Path Integral Quantum Field Theory

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1 Gaussian Integrals

The General Gaussian integral, for a complex, symmetric $n \times n$ matrix A such that $\operatorname{Re} A \geq 0$ and the eigenvalues a_i of A are nonzero, is given by

$$\boxed{Z_A(b) = \int d^n x \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j + \sum_{i=1}^{n} b_i x_i\right) = (2\pi)^{n/2} (\det A)^{-1/2} \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} b_i A^{-1}_{ij} b_j\right).} \tag{1.1}$$

Let μ be a measure in $\mathbb{R}^n$; we define the expectation value

$$\begin{aligned} \langle F \rangle_\mu &= \int d\mu(x) F(x) \\ &= \int d^n x \, \Omega(x) F(x). \end{aligned} \tag{1.2}$$

The measure is normalized so that

$$\int d\mu(x) = 1. \tag{1.3}$$

We define the generating function

$$Z_\mu(b) = \left\langle e^{(b,x)} \right\rangle_\mu = \int d\mu(x) \exp\left(\sum_{i=1}^{n} b_i x_i\right), \tag{1.4}$$

which is a function of the n-dimensional vector b and the measure μ. The integrand can be expanded

$$\exp\left(\sum_{i=1}^{n} b_i x_i\right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{i_1 \cdots i_\ell = 1}^{n} b_{i_1} \cdots b_{i_\ell} x_{i_1} \cdots x_{i_\ell}. \tag{1.5}$$

Therefore, substituting the definition of the correlator

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu = \int d\mu(x) x_{i_1} \cdots x_{i_\ell} \tag{1.6}$$

we obtain

$$Z_\mu(b) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{i_1 \cdots i_\ell = 1}^{n} b_{i_1} \cdots b_{i_\ell} \langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu. \tag{1.7}$$

Furthermore, it is useful to notice that

$$\frac{\partial}{\partial b_k} Z_\mu(b) = \int d\mu(x) x_k \exp\left(\sum_{i=1}^{n} b_i x_i\right), \tag{1.8}$$

which allows the correlaters to be written as

$$\langle x_{i_1} \cdots x_{i_\ell} \rangle_\mu = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} Z_\mu(b) \Big|_{b=0} . \tag{1.9}$$

Let us consder the Gaussian measure

$$d\mu_0(x) = d^n x \, \Omega_0(x) = d^n x \, \mathcal{N}_0 \exp\left(-\frac{1}{2} \sum_{i,j=0}^{n} x_i A_{ij} x_j \right), \tag{1.10}$$

where the normalization $\mathcal{N}_0$ is fixed by the normalization of the measure:

$$\mathcal{N}_0 = (2\pi)^{-n/2} (\det A)^{1/2}. \tag{1.11}$$

The generating function in this case can be readily computed using the boxed equation above:

$$Z(b) = \frac{Z_A(b)}{Z_A(0)} = \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i A^{-1}_{ij} b_j \right). \tag{1.12}$$

In the future, we will drop any subscript A when referring to a Gaussian measure for simplicity. And therefore, defining $\Delta_{ij} = A^{-1}_{ij}$,

$$\boxed{\langle x_{i_1} \cdots x_{i_\ell} \rangle_0 = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} .} \tag{1.13}$$

1.1 Wick's Theorem

Let us start with a couple explicit examples, which we will compute in full detail in order to get familiar with the algebraic manipulations.

One-point function: Let k be an integer between 1 and n, we have

$$\begin{aligned}
\langle x_k \rangle_0 &= \frac{\partial}{\partial b_k} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= \left(\frac{1}{2} \sum_{j=1}^{n} \Delta_{kj} b_j + \frac{1}{2} \sum_{i=1}^{n} b_i \Delta_{ik} \right) \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= \left(\sum_{j=1}^{n} \Delta_{kj} b_j \right) \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= 0,
\end{aligned} \tag{1.14}$$

where the last equality comes from the fact that the expression is linear in b, and we need to set $b = 0$.

Two-point function: We now consider a pair of indices k, l, and compute

$$\begin{aligned}
\langle x_k x_l \rangle_0 &= \frac{\partial}{\partial b_l} \frac{\partial}{\partial b_k} \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= \left[\Delta_{kl} + \left(\sum_{j=1}^{n} \Delta_{kj} b_j \right) \left(\sum_{m=1}^{n} \Delta_{lm} b_m \right) \right] \exp\left(\frac{1}{2} \sum_{i,j=1}^{n} b_i \Delta_{ij} b_j \right) \Bigg|_{b=0} \\
&= \Delta_{kl}
\end{aligned} \tag{1.15}$$

Thus, the generating function $Z(b)$ provides a systematic way to compute all correlators for a multi-dimensional Gaussian distribution. Having understood the general rule for such a process, we can generate a recipe to compute $\langle x_{i_1} \cdots x_{i_\ell} \rangle_0$, known as *Wick's Theorem*:

- Write down each $x_{i_1} \cdots x_{i_\ell}$ and organize them pairwise (i_p, i_q). Note that if ℓ must be even for the correlator to be non-zero.
- There are $(\ell-1) \times (\ell-3) \times \cdots \times 3 \times 1$ ways of doing this. Sum over all of these possible pairings.
- To each pair (i_p, i_q) associate a factor $\Delta_{i_p i_q}$.

Let us revisit our result for the **two-point function**: for $\langle x_{i_1} x_{i_2} \rangle_0$, there is only one possible pairing (i_1, i_2). Therefore

$$\langle x_{i_1} x_{i_2} \rangle_0 = \Delta_{i_1 i_2}. \tag{1.16}$$

Four-point function: For $\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_0$, there are three different pairings

$$P = \{\{(i_1, i_2), (i_3, i_4)\}, \{(i_1, i_3), (i_2, i_4)\}, \{(i_1, i_4), (i_2, i_3)\}\} \tag{1.17}$$

Wick's theorem then yields

$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_0 = \Delta_{i_1 i_2} \Delta_{i_3 i_4} + \Delta_{i_1 i_3} \Delta_{i_2 i_4} + \Delta_{i_1 i_4} \Delta_{i_2 i_3}. \tag{1.18}$$

We can also represent these pairings, *Wick contractions*, as should be familiar from canonical quantum field theory:

$$\langle x_i x_j \rangle_0 = \Delta_{ij} = \overline{x_i x_j} \tag{1.19}$$

1.2 Perturbed Gaussian Measure

Let us now consider a more complicated measure,

$$\Omega(x) = \frac{1}{Z(\lambda)} e^{-S(x,\lambda)}, \tag{1.20}$$

where the normalization is given as usual by

$$Z(\lambda) = \int d^n x e^{-S(x,\lambda)} \tag{1.21}$$

and

$$\begin{aligned} S(x,\lambda) &= \frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j + \lambda V(x) \\ &= S_0(x) + \lambda V(x). \end{aligned} \tag{1.22}$$

We call $V(\lambda)$ the potential term, foreshadowing the physics we will be doing using the perturbed Gaussian measure. Furthermore

$$\begin{aligned} Z(\lambda) &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int d^n x \, V^k(x) \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j\right) \\ &= Z(0) \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left\langle V^k(x) \right\rangle_0, \end{aligned} \tag{1.23}$$

where in the first line we have expanded the exponential term containing $e^V(x)$ and in the second line used

$$\langle F(x) \rangle_0 = \frac{1}{Z(0)} \int d^n x \exp\left(-\frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j\right) F(x), \tag{1.24}$$

and $1/Z(0)$ referring to the normalization constant $\mathcal{N}_0$ from Eq. 1.10. Recalling Eq. 1.13, given that $F(x)$ admits a Taylor expansion

$$F(x) = \sum_{\ell=0}^{\infty} \sum_{i_1 \cdots i_\ell = 1}^{n} F_{i_1 \cdots i_\ell} x_{i_1} \cdots x_{i_\ell}, \tag{1.25}$$

we can define the expectation value of F for a generic measure μ as

$$\langle F(x)\rangle_\mu = \sum_{\ell=0}^{\infty} \sum_{i_1\cdots i_\ell=1}^{n} F_{i_1\cdots i_\ell} \langle x_{i_1}\cdots x_{i_\ell}\rangle_\mu \,; \tag{1.26}$$

for the case of a Gaussian measure we obtain

$$\begin{aligned} \langle F(x)\rangle_0 &= \sum_{\ell=0}^{\infty} \sum_{i_1\cdots i_\ell=1}^{n} F_{i_1\cdots i_\ell} \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_\ell}} \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right)\Bigg|_{b=0} \\ &= F\left(\frac{\partial}{\partial b}\right) \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right)\Bigg|_{b=0} . \end{aligned} \tag{1.27}$$

Thus, returning to our potential term,

$$\begin{aligned} \frac{Z(\lambda)}{Z(0)} &= \left\langle e^{-\lambda V(x)}\right\rangle_0 \\ &= \exp\left[-\lambda V\left(\frac{\partial}{\partial b}\right)\right] \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} b_i \Delta_{ij} b_j\right)\Bigg|_{b=0} \end{aligned} \tag{1.28}$$