The outer automorphism of PGL(2,Z) and the induced 'modular' involution of $\mathbf{R}$.

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Students

Foreword

In a paper of his on binary quadratic forms, Poincaré states:

"it is not possible, for the indefinite quadratic forms to find invariants, in the sense that we gave to this word..."

Several attempts have been made since then...

Our study can be understood as another attempt to see what can be done by modifying the meaning of the word "invariant"....

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2. Dynamics
3. Tree automorphisms and Lebesgue's measure

PART I

Definition of Jimm and functional equations

There are two fundamental involutions of the real line $\mathbf{R}$:

$$V : x \in \mathbf{R} \to -x \in \mathbf{R}$$

$$K : x \in \mathbf{R} \to 1 - x \in \mathbf{R}$$

and a third one if we add the point at infinity:

$$U : x \in \mathbf{R} \to 1/x \in \mathbf{R}$$

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$$U : x \in \mathbf{R} \to 1/x \in \mathbf{R}$$

... together they generate the group

$$\mathrm{PGL}_2(\mathbf{Z}) = \left\{ \frac{px+q}{rx+s} \mid ps - qr = \pm 1, p, q, r, s \in \mathbf{Z} \right\}$$

$$\simeq \langle U, V, K \mid U^2 = V^2 = K^2 = (UV)^2 = (KU)^3 = 1 \rangle$$

Our aim here is to introduce a fourth involution, which we call Jimm

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Our aim here is to introduce a fourth involution, which we call Jimm

Notation

Every $x \in \mathbf{R}$ can be written as a continued fraction

$$[n_0, n_1, n_2, \ldots] = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\cdots}}}$$

($n_0 \in \mathbf{Z}$, $n_i \in \mathbf{Z}_{>0}$ for $i > 0$), uniquely if x is irrational.

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By 1_k we denote the sequence $1, 1, \ldots, 1$ of length k.

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Notation

By 1_k we denote the sequence $1, 1, \dots, 1$ of length k.

We introduce a 'singular' function $\mathbf{R} \to \mathbf{R}$:

Definition

$$\text{ج}([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

This is a kind of 'real' modular function, as we shall see.

But let us consider some examples first...

Definition (Recall)

$$Ꞇ([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

Examples

$$Ꞇ([3, 3, 3, \dots] = [1_{3-1}, 2, 1_{3-2}, 2, 1_{3-2}, 2 \dots] = [1, 1, 2, 1, 2, 1, 2, \dots]$$

$$Ꞇ([5, 5, 5, \dots] = [1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, \dots]$$

Definition (Recall)

$$\mathfrak{Z}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

This definition works only if $n_k \geq 2$. To make it work for $n_k = 2$, use

RULE I

$$\ldots, n, 1_0, m, \cdots = \ldots, n, m, \ldots$$

Examples

$$\mathfrak{Z}([2, 2, 2, \ldots]) = [1, 2, 1_0, 2, 1_0, 2 \ldots] = [1, 2, 2, 2, \ldots]$$

$$\mathfrak{Z}([2, 3, 2, 3 \ldots]) = [1, 2, 1, 2, 2, 1, 2, 2, 1, \ldots]$$

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$$\text{ج}([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

To make it work also when $n_k = 1$, use

RULE II

$$\dots, n, 1_{-1}, m, \dots = \dots, n+m-1, \dots$$

Examples

$$\text{ج}([1,1,2,1,2,1,2,\dots]) =$$

$$[1_0, \underbrace{2, 1_{-1}, 2}_{3}, 1_0, \underbrace{2, 1_{-1}, 2}_{3}, 1_0, \underbrace{2, 1_{-1}, 2}_{3}, \dots] =$$

$$= [3,3,3,\dots]$$

remember?

Definition (Recall)

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$$= [3, 3, 3, \dots]$$

remember?

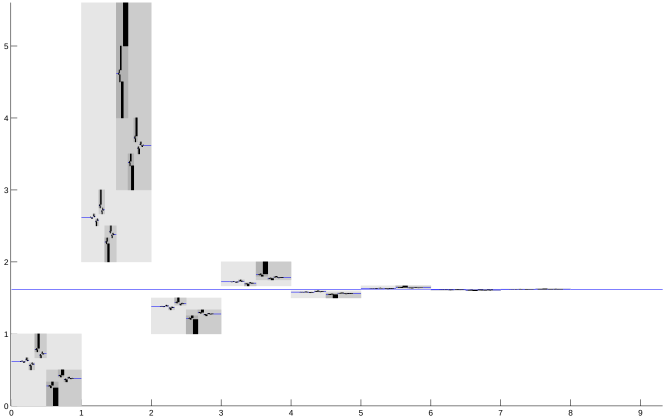
Definition (Recall)

$$\text{ج}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

With these two rules, ج becomes well-defined on $\mathbf{R} \setminus \mathbf{Q}$ and it is involutive:

$$\text{ج}(\text{ج}(x)) = x$$

Here is the plot of ج (the graph lies inside the darker boxes)



Some continuity properties of jimm

It can be shown that..

- ج is continuous on $\mathbf{R}\backslash\mathbf{Q}$
- have jump discontinuities on $\mathbf{Q}$
- ج is differentiable almost everywhere
- its derivative vanish almost everywhere
- admits a natural extension to $\mathbf{Q}$.

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Now consider....

Example

$$\begin{aligned}\text{ج}(1+[3,3,3\ldots]) = \text{ج}([4,3,3\ldots]) &= [1,1,1,2,1,2,1,\ldots] \\ &= 1 + \frac{1}{\underbrace{[1,1,2,1,2,1,\ldots]}_{=\text{ج}([3,3,3,\ldots])}}\end{aligned}$$

We have, in general

FUNCTIONAL EQUATION (*)

$$\text{ج}(1+x) = 1 + \frac{1}{\text{ج}(x)}$$

The functional equation (*) can be derived from the following fundamental set of functional equations

$$\text{Ç}(\text{Ç}(x)) = x \qquad \text{(involutivity)}$$

$$\text{Ç}\left(\frac{1}{x}\right) = \frac{1}{\text{Ç}(x)} \qquad \text{commutativity}$$

$$\text{Ç}(-x) = -\frac{1}{\text{Ç}(x)} \qquad \textbf{not}\ \text{commutativity}$$

$$\text{Ç}(1-x) = 1 - \text{Ç}(x) \qquad \text{commutativity}$$

Two-variable form of functional equations

$$\text{Ӡ}(x) = y \iff \text{Ӡ}(y) = x$$

$$xy = 1 \iff \text{Ӡ}(x)\text{Ӡ}(y) = 1$$

$$x + y = 0 \iff \text{Ӡ}(x)\text{Ӡ}(y) = -1$$

$$x + y = 1 \iff \text{Ӡ}(x) + \text{Ӡ}(y) = 1$$

$$\frac{1}{x} + \frac{1}{y} = 1 \iff \frac{1}{\text{Ӡ}(x)} + \frac{1}{\text{Ӡ}(y)} = 1$$

$\implies$ Ӡ preserves harmonic pairs of numbers.

Recall that

$$Ux := \frac{1}{x}, \quad Vx := -x, \quad Kx := 1 - x$$

The functional equations say

$$\text{Ꝅ}U = U\text{Ꝅ}, \quad \text{Ꝅ}K = K\text{Ꝅ}, \quad \text{Ꝅ}V = UV\text{Ꝅ}$$

$\Longrightarrow$ Ꝅ **is Dyer's outer automorphism of** $\mathrm{PGL}_2(\mathbf{Z})$.

(one has $Out(\mathrm{PGL}_2(\mathbf{Z})) \simeq \mathbf{Z}/2\mathbf{Z}$)

The most general functional equation has the form

$$\text{Ꝅ}(Mx) = \text{Ꝅ}(M)\text{Ꝅ}(x), \quad M \in \mathrm{PGL}_2(\mathbf{Z})$$

(where Ꝅ(M) is the image of M under Dyer's automorphism).

This equation says that Ꝅ is a "kind of" **covariant** function.

Note: f is said to be strictly covariant (or equivariant) if

$$f\left(\frac{pz+q}{rz+s}\right) = \frac{pf(z)+1}{rf(z)+s} \quad \forall \frac{pz+q}{rz+s} \in \mathrm{PGL}_2(\mathbf{Z})$$

Such analytic functions on the upper half plane can be obtained from modular forms.

Fact I

ج sends ultimately periodic continued fractions to ultimately periodic continued fractions.

$$\Longrightarrow$$

ج sends **quadratic irrationals** to **quadratic irrationals**
i.e. ج preserves the "real multiplication-set."

(it does not preserve nor respect the trace, norm, signature, etc)

Examples

$$\text{ج}(\sqrt{2}) = \text{ج}([1, 2, 2, \ldots]) = 1 + \sqrt{2}$$

Not so simple in general:

$$\text{ج}(\sqrt{11}) = \frac{15 + \sqrt{901}}{26}, \quad \text{ج}(-\sqrt{11}) = \frac{15 - \sqrt{901}}{26}$$

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Fact II

ج respects ends of continued fractions (i.e. if x, y has continued fractions that eventually coincide, then so does $\text{ج}(x)$ and $\text{ج}(y)$).

$$\Longleftrightarrow$$

ج **respects the** $\mathrm{PGL}_2(\mathbf{Z})$**-action** (i.e. if x and y are in the same $\mathrm{PGL}_2(\mathbf{Z})$-orbit, then so are $\text{ج}(x)$ and $\text{ج}(y)$.)

More precisely

$$\text{ج}(Mx) = \text{ج}(M)\text{ج}(x) \quad M \in \mathrm{PGL}_2(\mathbf{Z}), x \in \mathbf{R}$$

so that

$$x = My \implies \text{ج}(x) = \text{ج}(M)\text{ج}(y), \quad \text{ج}(M) \in \mathrm{PGL}_2(\mathbf{Z})$$

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Facts I&II together imply:

Fact III

ج induces an involution of the "moduli space of degenerate rank-2 lattices" inside **R**, preserving setwise the "real-multiplication" locus.

$$\text{ج} \circlearrowright \mathbf{R}/\mathrm{PGL}_2(\mathbf{Z})$$

The facts imply...

ج is really a modular function.

Furthermore, one has ....

Fact IV

ج **commutes with the Galois conjugation** on quadratic irrationals, i.e.

$$ج(a+\sqrt{b}) = A+\sqrt{B}$$

$$\Longleftrightarrow$$

$$ج(a-\sqrt{b}) = A-\sqrt{B}$$

Now go back to the two-variable functional equations....

$$\boxed{\begin{aligned} xy = 1 &\iff \text{ج}(x)\text{ج}(y) = 1 \\ x + y = 0 &\iff \text{ج}(x)\text{ج}(y) = -1 \\ x + y = 1 &\iff \text{ج}(x) + \text{ج}(y) = 1 \\ \frac{1}{x} + \frac{1}{y} = 1 &\iff \frac{1}{\text{ج}(x)} + \frac{1}{\text{ج}(y)} = 1 \end{aligned}}$$

...and set $y = \bar{x}$, where $x = a + \sqrt{b}$ is a quadratic irrational:

$$x\bar{x} = 1 \iff \daleth(x)\daleth(\bar{x}) = 1$$

$$x + \bar{x} = 0 \iff \daleth(x)\daleth(\bar{x}) = -1$$

$$x + \bar{x} = 1 \iff \daleth(x) + \daleth(\bar{x}) = 1$$

$$\frac{1}{x} + \frac{1}{\bar{x}} = 1 \iff \frac{1}{\daleth(x)} + \frac{1}{\daleth(\bar{x})} = 1$$

Recall from number theory

If $x = a + \sqrt{b}$ $(a, b \in \mathbf{Q},\ b > 0)$, then

norm of x is $N(x) := x\bar{x} \iff N(a + \sqrt{b}) = a^2 - b$

trace of x is $T(x) := x + \bar{x} \iff T(a + \sqrt{b}) = 2a$

Example

$$N(1 + \sqrt{2}) = -1, \quad T(1 + \sqrt{2}) = 2$$

We get...

Correspondence I

$$x\bar{x} = 1 \iff \text{ج}(x)\text{ج}(\bar{x}) = 1; \text{ i.e. } N(x) = 1 \iff N(\text{ج}(x)) = 1$$

$$\Longrightarrow$$

ج restricts to an involution of the set of **units of norm +1** of the rings of integers in quadratic number fields.

$$\text{ج} \circlearrowright \{a + \sqrt{a^2 - 1} \,|\, 1 < a \in \mathbf{Q}\}$$

We get...

Correspondence II

$$x + \bar{x} = 0 \iff ͡\text{ج}(x)\text{ج}(\bar{x}) = -1; \text{ i.e. } T(x) = 0 \iff N(\text{ج}(x)) = -1.$$

$\implies$ ج establishes a bijection between the set of **square roots of positive rationals** and the set of **units of norm -1** of the rings of integers of quadratic number fields.

$$\text{ج} : \{\sqrt{q} \mid q \in \mathbf{Q}\} \to \{a + \sqrt{a^2 + 1} \mid a \in \mathbf{Q}\}$$

.... and these correspondences are far from being trivial:

Correspondence II-Example

$$\sqrt{\frac{39}{17}} = [1, \overline{1, 1, 16, 1, 1, 2}] \implies$$

$$\text{ج}\left(\sqrt{\frac{39}{17}}\right) = [4, \overline{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 4, 4}] = A \implies$$

$$N(A) = N\left(\frac{7663 + \sqrt{70845893}}{3482}\right) = -1.$$

Correspondence II-More Examples

$$
\begin{array}{lcl}
\sqrt{N} & \to & \text{ج}(\sqrt{N}) \\
\sqrt{3} & \to & \frac{1}{2}(\sqrt{13}+3) \\
\sqrt{5} & \to & \frac{1}{3}(\sqrt{10}+1) \\
\sqrt{6} & \to & \frac{1}{14}(\sqrt{221}+5) \\
\sqrt{7} & \to & \frac{1}{6}(\sqrt{37}+1) \\
\sqrt{8} & \to & \frac{1}{4}(\sqrt{17}+1) \\
\sqrt{10} & \to & \frac{1}{7}(\sqrt{65}+4) \\
\sqrt{11} & \to & \frac{1}{26}(\sqrt{901}+15) \\
\sqrt{12} & \to & \frac{1}{34}(\sqrt{1517}+19) \\
\sqrt{13} & \to & \frac{1}{3}(\sqrt{13}+2) \\
\sqrt{14} & \to & \frac{1}{5}(\sqrt{34}+3) \\
\sqrt{15} & \to & \frac{1}{18}(\sqrt{445}+11) \\
\sqrt{17} & \to & \frac{1}{19}(\sqrt{442}+9)
\end{array}
$$

We get...

Correspondence III

$$x + y = 1 \iff ج(x) + ج(\bar{x}) = 1; \text{ i.e. } T(x) = 1 \iff T(ج(x)) = 1$$

$$ج \circlearrowright \{\frac{1}{2} + \sqrt{a} \,|\, 0 < a \in \mathbf{Q}\}$$

We get...

Correspondence IV

$$\frac{1}{x}+\frac{1}{\bar{x}}=1 \iff \frac{1}{\text{ج}(x)}+\frac{1}{\text{ج}(\bar{x})}=1; \text{ i.e. } T(\frac{1}{x})=1 \iff T(\frac{1}{\text{ج}(x)})=1$$

$$T(x)=N(x) \iff T(\text{ج}x)=N(\text{ج}x)$$

Equivalently,

$$\text{ج} \circlearrowright \{a+\sqrt{a^2-2a} \,|\, 1<a\in \mathbf{Q}\}$$

... and there are more correspondences of this type

What about algebraic numbers of higher degree?

Conjecture

If x is algebraic of degree > 2, then $\check{\jmath}(x)$ is transcendental[a]

[a] Testing the transcendence conjecture of Jimm and its continued fraction statistics (joint with H. Ayral, to appear)

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[a] Testing the transcendence conjecture of Jimm and its continued fraction statistics (joint with H. Ayral, to appear)

$$\begin{aligned}ꞔ(\sqrt[3]{2}) &= ꞔ([1;3,1,5,1,1,4,1,1,8,1,14,1,10,2,1,4,\ldots]) \\ &= [2,1,3,1,1,1,4,1,1,4,1_6,3,1_{12},3,1_8,2,3,1,1,2,\ldots] \\ &= 2.78473155866272 3\ldots\end{aligned}$$

$$ꞔ(\pi) = ꞔ([3,7,15,1,292,1,1,1,2,1,3,\ldots]) = \\ [1_2,2,1_5,2,1_{13},3,1_{290},5,3,\ldots] \\ = 1.7237707925480276079699326494931025145558144289232\ldots$$

$$ꞔ(e) = ꞔ([2,1,2,1,1,4,1,1,6,1,1,8,\ldots]) = \\ [1,3,4,1,1,4,1,1,1,1,\ldots,\overline{4,1_{2n}}] \\ = 1.3105752928466255215822495496939143349712038085627\ldots$$

(We tried to recognize these numbers by the PSLQ-algorithm with various sets of constants–we couldn't get any results)

PART II

Dynamics

Dynamics

Fact

$\daleth$ conjugates the Gauss map to the "Fibonacci map"

$$T_{Gauss} : [0, n_1, n_2, n_3, \dots] \in [0,1] \longrightarrow [0, n_2, n_3, n_4, \dots] \in [0,1]$$

$$\Longrightarrow$$

$$T_{Fibonacci} = \daleth T_{Gauss} \daleth : [0, 1_k, n_{k+1}, n_{k+2}, \dots] \rightarrow [0, n_{k+1} - 1, n_{k+2}, \dots]$$

Example

$$T_{Fibonacci}([0, 1, 1, 1, 5, 13, 7, \dots]) = [0, 4, 13, 7, \dots],$$

$$T_{Fibonacci}([0, 4, 13, 7, \dots]) = [0, 3, 13, 7, \dots], \dots$$

Dynamics

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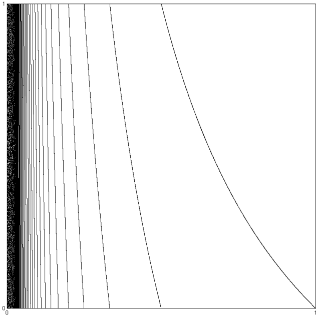
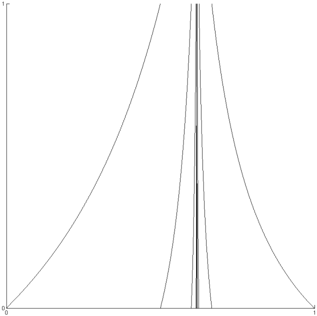


$\longrightarrow$



Gauss map $\longrightarrow$ **Fibonacci map**

Dynamics of these two maps are closely related (Isola et al).
The transfer operator of the Fibonacci map is

$$(\mathscr{L}_s^{Fib}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(F_{k+1}y + F_k)^{2s}} \psi\left(\frac{F_k y + F_{k-1}}{F_{k+1}y + F_k}\right) \tag{1}$$

Recall that the transfer operator of the Gauss map is

$$(\mathscr{L}_s^{Gauss}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^{2s}} \psi\left(\frac{1}{k+x}\right) \tag{2}$$

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$$(\mathscr{L}_s^{Fib}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(F_{k+1}y + F_k)^{2s}} \psi\left(\frac{F_k y + F_{k-1}}{F_{k+1}y + F_k}\right) \tag{1}$$

Recall that the transfer operator of the Gauss map is

$$(\mathscr{L}_s^{Gauss}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^{2s}} \psi\left(\frac{1}{k+x}\right) \tag{2}$$

Invariant measures

$$T_{Fibonacci} \leftrightarrow \frac{1}{x(x+1)} \text{ (infinite)}, \qquad T_{Gauss} \leftrightarrow \frac{1}{x+1}$$

Zeta functions (the transfer operator evaluated at Lebesgue's measure)

$$T_{Fibonacci} \leftrightarrow (\mathscr{L}_s^{Fib}\psi)(\mathbf{1}) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \qquad \text{(“Fibonacci zeta”)}$$

$$T_{Gauss} \leftrightarrow (\mathscr{L}_s^{Gauss}\psi)(\mathbf{1}) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{(“Riemann zeta”)}$$

Eigenfunctions of the Fibonacci transfer operator satisfies the three-term functional equation

$$\psi(y) = \frac{1}{y^{2s}}\psi\left(\frac{y+1}{y}\right) + \frac{1}{\lambda}\frac{1}{(y+1)^{2s}}\psi\left(\frac{y}{y+1}\right) \tag{3}$$

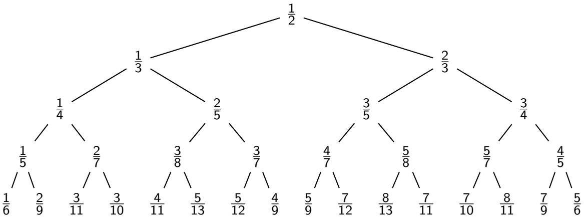
(Equivalent to three-term functional equation studied by Lewis and Zagier)

PART III

Tree automorphisms and Lebesgue's measure

ج as a tree automorphism

The Farey tree

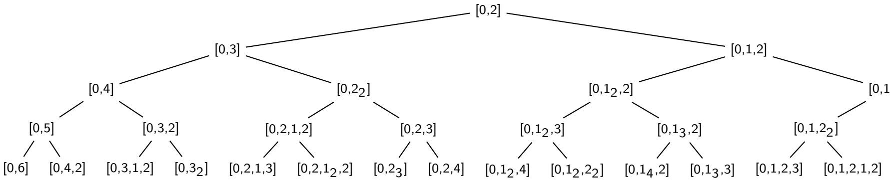


Produced by the Farey sum rule:

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$$

ج as a tree automorphism

The Farey tree by continued fractions



The boundary $\partial\mathcal{F}$ is the set of all infinite paths based at the root.

Fact

The map $\partial\mathcal{F} \to [0,1]$ sending path to its continued fraction, parametrize irrationals in $[0,1]$ (and is 2-to-1 over the rationals).

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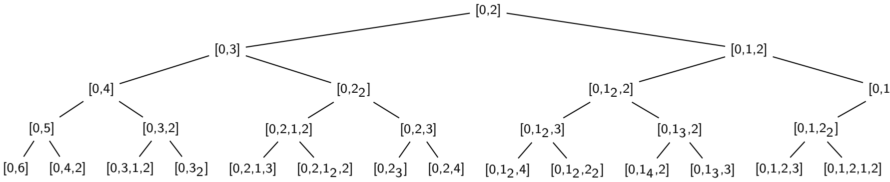

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ج as a tree automorphism

The automorphism group $\mathbf{Aut}(\mathcal{F})$ naturally acts on $\partial\mathcal{F}$.
$\implies$ $\mathbf{Aut}(\mathcal{F})$ acts on continued fractions via the above identification.
(ignoring a countable set of numbers for each automorphism).

ج as a tree automorphism

Shuffle description of $\mathbf{Aut}(\mathcal{F})$.

$\Longrightarrow$ ج is the automorphism which shuffles every vertex.

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Twist description of $\mathbf{Aut}(\mathcal{F})$

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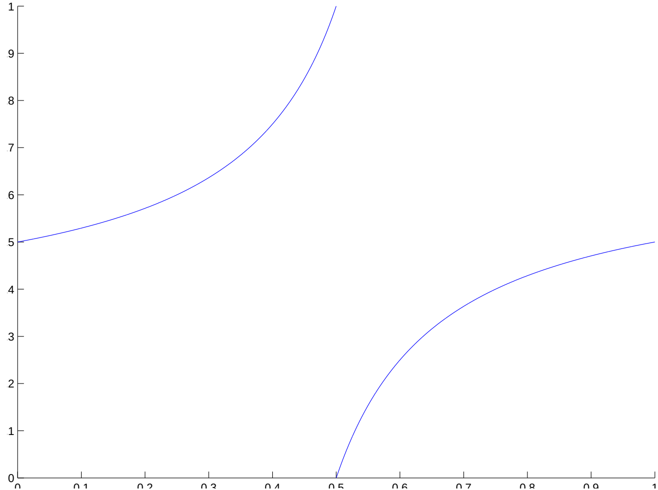
Looking at the boundary actions of shuffles (or twists), yields a presentation of ج as a limit of piecewise-$\mathrm{PGL}_2(\mathbf{Z})$ maps....

Jimm as a limit of piecewise-$\mathrm{PGL}_2(\mathbf{Z})$ maps



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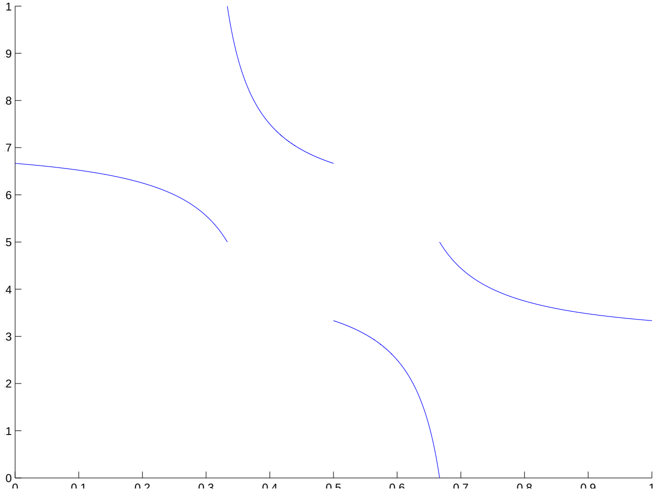

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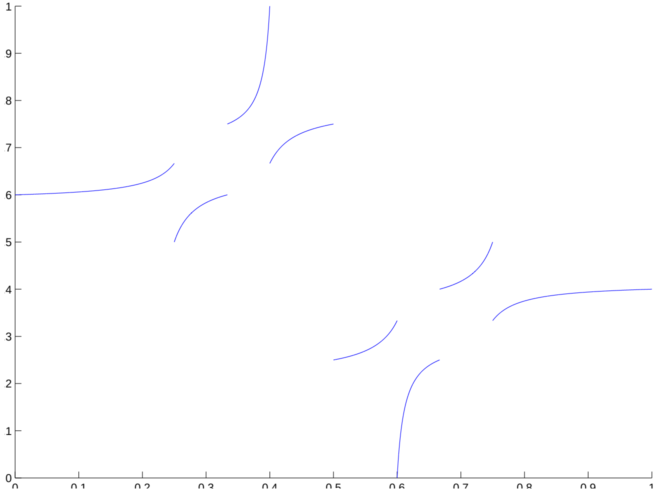

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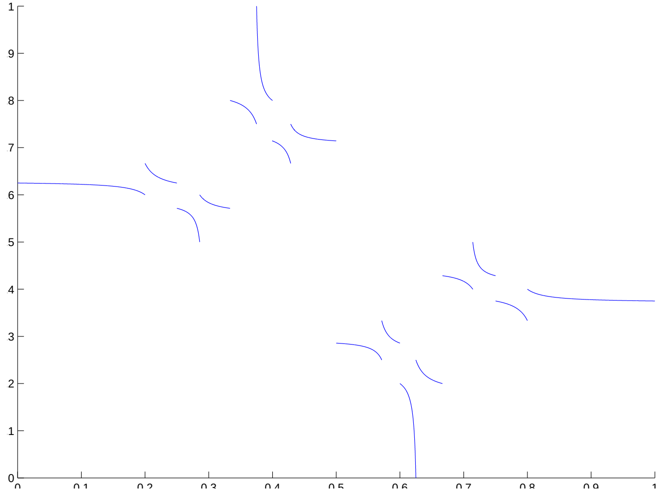

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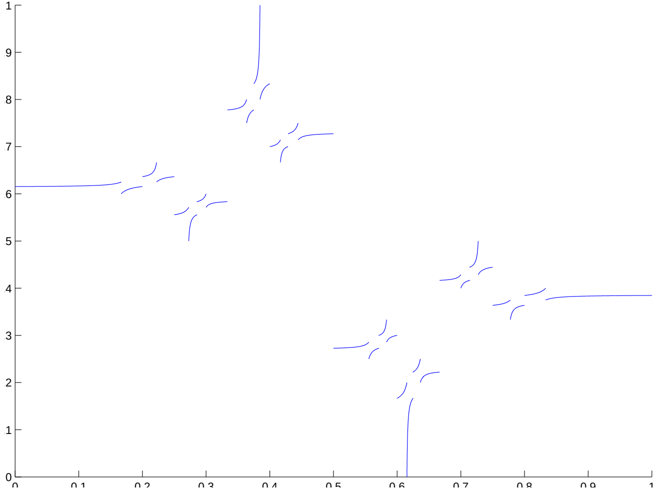

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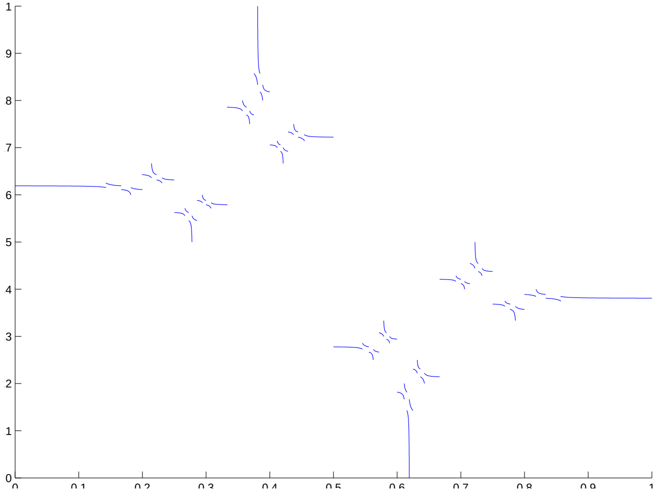

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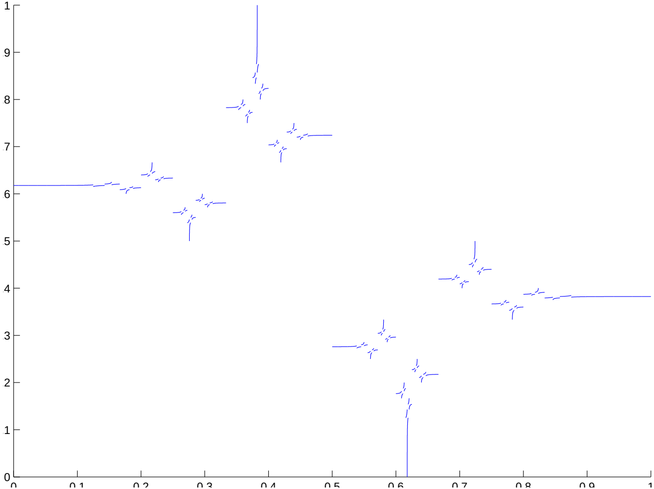

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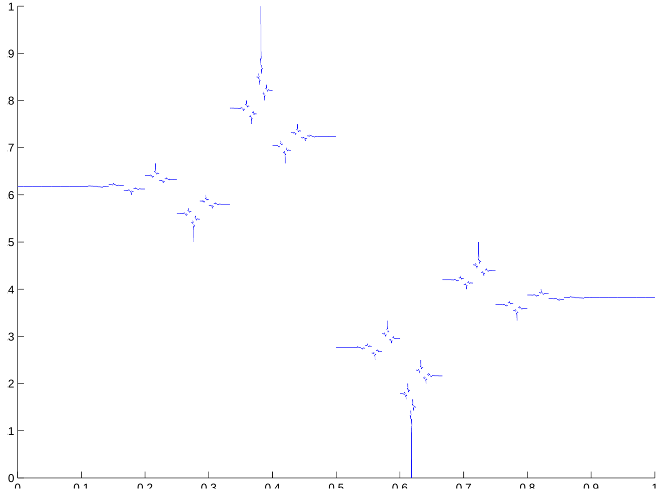
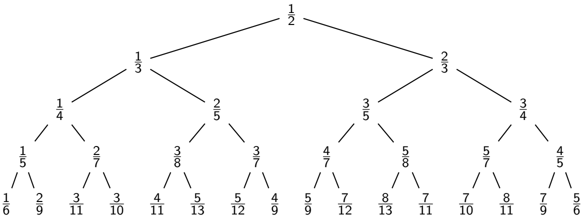


ج as a symmetry of Lebesgue's measure

Let's turn back to the Farey tree...



A random walker starts from the root vertex. For each vertex x, we are given the probability $\pi(x)$ of **arriving** to that vertex from its parent.

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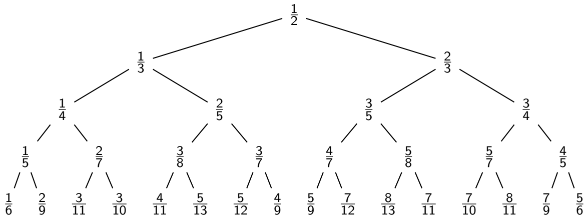

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If we set $\pi(x) \equiv 1/2$, then the c.d.f. of the induced measure on $[0,1]$ is the Minkowski-Denjoy question mark function.

Question

Which 'arrival' probability function $\pi_{Leb}(x)$ induce the Lebesgue measure?

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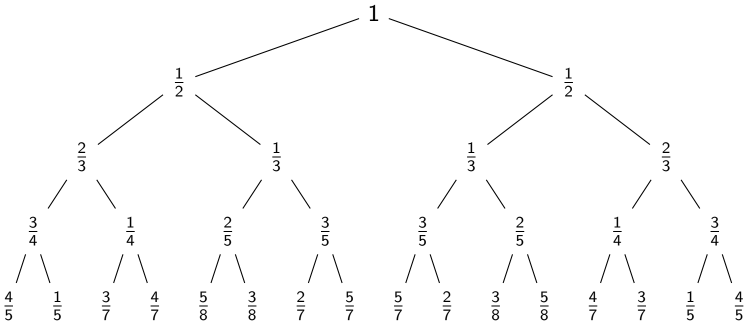

The "Lebesgue tree" $\mathcal{L}$.

Fact

Assume $n_k > 1$. Then the arrival probabilities

$$\pi_{Leb}([0, n_1, n_2, \ldots, n_{k-1}, n_k]) = 1 - [0, n_k - 1, n_{k-1}, \ldots, n_2, n_1]$$

induces the Lebesgue measure on $[0, 1]$.

A subtle symmetry of Lebesgue's measure:

$$\pi_{Leb}\text{ج}(x) = \text{ج}\pi_{Leb}(x)$$

(On the left hand side ج acts on the tree whereas on the right it acts on the rationals)

References

- Sur un mode nouveau de représentation géométrique des formes quadratiques binéaires définies ou indéfinies. M. H. Poincaré.
- *Jimm, a Fundamental Involution.* (with H. Ayral) arXiv:1501.03787
- *On the involution of the real line induced by Dyer's outer automorphism of PGL(2,Z).* (with H. Ayral) arXiv:1605.03717
- *A subtle symmetry of Lebesgue's measure.* (with H. Ayral) arXiv:1605.07330
- *Testing the transcendence conjecture of Jimm and its continued fraction statistics.* (with H. Ayral, to appear)
- *An involution of reals, discontinuous on rationals and whose derivative vanish almost everywhere.* (with H. Ayral, to appear)

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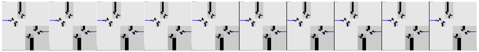
THANKS...



BONUS MATERIAL

ج acts on..

- Binary quadratic forms (tears apart class groups)
- Beatty partitions of $\mathbf{N}$.

$$r \in \mathbf{R} \setminus \mathbf{Q} \rightsquigarrow \mathcal{B}_r = \lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, \ldots = (\lfloor nr \rfloor)_{n \geq 1}$$

  If $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$ then $\mathcal{B}_r \cup \mathcal{B}_s = \mathbf{N}$. Hence ج induce a duality of Beatty partitions of $\mathbf{N}$.
- Trivalent ribbon graphs $\simeq$ dessins $\simeq$ decorated TM spaces. $\implies$ ج induces a duality of punctured Riemann surfaces.
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Example.

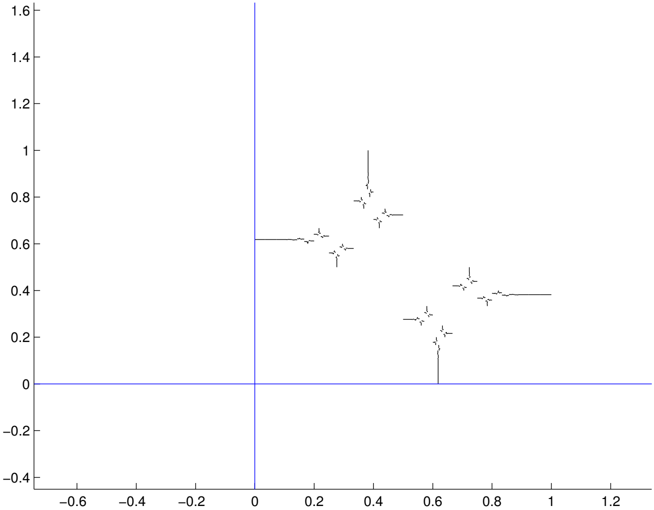
$$\text{ج}([0;\overline{1_{n-1},a}]) = [0;n,\overline{1_{a-2},n+1}] \implies$$

$$\text{ج}\left(\frac{a}{2}\left[\sqrt{1+4\frac{aF_{n-1}+F_{n-2}}{a^2F_n}}-1\right]\right)$$

$$= \frac{1}{n+\frac{n+1}{2}\left(\sqrt{1+4\frac{(n+1)F_{a-2}+F_{a-3}}{(n+1)^2F_{a-1}}}-1\right)}$$

(notice the exchange $(a, F_n) \leftrightarrow (F_a, n)$)

Extra Slides - Zoom



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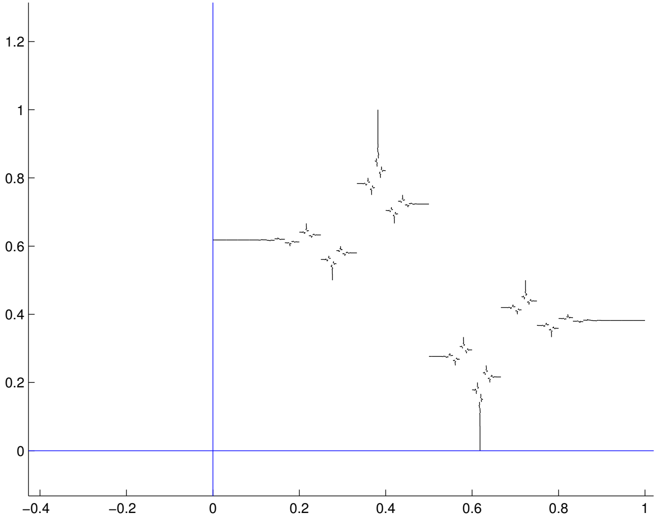

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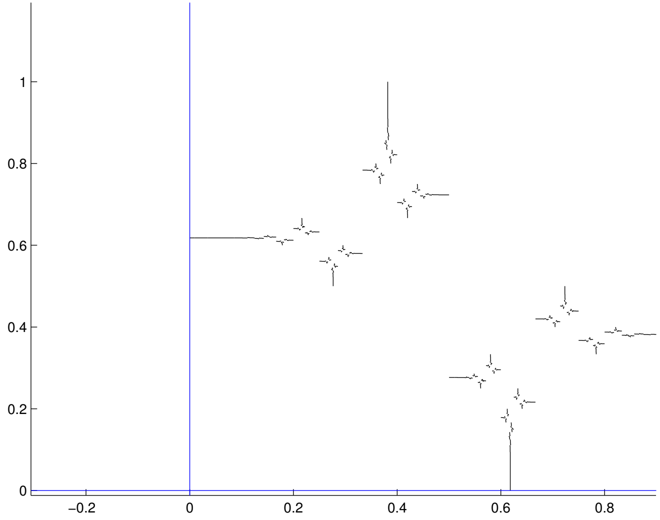

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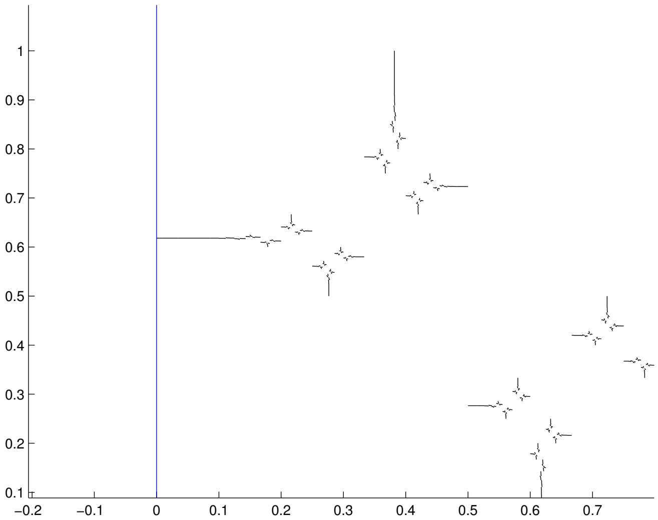

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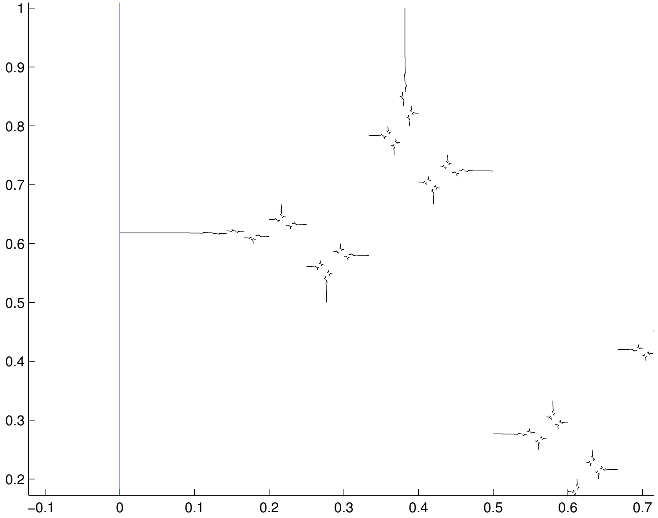

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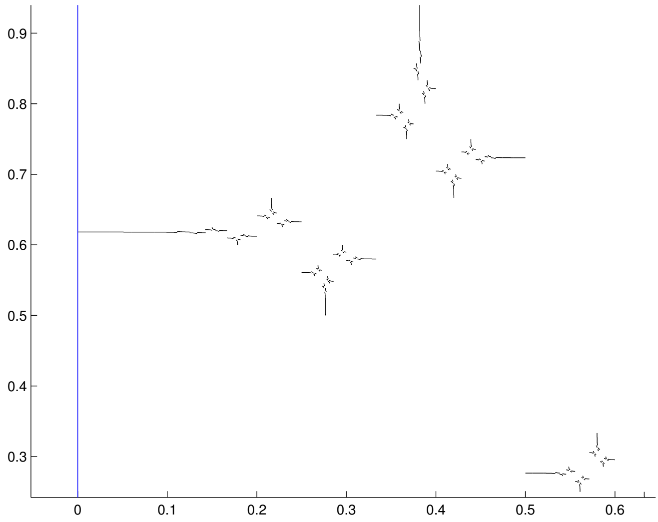

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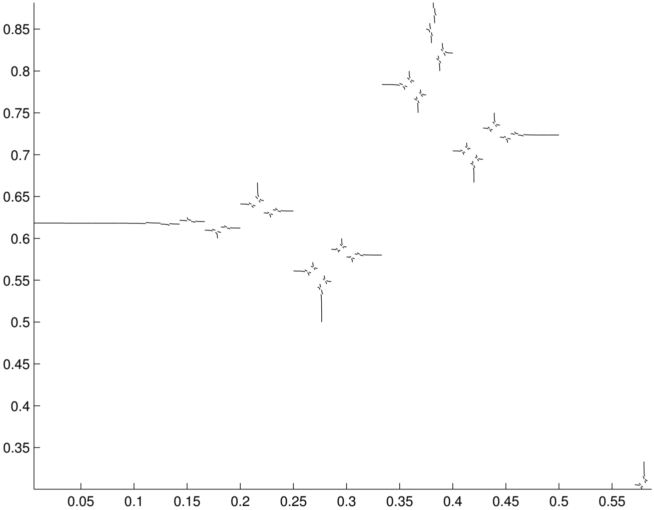

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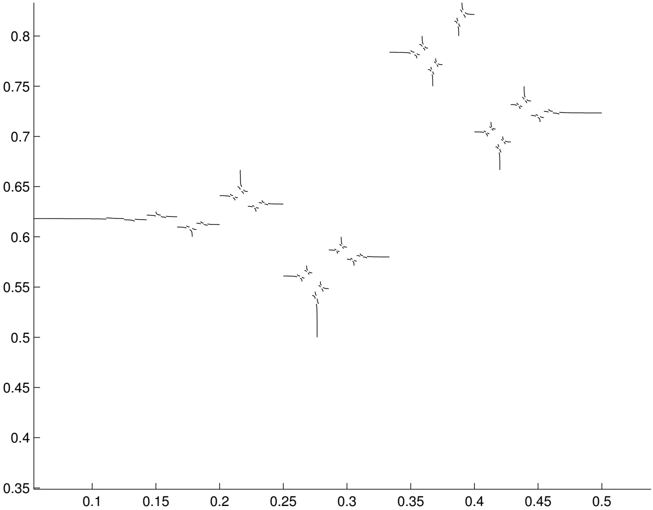

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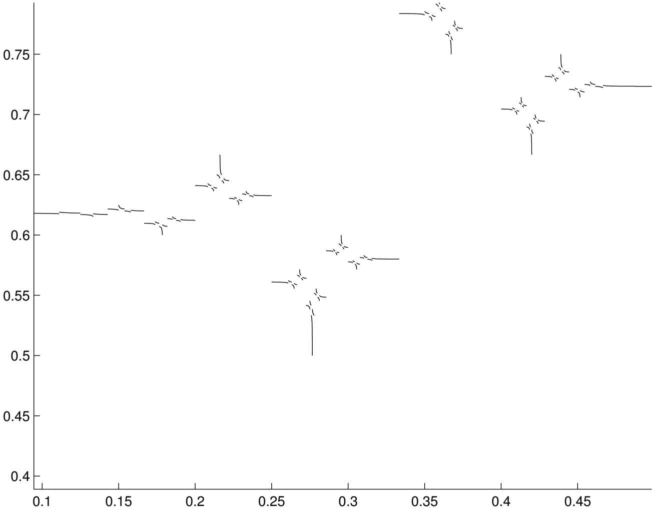

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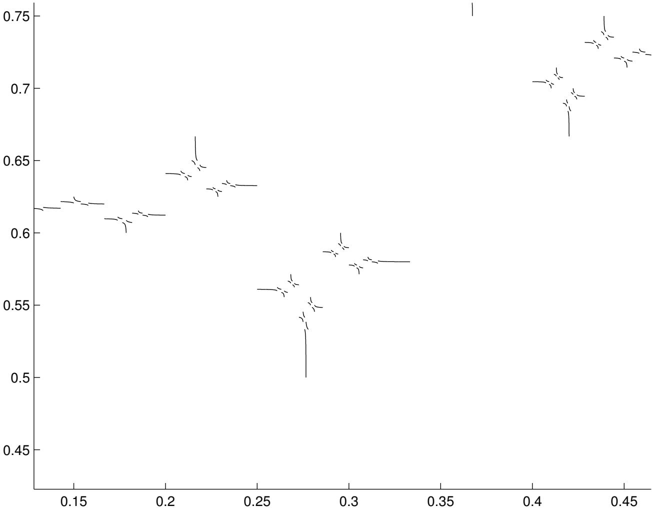

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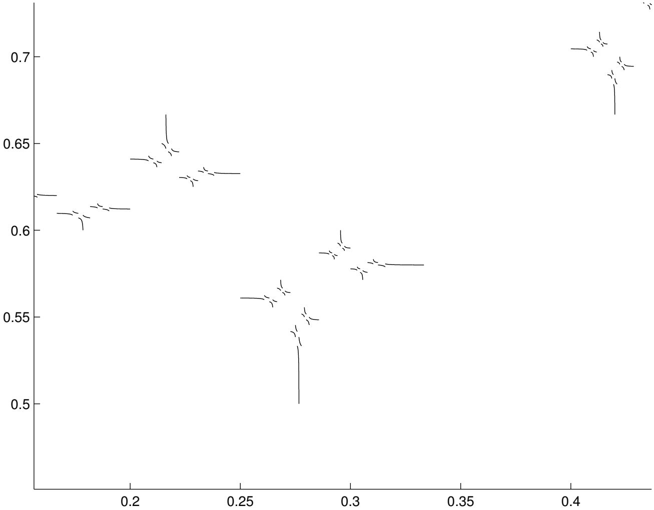

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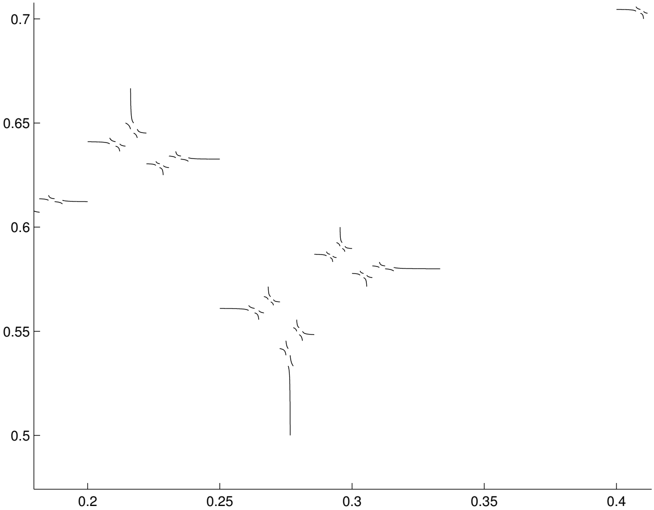

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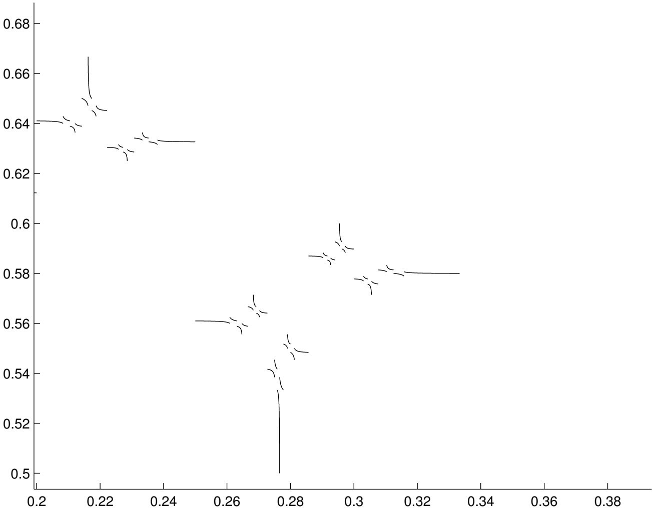

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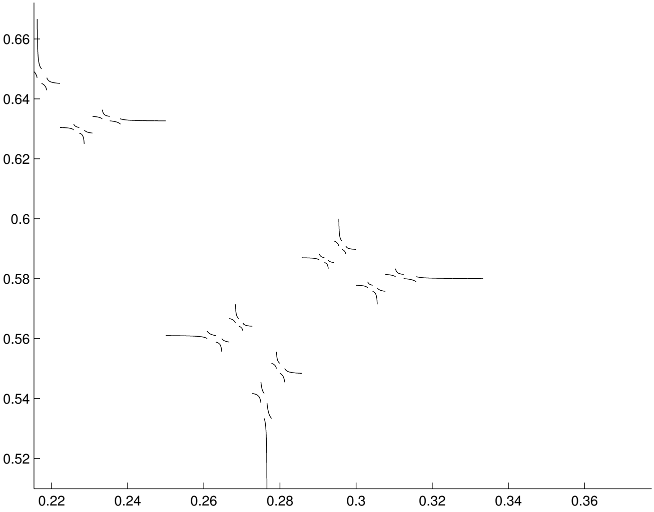

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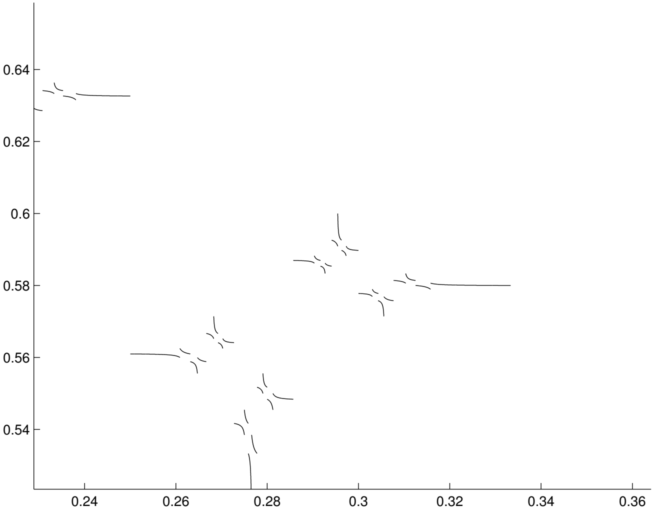

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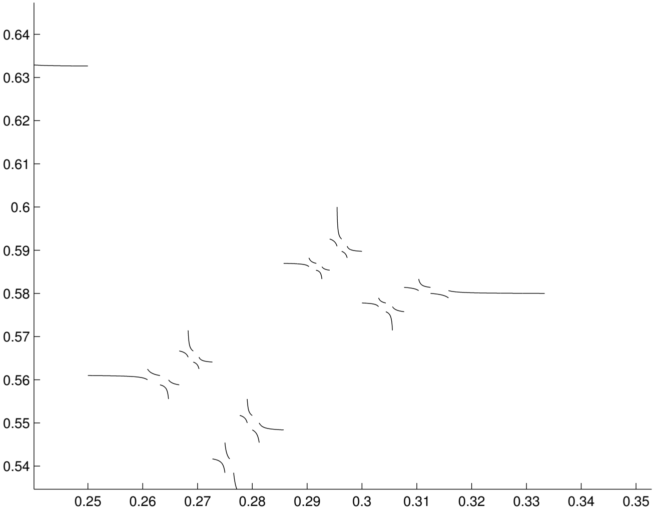

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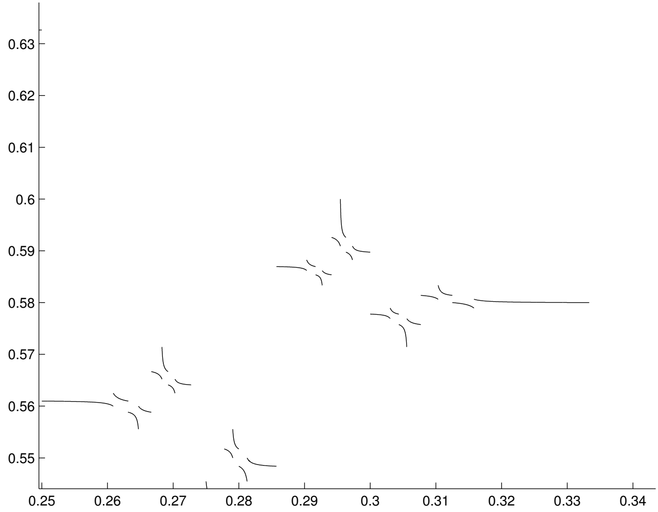

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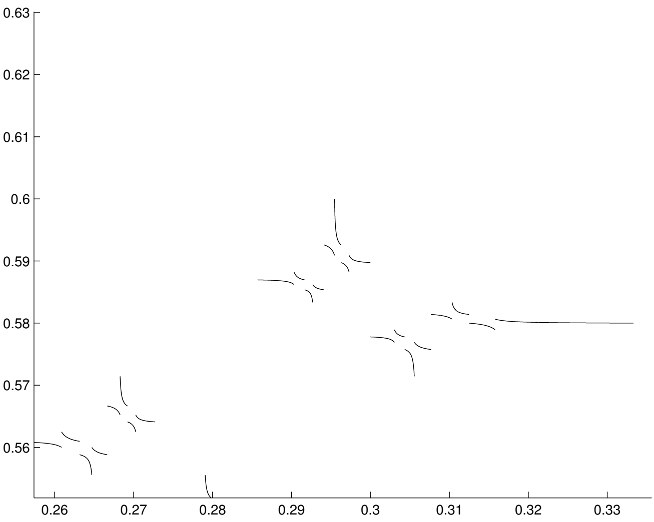

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