236201
Introduction to data processing and representation

Tutorial 5: Fourier family

Winter 2024-2025

Fourier Family

Consider the Fourier family of orthonormal functions

$$\{\ldots, \beta^F_{-2}(t), \beta^F_{-1}(t), \beta^F_0(t), \beta^F_1(t), \beta^F_2(t)\ldots\}$$

where for $k \in \mathbb{Z}$: $\beta^F_k(t) := \exp(i2\pi kt)$ for $t \in [0,1)$.

Recall that $i^2 := -1$ and that $\exp(i2\pi kt) = e^{i2\pi kt} = \cos(2\pi kt) + i\sin(2\pi kt)$

Are the functions orthonormal?

Fourier Family

Are the functions orthonormal?

Note that the inner-product is defined as

$$\langle \beta_k^F, \beta_l^F \rangle = \int_0^1 \beta_k^{F*}(t)\beta_l^F(t)dt$$

When $k = l$:

$$\langle \beta_k^F, \beta_l^F \rangle = \int_0^1 \beta_k^{F*}(t)\beta_k^F(t)dt = \int_0^1 e^{-i2\pi kt}e^{i2\pi kt}dt = \int_0^1 \exp(0)dt = 1$$

Fourier Family

When $k \neq l$:

$$\langle \beta_k^F, \beta_l^F \rangle = \int_0^1 e^{-i2\pi kt} e^{i2\pi lt} dt = \int_0^1 e^{i2\pi(l-k)t} dt$$

$$= \int_0^1 \cos(2\pi(l-k)t) + i\sin(2\pi(l-k)t) dt$$

$$= 0$$

Fourier Family

Consider the Fourier family of orthonormal functions

$$\{\ldots, \beta^F_{-2}(t), \beta^F_{-1}(t), \beta^F_0(t), \beta^F_1(t), \beta^F_2(t)\ldots\}$$

where for $k \in \mathbb{Z}$: $\beta^F_k(t) := \exp(i2\pi kt)$ for $t \in [0,1)$.

The above functions are **orthonormal:**

$$\langle \beta^F_k(t), \beta^F_l(t) \rangle = \delta_{k,l} = \begin{cases} 1, & k = l \\ 0, & \text{O.W.} \end{cases}$$

Signal Representation

Consider a function $\phi : [0,1) \to \mathbb{R}$.

Its optimal representation using the Fourier family is given by

$$\widehat{\phi}^F(t) = \sum_{k=-N}^{N} \widetilde{\phi}_k^F \beta_k^F(t)$$

The corresponding optimal coefficients (in the sense of minimal MSE) are:

$$\widetilde{\phi}_k^F = \langle \beta_k^F, \phi \rangle = \int_0^1 \phi(t) e^{-i2\pi kt} dt$$

Signal Representation

The error term is defined by $\mathscr{E}(t) = \phi(t) - \widehat{\phi}^F(t)$

The MSE is defined as:

$$\|\mathscr{E}(t)\|^2 = \langle \mathscr{E}(t), \mathscr{E}(t) \rangle = \int_0^1 \left(\phi(t) - \widehat{\phi}^F(t)\right)\left(\phi(t) - \widehat{\phi}^F(t)\right)^* dt$$

The minimal MSE obtained using the optimal coefficients is (prove!)

$$\|\mathscr{E}(t)\|^2 = \int_0^1 \phi^2(t)dt - \sum_{k=-N}^{N} |\widetilde{\phi}_k^F|^2$$

Conjugate-Symmetry of Representation

Note that we consider a function $\phi : [0,1) \to \mathbb{R}$.

The k-th coefficient:

$$\widetilde{\phi}_k^F = \langle \beta_k^F, \phi \rangle = \int_0^1 \phi(t) e^{-i2\pi kt} dt$$

The $(-k)$-th coefficient:

$$\widetilde{\phi}_{-k}^F = \langle \beta_{-k}^F, \phi \rangle = \int_0^1 \phi(t) e^{i2\pi kt} dt$$

$$\widetilde{\phi}_k^F = (\widetilde{\phi}_{-k}^F)^*$$

Fourier Family

In general…..

The Fourier family of orthonormal functions

$$\{\ldots, \alpha^F_{-2}(t), \alpha^F_{-1}(t), \alpha^F_0(t), \alpha^F_1(t), \alpha^F_2(t)\ldots\}$$

where for $k \in \mathbb{Z}$: $\alpha^F_k(t) := \exp\left(\frac{i2\pi kt}{T}\right)$ for $t \in [0,T)$.

Its optimal representation using the Fourier family is given by

$$\widehat{\phi}^F(t) = \sum_{k=-N}^{N} \widetilde{\phi}^F_k \alpha^F_k(t) \qquad\qquad \widetilde{\phi}^F_k = \frac{1}{T}\int_0^T \phi(t) e^{-\frac{i2\pi kt}{T}} dt$$

Example

Consider Fourier family of orthonormal functions

$$\{\ldots, \alpha_{-2}^F(t), \alpha_{-1}^F(t), \alpha_0^F(t), \alpha_1^F(t), \alpha_2^F(t)\ldots\}$$

where for $k \in \mathbb{Z}$: $\alpha_k^F(t) := \exp\left(\dfrac{i2\pi kt}{2}\right)$ for $t \in [0,2)$.

Consider a real-valued signal $\phi(t) = t^2$ defined for $t = [0,2)$.

Its optimal representation using the Fourier family is given by

$$\widehat{\phi}^F(t) = \sum_{k=-\infty}^{\infty} \widetilde{\phi}_k^F \alpha_k^F(t)$$

Example

Assume the error signal is zero

$$\phi(t) = t^2 = \widehat{\phi}^F(t) = \sum_{k=-\infty}^{\infty} \widetilde{\phi}_k^F \alpha_k^F(t) \;\text{ for } t \in [0,2)$$

- What is the optimal coefficient $\widetilde{\phi}_0^F$? Average
- Show that the optimal coefficient $\widetilde{\phi}_k^F = \dfrac{2(1 + i\pi k)}{\pi^2 k^2}$ for $k \in \mathbb{Z}$ and $k \neq 0$.
- Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$$

Example

$\phi(t) = t^2$

- What is the optimal coefficient $\widetilde{\phi}_0^F$?

$$\widetilde{\phi}_0^F = \frac{1}{2}\int_0^2 \phi(t)e^{-\frac{i2\pi 0t}{2}}dt$$

$$= \frac{1}{2}\int_0^2 \phi(t)dt = \frac{1}{2}\int_0^2 t^2 dt = \frac{4}{3}$$

Example

- Show that the optimal coefficient $\widetilde{\phi}_k^F = \dfrac{2(1+i\pi k)}{\pi^2 k^2}$ for $k \in \mathbb{Z}$ and $k \neq 0$.

$$
\begin{aligned}
\widetilde{\phi}_k^F &= \frac{1}{2}\int_0^2 \phi(t)e^{-\frac{i2\pi k t}{2}}dt = \frac{1}{2}\int_0^2 t^2 e^{-i\pi k t}dt \\
&= \frac{1}{2}\left(\left[\frac{t^2 e^{-i\pi k t}}{-i\pi k}\right]_0^2 - \int_0^2 \frac{2}{-i\pi k} t e^{-i\pi k t} dt\right) \\
&= \frac{1}{2}\left(\frac{4}{-i\pi k} + \frac{2}{i\pi k}\frac{-2}{i\pi k}\right) = \frac{2(1+i\pi k)}{\pi^2 k^2}
\end{aligned}
$$

Example

- Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$$

Recall that the error signal is zero

$$\phi(t) = t^2 = \widehat{\phi}^F(t) = \sum_{k=-\infty}^{\infty} \widetilde{\phi}_k^F \alpha_k^F(t) \;\; \text{for } t \in [0,2)$$

Example

$$\phi(t) = t^2 = \widehat{\phi}^F(t) = \sum_{k=-\infty}^{\infty} \widetilde{\phi}_k^F \alpha_k^F(t) \ \text{ for } t \in [0,2)$$

Observe when $t = 1$

$$\begin{aligned} 1 &= \frac{4}{3} + \sum_{k=1}^{\infty} \frac{2(1+i\pi k)}{\pi^2 k^2} e^{i\pi k} + \sum_{k=-\infty}^{-1} \frac{2(1+i\pi k)}{\pi^2 k^2} e^{i\pi k} \\ &= \frac{4}{3} + \sum_{k=1}^{\infty} \frac{2(1+i\pi k)}{\pi^2 k^2} (-1)^k + \sum_{k=1}^{\infty} \frac{2(1-i\pi k)}{\pi^2 k^2} (-1)^k \\ &= \frac{4}{3} + \sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2} (-1)^k \end{aligned}$$

Example

$$\sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2}(-1)^k = -\frac{1}{3}$$

$$\sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2}(-1)^{k+1} = \frac{1}{3}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$$

Fourier Transformation

$$\widetilde{\phi}_k^F = \frac{1}{T}\int_0^T \phi(t)e^{-\frac{i2\pi kt}{T}}dt$$

Signal	Fourier representation
$\phi(t)$	$\widetilde{\phi}_k^F$
$f(t) = \phi(t-a)$	$\tilde{f}_k^F = ?$
$g(t) = \phi(at)$	$\tilde{g}_k^F = ?$
$h(t) = \phi(t)\exp(iat)$	$\tilde{h}_k^F = ?$