Introducing the numerical values from the table, we have

$$\left(\frac{dE}{dt}\right)_{\text{syst}} = -(300\text{ J/kg})(800\text{ kg/m}^3)(2\text{ m}^2)(5\text{ m/s}) - 100(800)(3)(8) + 150(800)(2)(17)$$

$$= (-2{,}400{,}000 - 1{,}920{,}000 + 4{,}080{,}000)\text{ J/s}$$

$$= -240{,}000\text{ J/s} = -0.24\text{ MJ/s} \qquad \textit{Ans.}$$

Thus the system is losing energy at the rate of 0.24 MJ/s = 0.24 MW. Since we have accounted for all fluid energy crossing the boundary, we conclude from the first law that there must be heat loss through the control surface or the system must be doing work on the environment through some device not shown. Notice that the use of SI units leads to a consistent result in joules per second without any conversion factors. We promised in Chap. 1 that this would be the case.

Note: This problem involves energy, but suppose we check the balance of mass also. Then B = mass m, and $B = dm/dm$ = unity. Again the volume integral vanishes for steady flow, and Eq. (3.17) reduces to

$$\left(\frac{dm}{dt}\right)_{\text{syst}} = \int_{CS} \rho(\mathbf{V}\cdot\mathbf{n})\,dA = -\rho_1 A_1 V_1 - \rho_2 A_2 V_2 + \rho_3 A_3 V_3$$

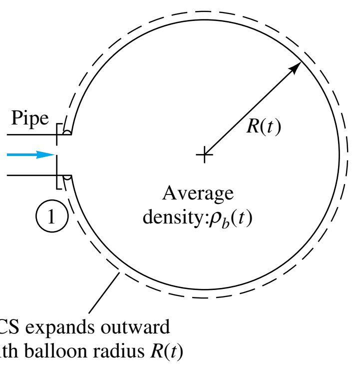
$$= -(800\text{ kg/m}^3)(2\text{ m}^2)(5\text{ m/s}) - 800(3)(8) + 800(17)(2)$$

$$= (-8000 - 19{,}200 + 27{,}200)\text{ kg/s} = 0\text{ kg/s}$$

Thus the system mass does not change, which correctly expresses the law of conservation of system mass, Eq. (3.1).

EXAMPLE 3.2

The balloon in Fig. E3.2 is being filled through section 1, where the area is A_1, velocity is V_1, and fluid density is ρ_1. The average density within the balloon is $\rho_b(t)$. Find an expression for the rate of change of system mass within the balloon at this instant.



E3.2

Solution

It is convenient to define a deformable control surface just outside the balloon, expanding at the same rate $R(t)$. Equation (3.16) applies with $V_r = 0$ on the balloon surface and $V_r = V_1$ at the pipe entrance. For mass change, we take $B = m$ and $\beta = dm/dm = 1$. Equation (3.16) becomes

$$\left(\frac{dm}{dt}\right)_{\text{syst}} = \frac{d}{dt}\left(\int_{CS} \rho\, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V}_r\cdot\mathbf{n})\,dA$$

Mass flux occurs only at the inlet, so that the control-surface integral reduces to the single negative term $-\rho_1 A_1 V_1$. The fluid mass within the control volume is approximately the average density times the volume of a sphere. The equation thus becomes

$$\left(\frac{dm}{dt}\right)_{\text{syst}} = \frac{d}{dt}\left(\rho_b \frac{4}{3}\pi R^3\right) - \rho_1 A_1 V_1 \qquad \textit{Ans.}$$

This is the desired result for the system mass rate of change. Actually, by the conservation law

(3.1), this change must be zero. Thus the balloon density and radius are related to the inlet mass flux by

$$\frac{d}{dt}(\rho_b R^3) = \frac{3}{4\pi}\rho_1 A_1 V_1$$

This is a first-order differential equation which could form part of an engineering analysis of balloon inflation. It cannot be solved without further use of mechanics and thermodynamics to relate the four unknowns ρ_b, ρ_1, V_1, and R. The pressure and temperature and the elastic properties of the balloon would also have to be brought into the analysis.

For advanced study, many more details of the analysis of deformable control volumes can be found in Hansen [4] and Potter and Foss [5].

3.3 Conservation of Mass

The Reynolds transport theorem, Eq. (3.16) or (3.17), establishes a relation between system rates of change and control-volume surface and volume integrals. But system derivatives are related to the basic laws of mechanics, Eqs. (3.1) to (3.5). Eliminating system derivatives between the two gives the control-volume, or *integral*, forms of the laws of mechanics of fluids. The dummy variable B becomes, respectively, mass, linear momentum, angular momentum, and energy.

For conservation of mass, as discussed in Examples 3.1 and 3.2, $B = m$ and $\beta = dm/dm = 1$. Equation (3.1) becomes

$$\left(\frac{dm}{dt}\right)_{\text{syst}} = 0 = \frac{d}{dt}\left(\int_{CV} \rho \, d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n}) \, dA \tag{3.20}$$

This is the integral mass-conservation law for a deformable control volume. For a fixed control volume, we have

$$\int_{CV} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \int_{CS} \rho(\mathbf{V} \cdot \mathbf{n}) \, dA = 0 \tag{3.21}$$

If the control volume has only a number of one-dimensional inlets and outlets, we can write

$$\int_{CV} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{\text{out}} - \sum_i (\rho_i A_i V_i)_{\text{in}} = 0 \tag{3.22}$$

Other special cases occur. Suppose that the flow within the control volume is steady; then $\partial\rho/\partial t \equiv 0$, and Eq. (3.21) reduces to

$$\int_{CS} \rho(\mathbf{V} \cdot \mathbf{n}) \, dA = 0 \tag{3.23}$$

This states that in steady flow the mass flows entering and leaving the control volume must balance exactly.[6] If, further, the inlets and outlets are one-dimensional, we have

[6]Throughout this section we are neglecting *sources* or *sinks* of mass which might be embedded in the control volume. Equations (3.20) and (3.21) can readily be modified to add source and sink terms, but this is rarely necessary.

for steady flow

$$\sum_i (\rho_i A_i V_i)_{\text{in}} = \sum_i (\rho_i A_i V_i)_{\text{out}} \tag{3.24}$$

This simple approximation is widely used in engineering analyses. For example, referring to Fig. 3.6, we see that if the flow in that control volume is steady, the three outlet mass fluxes balance the two inlets:

$$\text{Outflow} = \text{inflow}$$

$$\rho_2 A_2 V_2 + \rho_3 A_3 V_3 + \rho_5 A_5 V_5 = \rho_1 A_1 V_1 + \rho_4 A_4 V_4 \tag{3.25}$$

The quantity ρAV is called the *mass flow* $\dot{m}$ passing through the one-dimensional cross section and has consistent units of kilograms per second (or slugs per second) for SI (or BG) units. Equation (3.25) can be rewritten in the short form

$$\dot{m}_2 + \dot{m}_3 + \dot{m}_5 = \dot{m}_1 + \dot{m}_4 \tag{3.26}$$

and, in general, the steady-flow–mass-conservation relation (3.23) can be written as

$$\sum_i (\dot{m}_i)_{\text{out}} = \sum_i (\dot{m}_i)_{\text{in}} \tag{3.27}$$

If the inlets and outlets are not one-dimensional, one has to compute $\dot{m}$ by integration over the section

$$\dot{m}_{\text{cs}} = \int_{\text{cs}} \rho(\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.28}$$

where "cs" stands for cross section. An illustration of this is given in Example 3.4.

Incompressible Flow

Still further simplification is possible if the fluid is incompressible, which we may define as having density variations which are negligible in the mass-conservation requirement.[7] As we saw in Chap. 1, all liquids are nearly incompressible, and gas flows can *behave* as if they were incompressible, particularly if the gas velocity is less than about 30 percent of the speed of sound of the gas.

Again consider the fixed control volume. If the fluid is nearly incompressible, $\partial\rho/\partial t$ is negligible and the volume integral in Eq. (3.21) may be neglected, after which the density can be slipped outside the surface integral and divided out since it is nonzero. The result is a conservation law for incompressible flows, whether steady or unsteady:

$$\int_{\text{CS}} (\mathbf{V} \cdot \mathbf{n})\, dA = 0 \tag{3.29}$$

If the inlets and outlets are one-dimensional, we have

$$\sum_i (V_i A_i)_{\text{out}} = \sum_i (V_i A_i)_{\text{in}} \tag{3.30}$$

or

$$\sum Q_{\text{out}} = \sum Q_{\text{in}}$$

where $Q_i = V_i A_i$ is called the *volume flow* passing through the given cross section.

[7] Be warned that there is subjectivity in specifying incompressibility. Oceanographers consider a 0.1 percent density variation very significant, while aerodynamicists often neglect density variations in highly compressible, even hypersonic, gas flows. Your task is to justify the incompressible approximation when you make it.

Again, if consistent units are used, $Q = VA$ will have units of cubic meters per second (SI) or cubic feet per second (BG). If the cross section is not one-dimensional, we have to integrate

$$Q_{CS} = \int_{CS} (\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.31}$$

Equation (3.31) allows us to define an *average velocity* V_{av} which, when multiplied by the section area, gives the correct volume flow

$$V_{av} = \frac{Q}{A} = \frac{1}{A} \int (\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.32}$$

This could be called the *volume-average velocity*. If the density varies across the section, we can define an average density in the same manner:

$$\rho_{av} = \frac{1}{A} \int \rho\, dA \tag{3.33}$$

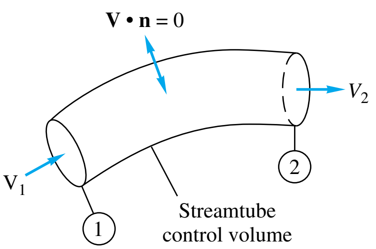
But the mass flow would contain the product of density and velocity, and the average product $(\rho V)_{av}$ would in general have a different value from the product of the averages

$$(\rho V)_{av} = \frac{1}{A} \int \rho(\mathbf{V} \cdot \mathbf{n})\, dA \approx \rho_{av} V_{av} \tag{3.34}$$

We illustrate average velocity in Example 3.4. We can often neglect the difference or, if necessary, use a correction factor between mass average and volume average.

EXAMPLE 3.3

Write the conservation-of-mass relation for steady flow through a streamtube (flow everywhere parallel to the walls) with a single one-dimensional exit 1 and inlet 2 (Fig. E3.3).



E3.3

Solution

For steady flow Eq. (3.24) applies with the single inlet and exit

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \text{const}$$

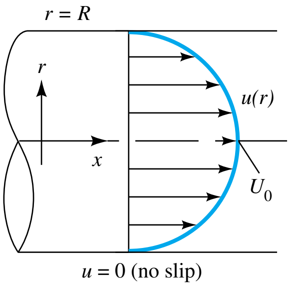
Thus, in a streamtube in steady flow, the mass flow is constant across every section of the tube. If the density is constant, then

$$Q = A_1 V_1 = A_2 V_2 = \text{const} \qquad \text{or} \qquad V_2 = \frac{A_1}{A_2} V_1$$

The volume flow is constant in the tube in steady incompressible flow, and the velocity increases as the section area decreases. This relation was derived by Leonardo da Vinci in 1500.

EXAMPLE 3.4

For steady viscous flow through a circular tube (Fig. E3.4), the axial velocity profile is given approximately by



E3.4

$$u = U_0\left(1 - \frac{r}{R}\right)^m$$

so that u varies from zero at the wall ($r = R$), or no slip, up to a maximum $u = U_0$ at the centerline $r = 0$. For highly viscous (laminar) flow $m \approx \frac{1}{2}$, while for less viscous (turbulent) flow $m \approx \frac{1}{7}$. Compute the average velocity if the density is constant.

Solution

The average velocity is defined by Eq. (3.32). Here $\mathbf{V} = \mathbf{i}u$ and $\mathbf{n} = \mathbf{i}$, and thus $\mathbf{V} \cdot \mathbf{n} = u$. Since the flow is symmetric, the differential area can be taken as a circular strip $dA = 2\ \pi r\ dr$. Equation (3.32) becomes

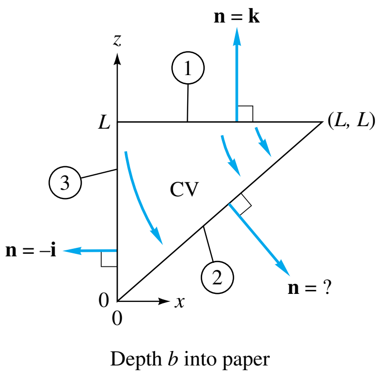
$$V_{av} = \frac{1}{A}\int u\ dA = \frac{1}{\pi R^2}\int_0^R U_0\left(1 - \frac{r}{R}\right)^m 2\pi r\ dr$$

or

$$V_{av} = U_0\frac{2}{(1+m)(2+m)} \qquad \textit{Ans.}$$

For the laminar-flow approximation, $m \approx \frac{1}{2}$ and $V_{av} \approx 0.53U_0$. (The exact laminar theory in Chap. 6 gives $V_{av} = 0.50U_0$.) For turbulent flow, $m \approx \frac{1}{7}$ and $V_{av} \approx 0.82U_0$. (There is no exact turbulent theory, and so we accept this approximation.) The turbulent velocity profile is more uniform across the section, and thus the average velocity is only slightly less than maximum.

EXAMPLE 3.5



E3.5

Consider the constant-density velocity field

$$u = \frac{V_0 x}{L} \qquad \upsilon = 0 \qquad w = -\frac{V_0 z}{L}$$

similar to Example 1.10. Use the triangular control volume in Fig. E3.5, bounded by (0, 0), (L, L), and (0, L), with depth b into the paper. Compute the volume flow through sections 1, 2, and 3, and compare to see whether mass is conserved.

Solution

The velocity field everywhere has the form $\mathbf{V} = \mathbf{i}u + \mathbf{k}w$. This must be evaluated along each section. We save section 2 until last because it looks tricky. Section 1 is the plane $z = L$ with depth b. The unit outward normal is $\mathbf{n} = \mathbf{k}$, as shown. The differential area is a strip of depth b varying with x: $dA = b\ dx$. The normal velocity is

$$(\mathbf{V} \cdot \mathbf{n})_1 = (\mathbf{i}u + \mathbf{k}w) \cdot \mathbf{k} = w\big|_1 = -\frac{V_0 z}{L}\bigg|_{z=L} = -V_0$$

The volume flow through section 1 is thus, from Eq. (3.31),

$$Q_1 = \int_1 (\mathbf{V} \cdot \mathbf{n})\ dA = \int_0^L (-V_0) b\ dx = -V_0 bL \qquad \textit{Ans. 1}$$

Since this is negative, section 1 is a net inflow. Check the units: V_0bL is a velocity times an area; OK.

Section 3 is the plane $x = 0$ with depth b. The unit normal is $\mathbf{n} = -\mathbf{i}$, as shown, and $dA = b\,dz$. The normal velocity is

$$(\mathbf{V} \cdot \mathbf{n})_3 = (\mathbf{i}u + \mathbf{k}w) \cdot (-\mathbf{i}) = -u\big|_3 = -\frac{V_0x}{L}\bigg|_{s=0} = 0 \qquad \textit{Ans. 3}$$

Thus $V_n \equiv 0$ all along section 3; hence $Q_3 = 0$.

Finally, section 2 is the plane $x = z$ with depth b. The normal direction is to the right $\mathbf{i}$ and down $-\mathbf{k}$ but must have *unit* value; thus $\mathbf{n} = (1/\sqrt{2})(\mathbf{i} - \mathbf{k})$. The differential area is either $dA = \sqrt{2}b\,dx$ or $dA = \sqrt{2}b\,dz$. The normal velocity is

$$(\mathbf{V} \cdot \mathbf{n})_2 = (\mathbf{i}u + \mathbf{k}w) \cdot \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k}) = \frac{1}{\sqrt{2}}(u - w)_2$$

$$= \frac{1}{\sqrt{2}}\left[V_0\frac{x}{L} - \left(-V_0\frac{z}{L}\right)\right]_{x=z} = \frac{\sqrt{2}V_0x}{L} \quad \text{or} \quad \frac{\sqrt{2}V_0z}{L}$$

Then the volume flow through section 2 is

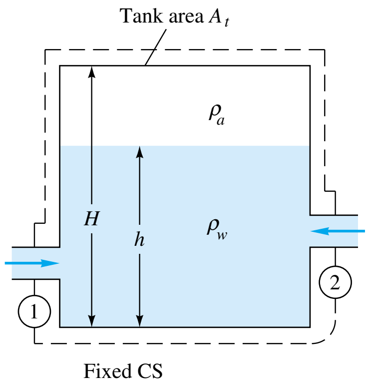
$$Q_2 = \int_2 (\mathbf{V} \cdot \mathbf{n})\,dA = \int_0^L \frac{\sqrt{2}V_0x}{L}(\sqrt{2}b\,dx) = V_0bL \qquad \textit{Ans. 2}$$

This answer is positive, indicating an outflow. These are the desired results. We should note that the volume flow is zero through the front and back triangular faces of the prismatic control volume because $V_n = v = 0$ on those faces.

The sum of the three volume flows is

$$Q_1 + Q_2 + Q_3 = -V_0bL + V_0bL + 0 = 0$$

Mass is conserved in this constant-density flow, and there are no net sources or sinks within the control volume. This is a very realistic flow, as described in Example 1.10



E3.6

EXAMPLE 3.6

The tank in Fig. E3.6 is being filled with water by two one-dimensional inlets. Air is trapped at the top of the tank. The water height is h. (a) Find an expression for the change in water height dh/dt. (b) Compute dh/dt if $D_1 = 1$ in, $D_2 = 3$ in, $V_1 = 3$ ft/s, $V_2 = 2$ ft/s, and $A_t = 2$ ft^2, assuming water at 20°C.

Solution

Part (a) A suggested control volume encircles the tank and cuts through the two inlets. The flow within is unsteady, and Eq. (3.22) applies with no outlets and two inlets:

$$\frac{d}{dt}\left(\int_{CV} \rho\,d\mathcal{V}\right) - \rho_1A_1V_1 - \rho_2A_2V_2 = 0 \qquad (1)$$

Now if A_t is the tank cross-sectional area, the unsteady term can be evaluated as follows:

$$\frac{d}{dt}\left(\int_{CV} \rho\,d\mathcal{V}\right) = \frac{d}{dt}(\rho_wA_th) + \frac{d}{dt}[\rho_aA_t(H - h)] = \rho_wA_t\frac{dh}{dt} \qquad (2)$$

The ρ_a term vanishes because it is the rate of change of air mass and is zero because the air is trapped at the top. Substituting (2) into (1), we find the change of water height

$$\frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho_w A_t} \qquad \textit{Ans. (a)}$$

For water, $\rho_1 = \rho_2 = \rho_w$, and this result reduces to

$$\frac{dh}{dt} = \frac{A_1 V_1 + A_2 V_2}{A_t} = \frac{Q_1 + Q_2}{A_t} \tag{3}$$

Part (b) The two inlet volume flows are

$$Q_1 = A_1 V_1 = \tfrac{1}{4}\pi(\tfrac{1}{12}\text{ ft})^2(3\text{ ft/s}) = 0.016\text{ ft}^3/\text{s}$$

$$Q_2 = A_2 V_2 = \tfrac{1}{4}\pi(\tfrac{3}{12}\text{ ft})^2(2\text{ ft/s}) = 0.098\text{ ft}^3/\text{s}$$

Then, from Eq. (3),

$$\frac{dh}{dt} = \frac{(0.016 + 0.098)\text{ ft}^3/\text{s}}{2\text{ ft}^2} = 0.057\text{ ft/s} \qquad \textit{Ans. (b)}$$

Suggestion: Repeat this problem with the top of the tank open.

An illustration of a mass balance with a deforming control volume has already been given in Example 3.2.

The control-volume mass relations, Eq. (3.20) or (3.21), are fundamental to all fluid-flow analyses. They involve only velocity and density. Vector directions are of no consequence except to determine the normal velocity at the surface and hence whether the flow is *in* or *out*. Although your specific analysis may concern forces or moments or energy, you must always make sure that mass is balanced as part of the analysis; otherwise the results will be unrealistic and probably rotten. We shall see in the examples which follow how mass conservation is constantly checked in performing an analysis of other fluid properties.

3.4 The Linear Momentum Equation

In Newton's law, Eq. (3.2), the property being differentiated is the linear momentum $m\mathbf{V}$. Therefore our dummy variable is $\mathbf{B} = m\mathbf{V}$ and $\boldsymbol{\beta} = d\mathbf{B}/dm = \mathbf{V}$, and application of the Reynolds transport theorem gives the linear-momentum relation for a deformable control volume

$$\frac{d}{dt}(m\mathbf{V})_{\text{syst}} = \sum \mathbf{F} = \frac{d}{dt}\left(\int_{\text{CV}} \mathbf{V}\rho\, d\mathcal{V}\right) + \int_{\text{CS}} \mathbf{V}\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \tag{3.35}$$

The following points concerning this relation should be strongly emphasized:

1. The term $\mathbf{V}$ is the fluid velocity relative to an *inertial* (nonaccelerating) coordinate system; otherwise Newton's law must be modified to include noninertial relative-acceleration terms (see the end of this section).
2. The term $\Sigma\, \mathbf{F}$ is the *vector* sum of all forces acting on the control-volume material considered as a free body; i.e., it includes surface forces on all fluids and

solids cut by the control surface plus all body forces (gravity and electromagnetic) acting on the masses within the control volume.

3. The entire equation is a vector relation; both the integrals are vectors due to the term **V** in the integrands. The equation thus has three components. If we want only, say, the x component, the equation reduces to

$$\sum F_x = \frac{d}{dt}\left(\int_{CV} u\rho \, d\mathcal{V}\right) + \int_{CS} u\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \tag{3.36}$$

and similarly, ΣF_y and ΣF_z would involve v and w, respectively. Failure to account for the vector nature of the linear-momentum relation (3.35) is probably the greatest source of student error in control-volume analyses.

For a fixed control volume, the relative velocity $\mathbf{V}_r \equiv \mathbf{V}$, and

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.37}$$

Again we stress that this is a vector relation and that **V** must be an inertial-frame velocity. Most of the momentum analyses in this text are concerned with Eq. (3.37).

One-Dimensional Momentum Flux

By analogy with the term *mass flow* used in Eq. (3.28), the surface integral in Eq. (3.37) is called the *momentum-flux term.* If we denote momentum by **M**, then

$$\dot{\mathbf{M}}_{CS} = \int_{sec} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n})\, dA \tag{3.38}$$

Because of the dot product, the result will be negative for inlet momentum flux and positive for outlet flux. If the cross section is one-dimensional, **V** and ρ are uniform over the area and the integrated result is

$$\dot{\mathbf{M}}_{seci} = \mathbf{V}_i(\rho_i V_{ni} A_i) = \dot{m}_i \mathbf{V}_i \tag{3.39}$$

for outlet flux and $-\dot{m}_i\mathbf{V}_i$ for inlet flux. Thus if the control volume has only one-dimensional inlets and outlets, Eq. (3.37) reduces to

$$\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \sum(\dot{m}_i\mathbf{V}_i)_{out} - \sum(\dot{m}_i\mathbf{V}_i)_{in} \tag{3.40}$$

This is a commonly used approximation in engineering analyses. It is crucial to realize that we are dealing with vector sums. Equation (3.40) states that the net vector force on a fixed control volume equals the rate of change of vector momentum within the control volume plus the vector sum of outlet momentum fluxes minus the vector sum of inlet fluxes.

Net Pressure Force on a Closed Control Surface

Generally speaking, the surface forces on a control volume are due to (1) forces exposed by cutting through solid bodies which protrude through the surface and (2) forces due to pressure and viscous stresses of the surrounding fluid. The computation of pressure force is relatively simple, as shown in Fig. 3.7. Recall from Chap. 2 that the external pressure force on a surface is normal to the surface and *inward*. Since the unit vector **n** is defined as *outward*, one way to write the pressure force is

$$\mathbf{F}_{press} = \int_{CS} p(-\mathbf{n})\, dA \tag{3.41}$$

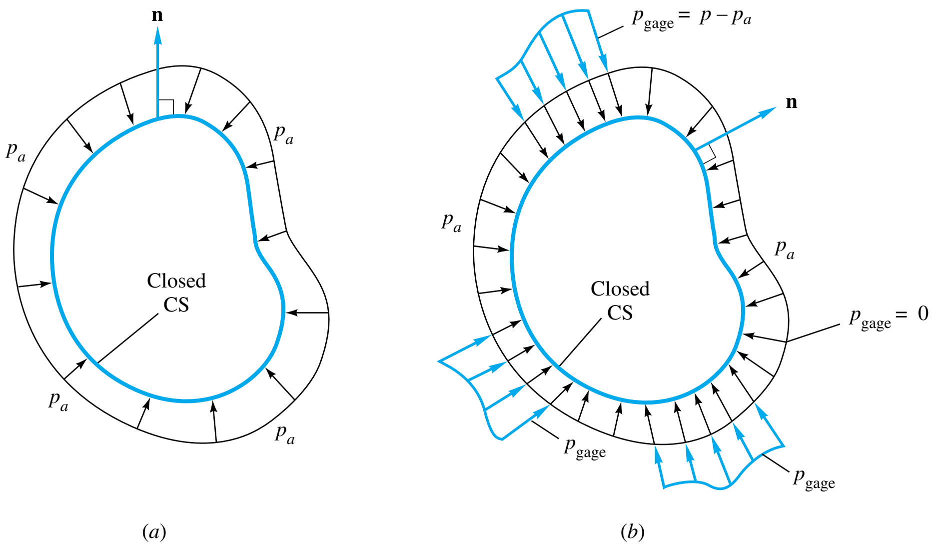


Fig. 3.7 Pressure-force computation by subtracting a uniform distribution: (*a*) uniform pressure, $\mathbf{F} = -p_a \int \mathbf{n}\, dA \equiv 0$; (*b*) nonuniform pressure, $\mathbf{F} = -\int (p - p_a)\mathbf{n}\, dA$.

Now if the pressure has a uniform value p_a all around the surface, as in Fig. 3.7*a*, the net pressure force is zero

$$\mathbf{F}_{\text{UP}} = \int p_a(-\mathbf{n})\, dA = -p_a \int \mathbf{n}\, dA \equiv 0 \tag{3.42}$$

where the subscript UP stands for uniform pressure. This result is *independent of the shape of the surface*[8] as long as the surface is closed and all our control volumes are closed. Thus a seemingly complicated pressure-force problem can be simplified by subtracting any convenient uniform pressure p_a and working only with the pieces of gage pressure which remain, as illustrated in Fig. 3.7*b*. Thus Eq. (3.41) is entirely equivalent to

$$\mathbf{F}_{\text{press}} = \int_{\text{CS}} (p - p_a)(-\mathbf{n})\, dA = \int_{\text{CS}} p_{\text{gage}}(-\mathbf{n})\, dA$$

This trick can mean quite a saving in computation.

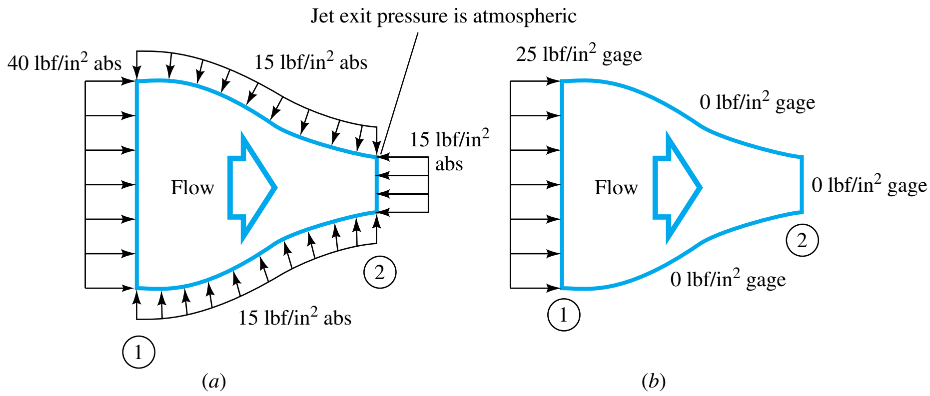
EXAMPLE 3.7

A control volume of a nozzle section has surface pressures of 40 lbf/in^{2}absolute at section 1 and atmospheric pressure of 15 lbf/in^{2}absolute at section 2 and on the external rounded part of the nozzle, as in Fig. E3.7*a*. Compute the net pressure force if $D_1 = 3$ in and $D_2 = 1$ in.

Solution

We do not have to bother with the outer surface if we subtract 15 lbf/in^2 from all surfaces. This leaves 25 lbf/in^{2}gage at section 1 and 0 lbf/in^2 gage everywhere else, as in Fig. E3.7*b*.

[8]Can you prove this? It is a consequence of Gauss' theorem from vector analysis.



E3.7

Then the net pressure force is computed from section 1 only

$$\mathbf{F} = p_{g1}(-\mathbf{n})_1 A_1 = (25 \text{ lbf/in}^2)\frac{\pi}{4}(3 \text{ in})^2\mathbf{i} = 177\mathbf{i} \text{ lbf} \qquad \textit{Ans.}$$

Notice that we did not change inches to feet in this case because, with pressure in pounds-force per square inch and area in square inches, the product gives force directly in pounds. More often, though, the change back to standard units is necessary and desirable. *Note:* This problem computes pressure force only. There are probably other forces involved in Fig. E3.7, e.g., nozzle-wall stresses in the cuts through sections 1 and 2 and the weight of the fluid within the control volume.

Pressure Condition at a Jet Exit

Figure E3.7 illustrates a pressure boundary condition commonly used for jet exit-flow problems. When a fluid flow leaves a confined internal duct and exits into an ambient "atmosphere," its free surface is exposed to that atmosphere. Therefore the jet itself will essentially be at atmospheric pressure also. This condition was used at section 2 in Fig. E3.7.

Only two effects could maintain a pressure difference between the atmosphere and a free exit jet. The first is surface tension, Eq. (1.31), which is usually negligible. The second effect is a *supersonic* jet, which can separate itself from an atmosphere with expansion or compression waves (Chap. 9). For the majority of applications, therefore, we shall set the pressure in an exit jet as atmospheric.

EXAMPLE 3.8

A fixed control volume of a streamtube in steady flow has a uniform inlet flow (ρ_1, A_1, V_1) and a uniform exit flow (ρ_2, A_2, V_2), as shown in Fig. 3.8. Find an expression for the net force on the control volume.

Solution

Equation (3.40) applies with one inlet and exit

$$\sum \mathbf{F} = \dot{m}_2\mathbf{V}_2 - \dot{m}_1\mathbf{V}_1 = (\rho_2 A_2 V_2)\mathbf{V}_2 - (\rho_1 A_1 V_1)\mathbf{V}_1$$

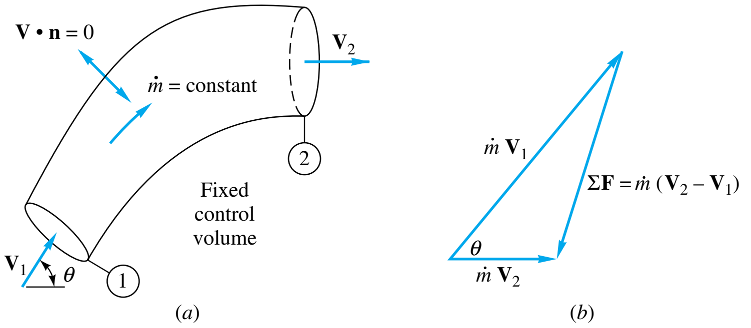


Fig. 3.8 Net force on a one-dimensional streamtube in steady flow: (*a*) streamtube in steady flow; (*b*) vector diagram for computing net force.

The volume-integral term vanishes for steady flow, but from conservation of mass in Example 3.3 we saw that

$$\dot{m}_1 = \dot{m}_2 = \dot{m} = \text{const}$$

Therefore a simple form for the desired result is

$$\sum \mathbf{F} = \dot{m}\,(\mathbf{V}_2 - \mathbf{V}_1) \qquad \textit{Ans.}$$

This is a *vector* relation and is sketched in Fig. 3.8*b*. The term Σ **F** represents the net force acting on the control volume due to all causes; it is needed to balance the change in momentum of the fluid as it turns and decelerates while passing through the control volume.

EXAMPLE 3.9

As shown in Fig. 3.9*a*, a fixed vane turns a water jet of area *A* through an angle θ without changing its velocity magnitude. The flow is steady, pressure is p_a everywhere, and friction on the vane is negligible. (*a*) Find the components F_x and F_y of the applied vane force. (*b*) Find expressions for the force magnitude *F* and the angle ϕ between *F* and the horizontal; plot them versus θ.

Solution

Part (a) The control volume selected in Fig. 3.9*a* cuts through the inlet and exit of the jet and through the vane support, exposing the vane force **F**. Since there is no cut along the vane-jet interface,

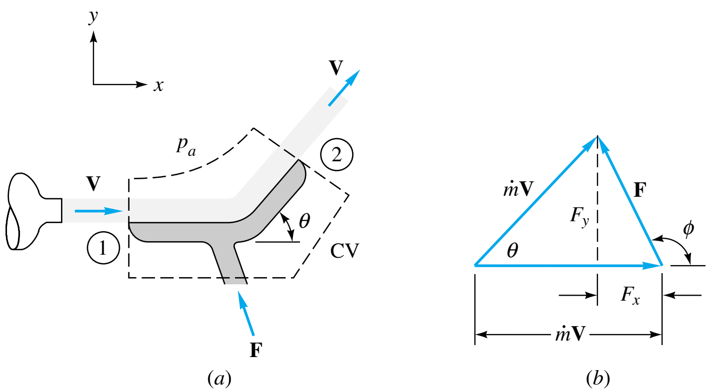


Fig. 3.9 Net applied force on a fixed jet-turning vane: (*a*) geometry of the vane turning the water jet; (*b*) vector diagram for the net force.

vane friction is internally self-canceling. The pressure force is zero in the uniform atmosphere. We neglect the weight of fluid and the vane weight within the control volume. Then Eq. (3.40) reduces to

$$\mathbf{F}_{\text{vane}} = \dot{m}_2\mathbf{V}_2 - \dot{m}_1\mathbf{V}_1$$

But the magnitude $V_1 = V_2 = V$ as given, and conservation of mass for the streamtube requires $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho AV$. The vector diagram for force and momentum change becomes an isosceles triangle with legs $\dot{m}\mathbf{V}$ and base $\mathbf{F}$, as in Fig. 3.9*b*. We can readily find the force components from this diagram

$$F_x = \dot{m}V(\cos\,\theta - 1) \qquad F_y = \dot{m}V\sin\,\theta \qquad \textit{Ans. (a)}$$

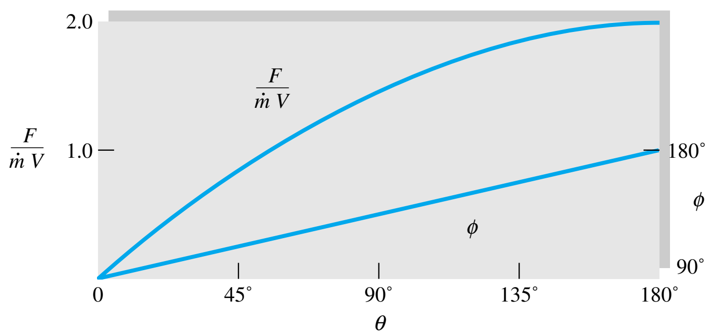
where $\dot{m}V = \rho AV^2$ for this case. This is the desired result.

Part (b) The force magnitude is obtained from part (*a*):

$$F = (F_x^2 + F_y^2)^{1/2} = \dot{m}V[\sin^2\theta + (\cos\,\theta - 1)^2]^{1/2} = 2\dot{m}V\sin\frac{\theta}{2} \qquad \textit{Ans. (b)}$$

From the geometry of Fig. 3.9*b* we obtain

$$\phi = 180° - \tan^{-1}\frac{F_y}{F_x} = 90° + \frac{\theta}{2} \qquad \textit{Ans. (b)}$$



E3.9

These can be plotted versus θ as shown in Fig. E3.9. Two special cases are of interest. First, the maximum force occurs at $\theta = 180°$, that is, when the jet is turned around and thrown back in the opposite direction with its momentum completely reversed. This force is $2\dot{m}V$ and acts to the *left*; that is, $\phi = 180°$. Second, at very small turning angles ($\theta < 10°$) we obtain approximately

$$F \approx \dot{m}V\theta \qquad \phi \approx 90°$$

The force is linearly proportional to the turning angle and acts nearly normal to the jet. This is the principle of a lifting vane, or airfoil, which causes a slight change in the oncoming flow direction and thereby creates a lift force normal to the basic flow.

EXAMPLE 3.10

A water jet of velocity V_j impinges normal to a flat plate which moves to the right at velocity V_c, as shown in Fig. 3.10*a*. Find the force required to keep the plate moving at constant velocity if the jet density is 1000 kg/m^3, the jet area is 3 cm^2, and V_j and V_c are 20 and 15 m/s, re-

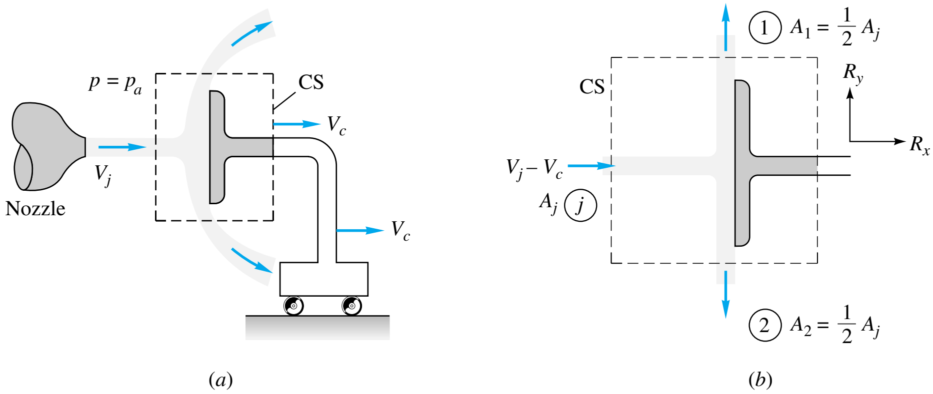


Fig. 3.10 Force on a plate moving at constant velocity: (*a*) jet striking a moving plate normally; (*b*) control volume fixed relative to the plate.

spectively. Neglect the weight of the jet and plate, and assume steady flow with respect to the moving plate with the jet splitting into an equal upward and downward half-jet.

Solution

The suggested control volume in Fig. 3.10*a* cuts through the plate support to expose the desired forces R_x and R_y. This control volume moves at speed V_c and thus is fixed relative to the plate, as in Fig. 3.10*b*. We must satisfy both mass and momentum conservation for the assumed steady-flow pattern in Fig. 3.10*b*. There are two outlets and one inlet, and Eq. (3.30) applies for mass conservation

$$\dot{m}_{\text{out}} = \dot{m}_{\text{in}}$$

or

$$\rho_1 A_1 V_1 + \rho_2 A_2 V_2 = \rho_j A_j (V_j - V_c) \tag{1}$$

We assume that the water is incompressible $\rho_1 = \rho_2 = \rho_j$, and we are given that $A_1 = A_2 = \frac{1}{2}A_j$. Therefore Eq. (1) reduces to

$$V_1 + V_2 = 2(V_j - V_c) \tag{2}$$

Strictly speaking, this is all that mass conservation tells us. However, from the symmetry of the jet deflection and the neglect of fluid weight, we conclude that the two velocities V_1 and V_2 must be equal, and hence (2) becomes

$$V_1 = V_2 = V_j - V_c \tag{3}$$

For the given numerical values, we have

$$V_1 = V_2 = 20 - 15 = 5 \text{ m/s}$$

Now we can compute R_x and R_y from the two components of momentum conservation. Equation (3.40) applies with the unsteady term zero

$$\sum F_x = R_x = \dot{m}_1 u_1 + \dot{m}_2 u_2 - \dot{m}_j u_j \tag{4}$$

where from the mass analysis, $\dot{m}_1 = \dot{m}_2 = \frac{1}{2}\dot{m}_j = \frac{1}{2}\rho_j A_j (V_j - V_c)$. Now check the flow directions at each section: $u_1 = u_2 = 0$, and $u_j = V_j - V_c = 5$ m/s. Thus Eq. (4) becomes

$$R_x = -\dot{m}_j u_j = -[\rho_j A_j (V_j - V_c)](V_j - V_c) \tag{5}$$

For the given numerical values we have

$$R_x = -(1000\ \text{kg/m}^3)(0.0003\ \text{m}^2)(5\ \text{m/s})^2 = -7.5\ (\text{kg}\cdot\text{m})/\text{s}^2 = -7.5\ \text{N} \qquad \textit{Ans.}$$

This acts to the *left*; i.e., it requires a restraining force to keep the plate from accelerating to the right due to the continuous impact of the jet. The vertical force is

$$F_y = R_y = \dot{m}_1 v_1 + \dot{m}_2 v_2 - \dot{m}_j v_j$$

Check directions again: $v_1 = V_1$, $v_2 = -V_2$, $v_j = 0$. Thus

$$R_y = \dot{m}_1(V_1) + \dot{m}_2(-V_2) = \tfrac{1}{2}\dot{m}_j(V_1 - V_2) \tag{6}$$

But since we found earlier that $V_1 = V_2$, this means that $R_y = 0$, as we could expect from the symmetry of the jet deflection.[9] Two other results are of interest. First, the relative velocity at section 1 was found to be 5 m/s up, from Eq. (3). If we convert this to absolute motion by adding on the control-volume speed $V_c = 15$ m/s to the right, we find that the absolute velocity $\mathbf{V}_1 = 15\mathbf{i} + 5\mathbf{j}$ m/s, or 15.8 m/s at an angle of 18.4° upward, as indicated in Fig. 3.10*a*. Thus the absolute jet speed changes after hitting the plate. Second, the computed force R_x does not change if we assume the jet deflects in all radial directions along the plate surface rather than just up and down. Since the plate is normal to the x axis, there would still be zero outlet x-momentum flux when Eq. (4) was rewritten for a radial-deflection condition.

EXAMPLE 3.11

The previous example treated a plate at normal incidence to an oncoming flow. In Fig. 3.11 the plate is parallel to the flow. The stream is not a jet but a broad river, or *free stream*, of uniform velocity $\mathbf{V} = U_0\mathbf{i}$. The pressure is assumed uniform, and so it has no net force on the plate. The plate does not block the flow as in Fig. 3.10, so that the only effect is due to boundary shear, which was neglected in the previous example. The no-slip condition at the wall brings the fluid there to a halt, and these slowly moving particles retard their neighbors above, so that at the end of the plate there is a significant retarded shear layer, or *boundary layer*, of thickness $y = \delta$. The

[9]Symmetry can be a powerful tool if used properly. Try to learn more about the uses and misuses of symmetry conditions. Here we doggedly computed the results without invoking symmetry.

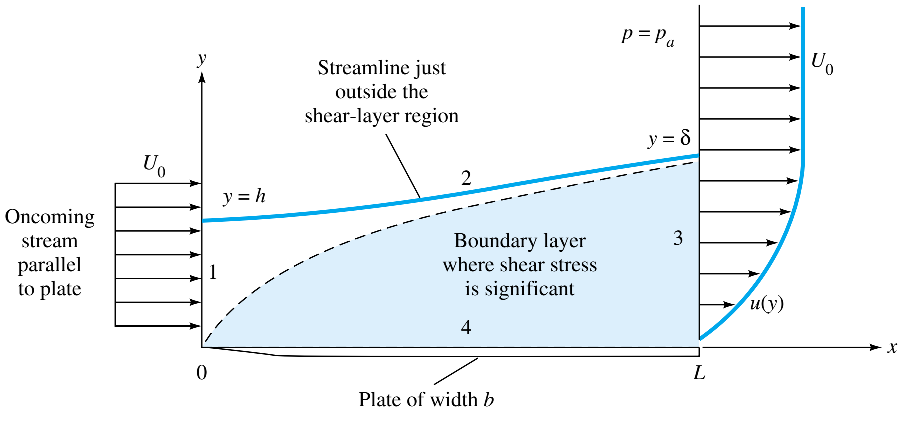


Fig. 3.11 Control-volume analysis of drag force on a flat plate due to boundary shear.

viscous stresses along the wall can sum to a finite drag force on the plate. These effects are illustrated in Fig. 3.11. The problem is to make an integral analysis and find the drag force D in terms of the flow properties ρ, U_0, and δ and the plate dimensions L and b.†

Solution

Like most practical cases, this problem requires a combined mass and momentum balance. A proper selection of control volume is essential, and we select the four-sided region from 0 to h to δ to L and back to the origin 0, as shown in Fig. 3.11. Had we chosen to cut across horizontally from left to right along the height $y = h$, we would have cut through the shear layer and exposed unknown shear stresses. Instead we follow the streamline passing through $(x, y) = (0, h)$, which is outside the shear layer and also has no mass flow across it. The four control-volume sides are thus

1. From (0, 0) to (0, h): a one-dimensional inlet, $\mathbf{V} \cdot \mathbf{n} = -U_0$
2. From (0, h) to (L, δ): a streamline, no shear, $\mathbf{V} \cdot \mathbf{n} \equiv 0$
3. From (L, δ) to (L, 0): a two-dimensional outlet, $\mathbf{V} \cdot \mathbf{n} = +u(y)$
4. From (L, 0) to (0, 0): a streamline just above the plate surface, $\mathbf{V} \cdot \mathbf{n} = 0$, shear forces summing to the drag force $-D\mathbf{i}$ acting from the plate onto the retarded fluid

The pressure is uniform, and so there is no net pressure force. Since the flow is assumed incompressible and steady, Eq. (3.37) applies with no unsteady term and fluxes only across sections 1 and 3:

$$\sum F_x = -D = \rho \int_1 u(\mathbf{V} \cdot \mathbf{n})\, dA + \rho \int_3 u(\mathbf{V} \cdot \mathbf{n})\, dA$$

$$= \rho \int_0^h U_0(-U_0) b\, dy + \rho \int_0^\delta u(+u) b\, dy$$

Evaluating the first integral and rearranging give

$$D = \rho U_0^2 b h - \rho b \int_0^\delta u^2 dy \tag{1}$$

This could be considered the answer to the problem, but it is not useful because the height h is not known with respect to the shear-layer thickness δ. This is found by applying mass conservation, since the control volume forms a streamtube

$$\rho \int_{CS} (\mathbf{V} \cdot \mathbf{n})\, dA = 0 = \rho \int_0^h (-U_0) b\, dy + \rho \int_0^\delta u b\, dy$$

or

$$U_0 h = \int_0^\delta u\, dy \tag{2}$$

after canceling b and ρ and evaluating the first integral. Introduce this value of h into Eq. (1) for a much cleaner result

$$D = \rho b \int_0^\delta u(U_0 - u)\, dy \Big|_{x=L} \qquad \textit{Ans.} \ (3)$$

This result was first derived by Theodore von Kármán in 1921.[10] It relates the friction drag on

†The general analysis of such wall-shear problems, called *boundary-layer theory*, is treated in Sec. 7.3.

[10]The autobiography of this great twentieth-century engineer and teacher [2] is recommended for its historical and scientific insight.

one side of a flat plate to the integral of the *momentum defect* $u(U_0 - u)$ across the trailing cross section of the flow past the plate. Since $U_0 - u$ vanishes as y increases, the integral has a finite value. Equation (3) is an example of *momentum-integral theory* for boundary layers, which is treated in Chap. 7. To illustrate the magnitude of this drag force, we can use a simple parabolic approximation for the outlet-velocity profile $u(y)$ which simulates low-speed, or *laminar*, shear flow

$$u \approx U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \qquad \text{for } 0 \le y \le \delta \tag{4}$$

Substituting into Eq. (3) and letting $\eta = y/\delta$ for convenience, we obtain

$$D = \rho b U_0^2 \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta = \tfrac{2}{15}\rho U_0^2 b\delta \tag{5}$$

This is within 1 percent of the accepted result from laminar boundary-layer theory (Chap. 7) in spite of the crudeness of the Eq. (4) approximation. This is a happy situation and has led to the wide use of Kármán's integral theory in the analysis of viscous flows. Note that D increases with the shear-layer thickness δ, which itself increases with plate length and the viscosity of the fluid (see Sec. 7.4).

Momentum-Flux Correction Factor

For flow in a duct, the axial velocity is usually nonuniform, as in Example 3.4. For this case the simple momentum-flux calculation $\int u\rho(\mathbf{V}\cdot\mathbf{n})\, dA = \dot{m}V = \rho A V^2$ is somewhat in error and should be corrected to $\beta\rho A V^2$, where β is the dimensionless momentum-flux correction factor, $\beta \ge 1$.

The factor β accounts for the variation of u^2across the duct section. That is, we compute the exact flux and set it equal to a flux based on average velocity in the duct

$$\rho\int u^2 dA = \beta\dot{m}V_{av} = \beta\rho A V_{av}^2$$

or

$$\beta = \frac{1}{A}\int\left(\frac{u}{V_{av}}\right)^2 dA \tag{3.43a}$$

Values of β can be computed based on typical duct velocity profiles similar to those in Example 3.4. The results are as follows:

Laminar flow:

$$u = U_0\left(1 - \frac{r^2}{R^2}\right) \qquad \beta = \frac{4}{3} \tag{3.43b}$$

Turbulent flow:

$$u \approx U_0\left(1 - \frac{r}{R}\right)^m \qquad \frac{1}{9} \le m \le \frac{1}{5}$$

$$\beta = \frac{(1+m)^2(2+m)^2}{2(1+2m)(2+2m)} \tag{3.43c}$$

The turbulent correction factors have the following range of values:

Turbulent flow:

m	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
β	1.037	1.027	1.020	1.016	1.013