Dyer's outer automorphism of PGL(2,Z) and the codenominator.

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(with several co-authors over time: Ismail Özkaraca, Hakan Ayral, Buket Eren Gökmen)

Abstract

$$\begin{array}{ccccccccccc} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & \dots \\ 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & 55, & \dots \end{array}$$

In this talk we give an answer to the question:

What is the qth Fibonacci number, where q is rational?

and finish with some more questions.

Spoiler: The q^{th} Fibonacci number will be the codenominator $F(x)$, which is always an integer.

For example, the $\frac{23}{31}^{th}$ Fibonacci number is $F(\frac{23}{31}) = 107$.

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For example, the $\frac{23}{31}^{th}$ Fibonacci number is $\mathsf{F}(\frac{23}{31}) = 107$.

The numerator function

Let num be the numerator function num : $\mathbf{Q}^+ \to \mathbf{Z}^+$ defined by

$$\text{num} : x = \frac{p}{q} \in \mathbf{Q}^+ \to p \in \mathbf{Z}^+,$$

with $p, q > 0$, $\gcd(p, q) = 1$.

It satisfies the functional equations

$$\text{num}(1+x) = \text{num}(x) + \text{num}(1/x),$$
$$\text{num}\left(\frac{x}{1+x}\right) = \text{num}(x)$$

and the initial condition $\text{num}(1) := 1$.

These equations determine the function num completely on $\mathbf{Q}^+$, and num satisfies the additional equation

$$\text{num}(x) = x\,\text{num}(1/x)$$

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Let us check the equations for the numerator:

Set $x = p/q > 0$ with p and q coprime $\implies$

$\mathrm{num}(x+1) =$

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Now consider the function $\mathrm{con} : \mathbf{Q}^+ \to \mathbf{Z}^+$ defined as the solution of the system

$$f(1+x) = f(x) + f(1/x), \quad (*)$$
$$f\left(\frac{1}{1+x}\right) = f(x) \quad (**)$$

which is unique under the condition $f(1) := 1$.

The conumerator

$$f(1+x) = f(x) + f(1/x), \quad (*)$$
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- One can show that this system is coherent and f can be computed in terms of $f(1)$.
- The solution is unique if we fix $f(1) = 1$.
- We call this solution the conumerator and denote as $\text{con} : \mathbf{Q}^+ \to \mathbf{Z}^+$.

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The codenominator

The codenominator function $\mathsf{F} : \mathbf{Q}^+ \to \mathbf{Z}^+$ is

$$\mathsf{F}(x) := \mathrm{con}(1/x).$$

It is defined by the system

$$\mathsf{F}(1+1/x) = \mathsf{F}(x) \iff \mathsf{F}(1+x) = \mathsf{F}(1/x) \tag{1}$$

$$\mathsf{F}\left(\frac{1}{1+x}\right) = \mathsf{F}(x) + \mathsf{F}(1/x) \tag{2}$$

with $\mathsf{F}(1) := 1$.

Computing $\mathsf{F}(x+2)$ by (1-2) we get

$$\begin{aligned} \mathsf{F}(x+2) &= \mathsf{F}(1/(x+1)) && \text{by (1)} \\ &= \mathsf{F}(x) + \mathsf{F}(1/x) && \text{by (2)} \\ &= \mathsf{F}(x) + \mathsf{F}(1+x) && \text{by (1)} \end{aligned}$$

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Connection with the Fibonacci Sequence

$\Longrightarrow$ F extends the Fibonacci sequence to $\mathbf{Q}^+$:

$$\boxed{\mathsf{F}(n) = F_n}$$

Here F_n is the usual Fibonacci sequence

$$F_0 = 0,$$
$$F_1 = 1,$$
$$F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbf{Z}^+)$$

$\mathrm{con}(x) = \mathsf{F}(1/x) = \mathsf{F}(1+x)$

$\Longrightarrow$ $\mathrm{con}(n) = F_{n+1}$ is the Fibonacci sequence shifted by one.

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Connection with the Fibonacci Sequence

- The codenominator extends the Fibonacci sequence to positive rational arguments.
- The codenominator is an integer-valued function on $\mathbf{Q}^+$.
- For every rational $x \in \mathbf{Q}^+$, the sequence $G_n := F(x+n)$ forms the **Gibonacci sequence** defined by:

$$\begin{aligned} G_0 &= F(x), \\ G_1 &= F(1+x), \\ G_{n+2} &= G_n + G_{n-1}. \end{aligned}$$

Examples of some values of F

Examples (n is a positive integer):

#	Formula
1	$F(n) = F_n$
2	$F(1/2) = 2$, more generally $F(1/n) = F(n+1) = F_{n+1}$
3	$F(n+1/2) = L_n = F_{n+1} + F_{n-1}$ (Lucas sequence)
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Further examples of values of F

A list of values $F(41/n)$:

1	59369×2789	51	$3 \times 5 \times 59$	101	2×137	151	5×73
2	7×2161	52	47	102	$2 \times 3 \times 67 \times 149$	152	7×19
3	$5 \times 7 \times 19$	53	5×13	103	$2 \times 31 \times 613$	153	$3^2 \times 11$
4	5×67	54	3×59	104	$2 \times 3 \times 29$	154	41^2
5	11×19	55	19×101	105	5×7	155	3×7^2
6	$3 \times 5 \times 7$	56	53	106	5×13	156	1069
7	103	57	$2 \times 3 \times 7$	107	31	157	$2^3 \times 67$
8	$3^2 \times 23$	58	$2 \times 5 \times 7$	108	$2^2 \times 11$	158	$2 \times 5 \times 53$
9	31	59	$2^3 \times 5$	109	$2 \times 3 \times 257$	159	1063
10	7^3	60	193	110	2×103	160	$5 \times 11 \times 31$
11	17	61	42187	111	3×5^2	161	5×677
12	$2^3 \times 3$	62	7×4463	112	$2^3 \times 7$	162	13×5923
13	5×13	63	11×13	113	$2^2 \times 5 \times 47$	163	72043×11699
14	3×281	64	29	114	2×41	164	5
15	23	65	53	115	$2^3 \times 3 \times 5^2$	165	1631643593
16	19	66	3^3	116	7×43	166	23×6481
17	29	67	37	117	$3^3 \times 11$	167	6553
18	17	68	$7 \times 11 \times 17$	118	$2^2 \times 149$	168	3301
19	3^4	69	3×53	119	$2^2 \times 239$	169	29×71
20	89×199	70	2×29	120	$2 \times 13 \times 73$	170	$2^3 \times 3 \times 43$

In particular,

$$\begin{aligned} \mathsf{F}(n+1/2) &= F_n F_2 + F_{n-1} F_3 \\ &= F_{n-1} + F_{n+1} \\ &= L_n \end{aligned}$$

is the **Lucas sequence.**

Hence we extend the Lucas sequence to the *Lucas function* on $\mathbf{Q}^+$ as

$$\mathsf{L}(x) := 2\mathsf{F}(1/x) - \mathsf{F}(x)$$

Iterating the functional equations yields the following result:

Fibonacci invariance

For all $n \in \mathbf{Z}^+$ and $x \in \mathbf{Q}^+$ one has

$$\mathsf{F}\left(\frac{F_n + F_{n+1}x}{F_{n-1} + F_n x}\right) = \mathsf{F}(x)$$

In particular

$$\mathsf{F}\left(\frac{F_n}{F_{n+1}}\right) = n, \quad \mathsf{F}\left(\frac{F_{n+1}}{F_n}\right) = 1, \quad \mathsf{F}\left(\frac{1}{n}\right) = \mathsf{F}(1+n) = F_{n+1}.$$

Properties of the codenominator-I (Fibonacci invariance)

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Properties of the codenominator-II (Fibonacci recursion)

Iterating the functional equations yields

Fibonacci recursion

For all $n \in \mathbf{Z}^+$ and $x \in \mathbf{Q}^+$ one has

$$\mathsf{F}(n+x) = F_n \mathsf{F}(1+x) + F_{n-1} \mathsf{F}(x)$$

Properties of the codenominator-II (Fibonacci recursion)

Any real number x can be written as a continued fraction

$$x = [n_0, n_1, \dots, n_k] := n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\dots}}},$$

which is unique if x is irrational or else $n_k > 1$.

We can use the recursion property to compute $\mathrm{F}(x) = \mathrm{F}[n_0, n_1, \dots, n_k]$

Continued fraction recursion

$$\mathrm{F}[n_0, \dots, n_k] = F_{n_0}\mathrm{F}[n_1, n_2, \dots, n_k] + F_{n_0 - 1}\mathrm{F}[n_1 + 1, \dots, n_k]$$

Any real number x can be written as a continued fraction

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Properties of the codenominator-II (Fibonacci recursion)

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Properties of the codenominator-III (Splitting)

The following property is an analogue of a property of continuants and is a generalization of the recursion property above:

Splitting

$$\begin{aligned}
&F[m_0, \ldots, m_r] = \\
&F[m_0, \ldots, m_l]F[m_{l+1}, \ldots, m_r] \\
&+ F[m_0, \ldots, m_s - 1]F[m_{l+1} + 1, \ldots, m_r],
\end{aligned}$$

where s is the least index such that $m_s = \cdots = m_l = 1$.
(If $m_0 = \cdots = m_l = 1$, then set $F[m_0, \ldots, m_s - 1] = 0$.)

Properties of the codenominator-IV (Symmetry)

Symmetry

For all $x \in \mathbf{Q}^{+} \cap (0,1)$ one has

$$F(1-x) = F(x)$$

As an analogue of Euler's reversion formula for continued fractions, we have

Reversion

$$\mathrm{F}[0, n_1, \ldots, n_k] = F[0, n_k, \ldots, n_1]$$

The Fibonacci sequence (F_n) is periodic modulo m for any positive integer m. This period is called the Pisano period and denoted by $\pi(m)$.

We have for the codemoninator F:

Periodicity

For any positive integer m:

- $(\mathrm{F}[n_0, n_1, \ldots, n_k] \mod m)_{n_j}$ is periodic for each j with period divisible by $\pi(m)$.
- $(\mathrm{F}(k/N) \mod m)_k$ is periodic for $\forall N$, with period divisible by N.
- $(\mathrm{F}(N/k) \mod m)_k$ is periodic for $\forall N$, with period divisible by N.
- $(\mathrm{F}(k+x) \mod m)_k$ is periodic for $\forall x$, with period?
- $(\mathrm{F}(kx) \mod m)_k$ is periodic for $\forall x$, with period $\mathrm{den}(x)\pi(m)$?

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It is known that the Fibonacci sequence satisfies for each m, n:

$$\frac{F_{mn}}{F_m} \in \mathbf{Z}$$

For the codenominator one has

Divisibility

$$\frac{F[mn_0, mn_1, \ldots, mn_k]}{F(m)} \in \mathbf{Z}$$

$$\frac{\mathrm{F}[0, n_0, n_1, \ldots, n_k, n_0, n_1, \ldots, n_k, \ldots n_k]}{\mathrm{F}[n_0, n_1, \ldots, n_k]} \in \mathbf{Z}$$

Involutivity

For every $x \in \mathbf{Q}^+$ one has

$$\frac{\mathrm{F}\left(\dfrac{\mathrm{F}(x)}{\mathrm{F}(1/x)}\right)}{\mathrm{F}\left(\dfrac{\mathrm{F}(1/x)}{\mathrm{F}(x)}\right)} = x$$

This is a consequence of the fact that

$$\mathrm{num}(x) = \mathrm{F}\left(\frac{\mathrm{F}(x)}{\mathrm{F}(1/x)}\right),$$

i.e. the numerator can be expressed in terms of the codenominator.

The codiscriminant function

We define the *codiscriminant* function for $x \in \mathbf{Q}^+$ as

$$\mathrm{cds}(x) := \mathrm{F}(1/x)^2 - \mathrm{F}(x)\mathrm{F}(1/x) - \mathrm{F}(x)^2$$

The codiscriminant

- cds is 2-periodic on $\mathbf{Q}^+$. In fact,

$$\mathrm{cds}(1+x) = -\,\mathrm{cds}(x).$$

- For $x \in (0,1) \cap \mathbf{Q}$ one has

$$\mathrm{cds}(1-x) = \mathrm{cds}(x).$$

Hence, $\mathrm{cds}(n-x) = (-1)^{n+1}\,\mathrm{cds}(x)$ for $n > x$, $n \in \mathbf{Z}$.

In particular, for $x = n \in \mathbf{Z}^+$ this reduces to the Cassini identity

$$\mathrm{cds}(n) = F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n.$$

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Generalizing Fibonacci identities: Examples

Among the myriad Fibonacci identities in the literature, many admit a codenominator interpretation.

The idea is to replace $F_n \leftrightarrow \mathrm{F}(x)$ and $(-1)^n \leftrightarrow \mathrm{cds}(x)$ in the formula.

For example:

Theorem

If at least two among $x, y, z \in \mathbf{Q}^+$ are integral, then

$$\mathrm{F}(x+y)\mathrm{F}(x+z) - \mathrm{F}(x)\mathrm{F}(x+y+z) = \mathrm{cds}(x)\mathrm{F}(y)\mathrm{F}(z) \quad (3)$$

This reduces to Taguiri's identity when $x, y, z \in \mathbf{Z}$.

D'Ocagne's identity and the Catalan identity are instances of this formula

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Generalizing Fibonacci identities: Further examples

For $x \in \mathbf{Q}^+$ and $n \in \mathbf{Z}^+$ one has

$$\sum_{k=0}^{n} F(x+k) = F(x+k+2) - F(1+x).$$

$$\sum_{k=0}^{n} \binom{n}{i} F(i+x) = F(2n+x)$$

$$\sum_{k=0}^{n} \sum_{\ell=1}^{n} \binom{n}{k} \binom{n}{l} 2^{k+\ell} F[k,\ell] = F[3n,3n] - F(3n-1)$$

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Exercise

If you want to have some fun

Take your favorite Fibonacci identity

and

generalize it

to the codenominator

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Codenominator identities

$$\sum_{k=0}^{\infty} \frac{F(q+k)}{2^k} = 2F(q+2)$$

$$\sum_{k=0}^{\infty} \frac{kF(q+k)}{2^k} = 2F(q+5)$$

$$\sum_{k=0}^{\infty} \frac{k^2F(q+k)}{2^k} = 2\left(F(q+7)+F(q+9)\right)$$

$$\sum_{k=0}^{\infty} \frac{k^3F(q+k)}{2^k} = 2\left(3F(q+12)+F(q+13)\right)$$

$$\sum_{k=0}^{\infty} \frac{F(q+k)}{3^k} = \frac{3}{5}\left(F(q+2)+F(q)\right)$$

$$\sum_{k=0}^{\infty} \frac{kF(q+k)}{3^k} = \frac{3}{5}F(q+3)$$

$$\sum_{k=0}^{\infty} \frac{k^2F(q+k)}{3^k} = \frac{3}{25}\left(7F(q+5)-F(q+4)\right)$$

$$\sum_{k=0}^{\infty} \frac{F(q+2k)}{3^k} = 3F(q+2)$$

$$\sum_{k=0}^{\infty} \frac{F(q+2k)}{3^k} = 3F(q+6)$$

$$\sum_{k=0}^{\infty} \frac{k^2F(q+2k)}{3^k} = 3\left(4F(q+8)+F(q+9)\right)$$

$$\sum_{k=0}^{\infty} \frac{(-1)^kF(q+2k)}{3^k} = \frac{3}{19}\left(3F(q)+F(q-2)\right)$$

Codenominator identities

$$\sum_{k=0}^{n}\binom{n}{k}F(q+k)=F(q+2n),\quad \sum_{k=0}^{n}\binom{n}{k}kF(q+k)=nF(q+2n-1),$$

$$\sum_{k=0}^{n}\binom{n}{k}k^2F(q+k)=n\big(nF(q+2n-2)+F(q+2n-3)\big)$$

$$\sum_{k=0}^{n}\binom{n}{k}k^3F(q+k)=n\big(n^2F(q+2n-3)+3nF(q+2n-4)-F(q+2n-6)\big)$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}k^4F(q+k)=n\big[&n^3F(q+2n-4)+6n^2F(q+2n-5)-n\big(F(q+2n-9)-2F(q+2n-8)\big)\\&-3F(q+2n-8)-F(q+2n-7)\big]\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}k^5F(q+k)=n\big[&n^4F(q+2n-5)+10n^3F(q+2n-6)+5n^2\big(3F(q+2n-9)+F(q+2n-8)\big)\\&-5n\big(5F(q+2n-9)+F(q+2n-8)\big)-2F(q+2n-10)+9F(q+2n-9)\big]\end{aligned}$$

$$\sum_{k=0}^{n}\binom{n}{k}F(q+2k)=5^{\lfloor n/2\rfloor}\big(F(q+n+1)-(-1)^nF(q+n-1)\big)$$

$$\sum_{k=0}^{n}\binom{n}{k}kF(q+2k)=5^{\lfloor (n-1)/2\rfloor}n\big(F(q+n+2)+(-1)^nF(q+n)\big)$$

$$\sum_{k=0}^{n}\binom{n}{k}k^2F(q+2k)=n\big[5^{\lfloor (n-1)/2\rfloor}\big(F(q+n+2)+(-1)^nF(q+n)\big)+5^{\lfloor n/2\rfloor-1}(n-1)\big(F(q+n+3)-(-1)^nF(q+n+1)\big)\big]$$

$$\begin{aligned}\sum_{k=0}^{\infty}x^kF(p+k)F(q+k)=&\frac{1}{1-3x+x^2}\big(F(p)F(q)-xF(p-1)F(q-1)\big)\\&-\frac{x}{(1+x)(1-3x+x^2)}\big(3F(p)F(q)-F(p-1)F(q-1)-F(p+1)F(q+1)\big)\end{aligned}$$

Codenominator and Riemann's Zeta (bait)

$$\sum_{x\in\mathbf{Q}^+} \frac{1}{\mathrm{F}(x)^s\mathrm{F}(1/x)^s} = \frac{\zeta(s)^2}{\zeta(2s)},$$

$$\sum_{x\in\mathbf{Q}^+} \frac{(-1)^{\mathrm{F}(x)+\mathrm{F}(1/x)}}{\mathrm{F}(x)^s\mathrm{F}(1/x)^s} = \frac{(2^{1-s}-1)^2\zeta(s)^2}{(2^{1-2s}-1)\zeta(2s)}.$$

$$\sum_{q\in\mathbf{Q}^+\cap[0,1]} \frac{1}{F(q)^s} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

For the simple reason that

$$(p,q)\in(\mathbf{Z}^+)^2 \to \gcd(p,q)(\mathrm{F}(p/q),\mathrm{F}(q/p))\in(\mathbf{Z}^+)^2$$

is bijective, i.e. it gives an alternative indexing of the first quadrant of $\mathbf{Z}^2$.

The involution Jimm

The function below is called Jimm:

$$\text{ج} : x \in \mathbf{Q}^+ \to \frac{F(1/x)}{F(x)} \in \mathbf{Q}^+$$

$$\iff \text{ج}(x) = \frac{F(x+1)}{F(x)}$$

The involution Jimm

$$\frac{\text{numerator}(x)}{\text{denominator}(x)} = x \quad \text{`rational'}$$

$$\frac{\text{conumerator}(x)}{\text{codenominator}(x)} = \text{ج}(x) \quad \text{`corational'}$$

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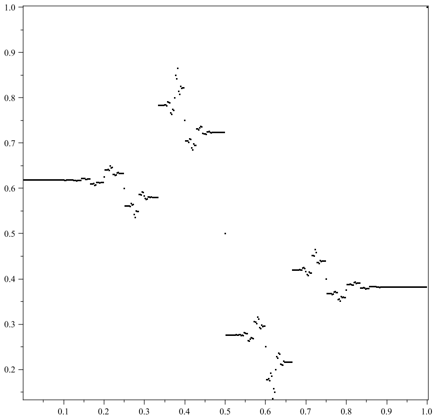

Plot of Jimm on the unit interval (more on the plot later)

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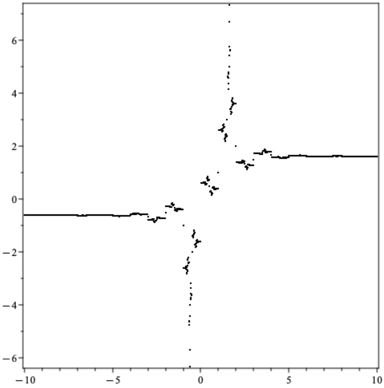

Plot of Jimm on the real line (more on the plot later)

Properties of Jimm - Involutivity

Involutivity

$$ج(ج(x)) = x$$

Properties of Jimm - Covariance

Covariance (commutativity) with $1/x$

$$ج\left(\frac{1}{x}\right) = \frac{1}{ج(x)}$$

Properties of Jimm - Covariance

Covariance (commutativity) with $1 - x$

For $x \in \mathbf{Q}^+ \cap (0,1)$ one has

$$\text{ج}(1 - x) = 1 - \text{ج}(x)$$

Properties of Jimm - Covariance

We can extend ج to $\mathbf{Q}\backslash\{0\}$ via $\text{ج}(-x) = -1/\text{ج}(x)$ so that it satisfies

Twisted covariance with $-x$

$$\text{ج}(-x) = \frac{-1}{\text{ج}(x)}$$

Properties of Jimm - Covariance

From the above functional equations we deduce

Golden connection

$$\text{ج}(1+x) = 1 + \frac{1}{\text{ج}(x)}$$

How to compute Jimm

If x is given as a continued fraction, then by using the functional equations we can easily compute ج(x)

Computation of Jimm

Let $x = [n_0, n_1, \ldots, n_k] \in \mathbf{Q}^+$.
Let 1_k the sequence $1, 1, \ldots, 1$ of length k. $\Longrightarrow$

ج$(x) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, 2, \ldots 2, 1_{n_{k-1}-2}, 2, 1_{n_k-1}]$

with the rules:
$[\ldots, n, 1_0, m, \ldots] := [\ldots, n, m, \ldots]$, and
$[\ldots, n, 1_{-1}, m, \ldots] := [\ldots, n+m-1, \ldots]$.

Extending Jimm to $\mathbf{R} \setminus \{0\}$

Extend ج to $\mathbf{R} \setminus \{0\}$ via

$$\text{ج}(y) = \lim_{x \in \mathbf{Q}^*, x \to y} \text{ج}(x),$$

Then the extension is also involutive and satisfies

$$\text{ج}(\frac{1}{x}) = \frac{1}{\text{ج}(x)}, \quad \text{ج}(1-x) = 1 - \text{ج}(x), \quad \text{ج}(-x) = -\frac{1}{\text{ج}(x)},$$

One has

$$\mathrm{PGL}_2(\mathbf{Z}) = \langle -x, 1/x, 1-x \rangle$$

and these functional equations shows that ج acts as the outer automorphism of $\mathrm{PGL}_2(\mathbf{Z})$.

Alternatively, ج is an equivariant function for the $\mathrm{PGL}_2(\mathbf{Z})$-action on $\mathbf{R}$.

Extending Jimm to $\mathbf{R} \setminus \{0\}$

Definition (Recall)

$$\text{ج}([n_0, n_1, n_2, \ldots]) = [1_{n_0 - 1}, 2, 1_{n_1 - 2}, 2, 1_{n_2 - 2}, \ldots]$$

Examples

$$\text{ج}([3, 3, 3, \ldots] = [1_{3-1}, 2, 1_{3-2}, 2, 1_{3-2}, 2 \ldots] = [1, 1, 2, 1, 2, 1, 2, \ldots]$$

$$\text{ج}([5, 5, 5, \ldots] = [1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, \ldots]$$

Extending Jimm to $\mathbf{R} \setminus \{0\}$

Computation (Recall)

$$\text{ج}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

This definition works only if $n_k \geq 2$. To make it work for $n_k = 2$, use

RULE I

$$\ldots, n, 1_0, m, \cdots = \ldots, n, m, \ldots$$

Examples

$$\text{ج}([2, 2, 2, \ldots]) = [1, 2, 1_0, 2, 1_0, 2 \ldots] = [1, 2, 2, 2, \ldots]$$

$$\text{ج}([2, 3, 2, 3 \ldots]) = [1, 2, 1, 2, 2, 1, 2, 2, 1, \ldots]$$

Extending Jimm to $\mathbf{R} \setminus \{0\}$

Computation (Recall)

$$ج([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

This definition works only if $n_k \geq 2$. To make it work for $n_k = 2$, use

RULE I

$$\ldots, n, 1_0, m, \cdots = \ldots, n, m, \ldots$$

Examples

$$ج([2, 2, 2, \ldots]) = [1, 2, 1_0, 2, 1_0, 2 \ldots] = [1, 2, 2, 2, \ldots]$$

$$ج([2, 3, 2, 3 \ldots]) = [1, 2, 1, 2, 2, 1, 2, 2, 1, \ldots]$$

Extending Jimm to $\mathbf{R} \setminus \{0\}$

Computation (Recall)

$$ج([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

To make it work also when $n_k = 1$, use

RULE II

$$\ldots, n, 1_{-1}, m, \cdots = \ldots, n+m-1, \ldots$$

Examples

$$ج([1, 1, 2, 1, 2, 1, 2, \ldots]) =$$

$$[1_0, \underbrace{2, 1_{-1}, 2}_{3}, 1_0, \underbrace{2, 1_{-1}, 2}_{3}, 1_0, \underbrace{2, 1_{-1}, 2}_{3}, \ldots] =$$

$$= [3, 3, 3, \ldots]$$

remember?

Extending Jimm to $\mathbf{R} \setminus \{0\}$

Computation (Recall)

$$\text{ج}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

To make it work also when $n_k = 1$, use

RULE II

$$\ldots, n, 1_{-1}, m, \cdots = \ldots, n+m-1, \ldots$$

Examples

$$\text{ج}([1, 1, 2, 1, 2, 1, 2, \ldots]) =$$

$$[1_0, \underbrace{2, 1_{-1}, 2}_{3}, 1_0, \underbrace{2, 1_{-1}, 2}_{3}, 1_0, \underbrace{2, 1_{-1}, 2}_{3}, \ldots] =$$

$$= [3, 3, 3, \ldots]$$

remember?

Extending Jimm to $\mathbf{R} \setminus \{0\}$

Computation (Recall)

$$\text{ج}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

Example

$$\text{ج}([\ldots, 7, 1, 1, 1, 1, 13, \ldots]) =$$

$$[\ldots 1_5, \underbrace{2, 1_{-1}, 2}, 1_{-1}, 2, 1_{-1}, 2, 1_{-1}, 2, 1_{11}, \ldots]$$

$$[\ldots 1_5, \underbrace{3, 1_{-1}, 2}, 1_{-1}, 2, 1_{-1}, 2, 1_{11}, \ldots]$$

$$[\ldots 1_5, \underbrace{4, 1_{-1}, 2}, 1_{-1}, 2, 1_{11}, \ldots]$$

$$[\ldots 1_5, \underbrace{5, 1_{-1}, 2}, 1_{11}, \ldots]$$

$$[\ldots 1_5, 6, 1_{11}, \ldots]$$

Jimm is a covariant modular function

We have

$$\text{ج}(Mx) = \alpha(M)\text{ج}(x),$$

where $\alpha : \mathrm{PGL}_2(\mathbf{Z}) \to \mathrm{PGL}_2(\mathbf{Z})$ is Dyer's outer automorphism

$$\begin{aligned} \alpha(1/x) &= 1/x \\ \alpha(1-x) &= 1-x \\ \alpha(-x) &= -1/x \end{aligned}$$

Note that this implies, by involutivity of ج,

$$\text{ج}(M\text{ج}(x)) = \alpha(M)(x),$$

i.e. Dyers's involution can be written in terms of ج.

Since ج is covariant, it respects the $\mathrm{PGL}_2(\mathbf{Z})$-action:

ج sends $\mathrm{PGL}_2(\mathbf{Z})$-orbits to $\mathrm{PGL}_2(\mathbf{Z})$-orbits.

In other words, ج respects ends of continued fractions:

If $x = [n_0, n_1, \ldots]$ and $y = [m_0, m_1, \ldots]$ have the same end, then so does $\text{ج}(x)$ and $\text{ج}(y)$.

Therefore ج induces an involution of the moduli space of pseudolattices

$$\text{ج} \circlearrowleft \mathrm{PGL}_2(\mathbf{Z}) \backslash (\mathbf{R} \cup \{\infty\})$$

Analytic properties of Jimm

- ج is continuous on $\mathbf{R} \backslash \mathbf{Q}$
- ج is differentiable almost everywhere
- its derivative vanish almost everywhere
- has jump discontinuities on $\mathbf{Q}$

Jumps of Jimm

Let $n_0 > 1$. Then the jump of ج at $[n_0, n_1, \ldots, n_k]$ is

$$\delta([n_0, n_1, \ldots, n_k]) = \frac{(-1)^{n_0+\cdots+n_k}\sqrt{5}}{\mathrm{cds}([0, n_k, n_{k-1}, \ldots, n_1, n_0 - 1])}.$$

Question. Are there any points where ج$'(x)$ exists but $\neq 0, \infty$? Can you classify those points?

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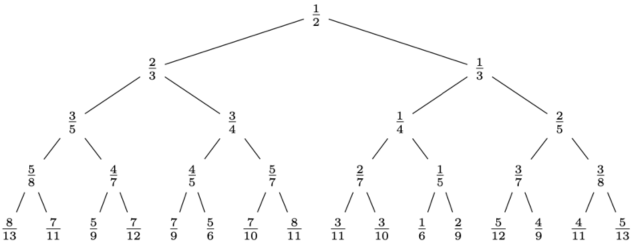
Question. Are there any points where ج$'(x)$ exists but $\neq 0, \infty$? Can you classify those points?

Jimm and the Stern-Brocot tree

In fact, Jimm is the boundary action of an automorphism of the Stern-Brocot tree induced by Dyer's automorphism.

This action is by homeomorphism of the boundary.

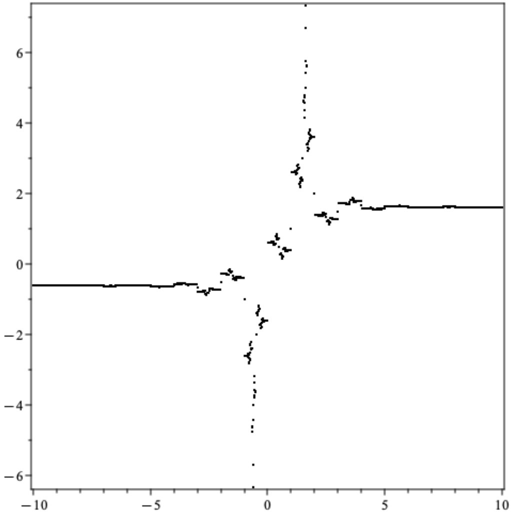
Applying ج to the nodes of the Stern-Brocot tree defines a new tree called Bird's tree.



Bird's tree

Analytic properties of Jimm: Golden ratio

The plot of ج is full of golden ratios



Plot of Jimm on the real line

Analytic properties of Jimm: Golden ratio

$\lim_{x \to +\infty} \text{ج}(x) = \varphi$	$\text{ج}(\varphi) = +\infty$
$\lim_{x \to -\infty} \text{ج}(x) = \bar{\varphi}$	$\text{ج}(\bar{\varphi}) = -\infty$
$\lim_{x \to 0^+} \text{ج}(x) = \varphi^{-1}$	$\text{ج}(\varphi^{-1}) = 0$
$\lim_{x \to 0^-} \text{ج}(x) = \bar{\varphi}^{-1}$	$\text{ج}(\bar{\varphi}^{-1}) = 0$
$\lim_{x \to 1^+} \text{ج}(x) = 1 + \varphi$	$\text{ج}(1 + \varphi) = 1$
$\lim_{x \to 1^-} \text{ج}(x) = 1 + \bar{\varphi}$	$\text{ج}(1 + \bar{\varphi}) = 1$

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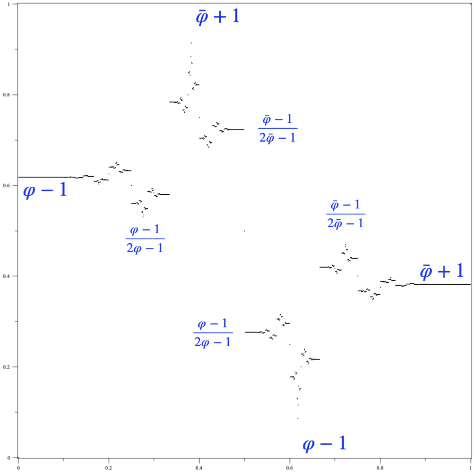

Plot of Jimm on the unit interval

Arithmetic properties: Jimm on real quadratic irrationals.

- ج sends real quadratic irrationals to real quadratic irrationals. Hence, ج defines an involution of the set of real quadratic irrationals $\sqrt{\mathbf{Q}^+} := \{a + \sqrt{b} \,:\, a \in \mathbf{Q}, b \in \mathbf{Q}^+\}$.
- The ج-action on $\sqrt{\mathbf{Q}^+}$ is compatible with the $\mathrm{PGL}_2(\mathbf{Z})$-action: i.e. ج sends $\mathrm{PGL}_2(\mathbf{Z})$-orbits of real quadratic irrationals to $\mathrm{PGL}_2(\mathbf{Z})$-orbits of real quadratic irrationals.
- ج commutes with the Galois conjugation on quadratic irrationals.

Hence, ج induces an involution of the moduli space Π of pseudolattices "with real multiplication"; commuting with the Galois-action on Π:

$$\text{ج} \circlearrowright \mathrm{PGL}_2(\mathbf{Z}) \backslash \sqrt{\mathbf{Q}^+} =: \Pi$$

We can identify Π with the set of periods (cycles):

$$\Pi := \{(n_1, n_2, \ldots, n_k) \,:\, n_i \in \mathbf{Z}_{>0}\},$$

and ج acts on Π by preserving $\sum n_i$ but can change the cycle length k. The Galois action is the cycle reversal.

and at this point we notice a connection with the Galois action and the Lebesgue measure

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Arithmetic properties: Jimm on quadratic surds

$$
\begin{array}{lcl}
\sqrt{N} & \to & \text{ج}(\sqrt{N}) \\
\sqrt{3} & \to & \frac{1}{2}(\sqrt{13}+3) \\
\sqrt{5} & \to & \frac{1}{3}(\sqrt{10}+1) \\
\sqrt{6} & \to & \frac{1}{14}(\sqrt{221}+5) \\
\sqrt{7} & \to & \frac{1}{6}(\sqrt{37}+1) \\
\sqrt{8} & \to & \frac{1}{4}(\sqrt{17}+1) \\
\sqrt{10} & \to & \frac{1}{7}(\sqrt{65}+4) \\
\sqrt{11} & \to & \frac{1}{26}(\sqrt{901}+15) \\
\sqrt{12} & \to & 134(\sqrt{1517}+19) \\
\sqrt{13} & \to & \frac{1}{3}(\sqrt{13}+2) \\
\sqrt{14} & \to & \frac{1}{5}(\sqrt{34}+3) \\
\sqrt{15} & \to & \frac{1}{18}(\sqrt{445}+11) \\
\sqrt{17} & \to & \frac{1}{19}(\sqrt{442}+9)
\end{array}
$$

Arithmetic properties: Jimm on Markov irrationals

Jimm of a Markov irrational x is much simpler than x!

Markov number	**Markov irrational** x	ج(x)
1	$\frac{1+\sqrt{5}}{2}$	∞
2	$1+\sqrt{2}$	$\sqrt{2}$
5	$\frac{9+\sqrt{221}}{10}$	$\sqrt{6}-1$
13	$\frac{23+\sqrt{1517}}{26}$	$\sqrt{12}-2$
29	$\frac{53+\sqrt{7565}}{58}$	$\sqrt{35/6}-1$
34	$\frac{15+5\sqrt{26}}{17}$	$\sqrt{20}-3$
89	$\frac{157+\sqrt{71285}}{178}$	$\sqrt{30}-4$
169	$\frac{309+\sqrt{257045}}{338}$	$\sqrt{204/35}-1$
194	$\frac{86+\sqrt{21170}}{97}$	$\sqrt{119/10}-2$
233	$\frac{411+\sqrt{488597}}{466}$	$\sqrt{42}-5$
433	$\frac{791+\sqrt{1687397}}{866}$	$\frac{12\sqrt{143}-60}{59}$
610	$\frac{269+\sqrt{209306}}{305}$	$\sqrt{56}-6$
985	$\frac{1801+\sqrt{8732021}}{1970}$	$\sqrt{1189/204}-1$

Arithmetic properties: Jimm on real quadratic irrationals

The functional equations of ج can be written as

$$y = \text{ج}(x) = 1 \iff \text{ج}(y) = x \quad \text{(involutivitiy)}$$

$$xy = 1 \iff \text{ج}(x)\text{ج}(y) = 1 \quad \text{(covariance)}$$

$$x + y = 0 \iff \text{ج}(x)\text{ج}(y) = -1 \quad \text{(covariance)}$$

$$x + y = 1 \iff \text{ج}(x) + \text{ج}(y) = 1 \quad \text{(covariance)}$$

$$\frac{1}{x} + \frac{1}{y} = 1 \iff \frac{1}{\text{ج}(x)} + \frac{1}{\text{ج}(y)} = 1$$

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Arithmetic properties: Jimm on real quadratic irrationals

Now set $y = \bar{x}$, where $x = a + \sqrt{b}$ is a quadratic irrational:

$$x\bar{x} = 1 \iff \text{ج}(x)\text{ج}(\bar{x}) = 1$$

$$x + \bar{x} = 0 \iff \text{ج}(x)\text{ج}(\bar{x}) = -1$$

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$$\frac{1}{x} + \frac{1}{\bar{x}} = 1 \iff \frac{1}{\text{ج}(x)} + \frac{1}{\text{ج}(\bar{x})} = 1$$

Recall from number theory

If $x = a + \sqrt{b}$ $(a, b \in \mathbf{Q},\ b > 0)$, then

norm of x is $N(x) := x\bar{x} \iff N(a + \sqrt{b}) = a^2 - b$

trace of x is $T(x) := x + \bar{x} \iff T(a + \sqrt{b}) = 2a$

Example

$$N(1 + \sqrt{2}) = -1, \quad T(1 + \sqrt{2}) = 2$$

Arithmetic properties: Jimm on real quadratic irrationals

The functional equations means

$$N(x) = x\bar{x} = 1 \iff \text{ج}(x)\text{ج}(\bar{x}) = 1 = N(\text{ج}x)$$

$$Tr(x) = x + \bar{x} = 0 \iff \text{ج}(x)\text{ج}(\bar{x}) = -1 = N(\text{ج}x)$$

$$Tr(x) = x + \bar{x} = 1 \iff \text{ج}(x) + \text{ج}(\bar{x}) = 1 = Tr(\text{ج}x)$$

$$\frac{Tr(x)}{N(x)} = \frac{1}{x} + \frac{1}{\bar{x}} = 1 \iff \frac{1}{\text{ج}(x)} + \frac{1}{\text{ج}(\bar{x})} = 1 = \frac{Tr(\text{ج}x)}{N(\text{ج}x)}$$

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$$Tr(x) = x + \bar{x} = 0 \iff \text{ج}(x)\text{ج}(\bar{x}) = -1 = N(\text{ج}x)$$

$$Tr(x) = x + \bar{x} = 1 \iff \text{ج}(x) + \text{ج}(\bar{x}) = 1 = Tr(\text{ج}x)$$

$$\frac{Tr(x)}{N(x)} = \frac{1}{x} + \frac{1}{\bar{x}} = 1 \iff \frac{1}{\text{ج}(x)} + \frac{1}{\text{ج}(\bar{x})} = 1 = \frac{Tr(\text{ج}x)}{N(\text{ج}x)}$$

Arithmetic properties: Jimm on real quadratic irrationals

We get...

Correspondence I

$$x\bar{x} = 1 \iff \text{ج}(x)\text{ج}(\bar{x}) = 1; \text{ i.e. } N(x) = 1 \iff N(\text{ج}(x)) = 1$$

$$\Longrightarrow$$

ج restricts to an involution of the set of **elements of norm +1** of the rings of integers in quadratic number fields.

$$\text{ج} \circlearrowright \{a + \sqrt{a^2 - 1} \,|\, 1 < a \in \mathbf{Q}\}$$

Problem. Find the ج-action on $a \in \mathbf{Q}_{>1}$.

Arithmetic properties: Jimm on real quadratic irrationals

We get...

Correspondence II

$$x + \bar{x} = 0 \iff \text{ج}(x)\text{ج}(\bar{x}) = -1; \text{ i.e. } T(x) = 0 \iff N(\text{ج}(x)) = -1.$$

$\Longrightarrow$ ج establishes a bijection between the set of **square roots of positive rationals** and the set of **elements of norm -1** of the rings of integers of real quadratic number fields.

$$\text{ج} : \{\sqrt{q} \mid q \in \mathbf{Q}\} \to \{a + \sqrt{a^2+1} \mid a \in \mathbf{Q}\}$$

.... and these correspondences are far from being trivial:

Correspondence II-Example

$$\sqrt{\frac{39}{17}} = [1, \overline{1, 1, 16, 1, 1, 2}] \implies$$

$$\text{ج}\left(\sqrt{\frac{39}{17}}\right) = [4, \overline{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 4, 4}] = A \implies$$

$$N(A) = N\left(\frac{7663 + \sqrt{70845893}}{3482}\right) = -1.$$

Arithmetic properties: Jimm on real quadratic irrationals

We get...

Correspondence III

$$x + y = 1 \iff \text{ج}(x) + \text{ج}(\bar{x}) = 1; \text{ i.e. } T(x) = 1 \iff T(\text{ج}(x)) = 1$$

$$\text{ج} \circlearrowright \{\frac{1}{2} + \sqrt{a} \,|\, 0 < a \in \mathbf{Q}\}$$

Problem. Find the ج-action on $a \in \mathbf{Q}_{>1}$.

Arithmetic properties: Jimm on real quadratic irrationals

We get...

Correspondence IV

$$\frac{1}{x}+\frac{1}{\bar{x}}=1 \iff \frac{1}{\text{ج}(x)}+\frac{1}{\text{ج}(\bar{x})}=1;\ \text{i.e. }\ T\left(\frac{1}{x}\right)=1 \iff T\left(\frac{1}{\text{ج}(x)}\right)=1$$

$$T(x)=N(x) \iff T(\text{ج}x)=N(\text{ج}x)$$

Equivalently,

$$\text{ج} \circlearrowright \left\{a+\sqrt{a^2-2a}\,\middle|\,1<a\in\mathbf{Q}\right\}$$

... and there are more correspondences of this type

Jimm and dynamics

Jimm conjugates the Gauss map

$$G([0, n_1, n_2, n_3 \dots] \to [0, n_2, n_3, \dots]$$

to the so-called Fibonacci map Φ, i.e. $\Phi = J \circ G \circ J$.

The expression of Jimm in terms of continued fractions shows that, if a real number x obeys the Gauss-Kuzmin distribution, then the asymptotic density of 1's among the partial quotients of ج(x) is one, i.e. ج(x) does not obey the Gauss-Kuzmin statistics.

This argument also shows that ج sends the set of real numbers obeying the Gauss-Kuzmin statistics, which is of full measure, to a set of null measure.

For example

Example: Gauss-Kuzmin statistics and Jimm I

$2^{1/3}$ = [1,3,1,5,1,1,4,1,1,8,1,14,1,10,2,1,4,12,2,3,2,1,3,4,1,1,2,14,3,12,1,15,3,
1,4,534,1,1,5,1,1,121,1,2,2,4,10,3,2,2,41,1,1,1,3,7,2,2,9,4,1,3,7,6,1,1,2,2,9,3,1,
1,69,4,4,5,12,1,1,5,15,1,4,1,1,1,1,1,89,1,22,186,6,2,3,1,3,2,1,1,5,1,3,1,8,9,1,26,
1,7,1,18,6,1,372,3,13,1,1,14,2,2,2,1,1,4,3,2,2,1,1,9,1,6,1,38,1,2,25,1,4,2,44,1,
22,2,12,11,1,1,49,2,6,8,2,3,2,1,3,5,1,1,1,3,1,2,1,2,4,1,1,3,2,1,9,4,1,4,1,2,1,27,1,
1,5,5,1,3,2,1,2,2,3,1,4,2,2,8,4,1,6,1,1,1,36,9,13,9,3,6,2,5,1,1,1,2,10,21,1,1,1,2,1,
2,6,2,1,6,19,1,1,18,1,2,1,1,1,27,1,1,10,3,11,38,7,1,1,1,3,1,8,1,5,1,5,4,4,4,7,2,1,
21,1,1,5,10,3,1,72,6,9,1,3,3,2,1,4,2,1,1,1,1,2,1,7,8,1,2,1,8,1,8,3,1,1,3,2,1,8,1,1,
1,1,1,6,1,4,3,4,1,1,1,4,30,39,2,1,3,8,1,1,2,1,3,1,9,1,4,1,2,2,1,6,2,1,1,3,1,4,1,2,1,
1,5,1,2,10,1,5,4,1,1,4,1,2,1,1,2,12,2,1,8,3,2,6,1,3,10,1,2,20,1,6,1,2,186,2,2,1,2,
47,1,19,2,2,1,1,1,2,1,1,3,2,8,1,18,3,5,39,1,2,1,1,1,1,4,1,5,2,6,3,1,1,1,4,2,1,6,1,1,
220,1,3,1,3,1,4,5,1,2,1,13,2,2,2,1,1,1,1,7,2,1,7,1,3,1,1,11,1,2,2,4,2,33,3,1,1,2,6,
3,1,1,3,6,8,3,4,84,1,1,2,1,10,2,2,20,1,3,1,7,13,14,1,29,1,1,5,1,7,1,1,2,1,56,1,3,2,
1,13,2,1,...]

Example: Gauss-Kuzmin statistics and Jimm I

$$
\text{ج}(2^{1/3}) = [2,1,3,1,1,1,4,1,1,4,1,1,1,1,1,1,3,1,1,1,1,1,1,1,1,1,1,1,1,3,1,1,1,1,1,
1,1,1,2,3,1,1,2,1,1,1,1,1,1,1,1,1,1,2,2,1,2,3,1,2,1,1,4,2,1,1,1,1,1,1,1,1,1,1,1,1,2,
1,2,1,1,1,1,1,1,1,1,1,1,3,1,1,1,1,1,1,1,1,1,1,1,1,1,2,1,3,1,1,2,1,1,1,1,1,1,1,1,1,1,
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1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4,1,1,1,4,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,\ldots
$$

Arithmetic properties: transcendence

What about algebraic numbers of degree > 2?

Transcendence Conjecture

ج sends algebraic numbers of degree > 2 to transcendental numbers.

Why?: It is believed that if x is an algebraic number of degree > 2, then it obeys the Gauss-Kuzmin statistics. By the previous remark, this implies that ج(x) violates the Gauss-Kuzmin statistics. Hence, according to the same belief, ج(x) must be transcendental. This is the basis of the conjecture that Jimm sends algebraic numbers of degree > 2 to transcendental numbers.

Strong Transcendence Conjecture

Any two algebraically related ج(x), ج(y) are in the same $\mathrm{PGL}_2(\mathbf{Z})$-orbit, if x, y are both algebraic of degree > 2.

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A few examples...

$$
\begin{aligned}
\text{ج}(\sqrt[3]{2}) &= \text{ج}([1;3,1,5,1,1,4,1,1,8,1,14,1,10,2,1,4,\dots]) \\
&= [2,1,3,1,1,1,4,1,1,4,1_6,3,1_{12},3,1_8,2,3,1,1,2,\dots] \\
&= 2.78473155866272\,3\dots
\end{aligned}
$$

$$
\begin{aligned}
\text{ج}(\pi) = \text{ج}([3,7,15,1,292,1,1,1,2,1,3,\dots]) = \\
[1_2,2,1_5,2,1_{13},3,1_{290},5,3,\dots] \\
= 1.7237707925480276079699326494931025145558144289232\dots
\end{aligned}
$$

$$
\begin{aligned}
\text{ج}(e) = \text{ج}([2,1,2,1,1,4,1,1,6,1,1,8,\dots]) = \\
[1,3,4,1,1,4,1,1,1,1,\dots,\overline{4,1_{2n}}] \\
= 1.3105752928466255215822495496939143349712038085627\dots
\end{aligned}
$$

(We tried to recognize these numbers by the PSLQ-algorithm with various sets of constants–we couldn't get any results)

Further study: Other triangle groups

Triangle groups:
Cusps, congruence and chaos

Curtis T. McMullen

28 January 2024

This paper studies lattices Δ_n isomorphic to $\mathbf{Z}_2 \star \mathbf{Z}_n$ inside $\mathrm{PSL}_2(\mathbf{R})$.
One can study the automorphism towers of Δ_n to get -like maps.

Further study: groups of automorphisms of groups

Group Actions on the Cubic Tree

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Received June 25, 1991; Revised May 27, 1992

Abstract. It is known that every group which acts transitively on the ordered edges of the cubic tree Γ_3, with finite vertex stabilizer, is isomorphic to one of seven finitely presented subgroups of the full automorphism group of Γ_3 — one of which is the modular group. In this paper a complete answer is given for the question (raised by Djoković and Miller) as to whether two such subgroups which intersect in the modular group generate their free product with the modular group amalgamated.

This paper studies 7 groups acting on the trivalent tree. $\mathrm{PSL}_2(\mathbf{Z})$, $\mathrm{PGL}_2(\mathbf{Z})$ and $\mathrm{Aut}(\mathrm{PGL}_2(\mathbf{Z}))$ are among them. The remaining groups will induce ج-like maps.

Conjecture: The group $\mathrm{Aut}(\mathrm{PGL}_2(\mathbf{Z}))$ is not linear.

$$\mathrm{Aut}(\mathrm{PGL}_2(\mathbf{Z})) \simeq \langle V, K, J \mid V^2 = K^2 = J^2 = (KJ)^2 = (VJ)^4 = (KVJVJ)^3 = 1\rangle$$

Further study: covariant functions

We are currently studying the functional equation systems of the form

$$f(1+x) = af(x) + bf(1/x), \quad (*)$$
$$f\left(\frac{1}{1+x}\right) = cf(x) + df(1/x), \quad (**)$$

where a, b, c, d are elements of some ring, possibly depending on x. These lead to covariant functions with respect to an action of $\mathrm{PSL}_2(\mathbf{Z})$ or some of its submonoids.

The values

$$g(y) := \lim_{x \to y} \frac{f(x)}{f(1/x)}$$

can be viewed as 'quantizations' of the real number y.

Further study: arithmetic

How are the arithmetic properties of x and $\text{ج}(x)$ related?

How are the arithmetic properties of x and $\mathsf{F}(x)$ related?

Find and study functions analytic on the upper half plane satisfying (an appropriate variant of) functional equations for ج and for the codenominator.

(Their Schwartizans will be modular forms)

Further study: measure theory and dynamics

Study the push-forward of the Lebesgue measure by ج

This measure (or rather its c.d.f.) puts into context the Gauss sums

$$\sum \frac{1}{A_k n + B_k},$$

where A_k, B_k are defined by a linear recurrence relation.

Further study: Graphs and Teichmüller theory

Dyer's automorphism acts on the set of bipartite trivalent graphs.

These graphs (when metrized) parametrize the Teichmüller spaces of Riemann surfaces.

Hence Dyer's automorphism induce a duality of Teichmüller spaces.

Further study: codes

Maple codes are available upon request

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