At infinity: † = *U*∞*y* + const (4.90) At the body: † = const

It is well within our capability to find some useful solutions to Eqs. (4.89) and (4.90), which we shall do in Chap. 8.

## Geometric Interpretation of †

The fancy mathematics above would serve by itself to make the stream function im- mortal and always useful to engineers. Even better, though, † has a beautiful geomet- ric interpretation: Lines of constant † are *streamlines* of the flow. This can be shown as follows. From Eq. (1.41) the definition of a streamline in two-dimensional flow is

*dx dy*

— = —

*u* v

or *u dy* — v *dx* = 0 streamline (4.91) Introducing the stream function from Eq. (4.85), we have

† †

— *dx* + — *dy* = 0 = *d*† (4.92)

*x y*

Thus the change in † is zero along a streamline, or

† = const along a streamline (4.93)

Having found a given solution †(*x*, *y*), we can plot lines of constant † to give the streamlines of the flow.

There is also a physical interpretation which relates † to volume flow. From Fig. 4.8, we can compute the volume flow *dQ* through an element *ds* of control surface of unit depth

† †

*dQ* = (**V** · **n**) *dA* = (**i**

**j** )

— — — ∙

*y x*

*dy dx*

— — — *ds*(1)

(**i**

**j** )

*ds ds*

† †

= — *dx* + — *dy* = *d*† (4.94)

*x y*



*dQ* = (**V • n**) *dA* = *d*

Control surface (unit depth into paper)

**Fig. 4.8** Geometric interpretation of stream function: volume flow through a differential portion of a control surface.

*dy* **V** = *i***u** + *j***v**

*ds*

*dx*

**n** = *dy* **i** – *dx* **j**

*ds ds*

**Fig. 4.9** Sign convention for flow in terms of change in stream func- tion: (*a*) flow to the right if †*U* is greater; (*b*) flow to the left if †*L* is greater.

(*a*) (*b*)

 2  1

Flow

1

 2  1

Flow

1

Thus the change in † across the element is numerically equal to the volume flow through the element. The volume flow between any two points in the flow field is equal to the change in stream function between those points:

*Q*12

= 2 (**V**

1

∫

· **n**) *dA* =

2

*d*† = †2 — †1

∫

1

(4.95)

Further, the direction of the flow can be ascertained by noting whether † increases or decreases. As sketched in Fig. 4.9, the flow is to the right if †*U* is greater than †*L*, where the subscripts stand for upper and lower, as before; otherwise the flow is to the left.

Both the stream function and the velocity potential were invented by the French mathematician Joseph Louis Lagrange and published in his treatise on fluid mechan- ics in 1781.

### EXAMPLE 4.7

If a stream function exists for the velocity field of Example 4.5

*u* = *a*(*x*2 — *y*2) v = —2*axy w* = 0 find it, plot it, and interpret it.

### Solution

Since this flow field was shown expressly in Example 4.3 to satisfy the equation of continuity, we are pretty sure that a stream function does exist. We can check again to see if

*u* v

— + — = 0

*x y*

Substitute: 2*ax* + (—2*ax*) = 0 checks

Therefore we are certain that a stream function exists. To find †, we simply set

†

2

*u* = — = *ax*

*y*

†

— *ay*2

(1)

v = —— = —2*axy* (2)

*x*

and work from either one toward the other. Integrate (1) partially

2 *ay*3

† = *ax y* — — + *f*(*x*) (3)

3

Differentiate (3) with respect to *x* and compare with (2)

†

— = 2*axy* + *f*'(*x*) = 2*axy* (4)

*x*

Therefore *f*'(*x*) = 0, or *f* = constant. The complete stream function is thus found

( 2 *y*3 )

† = *a x y* — —

3

+ *C Ans.* (5)

To plot this, set *C* = 0 for convenience and plot the function

2 3 3†

3*x y* — *y* = —

*a*

(6)

for constant values of †. The result is shown in Fig. E4.7*a* to be six 60° wedges of circulating motion, each with identical flow patterns except for the arrows. Once the streamlines are labeled, the flow directions follow from the sign convention of Fig. 4.9. How can the flow be interpreted? Since there is slip along all streamlines, no streamline can truly represent a solid surface in a viscous flow. However, the flow could represent the impingement of three incoming streams at 60, 180, and 300°. This would be a rather unrealistic yet exact solution to the Navier-Stokes equation, as we showed in Example 4.5.

–2*a*



 = 2*a*

*a* 0

– *a*

*y*

60

60

60

60

 = 2*a*

*a*

60

– *a*

–2*a*

 = – 2*a*

– *a*

0 *a*

2 *a*

The origin is a stagnation point

Flow around a 60 corner

*x*

**E4.7a E4.7b**

Flow around a rounded 60 corner

Incoming stream impinging against a 120 corner

By allowing the flow to slip as a frictionless approximation, we could let any given stream- line be a body shape. Some examples are shown in Fig. E4.7*b*.

A stream function also exists in a variety of other physical situations where only two coordinates are needed to define the flow. Three examples are illustrated here.

**Steady Plane Compressible Flow** Suppose now that the density is variable but that *w* = 0, so that the flow is in the *xy*

plane. Then the equation of continuity becomes

— (p*u*) + — (pv) = 0 (4.96)

*x y*

We see that this is in exactly the same form as Eq. (4.84). Therefore a compressible- flow stream function can be defined such that

† p*u* = —

*y*

† pv = ——

*x*

(4.97)

Again lines of constant † are streamlines of the flow, but the change in † is now equal to the *mass* flow, not the volume flow

*dm˙* = p(**V** · **n**) *dA* = *d*†

∫

or *m˙*

2

12 =

1

p(**V**

· **n**) *dA*

= †2 — †1

(4.98)

The sign convention on flow direction is the same as in Fig. 4.9. This particular stream function combines density with velocity and must be substituted into not only mo- mentum but also the energy and state relations (4.58) and (4.59) with pressure and tem- perature as companion variables. Thus the compressible stream function is not a great victory, and further assumptions must be made to effect an analytical solution to a typ- ical problem (see, e.g., Ref. 5, chap. 7).

## Incompressible Plane Flow in Polar Coordinates

Suppose that the important coordinates are *r* and θ, with v*z* = 0, and that the density is constant. Then Eq. (4.82*b*) reduces to

1 1

— — (*r*v*r*) + — — (vθ) = 0 (4.99)

*r r r* θ

After multiplying through by *r*, we see that this is the same as the analogous form of Eq. (4.84)

— (—) + — (——) = 0 (4.100)

*r*

†

θ

θ

†

*r*

By comparison of (4.99) and (4.100) we deduce the form of the incompressible polar- coordinate stream function

v 1 † †

*r* = — — vθ = ——

(4.101)

*r* θ *r*

Once again lines of constant † are streamlines, and the change in † is the *volume flow Q*12 = †2 — †1*.* The sign convention is the same as in Fig. 4.9. This type of stream function is very useful in analyzing flows with cylinders, vortices, sources, and sinks (Chap. 8).

## Incompressible Axisymmetric Flow

As a final example, suppose that the flow is three-dimensional (*r*, *z*) but with no cir- cumferential variations, vθ = / θ = 0 (see Fig. 4.2 for definition of coordinates). Such

a flow is termed *axisymmetric*, and the flow pattern is the same when viewed on any meridional plane through the axis of revolution *z*. For incompressible flow, Eq. (4.82*b*) becomes

1

— — (*r*v*r*) + — (v*z*) = 0 (4.102)

*r r z*

This doesn’t seem to work: Can’t we get rid of the one *r* outside? But when we real- ize that *r* and *z* are independent coordinates, Eq. (4.102) can be rewritten as

— (*r*v*r*) + — (*r*v*z*) = 0 (4.103)

*r z*

By analogy with Eq. (4.84), this has the form

— (——) + — (—) = 0 (4.104)

*r*

†

*z*

*z*

†

*r*

By comparing (4.103) and (4.104), we deduce the form of an incompressible axisym- metric stream function †(*r*, *z*)

1 † 1 †

v*r* = —— — v*z* = — —

(4.105)

*r z r r*

Here again lines of constant † are streamlines, but there is a factor (2π) in the volume flow: *Q*12 = 2π(†2 — †1). The sign convention on flow is the same as in Fig. 4.9.

### EXAMPLE 4.8

Investigate the stream function in polar coordinates

*R*2

† = *U* sin θ (*r* — —) (1)

*r*

where *U* and *R* are constants, a velocity and a length, respectively. Plot the streamlines. What does the flow represent? Is it a realistic solution to the basic equations?

### Solution

The streamlines are lines of constant †, which has units of square meters per second. Note that

†/(*UR*) is dimensionless. Rewrite Eq. (1) in dimensionless form

† 1 *r*

— = sin θ ( — ——)  = —

*UR*

*R*

(2)

Of particular interest is the special line † = 0. From Eq. (1) or (2) this occurs when (*a*) θ = 0 or 180° and (*b*) *r* = *R*. Case (*a*) is the *x*-axis, and case (*b*) is a circle of radius *R*, both of which are plotted in Fig. E4.8.

For any other nonzero value of † it is easiest to pick a value of *r* and solve for θ:

†/(*UR*)

sin θ =

——

*r*/*R* — *R*/*r*

(3)

In general, there will be two solutions for θ because of the symmetry about the *y*-axis. For ex- ample take †/(*UR*) = +1.0:

4.8 Vorticity and Irrotationality **245**

Streamlines converge, high-velocity region

 = +1

*r* = *R*

–1

0

0 0

+1

0

*UR*

+ 1

2

0

– 1

2

–1

**E4.8**

Singularity at origin

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Guess *r*/*R* | 3.0 | 2.5 | 2.0 | 1.8 | 1.7 | 1.618 |
| Compute θ | 22°  158° | 28°  152° | 42°  138° | 54°  156° | 64°  116° | 90° |

This line is plotted in Fig. E4.8 and passes over the circle *r* = *R*. You have to watch it, though, because there is a second curve for †/(*UR*) = +1.0 for small *r* < *R* below the *x*-axis:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Guess *r*/*R* | 0.618 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 |
| Compute θ | —90° | —70° | —42° | —28° | —19° | —12° | —6° |
|  |  | —110° | —138° | —152° | —161° | —168° | —174° |

This second curve plots as a closed curve inside the circle *r* = *R*. There is a singularity of infi- nite velocity and indeterminate flow direction at the origin. Figure E4.8 shows the full pattern. The given stream function, Eq. (1), is an exact and classic solution to the momentum equa- tion (4.38) for frictionless flow. Outside the circle *r* = *R* it represents two-dimensional inviscid flow of a uniform stream past a circular cylinder (Sec. 8.3). Inside the circle it represents a rather

unrealistic trapped circulating motion of what is called a *line doublet*.

# Vorticity and Irrotationality

The assumption of zero fluid angular velocity, or irrotationality, is a very useful sim- plification. Here we show that angular velocity is associated with the curl of the local- velocity vector.

The differential relations for deformation of a fluid element can be derived by ex- amining Fig. 4.10. Two fluid lines *AB* and *BC*, initially perpendicular at time *t*, move and deform so that at *t* + *dt* they have slightly different lengths *A*'*B*' and *B*'*C*' and are slightly off the perpendicular by angles *d*α and *d*β. Such deformation occurs kinemat- ically because *A*, *B*, and *C* have slightly different velocities when the velocity field **V**

*u dy dt*

*y*



*A*

*dy* +  *dy dt*

*d*

 *y*

Time: *t* + *dt*

*C*

Line 2

*A*

*d*

  *dx dt*

*x*

*B*

*dx* + *u dx dt*

Time *t*

*x*

*dy*

V

*y*

*B*

Line 1

*dx*

*C*

0

*x*

**Fig. 4.10** Angular velocity and strain rate of two fluid lines de- forming in the *xy* plane.

has spatial gradients. All these differential changes in the motion of *A*, *B*, and *C* are noted in Fig. 4.10.

We define the angular velocity ω*z* about the *z* axis as the average rate of counter- clockwise turning of the two lines

 1 (*d*α *d* )

*z* = —

— — ——

(4.106)

2

*dt*

*dt*

But from Fig. 4.10, *d*α and *d*β are each directly related to velocity derivatives in the limit of small *dt*

*dt*0 [

]

—1 ( v / *x*) *dx dt* v

*dx* + ( *u*/ *x*) *dx dt x*

*d*α = lim

tan —— = — *dt*

(4.107)

[ —1

*d*β = lim

*dt*0

tan

——

*dy* + ( v/ *y*) *dy dt*

( *u*/ *y*) *dy dt* ] *u*

Combining Eqs. (4.106) and (4.107) gives the desired result:

= — *dt*

*y*

ω 1 ( v *u* )

*z* = —

— — —

(4.108)

2

*x*

*y*

In exactly similar manner we determine the other two rates:

ω*y* = —

— — —

(4.109)

2

*z*

*x*

ω 1 ( *w*

*x* = —

— — —

2

*y*

*z*

v )

1 ( *u*

*w* )

The vector ω = **i***sx* + **j***sy* + **k***sz* is thus one-half the curl of the velocity vector

 **i j k** 

ω 1 1

= — (curl **V**) = — — — —

(4.110)

2 2 *x y z*

 *u* v *w* 

Since the factor of —1— is annoying, many workers prefer to use a vector twice as large,

2

called the *vorticity*:

= 2*s* = curl **V** (4.111)

Many flows have negligible or zero vorticity and are called *irrotational*

curl **V** Ξ 0 (4.112)

The next section expands on this idea. Such flows can be incompressible or com- pressible, steady or unsteady.

We may also note that Fig. 4.10 demonstrates the *shear-strain rate* of the element, which is defined as the rate of closure of the initially perpendicular lines

*d*α *d*β v *u*

ϵ*˙xy* = — + — = — + — (4.113)

*dt dt x y*

When multiplied by viscosity µ, this equals the shear stress r*xy* in a newtonian fluid, as discussed earlier in Eqs. (4.37). Appendix E lists strain-rate and vorticity compo- nents in cylindrical coordinates.

# Frictionless Irrotational Flows

When a flow is both frictionless and irrotational, pleasant things happen. First, the mo- mentum equation (4.38) reduces to Euler’s equation

*d***V**

p

— = p**g** — a*p* (4.114)

*dt*

Second, there is a great simplification in the acceleration term. Recall from Sec. 4.1 that acceleration has two terms

*d***V V**

— = — + (**V** · a)**V** (4.2)

*dt t*

A beautiful vector identity exists for the second term [11]:

(**V** · a)**V** Ξ a(—1—*V*2) + × **V** (4.115)

2

where $ = curl **V** from Eq. (4.111) is the fluid vorticity.

Now combine (4.114) and (4.115), divide by p, and rearrange on the left-hand side.

Dot the entire equation into an arbitrary vector displacement *d***r**:

**V** 1 2 1

[— + a(— *V* ) + × **V** + — a*p* — **g**] ∙ *d***r** = 0 (4.116)

*t*

2

p

Nothing works right unless we can get rid of the third term. We want

( × **V**) ∙ (*d***r***)* Ξ 0 (4.117)

This will be true under various conditions:

* + 1. **V** is zero; trivial, no flow (hydrostatics).
    2. is zero; irrotational flow.
    3. *d***r** is perpendicular to × **V**; this is rather specialized and rare.
    4. *d***r** is parallel to **V**; we integrate *along a streamline* (see Sec. 3.7).

Condition 4 is the common assumption. If we integrate along a streamline in friction- less compressible flow and take, for convenience, **g** = —*g***k**, Eq. (4.116) reduces to

**V** (1 2) *dp*

— ∙ *d***r** + *d* — *V*

+ — + *g dz* = 0 (4.118)

*t*

2

p

Except for the first term, these are exact differentials. Integrate between any two points 1 and 2 along the streamline:

∫2 *V*

— *ds* +

∫2 *dp*

1 (*V* 2

*V* 2)

1

*g*(*z*

+

2 —

*z* ) 0 (4.119)

1

=

1 *t* 1 p 2

— + —

2 —

where *ds* is the arc length along the streamline. Equation (4.119) is Bernoulli’s equa- tion for frictionless unsteady flow along a streamline and is identical to Eq. (3.76). For incompressible steady flow, it reduces to

*p* 1 2

— + — *V* + *gz* = constant along streamline (4.120)

p 2

The constant may vary from streamline to streamline unless the flow is also irrotational (assumption 2). For irrotational flow $ = 0, the offending term Eq. (4.117) vanishes regardless of the direction of *d***r**, and Eq. (4.120) then holds all over the flow field with the same constant.

## Velocity Potential

Irrotationality gives rise to a scalar function ф similar and complementary to the stream function †. From a theorem in vector analysis [11], a vector with zero curl must be the gradient of a scalar function

If a× **V** Ξ 0 then **V** = aф (4.121)

where ф = ф (*x*, *y*, *z*, *t*) is called the *velocity potential function*. Knowledge of ф thus immediately gives the velocity components

ф ф ф

v

*w*

*u* = — = — = —

*x y z*

(4.122)

Lines of constant ф are called the *potential lines* of the flow.

Note that ф, unlike the stream function, is fully three-dimensional and not limited to two coordinates. It reduces a velocity problem with three unknowns *u*, v, and *w* to a single unknown potential ф; many examples are given in Chap. 8 and Sec. 4.10. The velocity potential also simplifies the unsteady Bernoulli equation (4.118) because if ф exists, we obtain

**V** ф

— ∙ *d***r** = — (a*$*) ∙ *d***r** = *d*(—) (4.123)

*t*

*t*

*t*

Equation (4.118) then becomes a relation between ф and *p*

— + ∫ — + — a*$*

+ *gz* = const (4.124)

ф *dp* 1 2

*t* p 2

This is the unsteady irrotational Bernoulli equation. It is very important in the analy- sis of accelerating flow fields (see, e.g., Refs. 10 and 15), but the only application in this text will be in Sec. 9.3 for steady flow.

## Orthogonality of Streamlines and Potential Lines

If a flow is both irrotational and described by only two coordinates, † and ф both ex- ist and the streamlines and potential lines are everywhere mutually perpendicular ex- cept at a stagnation point. For example, for incompressible flow in the *xy* plane, we would have

*u* † ф

= — = —

(4.125)

*y x*

† ф

v = —— = — (4.126)

*x y*

Can you tell by inspection not only that these relations imply orthogonality but also that ф and † satisfy Laplace’s equation?10 A line of constant ф would be such that the change in ф is zero

ф ф

*d*ф = — *dx* + — *dy* = 0 = *u dx* + v *dy* (4.127)

Solving, we have

*x y*

*dy u* 1

(—)

*dx*

ф=const

v

(*dy*/*dx*)† =const

= —— = ———

(4.128)

Equation (4.128) is the mathematical condition that lines of constant ф and † be mu- tually orthogonal. It may not be true at a stagnation point, where both *u* and v are zero, so that their ratio in Eq. (4.128) is indeterminate.

## Generation of Rotationality

This is the second time we have discussed Bernoulli’s equation under different circum- stances (the first was in Sec. 3.7). Such reinforcement is useful, since this is probably the most widely used equation in fluid mechanics. It requires frictionless flow with no shaft work or heat transfer between sections 1 and 2. The flow may or may not be ir- rotational, the latter being an easier condition, allowing a universal Bernoulli constant. The only remaining question is: *When* is a flow irrotational? In other words, when does a flow have negligible angular velocity? The exact analysis of fluid rotationality under arbitrary conditions is a topic for advanced study, e.g., Ref. 10, sec. 8.5; Ref. 9, sec. 5.2; and Ref. 5, sec. 2.10. We shall simply state those results here without proof.

A fluid flow which is initially irrotational may become rotational if

1. There are significant viscous forces induced by jets, wakes, or solid boundaries. In this case Bernoulli’s equation will not be valid in such viscous regions.

10 Equations (4.125) and (4.126) are called the *Cauchy-Riemann equations* and are studied in com- plex-variable theory.

Viscous regions where Bernoulli's equation fails:

*U*

Uniform approach flow (irrotational)

Laminar boundary layer

Turbulent boundary

layer Separated flow



(*a*)

Wake flow

Curved shock wave introduces rotationality

Viscous regions where Bernoulli is invalid:

Laminar Turbulent

boundary boundary

layer layer

Slight separated flow

Wake flow

**Fig. 4.11** Typical flow patterns il- lustrating viscous regions patched onto nearly frictionless regions:

(*a*) low subsonic flow past a body (*U a*); frictionless, irrotational potential flow outside the boundary layer (Bernoulli and Laplace equa- tions valid); (*b*) supersonic flow past a body (*U* > *a*); frictionless, rotational flow outside the bound- ary layer (Bernoulli equation valid, potential flow invalid).

*U*

Uniform supersonic approach (irrotational)

(*b*)

1. There are entropy gradients caused by curved shock waves (see Fig. 4.11*b*).
2. There are density gradients caused by *stratification* (uneven heating) rather than by pressure gradients.
3. There are significant *noninertial* effects such as the earth’s rotation (the Coriolis acceleration).

In cases 2 to 4, Bernoulli’s equation still holds along a streamline if friction is negli- gible. We shall not study cases 3 and 4 in this book. Case 2 will be treated briefly in Chap. 9 on gas dynamics. Primarily we are concerned with case 1, where rotation is induced by viscous stresses. This occurs near solid surfaces, where the no-slip condi- tion creates a boundary layer through which the stream velocity drops to zero, and in jets and wakes, where streams of different velocities meet in a region of high shear.

Internal flows, such as pipes and ducts, are mostly viscous, and the wall layers grow to meet in the core of the duct. Bernoulli’s equation does not hold in such flows un- less it is modified for viscous losses.

External flows, such as a body immersed in a stream, are partly viscous and partly inviscid, the two regions being patched together at the edge of the shear layer or bound- ary layer. Two examples are shown in Fig. 4.11. Figure 4.11*a* shows a low-speed

subsonic flow past a body. The approach stream is irrotational; i.e., the curl of a con- stant is zero, but viscous stresses create a rotational shear layer beside and downstream of the body. Generally speaking (see Chap. 6), the shear layer is laminar, or smooth, near the front of the body and turbulent, or disorderly, toward the rear. A separated, or deadwater, region usually occurs near the trailing edge, followed by an unsteady tur- bulent wake extending far downstream. Some sort of laminar or turbulent viscous the- ory must be applied to these viscous regions; they are then patched onto the outer flow, which is frictionless and irrotational. If the stream Mach number is less than about 0.3, we can combine Eq. (4.122) with the incompressible continuity equation (4.73).

a· **V** = a· (a*$*) = 0

2 2ф 2ф 2ф

or ∇ ф = 0 = —2 + —2 + —2

(4.129)

*x y z*

This is Laplace’s equation in three dimensions, there being no restraint on the number of coordinates in potential flow. A great deal of Chap. 8 will be concerned with solv- ing Eq. (4.129) for practical engineering problems; it holds in the entire region of Fig. 4.11*a* outside the shear layer.

Figure 4.11*b* shows a supersonic flow past a body. A curved shock wave generally forms in front, and the flow downstream is *rotational* due to entropy gradients (case 2). We can use Euler’s equation (4.114) in this frictionless region but not potential the- ory. The shear layers have the same general character as in Fig. 4.11*a* except that the separation zone is slight or often absent and the wake is usually thinner. Theory of sep- arated flow is presently qualitative, but we can make quantitative estimates of laminar and turbulent boundary layers and wakes.

### EXAMPLE 4.9

If a velocity potential exists for the velocity field of Example 4.5

*u* = *a*(*x*2 — *y*2) v = —2*axy w* = 0 find it, plot it, and compare with Example 4.7.

### Solution

Since *w* = 0, the curl of **V** has only one *z* component, and we must show that it is zero:

v *u* 2 2

(a× **V**)*z* = 2ω*z* = — — — = — (—2*axy*) — — (*ax* — *ay* )

*x y x y*

= —2*ay* + 2*ay* = 0 checks *Ans.*

The flow is indeed irrotational. A potential exists.

To find ф(*x*, *y*), set

ф 2 2

*u* = — = *ax*

*x*

ф

— *ay*

(1)

v = — = —2*axy* (2)

*y*

2 *a*

*a*

0

– *a*

 = –2*a*

 = –2 *a*

–*a*

*y* 0

*a*

2*a*

*x*

**E4.9**

 = 2 *a*

*a* 0 –*a* –2*a*

Integrate (1)

*ax*3

ф = — — *axy*2

3

+ *f*(*y*) (3)

Differentiate (3) and compare with (2)

ф

— = —2*axy* + *f* '(*y*) = —2*axy* (4)

*y*

Therefore *f* ' = 0, or *f* = constant. The velocity potential is

*ax*3

ф = — — *axy*2

3

+ *C Ans.*

Letting *C* = 0, we can plot the ф lines in the same fashion as in Example 4.7. The result is shown in Fig. E4.9 (no arrows on ф). For this particular problem, the ф lines form the same pattern as the † lines of Example 4.7 (which are shown here as dashed lines) but are displaced 30°. The ф and † lines are everywhere perpendicular except at the origin, a stagnation point, where they are 30° apart. We expected trouble at the stagnation point, and there is no general rule for de- termining the behavior of the lines at that point.

# Some Illustrative Plane Potential Flows

## Uniform Stream in the *x* Direction

Chapter 8 is devoted entirely to a detailed study of inviscid incompressible flows, es- pecially those which possess both a stream function and a velocity potential. As sketched in Fig. 4.11*a*, inviscid flow is valid away from solid surfaces, and this inviscid pattern is “patched” onto the near-wall viscous layers—an idea developed in Chap. 7. Various body shapes can be simulated by the inviscid-flow pattern. Here we discuss plane flows, three of which are illustrated in Fig. 4.12.

A uniform stream **V** = **i***U*, as in Fig. 4.12*a*, possesses both a stream function and a ve- locity potential, which may be found as follows:

*u* = *U* = ф = † v

— —

= 0 = ф = — †

*x y y x*

— —

*U*



*K*/*r*



*m*/*r*

**Fig. 4.12** Three elementary plane potential flows. Solid lines are streamlines; dashed lines are poten- tial lines.

(*a*)

(*b*)

(*c*)

We may integrate each expression and discard the constants of integration, which do not affect the velocities in the flow. The results are

Uniform stream **i***U*: † = *Uy* ф = *Ux* (4.130)

The streamlines are horizontal straight lines (*y* = const), and the potential lines are ver- tical (*x* = const), i.e., orthogonal to the streamlines, as expected.

## Line Source or Sink at the Origin

Suppose that the *z*-axis were a sort of thin-pipe manifold through which fluid issued at total rate *Q* uniformly along its length *b*. Looking at the *xy* plane, we would see a cylindrical radial outflow or *line source*, as sketched in Fig. 4.12*b*. Plane polar coor- dinates are appropriate (see Fig. 4.2), and there is no circumferential velocity. At any radius *r*, the velocity is

*Q m* 1 † ф † 1 ф

v*r* = — = — = — — = — vθ = 0 = —— = — —

2π*rb r*

*r* θ *r*

*r r* θ

where we have used the polar-coordinate forms of the stream function and the veloc- ity potential. Integrating and again discarding the constants of integration, we obtain the proper functions for this simple radial flow:

Line source or sink: † = *m*θ ф = *m* ln *r* (4.131)

where *m* = *Q*/(2π*b*) is a constant, positive for a source, negative for a sink. As shown in Fig. 4.12*b*, the streamlines are radial spokes (constant θ), and the potential lines are circles (constant *r*).

## Line Irrotational Vortex

A (two-dimensional) line vortex is a purely circulating steady motion, vθ = *f*(*r*) only, v*r* = 0. This satisfies the continuity equation identically, as may be checked from Eq. (4.12*b*). We may also note that a variety of velocity distributions vθ(*r*) satisfy the θ-momentum equation of a viscous fluid, Eq. (E.6). We may show, as a problem ex- ercise, that only one function vθ(*r*) is *irrotational*, i.e., curl **V** = 0, and that is vθ = *K*/*r*, where *K* is a constant. This is sometimes called a *free vortex*, for which the stream function and velocity may be found:

v = 0

1 † ф

*K* †

1 ф

*r* = — — = — vθ = — = —— = — —

*r* θ *r*

*r r*

*r* θ

We may again integrate to determine the appropriate functions:

† = —*K* ln *r* ф = *K*θ (4.132)

where *K* is a constant called the *strength* of the vortex. As shown in Fig. 4.12*c*, the streamlines are circles (constant *r*), and the potential lines are radial spokes (constant θ). Note the similarity between Eqs. (4.131) and (4.132). A free vortex is a sort of re- versed image of a source. The “bathtub vortex,” formed when water drains through a bottom hole in a tank, is a good approximation to the free-vortex pattern.

## Superposition: Source Plus an Equal Sink

Each of the three elementary flow patterns in Fig. 4.12 is an incompressible irrotational flow and therefore satisfies both plane “potential flow” equations ∇2† = 0 and

∇2ф = 0. Since these are linear partial differential equations, any *sum* of such basic solutions is also a solution. Some of these composite solutions are quite interesting and useful.

For example, consider a source +*m* at (*x*, *y*) = (—*a*, 0), combined with a sink of equal strength —*m*, placed at (+*a*, 0), as in Fig. 4.13. The resulting stream function is simply the sum of the two. In cartesian coordinates,

† = †source + †sink = *m* tan—1 Similarly, the composite velocity potential is

*y*

— —

*x* + *a*

*m* tan—1

*y*

—

*x* — *a*

ф = фsource + фsink

1

= — *m* ln [(*x* + *a*) + *y* ] 2

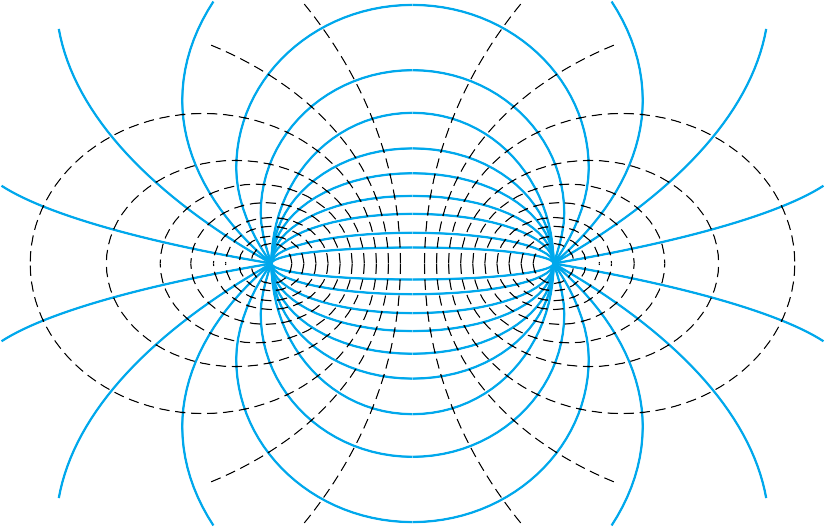
2 2

1

— — *m* ln [(*x* — *a*) + *y* ] 2

2 2

**Fig. 4.13** Potential flow due to a line source plus an equal line sink, from Eq. (4.133). Solid lines are streamlines; dashed lines are poten- tial lines.



By using trigonometric and logarithmic identities, these may be simplified to

2*ay*

Source plus sink: † = —*m* tan—1 —

2

*y*

*x*

+ —2

— *a*2

1 (*x* + *a*)2 + *y*2

(4.133)

ф = — *m* ln ——2 2

2 (*x* — *a*) + *y*

These lines are plotted in Fig. 4.13 and are seen to be two families of orthogonal circles, with the streamlines passing through the source and sink and the potential lines encircling them. They are harmonic (laplacian) functions which are exactly analogous in electromagnetic theory to the electric-current and electric-potential pat- terns of a magnet with poles at (±*a*, 0).

## Sink Plus a Vortex at the Origin

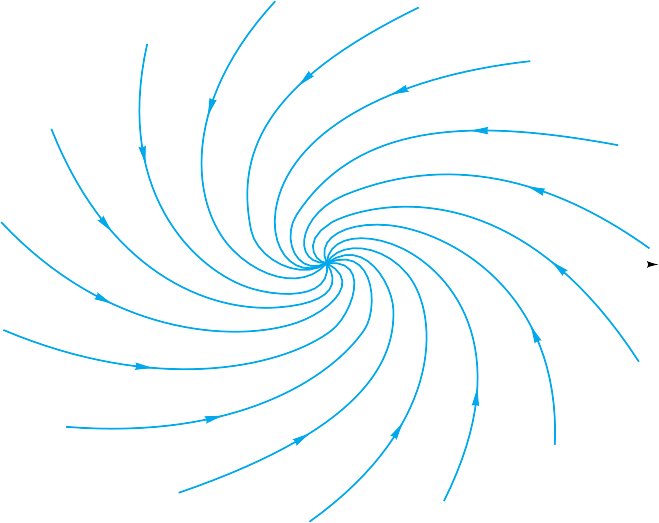
An interesting flow pattern, approximated in nature, occurs by superposition of a sink and a vortex, both centered at the origin. The composite stream function and velocity potential are

Sink plus vortex: † = *m*θ — *K* ln *r* ф = *m* ln *r* + *K*θ (4.134)

When plotted, these form two orthogonal families of logarithmic spirals, as shown in Fig. 4.14. This is a fairly realistic simulation of a tornado (where the sink flow moves up the *z*-axis into the atmosphere) or a rapidly draining bathtub vortex. At the center of a real (viscous) vortex, where Eq. (4.134) predicts infinite velocity, the actual cir- culating flow is highly *rotational* and approximates solid-body rotation vθ  *Cr*.

*y*

*x*



**Fig. 4.14** Superposition of a sink plus a vortex, Eq. (4.134), simu- lates a tornado.