$$V = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{1}{2}u_{\max} \tag{6.43}$$

For a horizontal tube ($\Delta z = 0$), Eq. (6.42) is of the form predicted by Hagen's experiment, Eq. (6.1):

$$\Delta p = \frac{8\mu LQ}{\pi R^4} \tag{6.44}$$

The wall shear is computed from the wall velocity gradient

$$\tau_w = \left|\mu \frac{du}{dr}\right|_{r=R} = \frac{2\mu u_{\max}}{R} = \frac{1}{2}R\left|\frac{d}{dx}(p + \rho gz)\right| \tag{6.45}$$

This gives an exact theory for laminar Darcy friction factor

$$f = \frac{8\tau_w}{\rho V^2} = \frac{8(8\mu V/d)}{\rho V^2} = \frac{64\mu}{\rho Vd}$$

or

$$f_{\text{lam}} = \frac{64}{\text{Re}_d} \tag{6.46}$$

This is plotted later in the Moody chart (Fig. 6.13). The fact that f drops off with increasing Re_d should not mislead us into thinking that shear decreases with velocity: Eq. (6.45) clearly shows that τ_w is proportional to $u_{\max}$; it is interesting to note that τ_w is independent of density because the fluid acceleration is zero.

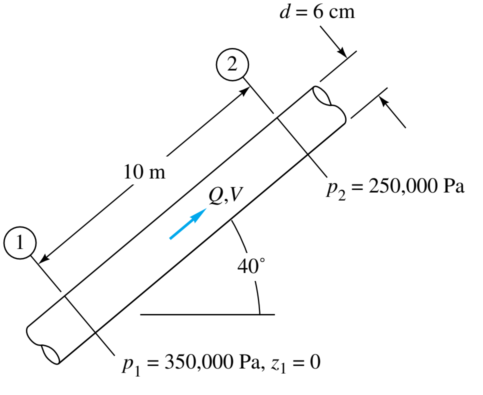
The laminar head loss follows from Eq. (6.30)

$$h_{f,\text{lam}} = \frac{64\mu}{\rho Vd}\frac{L}{d}\frac{V^2}{2g} = \frac{32\mu LV}{\rho gd^2} = \frac{128\mu LQ}{\pi\rho gd^4} \tag{6.47}$$

We see that laminar head loss is proportional to V.

EXAMPLE 6.4

An oil with $\rho = 900$ kg/m^3 and $\nu = 0.0002$ m^2/s flows upward through an inclined pipe as shown in Fig. E6.4. The pressure and elevation are known at sections 1 and 2, 10 m apart. Assuming



E6.4

steady laminar flow, (*a*) verify that the flow is up, (*b*) compute h_f between 1 and 2, and compute (*c*) Q, (*d*) V, and (*e*) Re_d. Is the flow really laminar?

Solution

Part (a) For later use, calculate

$$\mu = \rho\nu = (900 \text{ kg/m}^3)(0.0002 \text{ m}^2/\text{s}) = 0.18 \text{ kg/(m} \cdot \text{s)}$$

$$z_2 = \Delta L \sin 40° = (10 \text{ m})(0.643) = 6.43 \text{ m}$$

The flow goes in the direction of falling HGL; therefore compute the hydraulic grade-line height at each section

$$\text{HGL}_1 = z_1 + \frac{p_1}{\rho g} = 0 + \frac{350{,}000}{900(9.807)} = 39.65 \text{ m}$$

$$\text{HGL}_2 = z_2 + \frac{p_2}{\rho g} = 6.43 + \frac{250{,}000}{900(9.807)} = 34.75 \text{ m}$$

The HGL is lower at section 2; hence the flow is from 1 to 2 as assumed. *Ans. (a)*

Part (b) The head loss is the change in HGL:

$$h_f = \text{HGL}_1 - \text{HGL}_2 = 39.65 \text{ m} - 34.75 \text{ m} = 4.9 \text{ m} \qquad \textit{Ans. (b)}$$

Half the length of the pipe is quite a large head loss.

Part (c) We can compute Q from the various laminar-flow formulas, notably Eq. (6.47)

$$Q = \frac{\pi\rho g d^4 h_f}{128\mu L} = \frac{\pi(900)(9.807)(0.06)^4(4.9)}{128(0.18)(10)} = 0.0076 \text{ m}^3/\text{s} \qquad \textit{Ans. (c)}$$

Part (d) Divide Q by the pipe area to get the average velocity

$$V = \frac{Q}{\pi R^2} = \frac{0.0076}{\pi(0.03)^2} = 2.7 \text{ m/s} \qquad \textit{Ans. (d)}$$

Part (e) With V known, the Reynolds number is

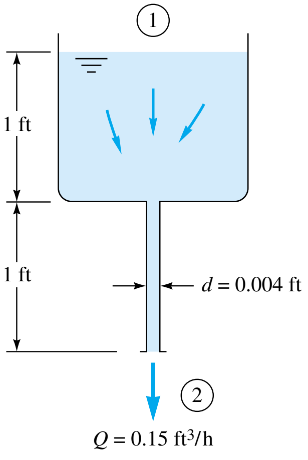
$$\text{Re}_d = \frac{Vd}{\nu} = \frac{2.7(0.06)}{0.0002} = 810 \qquad \textit{Ans. (e)}$$

This is well below the transition value $\text{Re}_d = 2300$, and so we are fairly certain the flow is laminar.

Notice that by sticking entirely to consistent SI units (meters, seconds, kilograms, newtons) for all variables we avoid the need for any conversion factors in the calculations.

EXAMPLE 6.5

A liquid of specific weight $\rho g = 58 \text{ lb/ft}^3$ flows by gravity through a 1-ft tank and a 1-ft capillary tube at a rate of $0.15 \text{ ft}^3/\text{h}$, as shown in Fig. E6.5. Sections 1 and 2 are at atmospheric pressure. Neglecting entrance effects, compute the viscosity of the liquid.



E6.5

Solution

Apply the steady-flow energy equation (6.24), including the correction factor α:

$$\frac{p_1}{\rho g} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_f$$

The average exit velocity V_2 can be found from the volume flow and the pipe size:

$$V_2 = \frac{Q}{A_2} = \frac{Q}{\pi R^2} = \frac{(0.15/3600)\text{ ft}^3/\text{s}}{\pi(0.002\text{ ft})^2} \approx 3.32\text{ ft/s}$$

Meanwhile $p_1 = p_2 = p_a$, and $V_1 \approx 0$ in the large tank. Therefore, approximately,

$$h_f \approx z_1 - z_2 - \alpha_2\frac{V_2^2}{2g} = 2.0\text{ ft} - 2.0\,\frac{(3.32\text{ ft/s})^2}{2(32.2\text{ ft/s}^2)} \approx 1.66\text{ ft}$$

where we have introduced $\alpha_2 = 2.0$ for laminar pipe flow from Eq. (3.72). Note that h_f includes the entire 2-ft drop through the system and not just the 1-ft pipe length.

With the head loss known, the viscosity follows from our laminar-flow formula (6.47):

$$h_f = 1.66\text{ ft} = \frac{32\mu LV}{\rho g d^2} = \frac{32\mu(1.0\text{ ft})(3.32\text{ ft/s})}{(58\text{ lbf/ft}^3)(0.004\text{ ft})^2} = 114{,}500\ \mu$$

or

$$\mu = \frac{1.66}{114{,}500} = 1.45\text{ E-5 slug/(ft}\cdot\text{s)} \qquad \textit{Ans.}$$

Note that L in this formula is the *pipe* length of 1 ft. Finally, check the Reynolds number:

$$\text{Re}_d = \frac{\rho V d}{\mu} = \frac{(58/32.2\text{ slug/ft}^3)(3.32\text{ ft/s})(0.004\text{ ft})}{1.45\text{ E-5 slug/(ft}\cdot\text{s)}} = 1650 \qquad \text{laminar}$$

Since this is less than 2300, we conclude that the flow is indeed laminar. Actually, for this head loss, there is a *second* (turbulent) solution, as we shall see in Example 6.8.

Turbulent-Flow Solution

For turbulent pipe flow we need not solve a differential equation but instead proceed with the logarithmic law, as in Example 6.3. Assume that Eq. (6.21) correlates the local mean velocity $u(r)$ all the way across the pipe

$$\frac{u(r)}{u^*} \approx \frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu} + B \qquad (6.48)$$

where we have replaced y by $R - r$. Compute the average velocity from this profile

$$V = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R u^*\left[\frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu} + B\right]2\pi r\,dr$$

$$= \frac{1}{2}u^*\left(\frac{2}{\kappa}\ln\frac{Ru^*}{\nu} + 2B - \frac{3}{\kappa}\right) \qquad (6.49)$$

Introducing $\kappa = 0.41$ and $B = 5.0$, we obtain, numerically,

$$\frac{V}{u^*} \approx 2.44\ln\frac{Ru^*}{\nu} + 1.34 \qquad (6.50)$$

This looks only marginally interesting until we realize that V/u^* is directly related to the Darcy friction factor

$$\frac{V}{u*} = \left(\frac{\rho V^2}{\tau_w}\right)^{1/2} = \left(\frac{8}{f}\right)^{1/2} \tag{6.51}$$

Moreover, the argument of the logarithm in (6.50) is equivalent to

$$\frac{Ru*}{\nu} = \frac{\frac{1}{2}Vd}{\nu}\frac{u*}{V} = \frac{1}{2}\mathrm{Re}_d\left(\frac{f}{8}\right)^{1/2} \tag{6.52}$$

Introducing (6.52) and (6.51) into Eq. (6.50), changing to a base-10 logarithm, and rearranging, we obtain

$$\frac{1}{f^{1/2}} \approx 1.99 \log\,(\mathrm{Re}_d f^{1/2}) - 1.02 \tag{6.53}$$

In other words, by simply computing the mean velocity from the logarithmic-law correlation, we obtain a relation between the friction factor and Reynolds number for turbulent pipe flow. Prandtl derived Eq. (6.53) in 1935 and then adjusted the constants slightly to fit friction data better

$$\frac{1}{f^{1/2}} = 2.0 \log\,(\mathrm{Re}_d f^{1/2}) - 0.8 \tag{6.54}$$

This is the accepted formula for a smooth-walled pipe. Some numerical values may be listed as follows:

Re_d	4000	10^4	10^5	10^6	10^7	10^8
f	0.0399	0.0309	0.0180	0.0116	0.0081	0.0059

Thus f drops by only a factor of 5 over a 10,000-fold increase in Reynolds number. Equation (6.54) is cumbersome to solve if Re_d is known and f is wanted. There are many alternate approximations in the literature from which f can be computed explicitly from Re_d

$$f = \begin{cases} 0.316\,\mathrm{Re}_d^{-1/4} & 4000 < \mathrm{Re}_d < 10^5 \quad \text{H. Blasius (1911)} \\ \left(1.8 \log \dfrac{\mathrm{Re}_d}{6.9}\right)^{-2} & \text{Ref. 9} \end{cases} \tag{6.55}$$

Blasius, a student of Prandtl, presented his formula in the first correlation ever made of pipe friction versus Reynolds number. Although his formula has a limited range, it illustrates what was happening to Hagen's 1839 pressure-drop data. For a horizontal pipe, from Eq. (6.55),

$$h_f = \frac{\Delta p}{\rho g} = f\frac{L}{d}\frac{V^2}{2g} \approx 0.316\left(\frac{\mu}{\rho V d}\right)^{1/4}\frac{L}{d}\frac{V^2}{2g}$$

or

$$\Delta p \approx 0.158\, L\rho^{3/4}\mu^{1/4}d^{-5/4}V^{7/4} \tag{6.56}$$

at low turbulent Reynolds numbers. This explains why Hagen's data for pressure drop begin to increase as the 1.75 power of the velocity, in Fig. 6.4. Note that Δp varies only slightly with viscosity, which is characteristic of turbulent flow. Introducing $Q = \frac{1}{4}\pi d^2 V$ into Eq. (6.56), we obtain the alternate form

$$\Delta p \approx 0.241 L\rho^{3/4}\mu^{1/4}d^{-4.75}Q^{1.75} \tag{6.57}$$

For a given flow rate Q, the turbulent pressure drop decreases with diameter even more sharply than the laminar formula (6.47). Thus the quickest way to reduce required pumping pressure is to increase the pipe size, although, of course, the larger pipe is more expensive. Doubling the pipe size decreases Δp by a factor of about 27 for a given Q. Compare Eq. (6.56) with Example 5.7 and Fig. 5.10.

The maximum velocity in turbulent pipe flow is given by Eq. (6.48), evaluated at $r = 0$

$$\frac{u_{\max}}{u^*} \approx \frac{1}{\kappa} \ln \frac{Ru^*}{\nu} + B \tag{6.58}$$

Combining this with Eq. (6.49), we obtain the formula relating mean velocity to maximum velocity

$$\frac{V}{u_{\max}} \approx (1 + 1.33\sqrt{f})^{-1} \tag{6.59}$$

Some numerical values are

Re_d	4000	10^4	10^5	10^6	10^7	10^8
$V/u_{\max}$	0.790	0.811	0.849	0.875	0.893	0.907

The ratio varies with the Reynolds number and is much larger than the value of 0.5 predicted for all laminar pipe flow in Eq. (6.43). Thus a turbulent velocity profile, as shown in Fig. 6.11, is very flat in the center and drops off sharply to zero at the wall.

Effect of Rough Walls

It was not known until experiments in 1800 by Coulomb [6] that surface roughness has an effect on friction resistance. It turns out that the effect is negligible for laminar pipe flow, and all the laminar formulas derived in this section are valid for rough walls also. But turbulent flow is strongly affected by roughness. In Fig. 6.9 the linear viscous sublayer only extends out to $y^+ = yu^*/\nu = 5$. Thus, compared with the diameter, the sublayer thickness y_s is only

$$\frac{y_s}{d} = \frac{5\nu/u^*}{d} = \frac{14.1}{\mathrm{Re}_d f^{1/2}} \tag{6.60}$$

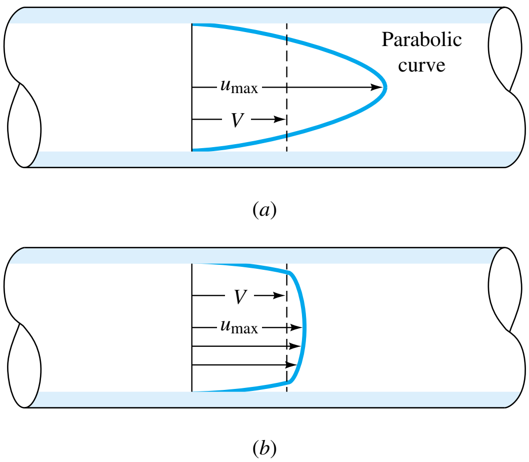
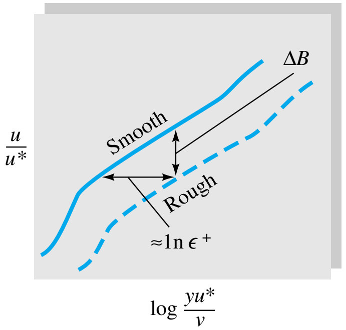
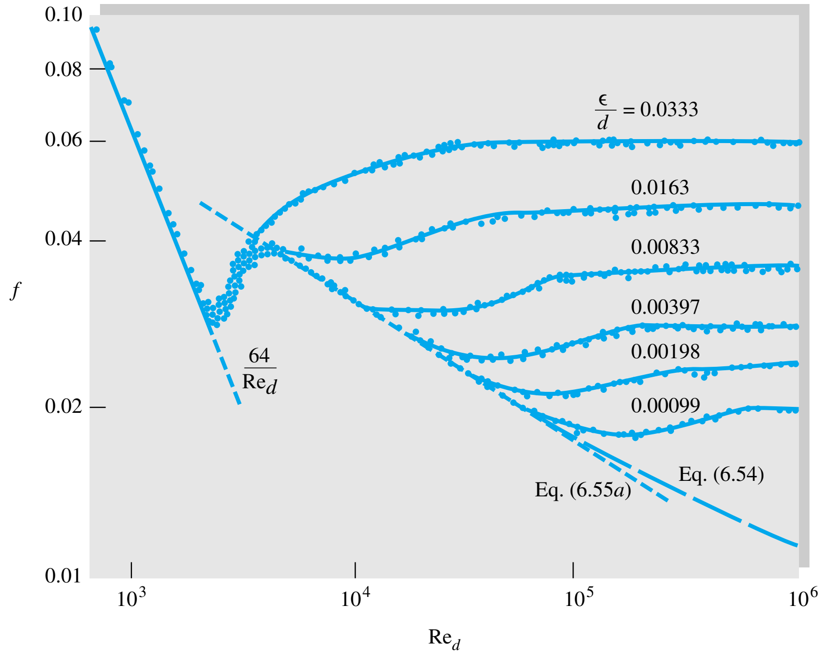


Fig. 6.11 Comparison of laminar and turbulent pipe-flow velocity profiles for the same volume flow: (*a*) laminar flow; (*b*) turbulent flow.



(*a*)



(*b*)

Fig. 6.12 Effect of wall roughness on turbulent pipe flow. (*a*) The logarithmic overlap-velocity profile shifts down and to the right; (*b*) experiments with sand-grain roughness by Nikuradse [7] show a systematic increase of the turbulent friction factor with the roughness ratio.

For example, at $\text{Re}_d = 10^5$, $f = 0.0180$, and $y_s/d = 0.001$, a wall roughness of about $0.001d$ will break up the sublayer and profoundly change the wall law in Fig. 6.9.

Measurements of $u(y)$ in turbulent rough-wall flow by Prandtl's student Nikuradse [7] show, as in Fig. 6.12*a*, that a roughness height ϵ will force the logarithm-law profile outward on the abscissa by an amount approximately equal to $\ln \epsilon^+$, where $\epsilon^+ = \epsilon u^*/\nu$. The slope of the logarithm law remains the same, $1/\kappa$, but the shift outward causes the constant B to be less by an amount $\Delta B \approx (1/\kappa) \ln \epsilon^+$.

Nikuradse [7] simulated roughness by gluing uniform sand grains onto the inner walls of the pipes. He then measured the pressure drops and flow rates and correlated friction factor versus Reynolds number in Fig. 6.12*b*. We see that laminar friction is unaffected, but turbulent friction, after an *onset* point, increases monotonically with the roughness ratio ϵ/d. For any given ϵ/d, the friction factor becomes constant *(fully rough)* at high Reynolds numbers. These points of change are certain values of $\epsilon^+ = \epsilon u^*/\nu$:

$\frac{\epsilon u^*}{\nu} < 5$: *hydraulically smooth* walls, no effect of roughness on friction

$5 \le \frac{\epsilon u^*}{\nu} \le 70$: *transitional* roughness, moderate Reynolds-number effect

$\frac{\epsilon u^*}{\nu} > 70$: *fully rough* flow, sublayer totally broken up and friction independent of Reynolds number

For fully rough flow, $\epsilon^+ > 70$, the log-law downshift ΔB in Fig. 6.12*a* is

$$\Delta B \approx \frac{1}{\kappa} \ln \epsilon^+ - 3.5 \tag{6.61}$$

and the logarithm law modified for roughness becomes

$$u^+ = \frac{1}{\kappa} \ln y^+ + B - \Delta B = \frac{1}{\kappa} \ln \frac{y}{\epsilon} + 8.5 \tag{6.62}$$

The viscosity vanishes, and hence fully rough flow is independent of the Reynolds number. If we integrate Eq. (6.62) to obtain the average velocity in the pipe, we obtain

$$\frac{V}{u^*} = 2.44 \ln \frac{d}{\epsilon} + 3.2$$

or

$$\frac{1}{f^{1/2}} = -2.0 \log \frac{\epsilon/d}{3.7} \qquad \text{fully rough flow} \tag{6.63}$$

There is no Reynolds-number effect; hence the head loss varies exactly as the square of the velocity in this case. Some numerical values of friction factor may be listed:

ϵ/d	0.00001	0.0001	0.001	0.01	0.05
f	0.00806	0.0120	0.0196	0.0379	0.0716

The friction factor increases by 9 times as the roughness increases by a factor of 5000. In the transitional-roughness region, sand grains behave somewhat differently from commercially rough pipes, so Fig. 6.12*b* has now been replaced by the Moody chart.

The Moody Chart

In 1939 to cover the transitionally rough range, Colebrook [9] combined the smooth-wall [Eq. (6.54)] and fully rough [Eq. (6.63)] relations into a clever interpolation formula

$$\frac{1}{f^{1/2}} = -2.0 \log \left(\frac{\epsilon/d}{3.7} + \frac{2.51}{\mathrm{Re}_d f^{1/2}} \right) \tag{6.64}$$

This is the accepted design formula for turbulent friction. It was plotted in 1944 by Moody [8] into what is now called the *Moody chart* for pipe friction (Fig. 6.13). The Moody chart is probably the most famous and useful figure in fluid mechanics. It is accurate to $\pm$ 15 percent for design calculations over the full range shown in Fig. 6.13. It can be used for circular and noncircular (Sec. 6.6) pipe flows and for open-channel flows (Chap. 10). The data can even be adapted as an approximation to boundary-layer flows (Chap. 7).

Equation (6.64) is cumbersome to evaluate for f if Re_d is known, although it easily yields to the EES Equation Solver. An alternate explicit formula given by Haaland [33] as

$$\frac{1}{f^{1/2}} \approx -1.8 \log \left[\frac{6.9}{\mathrm{Re}_d} + \left(\frac{\epsilon/d}{3.7} \right)^{1.11} \right] \tag{6.64a}$$

varies less than 2 percent from Eq. (6.64).

The shaded area in the Moody chart indicates the range where transition from laminar to turbulent flow occurs. There are no reliable friction factors in this range, $2000 < \mathrm{Re}_d < 4000$. Notice that the roughness curves are nearly horizontal in the fully rough regime to the right of the dashed line.

From tests with commercial pipes, recommended values for average pipe roughness are listed in Table 6.1.

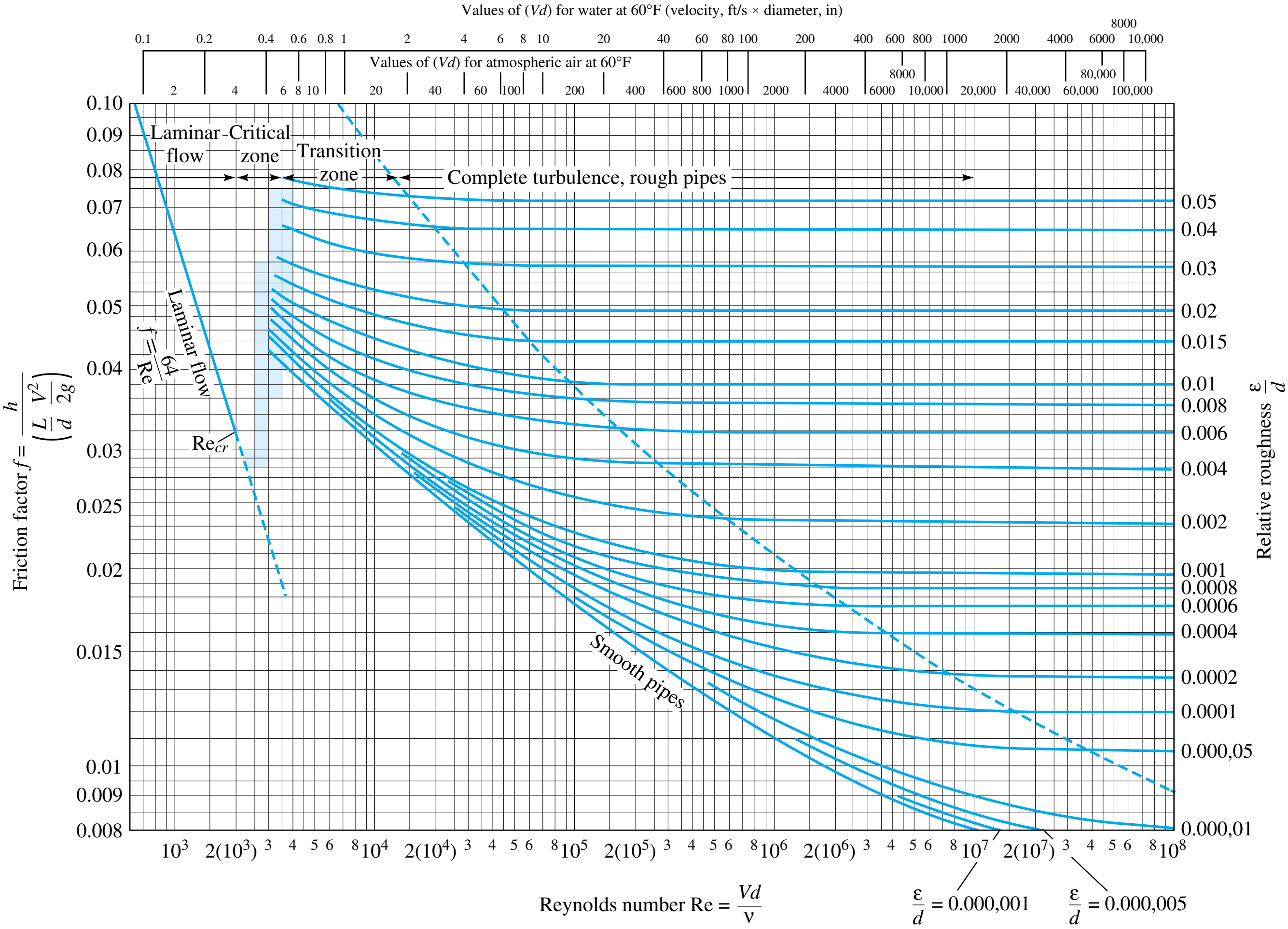


Fig. 6.13 The Moody chart for pipe friction with smooth and rough walls. This chart is identical to Eq. (6.64) for turbulent flow. *(From Ref. 8, by permission of the ASME.)*

Table 6.1 Recommended Roughness Values for Commercial Ducts

Material	Condition	ϵ ft	ϵ mm	Uncertainty, %
Steel	Sheet metal, new	0.00016	0.05	± 60
	Stainless, new	0.000007	0.002	± 50
	Commercial, new	0.00015	0.046	± 30
	Riveted	0.01	3.0	± 70
	Rusted	0.007	2.0	± 50
Iron	Cast, new	0.00085	0.26	± 50
	Wrought, new	0.00015	0.046	± 20
	Galvanized, new	0.0005	0.15	± 40
	Asphalted cast	0.0004	0.12	± 50
Brass	Drawn, new	0.000007	0.002	± 50
Plastic	Drawn tubing	0.000005	0.0015	± 60
Glass	—	Smooth	Smooth	
Concrete	Smoothed	0.00013	0.04	± 60
	Rough	0.007	2.0	± 50
Rubber	Smoothed	0.000033	0.01	± 60
Wood	Stave	0.0016	0.5	± 40

EXAMPLE 6.6[3]

Compute the loss of head and pressure drop in 200 ft of horizontal 6-in-diameter asphalted cast-iron pipe carrying water with a mean velocity of 6 ft/s.

Solution

One can estimate the Reynolds number of water and air from the Moody chart. Look across the top of the chart to V (ft/s) $\times$ d (in) = 36, and then look directly down to the bottom abscissa to find that $\text{Re}_d(\text{water}) \approx 2.7 \times 10^5$. The roughness ratio for asphalted cast iron ($\epsilon = 0.0004$ ft) is

$$\frac{\epsilon}{d} = \frac{0.0004}{\frac{6}{12}} = 0.0008$$

Find the line on the right side for $\epsilon/d = 0.0008$, and follow it to the left until it intersects the vertical line for $\text{Re} = 2.7 \times 10^5$. Read, approximately, $f = 0.02$ [or compute $f = 0.0197$ from Eq. (6.64a)]. Then the head loss is

$$h_f = f \frac{L}{d} \frac{V^2}{2g} = (0.02) \frac{200}{0.5} \frac{(6 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} = 4.5 \text{ ft} \qquad \textit{Ans.}$$

The pressure drop for a horizontal pipe ($z_1 = z_2$) is

$$\Delta p = \rho g h_f = (62.4 \text{ lbf/ft}^3)(4.5 \text{ ft}) = 280 \text{ lbf/ft}^2 \qquad \textit{Ans.}$$

Moody points out that this computation, even for clean new pipe, can be considered accurate only to about $\pm$ 10 percent.

EXAMPLE 6.7

Oil, with $\rho = 900$ kg/m^3 and $\nu = 0.00001$ m^2/s, flows at 0.2 m^3/s through 500 m of 200-mm-diameter cast-iron pipe. Determine (a) the head loss and (b) the pressure drop if the pipe slopes down at 10° in the flow direction.

Solution

First compute the velocity from the known flow rate

$$V = \frac{Q}{\pi R^2} = \frac{0.2 \text{ m}^3/\text{s}}{\pi (0.1 \text{ m})^2} = 6.4 \text{ m/s}$$

Then the Reynolds number is

$$\text{Re}_d = \frac{Vd}{\nu} = \frac{(6.4 \text{ m/s})(0.2 \text{ m})}{0.00001 \text{ m}^2/\text{s}} = 128{,}000$$

From Table 6.1, $\epsilon = 0.26$ mm for cast-iron pipe. Then

$$\frac{\epsilon}{d} = \frac{0.26 \text{ mm}}{200 \text{ mm}} = 0.0013$$

[3]This example was given by Moody in his 1944 paper [8].

Enter the Moody chart on the right at $\epsilon/d = 0.0013$ (you will have to interpolate), and move to the left to intersect with Re = 128,000. Read $f \approx 0.0225$ [from Eq. (6.64) for these values we could compute $f = 0.0227$]. Then the head loss is

$$h_f = f\frac{L}{d}\frac{V^2}{2g} = (0.0225)\frac{500\text{ m}}{0.2\text{ m}}\frac{(6.4\text{ m/s})^2}{2(9.81\text{ m/s}^2)} = 117\text{ m} \qquad \textit{Ans. (a)}$$

From Eq. (6.25) for the inclined pipe,

$$h_f = \frac{\Delta p}{\rho g} + z_1 - z_2 = \frac{\Delta p}{\rho g} + L\sin 10°$$

or $$\Delta p = \rho g[h_f - (500\text{ m})\sin 10°] = \rho g(117\text{ m} - 87\text{ m})$$

$$= (900\text{ kg/m}^3)(9.81\text{ m/s}^2)(30\text{ m}) = 265{,}000\text{ kg/(m}\cdot\text{s}^2) = 265{,}000\text{ Pa} \qquad \textit{Ans. (b)}$$

EXAMPLE 6.8

Repeat Example 6.5 to see whether there is any possible turbulent-flow solution for a smooth-walled pipe.

Solution

In Example 6.5 we estimated a head loss $h_f \approx 1.66$ ft, assuming laminar exit flow ($\alpha \approx 2.0$). For this condition the friction factor is

$$f = h_f\frac{d}{L}\frac{2g}{V^2} = (1.66\text{ ft})\frac{(0.004\text{ ft})(2)(32.2\text{ ft/s}^2)}{(1.0\text{ ft})(3.32\text{ ft/s})^2} \approx 0.0388$$

For laminar flow, $\text{Re}_d = 64/f = 64/0.0388 \approx 1650$, as we showed in Example 6.5. However, from the Moody chart (Fig. 6.13), we see that $f = 0.0388$ also corresponds to a *turbulent* smooth-wall condition, at $\text{Re}_d \approx 4500$. If the flow actually were turbulent, we should change our kinetic-energy factor to $\alpha \approx 1.06$ [Eq. (3.73)], whence the corrected $h_f \approx 1.82$ ft and $f \approx 0.0425$. With f known, we can estimate the Reynolds number from our formulas:

$$\text{Re}_d \approx 3250 \quad \text{[Eq. (6.54)]} \qquad \text{or} \qquad \text{Re}_d \approx 3400 \quad \text{[Eq. (6.55}b\text{)]}$$

So the flow *might* have been turbulent, in which case the viscosity of the fluid would have been

$$\mu = \frac{\rho V d}{\text{Re}_d} = \frac{1.80(3.32)(0.004)}{3300} = 7.2 \times 10^{-6}\text{ slug/(ft}\cdot\text{s)} \qquad \textit{Ans.}$$

This is about 55 percent less than our laminar estimate in Example 6.5. The moral is to keep the capillary-flow Reynolds number below about 1000 to avoid such duplicate solutions.

6.5 Three Types of Pipe-Flow Problems

The Moody chart (Fig. 6.13) can be used to solve almost any problem involving friction losses in long pipe flows. However, many such problems involve considerable iteration and repeated calculations using the chart because the standard Moody chart is essentially a *head-loss chart*. One is supposed to know all other variables, compute

Re_d, enter the chart, find f, and hence compute h_f. This is one of three fundamental problems which are commonly encountered in pipe-flow calculations:

1. Given d, L, and V or Q, ρ, μ, and g, compute the head loss h_f (head-loss problem).
2. Given d, L, h_f, ρ, μ, and g, compute the velocity V or flow rate Q (flow-rate problem).
3. Given Q, L, h_f, ρ, μ, and g, compute the diameter d of the pipe (sizing problem).

Only problem 1 is well suited to the Moody chart. We have to iterate to compute velocity or diameter because both d and V are contained in the ordinate *and* the abscissa of the chart.

There are two alternatives to iteration for problems of type 2 and 3: (a) preparation of a suitable new Moody-type chart (see Prob. 6.62 and 6.73); or (b) the use of *solver* software, especially the Engineering Equation Solver, known as EES [47], which gives the answer directly if the proper data are entered. Examples 6.9 and 6.11 include the EES approach to these problems.

Type 2 Problem: Find the Flow Rate

Even though velocity (or flow rate) appears in both the ordinate and the abscissa on the Moody chart, iteration for turbulent flow is nevertheless quite fast, because f varies so slowly with Re_d. Alternately, in the spirit of Example 5.7, we could change the scaling variables to (ρ, μ, d) and thus arrive at dimensionless head loss versus dimensionless *velocity*. The result is[4]

$$\zeta = \mathrm{fcn}(\mathrm{Re}_d) \qquad \text{where} \qquad \zeta = \frac{gd^3 h_f}{L\nu^2} = \frac{f\,\mathrm{Re}_d^2}{2} \tag{6.65}$$

Example 5.7 did this and offered the simple correlation $\zeta \approx 0.155\ \mathrm{Re}_d^{1.75}$, which is valid for turbulent flow with smooth walls and $\mathrm{Re}_d \le 1$ E5.

A formula valid for all turbulent pipe flows is found by simply rewriting the Colebrook interpolation, Eq. (6.64), in the form of Eq. (6.65):

$$\mathrm{Re}_d = -(8\zeta)^{1/2} \log\left(\frac{\epsilon/d}{3.7} + \frac{1.775}{\sqrt{\zeta}}\right) \qquad \zeta = \frac{gd^3 h_f}{L\nu^2} \tag{6.66}$$

Given ζ, we compute Re_d (and hence velocity) directly. Let us illustrate these two approaches with the following example.

EXAMPLE 6.9

Oil, with $\rho = 950\ \mathrm{kg/m^3}$ and $\nu = 2$ E-5 $\mathrm{m^2/s}$, flows through a 30-cm-diameter pipe 100 m long with a head loss of 8 m. The roughness ratio is $\epsilon/d = 0.0002$. Find the average velocity and flow rate.

Direct Solution

First calculate the dimensionless head-loss parameter:

$$\zeta = \frac{gd^3 h_f}{L\nu^2} = \frac{(9.81\ \mathrm{m/s^2})(0.3\ \mathrm{m})^3(8.0\ \mathrm{m})}{(100\ \mathrm{m})(2\ \mathrm{E\text{-}5\ m^2/s})^2} = 5.30\ \mathrm{E7}$$

[4]The parameter ζ was suggested by H. Rouse in 1942.

Now enter Eq. (6.66) to find the Reynolds number:

$$\mathrm{Re}_d = -[8(5.3\text{ E7})]^{1/2} \log\left(\frac{0.0002}{3.7} + \frac{1.775}{\sqrt{5.3\text{ E7}}}\right) = 72{,}600$$

The velocity and flow rate follow from the Reynolds number:

$$V = \frac{\nu\,\mathrm{Re}_d}{d} = \frac{(2\text{ E-5 m}^2/\text{s})(72{,}600)}{0.3\text{ m}} \approx 4.84\text{ m/s}$$

$$Q = V\frac{\pi}{4}d^2 = \left(4.84\frac{\text{m}}{\text{s}}\right)\frac{\pi}{4}(0.3\text{ m})^2 \approx 0.342\text{ m}^3/\text{s} \qquad \textit{Ans.}$$

No iteration is required, but this idea falters if additional losses are present.

Iterative Solution

By definition, the friction factor is known except for V:

$$f = h_f\frac{d}{L}\frac{2g}{V^2} = (8\text{ m})\left(\frac{0.3\text{ m}}{100\text{ m}}\right)\left[\frac{2(9.81\text{ m/s}^2)}{V^2}\right] \quad \text{or} \quad fV^2 \approx 0.471 \quad \text{(SI units)}$$

To get started, we only need to guess f, compute $V = \sqrt{0.471/f}$, then get Re_d, compute a better f from the Moody chart, and repeat. The process converges fairly rapidly. A good first guess is the "fully rough" value for $\epsilon/d = 0.0002$, or $f \approx 0.014$ from Fig. 6.13. The iteration would be as follows:

Guess $f \approx 0.014$, then $V = \sqrt{0.471/0.014} = 5.80$ m/s and $\mathrm{Re}_d = Vd/\nu \approx 87{,}000$. At $\mathrm{Re}_d = 87{,}000$ and $\epsilon/d = 0.0002$, compute $f_{\text{new}} \approx 0.0195$ [Eq. (6.64)].

New $f \approx 0.0195$, $V = \sqrt{0.481/0.0195} = 4.91$ m/s and $\mathrm{Re}_d = Vd/\nu = 73{,}700$. At $\mathrm{Re}_d = 73{,}700$ and $\epsilon/d = 0.0002$, compute $f_{\text{new}} \approx 0.0201$ [Eq. (6.64)].

Better $f \approx 0.0201$, $V = \sqrt{0.471/0.0201} = 4.84$ m/s and $\mathrm{Re}_d \approx 72{,}600$. At $\mathrm{Re}_d = 72{,}600$ and $\epsilon/d = 0.0002$, compute $f_{\text{new}} \approx 0.0201$ [Eq. (6.64)].

We have converged to three significant figures. Thus our iterative solution is

$$V = 4.84\text{ m/s}$$

$$Q = V\left(\frac{\pi}{4}\right)d^2 = (4.84)\left(\frac{\pi}{4}\right)(0.3)^2 \approx 0.342\text{ m}^3/\text{s} \qquad \textit{Ans.}$$

The iterative approach is straightforward and not too onerous, so it is routinely used by engineers. Obviously this repetitive procedure is ideal for a personal computer.

Engineering Equation Solver (EES) Solution

EES

In EES, one simply enters the data and the appropriate equations, letting the software do the rest. Correct units must of course be used. For the present example, the data could be entered as SI:

```
rho=950    nu=2E-5    d=0.3    L=100    epsod=0.0002    hf=8.0    g=9.81
```

The appropriate equations are the Moody formula (6.64) plus the definitions of Reynolds number, volume flow rate as determined from velocity, and the Darcy head-loss formula (6.30):

```
Re = V*d/nu
 Q = V*pi*d^2/4
 f = (-2.0*log10(epsod/3.7 + 2.51/Re/f^0.5))^(-2)
hf = f*L/d*V^2/2/g
```

EES understands that "pi" represents 3.141593. Then hit "SOLVE" from the menu. If errors have been entered, EES will complain that the system cannot be solved and attempt to explain why. Otherwise, the software will iterate, and in this case EES prints the correct solution:

```
Q=0.342    V=4.84    f=0.0201    Re=72585
```

The units are spelled out in a separate list as [m, kg, s, N]. This elegant approach to engineering problem-solving has one drawback, namely, that the user fails to check the solution for engineering viability. For example, are the data typed correctly? Is the Reynolds number turbulent?

EXAMPLE 6.10

Work Moody's problem (Example 6.6) backward, assuming that the head loss of 4.5 ft is known and the velocity (6.0 ft/s) is unknown.

Direct Solution

Find the parameter ζ, and compute the Reynolds number from Eq. (6.66):

$$\zeta = \frac{gd^3h_f}{L\nu^2} = \frac{(32.2\text{ ft/s}^2)(0.5\text{ ft})^3(4.5\text{ ft})}{(200\text{ ft})(1.1\text{ E-5 ft}^2/\text{s})^2} = 7.48\text{ E8}$$

Eq. (6.66): $$\text{Re}_d = -[8(7.48\text{ E8})]^{1/2}\log\left(\frac{0.0008}{3.7} + \frac{1.775}{\sqrt{7.48\text{ E8}}}\right) \approx 274{,}800$$

Then $$V = \nu\frac{\text{Re}_d}{d} = \frac{(1.1\text{ E-5})(274{,}800)}{0.5} \approx 6.05\text{ ft/s}$$ *Ans.*

We did not get 6.0 ft/s exactly because the 4.5-ft head loss was rounded off in Example 6.6.

Iterative Solution

As in Eq. (6.9) the friction factor is related to velocity:

$$f = h_f\frac{d}{L}\frac{2g}{V^2} = (4.5\text{ ft})\left(\frac{0.5\text{ ft}}{200\text{ ft}}\right)\left[\frac{2(32.2\text{ ft/s}^2)}{V^2}\right] \approx \frac{0.7245}{V^2}$$

or $$V = \sqrt{0.7245/f}$$

Knowing $\epsilon/d = 0.0008$, we can guess f and iterate until the velocity converges. Begin with the fully rough estimate $f \approx 0.019$ from Fig. 6.13. The resulting iterates are

$$f_1 = 0.019: \qquad V_1 = \sqrt{0.7245/f_1} = 6.18\text{ ft/s} \qquad \text{Re}_{d_1} = \frac{Vd}{\nu} = 280{,}700$$

$f_2 = 0.0198$: $\quad V_2 = 6.05$ ft/s $\quad \mathrm{Re}_{d_2} = 274{,}900$

$f_3 = 0.01982$: $\quad V_3 = 6.046$ ft/s *Ans.*

The calculation converges rather quickly to the same result as that obtained through direct computation.

Type 3 Problem: Find the Pipe Diameter

The Moody chart is especially awkward for finding the pipe size, since d occurs in all three parameters f, Re_d, and ϵ/d. Further, it depends upon whether we know the velocity or the flow rate. We cannot know both, or else we could immediately compute $d = \sqrt{4Q/(\pi V)}$.

Let us assume that we know the flow rate Q. Note that this requires us to redefine the Reynolds number in terms of Q:

$$\mathrm{Re}_d = \frac{Vd}{\nu} = \frac{4Q}{\pi d \nu} \tag{6.67}$$

Then, if we choose (Q, ρ, μ) as scaling parameters (to eliminate d), we obtain the functional relationship

$$\mathrm{Re}_d = \frac{4Q}{\pi d \nu} = \mathrm{fcn}\left(\frac{g h_f}{L \nu^5}, \frac{\epsilon \nu}{Q}\right) \tag{6.68}$$

and can thus solve d when the right-hand side is known. Unfortunately, the writer knows of no *formula* for this relation, nor is he able to rearrange Eq. (6.64) into the explicit form of Eq. (6.68). One could recalculate and *plot* the relation, and indeed an ingenious "pipe-sizing" plot is given in Ref. 13. Here it seems reasonable to forgo a plot or curve fitted formula and to simply set up the problem as an iteration in terms of the Moody-chart variables. In this case we also have to set up the friction factor in terms of the flow rate:

$$f = h_f \frac{d}{L} \frac{2g}{V^2} = \frac{\pi^2}{8} \frac{g h_f d^5}{L Q^2} \tag{6.69}$$

The following two examples illustrate the iteration.

EXAMPLE 6.11

Work Example 6.9 backward, assuming that $Q = 0.342$ m^3/s and $\epsilon = 0.06$ mm are known but that d (30 cm) is unknown. Recall $L = 100$ m, $\rho = 950$ kg/m^3, $\nu = 2$ E-5 m^2/s, and $h_f = 8$ m.

Iterative Solution

First write the diameter in terms of the friction factor:

$$f = \frac{\pi^2}{8} \frac{(9.81 \text{ m/s}^2)(8 \text{ m})d^5}{(100 \text{ m})(0.342 \text{ m}^3/\text{s})^2} = 8.28 d^5 \qquad \text{or} \qquad d \approx 0.655 f^{1/5} \tag{1}$$

in SI units. Also write the Reynolds number and roughness ratio in terms of the diameter:

$$\mathrm{Re}_d = \frac{4(0.342\ \mathrm{m^3/s})}{\pi(2\ \mathrm{E\text{-}5}\ \mathrm{m^2/s})d} = \frac{21{,}800}{d} \tag{2}$$

$$\frac{\epsilon}{d} = \frac{6\ \mathrm{E\text{-}5\ m}}{d} \tag{3}$$

Guess f, compute d from (1), then compute Re_d from (2) and ϵ/d from (3), and compute a better f from the Moody chart or Eq. (6.64). Repeat until (fairly rapid) convergence. Having no initial estimate for f, the writer guesses $f \approx 0.03$ (about in the middle of the turbulent portion of the Moody chart). The following calculations result:

$$f \approx 0.03 \qquad d \approx 0.655(0.03)^{1/5} \approx 0.325\ \mathrm{m}$$

$$\mathrm{Re}_d \approx \frac{21{,}800}{0.325} \approx 67{,}000 \qquad \frac{\epsilon}{d} \approx 1.85\ \mathrm{E\text{-}4}$$

Eq. (6.54): $$f_{\mathrm{new}} \approx 0.0203 \qquad \text{then} \qquad d_{\mathrm{new}} \approx 0.301\ \mathrm{m}$$

$$\mathrm{Re}_{d,\mathrm{new}} \approx 72{,}500 \qquad \frac{\epsilon}{d} \approx 2.0\ \mathrm{E\text{-}4}$$

Eq. (6.54): $$f_{\mathrm{better}} \approx 0.0201 \qquad \text{and} \qquad d = 0.300\ \mathrm{m}$$ *Ans.*

The procedure has converged to the correct diameter of 30 cm given in Example 6.9.

EES Solution

For an EES solution, enter the data and the appropriate equations. The diameter is unknown. Correct units must of course be used. For the present example, the data should use SI units:

```
rho=950    nu=2E-5    L=100    eps=6E-5    hf=8.0    g=9.81    Q=0.342
```

The appropriate equations are the Moody formula, the definition of Reynolds number, volume flow rate as determined from velocity, the Darcy head-loss formula, and the roughness ratio:

```
   Re = V*d/nu
    Q = V*pi*d^2/4
    f = (-2.0*log10(epsod/3.7 + 2.51/Re/f^0.5))^(-2)
   hf = f*L/d*V^2/2/g
epsod = eps/d
```

Hit *Solve* from the menu. Unlike Example 6.9, this time EES complains that the system *cannot* be solved and reports "logarithm of a negative number." The reason is that we allowed EES to assume that f could be a negative number. Bring down *Variable Information* from the menu and change the limits of f so that it cannot be negative. EES agrees and iterates to the solution:

```
d = 0.300    V = 4.84    f = 0.0201    Re = 72,585
```

The unit system is spelled out as (m, kg, s, N). As always when using software, the user should check the solution for engineering viability. For example, is the Reynolds number turbulent? (Yes)

EXAMPLE 6.12

Work Moody's problem, Example 6.6, backward to find the unknown (6 in) diameter if the flow rate $Q = 1.18$ ft^3/s is known. Recall $L = 200$ ft, $\epsilon = 0.0004$ ft, and $\nu = 1.1$ E-5 ft^2/s.

Solution

Write f, Re_d, and ϵ/d in terms of the diameter:

$$f = \frac{\pi^2}{8}\frac{gh_f d^5}{LQ^2} = \frac{\pi^2}{8}\frac{(32.2 \text{ ft/s}^2)(4.5 \text{ ft})d^5}{(200 \text{ ft})(1.18 \text{ ft}^3\text{/s})^2} = 0.642d^5 \qquad \text{or} \qquad d \approx 1.093 f^{1/5} \tag{1}$$

$$\mathrm{Re}_d = \frac{4(1.18 \text{ ft}^3\text{/s})}{\pi(1.1 \text{ E-5 ft}^2\text{/s})\, d} = \frac{136{,}600}{d} \tag{2}$$

$$\frac{\epsilon}{d} = \frac{0.0004 \text{ ft}}{d} \tag{3}$$

with everything in BG units, of course. Guess f; compute d from (1), Re_d from (2), and ϵ/d from (3); and then compute a better f from the Moody chart. Repeat until convergence. The writer traditionally guesses an initial $f \approx 0.03$:

$$f \approx 0.03 \qquad d \approx 1.093(0.03)^{1/5} \approx 0.542 \text{ ft}$$

$$\mathrm{Re}_d = \frac{136{,}600}{0.542} \approx 252{,}000 \qquad \frac{\epsilon}{d} \approx 7.38 \text{ E-4}$$

$$f_{\text{new}} \approx 0.0196 \qquad d_{\text{new}} \approx 0.498 \text{ ft} \qquad \mathrm{Re}_d \approx 274{,}000 \qquad \frac{\epsilon}{d} \approx 8.03 \text{ E-4}$$

$$f_{\text{better}} \approx 0.0198 \qquad d \approx 0.499 \text{ ft}$$ *Ans.*

Convergence is rapid, and the predicted diameter is correct, about 6 in. The slight discrepancy (0.499 rather than 0.500 ft) arises because h_f was rounded to 4.5 ft.

Table 6.2 Nominal and Actual Sizes of Schedule 40 Wrought-Steel Pipe*

Nominal size, in	Actual ID, in
$\frac{1}{8}$	0.269
$\frac{1}{4}$	0.364
$\frac{3}{8}$	0.493
$\frac{1}{2}$	0.622
$\frac{3}{4}$	0.824
1	1.049
$1\frac{1}{2}$	1.610
2	2.067
$2\frac{1}{2}$	2.469
3	3.068

*Nominal size within 1 percent for 4 in or larger.

In discussing pipe-sizing problems, we should remark that commercial pipes are made only in certain sizes. Table 6.2 lists standard water-pipe sizes in the United States. If the sizing calculation gives an intermediate diameter, the next largest pipe size should be selected.

6.6 Flow in Noncircular Ducts[5]

If the duct is noncircular, the analysis of fully developed flow follows that of the circular pipe but is more complicated algebraically. For laminar flow, one can solve the exact equations of continuity and momentum. For turbulent flow, the logarithm-law velocity profile can be used, or (better and simpler) the hydraulic diameter is an excellent approximation.

The Hydraulic Diameter

For a noncircular duct, the control-volume concept of Fig. 6.10 is still valid, but the cross-sectional area A does not equal πR^2 and the cross-sectional perimeter wetted by the shear stress $\mathcal{P}$ does not equal $2\pi R$. The momentum equation (6.26) thus becomes

$$\Delta p\, A + \rho g A\, \Delta L \sin\phi - \bar{\tau}_w \mathcal{P}\, \Delta L = 0$$

[5]This section may be omitted without loss of continuity.