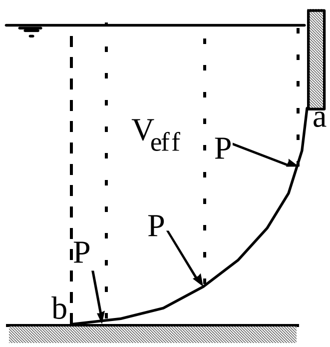
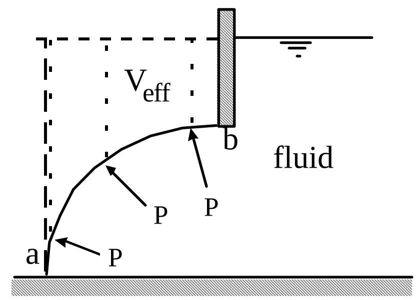
Thus to identify the effective volume - Veff:

1. Identify the curved surface in contact with the fluid.
2. Identify the pressure at each point on the curved surface.
3. Identify the height of fluid required to develop the pressure.
4. These collective heights combine to form V_{eff}.



Fluid above the surface



No fluid actually above surface

These two examples show two typical cases where this concept is used to determine V_{eff}.

The vertical force acts **vertically** through the <u>centroid</u> (center of mass) of the <u>effective</u> column of fluid. The vertical direction will be the direction of the vertical components of the pressure forces.

Therefore, to determine the vertical component of force on a curved surface in a hydrostatic fluid:

1. Identify the effective column of fluid necessary to cause the fluid pressure on the surface.
2. Determine the volume of the effective column of fluid.
3. Calculate the weight of the effective column of fluid - $F_V = \rho g V_{eff}$.
4. The location of F_V is through the centroid of V_{eff}.

Finding the Location of the Centroid

A second problem associated with the topic of curved surfaces is that of finding the location of the centroid of V_{eff}.

Recall:

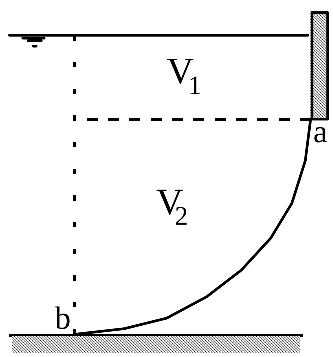
Centroid = the location where the first moment of a point area, volume, or mass equals the first moment of the distributed area, volume, or mass, e.g.

$$x_{cg}V_1 = \int_{V_1} x\,dV$$

This principle can also be used to determine the location of the centroid of complex geometries.

For example:

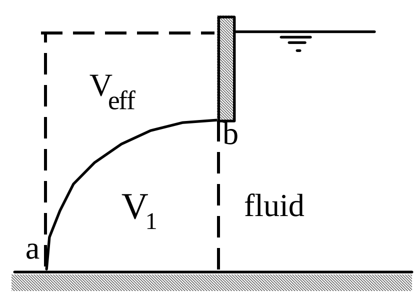
If $V_{eff} = V_1 + V_2$



then

$x_{cg}V_{eff} = x_1V_1 + x_2V_2$

or

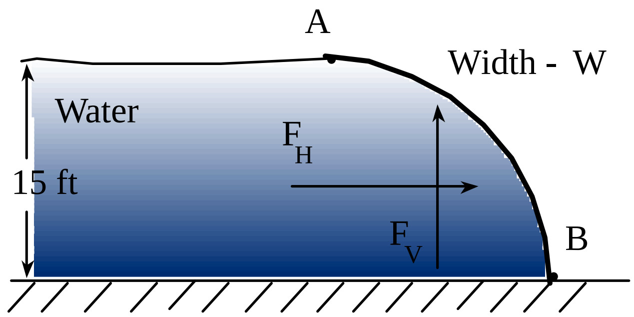


$V_T = V_1 + V_{eff}$

$x_TV_T = x_1V_1 + x_{cg}V_{eff}$

Note: In the figures shown above, each of the x values would be specified relative to a vertical axis through b since the cg of the quarter circle is most easily specified relative to this axis.

Example:

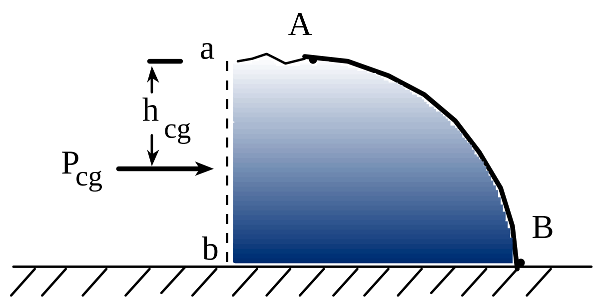


Gate AB holds back 15 ft of water. Neglecting the weight of the gate, determine the magnitude (per unit width) and location of the hydrostatic forces on the gate and the resisting moment about B.

a. Horizontal component $\gamma = \rho g = 62.4\ \text{lbf/ft}^3$

Rule: Project the curved surface into the vertical plane. Locate the centroid of the projected area. Find the pressure at the centroid of the vertical projection. $F = P_{cg}\ A_p$

Note: All calculations are done with the projected area. The curved surface is not used at all in the analysis.



The curved surface projects onto plane a - b and results in a **rectangle,** (not a quarter circle) 15 ft x W. For this rectangle:

$h_{cg} = 7.5$, $P_{cg} = \gamma h_{cg} = 62.4\ \text{lbf/ft}^3 * 7.5\ \text{ft} = 468\ \text{lbf/ft}^2$

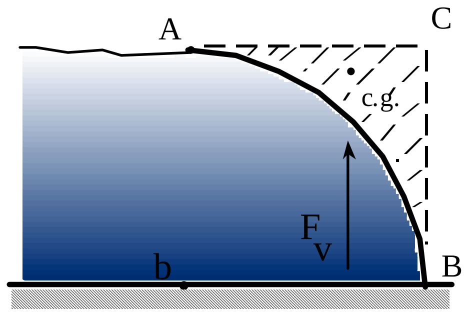
$F_h = P_{cg}\ A = 468\ \text{lbf/ft}^2 * 15\ \text{ft}*W = \underline{7020\ W\ \text{lbf}}$ $\longrightarrow$

Location: $I_{xx} = bh^3/12 = W * 15^3/12 = 281.25\ W\ \text{ft}^4$

$$y_{cp} = -\frac{I_{xx}\sin\theta}{h_{cg}A} = -\frac{281.25W\ ft^4 \sin 90^o}{7.5\ ft 15W\ ft^2} = -2.5\ ft$$

The location is 2.5 ft below the c.g. or 10 ft below the surface, 5 ft above the bottom.

b. Vertical force:



Rule: F_v equals the weight of the effective column of fluid above the curved surface.

Q: What is the effective volume of fluid above the surface?

What volume of fluid would result in the actual pressure distribution on the curved surface?

Vol = A - B - C

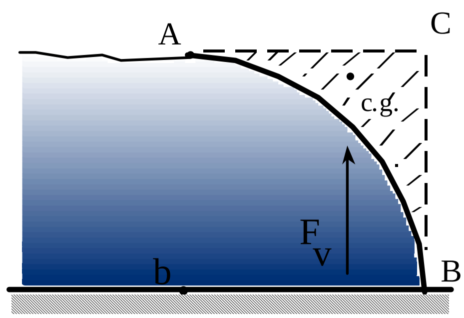
$V_{rec} = V_{qc} + V_{ABC}$, $\qquad V_{ABC} = V_{rec} - V_{qc}$

$V_{ABC} = V_{eff} = 15^2\ W - \pi\ 15^2/4*W = 48.29\ W\ ft^3$

$F_v = \rho g\ V_{eff} = 62.4\ lbf/ft^3 * 48.29\ ft^3 = \underline{3013\ lbf}$ $\uparrow$

Note: F_V is directed upward even though the effective volume is above the surface.

c. What is the location?



Rule: F_V will act through the centroid of the "effective volume causing the force.

We need the centroid of volume A-B-C. How do we obtain this centroid?

Use the concept which is the basis of the centroid, the "first moment of an area."

Since: $A_{rec} = A_{qc} + A_{ABC}$ $M_{rec} = M_{qc} + M_{ABC}$ $M_{ABC} = M_{rec} - M_{qc}$

Note: We are taking moments about the left side of the figure, ie., point b. **WHY?**

(The c.g. of the quarter circle is known to be $4R/3\pi$ w.r.t. b.)

$$x_{cg} A = x_{rec} A_{rec} - x_{qc} A_{qc}$$

$$x_{cg} \{15^2 - \pi*15^2/4\} = 7.5*15^2 - \{4*15/3/\pi\}* \pi*15^2/4$$

x_{cg} = 11.65 ft { distance to rt. of b to centroid }

Q: Do we need a y location? Why?

d. Calculate the moment about B
needed for equilibrium.

$\sum M_B = 0$ clockwise positive.

$$M_B + 5F_h + (15 - x_v)F_v = 0$$

$$M_B + 5\times 7020W + (15 - 11.65)3013W = 0 \quad M_B + 5\times 7020W + (15 - 11.65)3013W = 0$$

$$P_a \neq \rho g y \quad G \neq g$$

$$M_B + 35{,}100W + 10{,}093.6W = 0$$

$$\underline{M_B = -45{,}194W \ ft - lbf}$$ **Why negative?**

The hydrostatic forces will tend to roll the surface clockwise relative to B, thus a resisting moment that is counterclockwise is needed for static equilibrium.

Always review your answer (all aspects: magnitude, direction, units, etc.) to determine if it makes sense relative to physically what you understand about the problem. Begin to think like an engineer.

Buoyancy

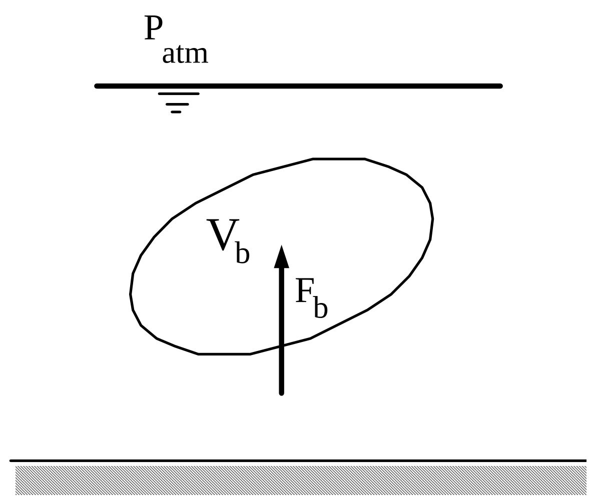
An important extension of the procedure for vertical forces on curved surfaces is that of the concept of buoyancy.

The basic principle was discovered by Archimedes.

It can be easily shown that (see text for detailed development) the buoyant force F_b is given by:

$$F_b = \rho g V_b$$

where V_b is the volume of the fluid displaced by the submerged body and ρg is the specific weight of the fluid displaced.



Thus, the **buoyant force** equals the **weight of the fluid displaced,** which is equal to the product of the specific weight times the volume of fluid displaced.

The location of the buoyant force is:

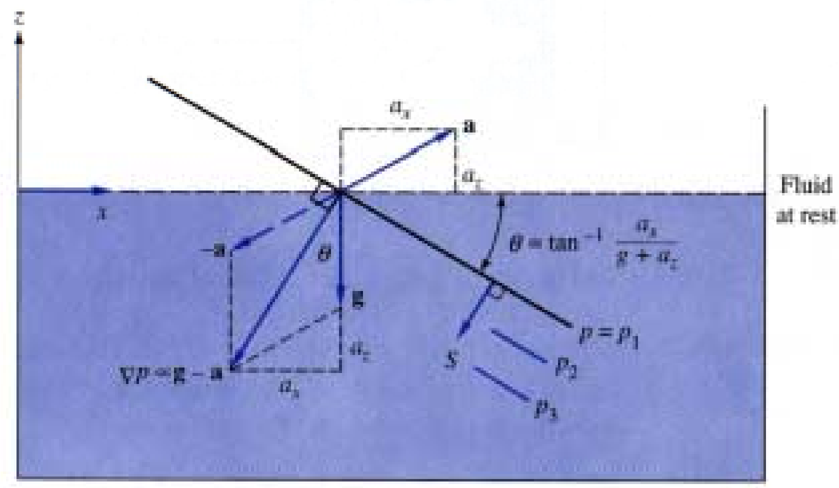
Through a vertical line of action, directed upward, which acts through the centroid of the volume of fluid displaced.

Review all text examples and material on buoyancy.

Pressure distribution in rigid body motion

All of the problems considered to this point were for static fluids. We will now consider an extension of our static fluid analysis to the case of rigid body motion, where the entire fluid mass moves and accelerates uniformly (as a rigid body).

The container of fluid shown below is accelerated uniformly up and to the right as shown.



From a previous analysis, the general equation governing fluid motion is

$$\overline{\nabla} P = \rho(\overline{g} - \overline{a}) + \mu \nabla^2 \, \overline{V}$$

For rigid body motion, there is no velocity gradient in the fluid, therefore

$$\mu \nabla^2 V = 0$$

The simplified equation can now be written as

$$\overline{\nabla} P = \rho(\overline{g} - \overline{a}) = \rho \overline{G}$$

where $\overline{G} = \overline{g} - \overline{a} \equiv$ the net acceleration vector acting on the fluid.

This result is similar to the equation for the variation of pressure in a hydrostatic fluid.

However, in the case of rigid body motion:

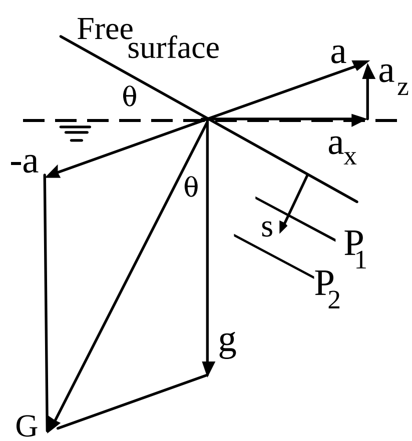
* $\overline{\nabla}P$ = f {fluid density & the **net** acceleration vector- $\overline{G} = \overline{g} - \overline{a}$ }

* $\overline{\nabla}P$ acts in the vector direction of $\overline{G} = \overline{g} - \overline{a}$

* Lines of constant pressure are perpendicular to $\overline{G}$. The new orientation of the free surface will also be perpendicular to $\overline{G}$.

The equations governing the analysis for this class of problems are most easily developed from an acceleration diagram.

Acceleration diagram:



For the indicated geometry:

$$\theta = \tan^{-1}\frac{a_x}{g + a_z} \quad \theta = \tan^{-1}\frac{a_x}{g + a_z}$$

$$\frac{dP}{ds} = \rho G \qquad \text{where } G = \left\{a_x^2 + (g + a_z)^2\right\}^{1/2}$$

and $P_2 - P_1 = \rho G(s_2 - s_1)$

Note: $P_2 - P_1 \neq \rho g(z_2 - z_1)$

and

$s_2 - s_1$ is not a vertical dimension

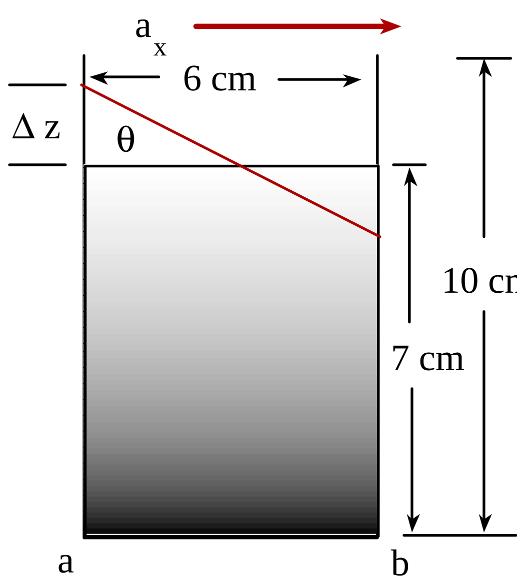
Note: s is the depth to a given point **perpendicular** to the free surface or **its extension.** s is aligned with $\overline{G}$.

In analyzing typical problems with rigid body motion:

1. Draw the acceleration diagram taking care to correctly indicate –a, g, and θ, the inclination angle of the free surface.
2. Using the previously developed equations, solve for G and θ.
3. If required, use geometry to determine $s_2 - s_1$ (the perpendicular distance from the free surface to a given point) and then the pressure at that point relative to the surface using $P_2 - P_1 = \rho\, G\, (s_2 - s_1)$.

Key Point: Do not use ρg to calculate $P_2 - P_1$, use ρ G.

Example 2.12



Given: A coffee mug, 6 cm x 6 cm square, 10 cm deep, contains 7 cm of coffee. Mug is accelerated to the right with $a_x = 7$ m/s^2 . Assuming rigid body motion. $\rho_c = 1010$ kg/m^3,

Determine: a. Will the coffee spill?

b. P_g at "a & b".

c. F_{net} on left wall.

a. First draw schematic showing original orientation and final orientation of the free surface.

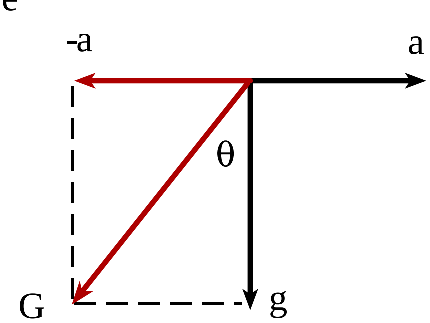
$\rho_c = 1010$ kg/m^3 $a_x = 7$m/s^2 $a_z = 0$ $g = 9.8907$ m/s^2

Have a new free surface angle θ where

$$\theta = \tan^{-1}\frac{a_x}{g + a_z}$$

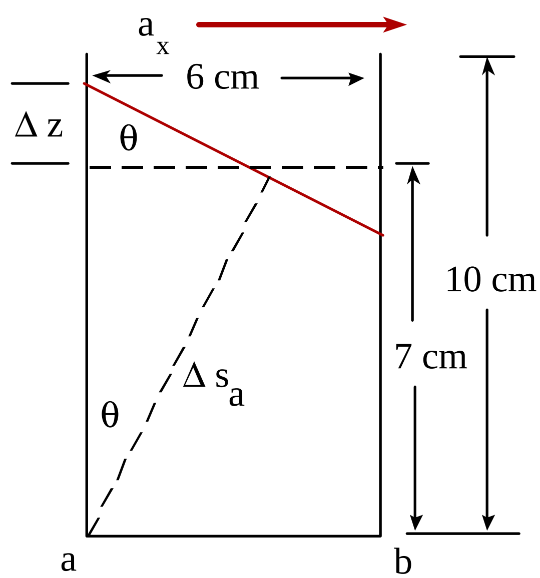
$$\theta = \tan^{-1}\frac{7}{9.807} = 35.5°$$

Δz = 3 tan 35.5 = 2.14 cm



$h_{max} = 7 + 2.14 = 9.14$ cm < 10 cm $\therefore$ **Will not spill**.

b. Pressure at " a & b."



$P_a = \rho\, G\, \Delta s_a$

$G = \{a^2_x + g^2\}^{.5} = \{7^2 + 9.807^2\}^{.5}$

$G = 12.05\ \text{m/s}^2$

$\Delta s_a = \{7 + z\} \cos\theta$

$\Delta s_a = 9.14\ \text{cm} \cos 35.5 = 7.44\ \text{cm}$

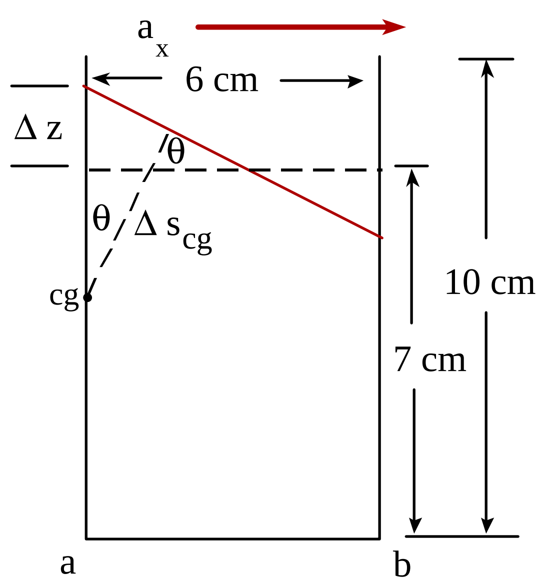
$P_a = 1010\ \text{kg/m}^3 * 12.05\text{m/s}^2 * 0.0744\ \text{m}$

$P_a = 906\ (\text{kg m/s}^2)/\text{m}^2 =$ **906 Pa**

Note: $P_a \neq \rho\, g\, y \quad G \neq g$

Q: **How would you find the pressure at b**, P_b?

c. What is the force on the left wall?



We have a plane surface, what is the rule?

Find cg, P_{cg}, $F = P_{cg.}\, A$

Vertical depth to cg is:

$z_{cg} = 9.14/2 = 4.57$ cm

$\Delta s_{cg} = 4.57 \cos 35.5 = 3.72$ cm

$P_{cg} = \rho\, G\, \Delta s_{cg}$

$P_{cg} = 1010\ \text{kg/m}^3 * 12.05\ \text{m/s}^2 * 0.0372\ \text{m}$

$P_{cg} = 452.7\ \text{N/m}^2$

$F = P_{cg}\, A = 452.7\ \text{N/m}^2 * 0.0914 * 0.06\text{m}^2$

F = 2.48 N $\leftarrow$

What is the direction?

Horizontal, perpendicular to the wall;

i.e., **Pressure <u>always</u> acts normal to a surface.**

Q: How would you find the force on the right wall?

III. Control Volume Relations for Fluid Analysis

From consideration of hydrostatics, we now move to problems involving fluid flow with the addition of effects due to fluid motion, e.g. inertia and convective mass, momentum, and energy terms.

We will present the analysis based on a control volume (not differential element) formulation, e.g. similar to that used in thermodynamics for the first law.

Basic Conservation Laws:

Each of the following basic conservation laws is presented in its most fundamental, **fixed mass** form. We will subsequently develop an equivalent expression for each law that includes the effects of the flow of mass, momentum, and energy (as appropriate) across a control volume boundary. These transformed equations will be the basis for the control volume analyses developed in this chapter.

Conservation of Mass:

Defining m as the mass of a fixed mass system, the mass for a control volume V is given by

$$m_{sys} = \int_{sys} \rho \, dV$$

The basic equation for conservation of mass is then expressed as

$$\left.\frac{dm}{dt}\right)_{sys} = 0$$

The time rate of change of mass for the control volume is zero since at this point we are still working with a fixed mass system.

Linear Momentum:

Defining $\overline{P}_{sys}$ as the linear momentum of a fixed mass, the linear momentum of a fixed mass control volume is given by:

$$\overline{P}_{sys} = m\overline{V} = \int_{sys} \overline{V} \rho dV$$

where $\overline{V}$ is the local fluid velocity and dV is a differential volume element in the control volume.

The basic linear momentum equation is then written as

$$\sum \overline{F} = \left.\frac{d\overline{P}}{dt}\right)_{sys} = \left.\frac{d\left(m\overline{V}\right)}{dt}\right)_{sys}$$

Moment of Momentum:

Defining $\overline{H}$ as the moment of momentum for a fixed mass, the moment of momentum for a fixed mass control volume is given by

$$\overline{H}_{sys} = \int_{sys} \overline{r} \times \overline{V} \rho dV$$

where $\overline{r}$ is the moment arm from an inertial coordinate system to the differential control volume of interest. The basic equation is then written as

$$\sum \overline{M}_{sys} = \sum \overline{r} \times \overline{F} = \left.\frac{d\overline{H}}{dt}\right)_{sys}$$

Energy:

Defining E_{sys} as the total energy of an element of fixed mass, the energy of a fixed mass control volume is given by

$$E_{sys} = \int_{sys} e \rho dV$$

where e is the total energy per unit mass (includes kinetic, potential, and internal energy) of the differential control volume element of interest.

The basic equation is then written as

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{sys} \qquad \text{(Note: written on a rate basis)}$$

It is again noted that each of the conservation relations as previously written applies only to fixed, constant mass systems.

However, since most fluid problems of importance are for open systems, we must transform each of these relations to an equivalent expression for a control volume which includes the effect of mass entering and/or leaving the system.

This is accomplished with the Reynolds transport theorem.

Reynolds Transport Theorem

We define a general, extensive property (an extensive property depends on the size or extent of the system) B_{sys} where

$$B_{sys} = \int_{sys} \beta \rho \, dV$$

B_{sys} could be total mass, total energy, total momentum, etc., of a system.

and B_{sys} per unit mass is defined as β or $\beta = \dfrac{dB}{dm}$

Thus, β is the intensive equivalent of B_{sys} .

Applying a general control volume formulation to the time rate of change of B_{sys} , we obtain the following (see text for detailed development):

$$\left.\frac{dB}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{cv} \beta \rho dV + \int_{A_e} \beta_e\, \rho_e\, V_e\, d A_e - \int_{A_i} \beta_i\, \rho_i\, V_i\, d A_i$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

System rate of change of B — **Rate of change of B in c.v.** — **Rate of B leaving c.v.** — **Rate of B entering c.v.**

$\downarrow$ **transient term** $\qquad\qquad$ $\downarrow$ **convective terms**

where B is any conserved quantity, e.g. **mass, linear momentum, moment of momentum, or energy**.

We will now apply this theorem to each of the basic conservation equations to develop their equivalent open system, control volume forms.

Conservation of mass

For conservation of mass, we have that

$$B = m \qquad \text{and} \qquad \beta = 1$$

From the previous statement of conservation of mass and these definitions, Reynolds transport theorem becomes

$$\frac{\partial}{\partial t}\int_{cv} \rho dV + \int_{A_e} \rho_e\, V_e\, d A_e - \int_{A_i} \rho_i\, V_i\, d A_i = 0$$

or

$$\frac{\partial}{\partial t}\int_{cv} \rho dV + \int_{A_e} \rho_e\, V_e\, d A_e - \int_{A_i} \rho_i\, V_i\, d A_i = 0$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

Rate of change of mass in c.v., $\downarrow$ **= 0 for steady-state**

Rate of mass leaving c.v., $\downarrow$ $\dot{m}_e$

Rate of mass entering c.v., $\downarrow$ $\dot{m}_i$

This can be simplified to

$$\left.\frac{dm}{dt}\right)_{cv} + \sum \dot{m}_e - \sum \dot{m}_i = 0$$

Note that the exit and inlet velocities V_e and V_i are the local components of fluid velocities at the exit and inlet boundaries **relative to an observer standing on the boundary**. Therefore, if the boundary is moving, the velocity is measured relative to the boundary motion. The location and orientation of a coordinate system for the problem are not considered in determining these velocities.

Also, the result of $\overline{V}_e \cdot d\overline{A}_e$ and $\overline{V}_i \cdot d\overline{A}_i$ is the product of the normal velocity component times the flow area at the exit or inlet, e.g.

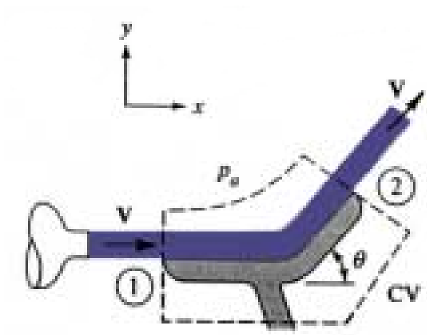
$$V_{e,n}\, dA_e \qquad \text{and} \qquad V_{i,n}\, dA_i$$

Special Case: For incompressible flow with a uniform velocity over the flow area, the previous integral expressions simplify to:

$$\dot{m} = \int_{cs} \rho V\, dA = \rho A V$$

Conservation of Mass Example

Water at a velocity of 7 m/s exits a stationary nozzle with D = 4 cm and is directed toward a turning vane with $\theta = 40^{o}$, Assume steady-state.



Determine:

a. Velocity and flow rate entering the c.v.
b. Velocity and flow rate leaving the c.v.