fourier

Table des matières

1 Introduction

1.1 Signals and Systems in Continuous Time

The input signal $x(t)$ is defined for $t \in (-\infty, \infty)$ and is a periodic extension of a signal defined over a limited interval with period T. The output signal $y(t) = \mathcal{H} \circ x(t)$ is also defined for $t \in (-\infty, \infty)$.

1.2 Properties of Systems

The system $\mathcal{H}$ is said to be *linear* if :

$$\mathcal{H}(k_1 x_1(t) + k_2 x_2(t)) = k_1 \mathcal{H}(x_1(t)) + k_2 \mathcal{H}(x_2(t)) = k_1 y_1(t) + k_2 y_2(t) \tag{1}$$

The system $\mathcal{H}$ is said to be *shift-invariant* if :

$$\mathcal{H}(\mathcal{T}_{t_0}(x(t))) = \mathcal{T}_{t_0}(y(t)) \quad \text{where} \quad \mathcal{T}_{t_0}(x(t)) = x(t - t_0) \tag{2}$$

1.3 Examples

System	Linear	Shift-Invariant
$y(t) = x^2(t)$		
$y(t) = x(t) + a$		
$y(t) = ax(t)$		
Optimal sampling		

2 Linear Shift-Invariant Systems

2.1 Input Signal and System

The input signal $x(t)$ is defined for $t \in (-\infty, \infty)$ and is a periodic extension of a signal defined over a limited interval with period 1.

The system $\mathcal{H}$ has an impulse response $\tilde{h}(t)$ with a basic period such that $\tilde{h} : [0, 1) \to R$. This impulse response is periodically extended into $h(t)$.

3 Convolution Systems

Consider an input signal $x(t)$ defined for $t \in [0, 1]$. The convolution system is given by :

$$y(t) = \int_0^1 x(\tau) h(t - \tau) d\tau \tag{3}$$

We use here the orthonormal family of Fourier functions, defined over $[0, 1)$ for $k \in Z$:

$$\beta_k^F(t) = \exp(i2\pi kt) \tag{4}$$

4 Fourier-based Analysis of Systems

Using the Fourier family, the representations are :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \beta_k^F(t) \tag{5}$$

$$a_k = \langle \beta_k^F, x \rangle = \int_0^1 x(t) \exp(-i2\pi kt) dt \tag{6}$$

$$h(t) = \sum_{k=-\infty}^{\infty} b_k \beta_k^F(t) \tag{7}$$

$$b_k = \langle \beta_k^F, h \rangle = \int_0^1 h(t) \exp(-i2\pi kt) dt \tag{8}$$

4.1 Convolution Systems

The convolution system is expressed as :

$$y(t) = \int_0^1 x(\tau) h(t - \tau) d\tau \tag{9}$$

$$= \sum_{k=-\infty}^{\infty} a_k b_k \beta_k^F(t) \tag{10}$$

$$c_k = a_k b_k \tag{11}$$

5 Derivation System

Consider an input signal $x(t)$ defined for $t \in [0, 1]$. The derivation system is :

$$y(t) = \frac{d}{dt} x(t) \tag{12}$$

with boundary values obeying the cyclic continuity : $x(0) = x(1)$.

$$c_k = i2\pi k a_k \tag{13}$$

6 DFT

The Discrete Fourier Transform (DFT) matrix is given by :

$$[\mathrm{DFT}] = \frac{1}{\sqrt{N}} \begin{bmatrix} W^{*0\cdot 0} & W^{*1\cdot 0} & \cdots & W^{*(N-1)\cdot 0} \\ W^{*0\cdot 1} & W^{*1\cdot 1} & \cdots & W^{*(N-1)\cdot 1} \\ \vdots & \vdots & \ddots & \vdots \\ W^{*0\cdot (N-1)} & W^{*1\cdot (N-1)} & \cdots & W^{*(N-1)\cdot (N-1)} \end{bmatrix} \tag{14}$$

where $W = \exp\left(\frac{i2\pi}{N}\right)$. Note that the DFT matrix is symmetric and unitary.

7 Representation of a Discrete Signal in the DFT Domain

The representation of the discrete signal $\phi \in R^N$ is :

$$\phi^F = [\mathrm{DFT}]\phi \tag{15}$$

Observe that :

$$[\mathrm{DFT}]^*\phi^F = [\mathrm{DFT}]^*[\mathrm{DFT}]\phi \tag{16}$$

$$[\mathrm{DFT}]^*\phi^F = \phi \tag{17}$$

Above is the inverse DFT procedure.

8 Example

Consider the following discrete signal of N samples :
For $n = 0, 1, 2, \ldots, (N-1)$, $\phi(n) = \cos\left(\frac{2\pi k_0}{N} n\right)$ where $k_0 \in \{0, 1, 2, \ldots, (N-1)\}$.
The k^{th} component of the DFT-domain representation of the above signal is :

$$\phi^F(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} W^{*k\cdot n}\phi(n) \tag{18}$$

$$= \frac{1}{\sqrt{N}} \left(\frac{1}{2} \sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n} + W^{-k_0\cdot n}) \right) \tag{19}$$

When $k = k_0$:

$$\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n}) = N \tag{20}$$

When $k \neq k_0$:

$$\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n}) = \frac{(W^{-(k-k_0)})^N - 1}{W^{-(k-k_0)} - 1} = 0 \tag{21}$$

Similarly :

$$\phi^F(k) = \frac{1}{\sqrt{N}}\left(\frac{\sqrt{N}}{2}(\delta_{k,k_0} + \delta_{k,-k_0})\right) \tag{22}$$

For example, for $N = 9$ and $k_0 = 3$, the vector of DFT coefficients is :

$$\left[0, 0, 0, \frac{3}{2}, 0, 0, \frac{3}{2}, 0, 0\right]^\top \tag{23}$$