Math 316 Project

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Problem 1

1. Suppose that $f : [0,2] \to \mathbb{R}$ is twice differentiable and $f(0) = 0,\ f(1) = 2,\ f(2) = 4$

   Prove : There is a number $c \in (0,2)$ where $f''(c) = 0$

   Proof:

   Suppose $f : [0,2] \to \mathbb{R}$ is twice differentiable and $f(0) = 0,\ f(1) = 2,\ f(2) = 4$. Since f is twice diferentiable we know that f' and f'' both exist and are continuous (differentiability implies continuity) on $[0,2]$. So we can apply the **Mean Value Theorem**, lets observe 2 subintervals of [0,2], [0,1] and [1,2]. There exists an $a \in [0,1]$ and $b \in [1,2]$ where $f'(a) = \frac{f(1)-f(0)}{1-0}$ and $f'(b) = \frac{f(2)-f(1)}{2-1}$.

   Then,

   $$f'(a) = \frac{2-0}{1-0} = 2$$

   $$f'(b) = \frac{4-2}{2-1} = 2$$

   Now we have $f'(a) = 2$ and $f'(b) = 2$. We can now apply the **Mean Value Theorem** again. So there exists a $c \in [0,2]$ such that $f''(c) = \frac{f'(b)-f'(a)}{b-a}$.

   Here we don't really need to know what b and a are just that they exist in the interval $[0,2]$.

   Now we see that,

   $$f''(c) = \frac{f'(b) - f'(a)}{b-a}$$

   $$f''(c) = \frac{2-2}{b-a}$$

   $$f''(c) = 0$$

   Which is what we set out to show.

2. Suppose $a > 0$ and $g[-a,a] \to \mathbb{R}$ is differentiable with $g'(x) \leq 1$ for all $x \in (-a,a)$.

   Prove : If $g(a) = a$ and $g(-a) = -a$ then $g(x) = x$ for all $x \in (-a,a)$

   Proof:

   Suppose g is defined as above and that $g(q) \neq q$ for some $q \in [-a,a]$. By the **Mean Value Theorem** there must exist some $c \in [-a,q]$ such that $g'(c) = \frac{g(q)-g(-a)}{q-(-a)} = \frac{g(q)+a}{q+a}$. By our assumptions $g(q) \neq q$, $q \geq -a$ and $g'(c) \leq 1$ therefore $g(q) \leq -a$. Furthermore there must also exist some $d \in [q,a]$ such that

$g'(d) = \frac{g(a)-g(q)}{a-q} = \frac{a-g(q)}{a-q}$, and by our assumptions $q < a$. Here we have a contradiction because we showed that $g(q) \leq -a$, but that means that $g'(d) > 1$. The contradiction arises because we assumed that $g(q) \neq q$ therefore $g(q) = q$ for all $q \in [-a, a]$

Problem 2

1. **Prove** : If C is a closed set of real numbers that contains the set of irrational numbers between 0 and 1 then C contains the interval [0,1].

   Proof :

   Let C be defined as above. By the definition of closed sets P=[0,1]$\cap$C^c must be an open set. By the definiton of an open set, for some $\epsilon > 0$, if a $\in$ P the N(a,ϵ) $\subseteq$ P, where N(a,ϵ)= (a-ϵ,a+ϵ). However P$\subseteq$[0,1] so a$\in$[0,1] and by the properties of the real number sytem there exists some irrational number between a and a+ϵ, therefore N(a,ϵ)= (a-ϵ,a+ϵ)$\not\subset$ P, so a $\notin$ P. Thus P=$\emptyset$, and [0,1] $\subseteq$ C.

2. **Prove** : The set of irrational numbers between 0 and 1 does not have measure 0

   Proof : Let C= {the set of irrational numbers between 0 and 1} and let P=[0,1]$\cap$C^c={the set of rational numbers between 0 and 1}. Let us assume that C has measure 0. Observe that P is countably infinite, and therefore has measure 0. So P+C must have measure 0, but P+C=[0,1] which does not have measure 0. Therefore C must not have measure 0.

Problem 3

1. Suppose c_0 is any real number and for any $n \geq 0$, $c_{n+1} = \cos(c_n)$.

   Prove : The sequence converges and the $\lim_{n\to\infty} c_n = c$ where c is the unique solution to the equation $\cos(x) = x$

   Proof : To make things a bit easier I broke this problem into four parts

   Definition: Cos(Cos(....C$os(A)$..) = C$os_n(a)$ where n is the number of consecutive Cos's

   Part 1

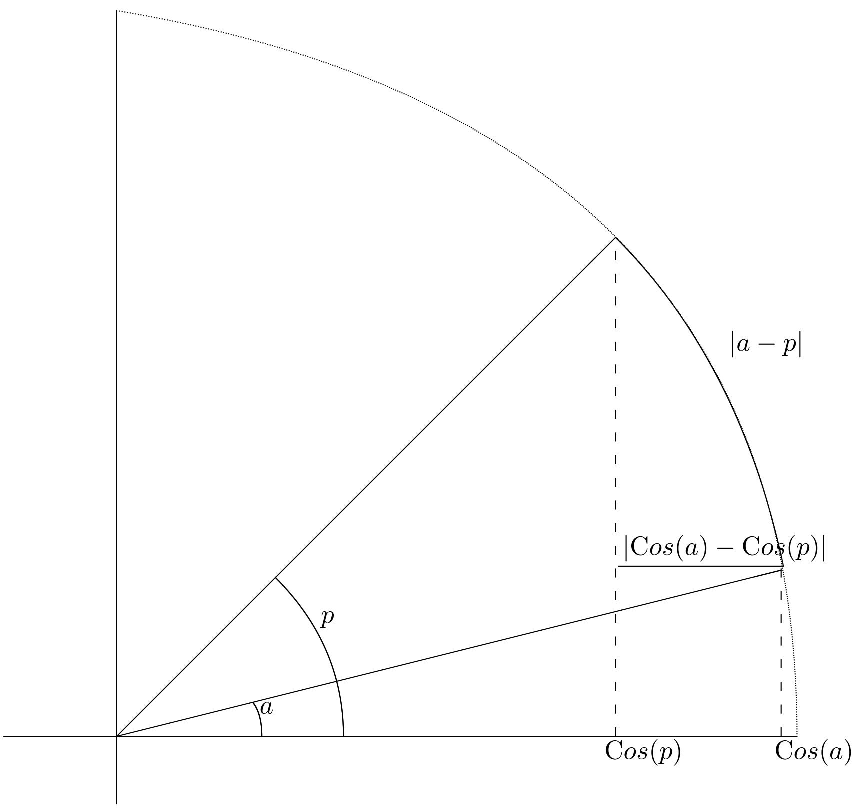
   The derivative of Cos(x) is -Sin(x), and -Sin(x) <0 in the interval [0,1]. Therefore if a,b$\in$[0,1] and a<b, then Cos(a)>Cos(b) and Cos$_2$(a)<Cos$_2$(b).

   Part 2

The functions $f(x) = x$ and $g(x) = \mathrm{C}os(x)$ are both continuous in $\mathbb{R}$ so the function $h(x) = \mathrm{C}os(x) - x$ is contiuous in $\mathbb{R}$. Since $h(0) > 0$ and $h(1) < 0$ by the **Intermediate Value Theorem** there must be a point p in the interval [0,1] such that $h(p) = 0$. Furthermore if $q \in \mathbb{R}$ such that $q \neq p$ and $h(q) = 0$ then $q > p$ or $q < p$. Without loss of generality allow $q > p$, it follows that $h(q) < h(p)$ because $h'(x) \leq 0$ for all $x \in \mathbb{R}$, and $h'(p) < 0$, but here is a contradiction because $h(q) = h(p) = 0$ therefore $q = p$.

Part 3

Let $\mathrm{C}os(p) = p$ and let $a \in [0, 1]$, by observation of the unit circle, since both p and $a \in [0, 1]$ they are both in the first quadrant. Furthermore if we let $|a - p|$ represents the difference in the angles a and p along the arc of the unit circle then $|\mathrm{C}os(a) - \mathrm{C}os(p)|$ represent the projection of $|a - p|$ onto the x-axis we see that $|\mathrm{C}os(a) - \mathrm{C}os(p)| = |\mathrm{C}os(a) - p| \leq |a - p|$. Graphically it looks like this model of 1^{st} quadrant of the unit circle



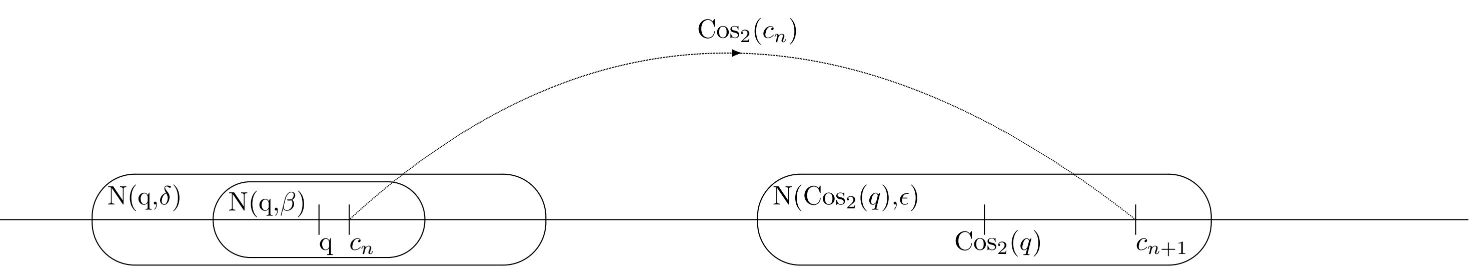
Part 4

Let $Cos(p) = p$ and let $\{a_n\}$ represent a seqence of Real numbers starting with $a_0 \in \mathbb{R}$ such that $a_{n+1} = Cos(a_n)$. By obersvation we can see that $Cos(a_0) \in [-1, 1]$ and $Cos_2(a_0) \in [0, 1]$ so $a_2 \in [0, 1]$. Furthermore since $a_2 \in [0, 1]$ then $|a_n - p| > |a_{n+1} - p|$ (by part 3) so for $n > 2$, $a_n \in [0, 1]$. Notice also that $\{a_n\}$ oscillates around p, (part 1), so we can separate the original sequence into 2 sequences, $\{c_n\}$ and $\{d_n\}$, such that if $a_r < p$ then $c_i = a_r$ and if $a_m \geq p$ then $d_i = a_m$ for any a_m, $a_r \in \{a_n\}$. Observe that $\{c_n\} \cup \{d_n\} = \{a_n\}$. By part 2 we see that $\{c_n\}$ is bounded above by p and $\{d_n\}$ is bounded below by p therefore $\{c_n\}$ and $\{d_n\}$ are bounded monotonic a sequences (part 3) , so they are both convergent. Furthermore $c_{n+1} = Cos_2(c_n)$ and $d_{n+1} = Cos_2(d_n)$(Part 1)

Observe that because $Cos_2(x)$ is continuous, it is ϵ/δ continuous. Therefore if $\epsilon > 0$ there must exist some δ such that $Cos_2(N(x, \delta)) \subseteq N(Cos_2(x), \epsilon)$ and as ϵ gets smaller δ must also get smaller, thus there exists some $\epsilon + \delta < |\frac{\text{Cos}(q)-q}{100}|$.

Now let $\epsilon + \delta < |\frac{\text{Cos}(q)-q}{100}|$, let $\beta > 0$ such that $\beta < \delta$ and suppose without loss of generality that $\{d_n\}$ converges to $q \neq p$ then $\lim_{n\to\infty} d_n = q$ and for β there exists some M such that if $n > M$ then $a_n \in N(q, \beta)$. Observe that $N(q, \beta) \subseteq N(q, \delta)$ so $\text{Cos}_2(d_n) \in N(\text{Cos}_2(q), \epsilon)$ however $N(\text{Cos}_2(q), \epsilon) \cap N(q, \delta) = \emptyset$ this means that $d_{n+1} \notin N(q, \beta)$ so q cannot be the limit. Therefore p must be the limit of $\{c_n\}$ and $\{d_n\}$, and since $\{c_n\}$ and $\{d_n\}$ both converge to p, $\{c_n\} \cup \{d_n\}$ must converge to p and so $\{a_n\}$ must converge to p.

Graphically it can be represented like this:



2. Suppose K is a compact subset of $\mathbb{R}$ and $\{x_n\}$ with $x_n \in$ K for all n.

(a) **Prove** : $\{x_n\}$ has a convergent subsequence.

Proof :

Assume K is a compact subset of $\mathbb{R}$ and $\{x_n\}$ is a sequence with $x_n \in$ K for all n. We know from the reverse of the **Heine-Borel Theorem** that if K is a compact subset of $\mathbb{R}$ then K is closed and bounded which means $\{x_n\}$ is a bounded sequence. Thus, by the **Bolzano-Weierstass Theorem**, since $\{x_n\}$ is bounded it has a convergent subsequence.

(b) **Prove** : If $\lim_{n\to\infty} x_n = a$ then $a \in$ K.

Proof :

Assume K is a compact subset of $\mathbb{R}$ and $\{x_n\}$ is a sequence with $x_n \in$ K for all n$\in \mathbb{N}$. Let $\lim_{n\to\infty} x_n = q$ with q$\notin$K. Since K is a compact subset of $\mathbb{R}$, K^c is an open set and if q $\notin$ K then q$\in$ K^c. By the properties of open sets N(q, ϵ) $\subseteq$ K^c. However since $x_n \in$ K for all n and the $\lim_{n\to\infty} x_n = q$, then for any δ >0, $x_n \in$ N(q, δ). But if we let $\delta < \epsilon$ we have a contradiction because $x_n \in$ K and $x_n \in$N(q,δ), this conflict arises because q $\notin$ K, therefore q must be an element of K.

(c) Suppose L is a subset of $\mathbb{R}$ with the property that if $\{x_n\}$ is a sequence with $x_n \in$ L for all n then $\{x_n\}$ has a subsequence that converges to an element of L.

Prove : L is compact.

Proof :

Consider M, a subsequence of the sequence $\{x_n\}$ that converges to an element in L. That is, there is a c $\in$ L where $\lim_{n\to\infty} m_n = c$. We know from properties of convergent sequences that if a subsequence converges it is bounded (both above and below). This means M is bounded, M $\subset$ [a, b] for some a and b. By the **Heine Borel Theorem**, [a, b] is compact. So, M is a closed subset of L which is a subset of $\mathbb{R}$. Thus M is compact. Observe that if L is not compact then L$\cap$M^c must be an open set. Furthermore if we let S={a—a$\in$L and a$\notin$ $\{x_n\}$ for any $\{x_n\}$ with a convergent subsequence} then N is an open set. Now let a$_0$$\in$S therefore for some ϵ_0 >0 N(a$_0$,ϵ_0)$\subseteq$N and furthermore we let a$_1$=a$_0$+$\frac{\epsilon_0}{2}$ then a$_1$ $\in$N. Let {a$_n$} represent the sequence a$_{n+1}$=a$_n$ + $\frac{\epsilon_n}{2}$. $\{x_n\}$ $\subseteq$L therefore it must have a convergent subsequence. This causes a contradiction because $x_n \in$ S for all n$\in \mathbb{N}$ and therefore S=$\emptyset$ and so L must be a union of closed and bounded intervals, and by the **Heine Borel Theorem** it must be compact.

Problem 4

Definition : If A is a set of real numbers, the closure of A, denoted K(A), is the intersection of the family of all closed sets L where A is a subset of L.

1. **Prove** : K((2,3))=[2,3]

   Proof :

   Observe that K((2,3))⊆[2,3], and K((2,3)) is a closed set therefore [2,3]∩K((2,3))^c must be an open set. Let [2,P) ∪ (Q,3] represent this set, observe that if $2 < P, Q < 3$ then P,Q∈(2,3) so P=2 and Q=3. If a∈[2,3] and a∉K((2,3)) then a ∈ [2,2) ∪ (3,3], and also for some ϵ >0, N(a,ϵ)⊆[2,2) ∪ (3,3] by the rules of open sets. However for any $\epsilon > 0$, 2+ϵ and 3-ϵ ∈(2,3) therefore a∉ [2,2)∪(3,3]. Thus [2,3]∩K$((2,3))^c$ =∅, and [2,3]=K((2,3)).

2. **Prove** K($\mathbb{Q} \cap (0,1)$) = $[0,1]$ where $\mathbb{Q}$ is the set of rational numbers.

   Proof :

   Let C=K($\mathbb{Q} \cap (0,1)$) observe that C⊆ $[0,1]$ and also that C is a closed set, by the rules of intersections of open and closed sets. If we let D=C^c∪[0,1], then D must be open, so for any $a \in$ C^c there must exist some $\epsilon > 0$ such that N$(a, \epsilon) \subset$ D. For any $\epsilon > 0$ the interval (a+$\epsilon, a - \epsilon$) must contain a rational number, so $a \notin$D. So D=∅ and because C+D=[0,1], it must be true that C=[0,1].

3. **Prove** : For any set of real numbers A, K(A)=A if and only if A is a closed set

   Proof :

   Let A be closed. A⊆A, therefore K(A)∩A=K(A) so K(A)⊆A and A⊆ K(A), thus K(A)=A.

   Now let K(A)=A. By the rules of closed sets the intersection of closed sets is closed therefore K(A)=A is a closed set.

Problem 5

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function

1. **Prove** : If A is a closed set of real numbers then $f^{-1}(A) = \{x : f(x) \in A\}$ is a closed set.

   Proof :

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Let A be a closed set of real numbers, by the rules of closed sets A^c must be an open set. Futhermore by the topological definition of continuity since A^c is open then $f^{-1}(A^c)$ must be open. Finally since $f^{-1}(A^c)$ is open$f^{-1}(A^c)^c$ must be closed, but $f^{-1}(A^c)^c = f^{-1}(A)$ so $f^{-1}(A)$ must be closed.

2. **Prove** : If $f(x+5) = f(x)$ for all x then f is uniformly continuous.

   Proof :

   Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $f(x+5) = f(x)$. By theorem 6.2.9 from http://pirate.shu.edu/ wachsmut/ira/cont/proofs/ctunifct.html we know that since [0,5] is a compact set and f is continous over $\mathbb{R}$ then$f$ must be uniformly continous over it [0,5]. Furthermore since every interval [5i,5i+5] for all i$\in$ $\mathbb{I}$is an exact image of [0,5] we know that f must be uniformly continuous over $\mathbb{R}$.