From the previous example for conservation of mass, we can again write

$$\dot{m} = \int_{cs} \rho V \, dA = \rho A V = 998 \text{ kg/m}^3 * 7 \text{ m/s} * \pi * 0.04^2/4$$

$\dot{m}_1$ = **8.78 kg/s** and V_1 = **7 m/s**

and for the exit:

$\dot{m}_2$ = **8.78 kg/s** and V_2 = **7 m/s** inclined 40° above the horizontal.

Substituting in the momentum equation, we obtain

$-F_b$ = 8.78 kg/s * 7 m/s *cos 40^o - 8.78 kg/s * 7 m/s

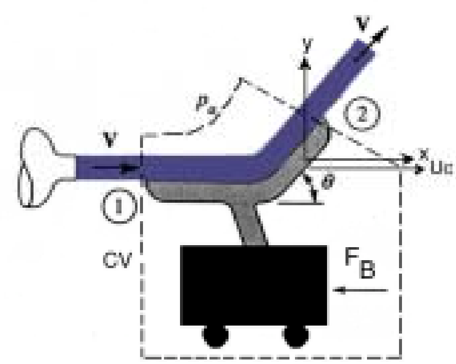
and $-F_b$ = - 14.4 kg m/s^2 or F_b = **14.4 N** ← **ans.**

Note: Since our final answer is positive, our original assumption of the applied force being to the left was correct. Had we assumed that the applied force was to the right, our answer would be negative, meaning that the direction of the applied force is opposite to what was assumed.

Modify Problem:

Now consider the same problem but with the cart moving to the right with a velocity U_c = 2 m/s. Again solve for the value of braking force F_b necessary to maintain a constant cart velocity of 2 m/s.

Note: The coordinate system for the problem has now been placed on the moving cart.



The transient term in the momentum equation is still zero. With the coordinate system on the cart, the momentum of the cart relative to the coordinate system is still zero. The fluid stream is still moving relative to the coordinate system, however, the flow is steady with constant velocity and the time rate of change of momentum of the fluid stream is therefore also zero. Thus

The momentum equation has the same form as for the previous problem (However the value of individual terms will be different.)

$$-F_b = \dot{m}_e V_e - \dot{m}_i V_i$$

$\dot{m}_1 = \rho_1 V_1 A_1 = 998\ kg/m^3 * 5\ m/s * \pi * 0.04^2/4 = 6.271\ kg/s = \dot{m}_2$

Now we must determine the momentum velocity at the inlet and exit. With the coordinate system on the moving control volume, the values of momentum velocity are

$V_1 = V_J - U_c = 7 - 2 = 5$ m/s and $V_2 = 5$ m/s inclined 40^o

The momentum equation (x - direction) now becomes

$$-F_b = 6.271\ kg/s * 5\ m/s * \cos 40^o - 6.271\ kg/s * 5\ m/s$$

and $-F_b = -7.34\ kg\ m/s^2$ or $F_b =$ **7.34 N** $\leftarrow$ **ans.**

Question: What would happen to the braking force F_b if the turning angle had been $> 90^o$, e.g., 130^o? Can you explain based on your understanding of change in momentum for the fluid stream?

Review and work examples for linear momentum with fixed and non-accelerating (moving at constant velocity) control volumes.

Accelerating Control Volume

The previous formulation applies only to an inertial coordinate system, i.e., fixed or moving at constant velocity (non-accelerating).

We will now consider problems with accelerating control volumes. For these problems we will again place the coordinate system on the accelerating control volume, thus making it a non-inertial coordinate system.

For coordinate systems placed on an accelerating control volume, we must account for the acceleration of the c.s. by correcting the momentum equation for this acceleration. This is accomplished by including the term as shown below:

$$\sum \bar{F} - \underbrace{\int_{cv} \bar{a}_{cv}\, dm_{cv}}_{\downarrow} = \frac{\partial}{\partial t}\int_{cv} \bar{V}\rho\, dV + \int_{A_e} \bar{V}\, d\dot{m}_e - \int_{A_i} \bar{V}\, d\dot{m}_i$$

integral sum of
the local c.v. (c.s.)
acceleration * the c.v. mass

The added term accounts for the acceleration of the control volume and allows the problem to be worked with the coordinate system placed on the accelerating c.v.

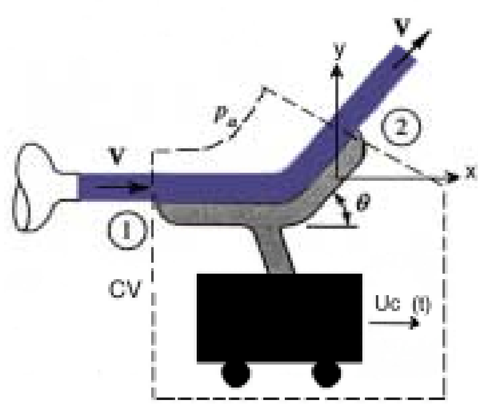
Note: Thus, all vector (momentum) velocities are then measured relative to an observer (coordinate system) on the accelerating control volume. For example, the velocity of a rocket as seen by an observer (c.s.) standing on the rocket is zero and the time rate of change of momentum is zero in this reference frame even if the rocket is accelerating.

Accelerating Control Volume Example

A turning vane with $\theta = 60^{o}$ accelerates from rest due to a jet of water ($V_J = 35$ m/s, $A_J = 0.003$ m^2). Assuming the mass of the cart m_c, is 75 kg and neglecting drag and friction effects, find:

a. Cart acceleration at t = 0.

b. U_c as a f(t)



Starting with the general equation shown above, we can make the following assumptions:

1. $\sum F_x = 0$, no friction or body forces.
2. The jet has uniform velocity and constant properties.
3. The entire cart accelerates uniformly over the entire control volume.
4. Neglect the relative momentum change of the jet stream that is within the control volume.

With these assumptions, the governing equation simplifies to

$$-a_c\, m_c = \dot{m}_e\, V_{x,e} - \dot{m}_i\, V_{x,i}$$

We thus have terms that account for the acceleration of the control volume, for the exit momentum, and for the inlet momentum (both of which change with time.)

Mass flow:

As with the previous example for a moving control volume, the mass flow terms are given by:

$$\dot{m}_i = \dot{m}_e = \dot{m} = \rho\, A_J\, (V_J - U_c)$$

Note that since the cart accelerates, U_c is not a constant but rather changes with time.

Momentum velocities:

$$U_{x,i} = V_J - U_c \qquad U_{x,e} = (V_J - U_c)\cos\theta$$

Substituting, we now obtain

$$-a_c\, m_c = \rho\, A_J\, (V_J - U_c)^2 \cos\theta - \rho\, A_J\, (V_J - U_c)$$

Solving for the cart acceleration, we obtain

$$a_c = \frac{\rho A_J\,(1 - \cos\theta)(V_J - U_c)^2}{m_c}$$

Substituting for the given values at t = 0, i.e., $U_c = 0$, we obtain

$$a_c\,(t = 0) = 24.45\ \text{m/s}^2 = 2.49\ \text{g's}$$

Note: The acceleration at any other time can be obtained once the cart velocity U_c at that time is known.

To determine the equation for cart velocity as a function of time, the equation for the acceleration must be written in terms of U_c (t) and integrated.

$$\frac{dU_c}{dt} = \frac{\rho A_J (1 - \cos\theta)(V_J - U_c)^2}{m_c}$$

Separating variables, we obtain

$$\int_0^{U_c(t)} \frac{dU_c}{(V_J - U_c)^2} = \int_0^t \frac{\rho A_J (1 - \cos\theta)}{m_c}\, dt$$

Completing the integration and rearranging the terms, we obtain a final expression of the form

$$\frac{U_c}{V_J} = \frac{V_J\, b\, t}{1 + V_J\, b\, t} \qquad \text{where} \qquad b = \frac{\rho A_J (1 - \cos\theta)}{m_c}$$

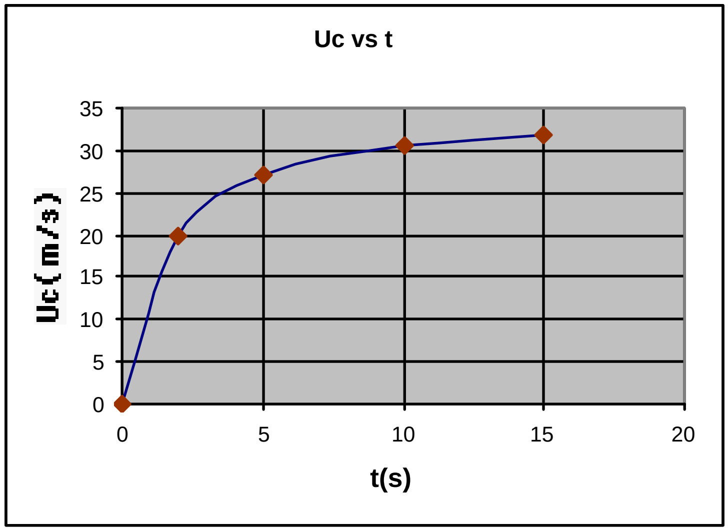
Substituting for known values, we obtain $V_J\, b = 0.699\ \text{s}^{-1}$

Thus the final equation for U_c is give by

$$\frac{U_c}{V_J} = \frac{0.699\, t}{1 + 0.699\, t}$$

The final results are now given as shown below:

t (s)	U_c/V_J	U_c (m/s)	a_c (m/s^2)
0	0.0	0.0	24.45
2	0.583	20.0	4.49
5	0.757	27.2	1.22
10	0.875	30.6	0.39
15	0.912	31.9	0.192
∞	1.0	35	0.0



Note that the limiting case occurs when the cart velocity reaches the jet velocity. At this point, the jet can impart no more momentum to the cart, the acceleration is now zero, and the terminal velocity has been reached.

Review the text example on accelerating control volumes.

Moment of Momentum (angular momentum)

For moment of momentum we have that

$$\overline{B} = \overline{H} = \overline{r} \times \left(m\,\overline{V}\right) \quad and \quad \overline{\beta} = \overline{r} \times \overline{V}$$

From the previous equation for moment of momentum and these definitions, Reynolds transport theorem becomes

$$\sum \bar{M} = \frac{\partial}{\partial t}\int_{cv} \bar{r} \times \bar{V} \rho dV + \int_{A_e} \bar{r} \times \bar{V} d\dot{m}_e - \int_{A_i} \bar{r} \times \bar{V} d\dot{m}_i$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

= the $\sum$ of all external moments acting on the c.v.	= the rate of change of moment of momentum in the c.v. = 0 for steady state	= the rate of moment of momentum leaving the c.v.	= the rate of moment of momentum entering the c.v.

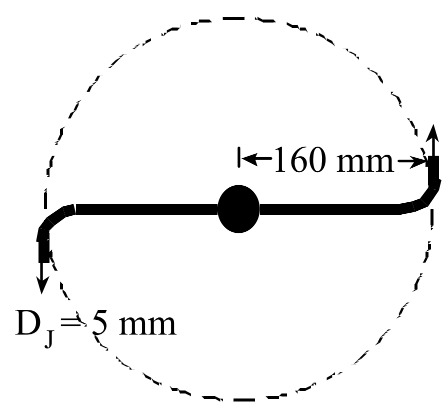
For the special case of steady-state, steady-flow and uniform properties at any exit or inlet, the equation becomes

$$\sum \bar{M} = \sum \dot{m}_e \, \bar{r} \times V_e - \sum \dot{m}_i \, \bar{r} \times V_i$$

For moment of momentum problems, we must be careful to correctly evaluate the moment of all applied forces and all inlet and exit momentum flows, with particular attention to the signs.

Moment of Momentum Example:

A small lawn sprinkler operates as indicated. The inlet flow rate is 9.98 kg/min with an inlet pressure of 30 kPa. The two exit jets direct flow at an angle of 40° above the horizontal.



For these conditions, determine the following:

a. Jet velocity relative to the nozzle.

b. Torque required to hold the arm stationary.

c. Friction torque if the arm is rotating at 35 rpm.

d. Maximum rotational speed if we neglect friction.

a. R = 160 mm, D_J = 5 mm, Therefore, for each of the two jets:

Q_J = 0.5* 9.98 kg/min/998 kg/m^3 = 0.005 m^3/min

A_J = π π 0.0025^2 = 1.963*10^{-5} m^2

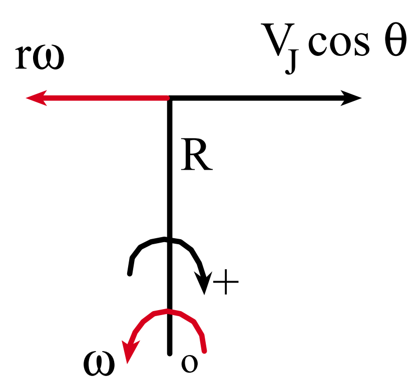
V_J = 0.005 m^3/min / 1.963*10^{-5} m^2 /60 s/min

V_J = 4.24 m/s relative to the nozzle exit **ans.**

b. Torque required to hold the arm stationary.

First develop the governing equations and analysis for the general case of the arm rotating.

With the coordinate system at the center of rotation of the arm, a general velocity diagram for the case when the arm is rotating is shown in the adjacent schematic.



Taking the moment about the center of rotation, the moment of the inlet flow is zero since the moment arm is zero for the inlet flow.

The basic equation then becomes

$$T_0 = 2\dot{m}_e R\left(V_J \cos\alpha - R\,\omega\right)$$

Note that the net momentum velocity is the difference between the tangential component of the jet exit velocity and the rotational speed of the arm. Also note that the direction of positive moments was taken as the same as for V_J and opposite to the direction of rotation.

For a stationary arm R ω = 0. We thus obtain for the stationary torque

$T_o = 2\ \rho\ Q_J\ R\ V_J \cos\alpha$

$$T_o = 2 * 998 \frac{kg}{m^3} .005 \frac{m^3}{\min} \frac{1\min}{60} m * 4.24 \frac{m}{s} \cos 4.160^o$$

$T_o = 0.0864$ N m clockwise. **ans.**

A resisting torque of 0.0864 N m must be applied in the clockwise direction to keep the arm from rotating in the counterclockwise direction.

c. At $\omega = 30$ rpm, calculate the friction torque T_f

$$\omega = 30 \frac{rev}{\min} 2\pi \frac{rad}{rev} \frac{1\min}{60} = \pi \frac{rad}{s}$$

$$T_o = 2 * 998 \frac{kg}{m^3} 0.005 \frac{m^3}{\min} \frac{1\min}{60} 0.16m \left[4.24 \frac{m}{s} \cos 40^o - .16m * \pi \frac{rad}{s} \right]$$

ans.

Note; The resisting torque decreases as the speed increases.

d. Find the maximum rotational speed.

The maximum rotational speed occurs when the opposing torque is zero and all the moment of momentum goes to the angular rotation. For this case,

$$V_J \cos\theta - R\omega = 0$$

$$\omega = \frac{V_J \cos\theta}{R} = \frac{4.2 \frac{rad}{s} = 193.8\,rpm 4\,m/s \bullet \cos 40}{0.16\,m} = 20.3 \frac{rad}{s} = 193.8\,rpm$$ **ans.**

Review material and examples on moment of momentum.

Energy Equation (Extended Bernoulli Equation)

For energy, we have that

$$B = E = \int_{cv} e\,\rho\, dV \qquad \text{and} \qquad \beta = e = u + \frac{1}{2}V^2 + g\,z$$

From the previous statement of conservation of energy and these definitions, Reynolds transport theorem becomes:

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{cv} e\,\rho\, dV + \int_{A_e} e_e\,\rho_e\,\bar{V}_e \cdot dA_e - \int_{A_i} e_i\,\rho_i\,\bar{V}_i \cdot dA_i$$

After extensive algebra and simplification (see text for detailed development), we obtain:

$$\frac{P_1 - P_2}{\rho\, g} = \frac{V_2^2 - V_1^2}{2\,g} + Z_2 - Z_1 + h_{f,1-2} - h_p$$

$\frac{P_1 - P_2}{\rho g}$ ↓	$\frac{V_2^2 - V_1^2}{2g}$ ↓	$Z_2 - Z_1$ ↓	$h_{f,1-2}$ ↓	h_p ↓
Pressure drop from 1 – 2, in the flow direction	**Pressure drop due to acceleration of the fluid**	**Pressure drop due to elevation change**	**Pressure drop due to frictional head loss**	**Pressure drop due to mechanical work on fluid**

Note: this formulation must be written in the flow direction from 1 - 2 to be consistent with the sign of the mechanical work term and so that $h_{f,1\text{-}2}$ is always a positive term. Also note the following:

- The points 1 and 2 must be specific points along the flow path
- Each term has units of linear dimension, e.g., ft or meters, and $z_2 - z_1$ is positive for z_2 above z_1
- The term $h_{f,1\text{-}2}$ is always positive when written in the flow direction and for internal, pipe flow includes pipe or duct friction losses and fitting or piping component (valves, elbows, etc.) losses,

- The term h_p is positive for pumps and fans (i.e., pumps increase the pressure in the flow direction) and negative for turbines (turbines decrease the pressure in the flow direction)
- For pumps:

$$h_p = \frac{w_s}{g}$$ where w_S = the useful work per unit mass to the fluid

Therefore: $w_S = g\,h_p$ and $\dot{W}_f = \dot{m}w_s = \rho Q g\,h_p$

where $\dot{W}_f$ = the useful power delivered to the fluid

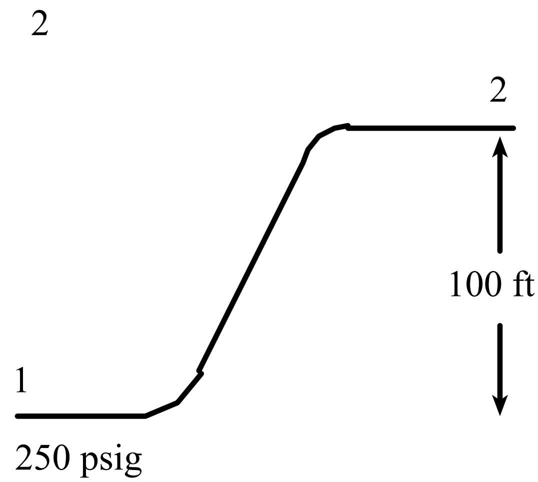
and $\dot{W}_p = \dfrac{\dot{W}_f}{\eta_p}$ where η_p is the pump efficiency

Example

Water flows at 30 ft/s through a 1000 ft length of 2 in diameter pipe. The inlet pressure is 250 psig and the exit is 100 ft higher than the inlet.

Assuming that the frictional loss is given by 18 $V^2/2g$,

Determine the exit pressure.



Given: $V_1 = V_2 = 30$ ft/s, L = 1000 ft, $Z_2 - Z_1 = 100$ ft, P_1 = 250 psig

Also, since there is no mechanical work in the process, the energy equation simplifies to

$$\frac{P_1 - P_2}{\rho g} = Z_2 - Z_1 \quad + \quad h_f$$

$$\frac{P_1 - P_2}{\rho g} = 100\ ft + 18\frac{30^2\ ft^2/s^2}{64.4\ ft/s^2} \quad = \quad 351.8\ ft$$

$$P_1 - P_2 = 62.4\ \text{lbf/ft}^3\ 351.8\ \text{ft} = 21{,}949\ \text{psf} = 152.4\ \text{psi}$$

$$P_2 = 250 - 152.4 = 97.6\ \text{psig} \quad \textbf{ans.}$$

Problem Extension

A pump driven by an electric motor is now added to the system. The motor delivers 10.5 hp. The flow rate and inlet pressure remain constant and the pump efficiency is 71.4 %, determine the new exit pressure.

$$Q = AV = \pi\ \pi\ (1/12)^2\ \text{ft}^2 * 30\ \text{ft/s} = 0.6545\ \text{ft}^3/\text{s}$$

$$W_f = \eta_p\ W_p = \rho\ Q\ g\ h_p$$

$$h_p = \frac{0.714 * 10.5\,hp * 550 ft - lbf/s/hp}{62.4\,lbm/ft^3 * 0.6545 ft^3/s} \quad = 101 ft$$

The pump adds a head increase equal to 101 ft to the system and the exit pressure should increase.

Substituting in the energy equation, we obtain

$$\frac{P_1 - P_2}{\rho g} = 100\ ft + 18\frac{30^2\ ft^2/s^2}{64.4\ ft/s^2} \quad - 101\ ft = \quad 250.8\ ft$$

$$P_1 - P_2 = 62.4\ \text{lbf/ft}^3\ 250.8\ \text{ft} = 15{,}650\ \text{psf} = 108.7\ \text{psi}$$

$$P_2 = 250 - 108.7 = 141.3\ \text{psig} \quad \textbf{ans.}$$

Review examples for the use of the energy equation

Ch. IV Differential Relations for a Fluid Particle

This chapter presents the development and application of the basic differential equations of fluid motion. Simplifications in the general equations and common boundary conditions are presented that allow exact solutions to be obtained. Two of the most common simplifications are 1). steady flow and 2). incompressible flow.

The Acceleration Field of a Fluid

A general expression of the flow field velocity vector is given by:

$$\overline{V}(\bar{r},t) = \hat{i}\,u(x,y,z,t) + \hat{j}\,\mathrm{v}(x,y,z,t) + \hat{k}\,w(x,y,z,t)$$

One of two reference frames can be used to specify the flow field characteristics:

eulerian – the coordinates are fixed and we observe the flow field characteristics as it passes by the fixed coordinates.

lagrangian - the coordinates move through the flow field following individual particles in the flow.

Since the primary equation used in specifying the flow field velocity is based on Newton's second law, the acceleration vector is an important solution parameter. In cartesian coordinates, this is expressed as

$$\bar{a} = \underbrace{\frac{d\overline{V}}{dt}}_{\text{total}} = \underbrace{\frac{\partial\overline{V}}{\partial t}}_{\text{local}} + \underbrace{\left(u\frac{\partial\overline{V}}{\partial x} + \mathrm{v}\frac{\partial\overline{V}}{\partial y} + w\frac{\partial\overline{V}}{\partial z}\right)}_{\text{convective}} = \frac{\partial\overline{V}}{\partial t} + \left(\overline{V}\cdot\overline{\nabla}\right)\overline{V}$$

The acceleration vector is expressed in terms of three types of derivatives:

Total acceleration = total derivative of velocity vector

= local derivative + convective derivative of velocity vector

Likewise, the total derivative (also referred to as the substantial derivative) of other variables can be expressed in a similar form, e.g.,

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \left(u\frac{\partial P}{\partial x} + \mathrm{v}\frac{\partial P}{\partial y} + w\frac{\partial P}{\partial z} \right) = \frac{\partial P}{\partial t} + \left(\overline{\mathrm{V}} \cdot \overline{\nabla} \right) P$$

Example 4.1

Given the eulerian velocity-vector field

$$\overline{V} = 3\,t\,\hat{i} + x\,z\,\hat{j} + t\,y^2\,\hat{k}$$

find the acceleration of the particle.

For the given velocity vector, the individual components are

$$u = 3\,t \qquad \mathrm{v} = x\,z \qquad w = t\,y^2$$

Evaluating the individual components, we obtain

$$\frac{\partial \overline{V}}{\partial t} = 3\,\mathbf{i} + y^2\,\mathbf{k}$$

$$\frac{\partial \overline{V}}{\partial x} = z\,\mathbf{j} \qquad \frac{\partial \overline{V}}{\partial y} = 2\,t\,y\,\mathbf{k} \qquad \frac{\partial \overline{V}}{\partial z} = x\,\mathbf{j}$$

Substituting, we obtain

$$\frac{d\overline{V}}{dt} = (3\,\mathbf{i} + y^2\,\mathbf{k}) + (3\,t)\,(z\,\mathbf{j}) + (x\,z)\,(2\,t\,y\,\mathbf{k}) + (t\,y^2)\,(x\,\mathbf{j})$$

After collecting terms, we have

$$\frac{d\overline{V}}{dt} = 3\,\mathbf{i} + (3\,t\,z + t\,x\,y^2)\,\mathbf{j} + (2\,x\,y\,z\,t + y^2)\,\mathbf{k} \quad \textbf{ans.}$$

The Differential Equation of Conservation of Mass

If we apply the basic concepts of conservation of mass to a differential control volume, we obtain a differential form for the continuity equation in cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \mathrm{v}) + \frac{\partial}{\partial z}(\rho w) = 0$$

and in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho \mathrm{v}_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \mathrm{v}_\theta) + \frac{\partial}{\partial z}(\rho \mathrm{v}_z) = 0$$

Steady Compressible Flow

For steady flow, the term $\frac{\partial}{\partial t} = 0$ and all properties are function of position only.

The previous equations simplify to

Cartesian: $$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \mathrm{v}) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}(r\rho \mathrm{v}_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \mathrm{v}_\theta) + \frac{\partial}{\partial z}(\rho \mathrm{v}_z) = 0$$

Incompressible Flow

For incompressible flow, density changes are negligible, ρ = const., and $\frac{\partial \rho}{\partial t} = 0$

In the two coordinate systems, we have

Cartesian: $$\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\mathrm{v}_r\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\mathrm{v}_\theta\right)+\frac{\partial}{\partial z}\left(\mathrm{v}_z\right)=0$$

Key Point:

It is noted that the assumption of incompressible flow is not restricted to fluids which cannot be compressed, e.g. liquids. Incompressible flow is valid for (1) when the fluid is essentially incompressible (liquids) and (2) for compressible fluids for which compressibility effects are not significant for the problem being considered.

The second case is assumed to be met when the Mach number is less than 0.3:

$Ma = V/c < 0.3$ Gas flows can be considered incompressible

The Differential Equation of Linear Momentum

If we apply Newton's Second Law of Motion to a differential control volume we obtain the three components of the differential equation of linear momentum. In cartesian coordinates, the equations are expressed in the form:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Inviscid Flow: Euler's Equation

If we assume the flow is frictionless, all of the shear stress terms drop out. The resulting equation is known as Euler's equation and in vector form is given by:

$$\rho\mathbf{g} - \nabla \mathrm{P} = \rho\frac{d\mathbf{V}}{dt}$$