(2) $\boldsymbol{\frac{a^m}{a^n} = a^{m-n}}$ For example, $\frac{c^5}{c^2} = c^{5-2} = c^3$

(3) $\boldsymbol{(a^m)^n = a^{mn}}$ For example, $\left(d^2\right)^3 = d^{2\times3} = d^6$

(4) $\boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m}}$ For example, $x^{\frac{4}{3}} = \sqrt[3]{x^4}$

(5) $\boldsymbol{a^{-n} = \frac{1}{a^n}}$ For example, $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(6) $\boldsymbol{a^0 = 1}$ For example, $17^0 = 1$

Here are some worked examples to demonstrate these laws of indices.

Problem 17. Simplify $a^2b^3c \times ab^2c^5$

$$a^2b^3c \times ab^2c^5 = a^2 \times b^3 \times c \times a \times b^2 \times c^5$$
$$= a^2 \times b^3 \times c^1 \times a^1 \times b^2 \times c^5$$

Grouping together like terms gives

$$a^2 \times a^1 \times b^3 \times b^2 \times c^1 \times c^5$$

Using law (1) of indices gives

$$a^{2+1} \times b^{3+2} \times c^{1+5} = a^3 \times b^5 \times c^6$$

i.e. $\boldsymbol{a^2b^3c \times ab^2c^5 = a^3b^5c^6}$

Problem 18. Simplify $a^{\frac{1}{3}}b^{\frac{3}{2}}c^{-2} \times a^{\frac{1}{6}}b^{\frac{1}{2}}c$

Using law (1) of indices,

$$a^{\frac{1}{3}}b^{\frac{3}{2}}c^{-2} \times a^{\frac{1}{6}}b^{\frac{1}{2}}c = a^{\frac{1}{3}+\frac{1}{6}} \times b^{\frac{3}{2}+\frac{1}{2}} \times c^{-2+1}$$
$$= \boldsymbol{a^{\frac{1}{2}}b^2c^{-1}}$$

Problem 19. Simplify $\frac{x^5y^2z}{x^2yz^3}$

$$\frac{x^5y^2z}{x^2yz^3} = \frac{x^5 \times y^2 \times z}{x^2 \times y \times z^3}$$
$$= \frac{x^5}{x^2} \times \frac{y^2}{y^1} \times \frac{z}{z^3}$$
$$= x^{5-2} \times y^{2-1} \times z^{1-3} \quad \text{by law (2) of indices}$$
$$= x^3 \times y^1 \times z^{-2}$$
$$= \boldsymbol{x^3yz^{-2}} \text{ or } \boldsymbol{\frac{x^3y}{z^2}}$$

Problem 20. Simplify $\frac{a^3b^2c^4}{abc^{-2}}$ and evaluate when $a = 3, b = \frac{1}{4}$ and $c = 2$

Using law (2) of indices,

$$\frac{a^3}{a} = a^{3-1} = a^2, \frac{b^2}{b} = b^{2-1} = b \text{ and}$$
$$\frac{c^4}{c^{-2}} = c^{4--2} = c^6$$

Thus, $\frac{a^3b^2c^4}{abc^{-2}} = \boldsymbol{a^2bc^6}$

When $a = 3, b = \frac{1}{4}$ and $c = 2$,

$$a^2bc^6 = (3)^2\left(\frac{1}{4}\right)(2)^6 = (9)\left(\frac{1}{4}\right)(64) = \mathbf{144}$$

Problem 21. Simplify $(p^3)^2(q^2)^4$

Using law (3) of indices gives

$$(p^3)^2(q^2)^4 = p^{3\times2} \times q^{2\times4}$$
$$= \boldsymbol{p^6q^8}$$

Problem 22. Simplify $\frac{(mn^2)^3}{(m^{1/2}n^{1/4})^4}$

The brackets indicate that each letter in the bracket must be raised to the power outside. Using law (3) of indices gives

$$\frac{(mn^2)^3}{(m^{1/2}n^{1/4})^4} = \frac{m^{1\times3}n^{2\times3}}{m^{(1/2)\times4}n^{(1/4)\times4}} = \frac{m^3n^6}{m^2n^1}$$

Using law (2) of indices gives

$$\frac{m^3n^6}{m^2n^1} = m^{3-2}n^{6-1} = \boldsymbol{mn^5}$$

Problem 23. Simplify $(a^3b)(a^{-4}b^{-2})$, expressing the answer with positive indices only

Using law (1) of indices gives $a^{3+-4}b^{1+-2} = a^{-1}b^{-1}$

Using law (5) of indices gives $a^{-1}b^{-1} = \frac{1}{a^{+1}b^{+1}} = \boldsymbol{\frac{1}{ab}}$

Problem 24. Simplify $\frac{d^2e^2f^{1/2}}{(d^{3/2}ef^{5/2})^2}$ expressing the answer with positive indices only

Using law (3) of indices gives

$$\frac{d^2e^2f^{1/2}}{(d^{3/2}e\,f^{5/2})^2} = \frac{d^2e^2f^{1/2}}{d^3e^2f^5}$$

Using law (2) of indices gives

$$d^{2-3}e^{2-2}f^{\frac{1}{2}-5} = d^{-1}e^0f^{-\frac{9}{2}}$$

$$= d^{-1}f^{-\frac{9}{2}} \quad \text{since } e^0 = 1 \text{ from law (6) of indices}$$

$$= \mathbf{\frac{1}{df^{9/2}}} \quad \text{from law (5) of indices}$$

Now try the following Practice Exercise

Practice Exercise 37 Laws of indices (answers on page 343)

In problems 1 to 18, simplify the following, giving each answer as a power.

1. $z^2 \times z^6$
2. $a \times a^2 \times a^5$
3. $n^8 \times n^{-5}$
4. $b^4 \times b^7$
5. $b^2 \div b^5$
6. $c^5 \times c^3 \div c^4$
7. $\dfrac{m^5 \times m^6}{m^4 \times m^3}$
8. $\dfrac{(x^2)(x)}{x^6}$
9. $(x^3)^4$
10. $(y^2)^{-3}$
11. $(t \times t^3)^2$
12. $(c^{-7})^{-2}$
13. $\left(\dfrac{a^2}{a^5}\right)^3$
14. $\left(\dfrac{1}{b^3}\right)^4$
15. $\left(\dfrac{b^2}{b^7}\right)^{-2}$
16. $\dfrac{1}{(s^3)^3}$
17. $p^3qr^2 \times p^2q^5r \times pqr^2$
18. $\dfrac{x^3y^2z}{x^5yz^3}$
19. Simplify $(x^2y^3z)(x^3yz^2)$ and evaluate when $x = \frac{1}{2}$, $y = 2$ and $z = 3$.
20. Simplify $\dfrac{a^5bc^3}{a^2b^3c^2}$ and evaluate when $a = \frac{3}{2}$, $b = \frac{1}{2}$ and $c = \frac{2}{3}$

Here are some further worked examples on the laws of indices

Problem 25. Simplify $\dfrac{p^{1/2}q^2r^{2/3}}{p^{1/4}q^{1/2}r^{1/6}}$ and evaluate when $p = 16$, $q = 9$ and $r = 4$, taking positive roots only

Using law (2) of indices gives $p^{\frac{1}{2}-\frac{1}{4}}q^{2-\frac{1}{2}}r^{\frac{2}{3}-\frac{1}{6}}$

$$p^{\frac{1}{2}-\frac{1}{4}}q^{2-\frac{1}{2}}r^{\frac{2}{3}-\frac{1}{6}} = \mathbf{p^{\frac{1}{4}}q^{\frac{3}{2}}r^{\frac{1}{2}}}$$

When $p = 16$, $q = 9$ and $r = 4$,

$$p^{\frac{1}{4}}q^{\frac{3}{2}}r^{\frac{1}{2}} = 16^{\frac{1}{4}}9^{\frac{3}{2}}4^{\frac{1}{2}}$$

$$= (\sqrt[4]{16})(\sqrt{9^3})(\sqrt{4}) \text{ from law (4) of indices}$$

$$= (2)(3^3)(2) = \mathbf{108}$$

Problem 26. Simplify $\dfrac{x^2y^3 + xy^2}{xy}$

Algebraic expressions of the form $\dfrac{a+b}{c}$ can be split into $\dfrac{a}{c} + \dfrac{b}{c}$. Thus,

$$\frac{x^2y^3 + xy^2}{xy} = \frac{x^2y^3}{xy} + \frac{xy^2}{xy} = x^{2-1}y^{3-1} + x^{1-1}y^{2-1}$$

$$= \mathbf{xy^2 + y}$$

(since $x^0 = 1$, from law (6) of indices).

Problem 27. Simplify $\dfrac{x^2y}{xy^2 - xy}$

The highest common factor (HCF) of each of the three terms comprising the numerator and denominator is xy. Dividing each term by xy gives

$$\frac{x^2y}{xy^2 - xy} = \frac{\dfrac{x^2y}{xy}}{\dfrac{xy^2}{xy} - \dfrac{xy}{xy}} = \mathbf{\frac{x}{y-1}}$$

Problem 28. Simplify $\dfrac{a^2b}{ab^2 - a^{1/2}b^3}$

The HCF of each of the three terms is $a^{1/2}b$. Dividing each term by $a^{1/2}b$ gives

$$\frac{a^2b}{ab^2-a^{1/2}b^3}=\frac{\dfrac{a^2b}{a^{1/2}b}}{\dfrac{ab^2}{a^{1/2}b}-\dfrac{a^{1/2}b^3}{a^{1/2}b}}=\frac{\boldsymbol{a^{3/2}}}{\boldsymbol{a^{1/2}b-b^2}}$$

Problem 29. Simplify $(a^3\sqrt{b}\sqrt{c^5})(\sqrt{a}\sqrt[3]{b^2}c^3)$ and evaluate when $a=\dfrac{1}{4}$, $b=6$ and $c=1$

Using law (4) of indices, the expression can be written as

$$(a^3\sqrt{b}\sqrt{c^5})(\sqrt{a}\sqrt[3]{b^2}c^3)=\left(a^3\,b^{\frac{1}{2}}\,c^{\frac{5}{2}}\right)\left(a^{\frac{1}{2}}\,b^{\frac{2}{3}}\,c^3\right)$$

Using law (1) of indices gives

$$\left(a^3\,b^{\frac{1}{2}}\,c^{\frac{5}{2}}\right)\left(a^{\frac{1}{2}}\,b^{\frac{2}{3}}\,c^3\right)=a^{3+\frac{1}{2}}\,b^{\frac{1}{2}+\frac{2}{3}}\,c^{\frac{5}{2}+3}$$

$$=a^{\frac{7}{2}}\,b^{\frac{7}{6}}\,c^{\frac{11}{2}}$$

It is usual to express the answer in the same form as the question. Hence,

$$a^{\frac{7}{2}}\,b^{\frac{7}{6}}\,c^{\frac{11}{2}}=\boldsymbol{\sqrt{a^7}\sqrt[6]{b^7}\sqrt{c^{11}}}$$

When $a=\dfrac{1}{4}$, $b=64$ and $c=1$,

$$\sqrt{a^7}\sqrt[6]{b^7}\sqrt{c^{11}}=\sqrt{\left(\frac{1}{4}\right)^7}\left(\sqrt[6]{64^7}\right)\left(\sqrt{1^{11}}\right)$$

$$=\left(\frac{1}{2}\right)^7(2)^7(1)=\mathbf{1}$$

Problem 30. Simplify $\dfrac{(x^2y^{1/2})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{1/2}}$

Using laws (3) and (4) of indices gives

$$\frac{(x^2y^{1/2})\left(\sqrt{x}\sqrt[3]{y^2}\right)}{(x^5y^3)^{1/2}}=\frac{(x^2y^{1/2})\left(x^{1/2}y^{2/3}\right)}{x^{5/2}y^{3/2}}$$

Using laws (1) and (2) of indices gives

$$x^{2+\frac{1}{2}-\frac{5}{2}}\,y^{\frac{1}{2}+\frac{2}{3}-\frac{3}{2}}=x^0y^{-\frac{1}{3}}=\boldsymbol{y^{-\frac{1}{3}}}\text{ or }\boldsymbol{\frac{1}{y^{1/3}}}\text{ or }\boldsymbol{\frac{1}{\sqrt[3]{y}}}$$

from laws (5) and (6) of indices.

Now try the following Practice Exercise

Practice Exercise 38 Laws of indices (answers on page 343)

1. Simplify $\left(a^{3/2}bc^{-3}\right)\left(a^{1/2}b^{-1/2}c\right)$ and evaluate when $a=3$, $b=4$ and $c=2$.

In problems 2 to 5, simplify the given expressions.

2. $\dfrac{a^2b+a^3b}{a^2b^2}$
3. $(a^2)^{1/2}(b^2)^3\left(c^{1/2}\right)^3$
4. $\dfrac{(abc)^2}{(a^2b^{-1}c^{-3})^3}$
5. $\dfrac{p^3q^2}{pq^2-p^2q}$
6. $(\sqrt{x}\sqrt{y^3}\sqrt[3]{z^2})(\sqrt{x}\sqrt{y^3}\sqrt{z^3})$
7. $(e^2f^3)(e^{-3}f^{-5})$, expressing the answer with positive indices only.
8. $\dfrac{(a^3b^{1/2}c^{-1/2})(ab)^{1/3}}{(\sqrt{a^3}\sqrt{b}\,c)}$

Chapter 10

Further algebra

10.1 Introduction

In this chapter, the use of brackets and factorization with algebra is explained, together with further practice with the laws of precedence. Understanding of these topics is often necessary when solving and transposing equations.

10.2 Brackets

With algebra

(a) $2(a+b)=2a+2b$

(b) $$\begin{aligned}(a+b)(c+d) &= a(c+d)+b(c+d)\\ &= ac+ad+bc+bd\end{aligned}$$

Here are some worked examples to help understanding of brackets with algebra.

Problem 1. Determine $2b(a-5b)$

$$\begin{aligned}2b(a-5b) &= 2b\times a+2b\times -5b\\ &= 2ba-10b^2\\ &= \mathbf{2ab-10b^2}\end{aligned}$$
(note that $2ba$ is the same as $2ab$)

Problem 2. Determine $(3x+4y)(x-y)$

$$\begin{aligned}(3x+4y)(x-y) &= 3x(x-y)+4y(x-y)\\ &= 3x^2-3xy+4yx-4y^2\\ &= 3x^2-3xy+4xy-4y^2\end{aligned}$$
(note that $4yx$ is the same as $4xy$)
$$= \mathbf{3x^2+xy-4y^2}$$

Problem 3. Simplify $3(2x-3y)-(3x-y)$

$$3(2x-3y)-(3x-y)=3\times 2x-3\times 3y-3x--y$$

(Note that $-(3x-y)=-1(3x-y)$ and the -1 multiplies **both** terms in the bracket)

$$=6x-9y-3x+y$$
(Note: $-\times-=+$)
$$\begin{aligned}&=6x-3x+y-9y\\ &=\mathbf{3x-8y}\end{aligned}$$

Problem 4. Remove the brackets and simplify the expression $(a-2b)+5(b-c)-3(c+2d)$

$$\begin{aligned}&(a-2b)+5(b-c)-3(c+2d)\\ &\quad =a-2b+5\times b+5\times -c-3\times c-3\times 2d\\ &\quad =a-2b+5b-5c-3c-6d\\ &\quad =\mathbf{a+3b-8c-6d}\end{aligned}$$

Problem 5. Simplify $(p+q)(p-q)$

$$\begin{aligned}(p+q)(p-q) &= p(p-q)+q(p-q)\\ &= p^2-pq+qp-q^2\\ &= \mathbf{p^2-q^2}\end{aligned}$$

Problem 6. Simplify $(2x-3y)^2$

$$\begin{aligned}(2x-3y)^2 &= (2x-3y)(2x-3y)\\ &= 2x(2x-3y)-3y(2x-3y)\\ &= 2x\times 2x+2x\times -3y-3y\times 2x\\ &\qquad -3y\times -3y\\ &= 4x^2-6xy-6xy+9y^2\end{aligned}$$
(Note: $+\times-=-$ and $-\times-=+$)
$$= \mathbf{4x^2-12xy+9y^2}$$

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Problem 7. Remove the brackets from the expression and simplify $2[x^2 - 3x(y + x) + 4xy]$

$2[x^2 - 3x(y + x) + 4xy] = 2[x^2 - 3xy - 3x^2 + 4xy]$

(Whenever more than one type of brackets is involved, always **start with the inner brackets**)

$$= 2[-2x^2 + xy]$$
$$= -4x^2 + 2xy$$
$$= \mathbf{2xy - 4x^2}$$

Problem 8. Remove the brackets and simplify the expression $2a - [3\{2(4a - b) - 5(a + 2b)\} + 4a]$

(i) Removing the innermost brackets gives

$$2a - [3\{8a - 2b - 5a - 10b\} + 4a]$$

(ii) Collecting together similar terms gives

$$2a - [3\{3a - 12b\} + 4a]$$

(iii) Removing the 'curly' brackets gives

$$2a - [9a - 36b + 4a]$$

(iv) Collecting together similar terms gives

$$2a - [13a - 36b]$$

(v) Removing the outer brackets gives

$$2a - 13a + 36b$$

(vi) i.e. $\mathbf{-11a + 36b}$ or $\mathbf{36b - 11a}$

Now try the following Practice Exercise

Practice Exercise 39 Brackets (answers on page 344)

Expand the brackets in problems 1 to 28.

1. $(x + 2)(x + 3)$
2. $(x + 4)(2x + 1)$
3. $(2x + 3)^2$
4. $(2j - 4)(j + 3)$
5. $(2x + 6)(2x + 5)$
6. $(pq + r)(r + pq)$
7. $(a + b)(a + b)$
8. $(x + 6)^2$
9. $(a - c)^2$
10. $(5x + 3)^2$
11. $(2x - 6)^2$
12. $(2x - 3)(2x + 3)$
13. $(8x + 4)^2$
14. $(rs + t)^2$
15. $3a(b - 2a)$
16. $2x(x - y)$
17. $(2a - 5b)(a + b)$
18. $3(3p - 2q) - (q - 4p)$
19. $(3x - 4y) + 3(y - z) - (z - 4x)$
20. $(2a + 5b)(2a - 5b)$
21. $(x - 2y)^2$
22. $(3a - b)^2$
23. $2x + [y - (2x + y)]$
24. $3a + 2[a - (3a - 2)]$
25. $4[a^2 - 3a(2b + a) + 7ab]$
26. $3[x^2 - 2x(y + 3x) + 3xy(1 + x)]$
27. $2 - 5[a(a - 2b) - (a - b)^2]$
28. $24p - [2\{3(5p - q) - 2(p + 2q)\} + 3q]$

10.3 Factorization

The **factors** of 8 are 1, 2, 4 and 8 because 8 divides by 1, 2, 4 and 8.
The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24 because 24 divides by 1, 2, 3, 4, 6, 8, 12 and 24.
The **common factors** of 8 and 24 are 1, 2, 4 and 8 since 1, 2, 4 and 8 are factors of both 8 and 24.
The **highest common factor (HCF)** is the largest number that divides into two or more terms.
Hence, the HCF of 8 and 24 is 8, as explained in Chapter 1.
When two or more terms in an algebraic expression contain a common factor, then this factor can be shown outside of a bracket. For example,

$$df + dg = d(f + g)$$

which is just the reverse of

$$d(f + g) = df + dg$$

This process is called **factorization**.
Here are some worked examples to help understanding of factorizing in algebra.

Problem 9. Factorize $ab - 5ac$

a is common to both terms ab and $-5ac$. a is therefore taken outside of the bracket. What goes inside the bracket?

(i) What multiplies a to make ab? Answer: b

(ii) What multiplies a to make $-5ac$? Answer: $-5c$

Hence, $b-5c$ appears in the bracket. Thus,

$$\mathbf{ab-5ac=a(b-5c)}$$

Problem 10. Factorize $2x^2+14xy^3$

For the numbers 2 and 14, the highest common factor (HCF) is 2 (i.e. 2 is the largest number that divides into both 2 and 14).
For the x terms, x^2 and x, the HCF is x.
Thus, the HCF of $2x^2$ and $14xy^3$ is $2x$.
$2x$ is therefore taken outside of the bracket. What goes inside the bracket?

(i) What multiplies $2x$ to make $2x^2$? Answer: x

(ii) What multiplies $2x$ to make $14xy^3$? Answer: $7y^3$

Hence $x+7y^3$ appears inside the bracket. Thus,

$$\mathbf{2x^2+14xy^3=2x(x+7y^3)}$$

Problem 11. Factorize $3x^3y-12xy^2+15xy$

For the numbers 3, 12 and 15, the highest common factor is 3 (i.e. 3 is the largest number that divides into 3, 12 and 15).
For the x terms, x^3, x and x, the HCF is x.
For the y terms, y, y^2 and y, the HCF is y.
Thus, the HCF of $3x^3y$ and $12xy^2$ and $15xy$ is $3xy$.
$3xy$ is therefore taken outside of the bracket. What goes inside the bracket?

(i) What multiplies $3xy$ to make $3x^3y$? Answer: x^2

(ii) What multiplies $3xy$ to make $-12xy^2$? Answer: $-4y$

(iii) What multiplies $3xy$ to make $15xy$? Answer: 5

Hence, x^2-4y+5 appears inside the bracket. Thus,

$$\mathbf{3x^3y-12xy^2+15xy=3xy(x^2-4y+5)}$$

Problem 12. Factorize $25a^2b^5-5a^3b^2$

For the numbers 25 and 5, the highest common factor is 5 (i.e. 5 is the largest number that divides into 25 and 5).
For the a terms, a^2 and a^3, the HCF is a^2.
For the b terms, b^5 and b^2, the HCF is b^2.
Thus, the HCF of $25a^2b^5$ and $5a^3b^2$ is $5a^2b^2$.
$5a^2b^2$ is therefore taken outside of the bracket. What goes inside the bracket?

(i) What multiplies $5a^2b^2$ to make $25a^2b^5$? Answer: $5b^3$

(ii) What multiplies $5a^2b^2$ to make $-5a^3b^2$? Answer: $-a$

Hence, $5b^3-a$ appears in the bracket. Thus,

$$\mathbf{25a^2b^5-5a^3b^2=5a^2b^2(5b^3-a)}$$

Problem 13. Factorize $ax-ay+bx-by$

The first two terms have a common factor of a and the last two terms a common factor of b. Thus,

$$ax-ay+bx-by=a(x-y)+b(x-y)$$

The two newly formed terms have a common factor of $(x-y)$. Thus,

$$\mathbf{a(x-y)+b(x-y)=(x-y)(a+b)}$$

Problem 14. Factorize $2ax-3ay+2bx-3by$

a is a common factor of the first two terms and b a common factor of the last two terms. Thus,

$$2ax-3ay+2bx-3by=a(2x-3y)+b(2x-3y)$$

$(2x-3y)$ is now a common factor. Thus,

$$a(2x-3y)+b(2x-3y)=\mathbf{(2x-3y)(a+b)}$$

Alternatively, $2x$ is a common factor of the original first and third terms and $-3y$ is a common factor of the second and fourth terms. Thus,

$$2ax-3ay+2bx-3by=2x(a+b)-3y(a+b)$$

$(a+b)$ is now a common factor. Thus,

$$2x(a+b)-3y(a+b)=\mathbf{(a+b)(2x-3y)}$$

as before

Problem 15. Factorize x^3+3x^2-x-3

x^2 is a common factor of the first two terms. Thus,

$$x^3+3x^2-x-3=x^2(x+3)-x-3$$

-1 is a common factor of the last two terms. Thus,

$$x^2(x+3)-x-3=x^2(x+3)-1(x+3)$$

$(x+3)$ is now a common factor. Thus,

$$x^2(x+3)-1(x-3)=\mathbf{(x+3)(x^2-1)}$$

Now try the following Practice Exercise

Practice Exercise 40 Factorization (answers on page 344)

Factorize and simplify the following.

1. $2x+4$
2. $2xy-8xz$
3. $pb+2pc$
4. $2x+4xy$
5. $4d^2-12df^5$
6. $4x+8x^2$
7. $2q^2+8qn$
8. $rs+rp+rt$
9. $x+3x^2+5x^3$
10. $abc+b^3c$
11. $3x^2y^4-15xy^2+18xy$
12. $4p^3q^2-10pq^3$
13. $21a^2b^2-28ab$
14. $2xy^2+6x^2y+8x^3y$
15. $2x^2y-4xy^3+8x^3y^4$
16. $28y+7y^2+14xy$
17. $\dfrac{3x^2+6x-3xy}{xy+2y-y^2}$
18. $\dfrac{abc+2ab}{2c+4}-\dfrac{abc}{2c}$
19. $\dfrac{5rs+15r^3t+20r}{6r^2t^2+8t+2ts}-\dfrac{r}{2t}$
20. $ay+by+a+b$
21. $px+qx+py+qy$
22. $ax-ay+bx-by$
23. $2ax+3ay-4bx-6by$

10.4 Laws of precedence

Sometimes addition, subtraction, multiplication, division, powers and brackets can all be involved in an algebraic expression. With mathematics there is a definite order of precedence (first met in Chapter 1) which we need to adhere to.
With the **laws of precedence** the order is

Brackets

Order (or p**O**wer)

Division

Multiplication

Addition

Subtraction

The first letter of each word spells **BODMAS**.
Here are some examples to help understanding of BODMAS with algebra.

Problem 16. Simplify $2x+3x\times 4x-x$

$$\begin{aligned}2x+3x\times 4x-x &= 2x+12x^2-x && \text{(M)}\\ &= 2x-x+12x^2 &&\\ &= \mathbf{x+12x^2} && \text{(S)}\\ \text{or } & \mathbf{x(1+12x)} && \text{by factorizing}\end{aligned}$$

Problem 17. Simplify $(y+4y)\times 3y-5y$

$$\begin{aligned}(y+4y)\times 3y-5y &= 5y\times 3y-5y && \text{(B)}\\ &= \mathbf{15y^2-5y} && \text{(M)}\\ \text{or } & \mathbf{5y(3y-1)} && \text{by factorizing}\end{aligned}$$

Problem 18. Simplify $p+2p\times(4p-7p)$

$$\begin{aligned}p+2p\times(4p-7p) &= p+2p\times -3p && \text{(B)}\\ &= \mathbf{p-6p^2} && \text{(M)}\\ \text{or } & \mathbf{p(1-6p)} && \text{by factorizing}\end{aligned}$$

Problem 19. Simplify $t\div 2t+3t-5t$

$$\begin{aligned}t\div 2t+3t-5t &= \frac{t}{2t}+3t-5t && \text{(D)}\\ &= \frac{1}{2}+3t-5t && \text{by cancelling}\\ &= \mathbf{\frac{1}{2}-2t} && \text{(S)}\end{aligned}$$

Problem 20. Simplify $x\div(4x+x)-3x$

$$\begin{aligned}x\div(4x+x)-3x &= x\div 5x-3x && \text{(B)}\\ &= \frac{x}{5x}-3x && \text{(D)}\\ &= \mathbf{\frac{1}{5}-3x} && \text{by cancelling}\end{aligned}$$

Problem 21. Simplify $2y\div(6y+3y-5y)$

$$\begin{aligned}2y\div(6y+3y-5y) &= 2y\div 4y && \text{(B)}\\ &= \frac{2y}{4y} && \text{(D)}\\ &= \mathbf{\frac{1}{2}} && \text{by cancelling}\end{aligned}$$

Problem 22. Simplify
$$5a + 3a \times 2a + a \div 2a - 7a$$

$$\begin{aligned}
&5a + 3a \times 2a + a \div 2a - 7a \\
&\quad = 5a + 3a \times 2a + \frac{a}{2a} - 7a && \text{(D)} \\
&\quad = 5a + 3a \times 2a + \frac{1}{2} - 7a && \text{by cancelling} \\
&\quad = 5a + 6a^2 + \frac{1}{2} - 7a && \text{(M)} \\
&\quad = -2a + 6a^2 + \frac{1}{2} && \text{(S)} \\
&\quad = \mathbf{6a^2 - 2a + \frac{1}{2}}
\end{aligned}$$

Problem 23. Simplify
$$(4y + 3y)2y + y \div 4y - 6y$$

$$\begin{aligned}
&(4y + 3y)2y + y \div 4y - 6y \\
&\quad = 7y \times 2y + y \div 4y - 6y && \text{(B)} \\
&\quad = 7y \times 2y + \frac{y}{4y} - 6y && \text{(D)} \\
&\quad = 7y \times 2y + \frac{1}{4} - 6y && \text{by cancelling} \\
&\quad = \mathbf{14y^2 + \frac{1}{4} - 6y} && \text{(M)}
\end{aligned}$$

Problem 24. Simplify
$$5b + 2b \times 3b + b \div (4b - 7b)$$

$$\begin{aligned}
&5b + 2b \times 3b + b \div (4b - 7b) \\
&\quad = 5b + 2b \times 3b + b \div -3b && \text{(B)} \\
&\quad = 5b + 2b \times 3b + \frac{b}{-3b} && \text{(D)} \\
&\quad = 5b + 2b \times 3b + \frac{1}{-3} && \text{by cancelling} \\
&\quad = 5b + 2b \times 3b - \frac{1}{3} \\
&\quad = \mathbf{5b + 6b^2 - \frac{1}{3}} && \text{(M)}
\end{aligned}$$

Problem 25. Simplify
$$(5p + p)(2p + 3p) \div (4p - 5p)$$

$$\begin{aligned}
&(5p + p)(2p + 3p) \div (4p - 5p) \\
&\quad = (6p)(5p) \div (-p) && \text{(B)} \\
&\quad = 6p \times 5p \div -p \\
&\quad = 6p \times \frac{5p}{-p} && \text{(D)} \\
&\quad = 6p \times \frac{5}{-1} && \text{by cancelling} \\
&\quad = 6p \times -5 \\
&\quad = \mathbf{-30p}
\end{aligned}$$

Now try the following Practice Exercise

Practice Exercise 41 Laws of precedence (answers on page 344)

Simplify the following.

1. $3x + 2x \times 4x - x$
2. $(2y + y) \times 4y - 3y$
3. $4b + 3b \times (b - 6b)$
4. $8a \div 2a + 6a - 3a$
5. $6x \div (3x + x) - 4x$
6. $4t \div (5t - 3t + 2t)$
7. $3y + 2y \times 5y + 2y \div 8y - 6y$
8. $(x + 2x)3x + 2x \div 6x - 4x$
9. $5a + 2a \times 3a + a \div (2a - 9a)$
10. $(3t + 2t)(5t + t) \div (t - 3t)$
11. $x \div 5x - x + (2x - 3x)x$
12. $3a + 2a \times 5a + 4a \div 2a - 6a$

Chapter 11

Solving simple equations

11.1 Introduction

$3x - 4$ is an example of an **algebraic expression**.
$3x - 4 = 2$ is an example of an **algebraic equation** (i.e. it contains an '$=$' sign).
An equation is simply a statement that two expressions are equal.
Hence, $A = \pi r^2$ (where A is the area of a circle of radius r)
$F = \frac{9}{5}C + 32$ (which relates Fahrenheit and Celsius temperatures)
and $y = 3x + 2$ (which is the equation of a straight line graph)
are all examples of equations.

11.2 Solving equations

To '**solve an equation**' means '**to find the value of the unknown**'. For example, solving $3x - 4 = 2$ means that the value of x is required.
In this example, $x = 2$. How did we arrive at $x = 2$? This is the purpose of this chapter – to show how to solve such equations.
Many equations occur in engineering and it is essential that we can solve them when needed.
Here are some examples to demonstrate how simple equations are solved.

Problem 1. Solve the equation $4x = 20$

Dividing each side of the equation by 4 gives

$$\frac{4x}{4} = \frac{20}{4}$$

i.e. $\boldsymbol{x = 5}$ by cancelling, which is the solution to the equation $4x = 20$.

The same operation **must** be applied to both sides of an equation so that the equality is maintained.
We can do anything we like to an equation, **as long as we do the same to both sides.** This is, in fact, the only rule to remember when solving simple equations (and also when transposing formulae, which we do in Chapter 12).

Problem 2. Solve the equation $\frac{2x}{5} = 6$

Multiplying both sides by 5 gives $5\left(\frac{2x}{5}\right) = 5(6)$
Cancelling and removing brackets gives $2x = 30$
Dividing both sides of the equation by 2 gives

$$\frac{2x}{2} = \frac{30}{2}$$

Cancelling gives $\boldsymbol{x = 15}$
which is the solution of the equation $\frac{2x}{5} = 6$.

Problem 3. Solve the equation $a - 5 = 8$

Adding 5 to both sides of the equation gives

$$a - 5 + 5 = 8 + 5$$

i.e. $a = 8 + 5$
i.e. $\boldsymbol{a = 13}$
which is the solution of the equation $a - 5 = 8$.
Note that adding 5 to both sides of the above equation results in the -5 moving from the LHS to the RHS, but the sign is changed to $+$.

Problem 4. Solve the equation $x + 3 = 7$

Subtracting 3 from both sides gives $x + 3 - 3 = 7 - 3$
i.e. $x = 7 - 3$
i.e. $\boldsymbol{x = 4}$
which is the solution of the equation $x + 3 = 7$.

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Note that subtracting 3 from both sides of the above equation results in the +3 moving from the LHS to the RHS, but the sign is changed to –. So, we can move straight from $x+3=7$ to $x=7-3$.
Thus, a term can be moved from one side of an equation to the other **as long as a change in sign is made**.

Problem 5. Solve the equation $6x+1=2x+9$

In such equations the terms containing x are grouped on one side of the equation and the remaining terms grouped on the other side of the equation. As in Problems 3 and 4, changing from one side of an equation to the other must be accompanied by a change of sign.

Since $\quad 6x+1=2x+9$

then $\quad 6x-2x=9-1$

i.e. $\quad 4x=8$

Dividing both sides by 4 gives $\quad \dfrac{4x}{4}=\dfrac{8}{4}$

Cancelling gives $\quad \boldsymbol{x=2}$

which is the solution of the equation $6x+1=2x+9$.
In the above examples, the solutions can be checked. Thus, in Problem 5, where $6x+1=2x+9$, if $x=2$, then

$$\text{LHS of equation} = 6(2)+1=13$$

$$\text{RHS of equation} = 2(2)+9=13$$

Since the left hand side (LHS) equals the right hand side (RHS) then $x=2$ must be the correct solution of the equation.
When solving simple equations, always check your answers by substituting your solution back into the original equation.

Problem 6. Solve the equation $4-3p=2p-11$

In order to keep the p term positive the terms in p are moved to the RHS and the constant terms to the LHS.
Similar to Problem 5, if $4-3p=2p-11$

then $\quad 4+11=2p+3p$

i.e. $\quad 15=5p$

Dividing both sides by 5 gives $\quad \dfrac{15}{5}=\dfrac{5p}{5}$

Cancelling gives $\quad \boldsymbol{3=p}$ or $\boldsymbol{p=3}$

which is the solution of the equation $4-3p=2p-11$.

By substituting $p=3$ into the original equation, the solution may be checked.

$$\text{LHS} = 4-3(3)=4-9=-5$$

$$\text{RHS} = 2(3)-11=6-11=-5$$

Since LHS = RHS, the solution $p=3$ must be correct.
If, in this example, the unknown quantities had been grouped initially on the LHS instead of the RHS, then $-3p-2p=-11-4$

i.e. $\quad -5p=-15$

from which, $\quad \dfrac{-5p}{-5}=\dfrac{-15}{-5}$

and $\quad \boldsymbol{p=3}$

as before.
It is often easier, however, to work with positive values where possible.

Problem 7. Solve the equation $3(x-2)=9$

Removing the bracket gives $\quad 3x-6=9$

Rearranging gives $\quad 3x=9+6$

i.e. $\quad 3x=15$

Dividing both sides by 3 gives $\quad \boldsymbol{x=5}$

which is the solution of the equation $3(x-2)=9$.
The equation may be checked by substituting $x=5$ back into the original equation.

Problem 8. Solve the equation $4(2r-3)-2(r-4)=3(r-3)-1$

Removing brackets gives

$$8r-12-2r+8=3r-9-1$$

Rearranging gives $\quad 8r-2r-3r=-9-1+12-8$

i.e. $\quad 3r=-6$

Dividing both sides by 3 gives $\quad \boldsymbol{r}=\dfrac{-6}{3}=\boldsymbol{-2}$

which is the solution of the equation

$4(2r-3)-2(r-4)=3(r-3)-1$.

The solution may be checked by substituting $r=-2$ back into the original equation.

LHS $=4(-4-3)-2(-2-4)=-28+12=-16$

RHS $=3(-2-3)-1=-15-1=-16$

Since LHS = RHS then $r=-2$ is the correct solution.

Now try the following Practice Exercise

Practice Exercise 42 Solving simple equations (answers on page 344)

Solve the following equations.

1. $2x+5=7$
2. $8-3t=2$
3. $\frac{2}{3}c-1=3$
4. $2x-1=5x+11$
5. $7-4p=2p-5$
6. $2.6x-1.3=0.9x+0.4$
7. $2a+6-5a=0$
8. $3x-2-5x=2x-4$
9. $20d-3+3d=11d+5-8$
10. $2(x-1)=4$
11. $16=4(t+2)$
12. $5(f-2)-3(2f+5)+15=0$
13. $2x=4(x-3)$
14. $6(2-3y)-42=-2(y-1)$
15. $2(3g-5)-5=0$
16. $4(3x+1)=7(x+4)-2(x+5)$
17. $11+3(r-7)=16-(r+2)$
18. $8+4(x-1)-5(x-3)=2(5-2x)$

Here are some further worked examples on solving simple equations.

Problem 9. Solve the equation $\frac{4}{x}=\frac{2}{5}$

The lowest common multiple (LCM) of the denominators, i.e. the lowest algebraic expression that both x and 5 will divide into, is $5x$.

Multiplying both sides by $5x$ gives

$$5x\left(\frac{4}{x}\right)=5x\left(\frac{2}{5}\right)$$

Cancelling gives $\quad 5(4)=x(2)$

i.e. $\quad 20=2x \qquad (1)$

Dividing both sides by 2 gives $\quad \frac{20}{2}=\frac{2x}{2}$

Cancelling gives $\quad \mathbf{10=x}$ or $\mathbf{x=10}$

which is the solution of the equation $\frac{4}{x}=\frac{2}{5}$

When there is just one fraction on each side of the equation as in this example, there is a quick way to arrive at equation (1) without needing to find the LCM of the denominators.

We can move from $\frac{4}{x}=\frac{2}{5}$ to $4\times 5=2\times x$ by what is called '**cross-multiplication**'.

In general, if $\frac{a}{b}=\frac{c}{d}$ then $ad=bc$

We can use cross-multiplication when there is one fraction only on each side of the equation.

Problem 10. Solve the equation $\frac{3}{t-2}=\frac{4}{3t+4}$

Cross-multiplication gives $\quad 3(3t+4)=4(t-2)$

Removing brackets gives $\quad 9t+12=4t-8$

Rearranging gives $\quad 9t-4t=-8-12$

i.e. $\quad 5t=-20$

Dividing both sides by 5 gives $\quad \mathbf{t}=\frac{-20}{5}=\mathbf{-4}$

which is the solution of the equation $\frac{3}{t-2}=\frac{4}{3t+4}$

Problem 11. Solve the equation

$$\frac{2y}{5}+\frac{3}{4}+5=\frac{1}{20}-\frac{3y}{2}$$

The lowest common multiple (LCM) of the denominators is 20; i.e., the lowest number that 4, 5, 20 and 2 will divide into.

Multiplying each term by 20 gives

$$20\left(\frac{2y}{5}\right)+20\left(\frac{3}{4}\right)+20(5)=20\left(\frac{1}{20}\right)-20\left(\frac{3y}{2}\right)$$

Cancelling gives $\quad 4(2y)+5(3)+100=1-10(3y)$

i.e. $\quad 8y+15+100=1-30y$

Rearranging gives $\quad 8y+30y=1-15-100$

i.e. $38y = -114$

Dividing both sides by 38 gives $\frac{38y}{38} = \frac{-114}{38}$

Cancelling gives $\boldsymbol{y = -3}$

which is the solution of the equation

$$\frac{2y}{5} + \frac{3}{4} + 5 = \frac{1}{20} - \frac{3y}{2}$$

Problem 12. Solve the equation $\sqrt{x} = 2$

Whenever square root signs are involved in an equation, both sides of the equation must be squared.

Squaring both sides gives $\left(\sqrt{x}\right)^2 = (2)^2$

i.e. $\boldsymbol{x = 4}$

which is the solution of the equation $\sqrt{x} = 2$.

Problem 13. Solve the equation $2\sqrt{d} = 8$

Whenever square roots are involved in an equation, the square root term needs to be isolated on its own before squaring both sides.

Cross-multiplying gives $\sqrt{d} = \frac{8}{2}$

Cancelling gives $\sqrt{d} = 4$

Squaring both sides gives $\left(\sqrt{d}\right)^2 = (4)^2$

i.e. $\boldsymbol{d = 16}$

which is the solution of the equation $2\sqrt{d} = 8$.

Problem 14. Solve the equation $\left(\frac{\sqrt{b}+3}{\sqrt{b}}\right) = 2$

Cross-multiplying gives $\sqrt{b} + 3 = 2\sqrt{b}$

Rearranging gives $3 = 2\sqrt{b} - \sqrt{b}$

i.e. $3 = \sqrt{b}$

Squaring both sides gives $\boldsymbol{9 = b}$

which is the solution of the equation $\left(\frac{\sqrt{b}+3}{\sqrt{b}}\right) = 2$.

Problem 15. Solve the equation $x^2 = 25$

Whenever a square term is involved, the square root of both sides of the equation must be taken.

Taking the square root of both sides gives $\sqrt{x^2} = \sqrt{25}$

i.e. $\boldsymbol{x = \pm 5}$

which is the solution of the equation $x^2 = 25$.

Problem 16. Solve the equation $\frac{15}{4t^2} = \frac{2}{3}$

We need to rearrange the equation to get the t^2 term on its own.

Cross-multiplying gives $15(3) = 2(4t^2)$

i.e. $45 = 8t^2$

Dividing both sides by 8 gives $\frac{45}{8} = \frac{8t^2}{8}$

By cancelling $5.625 = t^2$

or $t^2 = 5.625$

Taking the square root of both sides gives

$$\sqrt{t^2} = \sqrt{5.625}$$

i.e. $\boldsymbol{t = \pm 2.372}$

correct to 4 significant figures, which is the solution of the equation $\frac{15}{4t^2} = \frac{2}{3}$

Now try the following Practice Exercise

Practice Exercise 43 Solving simple equations (answers on page 344)

(answers on page 344)

Solve the following equations.

1. $\frac{1}{5}d + 3 = 4$
2. $2 + \frac{3}{4}y = 1 + \frac{2}{3}y + \frac{5}{6}$
3. $\frac{1}{4}(2x - 1) + 3 = \frac{1}{2}$
4. $\frac{1}{5}(2f - 3) + \frac{1}{6}(f - 4) + \frac{2}{15} = 0$
5. $\frac{1}{3}(3m - 6) - \frac{1}{4}(5m + 4) + \frac{1}{5}(2m - 9) = -3$
6. $\frac{x}{3} - \frac{x}{5} = 2$
7. $1 - \frac{y}{3} = 3 + \frac{y}{3} - \frac{y}{6}$