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1 Fourier Family

Consider the Fourier family of orthonormal functions :

$$\{\dots, \beta_{-2}^F(t), \beta_{-1}^F(t), \beta_0^F(t), \beta_1^F(t), \beta_2^F(t), \dots\}$$

where for $k \in \mathbb{Z}$:

$$\beta_k^F(t) := \exp(i2\pi kt) \quad \text{for } t \in [0, 1].$$

Recall that $i^2 := -1$ and that

$$\exp(i2\pi kt) = e^{i2\pi kt} = \cos(2\pi kt) + i \sin(2\pi kt).$$

Are the functions orthonormal ?

1.1 Orthonormality Check

Note that the inner-product is defined as :

$$\langle \beta_k^F, \beta_l^F \rangle = \int_0^1 \beta_k^{F*}(t) \beta_l^F(t) dt$$

When $k = l$:

$$\langle \beta_k^F, \beta_l^F \rangle = \int_0^1 \beta_k^{F*}(t) \beta_k^F(t) dt = \int_0^1 e^{-i2\pi kt} e^{i2\pi kt} dt = \int_0^1 \exp(0) dt = 1$$

When $k \neq l$:

$$\begin{aligned} \langle \beta_k^F, \beta_l^F \rangle &= \int_0^1 e^{-i2\pi kt} e^{i2\pi lt} dt = \int_0^1 e^{i2\pi(l-k)t} dt \\ &= \int_0^1 \cos(2\pi(l-k)t) + i \sin(2\pi(l-k)t) dt = 0 \end{aligned}$$

À retenir

Les fonctions de la famille de Fourier sont orthonormales si et seulement si :

$$\langle \beta_k^F(t), \beta_l^F(t) \rangle = \delta_{k,l} = \begin{cases} 1, & k = l \\ 0, & \text{sinon.} \end{cases}$$

2 Signal Representation

Consider a function $\phi : [0, 1) \rightarrow \mathbb{R}$.

Its optimal representation using the Fourier family is given by :

$$\hat{\phi}^F(t) = \sum_{k=-N}^N \tilde{\phi}_k^F \beta_k^F(t)$$

The corresponding optimal coefficients (in the sense of minimal MSE) are :

$$\tilde{\phi}_k^F = \langle \beta_k^F, \phi \rangle = \int_0^1 \phi(t) e^{-i2\pi kt} dt$$

The error term is defined by :

$$\mathcal{E}(t) = \phi(t) - \hat{\phi}^F(t)$$

The MSE is defined as :

$$\|\mathcal{E}(t)\|^2 = \langle \mathcal{E}(t), \mathcal{E}(t) \rangle = \int_0^1 \left(\phi(t) - \hat{\phi}^F(t) \right) \left(\phi(t) - \hat{\phi}^F(t) \right)^* dt$$

The minimal MSE obtained using the optimal coefficients is (prove!) :

$$\|\mathcal{E}(t)\|^2 = \int_0^1 \phi^2(t) dt - \sum_{k=-N}^N |\tilde{\phi}_k^F|^2$$

📖 Fiche Récapitulative

Objectif : Résumer la représentation optimale d'un signal.

Principe central : Utiliser la famille de Fourier pour représenter un signal de manière optimale.

Points essentiels : - Représentation optimale avec la famille de Fourier - Coefficients optimaux pour minimiser l'erreur quadratique moyenne - Définition de l'erreur et du MSE

Formules clés :

- $\hat{\phi}^F(t) = \sum_{k=-N}^N \tilde{\phi}_k^F \beta_k^F(t)$
- $\tilde{\phi}_k^F = \int_0^1 \phi(t) e^{-i2\pi kt} dt$

3 Conjugate-Symmetry of Representation

Note that we consider a function $\phi : [0, 1) \rightarrow \mathbb{R}$.

The k -th coefficient :

$$\tilde{\phi}_k^F = \langle \beta_k^F, \phi \rangle = \int_0^1 \phi(t) e^{-i2\pi kt} dt$$

The $(-k)$ -th coefficient :

$$\tilde{\phi}_{-k}^F = \langle \beta_{-k}^F, \phi \rangle = \int_0^1 \phi(t) e^{i2\pi kt} dt$$

$$\tilde{\phi}_k^F = (\tilde{\phi}_{-k}^F)^*$$

4 Fourier Family

In general, the Fourier family of orthonormal functions is :

$$\{\dots, \alpha_{-2}^F(t), \alpha_{-1}^F(t), \alpha_0^F(t), \alpha_1^F(t), \alpha_2^F(t), \dots\}$$

where for $k \in \mathbb{Z}$:

$$\alpha_k^F(t) := \exp\left(\frac{i2\pi kt}{T}\right) \quad \text{for } t \in [0, T].$$

Its optimal representation using the Fourier family is given by :

$$\begin{aligned} \hat{\phi}^F(t) &= \sum_{k=-N}^N \tilde{\phi}_k^F \alpha_k^F(t) \\ \tilde{\phi}_k^F &= \frac{1}{T} \int_0^T \phi(t) e^{-\frac{i2\pi kt}{T}} dt \end{aligned}$$

5 Example

Consider Fourier family of orthonormal functions :

$$\{\dots, \alpha_{-2}^F(t), \alpha_{-1}^F(t), \alpha_0^F(t), \alpha_1^F(t), \alpha_2^F(t), \dots\}$$

where for $k \in \mathbb{Z}$:

$$\alpha_k^F(t) := \exp\left(\frac{i2\pi kt}{2}\right) \quad \text{for } t \in [0, 2].$$

Consider a real-valued signal $\phi(t) = t^2$ defined for $t \in [0, 2]$.

Its optimal representation using the Fourier family is given by :

$$\hat{\phi}^F(t) = \sum_{k=-\infty}^{\infty} \tilde{\phi}_k^F \alpha_k^F(t)$$

Assume the error signal is zero :

$$\phi(t) = t^2 = \hat{\phi}^F(t) = \sum_{k=-\infty}^{\infty} \tilde{\phi}_k^F \alpha_k^F(t) \quad \text{for } t \in [0, 2]$$

What is the optimal coefficient $\tilde{\phi}_0^F$?

Show that the optimal coefficient :

$$\tilde{\phi}_k^F = \frac{2(1 + i\pi k)}{\pi^2 k^2} \quad \text{for } k \in \mathbb{Z} \text{ and } k \neq 0.$$

Show that :

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$$

5.1 Optimal Coefficient Calculation

What is the optimal coefficient $\tilde{\phi}_0^F$?

$$\begin{aligned}\tilde{\phi}_0^F &= \frac{1}{2} \int_0^2 \phi(t) e^{-\frac{i2\pi 0t}{2}} dt \\ &= \frac{1}{2} \int_0^2 \phi(t) dt = \frac{1}{2} \int_0^2 t^2 dt = \frac{4}{3}\end{aligned}$$

5.2 Show the Optimal Coefficient

Show that the optimal coefficient :

$$\begin{aligned}\tilde{\phi}_k^F &= \frac{2(1 + i\pi k)}{\pi^2 k^2} \quad \text{for } k \in \mathbb{Z} \text{ and } k \neq 0. \\ \tilde{\phi}_k^F &= \frac{1}{2} \int_0^2 \phi(t) e^{-\frac{i2\pi kt}{2}} dt = \frac{1}{2} \int_0^2 t^2 e^{-i\pi kt} dt \\ &= \frac{1}{2} \left(\left[\frac{t^2 e^{-i\pi kt}}{-i\pi k} \right]_0^2 - \int_0^2 \frac{2}{-i\pi k} t e^{-i\pi kt} dt \right) \\ &= \frac{1}{2} \left(\frac{4}{-i\pi k} + \frac{2}{i\pi k} \frac{-2}{i\pi k} \right) = \frac{2(1 + i\pi k)}{\pi^2 k^2}\end{aligned}$$

5.3 Show the Series Result

Show that :

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$$

Recall that the error signal is zero :

$$\phi(t) = t^2 = \hat{\phi}^F(t) = \sum_{k=-\infty}^{\infty} \tilde{\phi}_k^F \alpha_k^F(t) \quad \text{for } t \in [0, 2)$$

Observe when $t = 1$:

$$\begin{aligned} 1 &= \frac{4}{3} + \sum_{k=1}^{\infty} \frac{2(1 + i\pi k)}{\pi^2 k^2} e^{i\pi k} + \sum_{k=-\infty}^{-1} \frac{2(1 + i\pi k)}{\pi^2 k^2} e^{i\pi k} \\ &= \frac{4}{3} + \sum_{k=1}^{\infty} \frac{2(1 + i\pi k)}{\pi^2 k^2} (-1)^k + \sum_{k=1}^{\infty} \frac{2(1 - i\pi k)}{\pi^2 k^2} (-1)^k \\ &= \frac{4}{3} + \sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2} (-1)^k \\ &\quad \sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2} (-1)^k = -\frac{1}{3} \\ &\quad \sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2} (-1)^{k+1} = \frac{1}{3} \\ &\quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12} \end{aligned}$$

6 Fourier Transformation

$$\tilde{\phi}_k^F = \frac{1}{T} \int_0^T \phi(t) e^{-\frac{i2\pi kt}{T}} dt$$

Signal	Fourier representation
$\phi(t)$	$\tilde{\phi}_k^F$
$f(t) = \phi(t - a)$	$\tilde{f}_k^F = ?$
$g(t) = \phi(at)$	$\tilde{g}_k^F = ?$
$h(t) = \phi(t) \exp(iat)$	$\tilde{h}_k^F = ?$