Physics 223b: Advanced Condensed Matter Physics

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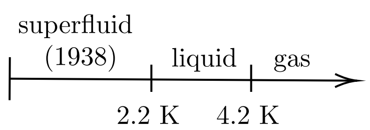
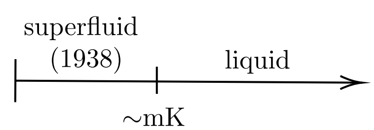
1 Course Overview: Superfluids and Super conductors

Superfluids and superconductors are considered by many to be the greatest all-time condensed matter discovery, which resulted in Nobel Prizes in the yeras 1913, 1962, 1972, 1972, 1973, 1978, 1987, 1996 and 2003!

He-4 (bosonic)



He-3 (fermionic) 0



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Besides in He-4 and He-3, superfluids have also been realized in ultracold bosonic/fermionic atoms (Rb, Li, Na, ...).

Superfluids and superconductors are deeply related macroscopic quantum phenomena with vast technological promise:

- Stable high $\vec{B}$-field generation (MRIs)
- Magnetic levitation
- Magnetometry (SQUIDs)
- Power transmission
- Dark matter detection
- Quantum computation (superconducting qubits)

There are many varieties, ranging from "conventional" to exotic (e.g., topological superconductors). Understanding them is key to many branches of modern quantum science!

This quarter: Superfluidity → superconductivity, emphasizing broader phenomenology, microscopic underpinnings, and some applications/frontier topics. Let us proceed with...

2 The Bose-Hubbard Model

Take bosons on a 2D[1] square lattice[2].

We use the canonical boson operators $b^\dagger_{\vec{r}}$, $b_{\vec{r}}$, with the commutation relations

[1]The dimensionality is important...
[2]...But the precise lattice is not so important.

$$\boxed{\begin{aligned} [b_{\vec r}, b^\dagger_{\vec r'}] = \delta_{\vec r, \vec r'}, \quad [b_{\vec r}, b_{\vec r'}] = [b^\dagger_{\vec r'}, b^\dagger_{\vec r'}] = 0 \\ n_{\vec r} = b^\dagger_{\vec r} b_{\vec r} = \text{boson occupation number} \\ b^\dagger_{\vec r} |\ldots n_{\vec r} \ldots\rangle = \sqrt{n_{\vec r}+1}\, |\ldots n_{\vec r}+1 \ldots\rangle \\ b_{\vec r} |\ldots n_{\vec r} \ldots\rangle = \sqrt{n_{\vec r}}\, |\ldots n_{\vec r}-1 \ldots\rangle \end{aligned}}$$

Then the Bose Hubbard model Hamiltonian is:

$$\boxed{H = \underbrace{-t \sum_{\langle \vec r \vec r' \rangle} \left(b^\dagger_{\vec r} b_{\vec r'} + \text{h.c.} \right)}_{\text{n.n. hopping}} + \sum_{\vec r} \Big[\underbrace{-\mu n_{\vec r}}_{\text{chem. pot.}} + \underbrace{\frac{U}{2} n_{\vec r}(n_{\vec r}-1)}_{\text{on-site repulsion}} \Big]}$$

Assume $t, U \geq 0$. This describes cold atoms in optical lattices. This has the following symmetries:

- Spatial (translation, rotation, reflection)
- Time reversal ($i \to -i, b_{\vec r} \to b_{\vec r}$)
- $U(1)$ particle conservation ($b_{\vec r} \to e^{i\alpha} b_{\vec r}, \alpha \in \mathbb{R}$)

Aside: $U \to \infty$ limit yields hard-core bosons, maps to spin-1/2 model:

$$\boxed{\begin{aligned} S^+_{\vec r} &= b^\dagger_{\vec r} \\ S^-_{\vec r} &= b_{\vec r} \\ S^z_{\vec r} &= n_{\vec r} - \frac{1}{2} \end{aligned}}$$

$$\boxed{\lim_{U\to\infty} H = \underbrace{-t \sum_{\langle \vec r \vec r' \rangle} \left(S^+_{\vec r} S^-_{\vec r'} + \text{h.c.} \right)}_{\text{Easy-plane FM exchange}} \underbrace{- \mu \sum_{\vec r} \left(S^z_{\vec r} + \frac{1}{2} \right)}_{\text{Zeeman field}}}$$

Let's build up the phase diagram for this model. First, explore the $t = 0$ limit, then explore the large t/U limit.

2.1 $t = 0$ limit

$$H \to \sum_{\vec r} \left[-\mu n_{\vec r} + \frac{U}{2} n_{\vec r}(n_{\vec r}-1) \right] = \sum_{\vec r} \Bigg\{ \underbrace{\frac{U}{2} \left[n_{\vec r} - \left(\frac{1}{2} + \frac{\mu}{U} \right) \right]^2}_{\text{parabolas centered around } n_{\vec r} - \frac{1}{2}} + \text{const.} \Bigg\}$$

2.2 $U = 0$ limit

$$H \to -t \sum_{\langle \vec{r}\vec{r}' \rangle} \left(b^\dagger_{\vec{r}} b_{\vec{r}'} + \text{h.c.} \right) - \mu \sum_{\vec{r}} n_{\vec{r}} \quad \text{Free bosons!}$$

Go to $\vec{k}$-space, where N_s is the number of sites:

$$b_{\vec{r}} = \frac{1}{\sqrt{N_s}} \sum_{\vec{k} \in \text{BZ}} e^{i\vec{k}\cdot\vec{r}} b_{\vec{k}}$$

$$\boxed{\begin{aligned} H &= \sum_{\vec{k} \in \text{BZ}} (\epsilon_{\vec{k}} - \mu) b^\dagger_{\vec{k}} b_{\vec{k}} \\ \epsilon_{\vec{k}} &= -2t[\cos(k_x a) + \cos(k_y a)] \end{aligned}}$$

Note: in the non-interacting limit, μ must sit <u>below</u> the band bottom so that $\frac{1}{\exp\left(\beta\left(\epsilon_{\vec{k}} - \mu\right)\right)} \geq 0$. For μ above the band bottom the energy decreases without bound by adding more and more bosons.

Let $|\psi\rangle$ be a ground state with average boson number N[3]. Consider the following:

$$\begin{aligned} \left\langle b^\dagger_{\vec{r}} b_{\vec{r}'} \right\rangle &= \frac{1}{N_s} \sum_{\vec{k}, \vec{k}' \in \text{BZ}} e^{-i\vec{k}\cdot\vec{r}} e^{i\vec{k}'\cdot\vec{r}'} \left\langle b^\dagger_{\vec{k}} b_{\vec{k}'} \right\rangle \\ &= \frac{1}{N_s} \left\langle b^\dagger_{\vec{k}=0} b_{\vec{k}=0} \right\rangle \\ &\boxed{= \frac{N}{N_s} \quad \text{even for } |\vec{r} - \vec{r}'| \to \infty!} \end{aligned}$$

This suggests that $\left\langle b^\dagger_{\vec{r}} b_{\vec{r}'} \right\rangle \to \left\langle b^\dagger_{\vec{r}} \right\rangle \left\langle b_{\vec{r}'} \right\rangle$ with

$$\boxed{\left\langle b_{\vec{r}} \right\rangle = \frac{1}{\sqrt{N_s}} \left\langle b_{\vec{k}=0} \right\rangle = \sqrt{\frac{N}{N_s}} e^{i\varphi}} \Rightarrow \text{Spontaneous U(1) symmetry breaking, violation of particle number conse} \tag{1}$$

This means that the condensate freely absorbs bosons. In the spin counterpart at $U \to \infty$ $\left\langle b_{\vec{r}} \neq 0 \right\rangle \Rightarrow \left\langle S^-_{\vec{r}} \right\rangle \neq 0$, i.e. ferromagnetic order in easy plane.

So the situation is qualitatively different from (symmetric) Mott phases. But what kind of states have $\left\langle b_{\vec{r}} \right\rangle \neq 0$?

[3]Think grand canonical here!

Coherent states

Recall Ph 223a Homework 1:

$$|\psi\rangle = e^{\frac{-|\psi|^2}{2}} e^{\psi b^\dagger} |0\rangle = e^{\frac{-|\psi|^2}{2}} \sum_{n=0}^{\infty} \frac{\psi^n}{n!} \left(b^\dagger\right)^n |0\rangle = e^{\frac{-|\psi|^2}{2}} \sum_{n=0}^{\infty} \frac{\psi^n}{\sqrt{n!}} |n\rangle$$

is a normalized coherent state obeying

$$b\,|\psi\rangle = \psi\,|\psi\rangle\,, \quad \langle\psi|\, b^\dagger = \langle\psi|\,\psi^*$$

Therefore we have the following expectation values for the operators $b, b^\dagger$, the average boson number N, and the boson number standard deviation ΔN:

$$\langle b \rangle = \psi, \langle b^\dagger \rangle = \psi^*$$

$$N = \langle b^\dagger b \rangle = |\psi|^2 = \langle b^\dagger \rangle \langle b \rangle$$

$$\begin{aligned}
\Delta N &= \left[\left\langle \left(b^\dagger b\right)^2 \right\rangle - \left\langle b^\dagger b \right\rangle^2\right]^{\frac{1}{2}} \\
&= \left[\left\langle b^\dagger \left(b^\dagger b + 1\right) b \right\rangle - \left\langle b^\dagger b \right\rangle^2\right]^{\frac{1}{2}} \\
&= \left[\left\langle b^\dagger b \right\rangle\right]^{\frac{1}{2}} = \sqrt{N} \ll N \quad \text{for} \quad N \gg 1
\end{aligned}$$

For the $U = 0$ problem,

$$\boxed{|\psi\rangle = e^{\frac{-|\psi|^2}{2}} e^{\psi b^\dagger_{\vec{k}=0}} |0\rangle}$$

is a ground state with

$$\boxed{\psi = \sqrt{N} e^{i\varphi}}$$

This gives Equation 1. It describes the essence of superfluids, but interactions are essential for generic behavior. $U = 0$ pathologies include infinite compressibility, k^2 dispersion, and no superflow.

2.3 Resurrecting $U \neq 0$

At $U \ll t$, small-$\vec{k}$ modes dominate. The goal is to derive a Hamiltonian that exploits macroscopic $\vec{k} = 0$ occupation, and we will study long-wavelength modes. Write:

$$b_{\vec{r}} = \frac{1}{\sqrt{N_2}} \sum_{\vec{k}\in\mathrm{BZ}} e^{i\vec{k}\cdot\vec{r}} b_{\vec{k}} \sim \underbrace{\frac{1}{\sqrt{N_s}} b_0}_{\substack{\text{pull out } \vec{k}=0 \text{ op.,} \\ \text{which is special}}} + \underbrace{\frac{1}{\sqrt{N_s}} \sum_{0<|\vec{q}|<\Lambda} e^{i\vec{q}\cdot\vec{r}} b_{\vec{q}}}_{\substack{\text{Throw away "large"} \\ \text{momenta}}}$$

and let $H = H_0 + H_{int}$, where H_0 is the kinetic energy and H_{int} is the U term. The kinetic energy then becomes

$$H_0 = -t \sum_{\langle \vec{r}\vec{r}' \rangle} \left(b^\dagger_{\vec{r}} b_{\vec{r}'} + \text{h.c.} \right) - \mu \sum_{\vec{r}} n_{\vec{r}}$$
$$\boxed{\sim \sum_{\vec{q}} \epsilon_q b^\dagger_{\vec{q}} b_{\vec{q}} \quad \text{w/} \quad \epsilon_q = \frac{\hbar^2 q^2}{2}}$$

Next, we expand the interaction:

$$H_{int} = \frac{U}{2} \sum_{\vec{r}} n_{\vec{r}} (n_{\vec{r}} - 1) = \frac{U}{2} \sum_{\vec{r}} \underbrace{: n^2_{\vec{r}} :}_{\text{“Normal ordering” -- kills self-interaction}}$$

assuming $b_{\vec{q}=0}$ bosons are dilute.

$$n_{\vec{r}} \sim \frac{1}{N_2} \left[b^\dagger_0 b_0 + \sum_{\vec{q}} \left(e^{i\vec{q}\cdot\vec{r}} b^\dagger_0 b_{\vec{q}} + \text{h.c.} \right) + \underbrace{\sum_{\vec{q},\vec{q}'}}_{\text{sums exclude } \vec{q}=0} e^{i\left(\vec{q}' - \vec{q}\right)\cdot\vec{r}} b^\dagger_{\vec{q}} b_{\vec{q}'} \right]$$

$$\Rightarrow H_{int} \sim \frac{U}{2} \frac{1}{N_s^2} \sum_{\vec{r}} \left[b^\dagger_0 b^\dagger_0 b_0 b_0 + 2 \sum_{\vec{q},\vec{q}'} e^{i\left(\vec{q}' - \vec{q}\right)\cdot\vec{r}} b^\dagger_0 b_0 b^\dagger_{\vec{q}} b_{\vec{q}'} \right.$$
$$\left. + \sum_{\vec{q},\vec{q}'} : \left(e^{i\vec{q}\cdot\vec{r}} b^\dagger_0 b_{\vec{q}} + \text{h.c.} \right) \left(e^{i\vec{q}'\cdot\vec{r}} b^\dagger_0 b_{\vec{q}'} + \text{h.c.} \right) : + \ldots \right]$$

where $\ldots$ includes terms that die after doing the $\vec{r}$ sum (momentum conservation), plus subleading 3- or 4-$b_{\vec{q}}$ interactions. Proceeding by doing the $\vec{r}$ sum:

$$\simeq \frac{U}{2N_s} \left[\left(b^\dagger_0 b_0 b_0 b^\dagger_0 - b^\dagger_0 b_0 \right) + 4 b^\dagger_0 b_0 \sum_{\vec{q}} b^\dagger_{\vec{q}} b_{\vec{q}} \right.$$
$$\left. + \sum_{\vec{q}} \left(b^\dagger_0 b^\dagger_0 b_{\vec{q}} b_{-\vec{q}} \right) + \text{h.c.} \right]$$

In particular, note the momentum and particle number conservation. We rewrite the first line using the equation for the total boson number N:

$$N = b^\dagger_0 b_0 + \sum_{\vec{q}} b^\dagger_{\vec{q}} b_{\vec{q}}$$

which enables factoring out a trivial part of the interaction energy that depends only on the number of bosons (vs. which levels they occupy).

$$\begin{aligned}\Rightarrow \left(b_0^\dagger b_0 b_0 b_0^\dagger - b_0^\dagger b_0\right) + 4 b_0^\dagger b_0 \sum_{\vec{q}} b_{\vec{q}}^\dagger b_{\vec{q}} \\ \simeq N^2 - N + (2 b_0^\dagger b_0 + 1) \sum_{\vec{q}} b_{\vec{q}}^\dagger b_{\vec{q}}\end{aligned}$$

dropping the 4-$b_{\vec{q}}$ terms and the negligible $+1$ term.

$$H_{int} \simeq \frac{U}{2N_s}\left\{ N(N-1) + \sum_{\vec{q}} \left[2 b_0^\dagger b_0 b_{\vec{q}}^\dagger b_{\vec{q}} + \left(b_0^\dagger b_0^\dagger b_{\vec{q}} b_{-\vec{q}} + \text{h.c.}\right)\right]\right\}$$

Note that the $N(N-1)$ term is the initial energy for N particles in the $\vec{k}=0$ level, which we will drop hereafter. The logic is similar to neglecting the Hartree energy in the previous term's treatment of ferromagnetism in a 3DEG – just a constant offset. Also note that we can't simply replace $b_0 \to \sqrt{N}$ in the original form of H_{int}! That would miss the fact that populating $b_{\vec{q}}$ modes must be compensated by a reduction in b_0 occupation.

Putting it all together:

$$H \sim \sum_{\vec{q}} \left[\left(\epsilon_q + \frac{U}{N_s} b_0^\dagger b_0 \right) b_{\vec{q}}^\dagger b_{\vec{q}} + \frac{U}{2N_s} \left(b_0^\dagger b_0^\dagger b_{\vec{q}} b_{-\vec{q}} + \text{h.c.}\right)\right]$$

This looks imposing, but remember that $b_0^\dagger, b_0^\dagger$ are approximately equal to classical variables here! Recall that at $U=0$ we found that

$$\left\langle b_{\vec{k}=0} \right\rangle = \sqrt{N} e^{i\varphi}$$

Choose $\varphi = 0$, replace $b_0^\dagger, b_0^\dagger \to \sqrt{N}$, and define $\nu = \frac{N}{N_s}$ to get

$$\boxed{H \sim \sum_{\vec{q}} \left[(\epsilon_q + \nu U) b_{\vec{q}}^\dagger b_{\vec{q}} + \frac{\nu U}{2} \underbrace{\left(b_{\vec{q}} b_{-\vec{q}} + \text{h.c.}\right)}_{\text{anomalous terms}}\right]}$$

which is a free boson problem! We will diagonalize this with a Bogoliubov transformation (see Homework 1):

$$\Rightarrow \boxed{H = \sum_{\vec{q}} E_q \beta_{\vec{q}}^\dagger \beta_{\vec{q}}} \quad \text{w/} \qquad \boxed{E_q = \sqrt{\epsilon_q(\epsilon_q + 2\nu U)}}$$

Take bosons in d-dimensional continuum with

$$H = H_0 + H_{int} \quad \text{(density-density interaction)}$$
$$H_0 = \int_{\vec{r}} b^\dagger_{\vec{r}} \left(-\frac{\hbar^2 \nabla^2}{2m} - \mu \right) b_{\vec{r}}$$

with $\left[b_{\vec{r}}, b^\dagger_{\vec{r}'} \right] = \delta(\vec{r} - \vec{r}')$, etc. The superfluid phase has $\left\langle b_{\vec{r}} \right\rangle = \psi(\vec{r}) \neq 0$, now allowing for $\vec{r}$-dependence, which is crucial for vortices and superflow explored below. This is valid for $T = 0$ in $d = 2$ and for $T < T_c$ in $d = 3$.

Plan:

- Develop phenomenological "Landau theory" for symmetry breaking order, which captures generic features of the superfluid phase. The connection to microscopics is fuzzy (not idea) but does not require weak interactions (virtue).
- Construct current in superfluids, vortices
- Two viewpoints on superflow

3 Landau Theory for Superfluidity

The general algorithm is:

1. Identify order parameter, symmetries
2. Expand free energy $F = -\frac{1}{\beta} \ln Z$
3. Minimize F to get optimal order parameter configuration

Applying this procedure to superfluidity...