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A Class of Copulas Associated with Brownian Motion Processes and Their Maxima

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Abstract

The copulas being introduced in this paper are derived from distributions associated with the Brownian motion and related processes. Useful background information is initially presented. Attention is focused on univariate Brownian motion processes having a drift parameter and their running maxima as well as correlated bivariate Brownian motion processes in conjunction with the cumulative maxima of one of their components. The copulas generated therefrom as well as the corresponding density functions are explicitly provided and graphically represented. The derivations are thorough, with the requisite preliminary results being stated beforehand. As well, various potential applications are suggested. A numerical example involving an actual data set consisting of daily stock prices is worked out in detail. The associated empirical copula density function is determined in terms of Bernstein's polynomials and compared to one of our proposed theoretical copula densities. The steps that are provided, including an initial transformation of the observations which likens them to a Wiener process, should enable one to efficiently apply the novel results introduced herein to data sets arising in other areas.

Keywords

Brownian motion, Copulas, Correlated Brownian processes, Cumulative maxima, Dependence, Wiener process

1. Introduction

The principal objective of this paper consists of expressing the dependence relationships existing between general Brownian motion (*BM*) processes and their cumulative maxima in terms of copulas. The main idea behind copulas is that the joint distribution of two or more random variables can be expressed in terms of their marginal distributions and a certain correlation structure. Copulas enable one to separate the effect due to the dependence between the variables from the contribution of each of the marginal variables.

In 1918, the mathematician Norbert Wiener gave a rigorous formulation of Brownian motion and established its existence; as a result, the alternative name, Wiener process, is also utilized in the literature. *BM* constitutes a useful modelling tool in various fields of scientific investigation such as Economics, Biology, Communications Theory, Business Administration and Quantitative Finance.

A thorough search of the statistical literature was carried out in connection with joint distributions that involve Brownian motion processes in various contexts. For example, the representations of the joint density function of a *BM* process and its minimum and maximum, which are given for instance in Borodin and Salminen [1], were shown to be convergent by Choi and Roh [2]. Upper and lower bounds for the distribution of the maximum of a two-parameter *BM* process were obtained by Cabaña and Wschebor [3]. Vardar-Acara et al. [4] provided explicit expressions for the correlation between the supremum and the infimum of a *BM* with drift. Kou and Zhong [5] studied the first-passage times of two-dimensional *BM* processes. Haugh [6] explained how to generate correlated Brownian motions and pointed out some applications involving security pricing and portfolio evaluation. Given the joint distribution of such stochastic processes, copulas could be developed.

This paper is organized as follows. Central results on copulas and Brownian motion are reviewed in Section 2. New copulas involving Brownian motion processes and their maxima are introduced in Section 3. In Section 4, we review some results on more general joint distributions and construct copulas associated with correlated bivariate Brownian motion processes and the cumulative maxima of one of their components. Potential applications of the proposed copulas are mentioned in Section 5 which also includes an illustrative numerical example. Pertinent concluding remarks are provided in Section 6.

The foremost contributions of this work which is principally based on a thesis written by the fourth co-author under the supervision of the first and second, could be summarized as follows: (1) The three new copulas specified in equations (17), (36) and (37), which were constructed from the joint distributions of a Brownian motion process and its maximum over certain intervals for the drifted case; (2) The copula given in equation (38) which is obtained from the joint distribution of a Brownian motion and the maximum of a correlated Brownian process; (3) The detailed derivations of the results, including those provided in the appendices; (4) The detailed numerical example involving an actual data set; (5) An explanation to the effect that the new copulas introduced in this paper are applicable to other stochastic processes via the time change technique, which opens up a wealth of applications.

2. Copulas and Brownian motion: foundation and classical results

This section formally defines copulas and specifies their chief properties. As well, a review of the main results that are currently available on copulas associated with Brownian processes and their maxima is presented.

We will denote the standard *BM* by $\{W_t\}_{t\geq 0}$ and its terminal value by W_T. Letting $M_t = \max_{0\leq s\leq t} W_s$ and $M_{(s,\,t)} = \max\{W_u, s\leq u\leq t\}$, we shall consider the joint distributions of

1. W_t and its maximum M_t,
2. W_T and M_t,
3. W_T and $M_{(s,t)}$,

which have previously been studied by Harrison [7] and Lee [8], among others. As mentioned in the Introduction, several further related results are available in the statistical literature.

Next, copulas are defined and their main properties are presented. Additional results are available from several authors including Cherubini et al. [9, 10], Denuit et al. [11], Joe [12], Nelsen [13], and Sklar [14]. We focus on the two-dimensional case in this paper. In this framework, a copula function is a bivariate distribution defined on the unit square $\mathbf{I}^2 = [0,\ 1]^2$ with uniformly distributed margins. Copulas can be formally defined as follows:

Definition 2.1. A function $C:\mathbf{I}^2 \mapsto \mathbf{I}$ is a bivariate copula if it satisfies the following properties:

1. For every $y,\ w\in \mathbf{I}$,

$$C(y,\ 1) = y \text{ and } C(1,w) = w;$$
$$C(y,\ 0) = C(0,w) = 0$$

2. For every $y_1, y_2, w_1, w_2 \in \mathbf{I}$ such that $y_1 \leq y_2$ and $w_1 \leq w_2$,

$$C(y_2,w_2) - C(y_2,w_1) - C(y_1,w_2) + C(y_1,w_1) \geq 0,$$

that is, the C-measure of the box vertices lying in $\mathbf{I}^2$ is nonnegative. In particular, the last inequality implies that $C(y,\ w)$ is increasing in both variables.

Copulas are useful for capturing the dependence structure of random distributions with arbitrary marginals. This statement is clarified by Sklar's theorem which is now stated for the bivariate case.

Theorem 2.1. Let $F(x_1,x_2)$ be the joint cumulative distribution function of random variables X_1 and X_2 having continuous marginal distributions $F_1(x_1)$ and $F_2(x_2)$. Then, there exists a unique bivariate copula $C:\mathbf{I}^2 \mapsto \mathbf{I}$ such that

$$F(x_1,x_2)=C(F_1(x_1),F_2(x_2)) \tag{1}$$

where $C(\cdot,\cdot)$ is a joint distribution function with uniform marginals. Conversely, for any continuous distribution functions $F_1(x_1)$ and $F_2(x_2)$ and any copula *C*, the function *F* defined in equation (1) is a joint distribution function with marginal distributions F_1 and F_2.

Sklar's theorem provides a scheme for constructing copulas. Indeed, the function

$$C(u_1,u_2)=F\left(F_1^{-1}(u_1),F_2^{-1}(u_2)\right) \tag{2}$$

is a bivariate copula, where the quasi-inverse F_i^{-1} for $i=1,2$ is defined by

$$F_i^{-1}(u)=\inf\{x \mid F_i(x)\ge u\} \quad \forall u\in(0,1). \tag{3}$$

It should be pointed out that copulas are invariant with respect to strictly increasing transformations. More specifically, let X_1 and X_2 be two continuous random variables with associated copula *C*. Now, letting α and β be two strictly increasing functions and denoting by $C_{\alpha,\beta}$ the copulas generated by $\alpha(X_1)$ and $\beta(X_2)$, it can be shown that for all $(u_1,u_2)\in\mathbf{I}^2$,

$$C_{\alpha,\beta}(u_1,u_2)=C(u_1,u_2). \tag{4}$$

We shall denote the density function corresponding to the copula $C(u_1,u_2)$ by

$$c(u_1,u_2)=\frac{\partial^2}{\partial u_1\partial u_2}C(u_1,u_2).$$

The following relationship between the joint density $f(\cdot,\cdot)$ and the copula density $c(\cdot,\cdot)$ can readily be obtained from equation (1):

$$f(x_1,x_2)=f_1(x_1)f_2(x_2)c(F_1(x_1),F_2(x_2)) \tag{5}$$

where $f_1(x_1)$ and $f_2(x_2)$ respectively denote the marginal density functions of X_1 and X_2. Thus, the copula density function can be expressed as follows:

$$c(u_1,u_2)=\frac{f\left(F_1^{-1}(u_1),F_2^{-1}(u_2)\right)}{f_1\left(F_1^{-1}(u_1)\right)f_2\left(F_2^{-1}(u_2)\right)} \tag{6}$$

Jaworski and Krzywda [15] and Bosc [16] determined the copulas corresponding to certain correlated Brownian motions. Lagerås [17] provided an explicit representation of the copulas associated with Brownian motion processes that are reflected at 0 and 1. Several recent books and articles point out the usefulness of correlated Brownian motions and promote the use of copulas generated therefrom in connection with various applications. For instance, Chen et al. [18] explain that correlated Brownian motions and their associated copulas can be utilized in the case of correlated assets occurring in risk management, pairs trading and multiassets derivative's pricing. Deschatre [19, 20] proposes to make use of asymmetric copulas generated from a Brownian motion and its reflection to model and control the distribution of their difference with applications to the energy market and the pricing of spread options. As measures of dependence, copulas have for instance found applications in signal processing, geodesy, hydrology, medicine and reliability theory. For instance, Ram [21] addressed a reliability problem occurring in a certain complex system by making use of a copula.

Darsow et al. [22] (Example 4.3) appear to have been the first to propose a copula in connection with the Brownian motion, which is given by

$$C(u,v)=\int_0^u \Phi\left(\frac{\sqrt{t}\Phi^{-1}(v)-\sqrt{s}\Phi^{-1}(x)}{\sqrt{t-s}}\right)\mathrm{d}x, \tag{7}$$

for $s<t$, where $\Phi(\cdot)$ is the distribution function of a standard normal random variable. In Section 4.2 of his PhD thesis, Schmitz [23] developed the copula for a Brownian motion and its supremum without mentioning any potential application. Incidentally, this copula is specified in equation (14) of this paper. Schmitz obtained the copula density function from the integral representation of the copula given in (7) as

$$c(u,v)=\sqrt{\frac{t}{t-s}}\frac{\phi\left(\left[\sqrt{t}\Phi^{-1}(v)-\sqrt{s}\Phi^{-1}(u)\right]/\sqrt{t-s}\right)}{\phi\left(\Phi^{-1}(v)\right)} \tag{8}$$

where $\phi(\cdot)$ is the density function of a standard normal random variable. Bibbona et al. [24] made use of (8) in their equations (24) and (26) to determine the copula for the Ornstein-Uhlenbeck process without mentioning that these equations had previously been developed by Schmitz [23]. Cherubini and Romagnoli [25] expressed (7) in the following conventional form of a Gaussian copula:

$$C(u,v)=\int_0^u\Phi\left(\frac{\Phi^{-1}(v)-\rho(s,t)\Phi^{-1}(x)}{\sqrt{1-\rho(s,t)^2}}\right)\mathrm{d}x, \tag{9}$$

where $\rho(s,t)=\sqrt{\frac{s}{t}}$. Thus, the correlation $\rho(s,t)$ tends to 1 as *s* converges to *t*, whereas it approaches zero as *t* tends to infinity. Alternatively, in Section 6.3 of their paper, Nadarajah et al. [26] pointed out that it follows from equation (7) that independence corresponds to $t-s\to\infty$, while full dependence corresponds to $t-s\to 0$.

Inspired by Schmitz [23], Cherubini et al. [25] emphasized the importance of Brownian copulas by pointing out that many non-Gaussian processes can be converted into a Brownian motion by means of the time change technique (Dambis [27], Dubins and Schwarz [28], Monroe [29]). Thus, any continuous local martingale can be transformed into a Brownian motion. This requires transforming a given stochastic process into a Brownian motion by an appropriate change in the time measure. Cherubini et al. also mentioned that the idea of applying the time change technique in connection with copulas was first introduced by Schmitz [23] in the case of deterministic time changes. Using the corollary in Item 5.59 of Schmitz [23], Cherubini et al. [25] provided the expression for the time-changing copula in their Proposition 3.11 which can be stated as follows: Consider an increasing stochastic process $h(t,\omega)$ such that one can construct a standard Brownian motion $W_{h(t,\omega)}$. Then, the copula function

$$C(u,v;\omega)=\int_0^u\Phi\left(\frac{\sqrt{h(t,\omega)}\Phi^{-1}(v)-\sqrt{h(s,\omega)}\Phi^{-1}(x)}{\sqrt{h(t,\omega)-h(s,\omega)}}\right)\mathrm{d}x \tag{10}$$

is called a time-changing copula. As explained in Schmitz [23], determining a copula for a stochastic process amounts to determining $\langle M\rangle_t$ where M is a local martingale canceling at zero, which is such that $\lim_{t\to\infty}\langle M\rangle_t\to\infty$ where $\langle M\rangle$ is the quadratic variation of the *M* process. Schmitz applied this idea for obtaining the copula associated with the Ornstein-Uhlenbeck process.

Accordingly, it ought to be kept in mind that the copulas derived in this paper for certain Brownian processes and their maxima can also be utilized in connection with other stochastic processes such as the Ornstein-Uhlenbeck process. This turns out to be quite an advantage as the underlying distributions of such processes needs not be known. For further considerations on the Ornstein-Uhlenbeck process, the time change technique applied to the Brownian motion, local martingales and quadratic variation, one may refer to Barndorff-Nielsen and Shiryaev [30], Karatzas and Shreve [31], Øksendal [32], Revuz and Yor [33], and Rogers and Williams [34, 35].

Sempi [36] focused on the coupled Brownian: $\left\{B_t=C_t\left(B_t^{(1)},B_t^{(2)}\right),t\ge 0\right\}$ establishing that it is a Markovian process but not a Gaussian one, and explaining that it is nevertheless a martingale. We believe that it will be useful, both for its own sake and in view of potential applications, to consider extensions to higher-dimensional settings involving *BM* processes. As well, the use of fractional Brownian motion processes which were recently utilized by Keddi et al. [37] to estimate a trend function, could be investigated. Furthermore, let us mention that Sempi [38] included in his equations (6) and (8), the copulas found by Darsow et al. [22] and Schmitz [23]. He also derived the copula for the Ornstein-Uhlenbeck process. Moreover, his paper included the copula associated with a Brownian motion and its maximum, which had previously been derived by Vachon [39].

3. Copulas associated with Brownian motion processes and their maxima

3.1 The standard case

As previously defined, $\{W_t\}_{t\ge 0}$ shall denote a standard *BM* and $M_t=\max_{0\le s\le t}W_s$, its maximum on the interval

$[0,t]$. It is well known (see for instance, Etheridge [40], Harrison [7], Karlin and Taylor [41], Revuz and Yor [42], Rogers and Williams [34]) that the joint distribution of (W_t, M_t) and the marginal distributions of M_t and W_t are respectively given by

$$\mathbb{P}\{W_t \le x, M_t \le a\} \equiv F_{W_t,M_t}(x,a) = \begin{cases} \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x-2a}{\sqrt{t}}\right) & \text{if } x \le a \\ 2\Phi\left(\frac{a}{\sqrt{t}}\right) - 1 & \text{if } x > a \end{cases} \tag{11}$$

$$\mathbb{P}\{W_t \le x\} \equiv F_{W_t}(x) = \Phi\left(\frac{x}{\sqrt{t}}\right), \tag{12}$$

and

$$\mathbb{P}\{M_t \le a\} \equiv F_{M_t}(a) = 2\Phi\left(\frac{a}{\sqrt{t}}\right) - 1, \tag{13}$$

$\forall t \in R_+$.

It follows from equation (2) that the copula $C_{W_t,M_t}(u,v)$ generated by a standard *BM* and its maximum is

$$C_{W_t,M_t}(u,v) = \begin{cases} u - \Phi\left(\Phi^{-1}(u) - 2\Phi^{-1}\left(\frac{v+1}{2}\right)\right) & \text{if } u \le \frac{v+1}{2} \\ v & \text{if } u > \frac{v+1}{2}, \end{cases} \tag{14}$$

its associated density function then being

$$\begin{aligned} c_{W_t,M_t}(u,v) &= \frac{\partial^2}{\partial u \partial v} C_{W_t,M_t}(u,v) \\ &= \frac{\left[2\Phi^{-1}\left(\frac{v+1}{2}\right) - \Phi^{-1}(u)\right]\phi\left(2\Phi^{-1}\left(\frac{v+1}{2}\right) - \Phi^{-1}(u)\right)}{\phi\left(\Phi^{-1}\left(\frac{v+1}{2}\right)\right)\phi\left(\Phi^{-1}(u)\right)} \end{aligned} \tag{15}$$

whenever $u \le \frac{v+1}{2}$, and zero otherwise. For more details, the reader is referred to Appendix B. This copula density is plotted in Figure 1.

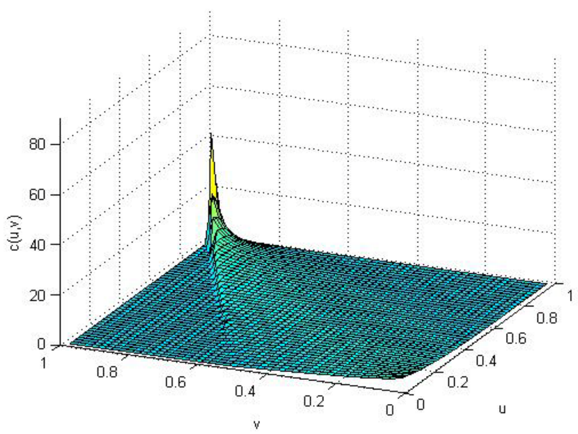

Figure 1. Density of the copula generated by W_t and Mt.

The copulas discussed in this paper which involve distinct marginal distributions, actually constitute a new type of copulas whose asymmetrical distributions conglomerate in the neighborhood of the point (1,1) and, to a lesser extent, near the origin, the associated copula density functions being equal to zero beyond a certain threshold that is dictated by certain relationships between the variables.

3.1.1 The drifted case

In light of the invariance properties of copulas, we consider a *BM* with $\sigma = 1$, since a $(\mu,\sigma) - BM$ can be derived from a $\left(\frac{\mu}{\sigma}, 1\right) - BM$ via a simple transformation (rescaling).

The next proposition provides the joint distribution of $\left\{W_t^{(\mu,\sigma)}\right\}_{t\geq 0}$, a *BM* with drift *μ* and variance σ^2, and $M_t^{(\mu,\sigma)}$, its maximum over the interval $0 \leq s \leq t$.

Proposition 3.1. *(Harrison [7])*

$$\mathbb{P}\{W_t^{(\mu,\sigma)} \leq x, M_t^{(\mu,\sigma)} \leq y\} \equiv F_{W_t,M_t}(x,\, y;\, \mu) = \begin{cases} \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}}\Phi\left(\frac{x-2y-\mu t}{\sigma\sqrt{t}}\right) & \text{if } x \leq y \\ \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}}\Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right) & \text{if } x > y. \end{cases}$$

The proof of this proposition is given in Appendix A. Moreover, the marginal distribution of $M_t^{(\mu,\sigma)}$ is given by

$$\mathbb{P}\{M_t^{(\mu,\sigma)} \leq y\} \equiv F_{M_t}(y;\, \mu) = \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu y}{\sigma^2}}\Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right), \tag{16}$$

which follows from Proposition 3.1 on noting that $\left\{M_t^{(\mu,\sigma)} \leq y\right\} \subset \left\{W_t^{(\mu,\sigma)} \leq x\right\}$ whenever $x > y$.

Let

$$F_{W_t}(x;\, \mu) = \mathbb{P}\{W_t^{(\mu,\,1)} \leq x\} = \Phi\left(\frac{x-\mu t}{\sqrt{t}}\right)$$

be the distribution function of a $(\mu,1) - BM$. For $y > 0$, the density function of $M_t^{(\mu,1)}$ is

$$f_{M_t}(y;\, \mu) = \frac{1}{\sqrt{t}}\phi\left(\frac{y-\mu t}{\sqrt{t}}\right) - e^{2\mu y}\left[2\mu\Phi\left(\frac{-y-\mu t}{\sqrt{t}}\right) - \frac{1}{\sqrt{t}}\phi\left(\frac{-y-\mu t}{\sqrt{t}}\right)\right].$$

Therefore, the copula $C_{W_t,\,M_t}(u,v;\,\mu)$ generated by $W_t^{(\mu,1)}$ and $M_t^{(\mu,1)}$ is

$$C_{W_t,\,M_t}(u,v;\,\mu) = \begin{cases} u - e^{2\mu\zeta(v)}\Phi\left(\Phi^{-1}(u) - \frac{2\zeta(v)}{\sqrt{t}}\right) & \text{if } u \leq \Phi\left(\frac{\zeta(v)-\mu t}{\sqrt{t}}\right) \\ v & \text{if } u > \Phi\left(\frac{\zeta(v)-\mu t}{\sqrt{t}}\right), \end{cases} \tag{17}$$

and the corresponding copula density is

$$c_{W_t,\,M_t}(u,v;\,\mu) = \frac{2e^{2\mu\zeta(v)}\phi\left(\Phi^{-1}(u) - \frac{2\zeta(v)}{\sqrt{t}}\right)}{f_{M_t}(\zeta(v);\,\mu)\,\phi\left(\Phi^{-1}(u)\right)} \times\left[\frac{1}{\sqrt{t}}\left(\frac{2\zeta(v)}{\sqrt{t}} - \Phi^{-1}(u)\right) - \mu\right] \quad \text{if } u \leq \Phi\left(\frac{\zeta(v)-\mu t}{\sigma\sqrt{t}}\right), \tag{18}$$

where $\zeta(v) = F_{M_t}^{-1}(v;\,\mu)$ and zero otherwise.

This density function is plotted in Figure 2 for increasing values of *μ* ($\mu = -2$, $\mu = 0$ and $\mu = 10$, respectively). Clearly, the strength of the dependence increases with μ; as expected, $C_{M_t}(u,v;\mu) \to C_{M_t}(u,v)$ as $\mu \to 0$.

4. A new general class of bivariate copulas

The results obtained in the previous section can be generalized by making use of the following properties of the multivariate normal distribution:

$$\Phi_2\left(z_1, z_2; \rho\right) = \Phi_2\left(z_2, z_1; \rho\right), \tag{19}$$

$$\Phi\left(z_1\right) - \Phi_2\left(z_1, z_2; \rho\right) = \Phi_2(z_1, -z_2; -\rho), \tag{20}$$

$$\begin{aligned}\Phi_3\left(z_1, z_2, z_3;\ \rho_{12}, \rho_{13}, \rho_{23}\right) &= \Phi_3\left(z_2, z_1, z_3;\ \rho_{12}, \rho_{23}, \rho_{13}\right) \\ &= \Phi_3\left(z_3, z_1, z_2; \rho_{13}, \rho_{23}, \rho_{12}\right)\end{aligned} \tag{21}$$

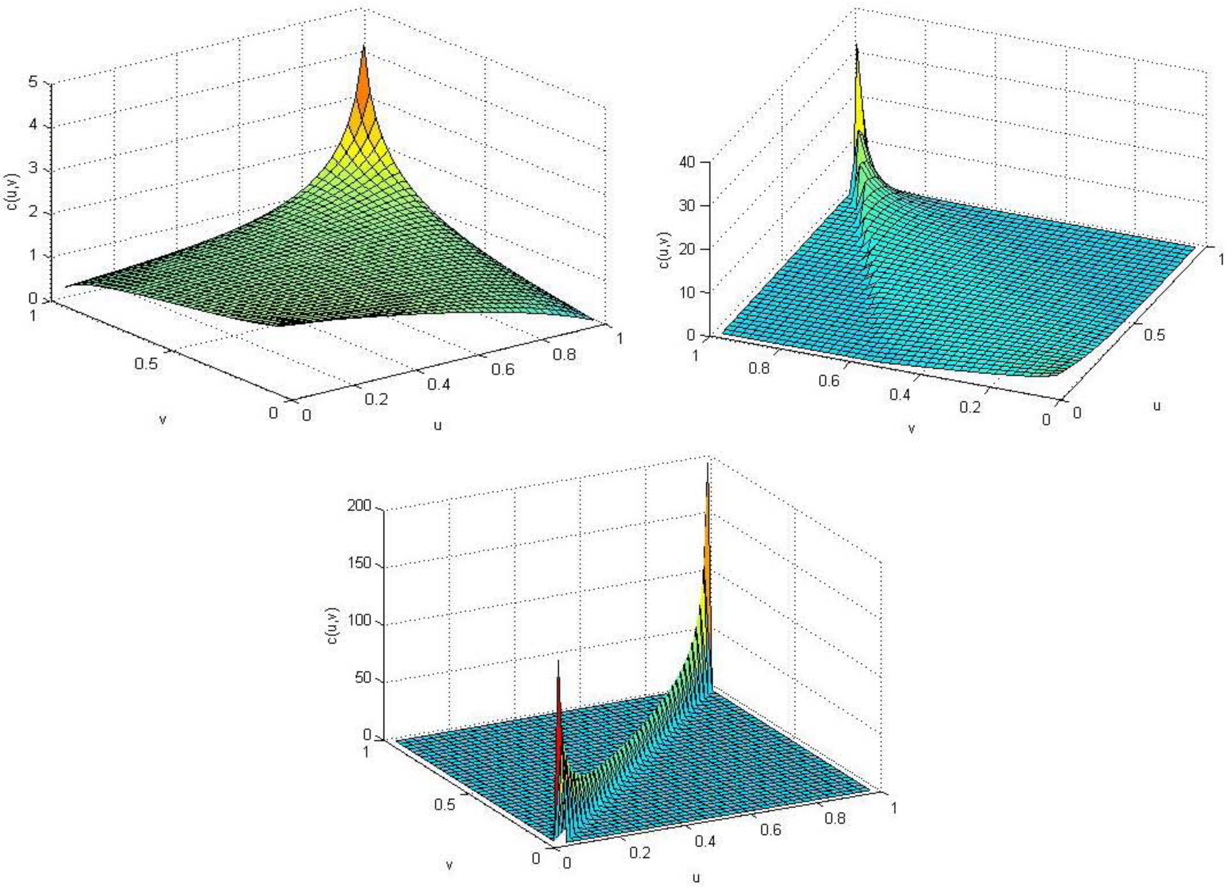

Figure 2. Density functions of the copulas generated by $W_t^{(\mu,\sigma)}$ and $M_t^{(\mu,\sigma)}$ for increasing values of μ.

and

$$\begin{aligned}&\Phi_2(z_2,\ z_3;\ \rho_{23}) - \Phi_3(z_1,\ z_2,\ z_3;\ \rho_{12},\ \rho_{13},\ \rho_{23}) \\ &= \ \Phi_3(-z_1,\ z_2,\ z_3;\ -\rho_{12},\ -\rho_{13},\ \rho_{23}).\end{aligned} \tag{22}$$

Lemma 4.1. Let z_1, z_2, z_3 be real constants and $\rho \geq 0$. If $z_1 = -\rho z_2 + \sqrt{1-\rho^2}\, z_3$, then

$$\Phi_2(z_1, z_2; -\rho) + \Phi_2\left(-z_1, z_3; -\sqrt{1-\rho^2}\right) = \Phi\left(z_2\right)\Phi\left(z_3\right) \tag{23}$$

and

$$\Phi_2(z_1, z_2; -\rho) + \Phi(-z_2)\Phi\left(z_3\right) = \Phi_2\left(z_1, z_3; \sqrt{1-\rho^2}\right). \tag{24}$$

Proof: Let Z_2 and Z_3 be two independent standard normal random variables, and Z_1 be a random variable defined by $Z_1 = -\rho Z_2 + 1 - \rho^2 Z_3$. Note that Z_1 has also a standard normal distribution, the random vectors $\left(Z_1, Z_2\right)$ and $\left(-Z_1, Z_3\right)$

have bivariate normal distributions with correlation coefficients given by $-\rho$ and $-\sqrt{1-\rho^2}$, respectively. Then,

$$\begin{aligned}
&\Phi_2(z_1,\, z_2;\, -\rho) + \Phi_2(-z_1,\, z_3;\, -\sqrt{1-\rho^2}) \\
&= \mathbb{P}\{Z_1 \le z_1,\, Z_2 \le z_2\} + \mathbb{P}\{-Z_1 \le -z_1,\, Z_3 \le z_3\} \\
&= \mathbb{P}\{Z_1 \le z_1,\, Z_2 \le z_2,\, Z_3 \le z_3\} + \mathbb{P}\{Z_1 \le z_1,\, Z_2 \le z_2,\, Z_3 \ge z_3\} \\
&\quad + \mathbb{P}\{Z_1 \ge z_1,\, Z_2 \le z_2,\, Z_3 \le z_3\} + \mathbb{P}\{Z_1 \ge z_1,\, Z_2 \ge z_2,\, Z_3 \le z_3\}.
\end{aligned}$$

We now replace Z_1 by $-\rho Z_2 + \sqrt{1-\rho^2}\,Z_3$ and z_1 by $-\rho z_2 + \sqrt{1-\rho}\,z_3$. Since the events $\{Z_1 \le z_1, Z_2 \le z_2, Z_3 \ge z_3\}$ and $\{Z_1 \ge z_1, Z_2 \ge z_2, Z_3 \le z_3\}$ are clearly empty, we obtain

$$\begin{aligned}
&\Phi_2(z_1,\, z_2;\, -\rho) + \Phi_2(-z_1,\, z_3;\, -\sqrt{1-\rho^2}) \\
&= \mathbb{P}\{Z_1 \le z_1,\, Z_2 \le z_2,\, Z_3 \le z_3\} + \mathbb{P}\{Z_1 \ge z_1,\, Z_2 \le z_2,\, Z_3 \le z_3\} \\
&= \mathbb{P}\{Z_2 \le z_2,\, Z_3 \le z_3\} = \Phi(z_2)\Phi(z_3).
\end{aligned}$$

It follows from equations (20) and (23) that

$$\begin{aligned}
&\Phi_2(z_1,\, z_2;\, -\rho) + \Phi_2(-z_1,\, z_3;\, -\sqrt{1-\rho^2}) = \Phi(z_2)\Phi(z_3) \\
&\Rightarrow\ \Phi_2(z_1,\, z_2;\, -\rho) + \Phi(z_3) - \Phi_2(z_1,\, z_3;\, \sqrt{1-\rho^2}) = (1-\Phi(-z_2))\Phi(z_3) \\
&\Rightarrow\ \Phi_2(z_1,\, z_2;\, -\rho) + \Phi(-z_2)\Phi(z_3) = \Phi_2(z_1,\, z_3;\, \sqrt{1-\rho^2}).
\end{aligned}$$

The joint distributions that will be considered further involve integrals for which closed form representations are given in the next proposition.

Proposition 4.1. Let $a,\ h,\ \theta_i,\ i=1,\ 2,\ 3,\ \delta_j$ and $\eta_j > 0,\ j=0,\ 1,\ 2,\ 3$ be constant, and $\mathbf{R} = \left[\rho_{ij}\right]_{i,\,j=1,\,2,\,3}$ be a correlation matrix, then

$$\begin{aligned}
&\int_{-\infty}^{a} \exp(hs)\Phi_3\left(\frac{\delta_1+\theta_1 s}{\eta_1},\, \frac{\delta_2+\theta_2 s}{\eta_2},\, \frac{\delta_3+\theta_3 s}{\eta_3};\mathbf{R}\right)\phi\left(\frac{s-\delta_0}{\eta_0}\right)\frac{\mathrm{d}s}{\eta_0} \\
&= \exp\left(h\delta_0 + \frac{h^2\eta_0^2}{2}\right)\Phi_4\left(\frac{a-\delta_0^*}{\eta_0},\, \frac{\delta_1+\theta_1\delta_0^*}{\kappa_1},\, \frac{\delta_2+\theta_2\delta_0^*}{\kappa_2},\, \frac{\delta_3+\theta_3\delta_0^*}{\kappa_3};\mathbf{R}^*\right)
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\int_{a}^{+\infty} \exp(hs)\Phi_3\left(\frac{\delta_1+\theta_1 s}{\eta_1},\, \frac{\delta_2+\theta_2 s}{\eta_2},\, \frac{\delta_3+\theta_3 s}{\eta_3};\mathbf{R}\right)\phi\left(\frac{s-\delta_0}{\eta_0}\right)\frac{\mathrm{d}s}{\eta_0} \\
&= \exp\left(h\delta_0 + \frac{h^2\eta_0^2}{2}\right)\Phi_4\left(\frac{-a+\delta_0^*}{\eta_0},\, \frac{\delta_1+\theta_1\delta_0^*}{\kappa_1},\, \frac{\delta_2+\theta_2\delta_0^*}{\kappa_2},\, \frac{\delta_3+\theta_3\delta_0^*}{\kappa_3};\mathbf{R}^{**}\right)
\end{aligned} \tag{26}$$

where $\delta_0^* = \delta_0 + h\eta_0^2; \kappa_i = \sqrt{\theta_i^2\eta_0^2 + \eta_i^2}$ for $i=1,\ 2,\ 3;\ \mathbf{R}^* = \left[\rho_{ij}^*\right]_{i,\,j=1,\,2,\,3,\,4}$ with $\rho_{1i+1}^* = -\theta_i\eta_0/\kappa_i$, $i=1,\ 2,\ 3$; $\rho_{2i+1}^* = \left(\rho_{1i}\eta_1\eta_i + \theta_1\theta_i\eta_0^2\right)/\left(\kappa_1\kappa_i\right)$, $i=2,\ 3$; $\rho_{34}^* = \left(\rho_{23}\eta_2\eta_3 + \theta_2\theta_3\eta_0^2\right)/\left(\kappa_2\kappa_3\right)$; and finally $\mathbf{R}^{**} = \left[\rho_{ij}^{**}\right]_{i,\,j=1,\,2,\,3,\,4}$ with $\rho_{1i}^{**} = -\rho_{1i}^*,\ i=2,\ 3,\ 4;\ \rho_{ij}^{**} = \rho_{ij}^*,\ i,\ j=2,\ 3,\ 4$.

These results are established by making use of properties of the conditional multivariate normal distribution. Note that this proposition is related to a result appearing in Lee [8] whose derivation relies on the Esscher transform.

Proof: Let $\mathbf{X} = \left(X_1, X_2, X_3, X_4\right)'$ be a normally distributed random vector such $E[X_i] = \mu_i,\ Var[X_i] = \sigma_i^2$ and $\mathbf{R}^* = \left[\rho_{ij}^*\right]$ for $i,\ j=1,\ 2,\ 3,\ 4$. Then the conditional distribution of $\left(X_2, X_3, X_4\right)$ given $X_1 = x_1$ is a trivariate normal distribution (Anderson [43]) with mean vector

$$\mu^{(1)} + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\left(\mathbf{x}^{(2)} - \mu^{(2)}\right) = \begin{pmatrix} \mu_2 + \frac{\sigma_2}{\sigma_1}\rho_{12}\left(x_1 - \mu_1\right) \\ \mu_3 + \frac{\sigma_3}{\sigma_1}\rho_{13}\left(x_1 - \mu_1\right) \\ \mu_4 + \frac{\sigma_4}{\sigma_1}\rho_{14}\left(x_1 - \mu_1\right) \end{pmatrix}$$

and covariance matrix

$$\mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} = \begin{pmatrix} \sigma_2^2(1-\rho_{12}^2) & \sigma_2\sigma_3(\rho_{23}-\rho_{12}\rho_{13}) & \sigma_2\sigma_4(\rho_{24}-\rho_{12}\rho_{14}) \\ \sigma_2\sigma_3(\rho_{23}-\rho_{12}\rho_{13}) & \sigma_3^2(1-\rho_{13}^2) & \sigma_3\sigma_4(\rho_{34}-\rho_{13}\rho_{14}) \\ \sigma_2\sigma_4(\rho_{24}-\rho_{12}\rho_{14}) & \sigma_3\sigma_4(\rho_{34}-\rho_{13}\rho_{14}) & \sigma_4^2(1-\rho_{14}^2) \end{pmatrix}.$$

Thus,

$$\begin{aligned}
&\Phi_4\left(\frac{x_1-\mu_1}{\sigma_1}, \frac{x_2-\mu_2}{\sigma_2}, \frac{x_3-\mu_3}{\sigma_3}, \frac{x_4-\mu_4}{\sigma_4}; \mathbf{R}^*\right) \\
&= \mathbb{P}\{X_1 \le x_1, X_2 \le x_2, X_3 \le x_3, X_4 \le x_4\} \\
&= \int_{-\infty}^{x_1} \mathbb{P}\{X_2 \le x_2, X_3 \le x_3, X_4 \le x_4 \,|\, X_1 = s\}\mathbb{P}\{X_1 \in \mathrm{d}s\} \\
&= \int_{-\infty}^{x_1} \Phi_3\left(\frac{x_2 - (\mu_2 + \rho_{12}^*\frac{\sigma_2}{\sigma_1}(s-\mu_1))}{\sigma_2\sqrt{1-(\rho_{12}^*)^2}}, \ldots, \frac{x_4 - (\mu_4 + \rho_{14}^*\frac{\sigma_4}{\sigma_1}(s-\mu_1))}{\sigma_4\sqrt{1-(\rho_{14}^*)^2}};\right. \\
&\qquad \left.\frac{\rho_{23}^* - \rho_{12}^*\rho_{13}^*}{\sqrt{1-(\rho_{12}^*)^2}\sqrt{1-(\rho_{13}^*)^2}}, \frac{\rho_{24}^* - \rho_{12}^*\rho_{14}^*}{\sqrt{1-(\rho_{12}^*)^2}\sqrt{1-(\rho_{14}^*)^2}}, \frac{\rho_{34}^* - \rho_{13}^*\rho_{14}^*}{\sqrt{1-(\rho_{13}^*)^2}\sqrt{1-(\rho_{14}^*)^2}}\right) \\
&\qquad \times \phi\left(\frac{s-\mu_1}{\sigma_1}\right)\frac{\mathrm{d}s}{\sigma_1}.
\end{aligned} \tag{27}$$

Now letting $x_1 = a$, $x_{i+1} = \delta_i$, $\mu_1 = \delta_0$, $\mu_{i+1} = -\theta_i\delta_0$, $\sigma_1 = \eta_0$, $\sigma_{i+1} = \kappa_i$ for $i = 1, 2, 3$ and replacing the elements of the matrix $\mathbf{R}^*$ with their respective values in equation (27), we obtain

$$\begin{aligned}
&\Phi_4\left(\frac{a-\delta_0}{\eta_0}, \frac{\delta_1+\theta_1\delta_0}{\kappa_1}, \frac{\delta_2+\theta_2\delta_0}{\kappa_2}, \frac{\delta_3+\theta_3\delta_0}{\kappa_3}; \mathbf{R}^*\right) \\
&= \int_{-\infty}^{a} \Phi_3\left(\frac{\delta_1+\theta_1 s}{\eta_1}, \frac{\delta_2+\theta_2 s}{\eta_2}, \frac{\delta_3+\theta_3 s}{\eta_3}; \mathbf{R}\right)\phi\left(\frac{s-\delta_0}{\eta_0}\right)\frac{\mathrm{d}s}{\eta_0}.
\end{aligned}$$

This establishes equation (25) for $h = 0$.

The case $h \neq 0$ follows from the last expression by completing the square in the exponent of $\exp(hs)\phi\left(\frac{s-\delta_0}{\eta_0}\right)$, so that

$$\exp(hs)\phi\left(\frac{s-\delta_0}{\eta_0}\right) = \exp\left(h\delta_0 + \frac{h^2\eta_0^2}{2}\right)\phi\left(\frac{s-\delta_0^*}{\eta_0}\right).$$

Finally, the last result, that is, equation (26) is similarly obtained on noting that

$$\begin{aligned}
&\Phi_4\left(\frac{-x_1+\mu_1}{\sigma_1}, \frac{x_2-\mu_2}{\sigma_2}, \frac{x_3-\mu_3}{\sigma_3}, \frac{x_4-\mu_4}{\sigma_4}; \mathbf{R}^{**}\right) \\
&= \mathbb{P}\{(-X_1) \le -x_1, X_2 \le x_2, X_3 \le x_3, X_4 \le x_4\} \\
&= \mathbb{P}\{X_1 \ge x_1, X_2 \le x_2, X_3 \le x_3, X_4 \le x_4\}.
\end{aligned}$$

Additionally, as $\delta_3 \to \infty$, we have

$$\begin{aligned}\int_{-\infty}^{a} & \exp(hs)\Phi_2\left(\frac{\delta_1+\theta_1 s}{\eta_1}, \frac{\delta_2+\theta_2 s}{\eta_2}; \rho_{12}\right) \\ & \times \phi\left(\frac{s-\delta_0}{\eta_0}\right)\frac{\mathrm{d}s}{\eta_0} \\ = & \exp\left(h\delta_0+\frac{h^2\eta_0^2}{2}\right)\Phi_3\left(\frac{a-\delta_0^*}{\eta_0}, \frac{\delta_1+\theta_1\delta_0^*}{\kappa_1}; \rho_{12}^*, \rho_{13}^*, \rho_{23}^*\right)\end{aligned} \tag{28}$$

and

$$\begin{aligned}\int_{a}^{+\infty} & \exp(hs)\Phi_2\left(\frac{\delta_1+\theta_1 s}{\eta_1}, \frac{\delta_2+\theta_2 s}{\eta_2}; \rho_{12}\right) \\ & \times \phi\left(\frac{s-\delta_0}{\eta_0}\right)\frac{\mathrm{d}s}{\eta_0} \\ = & \exp\left(h\delta_0+\frac{h^2\eta_0^2}{2}\right)\Phi_3\left(\frac{-a+\delta_0^*}{\eta_0}, \frac{\delta_1+\theta_1\delta_0^*}{\kappa_1}; -\rho_{12}^*, -\rho_{13}^*, \rho_{23}^*\right).\end{aligned} \tag{29}$$

Similarly, as $\delta_2 \to \infty$, it follows from equations (28) and (29) that

$$\begin{aligned}\int_{-\infty}^{a} & \exp(hs)\Phi\left(\frac{\delta_1+\theta_1 s}{\eta_1}\right)\phi\left(\frac{s-\delta}{\eta}\right)\frac{\mathrm{d}s}{\eta} \\ = & \exp\left(h\delta+\frac{h^2\eta^2}{2}\right)\Phi_2\left(\frac{a-\delta^*}{\eta}, \frac{\delta_1+\theta_1\delta^*}{\kappa_1}; -\frac{\theta_1\eta}{\kappa_1}\right)\end{aligned} \tag{30}$$

and

$$\begin{aligned}\int_{a}^{+\infty} & \exp(hs)\Phi\left(\frac{\delta_1+\theta_1 s}{\eta_1}\right)\phi\left(\frac{s-\delta}{\eta}\right)\frac{\mathrm{d}s}{\eta} \\ = & \exp\left(h\delta+\frac{h^2\eta^2}{2}\right)\Phi_2\left(\frac{-a+\delta^*}{\eta}, \frac{\delta_1+\theta_1\delta^*}{\kappa_1}; \frac{\theta_1\eta}{\kappa_1}\right).\end{aligned} \tag{31}$$

These results enable one to establish the distribution of $\left(W_T^{(\mu,\sigma)}, M_t^{(\mu,\sigma)}\right)$ within the interval $0 < t \le T$ as specified in the next proposition.

Proposition 4.2. *(Chuang [44] and Lee [8])*

$$\begin{aligned}\mathbb{P}\{W_T^{(\mu,\sigma)} \le x,\ M_t^{(\mu,\sigma)} \le y\} = & \Phi_2\left(\frac{x-\mu T}{\sigma\sqrt{T}}, \frac{y-\mu t}{\sigma\sqrt{t}}; \sqrt{\frac{t}{T}}\right) \\ & - e^{\frac{2\mu y}{\sigma^2}}\Phi_2\left(\frac{x-2y-\mu T}{\sigma\sqrt{T}}, \frac{-y-\mu t}{\sigma\sqrt{t}}; \sqrt{\frac{t}{T}}\right).\end{aligned} \tag{32}$$

The proof of this proposition is given in Appendix C. Note that as $t \to T$, $\mathbb{P}\{W_T^{(\mu,\sigma)} \le x,\ M_t^{(\mu,\sigma)} \le y\} \to \mathbb{P}\{W_T^{(\mu,\sigma)} \le x,\ M_T^{(\mu,\sigma)} \le y\}$.

Next, the joint distribution of $W_T^{(\mu,\sigma)}$ and $M_{(s,t)}^{(\mu,\sigma)}$, where $M_{(s,t)}^{(\mu,\sigma)} = \max_{s\le u\le t} W_u^{(\mu,\sigma)}$ and $0 < s < t \le T$, is considered.

Proposition 4.3. *(Lee [8])*

$$\begin{aligned}& \mathbb{P}\{W_T^{(\mu,\sigma)} \le x,\ M_{(s,t)}^{(\mu,\sigma)} \le y\} \\ & = \Phi_3\left(\frac{x-\mu T}{\sigma\sqrt{T}}, \frac{y-\mu t}{\sigma\sqrt{t}}, \frac{y-\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\ & \quad - e^{\frac{2\mu y}{\sigma^2}}\Phi_3\left(\frac{x-2y-\mu T}{\sigma\sqrt{T}}, \frac{-y-\mu t}{\sigma\sqrt{t}}, \frac{y+\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right)\end{aligned} \tag{33}$$

and

$$\mathbb{P}\{M_{(s,t)}^{(\mu,\sigma)} \le y\} = \Phi_2\left(\frac{y-\mu t}{\sigma\sqrt{t}}, \frac{y-\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{s}{t}}\right) - e^{\frac{2\mu y}{\sigma^2}}\Phi_2\left(\frac{-y-\mu t}{\sigma\sqrt{t}}, \frac{y+\mu s}{\sigma\sqrt{s}}; -\sqrt{\frac{s}{t}}\right). \tag{34}$$

The proof of this proposition is given in Appendix D. Note that as $s \to 0$, $\mathbb{P}\{W_T^{(\mu,\sigma)} \le x,\ M_{(s,t)}^{(\mu,\sigma)} \le y\} \to \mathbb{P}\{W_T^{(\mu,\sigma)} \le x,\ M_t^{(\mu,\sigma)} \le y\}$ and $\mathbb{P}\{M_{(s,t)}^{(\mu,\sigma)} \le y\} \to \mathbb{P}\{M_t^{(\mu,\sigma)} \le y\}$.

Now, consider $W = \left(W^1, W^2\right)'$ a Brownian vector where W^1 and W^2 are two independent standard *BM* processes. On letting $\left\{B_t^1 = \sigma_1\left(\rho W_t^2 + \sqrt{1-\rho^2}W_t^1\right) + \mu_1 t\right\}_{t\in R_+}$ and $\left\{B_t^2 = \sigma_2 W_t^2 + \mu_2 t\right\}_{t\in R_+}$, one can construct a correlated two-dimensional *BM* process. Then, it can be verified that $\{B_t^i\}_{t\in R_+}$ is a $(\mu_i, \sigma_i) - BM$ for $i = 1, 2$ and the correlation between B_t^1 and B_t^2 is equal to ρ. We say that $\left(B^1, B^2\right)'$ is a $(\mu,\ \Sigma) - BM$ with drift vector

$$\mu = (\mu_1, \mu_2)'$$

and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

Finally, we consider the joint distribution of $\left(B_t^1, M_{(s,t)}^2\right)$ for correlated BM_s where $M_{(s,t)}^2 = \max_{s\le u\le t} B_u^2$ and $0 < s < t \le T$.

Proposition 4.4. *(Lee [45])*

$$\begin{aligned}
&\mathbb{P}\{B_T^1 \le x,\ M_{(s,t)}^2 \le y\} \\
&= \Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, \rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\
&\quad - e^{2\mu_2 y/\sigma_2^2}\Phi_3\left(\frac{x-2\rho\frac{\sigma_1}{\sigma_2}y-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{-y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, -\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right).
\end{aligned} \tag{35}$$

The proof of this proposition is given in Appendix E. Note that as $\rho \to 0$,

$$\mathbb{P}\{B_T^1 \le x,\ M_{(s,t)}^2 \le y\} \to \mathbb{P}\{B_T^1 \le x\}\mathbb{P}\{M_{(s,t)}^2 \le y\}.$$

Additionally, when $\mu_1 = \mu_2$, $\sigma_1 = \sigma_2$ and $\rho = 1$,

$$\mathbb{P}\{B_T^1 \le x,\ M_{(s,t)}^2 \le y\} = \mathbb{P}\{W_T^{(\mu_1,\sigma_1)} \le x,\ M_{(s,t)}^{(\mu_1,\sigma_1)} \le y\}.$$

It follows from Propositions 4.2 and 4.3

$$\begin{aligned}
F_{W_T, M_t}(x,\ y;\ \mu) &= \mathbb{P}\{W_T^{(\mu,1)} \le x,\ M_t^{(\mu,1)} \le y\} \\
&= \Phi_2\left(\frac{x-\mu T}{\sqrt{T}}, \frac{y-\mu t}{\sqrt{t}}; \sqrt{\frac{t}{T}}\right) \\
&\quad - e^{2\mu y}\Phi_2\left(\frac{x-2y-\mu T}{\sqrt{T}}, \frac{-y-\mu t}{\sqrt{t}}; \sqrt{\frac{t}{T}}\right),
\end{aligned}$$

$$\begin{aligned} F_{W_T, M_{(s,t)}}(x, y; \mu) &= \mathbb{P}\{W_T^{(\mu,1)} \le x, M_{(s,t)}^{(\mu,1)} \le y\} \\ &= \Phi_3\left(\frac{x-\mu T}{\sqrt{T}}, \frac{y-\mu t}{\sqrt{t}}, \frac{y-\mu s}{\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\ &\quad - e^{2\mu y}\Phi_3\left(\frac{x-2y-\mu T}{\sqrt{T}}, \frac{-y-\mu t}{\sqrt{t}}, \frac{y+\mu s}{\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right), \end{aligned}$$

and

$$\begin{aligned} F_{M_{(s,t)}}(y; \mu) &= \mathbb{P}\{M_{(s,t)}^{(\mu,1)} \le y\} \\ &= \Phi_2\left(\frac{y-\mu t}{\sigma\sqrt{t}}, \frac{y-\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{s}{t}}\right) \\ &\quad - e^{\frac{2\mu y}{\sigma^2}}\Phi_2\left(\frac{-y-\mu t}{\sigma\sqrt{t}}, \frac{y+\mu s}{\sigma\sqrt{s}}; -\sqrt{\frac{s}{t}}\right). \end{aligned}$$

The copula $C_{W_T,M_t}(u, v; \mu)$ (resp. $C_{W_T,M_{(s,t)}}(u, v; \mu)$) describes the dependence structure induced by $W_T^{(\mu,1)}$ and its maximum value on the time interval [0, *t*] (resp. [*s*, *t*]). Invoking (2), we obtain

$$\begin{aligned} &C_{W_T, M_t}(u, v; \mu) \\ &= \Phi_2\left(\Phi^{-1}(u), \frac{\zeta_1(v)-\mu t}{\sqrt{t}}; \sqrt{\frac{t}{T}}\right) \\ &\quad - e^{2\mu\zeta_1(v)}\Phi_2\left(\Phi^{-1}(u) - \frac{2\zeta_1(v)}{\sqrt{T}}, \frac{-\zeta_1(v)-\mu t}{\sqrt{t}}; \sqrt{\frac{t}{T}}\right), \end{aligned} \tag{36}$$

and

$$\begin{aligned} &C_{W_T, M_{(s,t)}}(u, v; \mu) \\ &= \Phi_3\left(\Phi^{-1}(u), \frac{\zeta_2(v)-\mu t}{\sqrt{t}}, \frac{\zeta_2(v)-\mu s}{\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\ &\quad - e^{2\mu\zeta_2(v)}\Phi_3\left(\Phi^{-1}(u) - \frac{2\zeta_2(v)}{\sqrt{T}}, \frac{-\zeta_2(v)-\mu t}{\sqrt{t}}, \frac{\zeta_2(v)+\mu s}{\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right), \end{aligned} \tag{37}$$

where $\zeta_1(v) = F_{M_t}^{-1}(v; \mu)$ and $\zeta_2(v) = F_{M(s,t)}^{-1}(v; \mu)$.

Finally, consider (B^1, B^2) a $(\mu^*, \Sigma) - BM$ where

$$\mu^* = (0, \mu)'$$

and

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

The first *BM* has a zero drift because of the invariance property of copulas. Hence, in light of Proposition 4.4, we have

$$\begin{aligned} &F_{B_T^1, M_{(s,t)}^2}(x, y; \mu, \rho) \\ &= \mathbb{P}\{B_T^1 \le x, M_{(s,t)}^2 \le y\} \\ &= \Phi_3\left(\frac{x}{\sqrt{T}}, \frac{y-\mu t}{\sqrt{t}}, \frac{y-\mu s}{\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, \rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\ &\quad - e^{2\mu y}\Phi_3\left(\frac{x-2\rho y}{\sqrt{T}}, \frac{-y-\mu t}{\sqrt{t}}, \frac{y+\mu s}{\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, -\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right). \end{aligned}$$

Let us now denote by

$$F_{B_T^1}(x) = \Phi\left(\frac{x}{\sqrt{T}}\right)$$

the distribution function of the first *BM* and by $F_{M^2_{(s,t)}}(y;\mu)$ the distribution function of $M^2_{(s,t)}$ where $F_{M^2_{(s,t)}}(y;\mu) = F_{M_{(s,t)}}(y;\mu)$ for all $y > 0$.

The bivariate copula $C_{B_T^1, M^2_{(s,t)}}(u, v; \mu, \rho)$ generated by B_T^1 and $M^2_{(s,t)}$ is then defined by

$$\begin{aligned} &C_{B_T^1, M^2_{(s,t)}}(u, v; \mu, \rho) \\ &= \Phi_3\left(\Phi^{-1}(u), \frac{\zeta(v)-\mu t}{\sqrt{t}}, \frac{\zeta(v)-\mu s}{\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, \rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\ &\quad - e^{2\mu\zeta(v)}\Phi_3\left(\Phi^{-1}(u) - \frac{2\rho\zeta(v)}{\sqrt{T}}, \frac{-\zeta(v)-\mu t}{\sqrt{t}}, \frac{\zeta(v)+\mu s}{\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, -\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right) \end{aligned} \tag{38}$$

where $\zeta(v) = F^{-1}_{M^2_{(s,t)}}(v;\mu)$. This copula contains all the copulas considered in this section. Indeed, when ρ tends to 1, the copula specified by equation (38) converges to that generated by a *BM* with drift μ at T and its own maximum on the interval $[s, t]$, which is given in equation (37). From this result, we obtain the copula given in equation (36) by letting s tend to 0. Finally, as $\rho = 1$, $s \to 0$ and $t \to T$, the copula specified by equation (38) converges to that given in (17). Note that when $\rho = 0$, $C_{B_T^1, M^2_{(s,t)}}(u, v; \mu, \rho) = C_I(u,v)$ where C_I is the independent copula defined by $C_I(u,v) = uv$ for all $(u,v) \in \mathbf{I}^2$. The copula $C_{B_T^1, M^2_{(s,t)}}(u, v; \mu)$ is plotted in Figure 3 for $\rho = -0.99$, $\rho = 0$ and $\rho = 0.99$.

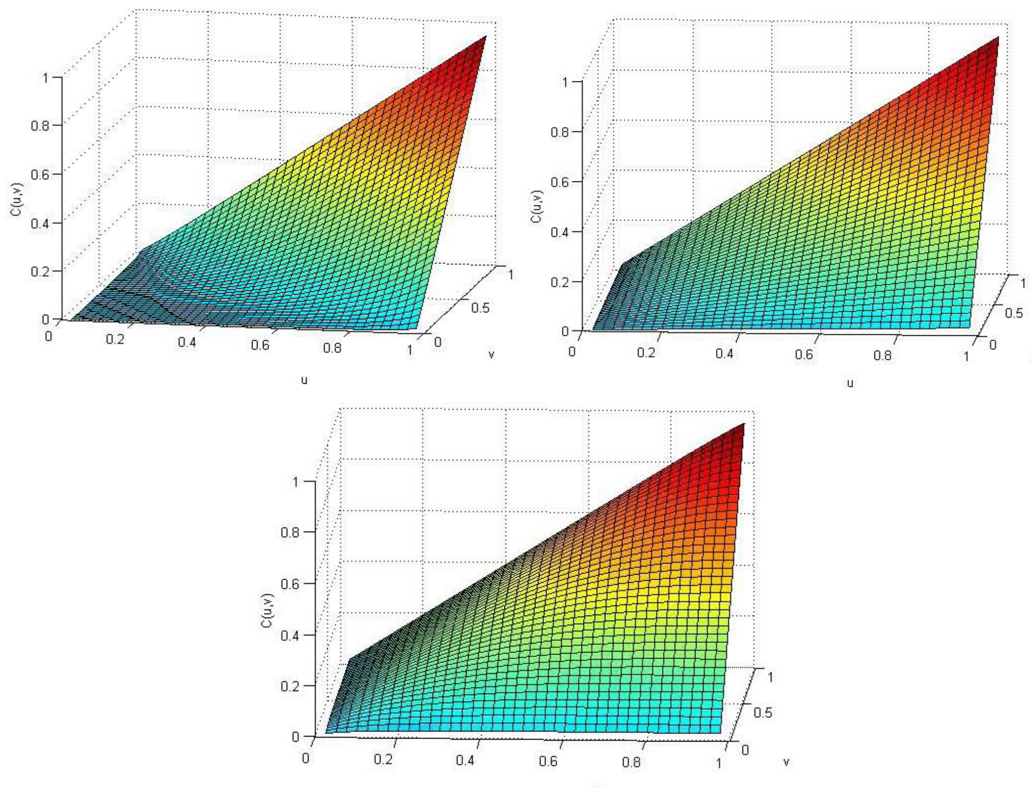

Figure 3. Copulas generated by B_T^1 and $M^2_{(s,t)}$ for increasing values of ρ.

5. Illustrative example and potential applications

As previously mentioned, Brownian motion processes have applications in several research areas. For instance, they come up in the Black-Scholes model in financial mathematics and the solution of the Schrödinger equation in quantum physics; they are also involved in the modeling of noise in electronics engineering as well as the study of the diffusion of particles and potential theory in physics.

Chuang [44] utilized certain distributional results pertaining to Brownian motion for pricing contingent claims with barriers on price processes and Cao [46] made use of correlated Brownian motions to solve an optimal investment-reinsurance problem. Cumulative maxima, which can be evaluated with the command "cummax" in the R language, have previously been utilized in connection with the price of an item over a certain number of months, the number of transactions occurring in a given period and stock prices in automated trading, to only name a few specific applications. Their utilization in the biosciences, reliability engineering, queueing theory, actuarial science, statistical physics and financial modeling are discussed for instance in the following monographs: Aven and Jensen [47], Barnes and Chu [48], Borokov [49], Demsper and Pliska [50], Grossi and Kunreuther [51], Krapivsky et al. [52], Lai and Xing [53], and Rolski et al. [54].

As a numerical illustration, consider the daily closing values of Air Canada's stock prices (from https://ca.finance.yahoo.com/quote/AC.TO/history?p=AC.TO) over the year 2019. First, the data are transformed so that it possesses the main characteristics of a Wiener process, that is, the first data point should be 0, the differences between successive observations should ideally often change signs and have a variance of one and there should be one unit of time between successive observations. Accordingly, the following transformation is applied: Let $U_1, U_2, \ldots, U_n$ denote the closing prices and $V_1, V_2, \ldots, V_{n-1}$ be their successive differences, that is, $V_i = U_{i+1} - U_i$. Denoting by σ_D the standard deviation of the differences $V_1, V_2, \ldots, V_{n-1}$, the following increasing affine transformation is applied

$$W_i = \frac{U_i - U_1}{\sigma_D}$$

and the resulting data are denoted by $W_1, W_2, \ldots, W_n$. Let Z_i be the i^{th} running maximum of the W_i's, that is, $Z_i = \max\{W_1,\ W_2, \ldots,\ W_i\}$, $i = 1,\ 2, \ldots, n$. Then, the bivariate data, (W_i, Z_i), $i = 1,\ 2, \ldots, n$, possesses the main features of a Brownian motion process and its cumulative maxima. The resulting prices W_i and their running maxima $Z_i, i = 1, \ldots,\ 250$, are respectively plotted in the left and right panels of Figure 4. A kernel density estimate of their joint density function is shown in Figure 5 along with a scatterplot. The theoretical copula density function of a Wiener process and its running maxima as specified in equation (15), is plotted in Figure 6. This copula is hinting at the type of dependence existing between the variables.

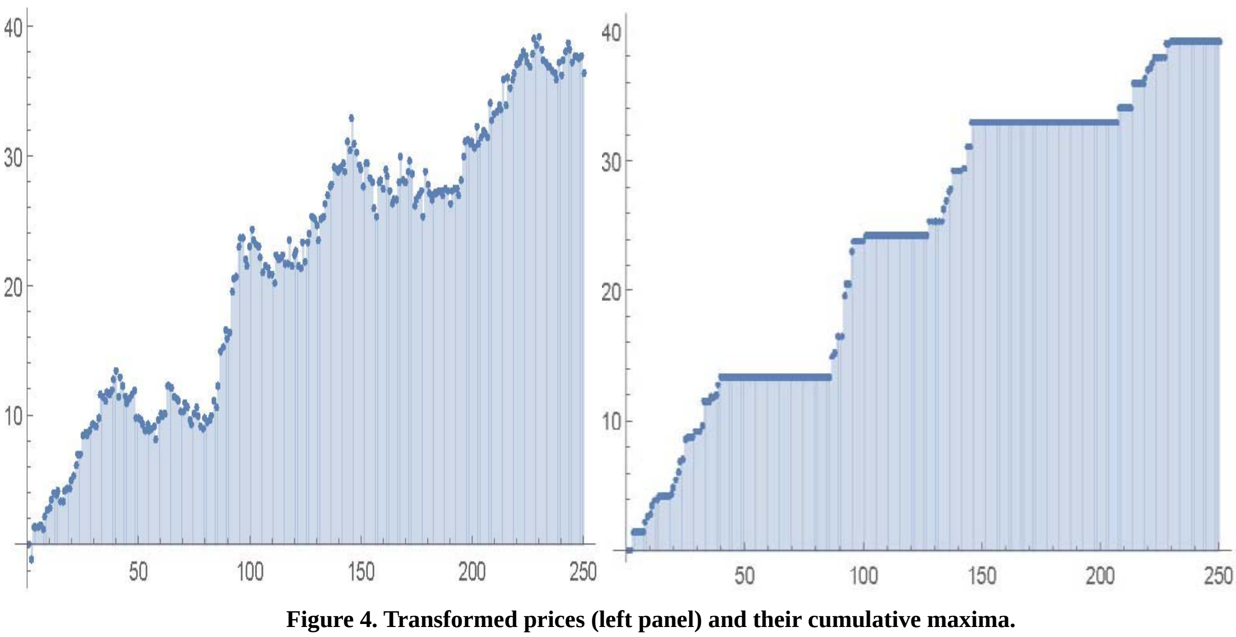

Figure 4. Transformed prices (left panel) and their cumulative maxima.

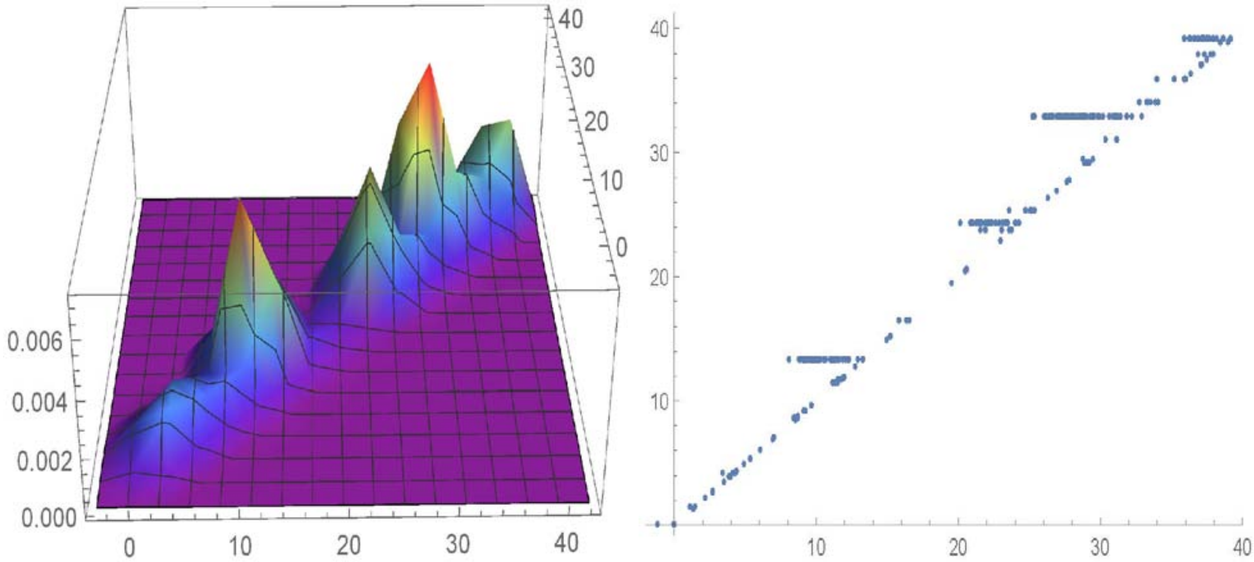

Figure 5. A kde of the joint distribution (left panel) and scatterplot.

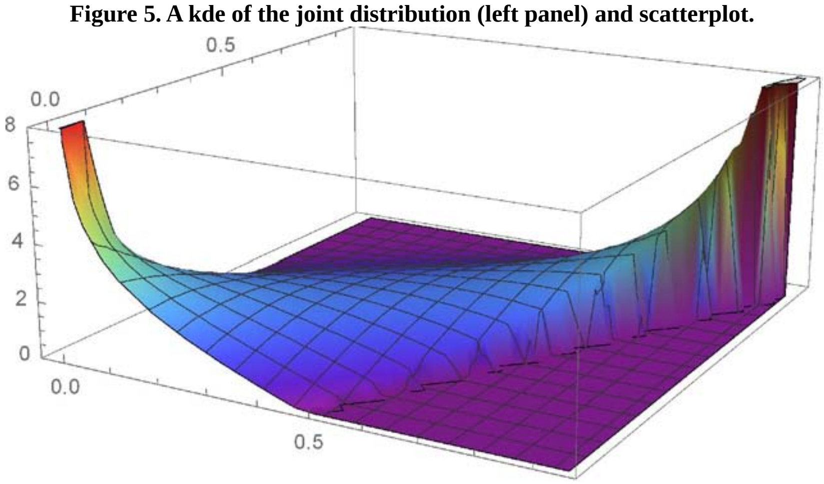

Figure 6. Theoretical copula density of a Wiener process and its running maxima.

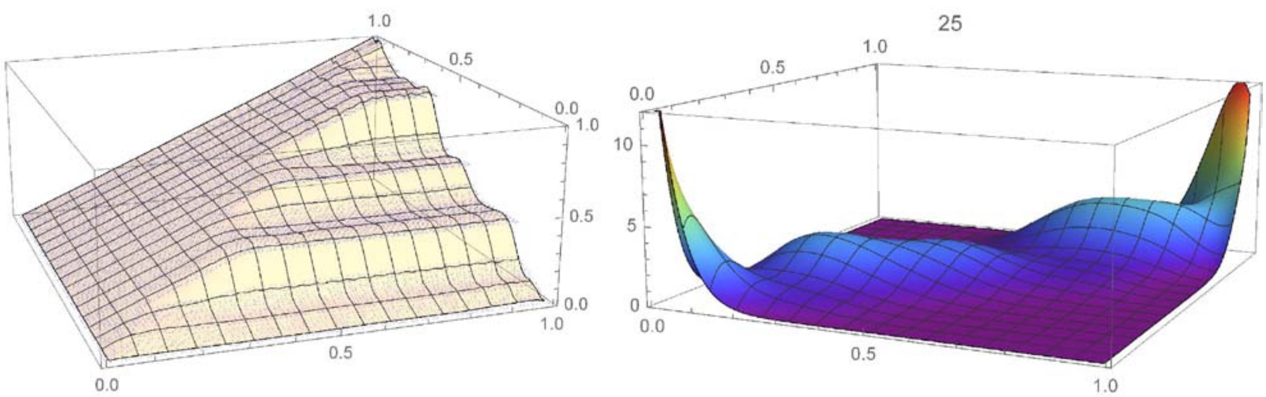

Figure 7. Deheuvels' copula (left panel) and Bernstein's empirical copula density.

On the basis of a bivariate data set of size n denoted by $(x_{i,1}, x_{i,2})$, $i = 1,\ldots, n$, an estimate of the associated copula, which was introduced in Deheuvels [55], is given by

$$C_n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n}[\mathbb{I}\{F_{1;n}(x_{i,1}) \le u_1\}\,\mathbb{I}\{F_{2;n}(x_{i,2}) \le u_2\}], \quad \text{for } \mathbf{u} = (u_1, u_2) \in [0,1]^2 \tag{39}$$

where I is the indicator function and $F_{j;n}$ is the empirical cumulative distribution function of the j^{th} component, $j = 1, 2$. Sancetta and Satchell [56] proposed and investigated the following density estimate referred to as Bernstein's empirical copula density:

$$\hat{c}_n(\mathbf{u}) = \sum_{v_1=0}^{k}\sum_{v_2=0}^{k} C_n\left(\frac{v_1}{k}, \frac{v_2}{k}\right)\prod_{j=1}^{2} P'_{v_j,k}(u_j). \tag{40}$$

where the order k plays the role of bandwidth parameter, $P_{vj,k}(u_j)$ is the binomial probability mass function,

$$P_{v_j,k}(u_j) = \binom{k}{v_j} u_j^{v_j}(1-u_j)^{k-v_j}, \tag{41}$$

and $P'_{vj,k}(u_j)$ is the derivative of $P_{vj,k}$ with respect to u_j.

For the example at hand, Deheuvels' empirical copula is shown in the left panel of Figure 7 whereas the Bernstein's empirical copula density or order 25 is plotted in the right panel of Figure 7. It can be observed that the behavior of this copula density exhibits several similarities with the distribution of theoretical copula associated with a standard *BM* and its cumulative maxima: it is concentrated around the diagonal, it vanishes to the right of the line (1/2, 0) − (1, 1) and increases in neighborhoods of the origin and the point (1,1).

6. Concluding remarks

Relevant background information on copulas and the Brownian motion process, including definitions, chief properties, standard results, current fields of applications and a wealth of references, is presented in the Introduction. The distributional properties of the standard, drifted and correlated cases of Brownian motion processes and their running maxima are then extensively discussed, and the corresponding copulas are formally derived. For the benefit of the reader, the proofs are thorough and the requisite preliminary results, provided.

Several potential applications of the new copulas introduced in this paper are pointed out and, as an illustrative example, the case of the daily closing prices of a certain stock over a year is completely worked out. A useful explanation is given in the matter of transforming the data so that it conforms to the characteristics of a Brownian motion process. Next, Deheuvels' empirical copula and Bernstein's empirical copula density are defined and plotted for the data set at hand. Actually, both of them can conveniently be applied to any set of bivariate observations. The empirical copula density is then compared with the theoretical copula density that was previously derived for a Wiener process and its cumulative maxima. These steps are applicable to the modeling of other types of data sets that could also benefit from the novel results presented herein. The related calculations were carried out with the symbolic computing package Mathematica, the code being available upon request.

Additionally, it should be mentioned that the results apply to other stochastic processes via a time change technique. The new class of copulas that has been introduced in this paper should constitute a welcomed addition in the field of applied probability. Indeed, in light of the numerous areas of applications wherein stochastic processes and their maxima arise, the proposed copulas ought to prove eminently suitable as models for characterizing their dependence structure.

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Appendices

A Proof of Proposition 3.1

We first consider the case $x \le y$. In light of equation (11), we have

$$\mathbb{P}\{W_t \in dx, M_t \le y\} = \frac{1}{\sqrt{t}}\left(\phi\left(\frac{x}{\sqrt{t}}\right) - \phi\left(\frac{x-2y}{\sqrt{t}}\right)\right)\mathrm{d}x,$$

where $\phi(\cdot)$ denotes the standard normal density function.

Now define

$$\mathbb{Q}(A) = \int_A L_t(\omega)\mathrm{d}\mathbb{P}(\omega), \quad A \in \mathfrak{F}_t^0$$

where $L_t = e^{\mu W_t - \frac{1}{2}\mu^2 t}$ is the Radon-Nikodym's derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ and $\mathfrak{F}_t^0 = \sigma(\{W_s, 0 \le s \le t\})$ for all $t \in \mathbb{R}_+$, is the smallest σ-algebra generated by the *BM* up to time *t*. It follows from Girsanov's theorem that $\{W_t\}_{t\ge 0}$ is a *BM* with drift μ under the new measure $\mathbb{Q}$. Therefore

$$\begin{aligned}
\mathbb{Q}\{W_t \le x, M_t \le y\} &= \int_{\{W_t\le x, M_t \le y\}} L_t(\omega)\mathrm{d}\mathbb{P}(\omega) \\
&= E^{\mathbb{P}}[\mathbf{1}_{\{W_t \le x, M_t\le y\}} L_t] \\
&= \int_{-\infty}^{x} e^{\mu z - \frac{\mu^2 t}{2}} \frac{1}{\sqrt{t}}\left(\phi\left(\frac{z}{\sqrt{t}}\right) - \phi\left(\frac{z-2y}{\sqrt{t}}\right)\right)\mathrm{d}z \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-\mu t)^2}{2t}}\mathrm{d}z - e^{2\mu y}\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-(2y+\mu t))^2}{2t}}\mathrm{d}z \\
&= \Phi\left(\frac{x-\mu t}{\sqrt{t}}\right) - e^{2\mu y}\Phi\left(\frac{x-2y-\mu t}{\sqrt{t}}\right).
\end{aligned}$$

B Establishing Equation (14)

Recall that

$$F_{M_t}(a) = \mathbb{P}\{M_t \le a\} = 2\Phi\left(\frac{a}{\sqrt{t}}\right) - 1, \quad \forall t \in \mathbb{R}_+,$$

$$F_{W_t}(x) = \mathbb{P}\{W_t \le x\} = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

and

$$\begin{aligned}
F_{W_t,M_t}(x,a) &= \mathbb{P}\{W_t \le x,\ M_t \le a\} \\
&= \begin{cases} \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x-2a}{\sqrt{t}}\right) & \text{if } x \le a \\ 2\Phi\left(\frac{a}{\sqrt{t}}\right) - 1 & \text{if } x > a. \end{cases}
\end{aligned}$$

Thus, we have

$$F_{Mt}^{-1}(v) = \Phi^{-1}\left(\frac{v+1}{2}\right)\sqrt{t}, \quad \forall v \in [0,1]$$

and $F_{W_t}^{-1}(u) = \Phi^{-1}(u)\sqrt{t}$ for all $u \in [0,1]$. Equation (14) is then derived by making use of Equation (2).

C Proof of Proposition 4.2

$$\begin{aligned}
&P\{W_T^{(\mu,\sigma)} \le x,\, M_t^{(\mu,\sigma)} \le y\} \\
&= \int_{-\infty}^{+\infty} P\{W_T^{(\mu,\sigma)} \le x,\, M_t^{(\mu,\sigma)} \le y | W_T^{(\mu,\,\sigma)} - W_t^{(\mu,\sigma)} = z\} \\
&\quad \times P\{W_T^{(\mu,\,\sigma)} - W_t^{(\mu,\sigma)} \in \mathrm{d}z\} \\
&= \int_{-\infty}^{+\infty} P\{W_t^{(\mu,\sigma)} \le x - z,\, M_t^{(\mu,\sigma)} \le y | W_T^{(\mu,\,\sigma)} - W_t^{(\mu,\sigma)} = z\} \\
&\quad \times \phi\left(\frac{z-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{T-t}} \\
&= \int_{-\infty}^{+\infty} P\{W_t^{(\mu,\sigma)} \le x - z,\, M_t^{(\mu,\sigma)} \le y\}\phi\left(\frac{z-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{T-t}} \\
&= \int_{-\infty}^{x-y} P\{W_t^{(\mu,\sigma)} \le x - z,\, M_t^{(\mu,\sigma)} \le y\}\phi\left(\frac{z-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{T-t}} \qquad \text{(C.1)} \\
&\quad + \int_{x-y}^{+\infty} P\{W_t^{(\mu,\sigma)} \le x - z,\, M_t^{(\mu,\sigma)} \le y\}\phi\left(\frac{z-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{T-t}}.
\end{aligned}$$

By substituting the result of Proposition 3.1 in equation (C.1), the following equality holds for the first part of the equation:

$$\begin{aligned}
&\int_{-\infty}^{x-y} P\{W_t^{(\mu,\sigma)} \le x - z,\, M_t^{(\mu,\sigma)} \le y\}\phi\left(\frac{z-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{T-t}} \\
&= \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right)\Phi\left(\frac{x-y-\mu(T-t)}{\sigma\sqrt{T-t}}\right) \qquad \text{(C.2)} \\
&\quad - e^{\frac{2\mu y}{\sigma^2}}\Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right)\Phi\left(\frac{x-y-\mu(T-t)}{\sigma\sqrt{T-t}}\right);
\end{aligned}$$

as for the second part,

$$\begin{aligned}
&\int_{x-y}^{+\infty} P\{W_t^{(\mu,\sigma)} \le x - z,\, M_t^{(\mu,\sigma)} \le y\}\phi\left(\frac{z-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{T-t}} \\
&= \int_{x-y}^{+\infty} \Phi\left(\frac{-z-(\mu t - x)}{\sigma\sqrt{t}}\right)\phi\left(\frac{z-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{T-t}} \\
&\quad - e^{\frac{2\mu y}{\sigma^2}}\int_{x-y}^{+\infty} \Phi\left(\frac{-z-(2y+\mu t - x)}{\sigma\sqrt{t}}\right)\phi\left(\frac{z-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{T-t}} \\
&= \Phi_2\left(\frac{y-x+\mu(T-t)}{\sigma\sqrt{T-t}}, \frac{x-\mu T}{\sigma\sqrt{T}}; -\sqrt{1-\frac{t}{T}}\right) \qquad \text{(C.3)} \\
&\quad - e^{\frac{2\mu y}{\sigma^2}}\Phi_2\left(\frac{y-x+\mu(T-t)}{\sigma\sqrt{T-t}}, \frac{x-2y-\mu T}{\sigma\sqrt{T}}; -\sqrt{1-\frac{t}{T}}\right)
\end{aligned}$$

where the last equality follows from equation (31). On combining the last two results and applying Lemma 4.1, we obtain

$$
\begin{aligned}
&P\{W_T^{(\mu,\sigma)} \leq x,\, M_t^{(\mu,\sigma)} \leq y\} \\
&= \Phi_2\left(\frac{x-\mu T}{\sigma\sqrt{T}},\, \frac{y-\mu t}{\sigma\sqrt{t}};\, \sqrt{\frac{t}{T}}\right) - e^{\frac{2\mu y}{\sigma^2}}\Phi_2\left(\frac{x-2y-\mu T}{\sigma\sqrt{T}},\, \frac{-y-\mu t}{\sigma\sqrt{t}};\, \sqrt{\frac{t}{T}}\right).
\end{aligned}
$$

D Proof of Proposition 4.3

Let us first consider the joint distribution of a *BM* and its maximum on the interval $[s, t]$. In that case,

$$
\begin{aligned}
&\mathbb{P}\{W_T^{(\mu,\sigma)} \leq x,\, M_{(s,t)}^{(\mu,\sigma)} \leq y\} \\
&= \int_{-\infty}^{y} \mathbb{P}\{W_T^{(\mu,\sigma)} \leq x,\, M_{(s,t)}^{(\mu,\sigma)} \leq y | W_s^{(\mu,\sigma)} = z\}\mathbb{P}\{W_s^{(\mu,\sigma)} \in \mathrm{d}z\} \\
&= \int_{-\infty}^{y} \mathbb{P}\{W_T^{(\mu,\sigma)} - W_s^{(\mu,\sigma)} \leq x-z,\, M_{(s,t)}^{(\mu,\sigma)} - W_s^{(\mu,\sigma)} \leq y-z | W_s^{(\mu,\sigma)} = z\} \\
&\quad \times \phi\left(\frac{z-\mu s}{\sigma\sqrt{s}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{s}} \\
&= \int_{-\infty}^{y} \mathbb{P}\{W_T^{(\mu,\sigma)} - W_s^{(\mu,\sigma)} \leq x-z,\, \max_{s\leq u\leq t}\{W_u^{(\mu,\sigma)} - W_s^{(\mu,\sigma)}\} \leq y-z\} \\
&\quad \times \phi\left(\frac{z-\mu s}{\sigma\sqrt{s}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{s}}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{y} \mathbb{P}\{W_{T-s}^{(\mu,\sigma)} \leq x-z,\, \max_{s\leq u\leq t} W_{u-s}^{(\mu,\sigma)} \leq y-z\}\phi\left(\frac{z-\mu s}{\sigma\sqrt{s}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{s}} \\
&= \int_{-\infty}^{y} \mathbb{P}\{W_{T-s}^{(\mu,\sigma)} \leq x-z,\, \max_{0\leq v\leq t-s} W_v^{(\mu,\sigma)} \leq y-z\}\phi\left(\frac{z-\mu s}{\sigma\sqrt{s}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{s}} \\
&= \int_{-\infty}^{y} \mathbb{P}\{W_{T-s}^{(\mu,\sigma)} \leq x-z,\, M_{t-s}^{(\mu,\sigma)} \leq y-z\}\phi\left(\frac{z-\mu s}{\sigma\sqrt{s}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{s}}. \qquad \text{(D.1)}
\end{aligned}
$$

On applying the result of Proposition 4.2 to the first term in the integrand of (D.1), we obtain

$$
\begin{aligned}
&\mathbb{P}\{W_T^{(\mu,\sigma)} \leq x,\, M_{(s,t)}^{(\mu,\sigma)} \leq y\} \\
&= \int_{-\infty}^{y} \Phi_2\left(\frac{x-z-\mu(T-s)}{\sigma\sqrt{T-s}},\, \frac{y-z-\mu(t-s)}{\sigma\sqrt{t-s}};\, \sqrt{\frac{t-s}{T-s}}\right) \\
&\quad \times \phi\left(\frac{z-\mu s}{\sigma\sqrt{s}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{s}} \\
&\quad - \int_{-\infty}^{y} e^{\frac{2\mu(y-z)}{\sigma^2}}\Phi_2\left(\frac{x+z-2y-\mu(T-s)}{\sigma\sqrt{T-s}},\, \frac{-y+z-\mu(t-s)}{\sigma\sqrt{t-s}};\, \sqrt{\frac{t-s}{T-s}}\right) \\
&\qquad \times \phi\left(\frac{z-\mu s}{\sigma\sqrt{s}}\right)\frac{\mathrm{d}z}{\sigma\sqrt{s}} \\
&= \Phi_3\left(\frac{x-\mu T}{\sigma\sqrt{T}},\, \frac{y-\mu t}{\sigma\sqrt{t}},\, \frac{y-\mu s}{\sigma\sqrt{s}};\, \sqrt{\frac{t}{T}},\, \sqrt{\frac{s}{T}},\, \sqrt{\frac{s}{t}}\right) \\
&\quad - e^{\frac{2\mu y}{\sigma^2}}\Phi_3\left(\frac{x-2y-\mu T}{\sigma\sqrt{T}},\, \frac{-y-\mu t}{\sigma\sqrt{t}},\, \frac{y+\mu s}{\sigma\sqrt{s}};\, \sqrt{\frac{t}{T}},\, -\sqrt{\frac{s}{T}},\, -\sqrt{\frac{s}{t}}\right)
\end{aligned}
$$

where the last equation follows from equation (28).

Finally, we obtain result (34) by letting x tend to $+\infty$ in equation (33).

E Proof of Proposition 4.4

Let $\{Z_t\}_{t\in\mathbb{R}_+}$ be a stochastic process defined by

$$Z_t = \frac{\sigma_2}{\sigma_1} B_t^1 - \rho B_t^2 \quad \forall t \in R_+ .$$

It follows from the construction of B^1 and B^2 that the process Z is a BM independent of B^2, with drift and variance parameters given by $\left(\frac{\sigma_2}{\sigma_1}\mu_1 - \rho\mu_2\right)$ and $\sigma_2^2\left(1-\rho^2\right)$, respectively. Thus,

$$\begin{aligned}
&\mathbb{P}\{B_T^1 \leq x,\ M_{(s,t)}^2 \leq y\} \\
&= \mathbb{P}\{\frac{\sigma_1}{\sigma_2}\left(Z_T + \rho B_T^2\right) \leq x,\ M_{(s,t)}^2 \leq y\} \\
&= \int_{-\infty}^{+\infty} \mathbb{P}\{\rho B_T^2 \leq \frac{\sigma_2}{\sigma_1}x - z,\ M_{(s,t)}^2 \leq y | Z_T = z\}\mathbb{P}\{Z_T \in \mathrm{d}z\} \\
&= \int_{-\infty}^{+\infty} \mathbb{P}\{\rho B_T^2 \leq \frac{\sigma_2}{\sigma_1}x - z,\ M_{(s,t)}^2 \leq y\} \\
&\quad \times \phi\left(\frac{z - \left(\frac{\sigma_2}{\sigma_1}\mu_1 - \rho\mu_2\right)T}{\sigma_2\sqrt{(1-\rho^2)T}}\right)\frac{\mathrm{d}z}{\sigma_2\sqrt{(1-\rho^2)T}}.
\end{aligned} \tag{E.1}$$

Define $z^* = 2x/\sigma_1 - z$ and consider first the case where $\rho < 0$. The following probability has to be determined:

$$\begin{aligned}
&\mathbb{P}\{\rho B_T^2 \leq \frac{\sigma_2}{\sigma_1}x - z,\ M_{(s,t)}^2 \leq y\} \\
&= \mathbb{P}\{B_T^2 \geq \frac{1}{\rho}z^*,\ M_{(s,t)}^2 \leq y\} \\
&= \mathbb{P}\{M_{(s,t)}^2 \leq y\} - \mathbb{P}\{B_T^2 \leq \frac{1}{\rho}z^*,\ M_{(s,t)}^2 \leq y\} \\
&= \left[\Phi_2\left(\frac{y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}}; \sqrt{\frac{s}{t}}\right)\right. \\
&\quad \left. - \Phi_3\left(\frac{\frac{1}{\rho}z^* - \mu_2 T}{\sigma_2\sqrt{T}}, \frac{y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right)\right] \\
&\quad - e^{\frac{2\mu_2 y}{\sigma_2^2}}\left[\Phi_2\left(\frac{-y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}}; -\sqrt{\frac{s}{t}}\right)\right. \\
&\quad \left. - \Phi_3\left(\frac{\frac{1}{\rho}z^* - 2y - \mu_2 T}{\sigma_2\sqrt{T}}, \frac{-y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right)\right],
\end{aligned}$$

which on applying equation (22) becomes

$$
\begin{aligned}
&\mathbb{P}\{\rho B_T^2 \le \frac{\sigma_2}{\sigma_1}x - z,\ M_{(s,t)}^2 \le y\} \\
&= \Phi_3\left(\frac{-(z^* - \rho\mu_2 T)}{\rho\sigma_2\sqrt{T}}, \frac{y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}}; -\sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\
&\quad - e^{\frac{2\mu_2 y}{\sigma_2^2}}\Phi_3\left(\frac{-(z^* - \rho(2y+\mu_2 T))}{\rho\sigma_2\sqrt{T}}, \frac{-y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}}; -\sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi_3\left(\frac{z^* - \rho\mu_2 T}{|\rho|\sigma_2\sqrt{T}}, \frac{y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}}; -\sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \qquad (\text{E.2}) \\
&\quad - e^{\frac{2\mu_2 y}{\sigma_2^2}}\Phi_3\left(\frac{z^* - \rho(2y+\mu_2 T)}{|\rho|\sigma_2\sqrt{T}}, \frac{-y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}}; -\sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right).
\end{aligned}
$$

Similarly, when $\rho > 0$, we have

$$
\begin{aligned}
&\mathbb{P}\{\rho B_T^2 \le \frac{\sigma_2}{\sigma_1}x - z,\ M_{(s,t)}^2 \le y\} \\
&= \mathbb{P}\{B_T^2 \le \frac{1}{\rho}z^*,\ M_{(s,t)}^2 \le y\} \\
&= \Phi_3\left(\frac{z^* - \rho\mu_2 T}{\rho\sigma_2\sqrt{T}}, \frac{y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \qquad (\text{E.3}) \\
&\quad - e^{\frac{2\mu_2 y}{\sigma_2^2}}\Phi_3\left(\frac{z^* - \rho(2y+\mu_2 T)}{\rho\sigma_2\sqrt{T}}, \frac{-y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right).
\end{aligned}
$$

On combining equations (E.2) and (E.3), we obtain the probability formula

$$
\begin{aligned}
&\mathbb{P}\{\rho B_T^2 \le \frac{\sigma_2}{\sigma_1}x - z,\ M_{(s,t)}^2 \le y\} \\
&= \Phi_3\left(\frac{z^* - \rho\mu_2 T}{|\rho|\sigma_2\sqrt{T}}, \frac{y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}}; s(\rho)\sqrt{\frac{t}{T}}, s(\rho)\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \qquad (\text{E.4}) \\
&\quad - e^{\frac{2\mu_2 y}{\sigma_2^2}}\Phi_3\left(\frac{z^* - \rho(2y+\mu_2 T)}{|\rho|\sigma_2\sqrt{T}}, \frac{-y-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}}; s(\rho)\sqrt{\frac{t}{T}}, -s(\rho)\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right)
\end{aligned}
$$

where $s(\rho) = 1$ if $\rho > 0$ and -1 otherwise.

In light of equation (E.4), the result given in equation (E.1) can be written as follows:

$$
\begin{aligned}
&\mathbb{P}\{B_T^1 \le x,\, M_{(s,t)}^2 \le y\} \\
&= \int_{-\infty}^{+\infty} \Phi_3\left(\frac{z^* - \rho\mu_2 T}{|\rho|\sigma_2\sqrt{T}},\, \frac{y-\mu_2 t}{\sigma_2\sqrt{t}},\, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}};\, s(\rho)\sqrt{\frac{t}{T}},\, s(\rho)\sqrt{\frac{s}{T}},\, \sqrt{\frac{s}{t}}\right) \\
&\qquad \times \phi\left(\frac{z - \left(\frac{\sigma_2}{\sigma_1}\mu_1 - \rho\mu_2\right)T}{\sigma_2\sqrt{(1-\rho^2)T}}\right)\frac{\mathrm{d}z}{\sigma_2\sqrt{(1-\rho^2)T}} \\
&- e^{\frac{2\mu_2 y}{\sigma_2^2}} \int_{-\infty}^{+\infty} \Phi_3\left(\frac{z^* - \rho(2y + \mu_2 T)}{|\rho|\sigma_2\sqrt{T}},\, \frac{-y-\mu_2 t}{\sigma_2\sqrt{t}},\, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}};\, s(\rho)\sqrt{\frac{t}{T}},\, s(\rho)\sqrt{\frac{s}{T}},\, \sqrt{\frac{s}{t}}\right) \\
&\qquad \times \phi\left(\frac{z - \left(\frac{\sigma_2}{\sigma_1}\mu_1 - \rho\mu_2\right)T}{\sigma_2\sqrt{(1-\rho^2)T}}\right)\frac{\mathrm{d}z}{\sigma_2\sqrt{(1-\rho^2)T}} \\
&= \Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}},\, \frac{y-\mu_2 t}{\sigma_2\sqrt{t}},\, \frac{y-\mu_2 s}{\sigma_2\sqrt{s}};\, \rho\sqrt{\frac{t}{T}},\, \rho\sqrt{\frac{s}{T}},\, \sqrt{\frac{s}{t}}\right) \\
&\qquad - e^{\frac{2\mu_2 y}{\sigma_2^2}}\, \Phi_3\left(\frac{x - 2\rho\frac{\sigma_1}{\sigma_2}y - \mu_1 T}{\sigma_1\sqrt{T}},\, \frac{-y-\mu_2 t}{\sigma_2\sqrt{t}},\, \frac{y+\mu_2 s}{\sigma_2\sqrt{s}};\, \rho\sqrt{\frac{t}{T}},\, -\rho\sqrt{\frac{s}{T}},\, -\sqrt{\frac{s}{t}}\right)
\end{aligned}
$$

where the last equality follows from Proposition 4.1 by letting a tend to $+\infty$.