac{1}{(4!)^2} \lambda^2 \left(\sum_{i=1}^{n} \langle x_i^4 \rangle_0 \right)^2 + \prime(\lambda^3) \tag{1.39}$$

This cancels all vacuum contributions.

1.4 Generating Functions for the Perturbed Gaussian Measure

We can now generalise the idea of a generating function to the case of a non-Gaussian measure. Introducing

$$Z(b, \lambda) = \int d^n x \exp[-S(x, \lambda) + b_i x_i], \qquad (1.40)$$

and remembering that we can also write

$$\langle e^{b_i x_i} \rangle = Z(b, \lambda)/Z(\lambda), \qquad (1.41)$$

the correlators in the perturbed measure are obtained by differentiation

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle = \left. \frac{1}{Z(\lambda)} \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} Z(b, \lambda) \right|_{b=0}. \qquad (1.42)$$

The logarithm of $Z(b, \lambda)$ is usually denoted $W(b, \lambda)$,

$$Z(b, \lambda) = e^{W(b, \lambda)}; \qquad (1.43)$$

$W(b, \lambda)$ is the generator of the connected ℓ-point correlators $W_{i_1 \cdots i_\ell}$, i.e. the correlators that can be represented as a single diagram, with ℓ open ends:

$$W_{i_1 \cdots i_\ell} = \left. \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} W(b, \lambda) \right|_{b=0}. \qquad (1.44)$$

In statistics, the $W_{i_1 \cdots i_\ell}$ are called the *cumulants* of the probability distribution $e^{-S(x, \lambda)}$.

2 Path Integral Quantum Mechanics

In this section, we will discuss the essentials of the path integral formulation of quantum mechanics, which will allow us to connect Gaussian integrals and the calculation of correlators to observables in quantum field theory. We consider a point particle in one dimension, with position operator $\hat{Q}$ such that

$$\hat{Q} |q\rangle = q |q\rangle \qquad (2.1)$$

with a completeness condition

$$\int dq |q\rangle \langle q| = 1. \qquad (2.2)$$

The aim of this section is to find an expression for the quantum amplitude $\langle q't'|qt\rangle = \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle$ for a Hamiltonian $\hat{H}$.

2.1 Setting up the Path Integral

In order to proceed with the calculation, let us define $T = t' - t$ to be the size of the time interval, and $\epsilon = T/n$, where n is an integer. Then

$$
\begin{aligned}
t_0 &= t \\
t_k &= t_0 = k\epsilon, k = 1, \cdots, n-1 \\
t_n &= t + n\epsilon = t'
\end{aligned}
\qquad (2.3)
$$

and

$$
\begin{aligned}
\langle q'|e^{-i\hat{H}(t'-t)}|q\rangle &= \langle q'|e^{-i\hat{H}T}|q\rangle \\
&= \langle q'|\left(e^{-i\hat{H}\epsilon}\right) \cdots \left(e^{-i\hat{H}\epsilon}\right)|q\rangle,
\end{aligned}
\qquad (2.4)
$$

where the expression in the second line contains n factors. Inserting the completeness relation $n-1$ times,

$$\langle q't'|qt\rangle \int \prod_{k=1}^{n-1} dq_k \, \langle q'|\left(e^{-i\hat{H}\epsilon}\right)|q_{n-1}\rangle \, \langle q_{n-1}|\left(e^{-i\hat{H}\epsilon}\right)|q_{n-2}\rangle \cdots \langle q_1|\left(e^{-i\hat{H}\epsilon}\right)|q\rangle . \tag{2.5}$$

For small ϵ we can expand the exponential to first order and evaluate the matrix elements:

$$e^{-i\hat{H}\epsilon} = 1 - i\hat{H}\epsilon + \mathcal{O}(\epsilon^2)$$
$$\hat{H} = \frac{1}{2}\hat{P} + V(\hat{Q}). \tag{2.6}$$

From the potential energy we get

$$\begin{aligned} \langle q_k|V(\hat{Q})|q_{k-1}\rangle &= V(q_{k-1}) \, \langle q_k|q_{k-1}\rangle \\ &\approx V\left(\frac{q_k + q_{k-1}}{2}\right) \delta(q_k - q_{k-1}) \\ &= \int \hat{d}p \, V\left(\frac{q_k + q_{k-1}}{2}\right) e^{ip(q_k - q_{k-1})}. \end{aligned} \tag{2.7}$$

In order to evaluate the contribution of the kinetic term, we introduce eigenstates of the momentum operator $|p\rangle$

$$\hat{P}|p\rangle = p|p\rangle \tag{2.8}$$
$$\langle q|p\rangle = e^{ipq}, \tag{2.9}$$

and use the completeness of the states $|p\rangle$:

$$\langle q_k|\hat{P}^2|q_{k-1}\rangle = \int \hat{d}p \, p^2 e^{ip(q_k - q_{k-1})}. \tag{2.10}$$