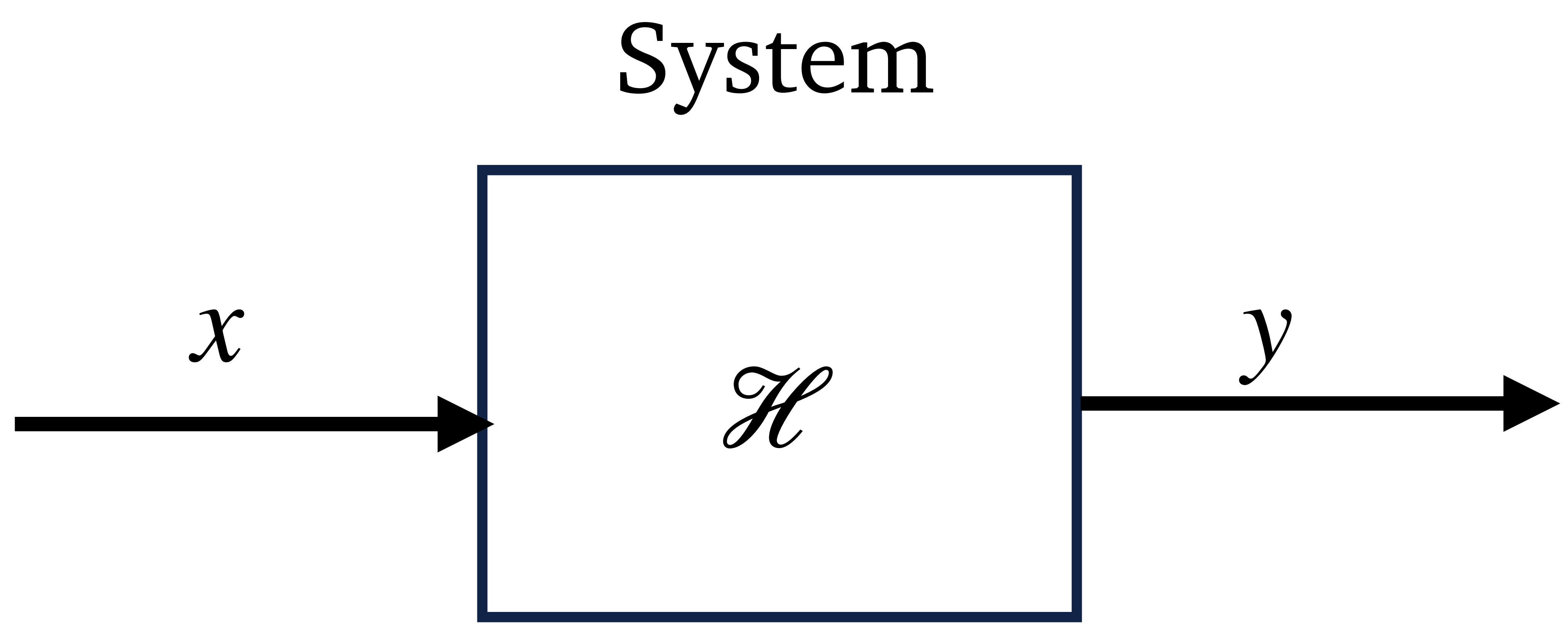
# 236201
# Introduction to data processing and representation

## Tutorial 6:
## Fourier Analysis of Linear Shift-Invariant Systems & DFT

Winter 2024-2025

# Signals and Systems in Continuous Time



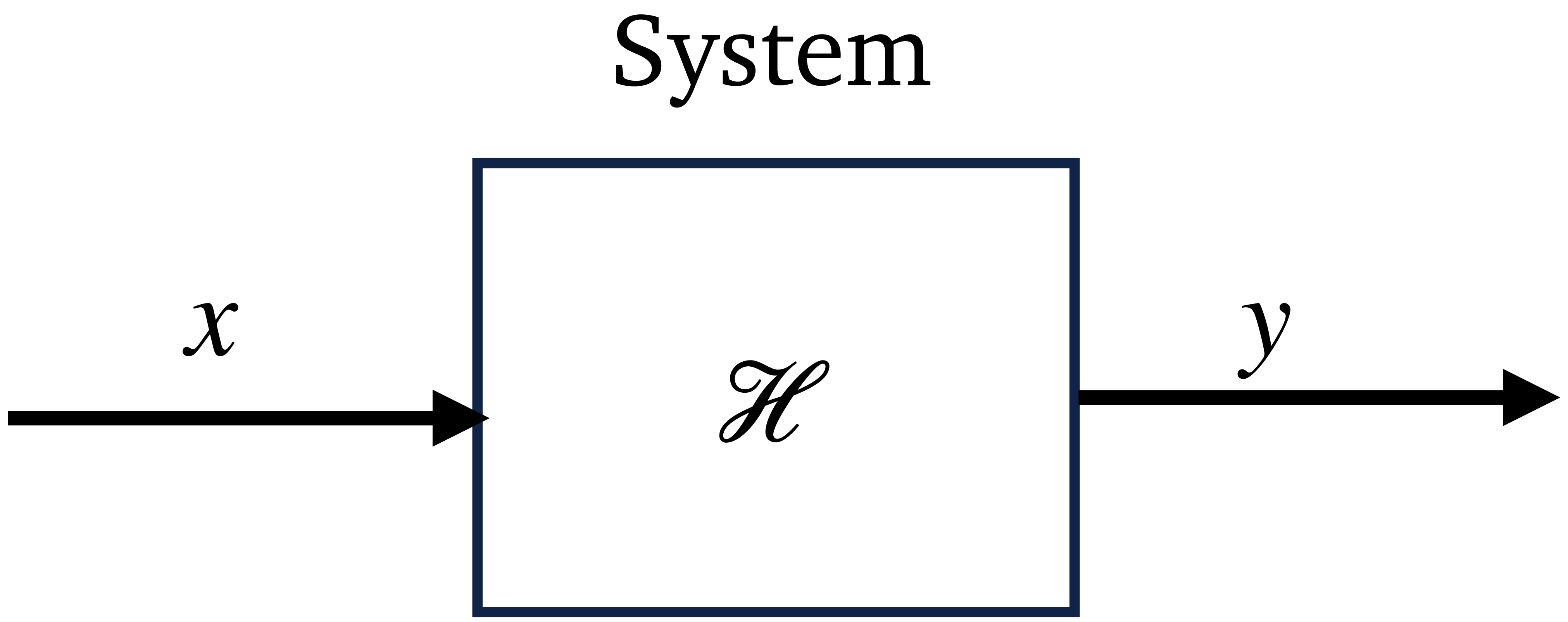

Input signal:

- $x(t)$ is defined for $t \in (-\infty, \infty)$
- $x(t)$ is periodic extension of a signal defined over a limited interval with period $T$

Output signal:

- $y(t) = \mathscr{H} \circ x(t)$ is defined for $t \in (-\infty, \infty)$

# Signals and Systems in Continuous Time




The system $\mathscr{H}$ is said to be ***linear*** if

$$\mathscr{H}(k_1x_1(t) + k_2x_2(t)) = k_1\mathscr{H}(x_1(t)) + k_2\mathscr{H}(x_2(t)) = k_1y_1(t) + k_2y_2(t)$$

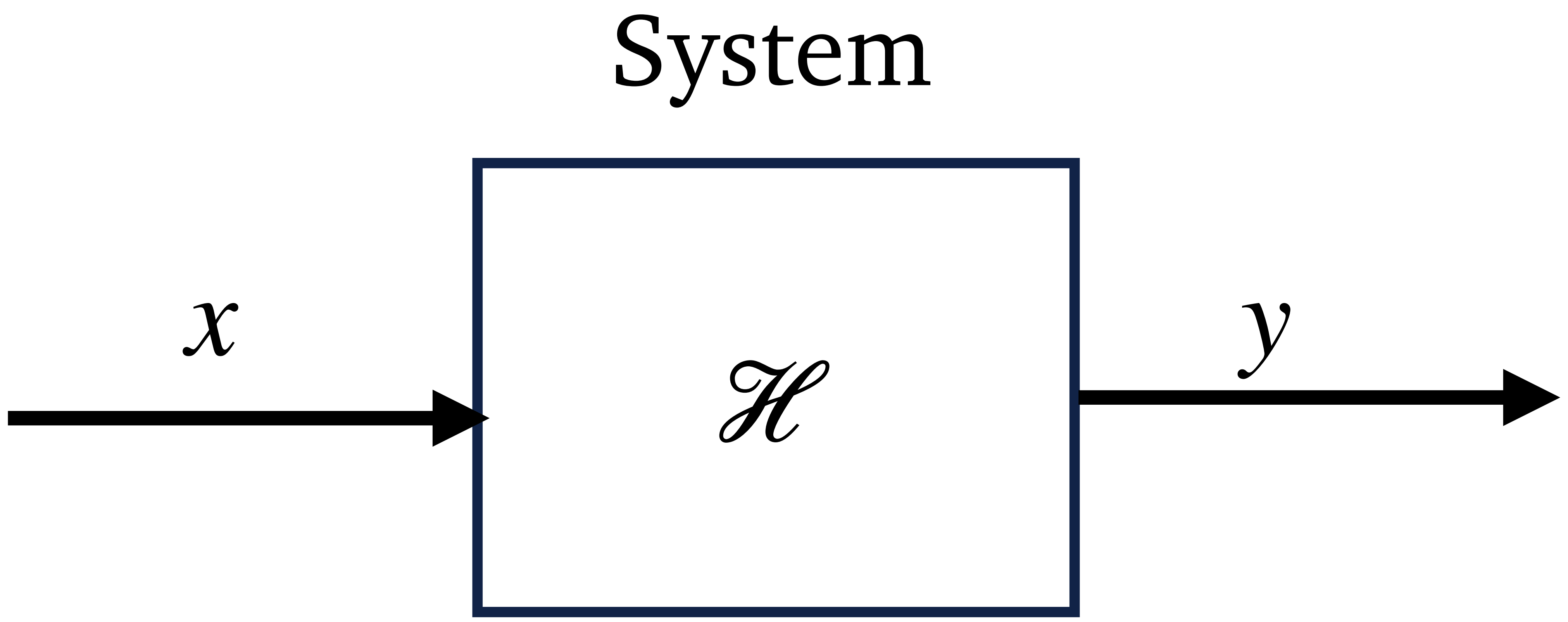
The system $\mathscr{H}$ is said to be ***shift-invariant*** if

$$\mathscr{H}(\mathscr{T}_{t_0}(x(t))) = \mathscr{T}_{t_0}(y(t)) \text{ where } \mathscr{T}_{t_0}(x(t)) = x(t - t_0)$$

# Examples

| System | Linear | Shift-Invariant |
|---|---|---|
| $y(t) = x^2(t)$ | | |
| $y(t) = x(t) + a$ | | |
| $y(t) = ax(t)$ | | |
| optimal sampling | | |

# Linear Shift-Invariant Systems




Input signal:

- $x(t)$ is defined for $t \in (-\infty, \infty)$
- $x(t)$ is periodic extension of a signal defined over a limited interval with period 1

System $\mathscr{H}$

- Let $\widetilde{h}(t)$ be the basic period of its **impulse response** such that $\widetilde{h} : [0,1) \to \mathbb{R}$
- Periodically extend $\widetilde{h}(t)$ into $h(t)$

# Convolution Systems

Consider an input signal $x(t)$ defined for $t \in [0,1)$

Consider the convolution system:

$$y(t) = \int_0^1 x(\tau)h(t-\tau)d\tau$$

We use here the orthonormal family of Fourier functions, defined over $[0,1)$ for $k \in \mathbb{Z}$

$$\beta_k^F(t) = \exp(i2\pi kt)$$

# Fourier-based Analysis of Systems

We use here the orthonormal family of Fourier functions, defined over $[0,1)$ for $k \in \mathbb{Z}$

$$\beta_k^F(t) = \exp(i2\pi kt)$$

We use this Fourier family for the following representations

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \beta_k^F(t) \qquad a_k = \langle \beta_k^F, x \rangle = \int_0^1 x(t)\exp(-i2\pi kt)dt$$

$$h(t) = \sum_{k=-\infty}^{\infty} b_k \beta_k^F(t) \qquad b_k = \langle \beta_k^F, h \rangle = \int_0^1 h(t)\exp(-i2\pi kt)dt$$

# Convolution Systems

The convolution system:

$$y(t) = \int_0^1 x(\tau)h(t-\tau)d\tau$$

$$= \int_0^1 \left( \sum_{k=-\infty}^{\infty} a_k \beta_k^F(\tau) \right) \left( \sum_{l=-\infty}^{\infty} b_l \beta_l^F(t-\tau) \right) d\tau = \int_0^1 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k b_l \beta_k^F(\tau) \beta_l^F(t-\tau) d\tau$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k b_l \int_0^1 \beta_k^F(\tau) \beta_l^F(t-\tau) d\tau = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k b_l \beta_l^F(t) \int_0^1 \exp(i2\pi(k-l)\tau) d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_k b_k \beta_k^F(t)$$

# Convolution Systems

The convolution system:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k b_k \beta_k^F(t)$$

$$\sum_{k=-\infty}^{\infty} c_k \beta_k^F(t) = \sum_{k=-\infty}^{\infty} a_k b_k \beta_k^F(t)$$

$$c_k = a_k b_k$$

# Derivation System

Consider an input signal $x(t)$ defined for $t \in [0,1)$

Consider the derivation system:

$$y(t) = \frac{d}{dt}x(t)$$

with boundary values obeying the cyclic continuity: $x(0) = x(1)$.

Is it linear? Is it shift invariant?

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We use here the orthonormal family of Fourier functions, defined over $[0,1)$ for $k \in \mathbb{Z}$

$$\beta_k^F(t) = \exp(i2\pi kt)$$

# Derivation System

Consider an input signal $x(t)$ defined for $t \in [0,1)$

Consider the derivation system:

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with boundary values obeying the cyclic continuity: $x(0) = x(1)$.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \beta_k^F(t) \qquad a_k = \int_0^1 x(t)\exp(-i2\pi kt)dt$$

$$y(t) = \sum_{k=-\infty}^{\infty} c_k \beta_k^F(t) \qquad c_k = \int_0^1 y(t)\exp(-i2\pi kt)dt$$

# Derivation System

$$c_k = \int_0^1 y(t)\exp(-i2\pi kt)dt = \int_0^1 \frac{dx}{dt}(t)\exp(-i2\pi kt)dt$$

$$= \Big[x(t)\exp(-i2\pi kt)\Big]_0^1 - \int_0^1 x(t)\left(\frac{d}{dt}\exp(-i2\pi kt)\right)dt$$

$$= -\int_0^1 x(t)(-i2\pi k)\exp(-i2\pi kt)dt = i2\pi k\int_0^1 x(t)\exp(-i2\pi kt)dt$$

$$c_k = i2\pi k a_k$$

# DFT

$$[\text{DFT}] = \frac{1}{\sqrt{N}} \begin{bmatrix} W^{*0\cdot 0}, & W^{*1\cdot 0}, & \dots, & W^{*(N-1)\cdot 0} \\ W^{*0\cdot 1}, & W^{*1\cdot 1}, & \dots, & W^{*(N-1)\cdot 1} \\ \vdots\,, & \vdots\,, & \dots, & \vdots \\ W^{*0\cdot(N-1)}, & W^{*1\cdot(N-1)}, & \dots, & W^{*(N-1)\cdot(N-1)} \end{bmatrix} \in \mathbb{C}^{N\times N}$$

where

$$W = \exp\left(\frac{i2\pi}{N}\right)$$

Note that the DFT matrix is symmetric and unitary.

# Representation of a Discrete Signal in the DFT Domain

The representation of the discrete signal $\phi \in \mathbb{R}^N$ is

$$\phi^F = [\mathrm{DFT}]\phi$$

Observe that

$$[\mathrm{DFT}]^*\phi^F = [\mathrm{DFT}]^*[\mathrm{DFT}]\phi$$

$$[\mathrm{DFT}]^*\phi^F = \phi$$

Above it's the inverse DFT procedure.

# Example

Consider the following discrete signal of $N$ samples

For $n = 0,1,2,\ldots,(N-1)$, $\phi(n) = \cos\left(\frac{2\pi k_0}{N}n\right)$ where $k_0 \in \{0,1,2,\ldots,(N-1)\}$

The $k^{\text{th}}$ component of the DFT-domain representation of the above signal is

$$\phi^F(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} W^{*k\cdot n}\phi(n) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} W^{*k\cdot n}\frac{1}{2}(W^{k_0\cdot n} + W^{-k_0\cdot n})$$

$$= \frac{1}{\sqrt{N}}(\frac{1}{2}\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n}) + \frac{1}{2}\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{-k_0\cdot n}))$$

# Example

$$\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n})$$

When $k = k_0$

$$\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n}) = N$$

When $k \neq k_0$

$$\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n}) = \sum_{n=0}^{N-1} (W^{-(k-k_0)})^n = \frac{(W^{-(k-k_0)})^N - 1}{(W^{-(k-k_0)}) - 1} = 0$$

# Example

$$\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n}) = N\delta_{k,k_0} = \begin{cases} 1, & k = k_0 \\ 0, & OW \end{cases}$$

Similarly,

$$\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{-k_0\cdot n}) = N\delta_{k,-k_0} = \begin{cases} 1, & k = -k_0 \\ 0, & OW \end{cases}$$

$$\phi^F(k) = \frac{1}{\sqrt{N}}(\frac{1}{2}\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n}) + \frac{1}{2}\sum_{n=0}^{N-1} W^{*k\cdot n}(W^{-k_0\cdot n})) = \frac{\sqrt{N}}{2}(\delta_{k,k_0} + \delta_{k,-k_0})$$

# Example

$$\phi^F(k) = \frac{1}{\sqrt{N}} \left( \frac{1}{2} \sum_{n=0}^{N-1} W^{*k\cdot n}(W^{k_0\cdot n}) + \frac{1}{2} \sum_{n=0}^{N-1} W^{*k\cdot n}(W^{-k_0\cdot n}) \right) = \frac{\sqrt{N}}{2}(\delta_{k,k_0} + \delta_{k,-k_0})$$

For example, for $N = 9$ and $k_0 = 3$, the vector of DFT coefficients is:

$$[0,0,0,\frac{3}{2},0,0,\frac{3}{2},0,0]^\top$$