Here are some worked examples to help us understand more about direct proportion.

**Problem 9.** 3 energy saving light bulbs cost £7.80. Determine the cost of 7 such light bulbs

(i) 3 light bulbs cost £7.80

(ii) Therefore, 1 light bulb costs $\frac{7.80}{3} = £2.60$

Hence, **7 light bulbs cost** $7 \times £2.60 =$ **£18.20**

**Problem 10.** If 56 litres of petrol costs £59.92, calculate the cost of 32 litres

(i) 56 litres of petrol costs £59.92

(ii) Therefore, 1 litre of petrol costs $\frac{59.92}{56} = £1.07$

Hence, **32 litres cost** $32 \times 1.07 =$ **£34.24**

**Problem 11.** Hooke's law states that stress, $\sigma$, is directly proportional to strain, $\varepsilon$, within the elastic limit of a material. When, for mild steel, the stress is 63 MPa, the strain is 0.0003. Determine (a) the value of strain when the stress is 42 MPa, (b) the value of stress when the strain is 0.00072

(a) Stress is directly proportional to strain.

(i) When the stress is 63 MPa, the strain is 0.0003

(ii) Hence, a stress of 1 MPa corresponds to a strain of $\frac{0.0003}{63}$

(iii) Thus, **the value of strain when the stress is 42 MPa** $= \frac{0.0003}{63} \times 42 = \mathbf{0.0002}$

(b) Strain is proportional to stress.

(i) When the strain is 0.0003, the stress is 63 MPa.

(ii) Hence, a strain of 0.0001 corresponds to $\frac{63}{3}$ MPa.

(iii) Thus, **the value of stress when the strain is** $\mathbf{0.00072 = \frac{63}{3} \times 7.2 = 151.2}$ **MPa**.

**Problem 12.** Charles's law states that for a given mass of gas at constant pressure, the volume is directly proportional to its thermodynamic temperature. A gas occupies a volume of 2.4 litres at 600 K. Determine (a) the temperature when the volume is 3.2 litres, (b) the volume at 540 K

(a) Volume is directly proportional to temperature.

(i) When the volume is 2.4 litres, the temperature is 600 K.

(ii) Hence, a volume of 1 litre corresponds to a temperature of $\frac{600}{2.4}$ K.

(iii) Thus, **the temperature when the volume is 3.2 litres** $= \frac{600}{2.4} \times 3.2 =$ **800 K**.

(b) Temperature is proportional to volume.

(i) When the temperature is 600 K, the volume is 2.4 litres.

(ii) Hence, a temperature of 1 K corresponds to a volume of $\frac{2.4}{600}$ litres.

(iii) Thus, **the volume at a temperature of 540 K** $= \frac{2.4}{600} \times 540 =$ **2.16 litres**.

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**Now try the following Practice Exercise**

**Practice Exercise 26 Direct proportion (answers on page 342)**

1. 3 engine parts cost £208.50. Calculate the cost of 8 such parts.
2. If 9 litres of gloss white paint costs £24.75, calculate the cost of 24 litres of the same paint.
3. The total mass of 120 household bricks is 57.6 kg. Determine the mass of 550 such bricks.
4. A simple machine has an effort : load ratio of 3 : 37. Determine the effort, in grams, to lift a load of 5.55 kN.
5. If 16 cans of lager weighs 8.32 kg, what will 28 cans weigh?
6. Hooke's law states that stress is directly proportional to strain within the elastic limit of a material. When, for copper, the stress is 60 MPa, the strain is 0.000625. Determine (a) the strain when the stress is 24 MPa and (b) the stress when the strain is 0.0005

7. Charles's law states that volume is directly proportional to thermodynamic temperature for a given mass of gas at constant pressure. A gas occupies a volume of 4.8 litres at 330 K. Determine (a) the temperature when the volume is 6.4 litres and (b) the volume when the temperature is 396 K.

Here are some further worked examples on direct proportion.

**Problem 13.** Some guttering on a house has to decline by 3 mm for every 70 cm to allow rainwater to drain. The gutter spans 8.4 m. How much lower should the low end be?

(i) The guttering has to decline in the ratio 3 : 700 or $\frac{3}{700}$

(ii) If $d$ is the vertical drop in 8.4 m or 8400 mm, then the decline must be in the ratio $d$ : 8400 or $\frac{d}{8400}$

(iii) Now $\frac{d}{8400} = \frac{3}{700}$

(iv) Cross-multiplying gives $700 \times d = 8400 \times 3$ from which, $d = \frac{8400 \times 3}{700}$

i.e. **$d$ = 36 mm, which is how much the lower end should be to allow rainwater to drain**.

**Problem 14.** Ohm's law state that the current flowing in a fixed resistance is directly proportional to the applied voltage. When 90 mV is applied across a resistor the current flowing is 3 A. Determine (a) the current when the voltage is 60 mV and (b) the voltage when the current is 4.2 A

(a) Current is directly proportional to the voltage.

(i) When voltage is 90 mV, the current is 3 A.

(ii) Hence, a voltage of 1 mV corresponds to a current of $\frac{3}{90}$ A.

(iii) Thus, **when the voltage is 60 mV, the current** $= 60 \times \frac{3}{90} =$ **2A**.

(b) Voltage is directly proportional to the current.

(i) When current is 3 A, the voltage is 90 mV.

(ii) Hence, a current of 1 A corresponds to a voltage of $\frac{90}{3}$ mV = 30 mV.

(iii) Thus, **when the current is 4.2 A, the voltage** = 30 × 4.2 = **126 mV**.

**Problem 15.** Some approximate imperial to metric conversions are shown in Table 6.1. Use the table to determine

(a) the number of millimetres in 12.5 inches

(b) a speed of 50 miles per hour in kilometres per hour

(c) the number of miles in 300 km

(d) the number of kilograms in 20 pounds weight

(e) the number of pounds and ounces in 56 kilograms (correct to the nearest ounce)

(f) the number of litres in 24 gallons

(g) the number of gallons in 60 litres

**Table 6.1**

| | |
|---|---|
| length | 1 inch = 2.54 cm |
| | 1 mile = 1.6 km |
| weight | 2.2 lb = 1 kg |
| | (1 lb = 16 oz) |
| capacity | 1.76 pints = 1 litre |
| | (8 pints = 1 gallon) |

(a) 12.5 inches = 12.5 × 2.54 cm = 31.75 cm
31.73 cm = 31.75 × 10 mm = **317.5 mm**

(b) 50 m.p.h. = 50 × 1.6 km/h = **80 km/h**

(c) 300 km $= \frac{300}{1.6}$ miles = **186.5 miles**

(d) 20 lb $= \frac{20}{2.2}$ kg = **9.09 kg**

(e) 56 kg = 56 × 2.2 lb = 123.2 lb

0.2 lb = 0.2 × 16 oz = 3.2 oz = 3 oz, correct to the nearest ounce.

Thus, 56 kg = **123 lb 3 oz**, correct to the nearest ounce.

(f) 24 gallons = 24 × 8 pints = 192 pints

192 pints $= \frac{192}{1.76}$ litres = **109.1 litres**

(g) 60 litres = 60 × 1.76 pints = 105.6 pints

$$105.6 \text{ pints} = \frac{105.6}{8} \text{ gallons} = \mathbf{13.2\ gallons}$$

**Problem 16.** Currency exchange rates for five countries are shown in Table 6.2. Calculate

(a) how many euros £55 will buy

(b) the number of Japanese yen which can be bought for £23

(c) the number of pounds sterling which can be exchanged for 6405 kronor

(d) the number of American dollars which can be purchased for £92.50

(e) the number of pounds sterling which can be exchanged for 2925 Swiss francs

**Table 6.2**

| | |
|---|---|
| France | £1 = 1.25 euros |
| Japan | £1 = 185 yen |
| Norway | £1 = 10.50 kronor |
| Switzerland | £1 = 1.95 francs |
| USA | £1 = 1.80 dollars |

(a) £1 = 1.25 euros, hence £55 = 55 × 1.25 euros = **68.75 euros**.

(b) £1 = 185 yen, hence £23 = 23 × 185 yen = **4255 yen**.

(c) £1 = 10.50 kronor, hence 6405 lira = £$\frac{6405}{10.50}$ = **£610**.

(d) £1 = 1.80 dollars, hence
£92.50 = 92.50 × 1.80 dollars = **$166.50**

(e) £1 = 1.95 Swiss francs, hence
2925 pesetas = £$\frac{2925}{1.95}$ = **£1500**

**Now try the following Practice Exercise**

**Practice Exercise 27 Further direct proportion (answers on page 342)**

1. Ohm's law states that current is proportional to p.d. in an electrical circuit. When a p.d. of 60 mV is applied across a circuit a current of 24 μA flows. Determine (a) the current flowing when the p.d. is 5 V and (b) the p.d. when the current is 10 mA.

2. The tourist rate for the Swiss franc is quoted in a newspaper as £1 = 1.92 fr. How many francs can be purchased for £326.40?

3. If 1 inch = 2.54 cm, find the number of millimetres in 27 inches.

4. If 2.2 lb = 1 kg and 1lb = 16 oz, determine the number of pounds and ounces in 38 kg (correct to the nearest ounce).

5. If 1 litre = 1.76 pints and 8 pints = 1 gallon, determine (a) the number of litres in 35 gallons and (b) the number of gallons in 75 litres.

6. Hooke's law states that stress is directly proportional to strain within the elastic limit of a material. When for brass the stress is 21 MPa, the strain is 0.00025. Determine the stress when the strain is 0.00035.

7. If 12 inches = 30.48 cm, find the number of millimetres in 23 inches.

8. The tourist rate for the Canadian dollar is quoted in a newspaper as £1 = 1.84 fr. How many Canadian dollars can be purchased for £550?

## 6.4 Inverse proportion

Two variables, $x$ and $y$, are in inverse proportion to one another if $y$ is proportional to $\frac{1}{x}$, i.e. $y\ \alpha\ \frac{1}{x}$ or $y = \frac{k}{x}$ or $k = xy$ where $k$ is a constant, called the **coefficient of proportionality**.

**Inverse proportion** means that, as the value of one variable increases, the value of another decreases, and that their product is always the same.

For example, the time for a journey is inversely proportional to the speed of travel. So, if at 30 m.p.h. a journey is completed in 20 minutes, then at 60 m.p.h. the journey would be completed in 10 minutes. Double the speed, half the journey time. (Note that 30 × 20 = 60 × 10.)

In another example, the time needed to dig a hole is inversely proportional to the number of people digging. So, if 4 men take 3 hours to dig a hole, then 2 men

(working at the same rate) would take 6 hours. Half the men, twice the time. (Note that $4 \times 3 = 2 \times 6$.)
Here are some worked examples on inverse proportion.

**Problem 17.** It is estimated that a team of four designers would take a year to develop an engineering process. How long would three designers take?

If 4 designers take 1 year, then 1 designer would take 4 years to develop the process. Hence, 3 designers would take $\frac{4}{3}$ years, i.e. **1 year 4 months**.

**Problem 18.** A team of five people can deliver leaflets to every house in a particular area in four hours. How long will it take a team of three people?

If 5 people take 4 hours to deliver the leaflets, then 1 person would take $5 \times 4 = 20$ hours. Hence, 3 people would take $\frac{20}{3}$ hours, i.e. $6\frac{2}{3}$ hours, i.e. **6 hours 40 minutes**.

**Problem 19.** The electrical resistance $R$ of a piece of wire is inversely proportional to the cross-sectional area $A$. When $A = 5\,\text{mm}^2$, $R = 7.02$ ohms. Determine (a) the coefficient of proportionality and (b) the cross-sectional area when the resistance is 4 ohms

(a) $R \alpha \frac{1}{A}$, i.e. $R = \frac{k}{A}$ or $k = RA$. Hence, when $R = 7.2$ and $A = 5$, the

**coefficient of proportionality**, $k = (7.2)(5) = \mathbf{36}$

(b) Since $k = RA$ then $A = \frac{k}{R}$. Hence, when $R = 4$,

**the cross sectional area,** $A = \frac{36}{4} = \mathbf{9\,mm^2}$

**Problem 20.** Boyle's law states that, at constant temperature, the volume $V$ of a fixed mass of gas is inversely proportional to its absolute pressure $p$. If a gas occupies a volume of $0.08\,\text{m}^3$ at a pressure of $1.5 \times 10^6$ pascals, determine (a) the coefficient of proportionality and (b) the volume if the pressure is changed to $4 \times 10^6$ pascals

(a) $V \propto \frac{1}{p}$ i.e. $V = \frac{k}{p}$ or $k = pV$. Hence, the

**coefficient of proportionality, $k$**

$$= (1.5 \times 10^6)(0.08) = \mathbf{0.12 \times 10^6}$$

(b) **Volume,** $V = \frac{k}{p} = \frac{0.12 \times 10^6}{4 \times 10^6} = \mathbf{0.03\,m^3}$

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**Now try the following Practice Exercise**

## Practice Exercise 28 Further inverse proportion (answers on page 342)

1. A 10 kg bag of potatoes lasts for a week with a family of 7 people. Assuming all eat the same amount, how long will the potatoes last if there are only two in the family?
2. If 8 men take 5 days to build a wall, how long would it take 2 men?
3. If $y$ is inversely proportional to $x$ and $y = 15.3$ when $x = 0.6$, determine (a) the coefficient of proportionality, (b) the value of $y$ when $x$ is 1.5 and (c) the value of $x$ when $y$ is 27.2
4. A car travelling at 50 km/h makes a journey in 70 minutes. How long will the journey take at 70 km/h?
5. Boyle's law states that, for a gas at constant temperature, the volume of a fixed mass of gas is inversely proportional to its absolute pressure. If a gas occupies a volume of $1.5\,\text{m}^3$ at a pressure of $200 \times 10^3$ pascals, determine (a) the constant of proportionality, (b) the volume when the pressure is $800 \times 10^3$ pascals and (c) the pressure when the volume is $1.25\,\text{m}^3$.

Chapter 7

# Powers, roots and laws of indices

## 7.1 Introduction

The manipulation of powers and roots is a crucial underlying skill needed in algebra. In this chapter, powers and roots of numbers are explained, together with the laws of indices.
Many worked examples are included to help understanding.

## 7.2 Powers and roots

### 7.2.1 Indices

The number 16 is the same as $2 \times 2 \times 2 \times 2$, and $2 \times 2 \times 2 \times 2$ can be abbreviated to $2^4$. When written as $2^4$, 2 is called the **base** and the 4 is called the **index** or **power**.
$2^4$ is read as '**two to the power of four**'.
Similarly, $3^5$ is read as '**three to the power of 5**'.
When the indices are 2 and 3 they are given special names; i.e. 2 is called 'squared' and 3 is called 'cubed'. Thus,
$4^2$ is called '**four squared**' rather than '4 to the power of 2' and
$5^3$ is called '**five cubed**' rather than '5 to the power of 3'
When no index is shown, the power is 1. For example, 2 means $2^1$.

**Problem 1.** Evaluate (a) $2^6$ (b) $3^4$

(a) $2^6$ means $2 \times 2 \times 2 \times 2 \times 2 \times 2$ (i.e. 2 multiplied by itself 6 times), and $2 \times 2 \times 2 \times 2 \times 2 \times 2 = \mathbf{64}$
i.e. $\mathbf{2^6 = 64}$

(b) $3^4$ means $3 \times 3 \times 3 \times 3$ (i.e. 3 multiplied by itself 4 times), and $3 \times 3 \times 3 \times 3 = \mathbf{81}$
i.e. $\mathbf{3^4 = 81}$

**Problem 2.** Change the following to index form: (a) 32 (b) 625

(a) (i) To express 32 in its lowest factors, 32 is initially divided by the lowest prime number, i.e. 2.

(ii) $32 \div 2 = 16$, hence $32 = 2 \times 16$.

(iii) 16 is also divisible by 2, i.e. $16 = 2 \times 8$. Thus, $32 = 2 \times 2 \times 8$.

(iv) 8 is also divisible by 2, i.e. $8 = 2 \times 4$. Thus, $32 = 2 \times 2 \times 2 \times 4$.

(v) 4 is also divisible by 2, i.e. $4 = 2 \times 2$. Thus, $32 = 2 \times 2 \times 2 \times 2 \times 2$.

(vi) Thus, $32 = \mathbf{2^5}$.

(b) (i) 625 is not divisible by the lowest prime number, i.e. 2. The next prime number is 3 and 625 is not divisible by 3 either. The next prime number is 5.

(ii) $625 \div 5 = 125$, i.e. $625 = 5 \times 125$.

(iii) 125 is also divisible by 5, i.e. $125 = 5 \times 25$. Thus, $625 = 5 \times 5 \times 25$.

(iv) 25 is also divisible by 5, i.e. $25 = 5 \times 5$. Thus, $625 = 5 \times 5 \times 5 \times 5$.

(v) Thus, $\mathbf{625 = 5^4}$.

DOI: 10.1016/B978-1-85617-697-2.00007-7

**Problem 3.** Evaluate $3^3 \times 2^2$

$$\begin{aligned}3^3 \times 2^2 &= 3 \times 3 \times 3 \times 2 \times 2\\ &= 27 \times 4\\ &= \mathbf{108}\end{aligned}$$

### 7.2.2 Square roots

When a number is multiplied by itself the product is called a square.
For example, the square of 3 is $3 \times 3 = 3^2 = 9$.
A square root is the reverse process; i.e., the value of the base which when multiplied by itself gives the number; i.e., the square root of 9 is 3.
The symbol $\sqrt{\ }$ is used to denote a square root. Thus, $\sqrt{9} = 3$. Similarly, $\sqrt{4} = 2$ and $\sqrt{25} = 5$.
Because $-3 \times -3 = 9$, $\sqrt{9}$ also equals $-3$. Thus, $\sqrt{9} = +3$ or $-3$ which is usually written as $\sqrt{9} = \pm 3$. Similarly, $\sqrt{16} = \pm 4$ and $\sqrt{36} = \pm 6$.
The square root of, say, 9 may also be written in index form as $9^{\frac{1}{2}}$.

$$9^{\frac{1}{2}} \equiv \sqrt{9} = \pm 3$$

**Problem 4.** Evaluate $\dfrac{3^2 \times 2^3 \times \sqrt{36}}{\sqrt{16} \times 4}$ taking only positive square roots

$$\begin{aligned}\frac{3^2 \times 2^3 \times \sqrt{36}}{\sqrt{16} \times 4} &= \frac{3 \times 3 \times 2 \times 2 \times 2 \times 6}{4 \times 4}\\ &= \frac{9 \times 8 \times 6}{16} = \frac{9 \times 1 \times 6}{2}\\ &= \frac{9 \times 1 \times 3}{1} \quad \text{by cancelling}\\ &= \mathbf{27}\end{aligned}$$

**Problem 5.** Evaluate $\dfrac{10^4 \times \sqrt{100}}{10^3}$ taking the positive square root only

$$\begin{aligned}\frac{10^4 \times \sqrt{100}}{10^3} &= \frac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10}\\ &= \frac{1 \times 1 \times 1 \times 10 \times 10}{1 \times 1 \times 1} \quad \text{by cancelling}\\ &= \frac{100}{1}\\ &= \mathbf{100}\end{aligned}$$

**Now try the following Practice Exercise**

**Practice Exercise 29 Powers and roots (answers on page 342)**

Evaluate the following without the aid of a calculator.

1. $3^3$
2. $2^7$
3. $10^5$
4. $2^4 \times 3^2 \times 2 \div 3$
5. Change 16 to index form.
6. $25^{\frac{1}{2}}$
7. $64^{\frac{1}{2}}$
8. $\dfrac{10^5}{10^3}$
9. $\dfrac{10^2 \times 10^3}{10^5}$
10. $\dfrac{2^5 \times 64^{\frac{1}{2}} \times 3^2}{\sqrt{144} \times 3}$ taking positive square roots only.

## 7.3 Laws of indices

There are six laws of indices.

(1) From earlier, $2^2 \times 2^3 = (2 \times 2) \times (2 \times 2 \times 2) = 32 = 2^5$

Hence, $2^2 \times 2^3 = 2^5$

or $2^2 \times 2^3 = 2^{2+3}$

This is the first law of indices, which demonstrates that **when multiplying two or more numbers having the same base, the indices are added**.

(2) $$\frac{2^5}{2^3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = \frac{1 \times 1 \times 1 \times 2 \times 2}{1 \times 1 \times 1} = \frac{2 \times 2}{1} = 4 = 2^2$$

Hence, $\dfrac{2^5}{2^3} = 2^2$ or $\dfrac{2^5}{2^3} = 2^{5-3}$

This is the second law of indices, which demonstrates that **when dividing two numbers having the same base, the index in the denominator is subtracted from the index in the numerator**.

(3) $(3^5)^2 = 3^{5\times 2} = 3^{10}$ and $(2^2)^3 = 2^{2\times 3} = 2^6$

This is the third law of indices, which demonstrates that **when a number which is raised to a power is raised to a further power, the indices are multiplied**.

(4) $3^0 = 1$ and $17^0 = 1$

This is the fourth law of indices, which states that **when a number has an index of 0, its value is 1**.

(5) $3^{-4} = \frac{1}{3^4}$ and $\frac{1}{2^{-3}} = 2^3$

This is the fifth law of indices, which demonstrates that **a number raised to a negative power is the reciprocal of that number raised to a positive power**.

(6) $8^{\frac{2}{3}} = \sqrt[3]{8^2} = (2)^2 = 4$ and

$25^{\frac{1}{2}} = \sqrt[2]{25^1} = \sqrt{25^1} = \pm 5$

(Note that $\sqrt{\ } \equiv \sqrt[2]{\ }$)

This is the sixth law of indices, which demonstrates that **when a number is raised to a fractional power the denominator of the fraction is the root of the number and the numerator is the power**.

Here are some worked examples using the laws of indices.

**Problem 6.** Evaluate in index form $5^3 \times 5 \times 5^2$

$5^3 \times 5 \times 5^2 = 5^3 \times 5^1 \times 5^2$ (Note that 5 means $5^1$)

$= 5^{3+1+2}$ from law (1)

$= \mathbf{5^6}$

**Problem 7.** Evaluate $\frac{3^5}{3^4}$

$\frac{3^5}{3^4} = 3^{5-4}$ from law (2)

$= 3^1$

$= \mathbf{3}$

**Problem 8.** Evaluate $\frac{2^4}{2^4}$

$\frac{2^4}{2^4} = 2^{4-4}$ from law (2)

$= 2^0$

But $\frac{2^4}{2^4} = \frac{2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2} = \frac{16}{16} = 1$

Hence, $\mathbf{2^0 = 1}$ from law (4)

**Any number raised to the power of zero equals 1**. For example, $6^0 = 1$, $128^0 = 1$, $13742^0 = 1$ and so on.

**Problem 9.** Evaluate $\frac{3 \times 3^2}{3^4}$

$\frac{3 \times 3^2}{3^4} = \frac{3^1 \times 3^2}{3^4} = \frac{3^{1+2}}{3^4} = \frac{3^3}{3^4} = 3^{3-4} = 3^{-1}$

from laws (1) and (2)

But $\frac{3^3}{3^4} = \frac{3 \times 3 \times 3}{3 \times 3 \times 3 \times 3} = \frac{1 \times 1 \times 1}{1 \times 1 \times 1 \times 3}$

(by cancelling)

$= \frac{1}{3}$

Hence, $\frac{3 \times 3^2}{3^4} = \mathbf{3^{-1}} = \mathbf{\frac{1}{3}}$ from law (5)

Similarly, $2^{-1} = \frac{1}{2}, 2^{-5} = \frac{1}{2^5}, \frac{1}{5^4} = 5^{-4}$ and so on.

**Problem 10.** Evaluate $\frac{10^3 \times 10^2}{10^8}$

$\frac{10^3 \times 10^2}{10^8} = \frac{10^{3+2}}{10^8} = \frac{10^5}{10^8}$ from law (1)

$= 10^{5-8} = 10^{-3}$ from law (2)

$= \frac{1}{10^{+3}} = \frac{1}{1000}$ from law (5)

Hence, $\frac{10^3 \times 10^2}{10^8} = \mathbf{10^{-3}} = \mathbf{\frac{1}{1000}} = \mathbf{0.001}$

Understanding powers of ten is important, especially when dealing with prefixes in Chapter 8. For example,

$$10^2 = 100, 10^3 = 1000, 10^4 = 10\,000,$$

$$10^5 = 100\,000, 10^6 = 1\,000\,000$$

$$10^{-1} = \frac{1}{10} = 0.1, 10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$$

and so on.

**Problem 11.** Evaluate (a) $5^2 \times 5^3 \div 5^4$ (b) $(3 \times 3^5) \div (3^2 \times 3^3)$

From laws (1) and (2):

(a) $5^2 \times 5^3 \div 5^4 = \frac{5^2 \times 5^3}{5^4} = \frac{5^{(2+3)}}{5^4}$

$= \frac{5^5}{5^4} = 5^{(5-4)} = 5^1 = \mathbf{5}$

(b) $(3 \times 3^5) \div (3^2 \times 3^3) = \dfrac{3 \times 3^5}{3^2 \times 3^3} = \dfrac{3^{(1+5)}}{3^{(2+3)}}$

$= \dfrac{3^6}{3^5} = 3^{6-5} = 3^1 = \mathbf{3}$

**Problem 12.** Simplify (a) $(2^3)^4$ (b) $(3^2)^5$, expressing the answers in index form

From law (3):

(a) $(2^3)^4 = 2^{3\times 4} = \mathbf{2^{12}}$

(b) $(3^2)^5 = 3^{2\times 5} = \mathbf{3^{10}}$

**Problem 13.** Evaluate: $\dfrac{(10^2)^3}{10^4 \times 10^2}$

From laws (1) to (4):

$$\frac{(10^2)^3}{10^4 \times 10^2} = \frac{10^{(2\times 3)}}{10^{(4+2)}} = \frac{10^6}{10^6} = 10^{6-6} = 10^0 = \mathbf{1}$$

**Problem 14.** Find the value of (a) $\dfrac{2^3 \times 2^4}{2^7 \times 2^5}$ (b) $\dfrac{(3^2)^3}{3 \times 3^9}$

From the laws of indices:

(a) $\dfrac{2^3 \times 2^4}{2^7 \times 2^5} = \dfrac{2^{(3+4)}}{2^{(7+5)}} = \dfrac{2^7}{2^{12}} = 2^{7-12}$

$= 2^{-5} = \dfrac{1}{2^5} = \mathbf{\dfrac{1}{32}}$

(b) $\dfrac{(3^2)^3}{3 \times 3^9} = \dfrac{3^{2\times 3}}{3^{1+9}} = \dfrac{3^6}{3^{10}} = 3^{6-10}$

$= 3^{-4} = \dfrac{1}{3^4} = \mathbf{\dfrac{1}{81}}$

**Problem 15.** Evaluate (a) $4^{1/2}$ (b) $16^{3/4}$ (c) $27^{2/3}$ (d) $9^{-1/2}$

(a) $4^{1/2} = \sqrt{4} = \mathbf{\pm 2}$

(b) $16^{3/4} = \sqrt[4]{16^3} = (2)^3 = \mathbf{8}$

(Note that it does not matter whether the 4th root of 16 is found first or whether 16 cubed is found first – the same answer will result.)

(c) $27^{2/3} = \sqrt[3]{27^2} = (3)^2 = \mathbf{9}$

(d) $9^{-1/2} = \dfrac{1}{9^{1/2}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{\pm 3} = \mathbf{\pm\dfrac{1}{3}}$

**Now try the following Practice Exercise**

### Practice Exercise 30 Laws of indices (answers on page 342)

Evaluate the following without the aid of a calculator.

1. $2^2 \times 2 \times 2^4$
2. $3^5 \times 3^3 \times 3$ in index form
3. $\dfrac{2^7}{2^3}$
4. $\dfrac{3^3}{3^5}$
5. $7^0$
6. $\dfrac{2^3 \times 2 \times 2^6}{2^7}$
7. $\dfrac{10 \times 10^6}{10^5}$
8. $10^4 \div 10$
9. $\dfrac{10^3 \times 10^4}{10^9}$
10. $5^6 \times 5^2 \div 5^7$
11. $(7^2)^3$ in index form
12. $(3^3)^2$
13. $\dfrac{3^7 \times 3^4}{3^5}$ in index form
14. $\dfrac{(9 \times 3^2)^3}{(3 \times 27)^2}$ in index form
15. $\dfrac{(16 \times 4)^2}{(2 \times 8)^3}$
16. $\dfrac{5^{-2}}{5^{-4}}$
17. $\dfrac{3^2 \times 3^{-4}}{3^3}$
18. $\dfrac{7^2 \times 7^{-3}}{7 \times 7^{-4}}$
19. $\dfrac{2^3 \times 2^{-4} \times 2^5}{2 \times 2^{-2} \times 2^6}$
20. $\dfrac{5^{-7} \times 5^2}{5^{-8} \times 5^3}$

Here are some further worked examples using the laws of indices.

**Problem 16.** Evaluate $\dfrac{3^3 \times 5^7}{5^3 \times 3^4}$

The laws of indices only apply to terms **having the same base**. Grouping terms having the same base and then applying the laws of indices to each of the groups independently gives

$$\frac{3^3 \times 5^7}{5^3 \times 3^4} = \frac{3^3}{3^4} \times \frac{5^7}{5^3} = 3^{(3-4)} \times 5^{(7-3)}$$

$$= 3^{-1} \times 5^4 = \frac{5^4}{3^1} = \frac{625}{3} = \mathbf{208\frac{1}{3}}$$

**Problem 17.** Find the value of $\dfrac{2^3 \times 3^5 \times (7^2)^2}{7^4 \times 2^4 \times 3^3}$

$$\begin{aligned}\frac{2^3 \times 3^5 \times (7^2)^2}{7^4 \times 2^4 \times 3^3} &= 2^{3-4} \times 3^{5-3} \times 7^{2\times 2-4}\\ &= 2^{-1} \times 3^2 \times 7^0\\ &= \frac{1}{2} \times 3^2 \times 1 = \frac{9}{2} = \mathbf{4\frac{1}{2}}\end{aligned}$$

**Problem 18.** Evaluate $\dfrac{4^{1.5} \times 8^{1/3}}{2^2 \times 32^{-2/5}}$

$$4^{1.5} = 4^{3/2} = \sqrt{4^3} = 2^3 = 8,\ 8^{1/3} = \sqrt[3]{8} = 2,$$

$$2^2 = 4,\ 32^{-2/5} = \frac{1}{32^{2/5}} = \frac{1}{\sqrt[5]{32^2}} = \frac{1}{2^2} = \frac{1}{4}$$

Hence, $\dfrac{4^{1.5} \times 8^{1/3}}{2^2 \times 32^{-2/5}} = \dfrac{8 \times 2}{4 \times \frac{1}{4}} = \dfrac{16}{1} = \mathbf{16}$

Alternatively,

$$\begin{aligned}\frac{4^{1.5} \times 8^{1/3}}{2^2 \times 32^{-2/5}} &= \frac{[(2)^2]^{3/2} \times (2^3)^{1/3}}{2^2 \times (2^5)^{-2/5}}\\ &= \frac{2^3 \times 2^1}{2^2 \times 2^{-2}} = 2^{3+1-2-(-2)} = 2^4 = \mathbf{16}\end{aligned}$$

**Problem 19.** Evaluate $\dfrac{3^2 \times 5^5 + 3^3 \times 5^3}{3^4 \times 5^4}$

Dividing each term by the HCF (highest common factor) of the three terms, i.e. $3^2 \times 5^3$, gives

$$\begin{aligned}\frac{3^2 \times 5^5 + 3^3 \times 5^3}{3^4 \times 5^4} &= \frac{\dfrac{3^2 \times 5^5}{3^2 \times 5^3} + \dfrac{3^3 \times 5^3}{3^2 \times 5^3}}{\dfrac{3^4 \times 5^4}{3^2 \times 5^3}}\\ &= \frac{3^{(2-2)} \times 5^{(5-3)} + 3^{(3-2)} \times 5^0}{3^{(4-2)} \times 5^{(4-3)}}\\ &= \frac{3^0 \times 5^2 + 3^1 \times 5^0}{3^2 \times 5^1}\\ &= \frac{1 \times 25 + 3 \times 1}{9 \times 5} = \mathbf{\frac{28}{45}}\end{aligned}$$

**Problem 20.** Find the value of $\dfrac{3^2 \times 5^5}{3^4 \times 5^4 + 3^3 \times 5^3}$

To simplify the arithmetic, each term is divided by the HCF of all the terms, i.e. $3^2 \times 5^3$. Thus,

$$\begin{aligned}\frac{3^2 \times 5^5}{3^4 \times 5^4 + 3^3 \times 5^3} &= \frac{\dfrac{3^2 \times 5^5}{3^2 \times 5^3}}{\dfrac{3^4 \times 5^4}{3^2 \times 5^3} + \dfrac{3^3 \times 5^3}{3^2 \times 5^3}}\\ &= \frac{3^{(2-2)} \times 5^{(5-3)}}{3^{(4-2)} \times 5^{(4-3)} + 3^{(3-2)} \times 5^{(3-3)}}\\ &= \frac{3^0 \times 5^2}{3^2 \times 5^1 + 3^1 \times 5^0}\\ &= \frac{1 \times 5^2}{3^2 \times 5 + 3 \times 1} = \frac{25}{45+3} = \mathbf{\frac{25}{48}}\end{aligned}$$

**Problem 21.** Simplify $\dfrac{7^{-3} \times 3^4}{3^{-2} \times 7^5 \times 5^{-2}}$ expressing the answer in index form with positive indices

Since $7^{-3} = \dfrac{1}{7^3}$, $\dfrac{1}{3^{-2}} = 3^2$ and $\dfrac{1}{5^{-2}} = 5^2$, then

$$\begin{aligned}\frac{7^{-3} \times 3^4}{3^{-2} \times 7^5 \times 5^{-2}} &= \frac{3^4 \times 3^2 \times 5^2}{7^3 \times 7^5}\\ &= \frac{3^{(4+2)} \times 5^2}{7^{(3+5)}} = \mathbf{\frac{3^6 \times 5^2}{7^8}}\end{aligned}$$

**Problem 22.** Simplify $\dfrac{16^2 \times 9^{-2}}{4 \times 3^3 - 2^{-3} \times 8^2}$ expressing the answer in index form with positive indices

Expressing the numbers in terms of their lowest prime numbers gives

$$\begin{aligned}\frac{16^2 \times 9^{-2}}{4 \times 3^3 - 2^{-3} \times 8^2} &= \frac{(2^4)^2 \times (3^2)^{-2}}{2^2 \times 3^3 - 2^{-3} \times (2^3)^2}\\ &= \frac{2^8 \times 3^{-4}}{2^2 \times 3^3 - 2^{-3} \times 2^6}\\ &= \frac{2^8 \times 3^{-4}}{2^2 \times 3^3 - 2^3}\end{aligned}$$

Dividing each term by the HCF (i.e. $2^2$) gives

$$\frac{2^8 \times 3^{-4}}{2^2 \times 3^3 - 2^3} = \frac{2^6 \times 3^{-4}}{3^3 - 2} = \mathbf{\frac{2^6}{3^4(3^3 - 2)}}$$

**Problem 23.** Simplify $\dfrac{\left(\frac{4}{3}\right)^3 \times \left(\frac{3}{5}\right)^{-2}}{\left(\frac{2}{5}\right)^{-3}}$ giving the answer with positive indices

Raising a fraction to a power means that both the numerator and the denominator of the fraction are raised to that power, i.e. $\left(\frac{4}{3}\right)^3 = \frac{4^3}{3^3}$

A fraction raised to a negative power has the same value as the inverse of the fraction raised to a positive power.

Thus, $\left(\frac{3}{5}\right)^{-2} = \dfrac{1}{\left(\frac{3}{5}\right)^2} = \dfrac{1}{\frac{3^2}{5^2}} = 1 \times \frac{5^2}{3^2} = \frac{5^2}{3^2}$

Similarly, $\left(\frac{2}{5}\right)^{-3} = \left(\frac{5}{2}\right)^3 = \frac{5^3}{2^3}$

Thus, $\dfrac{\left(\frac{4}{3}\right)^3 \times \left(\frac{3}{5}\right)^{-2}}{\left(\frac{2}{5}\right)^{-3}} = \dfrac{\frac{4^3}{3^3} \times \frac{5^2}{3^2}}{\frac{5^3}{2^3}}$

$$= \frac{4^3}{3^3} \times \frac{5^2}{3^2} \times \frac{2^3}{5^3} = \frac{(2^2)^3 \times 2^3}{3^{(3+2)} \times 5^{(3-2)}}$$

$$= \mathbf{\frac{2^9}{3^5 \times 5}}$$

**Now try the following Practice Exercise**

## Practice Exercise 31 Further problems on indices (answers on page 342)

In problems 1 to 4, simplify the expressions given, expressing the answers in index form and with positive indices.

1. $\dfrac{3^3 \times 5^2}{5^4 \times 3^4}$
2. $\dfrac{7^{-2} \times 3^{-2}}{3^5 \times 7^4 \times 7^{-3}}$
3. $\dfrac{4^2 \times 9^3}{8^3 \times 3^4}$
4. $\dfrac{8^{-2} \times 5^2 \times 3^{-4}}{25^2 \times 2^4 \times 9^{-2}}$

In Problems 5 to 15, evaluate the expressions given.

5. $\left(\dfrac{1}{3^2}\right)^{-1}$
6. $81^{0.25}$
7. $16^{-\frac{1}{4}}$
8. $\left(\dfrac{4}{9}\right)^{1/2}$
9. $\dfrac{9^2 \times 7^4}{3^4 \times 7^4 + 3^3 \times 7^2}$
10. $\dfrac{3^3 \times 5^2}{2^3 \times 3^2 - 8^2 \times 9}$
11. $\dfrac{3^3 \times 7^2 - 5^2 \times 7^3}{3^2 \times 5 \times 7^2}$
12. $\dfrac{(2^4)^2 - 3^{-2} \times 4^4}{2^3 \times 16^2}$
13. $\dfrac{\left(\frac{1}{2}\right)^3 - \left(\frac{2}{3}\right)^{-2}}{\left(\frac{3}{2}\right)^2}$
14. $\dfrac{\left(\frac{4}{3}\right)^4}{\left(\frac{2}{9}\right)^2}$
15. $\dfrac{(3^2)^{3/2} \times (8^{1/3})^2}{(3)^2 \times (4^3)^{1/2} \times (9)^{-1/2}}$

Chapter 8

# Units, prefixes and engineering notation

## 8.1 Introduction

Of considerable importance in engineering is a knowledge of units of engineering quantities, the prefixes used with units, and engineering notation.

We need to know, for example, that

$$80\,\text{kV} = 80 \times 10^3\,\text{V}, \text{ which means } 80\,000 \text{ volts}$$

and $25\,\text{mA} = 25 \times 10^{-3}\,\text{A}$, which means 0.025 amperes

and $50\,\text{nF} = 50 \times 10^{-9}\,\text{F}$, which means 0.000000050 farads

This is explained in this chapter.

## 8.2 SI units

The system of units used in engineering and science is the Système Internationale d'Unités (**International System of Units**), usually abbreviated to SI units, and is based on the metric system. This was introduced in 1960 and has now been adopted by the majority of countries as the official system of measurement.

The basic seven units used in the SI system are listed in Table 8.1 with their symbols.

There are, of course, many units other than these seven. These other units are called **derived units** and are defined in terms of the standard units listed in the table. For example, speed is measured in metres per second, therefore using two of the standard units, i.e. length and time.

**Table 8.1** Basic SI units

| Quantity | Unit | Symbol |
|---|---|---|
| Length | metre | m $(1\,\text{m} = 100\,\text{cm} = 1000\,\text{mm})$ |
| Mass | kilogram | kg $(1\,\text{kg} = 1000\,\text{g})$ |
| Time | second | s |
| Electric current | ampere | A |
| Thermodynamic temperature | kelvin | K $(\text{K} = {}^\circ\text{C} + 273)$ |
| Luminous intensity | candela | cd |
| Amount of substance | mole | mol |

Some derived units are given **special names**. For example, force = mass × acceleration has units of kilogram metre per second squared, which uses three of the base units, i.e. kilograms, metres and seconds. The unit of kg $\text{m/s}^2$ is given the special name of a **Newton**.

Table 8.2 contains a list of some quantities and their units that are common in engineering.

## 8.3 Common prefixes

SI units may be made larger or smaller by using prefixes which denote multiplication or division by a particular amount.

DOI: 10.1016/B978-1-85617-697-2.00008-9