Exercice 3 : Bayesian analysis of a one-way random effects model

1) En utilisant la formule de Bayes, on a :

$\mathbb{P}(X, \mu, \sigma^2, \tau^2 | Y) \propto \mathbb{P}(Y | X, \mu, \sigma^2, \tau^2) \mathbb{P}(X | \mu, \sigma^2) \pi(\mu, \sigma^2, \tau^2)$

(X ne dépend pas de τ^2)

. $\mathbb{P}(Y = y | X = x, \mu, \sigma^2, \tau^2) = \mathbb{P}(\varepsilon = y - x | \tau^2, \mu, \sigma^2)$

Donc on a finalement :

$$\boxed{\mathbb{P}(X, \mu, \sigma^2, \tau^2 | Y) \propto \tau^{-Nk} \exp\left(-\frac{\sum_{i=1}^{N}\sum_{j=1}^{k_i}(y_{ij} - x_i)^2}{2\tau^2}\right) \sigma^{-N} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i - \mu)^2\right) \times \sigma^{-2(1+\alpha)} \exp(-\beta/\sigma^2) \tau^{-2(1+\gamma)} \exp(-\beta/\tau^2)}$$

2) Déterminons les probabilités conditionnelles pour chaque paramètre :

. $\mathbb{P}(\sigma^2 | Y, X, \mu, \tau^2) \propto \sigma^{-(N + 2(\alpha+1))} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(X_i - \mu)^2 - \frac{\beta}{\sigma^2}\right)$

$$\Rightarrow \boxed{\sigma^2 | X, \mu \sim \mathrm{inv}\Gamma\left(\alpha + \frac{N}{2}, \beta + \frac{1}{2}\sum_{i=1}^{N}(X_i - \mu)^2\right)}$$

dans le sens "proportionnel"

. $\mathbb{P}(\tau^2 | Y, X, \mu, \sigma^2) \propto \tau^{-(Nk + 2(\gamma+1))} \exp\left(-\frac{1}{2\tau^2}\sum_{i=1}^{N}\sum_{j=1}^{k_i}(y_{ij} - X_i)^2 - \frac{\beta}{\tau^2}\right)$

$$\Rightarrow \boxed{\tau^2 | Y, X, \mu, \sigma^2 \sim \mathrm{inv}\Gamma\left(\gamma + \frac{Nk}{2}, \beta + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{k_i}(y_{ij} - X_i)^2\right)}$$

. $\mathbb{P}(\mu | X, \sigma^2) \propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(X_i - \mu)^2\right)$

$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(X_i^2 - 2\mu X_i + \mu^2)\right)$

$\propto \exp\left(-\frac{N}{2\sigma^2}\left[\left(\sum_{i=1}^{N}X_i/N\right)^2 - 2\mu\left(\sum_{i=1}^{N}X_i/N\right) + \mu^2\right]\right)$

$\mathbb{P}(\mu | X, \sigma^2) \propto \exp\left(-\frac{N}{2\sigma^2}\left(\sum_{i=1}^{N}X_i/N - \mu\right)^2\right)$

$$\Rightarrow \boxed{\mu \mid X, \sigma^2 \sim \mathcal{N}\left(\frac{1}{N}\sum_{i=1}^{N} X_i, \frac{\sigma^2}{N}\right)}$$

$$\cdot\ \mathbb{P}(X \mid Y, \mu, \sigma^2, \tau^2) \propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(X_i-\mu)^2 - \frac{1}{2\tau^2}\sum_{i=1}^{N}\sum_{j=1}^{k_i}(Y_{ij}-X_i)^2\right)$$

$$\propto \prod_{i=1}^{N} \exp\left(-\frac{(X_i^2 - 2\mu X_i + \mu^2)}{2\sigma^2} - \sum_{j=1}^{k_i}\frac{(Y_{ij}^2 - 2Y_{ij}X_i + X_i^2)}{2\tau^2}\right)$$

$$\propto \prod_{i=1}^{N} \exp\left(-X_i^2\left(\frac{1}{2\sigma^2} + \frac{k_i}{2\tau^2}\right) + X_i\left(\frac{\mu}{\sigma^2} + \frac{\sum_{j=1}^{k_i} Y_{ij}}{\tau^2}\right) - \left(\frac{\mu^2}{2\sigma^2} + \frac{\sum_{j=1}^{k_i} Y_{ij}^2}{2\tau^2}\right)\right)$$

$$\propto \prod_{i=1}^{N} \exp\left(-\frac{\left(X_i - \left(\frac{\tau^2\mu + \sigma^2\sum_{j=1}^{k_i} Y_{ij}}{\tau^2 + k_i\sigma^2}\right)\right)^2}{2\,\frac{\sigma^2\tau^2}{\tau^2+k_i\sigma^2}}\right)$$

$$\Rightarrow \boxed{X_i \mid Y, \mu, \sigma^2, \tau^2 \sim \mathcal{N}\left(\frac{\tau^2\mu + \sigma^2\sum_{j=1}^{k_i} Y_{ij}}{k_i\sigma^2+\tau^2}, \frac{\sigma^2\tau^2}{k_i\sigma^2+\tau^2}\right)}$$

car les X_i sont indépendants, donc chaque densité sera proportionnelle à une gaussienne.

3) Calculons la probabilité conditionnelle de X, μ :

$$\mathbb{P}(X, \mu \mid Y, \sigma^2, \tau^2) \propto \prod_{i=1}^{N} \exp\left(-\frac{1}{2\tau^2}\sum_{j=1}^{k_i}(X_i - Y_{ij})^2 - \frac{(X_i-\mu)^2}{2\sigma^2}\right)$$

$$\propto \prod_{i=1}^{N} \exp\left(-\frac{1}{2\tau^2}\sum_{j=1}^{k_i}(X_i^2 - 2Y_{ij}X_i + Y_{ij}^2) - \frac{1}{2\sigma^2}(X_i^2 - 2\mu X_i + \mu^2)\right)$$

$$\propto \prod_{i=1}^{N} \exp\left(-\frac{1}{2}\left(X_i^2\frac{(\sigma^2 k_i + \tau^2)}{\tau^2\sigma^2} - \frac{2X_i\sum_{j=1}^{k_i} Y_{ij}}{\tau^2} - \frac{2X_i\mu}{\sigma^2} + \frac{\mu^2}{\sigma^2}\right)\right)$$

$$\Rightarrow X, \mu \mid Y, \sigma^2, \tau^2 \sim \mathcal{N}(\mu, \Sigma)$$

avec
$$\Sigma = \begin{bmatrix} \frac{\sigma^2 k_1 + \tau^2}{\tau^2\sigma^2} & 0 & \cdots & 0 & -1/\sigma^2 \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \frac{\sigma^2 k_N + \tau^2}{\sigma^2\tau^2} & -1/\sigma^2 \\ -1/\sigma^2 & \cdots & \cdots & -1/\sigma^2 & \frac{N}{\sigma^2} \end{bmatrix}^{-1}$$

$$\mu = \Sigma x \left(\frac{\sum_{j=1}^{b_1} Y_{1,j}}{\tau^2}, \ldots \sum_{j=1}^{b_N} Y_{N,j} / \tau^2 \right)$$