Substituting $s = 42$ and $t = 2$ into $s = ut + \frac{1}{2}at^2$ gives

$$42 = 2u + \frac{1}{2}a(2)^2$$

i.e. $$42 = 2u + 2a \qquad (1)$$

Substituting $s = 144$ and $t = 4$ into $s = ut + \frac{1}{2}at^2$ gives

$$144 = 4u + \frac{1}{2}a(4)^2$$

i.e. $$144 = 4u + 8a \qquad (2)$$

Multiplying equation (1) by 2 gives

$$84 = 4u + 4a \qquad (3)$$

Subtracting equation (3) from equation (2) gives

$$60 = 0 + 4a$$

and $$\boldsymbol{a} = \frac{60}{4} = \mathbf{15}$$

Substituting $a = 15$ into equation (1) gives

$$42 = 2u + 2(15)$$
$$42 - 30 = 2u$$
$$\boldsymbol{u} = \frac{12}{2} = \mathbf{6}$$

Substituting $a = 15$ and $u = 6$ in equation (2) gives

$$\text{RHS} = 4(6) + 8(15) = 24 + 120 = 144 = \text{LHS}$$

Hence, **the initial velocity $u = 6$ m/s and the acceleration $a = 15$ m/s$^2$**.

Distance travelled after 3 s is given by $s = ut + \frac{1}{2}at^2$ where $t = 3$, $u = 6$ and $a = 15$.

Hence, $s = (6)(3) + \frac{1}{2}(15)(3)^2 = 18 + 67.5$

i.e. **distance travelled after 3 s = 85.5 m**.

**Problem 16.** The resistance $R\,\Omega$ of a length of wire at $t$°C is given by $R = R_0(1 + \alpha t)$, where $R_0$ is the resistance at 0°C and $\alpha$ is the temperature coefficient of resistance in /°C. Find the values of $\alpha$ and $R_0$ if $R = 30\,\Omega$ at 50°C and $R = 35\,\Omega$ at 100°C

Substituting $R = 30$ and $t = 50$ into $R = R_0(1 + \alpha t)$ gives

$$30 = R_0(1 + 50\alpha) \qquad (1)$$

Substituting $R = 35$ and $t = 100$ into $R = R_0(1 + \alpha t)$ gives

$$35 = R_0(1 + 100\alpha) \qquad (2)$$

Although these equations may be solved by the conventional substitution method, an easier way is to eliminate $R_0$ by division. Thus, dividing equation (1) by equation (2) gives

$$\frac{30}{35} = \frac{R_0(1 + 50\alpha)}{R_0(1 + 100\alpha)} = \frac{1 + 50\alpha}{1 + 100\alpha}$$

Cross-multiplying gives

$$30(1 + 100\alpha) = 35(1 + 50\alpha)$$
$$30 + 3000\alpha = 35 + 1750\alpha$$
$$3000\alpha - 1750\alpha = 35 - 30$$
$$1250\alpha = 5$$

i.e. $$\boldsymbol{\alpha} = \frac{5}{1250} = \frac{\mathbf{1}}{\mathbf{250}} \quad \text{or} \quad \mathbf{0.004}$$

Substituting $\alpha = \frac{1}{250}$ into equation (1) gives

$$30 = R_0\left\{1 + (50)\left(\frac{1}{250}\right)\right\}$$
$$30 = R_0(1.2)$$
$$\boldsymbol{R_0} = \frac{30}{1.2} = \mathbf{25}$$

Checking, substituting $\alpha = \frac{1}{250}$ and $R_0 = 25$ in equation (2), gives

$$\text{RHS} = 25\left\{1 + (100)\left(\frac{1}{250}\right)\right\}$$
$$= 25(1.4) = 35 = \text{LHS}$$

Thus, the solution is **$\alpha = 0.004$/°C** and **$R_0 = 25\,\Omega$**.

**Problem 17.** The molar heat capacity of a solid compound is given by the equation $c = a + bT$, where $a$ and $b$ are constants. When $c = 52$, $T = 100$ and when $c = 172$, $T = 400$. Determine the values of $a$ and $b$

When $c = 52$, $T = 100$, hence

$$52 = a + 100b \quad (1)$$

When $c = 172$, $T = 400$, hence

$$172 = a + 400b \quad (2)$$

Equation (2) – equation (1) gives

$$120 = 300b$$

from which, $$\boldsymbol{b} = \frac{120}{300} = \mathbf{0.4}$$

Substituting $b = 0.4$ in equation (1) gives

$$52 = a + 100(0.4)$$
$$\boldsymbol{a} = 52 - 40 = \mathbf{12}$$

Hence, $\boldsymbol{a = 12}$ and $\boldsymbol{b = 0.4}$

**Now try the following Practice Exercise**

**Practice Exercise 52 Practical problems involving simultaneous equations (answers on page 345)**

1. In a system of pulleys, the effort $P$ required to raise a load $W$ is given by $P = aW + b$, where $a$ and $b$ are constants. If $W = 40$ when $P = 12$ and $W = 90$ when $P = 22$, find the values of $a$ and $b$.

2. Applying Kirchhoff's laws to an electrical circuit produces the following equations:

$$5 = 0.2I_1 + 2(I_1 - I_2)$$
$$12 = 3I_2 + 0.4I_2 - 2(I_1 - I_2)$$

   Determine the values of currents $I_1$ and $I_2$

3. Velocity $v$ is given by the formula $v = u + at$. If $v = 20$ when $t = 2$ and $v = 40$ when $t = 7$, find the values of $u$ and $a$. Then, find the velocity when $t = 3.5$

4. Three new cars and 4 new vans supplied to a dealer together cost £97 700 and 5 new cars and 2 new vans of the same models cost £103 100. Find the respective costs of a car and a van.

5. $y = mx + c$ is the equation of a straight line of slope $m$ and $y$-axis intercept $c$. If the line passes through the point where $x = 2$ and $y = 2$, and also through the point where $x = 5$ and $y = 0.5$, find the slope and $y$-axis intercept of the straight line.

6. The resistance $R$ ohms of copper wire at $t$°C is given by $R = R_0(1 + \alpha t)$, where $R_0$ is the resistance at 0°C and $\alpha$ is the temperature coefficient of resistance. If $R = 25.44\,\Omega$ at 30°C and $R = 32.17\,\Omega$ at 100°C, find $\alpha$ and $R_0$

7. The molar heat capacity of a solid compound is given by the equation $c = a + bT$. When $c = 60$, $T = 100$ and when $c = 210$, $T = 400$. Find the values of $a$ and $b$.

8. In an engineering process, two variables $p$ and $q$ are related by $q = ap + b/p$, where $a$ and $b$ are constants. Evaluate $a$ and $b$ if $q = 13$ when $p = 2$ and $q = 22$ when $p = 5$.

9. In a system of forces, the relationship between two forces $F_1$ and $F_2$ is given by

$$5F_1 + 3F_2 + 6 = 0$$
$$3F_1 + 5F_2 + 18 = 0$$

   Solve for $F_1$ and $F_2$

## 13.6 Solving simultaneous equations in three unknowns

Equations containing three unknowns may be solved using exactly the same procedures as those used with two equations and two unknowns, providing that there are three equations to work with. The method is demonstrated in the following worked problem.

**Problem 18.** Solve the simultaneous equations.

$$x + y + z = 4 \quad (1)$$
$$2x - 3y + 4z = 33 \quad (2)$$
$$3x - 2y - 2z = 2 \quad (3)$$

There are a number of ways of solving these equations. One method is shown below.
The initial object is to produce two equations with two unknowns. For example, multiplying equation (1) by 4 and then subtracting this new equation from equation (2) will produce an equation with only $x$ and $y$ involved.

Multiplying equation (1) by 4 gives

$$4x + 4y + 4z = 16 \qquad (4)$$

Equation (2) – equation (4) gives

$$-2x - 7y = 17 \qquad (5)$$

Similarly, multiplying equation (3) by 2 and then adding this new equation to equation (2) will produce another equation with only $x$ and $y$ involved.

Multiplying equation (3) by 2 gives

$$6x - 4y - 4z = 4 \qquad (6)$$

Equation (2) + equation (6) gives

$$8x - 7y = 37 \qquad (7)$$

Rewriting equation (5) gives

$$-2x - 7y = 17 \qquad (5)$$

Now we can use the previous method for solving simultaneous equations in two unknowns.

Equation (7) – equation (5) gives $\quad 10x = 20$

from which, $\quad \boldsymbol{x = 2}$

(Note that $8x - -2x = 8x + 2x = 10x$)

Substituting $x = 2$ into equation (5) gives

$$-4 - 7y = 17$$

from which, $\quad -7y = 17 + 4 = 21$

and $\quad \boldsymbol{y = -3}$

Substituting $x = 2$ and $y = -3$ into equation (1) gives

$$2 - 3 + z = 4$$

from which, $\quad \boldsymbol{z = 5}$

Hence, the solution of the simultaneous equations is $\boldsymbol{x = 2}$, $\boldsymbol{y = -3}$ and $\boldsymbol{z = 5}$.

**Now try the following Practice Exercise**

**Practice Exercise 53 Simultaneous equations in three unknowns (answers on page 345)**

In problems 1 to 9, solve the simultaneous equations in 3 unknowns.

1. $x + 2y + 4z = 16$
   $2x - y + 5z = 18$
   $3x + 2y + 2z = 14$
2. $2x + y - z = 0$
   $3x + 2y + z = 4$
   $5x + 3y + 2z = 8$
3. $3x + 5y + 2z = 6$
   $x - y + 3z = 0$
   $2 + 7y + 3z = -3$
4. $2x + 4y + 5z = 23$
   $3x - y - 2z = 6$
   $4x + 2y + 5z = 31$
5. $2x + 3y + 4z = 36$
   $3x + 2y + 3z = 29$
   $x + y + z = 11$
6. $4x + y + 3z = 31$
   $2x - y + 2z = 10$
   $3x + 3y - 2z = 7$
7. $5x + 5y - 4z = 37$
   $2x - 2y + 9z = 20$
   $-4x + y + z = -14$
8. $6x + 7y + 8z = 13$
   $3x + y - z = -11$
   $2x - 2y - 2z = -18$
9. $3x + 2y + z = 14$
   $7x + 3y + z = 22.5$
   $4x - 4y - z = -8.5$
10. Kirchhoff's laws are used to determine the current equations in an electrical network and result in the following:

$$i_1 + 8i_2 + 3i_3 = -31$$

$$3i_1 - 2i_2 + i_3 = -5$$

$$2i_1 - 3i_2 + 2i_3 = 6$$

Determine the values of $i_1$, $i_2$ and $i_3$

11. The forces in three members of a framework are $F_1$, $F_2$ and $F_3$. They are related by following simultaneous equations.

$$1.4F_1 + 2.8F_2 + 2.8F_3 = 5.6$$

$$4.2F_1 - 1.4F_2 + 5.6F_3 = 35.0$$

$$4.2F_1 + 2.8F_2 - 1.4F_3 = -5.6$$

Find the values of $F_1$, $F_2$ and $F_3$

## Revision Test 5 : Transposition and simultaneous equations

This assignment covers the material contained in Chapters 12 and 13. *The marks available are shown in brackets at the end of each question.*

1. Transpose $p - q + r = a - b$ for $b$. (2)

2. Make $\pi$ the subject of the formula $r = \dfrac{c}{2\pi}$ (2)

3. Transpose $V = \dfrac{1}{3}\pi r^2 h$ for $h$. (2)

4. Transpose $I = \dfrac{E - e}{R + r}$ for $E$. (2)

5. Transpose $k = \dfrac{b}{ad - 1}$ for $d$. (4)

6. Make $g$ the subject of the formula $t = 2\pi\sqrt{\dfrac{L}{g}}$ (3)

7. Transpose $A = \dfrac{\pi R^2 \theta}{360}$ for $R$. (2)

8. Make $r$ the subject of the formula $x + y = \dfrac{r}{3 + r}$ (5)

9. Make $L$ the subject of the formula $m = \dfrac{\mu L}{L + rCR}$ (5)

10. The surface area $A$ of a rectangular prism is given by the formula $A = 2(bh + hl + lb)$. Evaluate $b$ when $A = 11\,750\,\text{mm}^2$, $h = 25\,\text{mm}$ and $l = 75\,\text{mm}$. (4)

11. The velocity $v$ of water in a pipe appears in the formula $h = \dfrac{0.03\,Lv^2}{2\,dg}$. Evaluate $v$ when $h = 0.384$, $d = 0.20$, $L = 80$ and $g = 10$. (3)

12. A formula for the focal length $f$ of a convex lens is $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$. Evaluate $v$ when $f = 4$ and $u = 20$. (4)

In problems 13 and 14, solve the simultaneous equations.

13. (a) $2x + y = 6$, $5x - y = 22$ (b) $4x - 3y = 11$, $3x + 5y = 30$ (9)

14. (a) $3a - 8 + \dfrac{b}{8} = 0$

$$b + \frac{a}{2} = \frac{21}{4}$$

(b) $\dfrac{2p + 1}{5} - \dfrac{1 - 4q}{2} = \dfrac{5}{2}$

$$\frac{1 - 3p}{7} + \frac{2q - 3}{5} + \frac{32}{35} = 0$$ (18)

15. In an engineering process two variables $x$ and $y$ are related by the equation $y = ax + \dfrac{b}{x}$, where $a$ and $b$ are constants. Evaluate $a$ and $b$ if $y = 15$ when $x = 1$ and $y = 13$ when $x = 3$. (5)

16. Kirchhoff's laws are used to determine the current equations in an electrical network and result in the following:

$$i_1 + 8\,i_2 + 3\,i_3 = -31$$
$$3\,i_1 - 2\,i_2 + i_3 = -5$$
$$2\,i_1 - 3\,i_2 + 2\,i_3 = 6$$

Determine the values of $i_1$, $i_2$ and $i_3$ (10)

# Chapter 14

# Solving quadratic equations

## 14.1 Introduction

As stated in Chapter 11, an **equation** is a statement that two quantities are equal and to '**solve an equation**' means 'to find the value of the unknown'. The value of the unknown is called the **root** of the equation.

A **quadratic equation** is one in which the highest power of the unknown quantity is 2. For example, $x^2 - 3x + 1 = 0$ is a quadratic equation.

There are four methods of **solving quadratic equations**. These are:

(a) by factorization (where possible),

(b) by 'completing the square',

(c) by using the 'quadratic formula', or

(d) graphically (see Chapter 19).

## 14.2 Solution of quadratic equations by factorization

Multiplying out $(x+1)(x-3)$ gives $x^2 - 3x + x - 3$ i.e. $x^2 - 2x - 3$. The reverse process of moving from $x^2 - 2x - 3$ to $(x+1)(x-3)$ is called **factorizing**.

If the quadratic expression can be factorized this provides the simplest method of solving a quadratic equation.

For example, if $x^2 - 2x - 3 = 0$, then, by factorizing

$$(x+1)(x-3) = 0$$

Hence, either $(x+1) = 0$, i.e. $x = -1$

or $(x-3) = 0$, i.e. $x = 3$

Hence, $x = -1$ and $x = 3$ are the roots of the quadratic equation $x^2 - 2x - 3 = 0$.

The technique of factorizing is often one of trial and error.

**Problem 1.** Solve the equation $x^2 + x - 6 = 0$ by factorization

The factors of $x^2$ are $x$ and $x$. These are placed in brackets: $(x \quad )(x \quad )$

The factors of $-6$ are $+6$ and $-1$, or $-6$ and $+1$, or $+3$ and $-2$, or $-3$ and $+2$.

The only combination to give a middle term of $+x$ is $+3$ and $-2$,

i.e. $\boldsymbol{x^2 + x - 6 = (x+3)(x-2)}$

The quadratic equation $x^2 + x - 6 = 0$ thus becomes

$$(x+3)(x-2) = 0$$

Since the only way that this can be true is for either the first or the second or both factors to be zero, then

either $(x+3) = 0$, i.e. $x = -3$

or $(x-2) = 0$, i.e. $x = 2$

Hence, **the roots of $\boldsymbol{x^2 + x - 6 = 0}$ are $\boldsymbol{x = -3}$** and $\boldsymbol{x = 2}$.

**Problem 2.** Solve the equation $x^2 + 2x - 8 = 0$ by factorization

The factors of $x^2$ are $x$ and $x$. These are placed in brackets: $(x \quad )(x \quad )$

The factors of $-8$ are $+8$ and $-1$, or $-8$ and $+1$, or $+4$ and $-2$, or $-4$ and $+2$.

The only combination to give a middle term of $+2x$ is $+4$ and $-2$,

i.e. $x^2 + 2x - 8 = (x+4)(x-2)$

(Note that the product of the two inner terms $(4x)$ added to the product of the two outer terms $(-2x)$ must equal the middle term, $+2x$ in this case.)

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The quadratic equation $x^2+2x-8=0$ thus becomes
$$(x+4)(x-2)=0$$
Since the only way that this can be true is for either the first or the second or both factors to be zero,

either $(x+4)=0$, i.e. $x=-4$

or $(x-2)=0$, i.e. $x=2$

Hence, **the roots of $x^2+2x-8=0$ are $x=-4$** and **$x=2$**.

**Problem 3.** Determine the roots of $x^2-6x+9=0$ by factorization

$x^2-6x+9=(x-3)(x-3)$, i.e. $(x-3)^2=0$

The LHS is known as a **perfect square**.
Hence, **$x=3$** is the only root of the equation $x^2-6x+9=0$.

**Problem 4.** Solve the equation $x^2-4x=0$

Factorizing gives $x(x-4)=0$

If $x(x-4)=0$,

either $x=0$ or $x-4=0$

i.e. **$x=0$** or **$x=4$**

These are the two roots of the given equation. Answers can always be checked by substitution into the original equation.

**Problem 5.** Solve the equation $x^2+3x-4=0$

Factorizing gives $(x-1)(x+4)=0$

Hence, either $x-1=0$ or $x+4=0$

i.e. **$x=1$** or **$x=-4$**

**Problem 6.** Determine the roots of $4x^2-25=0$ by factorization

The LHS of $4x^2-25=0$ is **the difference of two squares**, $(2x)^2$ and $(5)^2$.
By factorizing, **$4x^2-25=(2x+5)(2x-5)$**, i.e. $(2x+5)(2x-5)=0$

Hence, either $(2x+5)=0$, i.e. $\boldsymbol{x=-\frac{5}{2}=-2.5}$

or $(2x-5)=0$, i.e. $\boldsymbol{x=\frac{5}{2}=2.5}$

**Problem 7.** Solve the equation $x^2-5x+6=0$

Factorizing gives $(x-3)(x-2)=0$

Hence, either $x-3=0$ or $x-2=0$

i.e. **$x=3$** or **$x=2$**

**Problem 8.** Solve the equation $x^2=15-2x$

Rearranging gives $x^2+2x-15=0$

Factorizing gives $(x+5)(x-3)=0$

Hence, either $x+5=0$ or $x-3=0$

i.e. **$x=-5$** or **$x=3$**

**Problem 9.** Solve the equation $3x^2-11x-4=0$ by factorization

The factors of $3x^2$ are $3x$ and $x$. These are placed in brackets: $(3x\quad)(x\quad)$
The factors of $-4$ are $-4$ and $+1$, or $+4$ and $-1$, or $-2$ and 2.
Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is $+1$ and $-4$,

i.e. $\boldsymbol{3x^2-11x-4=(3x+1)(x-4)}$

The quadratic equation $3x^2-11x-4=0$ thus becomes $(3x+1)(x-4)=0$

Hence, either $(3x+1)=0$, i.e. $\boldsymbol{x=-\frac{1}{3}}$

or $(x-4)=0$, i.e. **$x=4$**

and both solutions may be checked in the original equation.

**Problem 10.** Solve the quadratic equation $4x^2+8x+3=0$ by factorizing

The factors of $4x^2$ are $4x$ and $x$ or $2x$ and $2x$.
The factors of 3 are 3 and 1, or $-3$ and $-1$.
Remembering that the product of the inner terms added to the product of the two outer terms must equal $+8x$, the only combination that is true (by trial and error) is

$$\boldsymbol{(4x^2+8x+3)=(2x+3)(2x+1)}$$

Hence, $(2x+3)(2x+1)=0$, from which either $(2x+3)=0$ or $(2x+1)=0$.

Thus, $2x=-3$, from which $\boldsymbol{x=-\frac{3}{2}}$ or $\mathbf{-1.5}$

or $2x=-1$, from which $\boldsymbol{x=-\frac{1}{2}}$ or $\mathbf{-0.5}$

which may be checked in the original equation.

**Problem 11.** Solve the quadratic equation $15x^2+2x-8=0$ by factorizing

The factors of $15x^2$ are $15x$ and $x$ or $5x$ and $3x$.
The factors of $-8$ are $-4$ are $+2$, or 4 and $-2$, or $-8$ and $+1$, or 8 and $-1$.
By trial and error the only combination that works is

$$\mathbf{15x^2+2x-8=(5x+4)(3x-2)}$$

Hence, $(5x+4)(3x-2)=0$, from which either $5x+4=0$ or $3x-2=0$.

Hence, $\boldsymbol{x=-\frac{4}{5}}$ or $\boldsymbol{x=\frac{2}{3}}$

which may be checked in the original equation.

**Problem 12.** The roots of a quadratic equation are $\frac{1}{3}$ and $-2$. Determine the equation in $x$

If the roots of a quadratic equation are, say, $\alpha$ and $\beta$, then $(x-\alpha)(x-\beta)=0$.

Hence, if $\alpha=\frac{1}{3}$ and $\beta=-2$,

$$\left(x-\frac{1}{3}\right)(x-(-2))=0$$

$$\left(x-\frac{1}{3}\right)(x+2)=0$$

$$x^2-\frac{1}{3}x+2x-\frac{2}{3}=0$$

$$\boldsymbol{x^2+\frac{5}{3}x-\frac{2}{3}=0}$$

or $$\mathbf{3x^2+5x-2=0}$$

**Problem 13.** Find the equation in $x$ whose roots are 5 and $-5$

If 5 and $-5$ are the roots of a quadratic equation then

$$(x-5)(x+5)=0$$

i.e. $$x^2-5x+5x-25=0$$

i.e. $$\mathbf{x^2-25=0}$$

**Problem 14.** Find the equation in $x$ whose roots are 1.2 and $-0.4$

If 1.2 and $-0.4$ are the roots of a quadratic equation then

$$(x-1.2)(x+0.4)=0$$

i.e. $$x^2-1.2x+0.4x-0.48=0$$

i.e. $$\mathbf{x^2-0.8x-0.48=0}$$

**Now try the following Practice Exercise**

**Practice Exercise 54 Solving quadratic equations by factorization (answers on page 346)**

In problems 1 to 30, solve the given equations by factorization.

| | |
|---|---|
| 1. $x^2-16=0$ | 2. $x^2+4x-32=0$ |
| 3. $(x+2)^2=16$ | 4. $4x^2-9=0$ |
| 5. $3x^2+4x=0$ | 6. $8x^2-32=0$ |
| 7. $x^2-8x+16=0$ | 8. $x^2+10x+25=0$ |
| 9. $x^2-2x+1=0$ | 10. $x^2+5x+6=0$ |
| 11. $x^2+10x+21=0$ | 12. $x^2-x-2=0$ |
| 13. $y^2-y-12=0$ | 14. $y^2-9y+14=0$ |
| 15. $x^2+8x+16=0$ | 16. $x^2-4x+4=0$ |
| 17. $x^2+6x+9=0$ | 18. $x^2-9=0$ |
| 19. $3x^2+8x+4=0$ | 20. $4x^2+12x+9=0$ |
| 21. $4z^2-\frac{1}{16}=0$ | 22. $x^2+3x-28=0$ |
| 23. $2x^2-x-3=0$ | 24. $6x^2-5x+1=0$ |
| 25. $10x^2+3x-4=0$ | 26. $21x^2-25x=4$ |
| 27. $8x^2+13x-6=0$ | 28. $5x^2+13x-6=0$ |
| 29. $6x^2-5x-4=0$ | 30. $8x^2+2x-15=0$ |

In problems 31 to 36, determine the quadratic equations in $x$ whose roots are

| | |
|---|---|
| 31. 3 and 1 | 32. 2 and $-5$ |
| 33. $-1$ and $-4$ | 34. 2.5 and $-0.5$ |
| 35. 6 and $-6$ | 36. 2.4 and $-0.7$ |

## 14.3 Solution of quadratic equations by 'completing the square'

An expression such as $x^2$ or $(x+2)^2$ or $(x-3)^2$ is called a **perfect square**.

If $x^2 = 3$ then $x = \pm\sqrt{3}$

If $(x+2)^2 = 5$ then $x+2 = \pm\sqrt{5}$ and $x = -2 \pm \sqrt{5}$

If $(x-3)^2 = 8$ then $x-3 = \pm\sqrt{8}$ and $x = 3 \pm \sqrt{8}$

Hence, if a quadratic equation can be rearranged so that one side of the equation is a perfect square and the other side of the equation is a number, then the solution of the equation is readily obtained by taking the square roots of each side as in the above examples. The process of rearranging one side of a quadratic equation into a perfect square before solving is called '**completing the square**'.

$$(x+a)^2 = x^2 + 2ax + a^2$$

Thus, in order to make the quadratic expression $x^2 + 2ax$ into a perfect square, it is necessary to add (half the coefficient of $x)^2$, i.e. $\left(\frac{2a}{2}\right)^2$ or $a^2$

For example, $x^2 + 3x$ becomes a perfect square by adding $\left(\frac{3}{2}\right)^2$, i.e.

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = \left(x + \frac{3}{2}\right)^2$$

The method of completing the square is demonstrated in the following worked problems.

**Problem 15.** Solve $2x^2 + 5x = 3$ by completing the square

The procedure is as follows.

(i) Rearrange the equation so that all terms are on the same side of the equals sign (and the coefficient of the $x^2$ term is positive). Hence,

$$2x^2 + 5x - 3 = 0$$

(ii) Make the coefficient of the $x^2$ term unity. In this case this is achieved by dividing throughout by 2. Hence,

$$\frac{2x^2}{2} + \frac{5x}{2} - \frac{3}{2} = 0$$

i.e. $$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

(iii) Rearrange the equations so that the $x^2$ and $x$ terms are on one side of the equals sign and the constant is on the other side. Hence,

$$x^2 + \frac{5}{2}x = \frac{3}{2}$$

(iv) Add to both sides of the equation (half the coefficient of $x)^2$. In this case the coefficient of $x$ is $\frac{5}{2}$

Half the coefficient squared is therefore $\left(\frac{5}{4}\right)^2$

Thus,

$$x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = \frac{3}{2} + \left(\frac{5}{4}\right)^2$$

The LHS is now a perfect square, i.e.

$$\left(x + \frac{5}{4}\right)^2 = \frac{3}{2} + \left(\frac{5}{4}\right)^2$$

(v) Evaluate the RHS. Thus,

$$\left(x + \frac{5}{4}\right)^2 = \frac{3}{2} + \frac{25}{16} = \frac{24+25}{16} = \frac{49}{16}$$

(vi) Take the square root of both sides of the equation (remembering that the square root of a number gives a $\pm$ answer). Thus,

$$\sqrt{\left(x + \frac{5}{4}\right)^2} = \sqrt{\left(\frac{49}{16}\right)}$$

i.e. $$x + \frac{5}{4} = \pm\frac{7}{4}$$

(vii) Solve the simple equation. Thus,

$$x = -\frac{5}{4} \pm \frac{7}{4}$$

i.e. $$x = -\frac{5}{4} + \frac{7}{4} = \frac{2}{4} = \frac{1}{2} \text{ or } 0.5$$

and $$x = -\frac{5}{4} - \frac{7}{4} = -\frac{12}{4} = -3$$

Hence, $\boldsymbol{x = 0.5}$ or $\boldsymbol{x = -3}$; i.e., **the roots of the equation $\boldsymbol{2x^2 + 5x = 3}$ are 0.5 and $\boldsymbol{-3}$**.

**Problem 16.** Solve $2x^2 + 9x + 8 = 0$, correct to 3 significant figures, by completing the square

Making the coefficient of $x^2$ unity gives

$$x^2 + \frac{9}{2}x + 4 = 0$$

Rearranging gives $$x^2 + \frac{9}{2}x = -4$$

Adding to both sides (half the coefficient of $x)^2$ gives

$$x^2+\frac{9}{2}x+\left(\frac{9}{4}\right)^2=\left(\frac{9}{4}\right)^2-4$$

The LHS is now a perfect square. Thus,

$$\left(x+\frac{9}{4}\right)^2=\frac{81}{16}-4=\frac{81}{16}-\frac{64}{16}=\frac{17}{16}$$

Taking the square root of both sides gives

$$x+\frac{9}{4}=\sqrt{\left(\frac{17}{16}\right)}=\pm 1.031$$

Hence, $\quad x=-\frac{9}{4}\pm 1.031$

i.e. $\boldsymbol{x=-1.22}$ or $\boldsymbol{-3.28}$, correct to 3 significant figures.

**Problem 17.** By completing the square, solve the quadratic equation $4.6y^2+3.5y-1.75=0$, correct to 3 decimal places

$$4.6y^2+3.5y-1.75=0$$

Making the coefficient of $y^2$ unity gives

$$y^2+\frac{3.5}{4.6}y-\frac{1.75}{4.6}=0$$

and rearranging gives $\quad y^2+\frac{3.5}{4.6}y=\frac{1.75}{4.6}$

Adding to both sides (half the coefficient of $y)^2$ gives

$$y^2+\frac{3.5}{4.6}y+\left(\frac{3.5}{9.2}\right)^2=\frac{1.75}{4.6}+\left(\frac{3.5}{9.2}\right)^2$$

The LHS is now a perfect square. Thus,

$$\left(y+\frac{3.5}{9.2}\right)^2=0.5251654$$

Taking the square root of both sides gives

$$y+\frac{3.5}{9.2}=\sqrt{0.5251654}=\pm 0.7246830$$

Hence, $\quad y=-\frac{3.5}{9.2}\pm 0.7246830$

i.e. $\quad \boldsymbol{y=0.344}$ or $\boldsymbol{-1.105}$

**Now try the following Practice Exercise**

**Practice Exercise 55 Solving quadratic equations by completing the square (answers on page 346)**

Solve the following equations correct to 3 decimal places by completing the square.

1. $x^2+4x+1=0$
2. $2x^2+5x-4=0$
3. $3x^2-x-5=0$
4. $5x^2-8x+2=0$
5. $4x^2-11x+3=0$
6. $2x^2+5x=2$

## 14.4 Solution of quadratic equations by formula

Let the general form of a quadratic equation be given by $ax^2+bx+c=0$, where $a$, $b$ and $c$ are constants.
Dividing $ax^2+bx+c=0$ by $a$ gives

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Rearranging gives $\quad x^2+\frac{b}{a}x=-\frac{c}{a}$

Adding to each side of the equation the square of half the coefficient of the term in $x$ to make the LHS a perfect square gives

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

Rearranging gives $\quad \left(x+\frac{b}{a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$

Taking the square root of both sides gives

$$x+\frac{b}{2a}=\sqrt{\left(\frac{b^2-4ac}{4a^2}\right)}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

Hence, $\quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$

i.e. the quadratic formula is $\quad \boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

(This method of obtaining the formula is completing the square – as shown in the previous section.)
In summary,

if $\boldsymbol{ax^2+bx+c=0}$ then $\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

This is known as the **quadratic formula**.

**Problem 18.** Solve $x^2 + 2x - 8 = 0$ by using the quadratic formula

Comparing $x^2 + 2x - 8 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1, b = 2$ and $c = -8$.
Substituting these values into the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)} = \frac{-2 \pm \sqrt{4 + 32}}{2}$$
$$= \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2}$$
$$= \frac{-2 + 6}{2} \text{ or } \frac{-2 - 6}{2}$$

Hence, $\boldsymbol{x = \frac{4}{2}}$ or $\frac{-8}{2}$, i.e. $\boldsymbol{x = 2}$ or $\boldsymbol{x = -4}$.

**Problem 19.** Solve $3x^2 - 11x - 4 = 0$ by using the quadratic formula

Comparing $3x^2 - 11x - 4 = 0$ with $ax^2 + bx + c = 0$ gives $a = 3, b = -11$ and $c = -4$. Hence,

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(3)(-4)}}{2(3)}$$
$$= \frac{+11 \pm \sqrt{121 + 48}}{6} = \frac{11 \pm \sqrt{169}}{6}$$
$$= \frac{11 \pm 13}{6} = \frac{11 + 13}{6} \text{ or } \frac{11 - 13}{6}$$

Hence, $\boldsymbol{x = \frac{24}{6}}$ or $\frac{-2}{6}$, i.e. $\boldsymbol{x = 4}$ or $\boldsymbol{x = -\frac{1}{3}}$

**Problem 20.** Solve $4x^2 + 7x + 2 = 0$ giving the roots correct to 2 decimal places

Comparing $4x^2 + 7x + 2 = 0$ with $ax^2 + bx + c$ gives $a = 4, b = 7$ and $c = 2$. Hence,

$$x = \frac{-7 \pm \sqrt{7^2 - 4(4)(2)}}{2(4)}$$
$$= \frac{-7 \pm \sqrt{17}}{8} = \frac{-7 \pm 4.123}{8}$$
$$= \frac{-7 + 4.123}{8} \text{ or } \frac{-7 - 4.123}{8}$$

Hence, $\boldsymbol{x = -0.36}$ or $\boldsymbol{-1.39}$, correct to 2 decimal places.

**Problem 21.** Use the quadratic formula to solve $\frac{x + 2}{4} + \frac{3}{x - 1} = 7$ correct to 4 significant figures

Multiplying throughout by $4(x - 1)$ gives

$$4(x - 1)\frac{(x + 2)}{4} + 4(x - 1)\frac{3}{(x - 1)} = 4(x - 1)(7)$$

Cancelling gives $\quad (x - 1)(x + 2) + (4)(3) = 28(x - 1)$

$$x^2 + x - 2 + 12 = 28x - 28$$

Hence, $\quad x^2 - 27x + 38 = 0$

Using the quadratic formula,

$$x = \frac{-(-27) \pm \sqrt{(-27)^2 - 4(1)(38)}}{2}$$
$$= \frac{27 \pm \sqrt{577}}{2} = \frac{27 \pm 24.0208}{2}$$

Hence, $\quad x = \frac{27 + 24.0208}{2} = 25.5104$

or $\quad x = \frac{27 - 24.0208}{2} = 1.4896$

Hence, $\boldsymbol{x = 25.51}$ or $\boldsymbol{1.490}$, correct to 4 significant figures.

**Now try the following Practice Exercise**

**Practice Exercise 56 Solving quadratic equations by formula (answers on page 346)**

Solve the following equations by using the quadratic formula, correct to 3 decimal places.

1. $2x^2 + 5x - 4 = 0$
2. $5.76x^2 + 2.86x - 1.35 = 0$
3. $2x^2 - 7x + 4 = 0$
4. $4x + 5 = \frac{3}{x}$
5. $(2x + 1) = \frac{5}{x - 3}$
6. $3x^2 - 5x + 1 = 0$