Groupes Fondamentaux d'une Famille de Courbes Rationnelles Cuspidales

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Je remercie tous ceux qui m'ont guidé durant ma marche en mathématiques, en commençant par Mehmet Çiftçi qui m'a transmis sa passion. Iossif Vladimir Ostrovskii m'a initié à la recherche et m'a appris la rigueur mathématique. Finalement, Mikhail Zaidenberg m'a accueilli en France et a bien voulu diriger ce travail.

J'adresse un remerciement particulier à S. Kaliman qui m'a évité une déconvenue en trouvant une erreur dans la preuve d'un de mes résultats.

Je remercie Alexandru Dimca et Enriques Artal Bartolo qui ont accepté de rapporter sur cette thèse, ainsi que pour leur participation au jury. Je remercie aussi José Bertin et Stepan Orevkov pour l'intérêt qu'ils ont porté à mon travail et d'avoir bien voulu siéger dans ce jury.

Je remercie également tous mes collègues et amis pour leur soutien et leur patience, ainsi que pour les nombreuses discussions qui ont permis d'enrichir ce travail.

Je remercie finalement ma famille qui, bien qu'ayant supporté quatre années de séparation, m'a toujours gardé son soutien.

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Chapter 1

Introduction

Le thème principal de ce travail est l'étude du groupe fondamental du complémentaire d'une courbe algébrique dans le plan projectif. Dans la suite, si $C \subset \mathbb{P}^2$ est une courbe plane, nous appelons $\pi_1(\mathbb{P}^2 \backslash C)$ le *groupe de* C. L'étude des groupes des courbes planes a été initiée par Zariski [100] dans les années trente. Il a développé une méthode permettant de trouver une présentation du groupe d'une courbe et a démontré que ce groupe est cyclique d'ordre d si la courbe est lisse de degré d. Dans le cas où la courbe est singulière, son groupe peut être abélien, non-abélien, fini ou infini. Cette technique est appelée méthode de Zariski-Van Kampen, car une démonstration topologique rigoureuse en a été donnée par ce dernier. Dans un autre article, Zariski a entrepris le calcul des groupes de plusieurs familles de courbes, en exploitant leur structure hautement symétrique [96].

Le groupe fondamental de $\mathbb{P}^2 \backslash C$ est un invariant très fin d'une courbe singulière C. Par exemple, il peut distinguer deux courbes singulières de même degré ayant les mêmes singularités. Le premier exemple d'un tel couple de courbes — appelé *couple de Zariski* — a été trouvé par Zariski. Mais, en utilisant la méthode de Zariski-Van Kampen, il est difficile de trouver une présentation du groupe d'une courbe donnée par une équation explicite. Depuis le travail de Zariski, il y a eu beaucoup de recherches sur le groupe des courbes, mais il y a cependant peu de calculs concrets. En outre, la plupart de ces calculs ne sont effectués que pour des courbes d'un type très particulier, et donnent donc peu d'indices sur les anomalies qui peuvent apparaître en général. Par ailleurs, la plupart des auteurs se contentent d'obtenir une présentation du groupe, alors que l'étude du groupe ne s'arrête pas là : une connaissance detaillée des sous-groupes distingués d'indice fini du groupe d'une courbe permet l'étude des revêtements galoisiens du plan ramifiés le long de cette courbe. Par le théorème de Grauert, ces revêtements ramifiés sont algébriques. Par conséquent, une étude approfondie du groupe d'une courbe est importante du point de vue de la théorie des surfaces projectives.

Dans ce travail, nous calculerons le groupe de courbes provenant de la classification des courbes rationnelles cuspidales ayant une singularité dite "profonde". Rappelons qu'une courbe $C \subset \mathbb{P}^2$ est dite *rationnelle* si elle est birationnelle à $\mathbb{P}^1$, et qu'elle est dite *cuspidale* si toutes ses singularités sont localement irréductibles. Par singularité "profonde", nous entendons une singularité dont la multiplicité est grande par rapport au degré de la courbe. Plus précisément, une courbe C est dite de *type* (d, n) si elle est de degré d, et la plus grande multiplicité de ses singularités est n. Donc, une courbe de type (d, n) a une singularité "profonde" si n est proche de d.

Les courbes rationnelles cuspidales de type (d, n) pour $n \geq d - 3$ admettent une classification. Flenner et Zaidenberg [26], [27] ont démontré qu'une courbe rationnelle cuspidale de type $(d, d - 2)$ ou de type $(d, d - 3)$ peut avoir au plus trois singularités, et ils ont donné une classification des telles courbes ayant exactement trois cusps. La classification pour les courbes de type $(d, d - 2)$ ayant un ou deux cusps est apparue dans des travaux indépendants de Sakai-Tono [84] et Fenske [23]. Ce dernier a aussi achevé la classification pour les courbes de type $(d, d - 3)$ en construisant toutes les courbes de type $(d, d - 3)$ ayant un ou deux cusps. Plus tard, il a donné une classification des courbes rationnelles 3-cuspidales de type $(d, d-4)$ sous une condition supplémentaire de "rigidité" [24].

Ce travail est organisé comme suit :

Le chapitre 2 contient un résumé des études précédentes sur le groupe des courbes. Les exemples principaux de groupes déjà connus sont exposés, afin de donner au lecteur une idée générale de leur structure. En l'absence de papier récapitulant les résultats publiés depuis l'article de Zariski, des questions motivant la recherche sur le sujet sont aussi présentées.

Le chapitre 3 commence par un rappel de la classification des courbes rationnelles cuspidales ayant une singularité profonde. Le reste du chapitre est consacré au calcul du groupe de ces courbes. Rappelons que le premier pas pour déterminer ces groupes avait été fait par Artal [2], qui avait donné une présentation du groupe des courbes rationnelles 3-cuspidales de type $(d, d-2)$. Dans son travail, il utilisait la construction explicite de ces courbes à partir de courbes simples par des transformations birationnelles, et il trouvait le groupe à l'aide du lemme de Fujita. On remarquera qu'auparavant, Degtyarev [15] avait déjà obtenu une classification assez fine du groupe d'une courbe quelconque de type $(d, d-2)$, mais les résultats de Artal sont plus précis dans le cas particulier des courbes rationnelles cuspidales de type $(d, d-2)$. En appliquant la méthode élaborée par Artal, nous complétons son travail en trouvant le groupe fondamental de toutes les autres courbes rationnelles cuspidales classifiées.

Une conséquence intéressante de cette méthode est le théorème suivant.

Théorème 1. *Si G est le groupe d'une courbe de degré d, alors pour chaque $n \in \mathbb{N}$, il existe une courbe de degré nd, dont le groupe fondamental est une extension centrale de G par le groupe cyclique d'ordre n.*

Le chapitre 4 contient la preuve de ce théorème, et des résultats qui peuvent s'en deduire. Par exemple, on peut facilement trouver des exemples de familles infinies de courbes ayant un groupe non-abélien fini à partir des exemples connus. La recherche de telles courbes avait été suggèrée par Oka [77]. La démonstration du théorème 1 permet aussi de construire des couples de Zariski à partir de couples connus; ce problème avait été proposé par Artal [3].

Le chapitre 5 donne une étude du groupe $\pi_1(\mathbb{P}^2 \backslash C)$ du point de vue de la théorie des groupes abstraits. Une fois qu'une représentation de ce groupe est trouvée, le problème de son analyse se pose. Étant donnée une présentation d'un groupe, il n'est pas évident de démontrer que le groupe est trivial, ou isomorphe à un groupe donné par une autre présentation. Il s'agit donc de développer des outils permettant de tirer du groupe des informations utilisables. Une des possibilités est de calculer le polynôme d'Alexander, dans certains cas on peut même l'obtenir sans connaître le groupe fondamental. Un autre invariant du groupe fondemantal est le groupe G/G'', où G'' est le deuxième sous-groupe des commutateurs de G. Nous donnons un algorithme très pratique pour trouver une présentation de ce groupe, à partir d'une présentation du groupe G d'une courbe irréductible. A la fin de ce chapitre, on peut également trouver un rappel des résultats récents sur les groupes de triangle généralisés que nous utilisons fréquemment dans ce travail.

Le chapitre 6 contient une brève introduction au problème de Fenchel dans le plan projectif : étant donné un diviseur D, quand peut-on trouver un revêtement de $\mathbb{P}^2$ ramifié en D ? Kato [43] et Namba [69] ont donné des solutions partielles pour des diviseurs dont les supports sont réductibles, mais il y a très peu de résultats pour les diviseurs de la forme $D = nC$, où C est une courbe plane irréductible. Nous obtenons le résultat suivant :

Théorème 2. *Soit $C \subset \mathbb{P}^2$ une courbe plane irréductible. S'il existe une surjection $\pi_1(\mathbb{P}^2 \backslash C) \twoheadrightarrow \mathbb{Z}_p * \mathbb{Z}_q$, alors pour chaque $n \in \mathbb{N}$ il existe un revêtement galoisien fini ramifié en nC.*

Le dernier chapitre contient des perspectives et des questions que nous trouvons intéressantes sur le sujet.

Une bibliographie longue, mais certainement pas exhaustive, se trouve à la fin de ce travail.

Nous avons utilisé le logiciel Maple pour établir la finitude de certains groupes que nous avons trouvés.

En suivant les conseils de mon directeur de thèse, j'ai inclus mes précédents travaux dans le texte; ils sont totalement indépendants de ma thèse. Le premier contient les résultats de mon travail de DEA à l'Institut Fourier. L'origine de ce travail est une question de Mikhail Zaidenberg sur l'existence de fonctions de Green sur les revêtements de surfaces de Riemann non-compactes. Après avoir résolu ce problème, certaines remarques de Mohan Ramachandran m'ont permis de découvrir une généralisation pour les variétés riemanniennes arbitraires. Enfin, une réponse à une question de Vladimir Lin sur l'existence de fonctions harmoniques bornées sur les revêtements de surfaces de Riemann complète cette partie.

La deuxième est mon "master's thesis" soutenu à l'Université de Bilkent en Turquie, sous la direction de Iossif Vladimir Ostrovskii, où j'ai étudié des problèmes de convolution de densités de probabilité entières. Un principe général énoncé par Paul Lévy est l'amélioration des propriétés de lissité des fonctions par l'opération de convolution. J'ai pu améliorer les découvertes de D. Raikov sur la non-validité de ce principe. Mes résultats ont donné lieu à une publication, et Alexander Ilinskii a démontré ensuite une conjecture formulée dans cet article.

Chapter 2

Overview

2.1 Zariski-Van Kampen Theorem

Given an algebraic curve $C \subset \mathbb{P}^2$, it is interesting to study the topology of its complement $\mathbb{P}^2 \backslash C$ from the point of view of the classification theory of algebraic curves. The idea is an analogue of the leading principle in the knot theory: in order to understand a knot $\mathcal{K} \subset S^3$, look at the topology of the knot complement $S^3 \backslash \mathcal{K}$. Another strong motivation for studying $\pi_1(\mathbb{P}^2 \backslash C)$ comes from the surface theory: A good knowledge of $\pi_1(\mathbb{P}^2 \backslash C)$ allows one to construct Galois coverings of $\mathbb{P}^2$ branched at C. Moreover, if $X \to \mathbb{P}^2$ is a branched covering, with C as the branching locus, one can hope to derive some invariants of X from the invariants of the topology of $\mathbb{P}^2 \backslash C$. Such a reasoning is justified by Kulikov's recent confirmation of a conjecture of Chisini [45]: In many cases, the branch curve of a generic projection $X \subset \mathbb{P}^n \to \mathbb{P}^2$ determines the surface X uniquely.

Let us begin by considering the first homology group of $\mathbb{P}^2 \backslash C$. At first glance, this question looks like a particular case of the one one may ask for the complement of an arbitrary subvariety V in $\mathbb{P}^n$. However, if V is of codimension ≥ 2, it is easily seen that $\mathbb{P}^2 \backslash V$ is simply connected. The group $H_1(\mathbb{P}^n \backslash V)$ is thus trivial, so that the question is interesting only when V is of codimension 1. The following well-known theorem shows that considering only the case $n = 2$ imposes no restrictions neither.

Theorem 2.1.1 ***(Zariski-Leftschetz hyperplane section theorem [97])*** *Let V be a hypersurface in $\mathbb{P}^n$. Then the inclusion homomorphism*

$$\pi_1(H \backslash V) \to \pi_1(\mathbb{P}^n \backslash V)$$

is an isomorphism for a generic plane $H = \mathbb{P}^2$ in $\mathbb{P}^n$.

For a proof of a more general assertion, precising the meaning of "generic", see Dimca's book [18], which is the standard reference for most of the results presented here. One may also consult the paper of Lamotke [53]. Generalizations in several directions can be found in [88], [90], [30], and [72].

Abelianizing the above isomorphism, we get $H_1(\mathbb{P}^2\backslash C) = H_1(\mathbb{P}^n\backslash V)$, where $C := \mathbb{P}^2 \cap V$. Now, if C is a reduced plane algebraic curve, with the irreducible components C_i of degree d_i for $1 \leq i \leq k$, then (as in the knot theory), the homology groups of $\mathbb{P}^2\backslash C$ are quite simple and do not give too much information: One has

$$H_1(\mathbb{P}^2\backslash C, \mathbb{Z}) = \mathbb{Z}^{k-1} \oplus (\mathbb{Z}/\text{g.c.d}(d_1, d_2, \dots, d_k)\mathbb{Z}),$$

whereas the fundamental group $\pi_1(\mathbb{P}^2\backslash C)$, turns out to be much more informative. Its study has been initiated by Zariski in the thirties [100]. Theorem 2.1.1 says that the study of $\pi_1(\mathbb{P}^2\backslash C)$ is equivalent to the study of $\pi_1(\mathbb{P}^n\backslash V)$ for a hypersurface V.

Although the prevailing convention in the literature is to call $\pi_1(\mathbb{P}^2\backslash C)$, by abuse of language, the fundamental group *of* C, following Degtyarev [15], we shall take the liberty to call it simply *the group of* C.

Before considering the groups of irreducible curves, a first insight on the structure of the group of a curve can be obtained by looking at some simple line arrangements. In what follows, we deal exclusively with reduced curves. Evidently, if $C = L$, a single line, then $\mathbb{P}^2\backslash C = \mathbb{C}^2$, which is simply connected. If $C = L_1 \cup L_2$ consists of two lines, then considering L_1 to be the line at infinity, one obtains $\mathbb{P}^2\backslash C = \mathbb{C}^2\backslash L_2 = \mathbb{C} \times \mathbb{C}\backslash\{0\}$, so that $\pi_1(\mathbb{P}^2\backslash C) = \mathbb{Z}$. In the case $C = L_1 \cup L_2 \cup L_3$, there are two possibilities: If the lines are in generic position, then, considering L_1 to be the line at infinity, one has $\mathbb{P}^2\backslash C = \mathbb{C}^2\backslash(L_2 \cup L_3)$. One may suppose L_2 to be the x-axis, and L_3 to be the y-axis. Hence, $\mathbb{P}^2\backslash = \mathbb{C}\backslash\{0\} \times \mathbb{C}\backslash\{0\}$. It follows that $\pi_1(\mathbb{P}^2\backslash C) = \mathbb{Z}^2$.

In the case of three lines passing through a common point, considering again one of them to be the line at infinity, one obtains two parallel lines in $\mathbb{C}^2$, and the complement can be identified with $\mathbb{C}\backslash\{0, 1\} \times \mathbb{C}$. Hence, in this case one has $\pi_1(\mathbb{P}^2\backslash C) = \mathbb{F}_2$, the free group of rank 2.

Reasoning this way, one can easily show the following: If C consists of a line L_1, plus m lines passing through a common point $p \in L_1$, and n lines passing through a common point $q \in L_1$, then $\pi_1(\mathbb{P}^2\backslash C) = \mathbb{F}_m \times \mathbb{F}_n$.

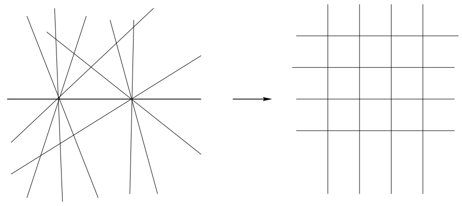


Figure 2.1

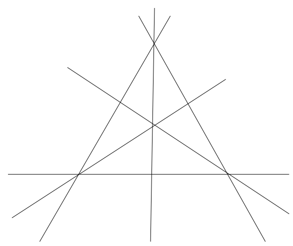


Figure 2.2

Indeed, taking L_1 to be the line at infinity leaves a copy of $\mathbb{C}\backslash\{m\, points\} \times \mathbb{C}\backslash\{n\, points\}$ (Figure (2.1))

However, this kind of reasoning cannot be applied to more complicated line arrangements e.g. as those in Figure (2.2), nor to irreducible curves, so one needs to develop more special techniques to study their groups. Before introducing them, let us present some known facts, which were already present in the pioneering article of Zariski [100]:

1. The group of a smooth curve of degree d is $\mathbb{Z}/d\mathbb{Z}$.

2. For a singular curve, this group can be non-abelian, finite or infinite, even if the curve is irreducible.

3. There are irreducible curves of the same degree and with same singularities, but with non-isomorphic groups.

4. The group of a generic line arrangement is abelian.

Perhaps the most useful facts for understanding these results are the following:

Fact 1. If a line $L \subset \mathbb{P}^2$ intersects C transversally, then the inclusion $L \backslash C \hookrightarrow \mathbb{P}^2 \backslash C$ induces a surjection $\pi_1(L \backslash C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \backslash C)$ of fundamental groups.

Fact 2. Let $C_1, C_2 \subset \mathbb{P}^2$ be two plane curves with C_1 irreducible, and Δ be a small analytic disc intersecting C_1 at a smooth point of $C_1 \cup C_2$. Pick a base point $*$, and define a *meridian* of C_1 in $\mathbb{P}^2 \backslash (C_1 \cup C_2)$ to be a positively oriented loop $\omega \cdot \partial\Delta \cdot \omega^{-1}$, where ω is a path in $\mathbb{P}^2 \backslash (C_1 \cup C_2)$ with $\omega(0) = *$ and $\omega(1) \in \partial\Delta$. Then any two meridians of C_1 are conjugate elements in $\pi_1(\mathbb{P}^2 \backslash (C_1 \cup C_2), *)$, and

$$\pi_1(\mathbb{P}^2 \backslash C_2) \simeq \pi_1(\mathbb{P}^2 \backslash (C_1 \cup C_2), *) / \ll \mu \gg .$$

Here, $\ll \mu \gg$ denotes the normal closure of μ in $\pi_1(\mathbb{P}^2 \backslash (C_1 \cup C_2), *)$. The above claim holds true if C_2 is absent, i.e. $\pi_1(\mathbb{P}^2 \backslash C_1, *)$ is generated by the meridians of C_1.

The main tool for the calculation of the fundamental group of $\mathbb{P}^2 \backslash C$ for a concretely given curve C has been developed by Zariski [100], its rigorous proof is due to Van Kampen [95]. (Other methods has been proposed in [51] or [90]). Below is a modern version, its proof in this form can be found in [18]. See also the detailed exposition of Chéniot [9].

Theorem 2.1.2 *Let $C \subset \mathbb{P}^2$ be a curve, $O \in \mathbb{P}^2$ be a point, and consider the linear projection $pr : \mathbb{P}^2 \backslash \{O\} \to \mathbb{P}^1$. Let n be the degree of the restriction of pr to C. Then if $L_1, L_2, \ldots, L_k$ are all the lines passing through O which intersect C in less then n points (there are finitely many such lines), then the restriction ϕ of pr to $E := \mathbb{P}^2 \backslash (C \cup_{i=1}^k L_i)$ is a locally trivial C^∞ fibration.*

Consequently, one has the exact sequence of the fibration

$$\pi_2(B) \longrightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{\phi_*} \pi_1(B) \longrightarrow \pi_0(F).$$

It is readily seen that $\pi_2(B) = \pi_0(F) = 1$, so that there is the short exact sequence

$$1 \longrightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{\phi_*} \pi_1(B) \longrightarrow 1.$$

This means that there is an action of $\pi_1(B)$ on $\pi_1(F)$, and $\pi_1(E)$ is the semi-direct product of $\pi_1(B)$ with $\pi_1(F)$ under this action. Hence, once this action

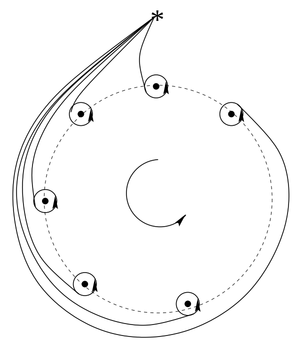


Figure 2.3

is known, the fundamental group of $E = \mathbb{P}^2 \backslash (C \cup_{i=1}^{k} L_i)$ can be found. Now one invokes Fact 2: If one can find meridians of L_i, then the fundamental group of $\mathbb{P}^2 \backslash C$ can be calculated as the quotient of $\pi_1(E)$ by these meridians.

Returning to the points **1-4** above, Zariski's proof of **1** is easy but tricky: Zariski first remarked the following:

Theorem 2.1.3 *The diffeomorphism type of $(\mathbb{P}^n, V_d)$ is constant for a smooth hypersurface V_d of degree d.*

Observe that the curve given by the equation $x^d + xy^{d-1} + z^d = 0$ is smooth, and the line $x = 0$ is a flex of order d of this curve. Hence, the first part of the proposition below shows that the group of a smooth curve is abelian by Theorem 2.1.3.

Proposition 2.1.1 *For a curve $C \subset \mathbb{P}^2$ of degree d, the group $\pi_1(\mathbb{P}^2 \backslash C)$ is abelian if one of the following two conditions is satisfied.*

(i) C has an inflectional line intersecting C with multiplicity d,

(ii) C has a singular point p of multiplicity $d-1$, and no linear component of C passes through p.

Notice that, the part (ii) of the proposition is not void, since e.g. the curve given by $x^n = y^{n-1}z$ satisfies the hypothesis.

To get an idea of the proof of (i), consider the following situation: If L is the flex line of C at the point p, choose the center of projection in 2.1.2 to be a point $O \in L \backslash C$. Clearly, L itself is a singular fiber of this projection.

Now choose a line L_∞ passing through O, and pass to affine coordinates in $\mathbb{C}^2 = \mathbb{P}^2 \backslash (C \cup L_\infty)$. One can choose p to be the origin, and L to be the y-axis in $\mathbb{C}^2 = L \times B$, where B is a line passing through p. Consider the projection $pr : \mathbb{C}^2 \to B$, and identify the fibers of pr via the projection $\mathbb{C}^2 \to L$. The local situation can be illustrated by taking C to be the curve $x = y^d$. The fiber F above $x = 1$ intersects C at d points $y_k = e^{2\pi i d/k}, \quad 0 \le k < 1$. Choose a base point $*$, and generators for $\pi_1(F \backslash C)$ as in Figure (2.3). Denote these loops by $a_1, a_2, \ldots, a_d$, ordered in the positive sense. Now consider the path $\alpha : x = e^{2\pi i t} \subset B$. Let F_t be the fiber above $\alpha(t)$. Then the points of intersection of F_t with C are $y_k e^{2\pi i t/d}$. Hence, when F follows the path α in the counterclockwise sense and return back to its original position, the points of intersection $F_t \cup C$ turns $2\pi i/d$ in the positive sense, and the loop a_i is mapped to the loop a_{i+1}, for $1 \le i < d$. Now it can be shown that this gives a homotopy between a_i and a_{i+1} in $\mathbb{C}^2 \backslash C$. Hence, $\pi_1(\mathbb{P}^2 \backslash (C \cup L_\infty))$ is generated by just one element $a = a_1 = \cdots = a_d$, and is abelian. Finally, note that the product $a^d = a_d a_{d-1} \cdots a_1$ encircles all the intersection points $C \cup F$, and can be deformed to a meridian of L_∞. Hence, one has the relation $a^d = 1$.

The second part of the proposition is proved by putting the center of projection at the singularity of C. A generic line issued from the singular point intersect C at a unique point, which shows that $\pi_1(\mathbb{P}^2 \backslash C)$ is generated by just one element, and hence is abelian.

The reader may amuse himself by showing that for C a plane curve of degree d, the group $\pi_1(\mathbb{P}^2 \backslash C)$ is abelian in each one of the following cases:
(1) C has a flex of order $d - 1$;
(2) C has an ordinary double point p, and there is a line intersecting C at p with multiplicity $d - 1$;
(3) C has a singular point p of order $d - 2$ and a node q, such that $\overline{pq}$ is not a linear component of C; or
(4) C has a singular point p of multiplicity m, and there is a flex line through p that intersect C with multiplicity $d - m$ at a smooth point of C.

2.2 Some examples of non-abelian curve groups

In the light of above examples, it seems like that the group of any irreducible curve is abelian. First examples of curves with non-abelian groups have appeared as branching curves of generic projections. If $X \subset \mathbb{P}^n$ is a smooth algebraic surface, and $\phi : X \to \mathbb{P}^2$ is a generic projection, then the singularities of the branching locus $C \subset \mathbb{P}^2$ of ϕ consist of *simple cusps* (locally given by $x^2 = y^3$) and *ordinary nodes* (locally given by $xy = 0$) only. Zariski calculated the group of C when X is the cubic surface in $\mathbb{P}^3$ and found it to be the group $\mathbb{Z}_2 * \mathbb{Z}_3$, the free product of the cyclic groups[1] $\mathbb{Z}_2$ and $\mathbb{Z}_3$. In this case, C is a sextic, i.e. it is of degree 6, and can be given by the equation $C : f(x,y,z)^2 - g(x,y,z)^3 = 0$, where f is a cubic form and g is a quadratic form in 3 variables. Let C_f, C_g be the cubic and conic curves defined by f and g. Then it is easily seen that if C_f and C_g intersect generically, then C has 6 simple cusps at their points of intersection.

A generalization of this is the following.

Theorem 2.2.1 ***(Moishezon [66])*** *Let $X \subset \mathbb{P}^3$ be a degree n hypersurface, and C be the branching curve of a generic projection $X \to \mathbb{P}^2$. Then the singular locus of C consists of $n(n-1)(n-2)$ simple cusps and $n(n-1)(n-2)(n-3)/2$ ordinary nodes, and $\mathbb{B}_n^p := \pi_1(\mathbb{P}^2 \backslash C)$ is the quotient of the braid group $\mathbb{B}_n$ by its center.*

Recall that the Artin braid group $\mathbb{B}_n$ on n admits the presentation

$$\mathbb{B}_n := \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1\,;\, [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| > 1 \rangle,$$

and its center is generated by $\delta := (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$. When $n = 3$, Theorem 2.2.1 gives the group

$$\pi_1(\mathbb{P}^2 \backslash C) = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad (\sigma_1\sigma_2)^3 = 1 \rangle. \tag{2.1}$$

Changing the generators by setting $x := \sigma_1\sigma_2\sigma_1$, $y := \sigma_1\sigma_2$, one obtains $\sigma_1 = y^{-1}x$ and $\sigma_2 = x^{-1}y^2$. The presentation 2.1 becomes

$$\pi_1(\mathbb{P}^2 \backslash C) = \langle x, y \mid x^2 = y^3 = 1 \rangle = \mathbb{Z}_2 * \mathbb{Z}_3,$$

i.e. one obtains the above-mentioned result of Zariski.

Another class of curves with non-abelian fundamental groups came from the hyperplane sections of the *discriminant hypersurfaces* $D_n \subset \mathbb{P}^n$, defined by the equation $D(a_0, a_1, \ldots, a_n) = 0$, where $D(a_0, a_1, \ldots, a_n)$ is the discriminant of the polynomial $a_0z^n + a_1z^{n-1} + \cdots + a_n$. The fundamental group of $\mathbb{P}^n \backslash D_n$ can be interpreted as the "braid group of the sphere on n strands", and will be denoted here by $\mathbb{B}_n^s$.

[1]In the sequel, we will use the notations $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ alternatively for the cyclic group of order n.

Theorem 2.2.2 ***(Zariski [98])*** *The group $\mathbb{B}_n^s := \pi_1(\mathbb{P}^n \backslash D_n)$ is the quotient of the braid group $\mathbb{B}_n$ by the relation $\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1 = 1$. A generic hyperplane section of D_n is a rational curve of degree $2(n-1)$ with $3n-6$ simple cusps and $2(n-2)(n-3)$ nodes. The group of any other rational curve with only ordinary nodes and simple cusps as singularities is cyclic, except the case of a degree 6 curve.*

For some other related results, see [96], [57], [19], [44].

Sometimes the equation of a curve is reflected in a very nice manner in the presentation of its group, as the next examples of groups calculated by Oka shows.

Theorem 2.2.3 ***(Oka [79])*** *Let $p, q \geq 2$ be two coprime integers.*

(i) The group of the affine curve $C \subset \mathbb{C}^2$ defined by the equation $x^p - y^q = 0$ has the presentation

$$\pi_1(\mathbb{C}^2 \backslash C) = \langle a, b \mid a^p = b^q \rangle.$$

(ii) The group of the projective curve $C \subset \mathbb{P}^2$ defined by the equation $(x^p + y^q)^q + (y^q + z^q)^p = 0$ has the presentation

$$\pi_1(\mathbb{P}^2 \backslash C) = \langle a, b \mid a^p = b^q = 1 \rangle = \mathbb{Z}_p * \mathbb{Z}_q.$$

Hence, the commutator subgroup of $\pi_1(\mathbb{P}^2 \backslash C)$ is the free group $\mathbb{F}_{(p-1)(q-1)}$ of rank $(p-1)(q-1)$.

For alternative calculations, and generalizations to the generic curve defined by $f^p + g^q$, f and g being forms of degree q and p respectively, see [94], [73] [47], [88].

Degtyarev classified the possible groups of curves of degree d with a singular point of multiplicity $d-2$. As a result, he obtained

Theorem 2.2.4 ***(Degtyarev [15])*** *Given four integers $p, q \geq 0$ and $a, b > 0$ with $ap > b(2q+1)$, there exists an irreducible curve C of degree $2b(2q+1) - ap$ with the fundamental group*

$$\langle x, y \mid x^p = y^p, \quad (xy)^q x = y(xy)^q, \quad x^{ap} = (xy)^{b(2q+1)} \rangle.$$

This group is abelian only if $q = 0$ or $p = 1$, otherwise, finite only if $p = 2$ or $(p, q) = (3, 1)$, $(3, 2)$, $(4, 1)$, or $(5, 1)$.

Groups of the curves of degree ≤ 5.

Degtyarev [15] calculated the groups of all the quintics (irreducible or not), so that, for the curves of low order, one can state the following result:

Theorem 2.2.5 *The only irreducible curves of degree ≤ 5 with a non-abelian group are rational. Such a curve is projectively equivalent to one of the curves described below.*

(a) ***(Zariski [100])*** *The three cuspidal quartic, whose group is finite non-abelian of order 12 with the presentation*

$$\langle a, b \mid aba = bab, \quad a^2b^2 = 1 \rangle.$$

(b) ***(Degtyarev [15])***

(i) The quintic with three double cusps (locally $x^2 = y^5$), whose group is finite non-abelian of order 320 *with the presentation*

$$\langle a, b \mid b = ab^4a, \quad a^2 = b^2a^3b^2 \rangle,$$

(ii) and the quintic with three simple cusps and a cusp of type $[2_3]$*, whose group is infinite and has the presentation*

$$\langle a, b \mid a = b^2a^2b^2, \quad b^2 = ab^5a \rangle.$$

Groups of the curves of degree 6 are not known, since, in the first place, a classification of curves of degree 6 is missing. A special class of sextics, namely the rational cuspidal ones, has been classified by Fenske [23], see Theorem 3.1.5. Their groups are given in Corollary 3.2.4.

2.3 Curves with abelian groups

A curve is said to be *nodal* if all its singularities are ordinary double points, that is, a transversal intersection of smooth branches. It can be seen easily that the moduli $V_{n,d}$ of nodal curves of degree d and with n nodes carries an algebraic structure. In [100], Zariski used a claim of Severi on the irreducibility of this moduli to show that the group of a nodal curve is abelian. He first showed by an inductive argument that the group of a generic line arrangement is abelian, and then remarked that in $V_{n,d}$ there exists curves C which can be "degenerated" (see Theorem 2.3.2) to such a line arrangement. He then used the "semicontinuity" of the fundamental group (see Theorem 2.3.2 below) to conclude that the group of C is abelian. The group $\pi_1(\mathbb{P}^2\backslash C)$ being constant for C belonging to an irreducible component of $V_{n,d}$, it follows that the group of a nodal curve is abelian by Severi's claim.

However, Severi's proof of irreducibility turned out to be incomplete. It has been proved in the 80's by Harris [36]. In between, much of the work on the groups of algebraic curves was concentrated on the problem abelianity of the group of a nodal curve, which came to be known as the *Zariski conjecture*. Abhyankar proved the Zariski conjecture in many particular cases [1]. Fulton completed Abhyankar's work and proved the Zariski conjecture for the algebraic fundamental group [31]. Deligne applied their ideas to show the validity of the Zariski conjecture for the topological fundamental group [14].

Since then, many other conditions for the abelianity of a curve group has been given, see e.g. [80], [86], [15]. The one we present below is a particular case of a theorem which was formulated for curves with arbitrary singularities in an arbitrary surface.

Theorem 2.3.1 ***(Nori [74])*** *Let $C \subset \mathbb{P}^2$ be a curve of degree d with k simple cusps and n ordinary nodes. Then $\pi_1(\mathbb{P}^2\backslash C)$ is abelian provided $6k + 4n < d^2$.*

The following theorem helps to deduce the abelianity of a curve group, once the abelianity of the group of a "limit curve" is known. Following Dimca [18], let us first recall that a family $\{C_t : t \in (0,1]\}$ of curves is said to be *equisingular* if the singularities of C_t can be indexed as $p_1(t), p_2(t), \dots, p_k(t)$ such that at each singular point p_i, the family $\{C_t, p_i(t)\}$ of germs has constant Milnor number. (The *Milnor number* of a curve germ $(C, 0)$ with equation $f(x, y) = 0$ is defined to be the dimension over $\mathbb{C}$ of the algebra $\mathbb{C}\{x, y\}/(\partial f/\partial x, \partial f/\partial y)$.) If $\{C_t : t \in (0,1]\}$ is a smooth equisingular family, it can be shown that the pair $(\mathbb{P}^2, C_t)$ is topologically constant, in particular the group $\pi_1(\mathbb{P}^2\backslash C_t)$ do not depend on t. Now given a smooth family $\{C_t : t \in [0,1]\}$ such that the family $\{C_t : t \in (0,1]\}$ is equisingular, then C_0 is said to be an *equisingular degeneration* of C_1.

Theorem 2.3.2 ***(Zariski [100])*** *An equisingular degeneration $C_1 \to C_0$ with a reduced limit curve C_0 induces a surjection of the fundamental groups*

$$\pi_1(\mathbb{P}^2\backslash C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2\backslash C_1),$$

which sends meridians to meridians.

The knowledge of curve groups is very helpful for the "degeneration problem", which can be stated as "for two given curves of the same degree, decide if there is an equisingular degeneration carrying one into the other".

2.4 Characterizing the curve groups

Hereafter a group will be called *big* if it has a subgroup isomorphic to $\mathbb{F}_2$, the free group with two generators. Since the commutator subgroup of $\mathbb{Z}_p * \mathbb{Z}_q$ is free of rank $(p-1)(q-1)$ for $(p,q) = 1$, it follows that these groups, which are curve groups by Theorem 2.2.3, are big. It can be shown that the braid groups $\mathbb{B}_n$ is big for $n \geq 3$, which implies that the groups of Theorem 2.2.1 are also big. On the other hand, the group of a smooth curve is finite cyclic. So the question arises whether there are intermediate examples. It turns out that there are such curve groups. Recall that a group G is said to be *almost solvable* (or virtually solvable) if G has a solvable subgroup of finite index. It is easy to see that an almost solvable group cannot be big. The first examples of curves with infinite but almost solvable groups has been discovered by Moishezon-Teicher [60], among the branch curves of generic projections to $\mathbb{P}^2$ of some surfaces in a projective space $\mathbb{P}^n$. They have further conjectured that almost solvability is a common feature for branch curves of generic projections of simply connected surfaces. The example below has been found later by Robb, we refer the reader to [91].

Theorem 2.4.1 ***(Robb)*** *Let $\mathbb{B}_n$ be the braid group with the standard generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ for $n \geq 4$. Put $\widetilde{\mathbb{B}}_n := \mathbb{B}_n / \ll [\sigma_2, (\sigma_1\sigma_3)^{-1}\sigma_2(\sigma_1\sigma_3)] \gg$. Let $V \subset \mathbb{P}^n$ be a non-degenerate smooth complete intersection of codimension ≥ 2. Then the groups of the affine and projective branch curves of a generic projection $V \subset \mathbb{P}^n$ are $\widetilde{\mathbb{B}}_n$, and a quotient of $\widetilde{\mathbb{B}}_n$ by a central element, respectively.*

It can be shown that there is a series

$$1 \lhd A_1 \lhd A_2 \lhd A_3 \lhd \widetilde{\mathbb{B}}_n$$

with

$$\widetilde{\mathbb{B}}_n / A_3 = \Sigma_n, \quad A_3/A_2 = \mathbb{Z}, \quad A_2/A_1 = \mathbb{Z}^{n-1}, \quad A_1 = \mathbb{Z}_2,$$

t where Σ_n is the symmetric group on a set of cardinality n (see [60]). Hence, $\widetilde{\mathbb{B}}_n$ is an infinite almost solvable group. By the way, this group gives an answer to a question of Oka in the negative: *An infinite fundamental group do not necessarily contain a free subgroup (of rank ≥ 1) of finite index.* Indeed, $\widetilde{\mathbb{B}}_n$ cannot contain a free subgroup of rank ≥ 2, since it is an almost solvable group. On the other hand, it can be seen by a simple growth argument that $\mathbb{Z}$ cannot be a finite index subgroup of $\widetilde{\mathbb{B}}_n$, nor that of the corresponding projective group.

Considering the solvability question in the opposite sense, Dethloff, Orevkov, Zaidenberg [17] have shown that, the fundamental group of the dual of an immersed curve of degree ≥ 4 is big. Recall that a curve is said to be *immersed* if all its singularities are intersection of smooth branches. Note that the fundamental group of an immersed curve need not to be abelian. The following example is due to Artal [3]: Consider Zariski's sextic defined as above by the equation $C_1 : f^2 - g^3 = 0$. Then one can deform the cubic C_f and the conic C_g such that C_f and C_g intersect at a unique point. The corresponding sextic C_0 is an immersed curve, which is an equisingular deformation of the initial sextic, whose group $\mathbb{Z}_2 * \mathbb{Z}_3$. By Theorem 2.3.2, there is a surjection $\pi_1(\mathbb{P}^2 \backslash C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \backslash C_1)$, which shows that the group of C_0 cannot be abelian.

Continuing our search for a characterization of the class of curve groups, let us recall the famous Tits' alternative, which asserts that a discrete subgroup of the group $\mathrm{GL}_n(\mathbb{C})$ is either big or almost solvable [92].

Question (Zaidenberg) *Is it true that the class of curve groups satisfies Tits' alternative?*

As a final remark on the abstract group-theoretical characterization of curve groups, notice that there are some obstructions for an abstract group to be a curve group, these obstructions are imposed principally by the Alexander polynomial, see [56] for a detailed discussion.

Another important fact in the understanding of the curve groups is the following theorem which was first proved by Oka and Sakamoto, and reproved later by Kaliman:

Theorem 2.4.2 ***(Oka-Sakamoto [78], Kaliman [42])*** *If $C_1, C_2 \subset \mathbb{C}^2$ are two affine curves intersecting at $deg(C_1) \cdot deg(C_2)$ distinct points, then*

$$\pi_1(\mathbb{C}^2 \backslash (C_1 \cup C_2)) = \pi_1(\mathbb{C}^2 \backslash C_1) \times \pi_1(\mathbb{C}^2 \backslash C_2).$$

2.5 Curves with finite non-abelian groups

Theorem 2.2.5 shows that the group of a curve can be finite non-abelian. To find general conditions which ensure that the group of a curve is a finite non-abelian one is a difficult question. At the current stage of research, one rather tries to construct explicit examples to see to what extent this is an "accident". Several infinite families of finite non-abelian curve groups found by Degtyarev are exhibited in Theorem 2.2.4. Oka suggested the problem of finding further examples and constructed an infinite series of such curves [76]. These curves were independently found also by Shimada [87], who, in a subsequent article, constructed another infinite family of curves with this property [88]. See Chapter 4.2. for a more detailed account of their results. In Chapter 4.3, we prove the following theorem:

Theorem 2.5.1 *If G is a curve group, then for each $n \in \mathbb{N}$, there is a central extension of G by the cyclic group of order n, which is also a curve group.*

It follows that if G is a finite non-abelian curve group of order m, then for each $n \in \mathbb{N}$, there is a finite non-abelian curve group of order nm. Note that all the curves with finite non-abelian groups constructed up to date are cuspidal. The curves constructed for the proof of Theorem 2.5.1 possess other kinds of singularities. Note also that Theorem 2.5.1 can also be applied to the groups $\widetilde{B_n}$ of Theorem 2.4.1 to find new examples of curves with infinite almost solvable groups.

2.6 Alexander polynomials

As in the knot theory, it is possible to define the Alexander polynomial of a plane curve. Many equivalent definitions exist, and the origins of the topic can be traced back to Zariski. We refer to Chapter 5.1 for a more detailed account of the topic. One of the most interesting results in this direction is the following. (For similar results see [83], [58].)

Theorem 2.6.1 ***(Kulikov [50])*** *If the degree of an irreducible curve C is a prime power, then the group G'/G'' is finite, where $G := \pi_1(\mathbb{P}^2 \backslash C)$, and G' is the commutator subgroup of G.*

This answers another question of Oka in the negative sense: Commutator subgroup of an infinite curve group need not be free. We obtain a presentation for the commutator subgroup of such a group in Chapter 5.1. Moreover, we describe a short-cut algorithm for finding the group G/G'' for the group G of an irreducible curve. It follows in particular that the groups $\mathbb{B}_n^p$ of Theorem 2.2.1, the groups $\mathbb{B}_n^s$ of Theorem 2.2.2, and the groups $\widetilde{\mathbb{B}_n}$ of Theorem 2.4.1 have perfect commutator subgroups for $n \geq 5$ (see also [33] and [57]). The group $\widetilde{B_n}$ also supplies an example of an infinite almost solvable group with a perfect commutator subgroup. For details see Chapter 5.1.

2.7 Zariski pairs

Zariski was the first to notice that, the group of a curve do not only depend on the singularities of the curve, but also on its global structure. After having shown that the group of the six cuspidal sextic C given by the equation $f^2 - g^3 = 1$ is $\mathbb{Z}_2 * \mathbb{Z}_3$, he noted that the six cusps of C lie on the conic defined by $g = 0$. Then he showed that if C' is another sextic with six cusps *not* lying on a conic, then the group of C' should be abelian. He then remarked that a sextic curve of this type must exist, because "it is highly improbable that the six cusps of a sextic should lie on a conic". An equation for such a sextic has been given much later by Oka in [77]. Hence, C and C' are two curves having same number of singularities of the same type, yet having different groups.

This is related with the following problem: Consider the moduli $\mathcal{M}$ of algebraic curves of degree d with given number and topological type of singularities. Is there an algebraic structure on $\mathcal{M}$? If yes, is $\mathcal{M}$ irreducible? It is readily seen that if there are curves in $\mathcal{M}$ with different fundamental groups, then $\mathcal{M}$ should be reducible.

Artal Bartolo called a pair of curves of the same degree and with the same singularities, but with non-homeomorphic complements in $\mathbb{P}^2$ a *Zariski pair* and suggested the problem of finding other Zariski pairs. For a more detailed account of the developments on this problem, we refer to Chapter 4.1. The strongest result up to date is the following one.

Theorem 2.7.1 ***(Kulikov [46])*** *For each $k \in \mathbb{N}$ there exists an infinite family of Zariski k-tuples $(C_1, C_2, \ldots, C_k)$.*

The curves C_i appearing in the above theorem are branch curves of generic projections of certain projective surfaces, thus, they only have simple cusps and ordinary nodes as singularities. In Chapter 4.3, we describe a recipe for finding new examples of Zariski pairs of curves with more complicated singularities starting from the known ones.

2.8 Fenchel's problem

One of the reasons for finding fundamental groups of plane curves is to construct Galois coverings $\pi : X \to \mathbb{P}^2$ branched along these curves. In case π is of finite degree, the surface X is projective by the Grauert-Remmert theorem [34]. Hence, this gives a means of constructing algebraic surfaces. On the other hand, any projective surface X can be considered as a branched covering of $\mathbb{P}^2$ by embedding X in a projective space $\mathbb{P}^n$ and then taking the restriction to X of a generic projection $\mathbb{P}^n \to \mathbb{P}^2$. (Note however that this is not a Galois covering in general).

In particular, the question arises whether it is possible to find a Galois covering of $\pi : X \to \mathbb{P}^2$ branched at a given curve C with the given ramification indexes for irreducible components of C. Note that this data defines a divisor D_π on $\mathbb{P}^2$, which is called the *branching divisor* of π, and π is said to be *branched at* D. The similar problem for the projective line $\mathbb{P}^1$ was posed by Fenchel. It is well known that (except the trivial cases) there is a finite covering of $\mathbb{P}^1$ (in fact, of any Riemann surface) branched at a given divisor, see [28], [8].

As for the Fenchel problem for $\mathbb{P}^2$, there are some partial results of Namba and Kato for certain families of reducible curves. We refer to Chapter 6.1 for a brief account of these solutions. It is convenient to single out the abelian coverings from the discussion, since their existence is rather easy to establish. For example, if C is a smooth irreducible curve of degree d, then $\pi_1(\mathbb{P}^2 \backslash C)$ is abelian, so that there is an abelian covering of $\mathbb{P}^2$ branched at $D := nC$ if and only if n divides d. For the non-abelian case, it can be said that we are almost at the point where Zariski [100] gave the complete solution to the problem for the three-cuspidal quartic, whose group is a finite non-abelian one, so that all the coverings of $\mathbb{P}^2$ branched at C can easily be described. Fenchel's problem for the curves with finite non-abelian groups discussed above should be considered to be solved in this context.

As for the irreducible curves with infinite group, we prove the following theorem, which is rather a group theoretical result. Recall that by Theorem 2.2.3, the group $\mathbb{Z}_p * \mathbb{Z}_q$ with $gcd\,(p, q) = 1$ is the group of an irreducible curve.

Theorem 2.8.1 *Let $C \subset \mathbb{P}^2$ be an irreducible curve. If for some $p, q \geq 2$ there is a surjection $\pi_1(\mathbb{P}^2 \backslash C) \twoheadrightarrow \mathbb{Z}_p * \mathbb{Z}_q$ then for each $n \in \mathbb{N}$ there exists a finite Galois covering of $\mathbb{P}^2$ branched at the divisor $D := nC$.*

The following theorem of Namba is interesting from the point of view of Fenchel's problem.

Theorem 2.8.2 ***(Namba [67])*** *For any projective manifold X and for any finite group G, there exists a finite branched Galois covering $\phi : Y \to X$ with G as the automorphism group.*

2.9 Line arrangements

In contrast with the irreducible curves, there is a fastly growing literature on the fundamental groups of the complements to line arrangements. The (topological) classification of line arrangements itself is a challenging problem. To give an idea of its complexity, recall that neither the Alexander polynomial determines the group of a line arrangement nor the group of an arrangement determines its topology. There are even Zariski pairs of line arrangements [4]. For a discussion of the recent developments in this topic, we refer the reader to [41] and references therein. This beautiful, and classical theory is not touched on in this work. However, we believe that *groups of divisors* whose supports are line arrangements can lead to some nice invariants: Loosely speaking, we define the group $Gr(D)$ of a divisor $D = m_1C_1 + m_2C_2 + \cdots + m_kC_k$ with the support $C := C_1 \cup C_2 \cup \cdots \cup C_k$ to be the quotient $\pi_1(\mathbb{P}^2 \backslash C)/ \ll \mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_k^{m_k} \gg$, where μ_i is a meridian in $\mathbb{P}^2 \backslash C$ of the irreducible component C_i of C. From the point of view of the Fenchel's problem, groups of divisors seems like the correct object of study, and there are some interesting questions on their structure, see the last chapter for a detailed discussion.

As a simple example, consider the divisor $D := 2L_1 + 3L_2 + 6L_3$. Then one has $Gr(D) = \langle a, b, c \,|\, a^2 = b^3 = c^6 = abc = 1 \rangle$, which is isomorphic to the triangle group $T_{2,3,6}$ and is solvable. Hence, one can effectively classify the Galois coverings of $\mathbb{P}^2$ ramified at D.

Chapter 3

Groups of rational cuspidal curves with a deep singularity

3.1 Classification of rational cuspidal curves with a deep singularity.

A projective plane curve $C \subset \mathbb{P}^2$ is said to be *cuspidal* if all its singularities are irreducible. It is said to be of *type* (d, m) if C is of degree d and the maximal multiplicity of its singularities is m, i.e. $m = \max_{p \in C}(\mathrm{mult}_p C)$. Rational cuspidal curves of type $(d, d-2)$ and those of type $(d, d-3)$, having at least three cusps has been classified by Flenner and Zaidenberg. It turns out that these curves have exactly three cusps, moreover, the rational 3-cuspidal curves of type $(d, d-2)$ can be obtained from the quadric $\{xy - z^2 = 0\}$, and the rational 3-cuspidal curves of type $(d, d-3)$ can be obtained from the cubic $\{xy^2 - z^3 = 0\}$ by means of Cremona transformations. We summarize these results in Theorems 3.1.1 and 3.1.2 below, where we use the following conventions settled in [27]: the multiplicity sequence of a cusp will be called the *type* of this cusp. Recall that, if

$$V_{n+1} \to V_n \to \cdots V_1 \to V_0 = \mathbb{C}^2$$

is a minimal resolution of an irreducible analytic curve singularity germ $(C, 0) \subset (\mathbb{C}^2, 0)$, and (C_i, P_i) denotes the proper transform of $(C, 0)$ in V_i, so that $(C_0, P_0) = (C, O)$, then the sequence $[m^{(n+1)}, m^{(n)}, \ldots, m^{(1)}, m^{(0)}]$, where $m^{(i)} := \mathrm{mult}_{P_i} C_i$, is called the *multiplicity sequence* of $(C, 0)$. Evidently, $m^{(i+1)} \leq m^{(i)}$, $m^{(n)} \geq 2$, and $m^{(n+1)} = 1$. The (sub)sequence

$$\underbrace{k, k, k, \ldots, k}_{m \text{ times}}$$

will be abbreviated by k_m. For instance, $[kn, k_{n+m}, k-1]$ is the sequence $[kn, \underbrace{k, k, k, \ldots, k}_{n+m \text{ times}}, k-1]$. Also, the last term of the multiplicity sequence will be omitted. Under this notation, $[2]$ corresponds to a simple cusp and $[2_3] = [2, 2, 2]$ corresponds to a ramphoid cusp.

Theorem 3.1.1 ***(Flenner-Zaidenberg [26])*** *A rational cuspidal curve of type $(d, d-2)$ with at least three cusps has exactly three cusps. For each $(d, n, m) \in \mathbb{N}^3$ such that $d \geq 4$, $n \geq m > 0$ and $n + m = d - 2$ there is exactly one (up to projective equivalence) such curve C, whose cusps are of types $[d-2]$, $[2_n]$, $[2, 2_m]$. There are no other rational cuspidal curves of type $(d, d-2)$ with number of cusps ≥ 3.*

Theorem 3.1.2 ***(Flenner-Zaidenberg [27])*** *A rational cuspidal curve of type $(d, d-3)$ with at least three cusps has exactly three cusps. For each $d = 2n + 3$, $n \geq 1$, there is exactly one (up to projective equivalence) such curve C_n, and the cusps of C_n are of types $[2n, 2_n]$, $[3_n]$, $[2]$. There are no other rational cuspidal curves of type $(d, d-2)$ with number of cusps ≥ 3.*

Following the methods employed by Flenner and Zaidenberg, these results are completed (independently) in the works of Fenske and Sakai-Tono, where the following theorems are shown:

Theorem 3.1.3 ***(Fenske [23], Sakai-Tono [84])*** *Let $n, m \in \mathbb{N}$ be such that $n \geq 1$ and $0 \leq m < n$.*

(i) A rational cuspidal curve of type $(d, d-2)$ with exactly one cusp exists if and only if $d = 2n+2$. Such a curve is unique up to projective equivalence, and the type of its cusp is $[2n, 2_{2n}]$.

(ii) A rational cuspidal curve of type $(d, d-2)$ with exactly two cusp exists if and only if

(a) $d = 2n + 3$, with types of cusps $[2n + 1, 2_n]$, $[2_{n+1}]$;

(b) $d = n + 2$, with types of cusps $[n]$, $[2_n]$;

(c) $d = 2n + 2$, with types of cusps $[2n, 2_{n+m}]$, $[2_{n-m}]$.

Moreover, all these curves are unique up to projective equivalence.

Theorem 3.1.4 ***(Fenske [23])***

Let $n, m \in \mathbb{N}$ be such that $n \geq 1$ and $0 \leq m < n$.

(i) A rational cuspidal curve of type $(d, d-3)$ with exactly one cusp exists if and only if

(a) $d = 3n + 3$, where the type of the cusp is $[3n, 3_{2n}, 2]$;

(b) $d = 5$, where the type of the cusp is $[2_6]$. These curves are unique up to projective equivalence.

(ii) The only existing rational cuspidal curves of type $(d, d-3)$ with exactly two cusps are the following ones:

	degree	*types of cusps*
1	7	$[4]$, $[3_3]$
2	6	$[3]$, $[3_2, 2]$
3	5	$[2_4]$, $[2_2]$
4	$2n + 3$	$[2n, 2_n]$, $[3_n, 2]$
5	$2n + 4$	$[2n + 1, 2_n]$, $[3_{n+1}]$
6	$2n + 3$	$[2n, 2_{n+1}]$, $[3_n]$
7	$3n + 3$	$[3n, 3_{2n}]$, $[2]$
8	$3n + 4$	$[3n + 1, 3_n]$, $[3_{n+1}]$
9	$3n + 3$	$[3n, 3_{n+m}, 2]$, $[3_{n-m}]$
10	$3n + 3$	$[3n, 3_{n+m}]$, $[3_{n-m}, 2]$
11	$3n + 3$	$[3n + 2, 3_n, 2]$, $[3_{n+1}, 2]$

These curves are unique up to projective equivalence.

Considering the cases $d = 6$ in the above theorems leads to a complete classification of rational cuspidal curves of degree 6 is given in [23]. We notice that the classification of sextics in general is an open problem. For a classification of the curves of lower degree, see [68].

Theorem 3.1.5 ***(Fenske [23])*** *Up to projective equivalence, rational cuspidal curves of degree 6 are the following ones:*

	types of cusps
1	$[5]$
2	$[4, 2_4]$
3	$[3_3, 2]$
4	$[3_3]$, $[2]$
5	$[3_2, 2]$, $[3]$
6	$[3_2]$, $[3, 2]$
7	$[4, 2_3]$, $[2]$
8	$[4, 2_2]$, $[2_2]$
9	$[4]$, $[2_4]$
10	$[4]$, $[2_3]$, $[2]$
11	$[4]$, $[2_2]$, $[2_2]$

In fact, in [23] the following more general list of curves is constructed.

Theorem 3.1.6 ***(Fenske [23])*** *Let $d \geq 2$ and $0 \leq m < n$ be integers. The following rational cuspidal plane curves exist:*

	type of curves (d, m)	*type of cusps*
1	$(kn + k, kn)$	$[kn, k_{n+m}, k-1]$, $[k_{n-m}]$
1a	$(kn + k, kn)$	$[kn, k_{2n}, k-1]$
2	$(kn + k, kn)$	$[kn, k_{n+m}]$, $[k_{n-m}, k-1]$
2a	$(kn + k, kn)$	$[kn, k_{2n}]$, $[k-1]$
3	$(kn + k + 1, kn + 1)$	$[kn + 1, k_n]$, $[k_{n+1}]$
4	$(kn + k + 1, kn)$	$[kn, k_{n+1}]$, $[(k+1)_n]$
5	$(kn + k + 1, kn)$	$[kn, k_n]$, $[(k+1)_n, k]$
6	$(kn + k + 2, kn + 1)$	$[kn + 1, k_n]$, $[(k+1)_{n+1}]$
7	$(kn + 2k - 1, kn + k - 1)$	$[kn + k - 1, k_n, k-1]$, $[k_{n+1}, k-1]$
8	$(n + 2, n)$	$[n]$, $[2_n]$

Beware of the following exception in the table above: In case $n = 1$, the curves (4) and (5) are of type $(kn+k+1, k+1)$, instead of $(kn+k+1, nk)$.

T. Fenske begun the classification of rational cuspidal curves of type $(d, d-4)$. Recall that a curve C is said to be *unobstructed* if $H^2(\Theta_V\langle D\rangle) = 0$, where $(V, D) \to (\mathbb{P}^2, C)$ is a minimal embedded resolution of singularities of C, and $\theta_V\langle D\rangle$ is the sheaf of holomorphic vector fields on V tangent along D.

Theorem 3.1.7 ***(Fenske [24])*** *For each $n \geq 1$ there exists a rational cuspidal plane curve of type $(d, d-4)$, where $d = deg C_n = 3n + 4$. This curve* *has three cusps of types* $[3n, 3_n]$, $[4_n, 2_2]$, $[2_1]$. *The curve* C_n *is rectifiable*[1] *and unique up to a projective equivalence. Moreover, any unobstructed rational cuspidal curve of type* $(d, d-4)$ *is projectively equivalent to a curve of this type.*

3.2 Groups of rational cuspidal curves with a deep singularity: Results

Note that an irreducible curve of type $(d, d-1)$ is rational, and has an abelian group as noted in Proposition 2.1.1. On the other hand, a general computation of groups of arbitrary curves of type $(d, d-2)$ was performed by Degtyarev [15]. For the case of rational cuspidal curves of type $(d, d-2)$ Artal gave more direct and precise results, using the explicit construction of these curves by means of Cremona transformations.

Theorem 3.2.1 ***(Artal [2])*** *Let* C *be as in Theorem 3.1.1. Then the group of* C *admits the presentation*

$$\pi_1(\mathbb{P}^2 \backslash C) = \langle a, b \mid (ba)^{d-1} = b^{d-2}, \quad (ba)^k b = a(ba)^k \rangle,$$

where $k \geq 0$ *and* $2k+1 = \gcd(2n+1, 2m+1)$. *Hence, the group depends only on* (d, k). *This group is abelian if and only if* $k = 0$, *finite of order* 12 *if* $(d, n, m) = (4, 1, 1)$, *finite of order* 840 *if* $(d, n, m) = (7, 4, 1)$, *otherwise it is a big group*[2].

Note that the case $(d, n, m) = (4, 1, 1)$ corresponds to the three cuspidal quartic, whose group had been calculated already by Zariski [100].

Taking for example C_1 with $(d, n, m) = (13, 10, 1)$ and C_2 with $(d, n, m) = (13, 7, 4)$, one has the following result.

Corollary 3.2.1 ***(Artal [2])*** *There exist infinitely many pairs* (C_1, C_2) *of curves with isomorphic (big) groups, but different homeomorphism types of the pairs* $(\mathbb{P}^2, C_1)$, $(\mathbb{P}^2, C_2)$.

Using the technique employed by Artal, we show that for the curves given in Theorem 3.1.2, one has the following result:

[1] i.e., it is equivalent to a line up to the action of the Cremona group $\mathbf{Bir}(\mathbb{P}^2)$ of birational transformations of the projective plane.

[2] Recall that we call a group *big* if it has a non-abelian free subgroup.

Theorem 3.2.2 *Let C_n be as in Theorem 3.1.2, where $d = 2n + 3$. Then the group of C_n admits the presentation*

$$\pi_1(\mathbb{P}^2 \backslash C_n) = \langle c, b \mid cbc = bcb, \quad b^n c^{n+2} = c^{n+2} b^n, \quad (b^{-n} cb^2)^{n+1} c^{n^2} = 1 \rangle.$$

This group is big when n is odd and ≥ 7, abelian when $n \in \{0, 1, 2\}$, and finite of order 8640 for $n = 3$, and finite of order 1560 for $n = 5$.

This is proved in 3.3.1 below.

Theorem 3.2.3 *The groups of the curves in Theorem 3.1.6 are as follows:*

1	$\langle \alpha, \beta, y \mid \alpha = y^{-k}\beta^{k-1}, \quad [\beta, y^k] = \alpha(\alpha\beta)^m = \beta(\alpha\beta)^{n-m} = 1 \rangle$
$1a$	*abelian*
2	$\langle \alpha, \beta, y \mid \alpha = y^{-k}\beta^{k-1}, \quad [\beta, y^k] = \alpha(\alpha\beta)^{n-m} = \beta(\alpha\beta)^m = 1 \rangle$
$2a$	$\langle \beta, y \mid y^k = \beta^{k-1}, \quad \beta^{n+1} = 1 \rangle$
3	*abelian*
4	$\langle x, y \mid (xy)^{n+1} y^{n^2} = [y, (xy)^k] = 1, \quad x^k y^n x = (xy)^k \rangle$
5	*abelian*
6	*abelian*
7	*abelian*
8	*abelian*

The groups (1) are central extensions of the group $\mathbb{Z}_k * \mathbb{Z}_j$, where $j :=$ g.c.d.$(mk + k - 1, n + 1)$. Thus, they are abelian if $j = 1$, and big if $j \geq 2$.

The groups (2) are central extensions of the group $\mathbb{Z}_k * \mathbb{Z}_j$, where $j :=$ g.c.d.$(1 + mk, n + 1)$. Thus, they are abelian if $j = 1$, and big if $j \geq 2$. The same conclusion is true for the groups (2a), where this time $j :=$ g.c.d.$(k - 1, n + 1)$.

The groups (4) are abelian if $j := (n + 1, k) = 1$ or $n = 1$. Otherwise, they are big with the following exceptions:

(i) If $(n, k) = (3, 2)$, then the group is finite non-abelian of order 72 (the degree of the curve is 9).

(ii) If $(n, k) = (5, 2)$, then the group is finite non-abelian of order 1560 (the degree of the curve is 13)

(iii) If (n,k)=(2,3), then the group is finite non-abelian of order 240 (the degree of the curve is 10).

This is proved in 3.3.2-3.3.8 below.

Corollary 3.2.2 *(i) The group of a rational unicuspidal curve of type* $(d, d-2)$ *is abelian,*

(ii) The group of a rational two-cuspidal curve C *of type* $(d, d-2)$ *is abelian unless* C *is one of the curves described in Theorem 3.1.3-(ii)c, and* $j := g.c.d.(2m+1, n+1) \neq 1$*. In this case, the group of* C *is the big group with the following presentation*

$$\langle y, \beta \,|\, [\beta, y^2] = y^{-2m-2}\beta^{2m+1} = y^{2m-2n}\beta^{2n-2m+1} = 1 \rangle.$$

This group is a central extension of $\mathbb{Z}_2 * \mathbb{Z}_j$*.*

Proof.

(i) This is the case (1a) with $k = 2$ in Theorem 3.2.3.

(ii) (a) This is the case (3) with $k = 2$ in Theorem 3.2.3.

(b) This is the case (8) in Theorem 3.2.3.

(c) This is the case (1) with $k = 2$ in Theorem 3.2.3. One obtains the presentation easily by substituting $\alpha = y^{-3}\beta$.

Corollary 3.2.3 *(i) The group of a rational unicuspidal curve of type* $(d, d-3)$ *is abelian.*

(ii) Groups of rational two-cuspidal curves of type $(d, d-3)$ *are given below:*

	degree	*group*
1	7	*abelian*
2	6	$\mathbb{Z}_2 * \mathbb{Z}_3$
3	5	*abelian*
4	$2n+3$	*abelian*
5	$2n+4$	*abelian*
6	$2n+3$	$\langle x, y \,\vert\, xy^n x = yxy, \quad (xy)^{n+1}y^{n^2} = [y, xyx] = 1 \rangle$
7	$3n+3$	$\langle y, \beta \,\vert\, \beta^2 = y^3, \quad \beta^{n+1} = 1 \rangle$
8	$3n+4$	*abelian*
9	$3n+3$	$\langle \alpha, \beta, y \mid \alpha = y^{-3}\beta^2, \quad [\beta, y^3] = \alpha(\alpha\beta)^m = \beta(\alpha\beta)^{n-m} = 1 \rangle$
10	$3n+3$	$\langle \alpha, \beta, y \mid \alpha = y^{-3}\beta^2, \quad [\beta, y^3] = \alpha(\alpha\beta)^{n-m} = \beta(\alpha\beta)^m = 1 \rangle$
11	$3n+5$	*abelian*

The groups (6) are abelian if n *is even or* $n = 1$*. Otherwise they are big unless* $n = 3$ *or* $n = 5$*, in these cases the group is finite of order* 72 *and* 1560 *respectively. The groups (7) are abelian if* n *is even, and big otherwise. The groups (9) are abelian if g.c.d.* $(3m+2, n+1) = 1$*, and big otherwise. The groups (10) are abelian if g.c.d.* $(3m+1, n+1) = 1$*, and big otherwise.*

Proof.

1. This is the case (4) in Theorem 3.2.3 with $k = 3$ an $n = 1$. Hence, the group has the presentation

$$\langle x, y \mid x^3yx = (xy)^3, \quad (xy)^2y = [y, (xy)^3] = 1\rangle,$$

which is easily seen to be abelian, by substituting $(xy)^2 = y^{-1}$ in the commutation relation.

2. This is the case (1) in Theorem 3.2.3 with $k = 3$, $n = 1$, and $m = 0$. The group is

$$\langle \alpha\beta, y \mid \alpha = y^{-3}\beta^2, \quad [\beta, y^3] = \alpha = \beta\alpha\beta = 1\rangle.$$

Thus, $y^3 = \beta^2 = 1$, i.e. the group is $\mathbb{Z}_2 * \mathbb{Z}_3$.

3. The group of this curve was found to be abelian by Degtyarev [15].

4. This is the case (5) with $k = 2$ in Theorem 3.2.3. The group is thus abelian.

5. This is the case (6) with $k = 2$ in Theorem 3.2.3, and the group is abelian.

6. This is the case (4) with $k = 2$ in Theorem 3.2.3.

7. This is the case (2a) with $k = 3$ in Theorem 3.2.3.

8. This is the case (3) with $k = 3$ in Theorem 3.2.3.

9. This is the case (1) with $k = 3$ in Theorem 3.2.3.

10. This is the case (2) with $k = 3$ in Theorem 3.2.3.

11. This is the case (1) with $k = 3$ in Theorem 3.2.3.

Corollary 3.2.4 *Groups of rational cuspidal sextics are listed below.*

	types of cusps	*group*
1	$[5]$	*abelian*
2	$[4, 2_4]$	*abelian*
3	$[3_3, 2]$	*abelian*
4	$[3_3], [2]$	$\mathbb{Z}_2 * \mathbb{Z}_3$
5	$[3_2, 2]\,,\,[3]$	$\mathbb{Z}_2 * \mathbb{Z}_3$
6	$[3_2]\,,\,[3, 2]$	$\mathbb{Z}_2 * \mathbb{Z}_3$
7	$[4, 2_3]\,,\,[2]$	$\mathbb{Z}_2 * \mathbb{Z}_3$
8	$[4, 2_2]\,,\,[2_2]$	*abelian*
9	$[4]\,,\,[2_4]$	*abelian*
10	$[4]\,,\,[2_3], [2]$	*abelian*
11	$[4]\,,\,[2_2], [2_2]$	$\langle a, b \mid (ab)^5 = b^4, \quad a(ba)^2 = (ba)^2b$

Proof.

1-2-3: These sextics are unicuspidal, hence their groups are abelian by Corollaries 3.2.2 and 3.2.3.

4. This is the case (7) in Corollary 3.2.3 with $n = 1$.

5. This is the case (2) in Corollary 3.2.3.

6. This is the case (10) in Corollary 3.2.3 with $n = 1$ and $m = 0$. Thus $j = 1$, and the group is abelian.

7. This is the curve in Corollary 3.2.2-(c) with $n = 2$ and $m = 1$. Thus $j = 3$, and the group has the presentation

$$\langle y, \beta \,|\, [\beta, y^2] = y^{-4}\beta^3 = y^{-2}\beta^3 = 1\rangle,$$

which easily seen to be isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$.

8. This is the curve in Corollary 3.2.2-(c) with $n = 2$ and $m = 0$. Thus $j = 1$, and the group is abelian.

9. This is the curve in Corollary 3.2.2-(b).

10. This is the curve in Theorem 3.2.1 with $(d, n, m) = (6, 3, 1)$. Thus, $k = 1$, and the group is abelian.

11. This is the curve in Theorem 3.2.1 with $(d, n, m) = (6, 3, 1)$, and $k = 2$.

Theorem 3.2.4 *Groups of the curves in Theorem 3.1.7 admit the presentation*

$$\pi_1(\mathbb{P}^2 \backslash \widetilde{C}) = \langle a, \gamma \,|\, a\gamma a = \gamma a^{n+1}\gamma, \quad [a^n, \gamma^3] = a^{3n^2+2n}(\gamma^3 a)^{n+1} = 1\rangle.$$

This group is big provided $3|(n+1)$ *and* $n > 6$.

The proof of this theorem is given in 3.3.9.

3.3 Groups of rational cuspidal curves with a deep singularity: Calculations

Conventions. Throughout this work, we shall use the following conventions: If $\alpha, \beta : [0, 1] \to T$ are two paths in a topological space T, then the product $\alpha \cdot \beta$ is defined provided that $\alpha(1) = \beta(0)$, and one has

$$\alpha \cdot \beta(t) := \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

If α is a path in T with $\alpha(0) = \alpha(1) = * \in T$, we shall take the freedom to talk about α as an element of $\pi_1(T, *)$, ignoring the fact that the elements of $\pi_1(T, *)$ are equivalence classes of such paths under the homotopy. Also, when this do not lead to a confusion, we shall write $\pi_1(T)$ instead of $\pi_1(T, *)$, omitting the base point.

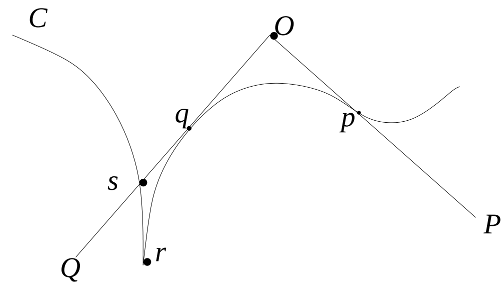


Figure 3.1

3.3.1 Groups of the curves in Theorem 3.1.2

Construction of the curves

Let C be the cubic defined by the equation $x^2z - y^3 = 0$. Then C has a unique, simple cusp at the point $r = [0 : 0 : 1]$, and a unique, simple inflection point at the point $p = [1 : 0 : 0]$. Denote by P the tangent line to C at p. In order to transform C to C_n by means of appropriate Cremona transformations, we begin by taking an arbitrary point $q \in C\backslash\{r, p\}$. Let Q be the tangent to C at q. Then Q intersects C at a second point s, and the lines P and Q intersect at a point $O \notin C$ (Figure 3.1).

By blowing-up the point O, we obtain a Hirzebruch surface X; let E be its exceptional section. Let $e := Q \cap E$. We apply an elementary transformation (or Nagata transformation) at the point e followed by an elementary transformation at the point s. Denote by X_1 the Hirzebruch surface so obtained, by Q_1, P_1 the fibers replacing Q, P and by $\tilde{C}_1$, E_1 the proper transforms of C, E respectively. Then $3p_1 := C_1.P_1$, $2q_1 + s_1 := C_1.Q_1$, $e_1 := E_1.Q_1$. Now we apply an elementary transformation at s_1 followed by an elementary transformation at e_1. We get another Hirzebruch surface X_2, with fibers Q_2, P_2 replacing Q_1, P_1 and with proper transforms $\tilde{C}_2$, E_2 of $\tilde{C}_1$, E_1 respectively. Iterating this procedure n times gives a Hirzebruch surface X_n with exceptional section E_n satisfying $E_n^2 = -1$. After the contraction of E_n, we turn back to $\mathbb{P}^2$. Then $C_n \subset \mathbb{P}^2$ is the image of $\tilde{C}_n$.

Composition of these birational maps gives a biholomorphism

$$\mathbb{P}^2\backslash(C \cup P \cup Q) \xrightarrow{\simeq} \mathbb{P}^2\backslash(C_n \cup P_n \cup Q_n),$$

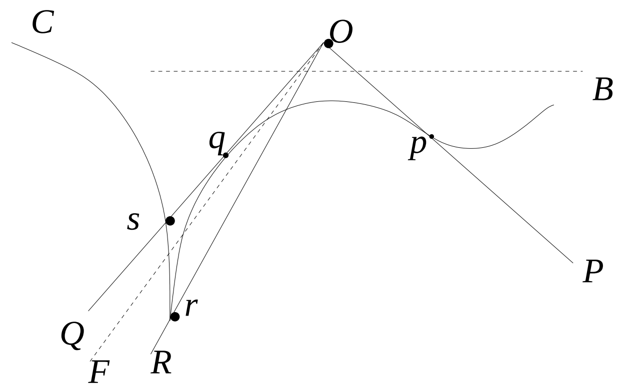


Figure 3.2

and hence an isomorphism

$$\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q)) \simeq \pi_1(\mathbb{P}^2 \backslash (C_n \cup P_n \cup Q_n)).$$

So, the group of C_n can be deduced from $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$ by adding the relations corresponding to the gluing of the lines P_n and Q_n.

Finding $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$**.**

Let T be the projective linear transformation

$$[x : y : z] \longrightarrow [x : y : x + z].$$

Then the equation of C reads, in the new coordinates, as $x^2(z - x) - y^3 = 0$, the point $r = [0 : 0 : 1]$ is the cusp and $p = [1 : 0 : 1]$ is the inflection point of C. Put $L_\infty := \{z = 0\}$, and pass to affine coordinates in $\mathbb{C}^2 = \mathbb{P}^2 \backslash L_\infty$. The real picture of C is shown in Figure (3.2). Let $q = (x_0, y_0) \in C$ be a point such that $x_0 > 0$ is sufficiently small, and let Q be the tangent to C at q. Let $O := P \cap Q$, and let R be the line $\overline{Or}$. Let $B := \{y = y_1\}$ be a line close to O but $O \notin B$. We shall apply the Zariski-Van Kampen method to the linear projection $pr : \mathbb{P}^2 \backslash O \to B$ with center O. Clearly, P, Q and R are singular fibers of this projection (see Figure (3.2)). That these constitute all the singular fibers can be seen by looking at the dual picture: the dual $C^\star$ of

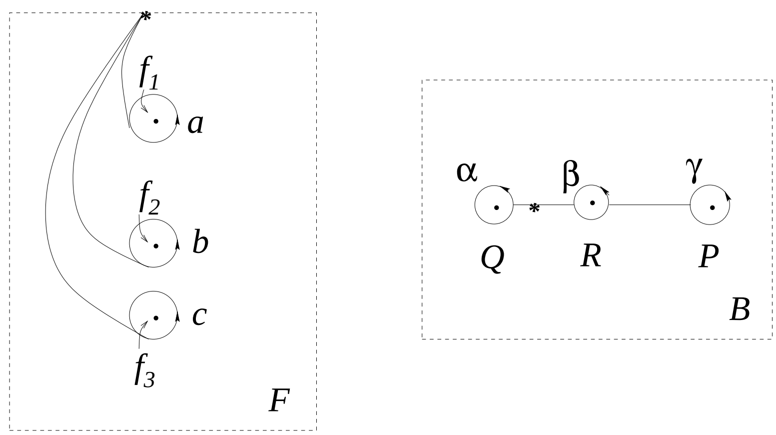


Figure 3.3

C is known to be the curve C itself (see [68]). The dual $P^\star$ of P is the cusp of $C^\star$, and $O^\star$ is the tangent line to $C^\star$ at $O^\star$. Since $\deg(C^\star)=\deg(C)=3$, $O^\star$ cuts $C^\star$ at another point, which is $Q^\star$. The singular line R corresponds to the intersection of $O^\star$ with the line $r^\star$.

Consider the restriction pr' of the projection pr to $\mathbb{P}^2\backslash(C\cup P\cup Q\cup R)\to B\backslash(P\cup Q\cup R)$. This is a locally trivial fibration . Put $F' := F\backslash(C\cup O)$ where F is a generic fiber of the projection pr, and let $B' := B\backslash(P\cup Q\cup R)$. Since $\pi_2(B')=\pi_0(F')=0$, there is the short exact sequence of the fibration

$$0\longrightarrow \pi_1(F')\longrightarrow \pi_1(\mathbb{P}^2\backslash(C\cup P\cup Q\cup R))\longrightarrow \pi_1(B')\longrightarrow 0.$$

To determine the group $\pi_1(\mathbb{P}^2\backslash(C\cup P\cup Q\cup R))$ it suffices therefore to find the *monodromy*, that is, the action of $\pi_1(B')\simeq\mathbb{F}_2$ on the group $\pi_1(F')\simeq\mathbb{F}_3$. Choose the base fiber F as shown in Figure (3.2). Denote by f_1, f_2, f_3 the intersection points $F\cap C$. Let $*=F\cap B$ be the base point.

One can identify the fibers with $\mathbb{C}$ e.g. by taking $*$ to be the origin in $\mathbb{C}^2$, F to be the $y-axis$, and B to be the $x-axis$. Then the projection $\mathbb{C}^2\to F$ gives the desired identification.

Choose positively oriented simple loops $a,b,c\in\pi_1(F',*)$ around f_1, f_2, f_3 and the loops $\alpha,\beta,\gamma\in\pi_1(B',*)$ as in Figure (3.3). Note that $\pi_1(B',*)=\langle\alpha,\beta\rangle$. The local monodromy of pr' around the points p, q, r and s is well known. The monodromy around q gives the relations

$$\alpha^{-1}a\alpha = b \quad (\mathcal{R}_1)$$
$$\alpha^{-1}b\alpha = bab^{-1} \quad (\mathcal{R}_2),$$

and the monodromy around s gives the relation

$$\alpha^{-1}c\alpha = c \Leftrightarrow [\alpha, c] = 1 \quad (\mathcal{R}_3).$$

One has $a = \alpha b\alpha^{-1}$; by $(\mathcal{R}_1)$, substituting this in the relation $(\mathcal{R}_2)$ we obtain

$$(\alpha b)^2 = (b\alpha)^2 \quad (\mathcal{R}_4).$$

The relation obtained from the monodromy around R gives the cusp relation

$$cbc = bcb \quad (\mathcal{R}_5)$$

(recall that we glue R back to $\mathbb{P}^2 \backslash (C \cup P \cup Q \cup R)$). Since $\pi_1(B') = \langle \alpha, \beta \rangle$, it is not necessary to calculate the relations obtained from the monodromy around P; these can be derived from the ones we have found. To sum up, we have the presentation

$$\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q)) =$$

$$\langle a, b, c, \alpha, \gamma \,|\, a = \alpha b\alpha^{-1},\ (\alpha b)^2 = (b\alpha)^2,\ cbc = bcb,\ [\alpha, c] = 1,\ cba\gamma\alpha = 1 \rangle,$$

where the last relation $cba\gamma\alpha = 1$ comes from the loop vanishing at infinity. This can be seen as follows: Clearly, cba is a loop in F surrounding the points f_1, f_2, f_3. Let Σ be a small disc in F containing the point O, and let σ be its boundary. Let $\mathbb{R}$ be the real line in F, put $h := \sigma \cap \mathbb{R}$, and let ω be the real line segment $\overline{*h}$. Define the positively oriented loop ρ as

$$\rho := \omega \cdot \sigma \cdot \omega^{-1}.$$

Then one has the relation $\rho cba = 1$ since $F = \mathbb{P}^1 = S^2$ is a sphere. Let U be a small neighborhood of O in $\mathbb{P}^2 \backslash (C \cup P \cup Q)$. Then clearly U is biholomorphic to $\Delta^* \times \Delta^*$, where Δ^* is the punctured disc (see Figure (3.4)). Hence, $\pi_1(U) = \mathbb{Z}^2 = \langle \alpha, \gamma \,|\, [\alpha, \gamma] = 1 \rangle$, and it is easy to see that ρ is homotopic to $\alpha\gamma$.

Note that using $(\mathcal{R}_1)$ and $[\alpha, \gamma] = 1$, the relation $cba\gamma\alpha = 1$ becomes

$$cb\alpha b\gamma = 1 \quad (\mathcal{R}_6).$$

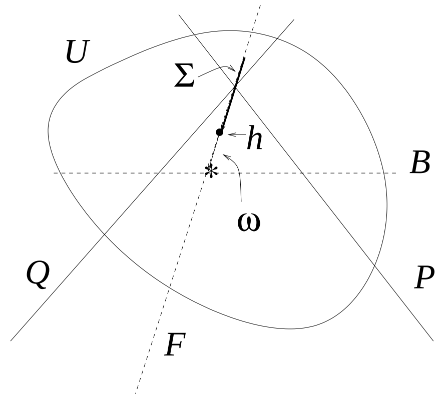


Figure 3.4

Eliminating the generators a and γ from the above presentation, we get

$$\pi_1(\mathbb{P}^2\backslash(C \cup P \cup Q)) = \langle b, c, \alpha \,|\, (\alpha b)^2 = (b\alpha)^2,\ cbc = bcb,\ [\alpha, c] = 1\rangle$$
$$= \pi_1(\mathbb{P}^2\backslash(C_n \cup P_n \cup Q_n)).$$

To obtain a presentation of the group $= \pi_1(\mathbb{P}^2\backslash C_n)$, it remains to find the relations corresponding to the gluing of the lines P_n and Q_n. To this end, we introduce the following concept.

Definition 3.3.1 ***(meridian)*** Let C be a curve in a surface X, and pick a base point $* \in X\backslash C$. Let Δ be a small analytic disc in X, intersecting C transversally at a unique point q of C. If q is a smooth point of C, a *meridian* of C in X with respect to the base point $*$ is a loop in $X\backslash C$ constructed as follows: Connect $*$ to a point $p \in \partial\Delta$ by means of a path $\omega \subset X\backslash C$ such that $\omega \cap \Delta = p$, and let

$$\mu := \omega^{-1} \cdot \delta \cdot \omega,$$

where $\delta := \partial\Delta$, oriented clockwise (Figure (3.5)). A loop μ_q given by the same construction will be called a *singular meridian at* q if q is a singular point of C.

Lemma 3.3.1 *Let $M \lhd \pi_1(X\backslash C, *)$ be the subgroup normally generated by the meridians of C. Then*

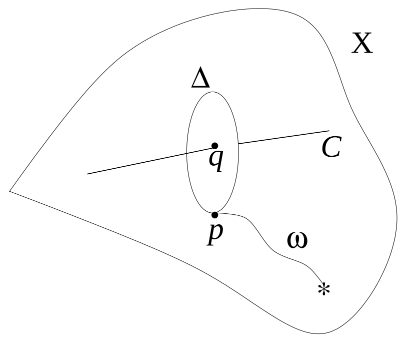


Figure 3.5

(i) $\pi_1(X) = \pi_1(X\backslash C, *)/M$.

(ii) If C *is irreducible, then any two meridians of* C *are conjugate in* $\pi_1(X\backslash C, *)$. *Hence,* $M = \ll \mu \gg$, *where*[3] μ *is any meridian of* C.

(iii) Any two singular meridians μ_q, $\tilde{\mu}_q$ *at the singular point* $q \in C$ *are conjugate.*

Proof. Parts *(i)-(ii)* are well known [53]. To show *(iii)*, assume that μ_q, $\tilde{\mu}_q$ are obtained from the discs Δ_q, $\widetilde{\Delta}_q$ intersecting C transversally at q. Let $\sigma : Y \to X$ be the blow-up of the surface X at the point q, and denote by $Q := \sigma^{-1}(q)$ the exceptional divisor of this blow-up. Then the proper transforms $\sigma^{-1}(\Delta_q)$, $\sigma^{-1}(\tilde{\Delta}_q)$ intersect Q transversally at distinct points of Q, and these points of intersection are smooth in $C \cup Q$. Hence, $\sigma^{-1}(\mu_q)$, $\sigma^{-1}(\tilde{\mu}_q)$ are meridians of Q. Since Q is irreducible, applying the part *(ii)* to the surface $Y\backslash C$ gives the desired result. $\square$

For a group G, denote by (g) the conjugacy class of $g \in G$, i.e. $(g) := \{hgh^{-1} : h \in G\}$. Lemma 3.3.1 implies that the group of a curve $C \subset \mathbb{P}^2$, supplied with the following data

(i) Conjugacy classes $(\mu_1), (\mu_2), \dots$ of meridians of C,

(ii) Conjugacy classes of singular meridians $(\mu_{q_1}), (\mu_{q_2}), \dots$ of C at singular points $q_1, q_2, \dots$ of C

is a richer invariant of the pair $(\mathbb{P}^2, C)$ than solely the group $\pi_1(\mathbb{P}^2\backslash C)$.

[3] Recall that by $\ll \mu \gg$ we denote the normal subgroup generated by μ.

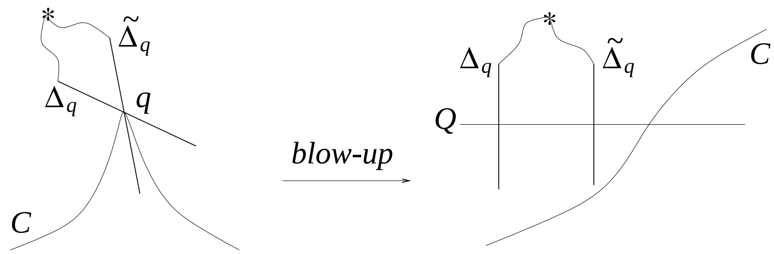


Figure 3.6

For the group $\pi_1(\mathbb{P}^2\backslash(C \cup P \cup Q))$ found above, there are clearly three classes of meridians (a), (α), (β), corresponding to the curves C, P, Q, respectively. Consider the loop cb in F, which surrounds the points f_2 and f_3. Pushing F over R, the points f_2, f_3 come together at the cusp r, and the loop cb becomes a loop in R surrounding the cusp r, that is, cb is a singular meridian of $C \cup P \cup Q$ at r, i.e. $(\mu_r) = (cb)$. (The classes (μ_O), (μ_p), (μ_q), (μ_s) are irrelevant so we will not find them.)

On the other hand, Lemma 3.3.1 implies that the relations corresponding to the gluing of the lines Q_n, P_n are of the form $\mu(Q_n) = 1$, $\mu(P_n) = 1$, where $\mu(Q_n)$, $\mu(P_n)$ are meridians of Q_n and P_n respectively. Finding these meridians will be achieved by Fujita's lemma, which we proceed to explain now.

Let $q \in C$ be an ordinary double point, and take a small neighborhood X' of q such that $X' \cap C$ consists of two branches C_1 and C_2 satisfying $C_1 \cap C_2 = \{q\}$. Pick an intermediate base point $*' \in X'\backslash C$, and take meridians μ_1' of C_1 in $X'\backslash C_2$ and μ_2' of C_2 in $X'\backslash C_1$ with respect to $*'$. Let ω be a path in $X\backslash C$ connecting $*$ to $*'$, and define

$$\mu_1 := \omega^{-1} \cdot \mu_1' \cdot \omega, \quad \mu_2 := \omega^{-1} \cdot \mu_2' \cdot \omega.$$

Clearly, μ_1, μ_2 are meridians of C in X with respect to $*$ and they commute. Moreover, $\mu_1 \cdot \mu_2$ is homotopic to a singular meridian of C at q. Hence, we have the following lemma:

Lemma 3.3.2 ***(Fujita [29])*** *Let $\sigma : X \mapsto Y$ be the blowing up of q, and put $Q := \sigma^{-1}(q)$. Identify $X\backslash\{Q\}$ with $Y\backslash\{q\}$. Then $\mu_1 \cdot \mu_2$ is a meridian*

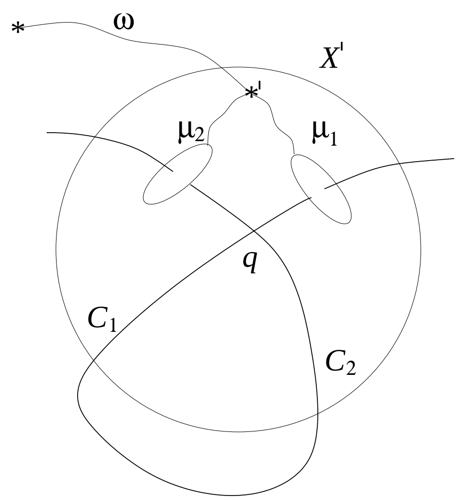


Figure 3.7

of Q in Y with respect to $*$. Moreover, as Q is irreducible, by Lemma 3.3.1 one has

$$\pi_1(X\backslash C, *) = \pi_1\left(Y\backslash(C \cup Q), *\right) / \ll \mu_1 \cdot \mu_2 \gg .$$

Meridians of P_n and Q_n

Turning back to our search for the relations introduced in $\pi_1(\mathbb{P}^2\backslash(C_n \cup P_n \cup Q_n))$ after the gluing of P_n, Q_n, we first note that one can apply Fujita's Lemma to the loops α, γ, which are meridians of P and Q respectively, and obtain a meridian of E. The blowing-up of the point $E \cap P$ will give $\gamma(\alpha\gamma)$ as a meridian of P_1, by induction we obtain $\mu(P_n) := \gamma(\alpha\gamma)^n$ as a meridian of P_n. Recalling that $[\alpha, \gamma] = 1$, this gives the relation

$$\alpha^n\gamma^{n+1} = 1.$$

Substituting γ from $(\mathcal{R}_6)$ we get,

$$\alpha^n(cb\alpha b)^{-(n+1)} = 1 \quad (\mathcal{R}_7).$$

The construction of C_n also shows that $\mu_p := \alpha^{n-1}\gamma^n$ is a singular meridian of C_n at p_n, since it is a meridian of the exceptional line of the blow-up

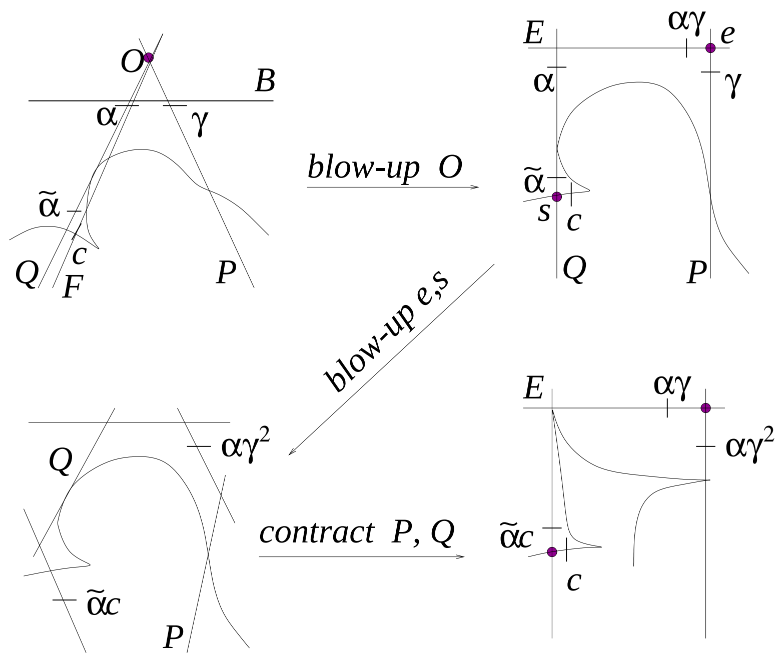


Figure 3.8

at p. Setting $\mu(P_n) = 1$ yields that $\mu_p = (\alpha\gamma)^{-1}$ is a singular meridian of C_n at p_n.

To find a meridian of Q_n, first define a meridian $\tilde{\alpha}$ of Q as shown in figure (3.8). That is, take a small disc Δ intersecting Q transversally above the point s, and take a path ω joining Δ to a neighborhood f_3, and continuing to $*$ in F' along the loop c. Let $\tilde{\alpha} := \omega \cdot \delta \cdot \omega^{-1}$. Then the blowing up of the point s will give $\tilde{\alpha}c$ as a meridian of Q_1. A recursive application of Fujita's Lemma gives $\mu(Q_n) := \tilde{\alpha}c^n$ as a meridian of Q_n.

Note that the construction of C_n also shows that $\mu_q := \alpha\gamma$ is a singular meridian of C_n at q_n, since μ_q is a meridian of the exceptional line of the blow-up of q_n.

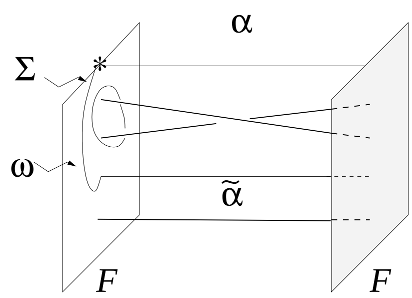


Figure 3.9

Finding $\tilde{\alpha}$

The final step in determining the fundamental group of C_n is to express $\tilde{\alpha}$ in terms of the above presentation of $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$. Let us show that $\tilde{\alpha}$ is in fact homotopic to α.

First, let $\widetilde{\Delta} := pr^{-1}(\Delta) \cap B$. Then α is clearly homotopic to a loop obtained by connecting (properly) $\partial\widetilde{\Delta}$ to the base point $*$. $\partial\widetilde{\Delta}$ intersect the real axis of B at two points, let $*'$ be the one on the right, which will be used as a temporary base point. Now push the fiber F over $*'$ along the real axis of B. As all the intersection points remains real, it is easy to see that the picture of F stays as in Figure (3.3).

Next, consider the restriction $pr' : pr^{-1}(\partial\widetilde{\Delta}) \to \partial\widetilde{\Delta}$ of pr to the border of the disc $\widetilde{\Delta}$. This is a locally trivial fibration, and it can be pictured as in Figure (3.9), where we have cut $\partial\widetilde{\Delta}$ at $*'$ to give a better picture. Let Σ be a disc in F, containing the upper intersection points f_1 and f_2 of C with F, we suppose that Σ avoids the loop c. Then, as the lower intersection point f_3 is a transversal intersection, we can suppose that the corresponding Leftschez homeomorphisms $F \to F_t$, $t \in \partial\widetilde{\Delta}$ are constant outside Σ. Hence, the following map gives a homotopy between α and $\tilde{\alpha}$.

$$H(s,t) := \begin{cases} (\alpha(0), \omega(t)), & 0 \le t \le s/3 \\ (\alpha\left(\frac{t-s/3}{1-2s/3}\right), \omega(s/3)), & s/3 \le t \le 1-s/3 \\ (\alpha(1), \omega(1-t)) & 1-s/3 \le t \le 1 \end{cases}$$

Consequently, the vanishing of the meridian of Q_n yields the relation

$$\mu(Q_n) = 1 \Rightarrow \alpha = c^{-n} \quad (\mathcal{R}_8).$$

Substituting α from $(\mathcal{R}_8)$, $(\mathcal{R}_7)$ becomes

$$c^{n^2}(cbc^{-n}b)^{n+1} = 1.$$

From the cusp relation $cbc = bcb$ it follows that $cbc^k = b^k cb$ for any $k \in \mathbb{Z}$. Using this in the above relation, we get

$$c^{n^2}(b^{-n}cb^2)^{n+1} = 1 \quad (\mathcal{R}_9).$$

To sum up, we have the presentation

$$= \pi_1(\mathbb{P}^2 \backslash C_n) = \langle b, c, \alpha \,|\, \mathcal{R}_3,\ \mathcal{R}_4,\ \mathcal{R}_5,\ \mathcal{R}_7,\ \mathcal{R}_9 \rangle.$$

Study of the group

Since $\alpha = c^{-n}$ by $(\mathcal{R}_7)$, the relation $(\mathcal{R}_3)$ is trivialized, and using the cusp relation $(\mathcal{R}_4)$ becomes

$$(bc^{-n})^2 = (c^{-n}b)^2 \Leftrightarrow [b^n, c^{n+2}] = 1.$$

Finally, we have obtained the presentation given in Theorem 3.2.2.

$$G_n := \pi_1(\mathbb{P}^2 \backslash C_n) = \langle b, c \,|\, cbc = bcb, \quad [b^n, c^{n+2}] = 1, \quad (b^{-n}cb^2)^{n+1}c^{n^2} = 1 \rangle.$$

Notice that (c) is the unique class of meridian in G_n. Also, the singular meridian $\mu_r = cb$ of C is unchanged during the birational transformations, so that μ_r is a singular meridian of C_n at r_n. Other singular meridians are, as found above, $\mu_q = \alpha\gamma$ and $\mu_p = (\alpha\gamma)^{-1}$. One has $(\alpha\gamma)^{-1} = cba = cb\alpha b\alpha^{-1} = cbc^{-n}bc^n$.

It is easily seen that $G_0 = \mathbb{Z}/3\mathbb{Z}$. Let us now show that $G_1 = \mathbb{Z}/5\mathbb{Z}$ and $G_2 = \mathbb{Z}/7\mathbb{Z}$. One has

$$G_1 = \langle c, b, \,|\, cbc = bcb,\ [b, c^3] = 1,\ (b^{-1}cb^2)^2 c = 1 \rangle.$$

Expanding the last relation, we get

$$b^{-1}c \cdot bcb^2 \cdot c = b^{-1}c \cdot c^2bc \cdot c = b^{-1}c^3bc^2 = c^5.$$

Thus,

$$[b, c^3] = 1 \Rightarrow [b, c^6] = [b, c^{-1}] = [b, c] = 1 \Rightarrow b = c.$$

Hence, G_1 is generated by b, and $G_1 = \mathbb{Z}/5\mathbb{Z}$. The group G_2 has the presentation

$$G_2 = \langle c, b \mid cbc = bcb, \quad [b^2, c^4] = 1, \quad (b^{-2}cb^2)^3c^4 = 1\rangle.$$

Again, expanding the last relation we get

$$b^{-2}c^3b^2c^4 = 1 \Rightarrow c^7 = 1.$$

Thus,

$$[b^2, c^4] = 1 \Rightarrow [b^6, c^8] = [b^{-1}, c] = [b, c] = 1 \Rightarrow c = b.$$

Hence, G_2 is generated by b, and one has $G_2 = \mathbb{Z}/7\mathbb{Z}$. The group G_3 is found to be finite of order 8640 by using the programme Maple. The order of the group G_5 is calculated to be 1560 by Artal by the help of the programme GAP.

Definition 3.3.2 ***(residual group)*** For an element $a \in G$ of an arbitrary group G, the group $G/\ll a \gg$ will be denoted by $G(a)$. If $G_C := \pi_1(\mathbb{P}^2 \backslash C)$ is the group of an irreducible curve C, with μ a meridian of C, then for $k \in \mathbb{N}$, we will call the group $G_C(\mu^k)$ a *residual group* of G_C and denote it by $G_C(k)$. The group $G_C(\mu_p^k)$, where μ_p is a singular meridian of C at a singular point $p \in C$, will be called a *residual group of* G_C *at* p and denoted by $G_C^p(k)$. Note that the groups $G_C(k)$ do not depend on the particular meridian μ chosen, since by Lemma 3.3.1(ii), for an irreducible curve, any two meridians are conjugate. In view of Lemma 3.3.1(iii), this is also true for the groups $G_C^p(k)$.

Proposition 3.3.1 *If n is odd, and $k|n$, then there is a surjection*

$$G_n(k) \twoheadrightarrow T_{2,3,k} = \langle x, y, z \mid x^2 = y^3 = z^k = xyz = 1\rangle$$

onto the triangle group $T_{2,3,k}$. Hence, G_n is big for odd $n \geq 7$.

Proof. For the last assertion, it is known that the group $T_{2,3,k}$ is big if $k \geq 7$. Putting $n = k$, we get that the groups $G_n(n)$ are big for $n \geq 7$ odd, so that G_n is big if $n \geq 7$ is odd.

Now let us establish the surjection claimed. If $b^k = 1$, then $c^n = b^n{=}1$ since $k|n$. This gives the presentation

$$G_n(k) = \langle b, c \mid cbc = bcb, \quad (cb^2)^{n+1} = 1, \quad b^k = 1\rangle.$$

Following an idea due to Artal [2], we apply the transformation $x = cbc$, $y = cb$ ($\Leftrightarrow c = y^{-1}x$, $b = x^{-1}y^2$) to obtain

$$G_n(k) = \langle x, y \mid x^2 = y^3, \quad (yx^{-1}y^2)^{n+1} = 1, \quad (y^{-1}x)^k = 1\rangle.$$

Note that $yx^{-1}y^2 = yxy^{-1}$ since $x^2 = y^3$. Thus,

$$G_n(k) = \langle x, y \mid x^2 = y^3, \quad x^{n+1} = 1, \quad (y^{-1}x)^k = 1\rangle.$$

Let H be the quotient of this group by the relation $x^2 = y^3 = 1$ (note that x^2 is central). Then the relation $x^{n+1} = 1$ is killed if n is odd, and we get the desired result:

$$H = \langle x, y \mid x^2 = y^3 = (y^{-1}x)^k = 1\rangle = T_{2,3,k}. \qquad \square$$

This completes the proof of Theorem 3.2.2.

3.3.2 Groups of the curves (1)-(1a) in Theorem 3.2.3

Construction of the curves

Before passing to the construction of the curves, let us fix some notations following [23].

Notation. Let $\sigma_O : X_1 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the point $O \in \mathbb{P}^2$. Denote by E_1 the exceptional divisor of this blow-up. For any curve $C \subset \mathbb{P}^2$, the proper preimage of C in the Hirzebruch surface X_1 will be denoted by C_1. Now let X_i be a Hirzebruch surface. Then X_i is a ruled surface, whose horizontal section is denoted by E_i. Let $p \in X_i$, and let L_i be the fiber of the ruling passing through p. The surface obtained from X_i by an elementary transformation at the point p will be denoted by X_{i+1}. Recall that this is a birational mapping which consists of blowing-up $p \in X_i$ followed by the contraction of L_i. The fiber replacing L_i will be denoted by L_{i+1}, and for any curve $C_i \in X_i$, the proper transform of C_i in X_{i+1} will be denoted by C_{i+1}.

Let $n, m \in \mathbb{N}$ be such that $0 \leq m < n$, and for $k \geq 2$, let C be the curve defined by the equation $F(x, y, z) := xy^{k-1} - z^k = 0$. Then for $k > 2$, the curve C has a unique singularity at $p := [1 : 0 : 0]$ which is a cusp and $q = [0 : 1 : 0]$ is an inflection point of C of order k. If $k = 2$, then p is a smooth point of C. The line $P := \{y = 0\}$ is the tangent to C at p and $Q := \{x = 0\}$ is the tangent at q, these tangents intersects at the point $O := P \cap Q = [0 : 0 : 1] \notin C$. Blowing-up the point O, we get a Hirzebruch surface X_1. Let E_1 be its horizontal section, and denote by P_1, Q_1 the proper transforms of P and Q respectively. For $i = 1, 2, \ldots, m$ we apply m elementary transformations at the points $E_i \cap P_i$, followed by elementary transformations applied at the points $E_i \cap Q_i$ for $i = m + 1, m + 2, \ldots, n$, and we arrive at the Hirzebruch surface X_{n+1} with $E_{n+1}^2 = -n - 1$ (see Figure (3.12)).

Performing elementary transformations at arbitrary points $s_i \in P_i \backslash E_i$ for $i = n + 1, \ldots, 2n$ we obtain the Hirzebruch surface X_{2n+1} with $E_{2n+1}^2 = -1$. Hence, one can contract E_{2n+1} and return to the projective plane $\mathbb{P}^2$. Let $\widetilde{C}, \widetilde{P}, \widetilde{Q}$ be the images of respectively $C_{2n+1}, P_{2n+1}, Q_{2n+1}$ under the contraction of E_{2n+1}. Then $\widetilde{C}$ is a curve of the family (1).

The curves (1a) are obtained in the same way, except that in this case one applies elementary transformations at the points $E_i \cap P_i$ for $i = 1, 2, \ldots, n$ followed by elementary transformations at some points $s_i \in E_i \cap Q_i$ for $i = n + 1, n + 2, \ldots, 2n$.

These birational morphisms provides a biholomorphism

$$\mathbb{P}^2 \backslash (C \cup P \cup Q) \xrightarrow{\simeq} \mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{P} \cup \widetilde{Q}).$$

One has the induced isomorphism

$$\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q)) \simeq \pi_1(\mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{P} \cup \widetilde{Q})).$$

Finding $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$

We will apply the Zariski-Van Kampen method to the projection $pr := \mathbb{P}^2 \backslash O \to \mathbb{P}^1$. Clearly, P, Q are singular fibers of this projection, and it is easy to see that these are the only ones. Indeed, a line passing through $O = [0 : 0 : 1]$ has an equation of the form $ax + by = 0$. Comparing with the equation $xy^{k-1} - z^k = 0$ of C, one obtains $by^k + az^k = 0$, which has multiple solutions if and only if $a = 0, b \neq 0$ or $a \neq 0, b = 0$, corresponding to the lines

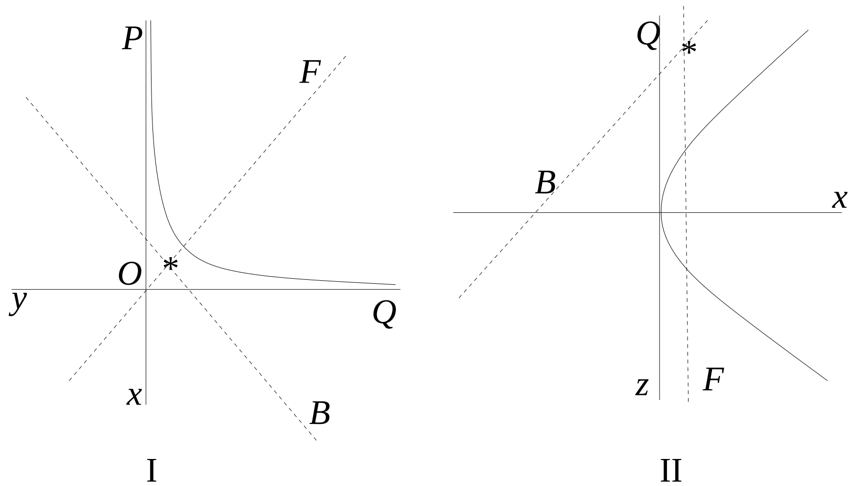


Figure 3.10

$P = \{y = 0\}$ and $Q = \{x = 0\}$. Let pr' be the restriction of the projection pr to $\mathbb{P}^2 \backslash (C \cup P \cup Q)$.

Let $L_\infty := \{z = 0\}$, and shift to the affine coordinates in $\mathbb{P}^2 \backslash L_\infty = \mathbb{C}^2$. Let B be the line $\{x + y = \epsilon\}$, where ϵ is a small real number and let $F := \{x = y\}$ be the base fiber (Figure (3.10)-I). Put $B' := B \backslash (P \cup Q)$ and $F' := F \backslash (\{O\} \cup C)$. If we choose $L_\infty := \{y = 0\}$ and pass to the affine coordinates in $\mathbb{P}^2 \backslash L_\infty = \mathbb{C}^2$, then the real picture of the configuration $C \cup P \cup Q$ will be as it is drawn in Figure (3.10)-II. Let $* := F \cap B$ be the base point. Identify the fibers with $\mathbb{C}$ via the projection to the z-axis, and take the generators of $\pi_1(B')$ and $\pi_1(F')$ as in Figure (3.11). The monodromy relations around the singular fiber Q are given by

$$\beta^{-1} a_i \beta = \begin{cases} a_{i+1}, & 1 \le i \le k-1, \\ \delta\ a_1\ \delta^{-1}, & i = k. \end{cases}$$

where $\delta := a_k a_{k-1} \ldots a_2 a_1$. Setting $a := a_1$, these relations can be expressed as

$$a_i = \beta^{-i+1} a \beta^{i-1}, \quad (\beta a)^k = (a\beta)^k.$$

Hence, one has the presentation

$$\pi_1(\mathbb{P}^2 \backslash (C \cup T \cup L)) = \langle \beta, a \quad | \quad (a\beta)^k = (\beta a)^k \rangle.$$

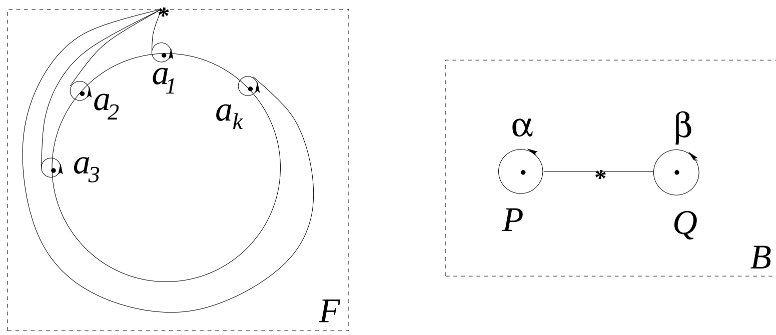


Figure 3.11

Note that $[\alpha,\beta] = 1$ and $\alpha\beta a_k \cdots a_2 a_1 = 1$. The change of generators $(\beta, \quad a) \Leftrightarrow (\beta, \quad y := \beta a)$ gives a more convenient presentation

$$\pi_1(\mathbb{P}^2 \backslash (C \cup T \cup L)) = \langle \beta, y \mid [\beta, y^n] = 1 \rangle.$$

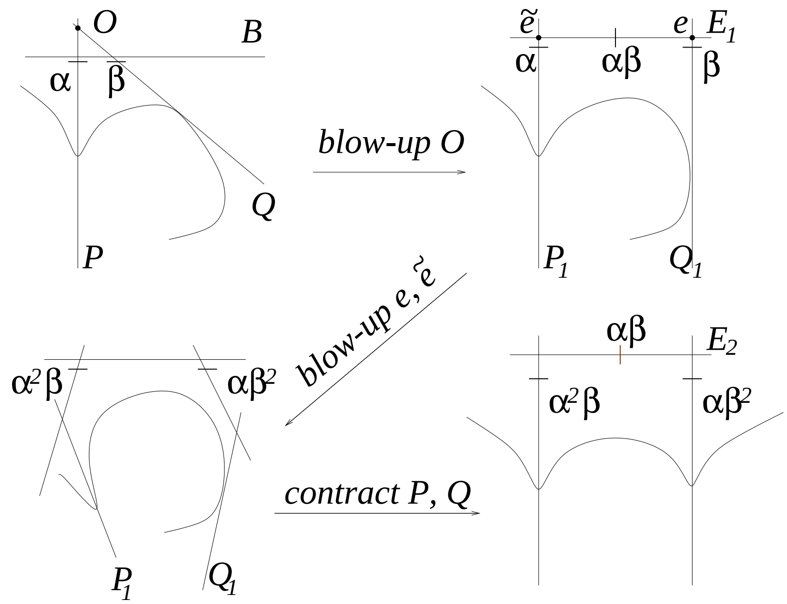
For the future applications, note that α can be expressed by using the relation

$$\alpha\beta a_k a_{k-1} \cdots a_1 = 1 \Leftrightarrow \alpha = y^{-k}\beta^{k-1}.$$

Note also that $[\alpha,\beta] = 1$. This can be derived either from the above presentation or by applying Fujita's lemma to the meridians α of P and β of Q, with respect to the point $O = P \cup Q$.

Meridians of $\widetilde{P}$ and $\widetilde{Q}$

An obvious application of Fujita's lemma yields that $\mu(P_{n+1}) := \alpha(\alpha\beta)^m$ is a meridian of P_{n+1} and $\mu(Q_{n+1}) := \beta(\alpha\beta)^{n-m}$ is a meridian of Q_{n+1} in the surface $X_{n+1} \backslash (C_{n+1} \cup P_{n+1} \cup Q_{n+1} \cup E_{n+1})$. (See Figure (3.12), where the situation is illustrated for $n = 2$, $m = 1$, beware that the elementary transformation applied at $e := E_1 \cap P_1$ and the elementary transformation applied at $\tilde{e} := E_2 \cap Q_2$ are shown simultaneously in the figure.) Recall that the subsequent transformations are applied at points $s_i \in E_i \backslash P_i$ for $i = n+1, \ldots, 2n$. Since the line Q_{n+1} is not affected by these transformations, $\mu(Q_{n+1})$ is a meridian of $\widetilde{Q}$ in $\mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{P} \cup \widetilde{Q})$, too.



O
B
α
β
Q
P
blow-up O
ẽ
e
E1
α
αβ
β
P1
Q1
blow-up e, ẽ
αβ
E2
α2β
αβ2
α2β
αβ2
contract P, Q
P1
Q1

Figure 3.12

On the other hand, $\mu(P_{n+1})$ stays to be a meridian of P_{n+i} after an elementary transformation applied at a point $s_i \in E_i \backslash P_i$. This can be seen e.g. by choosing $s_{n+1} = \Delta \cap P_{n+1}$, where Δ is the defining disc of $\mu(P_{n+1})$. Hence, $\mu(P_{n+1})$ is a meridian of $\widetilde{P}$, too.

Denote by $\tilde{p}$ and $\tilde{q}$ the cusps of $\widetilde{C}$. Then, by the construction of the curve, $\alpha\beta$ is a singular meridian at $\tilde{p}$, and $\beta(\alpha\beta)^{n-m-1}$ is a singular meridian at $\tilde{q}$.

Setting $\mu(P_{n+1}) = \mu(Q_{n+1}) = 1$, we obtain the presentation

$$G_{(1)} := \pi_1(\mathbb{P}^2 \backslash \widetilde{C}) =$$

$$\langle \alpha, \beta, y \,|\, \alpha = y^{-k}\beta^{k-1}, \quad [\beta, y^k] = \alpha(\alpha\beta)^m = \beta(\alpha\beta)^{n-m} = 1 \rangle.$$

A meridian of $\widetilde{C}$ can be given as $a = \beta^{-1}y$. After the obvious simplifications, one finds that $\alpha\beta = y^{-k}\beta^k$ is a singular meridian at $\tilde{p}$, and $(\alpha\beta)^{-1} = y^k\beta^{-k}$ is a singular meridian at $\tilde{q}$.

It is easy to see that the element y^k is central in the group $G_{(1)}$. Eliminating α in $G_{(1)}(y^k)$, one obtains

$$G_{(1)}(y^k) = \langle \beta, y \,|\, y^k = \beta^j = 1 \rangle = \mathbb{Z}_k * \mathbb{Z}_j,$$

where $j :=$ g.c.d.$(mk + k - 1, nk - mk + 1) =$ g.c.d.$(mk + k - 1, kn + k) =$ g.c.d.$(mk + k - 1, n + 1)$. Hence, this group is big if $j \geq 2$. Finally, the group $G_{(1)}$ is abelian when $j = 1$ by the following trivial lemma:

Lemma 3.3.3 *Let G be a group, and $z \in G$ be a central element. If $G(z)$ is cyclic, then G is abelian.*

As for the curves from the family (1a), the same procedure applies. The meridian β of Q stays to be a meridian of $\widetilde{Q}$, so that one has the relation $\beta = 1$, which implies that the fundamental group is generated by just one element $y (= a)$ and thus it is abelian.

Remark. In [17], the authors provide a long argument due to V. Lin, showing the bigness of the group given by the presentation $\langle a, b \,|\, (ab)^2 = (ba)^2 \rangle$. Here is a simpler proof of a more general assertion:

Proposition 3.3.2 *Let $n, m \in \mathbb{Z}$ such that $k := gcd\,(n, m)$ satisfy $|k| \geq 2$. Then the group*

$$\langle a, b \,|\, (ab)^n = (ba)^m \rangle$$

is big.

Proof. Put, as above, $x := ab$ and $y := b$. Then the above presentation is written in terms of x, y as

$$\langle x, y \,|\, x^n = yx^m y^{-1} \rangle.$$

Passing to the quotient by the relation $x^k = 1$, we get the group $\mathbb{Z}_k * \mathbb{Z}$, which is big. This can be seen as follows: Let $r \geq 3$ be an integer such that $gcd\,(r, k) = 1$. Passing once more to the quotient by the relation $y^r = 1$ gives the group $\mathbb{Z}_k * \mathbb{Z}_r$, and it is well known that the commutator subgroup of this group is the free group of rank $(k-1)(r-1)$. $\square$

Note that the group $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to the infinite dihedral group $\mathbb{D}_\infty$, whose commutator subgroup is $\mathbb{Z}$, hence this group is solvable and is not big. Also, it can be shown that the commutator subgroup of the group $\langle a, b \,|ab = (ba)^2 \rangle$ is abelian, but not finitely generated.

Groups of the curves (2)-(2a)

These curves are constructed as the curves (1)-(1a) with the following difference: One performs elementary transformations at the points $E_i \cap Q_i$ for $i = 1, 2, \ldots, m$ followed by elementary transformations at the points $E_i \cap P_i$ for $i = m+1, m+2, \ldots, n$. Finally, for $i = n, n+1 \ldots, 2n$ one applies elementary transformations at some points $s_i \in Q_i \backslash E_i$. The curves (2a) are obtained by setting $n = m$ in the above procedure.

The same reasoning as in the case of the curves (1)-(1a) shows that $\mu(Q_{m+1}) := \beta(\alpha\beta)^m$ is a meridian of $\widetilde{Q}$, and $\mu(P_{m+1}) := \alpha(\alpha\beta)^{m-n}$ is a meridian of $\widetilde{P}$. Setting $\mu(Q_{m+1}) = \mu(P_{m+1}) = 1$, we obtain the presentation

$$G_{(2)} := \pi_1(\mathbb{P}^2 \backslash \widetilde{C}) =$$

$$\langle \alpha, \beta, y \,|\, \alpha = y^{-k}\beta^{k-1}, \quad [\beta, y^k] = \beta(\alpha\beta)^m = \alpha(\alpha\beta)^{n-m} = 1 \rangle.$$

Obviously, $\alpha\beta$ and $(\alpha\beta)^{-1}$ are singular meridians at $\tilde{q}$ and $\tilde{p}$, respectively. A meridian of $\widetilde{C}$ can be given as $a = \beta^{-1}y$.

For the curves (2a) we obtain,

$$G_{(2a)} := \pi_1(\mathbb{P}^2 \backslash \widetilde{C}) =$$

$$\langle \alpha, \beta, y \,|\, \alpha = y^{-k}\beta^{k-1}, \quad [\beta, y^k] = \alpha^m \beta^{m+1} = \alpha = 1 \rangle.$$

$$= \langle \beta, y \,|\, y^{-k} = \beta^{k-1}, \quad \beta^{n+1} = 1 \rangle.$$

Meridians in this case can be obtained from those of the case (2) by putting $\alpha = 1$.

Again, the element y^k is central in the group G_2, and one has

$$G_{(2)}(y^k) = \langle y, \beta \,|\, y^k = 1, \quad \beta^j = 1 \rangle = \mathbb{Z}_k * \mathbb{Z}_j,$$

where this time $j := \text{g.c.d.}(1+mk, nk-mk+k-1) = \text{g.c.d.}(1+mk, nk+k) = \text{g.c.d.}(1+mk, n+1)$.

As for the curves (2a) we obtain,

$$G_{(2a)}(y^k) = \langle y, \beta \,|\, y^k = 1, \quad \beta^j = 1 \rangle = \mathbb{Z}_k * \mathbb{Z}_j,$$

where $j = \text{g.c.d.}(k-1, n+1)$.

3.3.3 Groups of the curves (3)

Let C be the curve defined by the equation $xy^k - z^{k+1} = 0$, the point p be its cusp, q its inflection point, and R be the line $\overline{pq} = \{z = 0\}$. Let, as in the case (1), P, Q be the tangent lines at p, q, respectively.

Blowing-up the inflection point $q = [0 : 1 : 0]$ we get the Hirzebruch surface X_1. For $i = 1, 2, \ldots, n$, we perform elementary transformations at the points $E_i \cap R_i$, followed by elementary transformations at some points $s_i \in Q_i \backslash E_i$, and we end up with a Hirzebruch surface X_{2n+1} with $E^2_{2n+1} = 1$. Let $\widetilde{C}$, $\widetilde{R}$, $\widetilde{Q}$ be the images of C_{2n+1} ,R_{2n+1}, Q_{2n+1} in $\mathbb{P}^2$ under the contraction of E_{2n+1}. Then $\widetilde{C}$ is a curve from the family (3). One has the biholomorphic map

$$\mathbb{P}^2 \backslash (C \cup R \cup Q) \xrightarrow{\simeq} \mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{R} \cup \widetilde{Q}),$$

inducing an isomorphism

$$\pi_1(\mathbb{P}^2 \backslash (C \cup R \cup Q)) \simeq \pi_1(\mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{R} \cup \widetilde{Q})).$$

In order to find $\pi_1(\mathbb{P}^2 \backslash (C \cup R \cup Q))$, we shall use the projection from the point $O := P \cap Q$, as before. As the only points of intersection $R \cap C$ are p and q, this projection has only two singular fibers, namely P and Q. Choose the base B and the generic fiber F as in Figure (3.10). Let $F' := F \backslash (C \cup \{O\} \cup R)$, and choose the generators $a_1, \ldots a_{k+1}, \gamma$ for $\pi_1(F')$ as in Figure (3.13). The monodromy relations around Q are given by

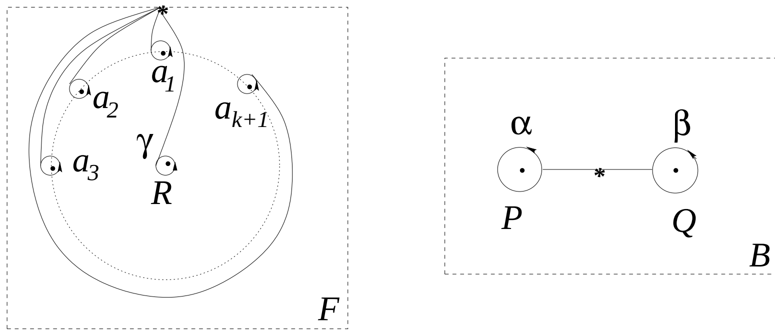


Figure 3.13

$$\begin{aligned} \beta^{-1} a_i \beta &= \begin{cases} a_{i+1}, & 1 \leq i \leq k, \\ (\delta\gamma)\ a_1\ (\delta\gamma)^{-1}, & i = k+1, \end{cases} \\ \beta^{-1}\gamma\beta &= \qquad a_1 \gamma a_1^{-1}. \end{aligned}$$

where $\delta := a_{k+1}a_k...a_2a_1$. Observe that, as β is a meridian of Q, and as the elementary transformations are applied at points $s_i \in Q_i \backslash E_i$, β is a meridian of Q_i, and thus it is a meridian of $\widetilde{Q}$. Imposing the relation $\beta = 1$ in the relations found above, we find that $a_1 = a_2 = \cdots = a_{k+1}$, and $[a_1, \gamma] = 1$. Since the group $\pi_1(\mathbb{P}^2 \backslash \widetilde{C})$ is generated by these elements we conclude that it is abelian.

3.3.4 Groups of the curves (4)

We begin by the curve $C := xy^k - z^{k+1}$. Let $Q := \{x = 0\}$ be the tangent to C at its flex $q := [0:1:0]$, and let P be a line intersecting C transversally at its cusp $p := [1:0:0]$, and such that $q \notin P$. Then by Bezout's theorem, P intersects C at one further point $r \neq q$. Let $O := P \cap Q$. Blowing-up O, we get the Hirzebruch surface X_1, with the horizontal section E_1 with $E_1^2 = -1$. For $i = 1, 2, \ldots, n$, apply elementary transformations at the points r_i followed by elementary transformations applied at the points $E_i \cap Q_i$. Then $E_{2n+1}^2 = -1$; contracting it, we turn back to the projective plane $\mathbb{P}^2$. Let $\widetilde{C}$, $\widetilde{P}$, $\widetilde{Q}$ be the images of C_{2n+1}, P_{2n+1} Q_{2n+1} under this contraction. Then $\widetilde{C}$

is a curve of the family (4), and one has the biholomorphism

$$\mathbb{P}^2\backslash(C \cup P \cup Q) \xrightarrow{\simeq} \mathbb{P}^2\backslash(\widetilde{C} \cup \widetilde{P} \cup \widetilde{Q}),$$

inducing an isomorphism of the fundamental groups.

To find the group of $C \cup P \cup Q$, we shall use the projection from the point O. In addition to P and Q, this projection has a third singular fiber R, which is a simple tangent to C at a unique, smooth point of C. That P, Q, R are the only singular fibers can be seen by looking at the dual picture. Indeed, by the class formula, one has

$$d^* = 2(g - 1 + k + 1) - (k - 1) = k + 1$$

where $g = 0$ is the genus of C, and d^* is the degree of the dual curve C^*. Now, the point $Q^* \in \mathbb{P}^{2^*}$ is a cusp of C^* with multiplicity k. Hence, the line O^* which passes through Q^* should intersect C^* transversally at a unique further point R^*, which is the dual of the simple tangent line R from O to C.

Now we apply the change of coordinates

$$[x : y : z] \Rightarrow [x + y : y : z].$$

In the new coordinates, the equation of C reads as $(x - y)y^k - z^{k+1} = 0$. Let L_∞ be the line $x = 0$, and pass to the affine coordinates $(y/x, z/x)$ in $\mathbb{C}^2 = \mathbb{P}^2\backslash L_\infty$. Recall that we have the freedom to choose O (or, equivalently, P). So let $O = (1, z_0)$, where z_0 is a big real number. The real picture of the configuration $C \cup P \cup Q \cup R$ is shown in Figure (3.14). Choose the base fiber F, and the base of the projection as in Figure (3.14). Put $F' := F\backslash(C \cup O)$ and $B' := B\backslash(P \cup Q \cup R)$. Take the generators $b, a_1, a_2, \ldots, a_k$ for $\pi_1(F')$ as in Figure (3.15)-I and the generators α, β, γ for $\pi_1(B')$ as in Figure (3.15)-III.

The monodromy around R yields (after setting $\gamma = 1$),

$$a_1 = b \quad (\mathcal{R}_1),$$

and the monodromy around P gives

$$\beta^{-1}a_i\beta = \begin{cases} \delta a_{i+1}\delta^{-1} & 1 \le i < k, \\ \delta^2 a_1 \delta^{-2} & i = k, \end{cases} \qquad (\mathcal{R}_2),$$

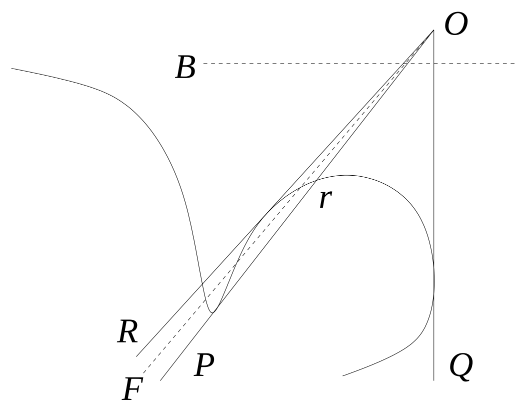


Figure 3.14

where $\delta := a_k a_{k-1} \cdots a_1$. Hence, we have the presentation

$$\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q \cup R)) = \langle \beta, b, a_1, a_2, \ldots, a_k \mid (\mathcal{R}_1),\ (\mathcal{R}_2) \rangle$$

Note that $\alpha\beta\delta b = 1$, and that $[\alpha, \beta] = 1$.

An obvious application of Fujita's Lemma shows that βb^n is a meridian of $\widetilde{P}$ and $\alpha(\alpha\beta)^n$ is a meridian of $\widetilde{Q}$. Therefore,

$$G_{(4)} := \pi_1(\mathbb{P}^2 \backslash \widetilde{C}) = \langle \beta, b, a_1, a_2, \ldots, a_k \mid \beta b^n = \alpha(\alpha\beta)^n = 1,\ (\mathcal{R}_1),\ (\mathcal{R}_2) \rangle.$$

Study of the group

By using $(\mathcal{R}_2)$, one can express the generators $a_1, a_2, \ldots, a_k$ in terms of b as follows

$$a_i = (\beta\delta)^{-i+1} b (\beta\delta)^{i-1} \qquad 1 \leq i \leq k.$$

Then, the last relation in $(\mathcal{R}_2)$ reads

$$(\beta\delta)^{-k} b (\beta\delta)^k = \delta b \delta^{-1} \quad (\mathcal{R}_3).$$

For δ, one has

$$\delta = a_k a_{k-1} \cdots a_1 = (\beta\delta)^{-k} (\beta\delta b)^k \quad (\mathcal{R}_4).$$

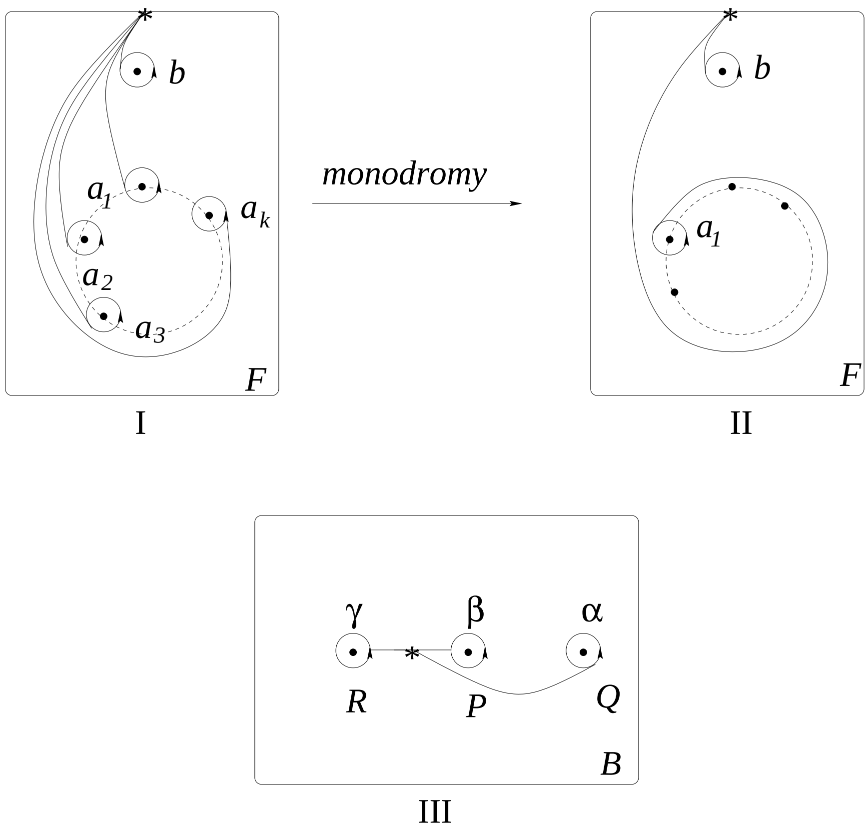


Figure 3.15

Since $\alpha = (\beta\delta b)^{-k}$, the relation $\alpha^{n+1}\beta^n = 1$ becomes

$$(\beta\delta b)^{n+1} b^{n^2} = 1 \quad (\mathcal{R}_5),$$

where we have used $\beta = b^{-n}$. This gives the presentation

$$G_{(4)} = \langle \beta, \delta, b \mid \delta = (\beta\delta)^{-k}(\beta\delta b)^k, \quad \beta = b^{-n}, \quad (\beta\delta b)^{n+1}b^{n^2} = 1, \quad (\beta\delta)^{-k}b(\beta\delta)^k = \delta b \delta^{-1}\rangle.$$

Now put $x := \beta\delta$ and $y := b$. Then $\delta = \beta^{-1}x = y^n x$, and one can rewrite the above presentation as

$$G_{(4)} = \langle x, y \mid (xy)^{n+1}y^{n^2} = [y, x^k y^n x] = 1, \quad x^k y^n x = (xy)^k\rangle,$$

since $(\mathcal{R}_3)$ becomes

$$x^{-k}yx^k = y^n xyx^{-1}y^{-n} \Leftrightarrow [y, x^k y^n x] = 1,$$

and for $(\mathcal{R}_4)$ one has

$$y^n x = x^{-k}(xy)^k \Leftrightarrow x^k y^n x = (xy)^k.$$

Finally, $(\mathcal{R}_5)$ is written as $(xy)^{n+1}y^{n^2} = 1$. Note that $y = b$ is a meridian of $\widetilde{C}$.

Obviously, the latter presentation is equivalent to the presentation

$$G_{(4)} = \langle x, y \mid (xy)^{n+1}y^{n^2} = [y, (xy)^k] = 1, \quad x^k y^n x = (xy)^k\rangle \qquad (*)$$

To simplify this presentation further, put $z := xy$. Then $x = zy^{-1}$, and one obtains the presentation

$$G_{(4)} = \langle z, y \mid z^{n+1}y^{n^2} = [y, z^k] = 1, \quad (zy^{-1})^k y^n zy^{-1} = z^k\rangle$$

It is readily seen from this presentation that the element z^k is central. Passing to the quotient by z^k gives

$$G_{(4)}(z^k) = \langle z, y \mid z^{n+1}y^{n^2} = 1, \quad (zy^{-1})^{k+1}y^n = z^k = 1\rangle.$$

Now put $j := gcd(n+1, k)$. Then one has

$$G_{(4)}(z^k)(y^n) = \langle z, y \mid z^j = y^n = (zy^{-1})^{k+1} = 1\rangle = T_{j,n,k+1},$$

so that this latter group is big if

$$\frac{1}{j} + \frac{1}{n} + \frac{1}{k+1} < 1. \tag{3.1}$$

Obviously, $G_{(4)}(z^k)$, and hence also $G_{(4)}$ is abelian if $j = 1$ or $n = 1$. So suppose that $n, j \geq 2$.

First we consider the case $k = 2$. Since $j \geq 2$, this forces $n \geq 3$ to be odd. If $n \geq 7$, then 3.1 is satisfied. In case $n = 3$ or $n = 5$, 3.1 is violated.

Now assume $k = 3$. Then $j = 3$ by the assumption $j \geq 2$, which forces $n+1 \geq 3$ to be a multiple of 3. For $n+1 \geq 6$, 3.1 is not violated. For $n = 2$, 3.1 is violated.

For $k \geq 4$, first assume that k is even. Then the least value that j can take is 2, and the least value that n can take is 3. If $n = 3$, then $j = 4$, and 3.1 is not violated. But if $n \geq 4$, then 3.1 is not violated neither.

If $k \geq 4$ is odd, then the least divisor of k is 3, hence $j \geq 3$, and $k \geq 6$. In this case 3.1 is never violated.

This leaves the cases $(d, n, k) = (9, 3, 2)$, $(d, n, k) = (13, 5, 2)$, and $(n, k) = (10, 2, 3)$ open. Calculations with Maple show that these are finite groups of order 72, 1560 and 240 respectively.

Notice that when $k = 2$, the degree of the curves (4) is $2n + 3$, so it is interesting to compare their groups with the groups in Theorem 3.2.2. For $k = 2$, the relation $x^2 y^n x = (xy)^k$ in the presentation (*) above becomes

$$x^2 y^n x = xyxy \quad \Leftrightarrow xy^n x = yxy \quad (**).$$

Now put

$$\alpha := y, \quad \beta := xy^{n-1}.$$

Then

$$y = \alpha, \quad x = \beta\alpha^{1-n},$$

and the relation (**) becomes the braid relation $\beta\alpha\beta = \alpha\beta\alpha$. On the other hand, the relation $[y, (xy)^k] = 1$ in the presentation (*) is written, in terms of α, β, as $[\alpha, \beta^n] = 1$. Finally, the relation $(xy)^{n+1} y^{n^2} = 1$ in (*) becomes

$$(\beta\alpha^{2-n})^{n+1}\alpha^{n^2} = 1.$$

Hence, the presentation

$$\langle \alpha, \beta \mid \beta\alpha\beta = \alpha\beta\alpha, \quad [\alpha, \beta^n] = (\beta\alpha^{2-n})^{n+1}\alpha^{n^2} = 1 \rangle.$$

3.3.5 Groups of the curves (5)

Let C be the curve $\{xy^{k-1} - z^k - z^{k-1}y\}$. Its unique singularity is a cusp at the point $p := [1 : 0 : 0]$, and it has a flex of order $k-1$ at the point $q := [0 : 1 : 0]$. Let P, Q be the tangents to C at p and q. By Bezout's theorem, Q intersect C at a third point r. Blowing-up the point $O := P \cap Q$, we get the Hirzebruch surface X_1. Perform n elementary transformations at $E_i \cap P_i$, followed by n elementary transformations at the points r_i. One obtains the Hirzebruch surface X_{2n+1} with $E_{2n+1}^2 = -1$. Contraction of E_{2n+1} gives the projective plane $\mathbb{P}^2$; denote by $\widetilde{C}$, $\widetilde{P}$, $\widetilde{Q}$ the images of C, P and Q. Then $\widetilde{C}$ is a curve of the family (5).

To find $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$, we shall use the projection from the point O. Evidently, P and Q are singular fibers of this projection. There is one further singular fiber say R, which is tangent to C at a unique smooth point of C. Indeed, by the class formula, one has $d^* = k+1$. The point $Q^* \in \mathbb{P}^{2*}$ is a cusp of multiplicity $k-1$. The line O^* intersect C^* at P^*, which is a smooth point of C^*, and at the cusp Q^*. By Bezout's theorem, O^* should intersect C^* at a third point R^* transversally, which is the point dual to the line R^*. This reasoning shows also that there are no other singular fibers.

In the affine coordinates (x, z), the equation of C reads $x = z^k + z^{k+1}$, and it is easy to see that the third singular fiber is the tangent line at $z = -k/(k+1)$. For k even, the situation is pictured in Figure (3.16). Choose the base B and the fiber F as in the Figure (3.16), define F', B' as usual, and choose the generators for their fundamental groups as in Figure (3.17).

Set $\delta := a_k a_{k-1} ... a_2 a_1$. Then the monodromy around Q yields the relations

$$\beta^{-1} a_i \beta = \begin{cases} a_{i+1}, & 1 \leq i \leq k-1, \\ \delta \; a_1 \; \delta^{-1}, & i = k, \end{cases} \qquad [\beta, b] = 1$$

and the monodromy around R gives

$$a_k = b$$

Now, the relation $\beta^{-1} a_{k-1} \beta = a_k = b$ implies $a_{k-1} = b$, since $[\beta, a_k] = 1$. Similarly, one obtains $a_1 = a_2 = \cdots = a_k = b$. This shows that $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$ is abelian, which implies that $\pi_1(\mathbb{P}^2 \backslash \widetilde{C})$ is abelian, too.

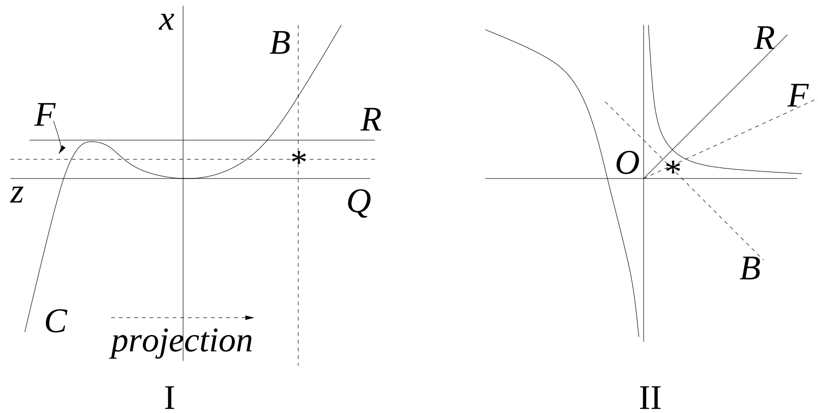


Figure 3.16

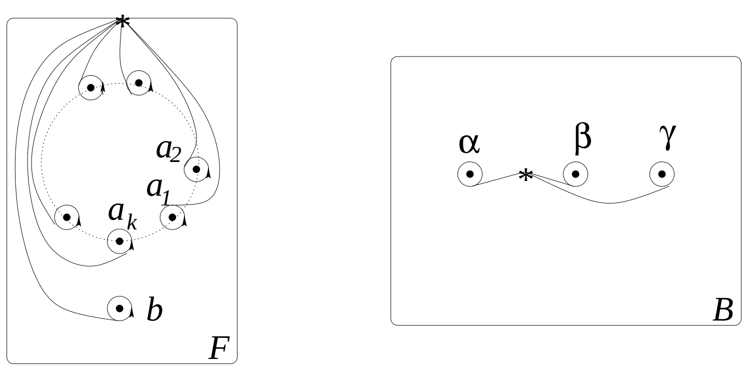


Figure 3.17

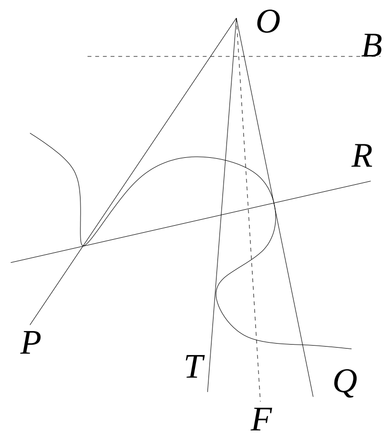


Figure 3.18

3.3.6 Groups of the curves (6) in Theorem 3.2.3

Let C be the curve $\{xy^{k+1} - z^{k+2} - z^{k+1}y = 0\}$, and let Q be its tangent line at the inflection point q. Denote by S the line $\overline{pq}$. The line Q intersects C at a second point r. Blowing-up the point q, we get the Hirzebruch surface X_1 with the horizontal section E_1. For $i = 1, 2, \ldots, n$, perform elementary transformations at $E_i \cap S_i$, followed by elementary transformations at the points r_i for $i = n+1, n+2, \ldots, 2n$. We end up with the Hirzebruch surface X_{2n+1}, with $E_{2n+1} = -1$. Contraction of E_{2n+1} gives the projective plane $\mathbb{P}^2$, denote by $\widetilde{S}$, $\widetilde{Q}$, $\widetilde{C}$ the images of S, Q, and C under this contraction.

The calculation of $\pi_1(\mathbb{P}^2 \backslash (C \cup S \cup Q))$ will be realized by using the projection from the point $O := P \cap Q$, where P is the tangent to C at its cusp as usual. The real picture is as in Figure (3.16), except that we should take the line $S = \overline{pq} = \{z = 0\}$ into consideration. Now, the only points of intersection of the line S with C are the points p and q. Hence, the only singular lines of the projection from the point O are P, Q, and R, where R is the simple tangent line from O to C, as shown in the previous section.

Choose a fiber F and a base B as in Figure (3.16)-I, and put $F' := F \backslash (C \cup S \cup \{O\})$, $B' := B \backslash (P \cup Q \cup R)$. The situation is illustrated in Figure (3.18). Choose the generators $a_1, a_2, \ldots, a_{k+1}, b, \gamma$ for $\pi_1(F')$, and the generators α, β for $\pi_1(B')$ as in Figure (3.19).

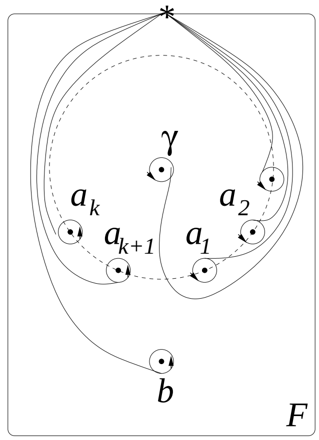


Figure 3.19

The monodromy around the singular fiber Q gives

$$\beta^{-1}a_i\beta = \begin{cases} a_{i+1}, & 1 \leq i \leq k, \\ (\delta\gamma)a_1(\delta\gamma)^{-1}, & i = k+1, \end{cases} \qquad [\beta, b] = 1,$$

where $\delta := a_{k+1}a_k \cdots a_1$. From the monodromy around R we get,

$$a_{k+1} = b.$$

These relations yields a presentation of the group $\pi_1(\mathbb{P}^2 \backslash (C \cup S \cup Q))$.

Applying Fujita's Lemma to the elementary transformations applied at the points r_i, we obtain the relation $\beta b^n = 1$. Imposing this relation on the relations found above, we find that $a_1 = a_2 = \cdots = a_{k+1} = b$, and $[b, \gamma] = 1$. We conclude that the group is abelian.

3.3.7 Groups of the curves (7) in Theorem 3.2.3

Let C be the curve $xy^{k-1} - z^k = 0$, with $p := [1:0:0]$ as its cusp, $q := [0:1:0]$ as its inflection point, and P, Q as the tangents to C at these points. Blow-up the point $O := P \cap Q$, to get the Hirzebruch surface X_1, with the horizontal section E_1. Perform an elementary transformation at q_1, followed by an elementary transformation at $E_2 \cap P_2$. Now for $i = 3, 4, \ldots, n+2$

perform elementary transformations applied at the points $E_i \cap P_i$, followed by elementary transformations applied at some points $s_i \in Q_i \backslash E_i$ for $i = n+3, n+4, \ldots, 2n+2$. We end up with a Hirzebruch surface X_{2n+3} with $E_{2n+3} = -1$. Contraction of E_{2n+3} gives the projective plane $\mathbb{P}^2$.

We shall use the projection from the point O to find the fundamental group. This projection has, in addition to P and Q, a third singular fiber R, which is a simple tangent from O to C at a unique point. The setting is same as in Case 4, see Figure (3.14). Choose a base fiber close to Q, and take the generators for the base and the fiber as in Figure (3.15). The monodromy around Q gives the relations

$$\beta^{-1} a_i \beta = \begin{cases} a_{i+1}, & 1 \leq i \leq k-1, \\ \delta a_1 \delta^{-1}, & i = k, \end{cases}$$

where $\delta := a_k a_{k-1} \cdots a_1$.

Now, without finding the monodromy around R or P, notice that an obvious application of Fujita's lemma gives $a_1\beta$ as a meridian of Q_1. Subsequent elementary transformations on Q_i are applied at some points $s_i \in Q_i \backslash E_i$, so that $a_1\beta$ stays to be a meridian of Q_i. Imposing the relation $a_1\beta = 1$ on the above relations gives $\beta^{-1} = a_1 = a_2 = \cdots = a_k$. But, the group $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$ is generated by the elements $a_1, a_2, \ldots, a_k, \beta$. Hence, the fundamental groups of the curves of the family (7) are abelian.

3.3.8 Groups of the curves (8) in Theorem 3.2.3

Let C be the curve $xy^2 - z^3 = 1$. Pick an arbitrary smooth point $q \in C$ which is not the inflection point of C, and let Q be the tangent of C at q. Put $P := \overline{pq}$, where p is the cusp of C.

By Bezout's theorem, the line Q intersect C at a second point, say r.

Blowing-up the point q, we get the Hirzebruch surface X_1, with the horizontal section E_1. For $i = 1, 2, \ldots, n-1$, perform elementary transformations at the points r_i, followed by elementary transformations performed at the points $E_i \cap P_i$ for $i = n, n+1, \ldots, 2n-2$. We end up with the Hirzebruch surface X_{2n-2} with $E^2_{2n-1} = -1$. Contraction of E_{2n+1} gives the projective plane $\mathbb{P}^2$ back. Denote as usual by $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{C}$ the images of P, Q, C under this contraction. Then $\widetilde{C}$ is a curve of the family (8).

Let us show that the group $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$ is abelian, thereby showing that the groups of the curves of the family (8) are abelian.

Consider the *singular* projection from the cusp p. A generic fiber of this projection intersects $C \cup P \cup Q$ at two points, i.e. $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$ is generated by two elements. These two points meet at the point r, which is a transversal intersection of Q with C. This implies that the corresponding generators commute. It follows that $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$ is abelian. $\square$

3.3.9 Groups of the curves in Theorem 3.1.7

Let C be the curve $(yz - x^2)^2 - x^3 y = 0$.

Lemma 3.3.4 *(Fenske [24]) C is a rational cuspidal quartic with cusps at the point $p := [0:0:1]$ of type $[2_2]$, and at the point $r := [0:1:0]$ of type $[2_1]$. This curve has an inflection point of order 3 at the point $q := [-576 : -4096 : 135]$.*

Let P be the tangent to C at the cusp p, and let Q be the inflectional tangent line at q. The cusp p is the only intersection point of P with C, whereas Q intersect C at a second further point, say s. It is clear that the point $O := P \cup Q$ does not lie on C. Blowing-up O, we get the Hirzebruch surface X_1, with a section E_1 such that $E_1^2 = -1$. Now for $i = 1, 2, \ldots, n$ perform elementary transformations at the points s_i, followed by elementary transformations applied at the points $E_i \cup P_i$ for $i = n+1, n+2, \ldots, 2n+1$. We end up with the Hirzebruch surface X_{2n+1} with $E_{2n+1}^2 = -1$. Contraction of $E_{2n+1}^2 = -1$ gives the projective plane $\mathbb{P}^2$. Denote by $\widetilde{C}$, $\widetilde{P}$, $\widetilde{Q}$ the images of C_{2n+1}, P_{2n+1}, Q_{2n+1} under this contraction. Then $\widetilde{C}$ is the desired curve of degree $3n+4$.

To find the fundamental group of $C \cup P \cup Q$, we shall use the projection from the center O. Let R be the line $\overline{Or}$. Then, clearly P, Q and R are singular fibers of this projection. That there are no other singular fibers can be seen by looking at the dual picture. By the class formula, the degree of the curve C^* dual to C is 4. The line O^* intersects C^* at its simple cusp Q^* with multiplicity 2. Since the line P intersects C with multiplicity 4 at the cusp p, O^* intersects C^* with multiplicity 2 at the cusp P^* of C^*. (That P^* is a cusp of C^* of multiplicity 2 can be seen by using the parameterization $[t^2 : t^4 : 1+t]$ of C.) By Bezout's theorem, this accounts for all the intersection points of the line O^* with C^*.

To get a better picture of the curve, we apply the transformation $[x : y : z :] \to [x : y : y+z]$, and then pass to the affine coordinate system in

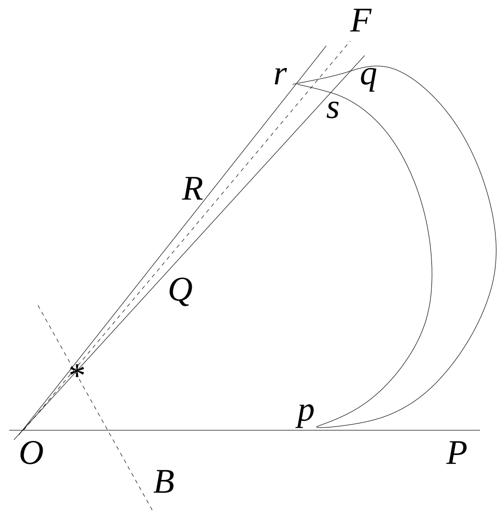


Figure 3.20

$\mathbb{C}^2 = \mathbb{P}^2 \backslash L_\infty$, where $L_\infty = \{z = 0\}$. In these coordinates, C is parameterized as $(t^2/(1+t+t^4), t^4/(1+t+t^4))$, the cusp p is the point $(0,0)$, the cusp r is the point $(0,1)$, and the flex q is the point $(576/3961, 4096/3961)$. It turns out that the point s, the second point of intersection of Q with C, is real. The configuration $C \cup P \cup Q$ is pictured in Figure (3.20). Pick F, B as in Figure (3.20), put $F' := F \backslash (C \cup \{O\})$, $B' := B \backslash (P \cup Q \cup R)$, and choose the generators a, b, c, d of F' and the generators α, β, γ of B' as in Figure (3.21).

The monodromy around Q gives the relations

$$\beta^{-1} b \beta = c, \quad \beta^{-1} c \beta = d, \quad \beta^{-1} d \beta = dcb(dc)^{-1},$$

and the relation

$$\beta^{-1} a \beta = a \Leftrightarrow [\beta, a] = 1.$$

Using these relations, one can express the generators c, d in terms of b and β, and one easily deduces the relation

$$(b\beta)^3 = (\beta b)^3.$$

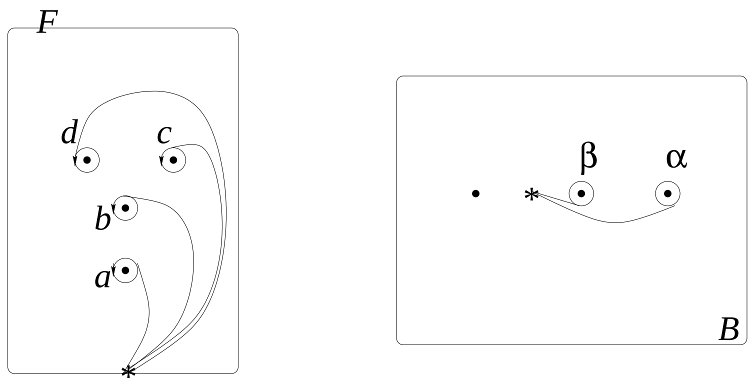


Figure 3.21

Finally, the monodromy around P gives the cusp relation

$$aba = bab.$$

This completes the presentation of $\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q))$. Note that the loop α can be expressed by using the relation

$$dcba\alpha\beta = 1 = \beta^{-2}(b\beta)^3 a\alpha,$$

which implies that $[\alpha, \beta] = 1$, this latter can also be derived by an application of Fujita's lemma at the point O.

By Fujita's lemma, $a^n\beta$ is a meridian of $\widetilde{Q}$, and $\alpha^{n+1}\beta^n$ is a meridian of $\widetilde{P}$. Imposing the corresponding relations on the above presentation we get

$$\pi_1(\mathbb{P}^2 \backslash \widetilde{C}) = \langle a, b, \alpha, \beta \mid \quad (b\beta)^3 = (\beta b)^3 \quad (\mathcal{R}_1), \quad aba = bab \quad (\mathcal{R}_2),$$
$$\beta a^n = 1 \quad (\mathcal{R}_3), \quad \alpha^{n+1}\beta^n = 1 \quad (\mathcal{R}_4), \quad \beta^{-2}(b\beta)^3 a\alpha = 1 \quad (\mathcal{R}_5) \rangle$$

Note that the relation $[\beta, a] = 1$ follows from the relation $(\mathcal{R}_3)$ and therefore is not written in the above presentation.

In order to simplify this presentation, put $b\beta =: \gamma$. Then the relation $(\mathcal{R}_1)$ can be expressed as $[\beta, \gamma^3] = 1$, or, substituting $\beta = a^{-n}$, as $[a^n, \gamma^3] = 1$. On the other hand, $b = \gamma\beta^{-1} = \gamma a^n$, so that the cusp relation $(\mathcal{R}_2)$ becomes

$$a\gamma a^n a = \gamma a^n a \gamma a^n \Leftrightarrow a\gamma a = \gamma a^{n+1}\gamma.$$

Using $(\mathcal{R}_5)$, we can express α as follows:

$$\alpha = a^{-1}\gamma^{-3}a^{-2n}.$$

Substituting this in $(\mathcal{R}_4)$ we get, using $[a^n, \gamma^3] = 1$,

$$(a^{-1}\gamma^{-3}a^{-2n})^{n+1}a^{n^2} = 1 \Leftrightarrow a^{3n^2+2n}(\gamma^3 a)^{n+1} = 1.$$

So we have obtained the presentation

$$\pi_1(\mathbb{P}^2\backslash\widetilde{C}) = \langle a, \gamma \mid a\gamma a = \gamma a^{n+1}\gamma, \quad [a^n, \gamma^3] = a^{3n^2+2n}(\gamma^3 a)^{n+1} = 1\rangle.$$

Put $G := \pi_1(\mathbb{P}^2\backslash\widetilde{C})$. Then one has

$$G(n) = G(a^n) = \langle a, \gamma \mid a\gamma a = \gamma a\gamma, \quad (\gamma^3 a)^n = 1\rangle.$$

Applying the transformation $x := a\gamma$, $y := a\gamma a$, and imposing the relation $x^3 = 1$ we get a surjection $G \twoheadrightarrow T$, where

$$T := \langle x, y \mid y^2 = x^3 = x^{2(n+1)} = (xy^{-1})^n = 1\rangle,$$

so that if $3|(n+1)$, then one has a surjection $G(n) \twoheadrightarrow T_{2,3,n}$ onto the triangle group, which implies that $G(n)$ is big for $n \geq 7$.

A more economical way of writing the last relation in the presentation of G is as follows: one has $a^{n+1} = \gamma^{-1}a\gamma a\gamma^{-1}$. Hence,

$$\begin{aligned} a^{3n^2+2n}(\gamma^3 a)^{n+1} &= a^{2n^2+n-1}a(\gamma^3 a^{n+1})^3 \\ &= (a^{n+1})^{2n-1}a\gamma^2(\gamma a^{n+1}\gamma\cdot\gamma\cdot\gamma a^{n+1}\cdots\gamma a^{n+1}\gamma)\gamma^{-1} \\ &= (\gamma^{-1}a\gamma a\gamma^{-1})^{2n-1}a\gamma^2(a\gamma a)^{n+1}\gamma^{-1}, \end{aligned}$$

so that one can replace the last relation by the relation

$$(\gamma^{-1}a\gamma a\gamma^{-1})^{2n-1}a\gamma(\gamma a)^{2n+2}\gamma^{-1}.$$

Remark. By the following lemma, the group G is actually a quotient of the braid group $\mathbb{B}_3$ on three strands.

Lemma 3.3.5 *For any $n \in \mathbb{Z}$, the group with the presentation*

$$\langle a, b \mid aba = ba^{n+1}b\rangle$$

is isomorphic to $\mathbb{B}_3$.

Proof. Applying the transformation $(a, b) \to (x := a, y := ba^n)$, with inverse $(x, y) \to (a = x, b = yx^{-n})$, the above relation becomes

$$x\,yx^{-n}\,x \;=\; yx^{-n}\,x^{n+1}\,yx^{-n},$$

which is nothing but the braid relation $xyx = yxy$. $\square$

Chapter 4

Finite non-abelian curve groups and Zariski pairs

4.1 Zariski pairs

In his pioneering paper on the fundamental groups of plane curve complements, Zariski [100] has shown that the group of a sextic C_1 with six cusps lying on a conic $\mathbb{Z}_2 * \mathbb{Z}_3$, the free product of cyclic groups of order 2 and 3. Such a curve can be given by an equation $h := f^2 - g^3 = 0$, where f is a cubic polynomial, and g is a quadratic polynomial. For generic f and g, the sextic curve defined by h has six simple cusps at the points of intersection of the curves defined by f and g. Zariski went on showing that if C_2 is a sextic with six simple cusps *not* lying on a conic, then $\pi_1(\mathbb{P}^2 \backslash C_2)$ is abelian. He then remarked that a sextic curve of this type must exist, because "it is highly improbable that the six cusps of a sextic should lie on a conic". An equation for such a sextic has been given much later by Oka in [77]. It follows from this example that the singularities of a curve do not determine its group.

Artal considered the problem of finding other examples of such pairs [3]. Following him, we give the definition below (see [5]).

Definition 4.1.1 A pair (C_1, C_2) of plane curves is said to be a *Zariski pair* if the following two conditions are satisfied:

(i) There is a degree-preserving correspondence α between the set of irreducible components of C_1 and C_2, and there exist tubular neighborhoods T_1 of C_1 and T_2 of C_2 such that the pairs (T_1, C_1) and (T_2, C_2) are homeomorphic via a map respecting α.

(ii) The pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic.

As noted in [5], under these conditions one can say that "the curves C_1 and C_2 has the same singularities", since the condition (i) implies the existence of a correspondence β between the branches at the singular points of C_1 and C_2 such that
(1) B and $\beta(B)$ are of same topological type for any branch B at a singular point of C_1,
(2) the intersection numbers of branches at singular points are preserved under β,
(3) if C_1' is the irreducible component of C_1 containing a branch B at a singular point, then $\alpha(C_1')$ is the irreducible branch of C_2 containing $\beta(B)$.

For a pair of curves satisfying the condition (i), one can use several invariants to show the condition (ii). The most obvious one is the fundamental group: If $\pi_1(\mathbb{P}^2\backslash C_1) \ncong \pi_1(\mathbb{P}^2\backslash C_2)$, then the pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ cannot be homeomorphic. Other, coarser invariants are the Alexander polynomial, which is an invariant of the fundamental group, and the b-invariant, defined as follows:

For a reduced curve $C \subset \mathbb{P}^2$ of degree d given by a homogeneous polynomial $F(x, y, z)$, consider a desingularization $\widetilde{X}$ of the surface X in $\mathbb{P}^3$ defined by the equation $F(x, y, z) - w^d$. As any two desingularizations of X are birationally equivalent, it follows that the dimension $b(C)$ of $H^1(\widetilde{X}, \mathbb{C})$ is an invariant of the pair $(\mathbb{P}^2, C)$. We propose the residual groups (introduced in Definition 3.3.2), or divisor groups (introduced in Chapter 6) as alternative invariants that can be used to distinguish such pairs.

Following Zariski's idea of using the b-invariants to distinguish pairs of curves with the same singularities, Artal gave examples of Zariski pairs of irreducible curves of degree 6 with a unique, (reducible) singularity of multiplicity 2 [3]. In the same year, Degtyarev [16] gave a classification of all irreducible sextics with non-trivial Alexander polynomials and discovered some new Zariski pairs of curves of degree six. Oka used some coverings of the plane to construct a Zariski pair of curves of order 12 in 1995 [76]. Following him, Shimada gave an example of an infinite family of Zariski pairs [89]. Some other Zariski pairs of sextics appeared in [6], [93]. Some examples of Zariski pairs (of reducible curves) not distinguished by their Alexander polynomials are exhibited by Artal and Ruber [5] in 1996. The strongest result up-to date is the following one:

Theorem 4.1.1 ***(Kulikov [45], [46])*** *For each $k \in \mathbb{N}$ there exists an infi-*

nite family of Zariski k-tuples $(C_1, C_2, \ldots, C_k)$.

Very recently, Oka gave examples of Alexander-equivalent Zariski pairs of irreducible sextics and discovered some Zariski triples [75].

Examples of arrangements of nine lines with the same singularities but different fundamental groups are given in [4]. Fan [22] discovered an example with seven lines, and showed that there are no such arrangements with less then seven lines.

In Section 3 below, we describe a recipe for finding new Zariski pairs from the known ones.

4.2 Curves with finite non-abelian groups

The first example of a curve with a finite non-abelian group (already known to Zariski) is the three cuspidal quartic. Degtyarev [15] discovered the next example: the group of the rational quintic with three double cusps is a finite group of order 320. In the same article, he also gave several infinite families of curves of type $(d, d-2)$ with finite non-abelian groups, see Theorem 2.2.4.

Oka suggested the problem of finding more examples of curves with finite non-abelian groups, and constructed another infinite family of such curves. These examples appeared in an independent work of Shimada [88], too. His curves are obtained from the three cuspidal quartic as pull-backs by some coverings.

Theorem 4.2.1 ***(Oka [76], Shimada [88])*** *For every* $n \in \mathbb{N}$ *there exists a curve* $C \subset \mathbb{P}^2$ *of degree* $4n$*, whose group is a finite non-abelian one of order* $12n$*. These groups are central extensions of the group of the three-cuspidal quartic.*

After Oka's work, Shimada has constructed another family of cuspidal curves with a finite non-abelian fundamental group.

Theorem 4.2.2 ***(Shimada [87])*** *For each* $q, p \in \mathbb{N}$ *with* $q > 1$ *odd, there exists a curve* $C_{p,q}$ *with singular locus consisting of* $(2q-3)p^2$ *singular points of type* $\mathbb{A}_{q-1}$ *(i.e. locally given by* $x^2 = y^q$*), such that* $\pi_1(\mathbb{P}^2 \backslash C_{p,q})$ *is a finite non-abelian group of order* $2pq(q-1)$*. Moreover,* $\pi_1(\mathbb{P}^2 \backslash C_{p,q})$ *is a central extension of the dihedral group* D_q *by the cyclic group* $\mathbb{Z}_{p(q-1)}$.

Artal [2] (see Theorem 3.2.1) discovered a rational cuspidal curve of degree 7 with a finite non-abelian group among the curves constructed in [26]. We have discovered four other finite fundamental groups of rational cuspidal plane curves (see Theorem 3.2.3).

The recipe described below allows one to construct new curves with finite groups from the known ones, see Corollary 4.3.1. It can also be applied to curves with almost solvable groups in Theorem 2.4.1 to find more examples of such groups.

4.3 Recipe

Theorem 4.3.1 *For a curve $C \subset \mathbb{P}^2$ and a line $Q \subset \mathbb{P}^2$, let μ be a meridian of the line Q in $\mathbb{P}^2 \backslash (C \cup Q)$. Then for each $n \in \mathbb{N}$, there exists a plane curve $\widetilde{C} \subset \mathbb{P}^2$ with*

$$\pi_1(\mathbb{P}^2 \backslash \widetilde{C}) = \pi_1(\mathbb{P}^2 \backslash (C \cup Q)) / \ll \mu^n \gg .$$

The curve $\widetilde{C}$ is obtained as the image of C by a Cremona transformation $\mathbb{P}^2 \to \mathbb{P}^2$.

Proof. Let $O \in Q \backslash C$ be a point and take another line P passing through O. Blowing-up the point O, we get the Hirzebruch surface X_1. Following the conventions fixed in Chapter 3.3.2, denote the exceptional section of this blow-up by E_1, and the proper images of C, P, Q by C_1, P_1, and Q_1. Now perform n elementary transformations at some points $s_i \in P_i \backslash C$, followed by n elementary transformations applied at the points $Q_i \cap E_i$. This gives the Hirzebruch surface X_{2n+1} with $E_{2n+1}^2 = -1$. Contraction of E_{2n+1} gives the projective space; denote the images of P_{2n+1}, Q_{2n+1}, C_{2n+1} under this contraction by $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{C}$. If the curve C is of degree d, then $\widetilde{C}$ is a singular plane curve of degree $d(n+1)$. Note that, besides the singularities of C, the curve $\widetilde{C}$ has two additional singularities of multiplicity d. If the lines P,Q intersect C transversally, and $n = 1$, one of these is an ordinary d-tuple point. The other singularity also has d smooth branches, but these branches do not intersect transversally. If $n > 1$ these singularities become more complicated.

The Cremona transformation of C to $\widetilde{C}$ yields a biholomorphism

$$\mathbb{P}^2 \backslash (C \cup P \cup Q) \stackrel{\simeq}{\to} \mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{P} \cup \widetilde{Q}),$$

which in turns induces an isomorphism

$$\pi_1(\mathbb{P}^2 \backslash (C \cup P \cup Q)) \simeq \pi_1(\mathbb{P}^2 \backslash (\widetilde{C} \cup \widetilde{P} \cup \widetilde{Q})).$$

Hence, the group $\pi_1(\mathbb{P}^2\backslash\widetilde{C})$ can be found from $\pi_1(\mathbb{P}^2\backslash(C\cup P\cup Q))$ by adding the relations which correspond to the gluing of the lines $\widetilde{P}$ and $\widetilde{Q}$.

Let α be a meridian of P and β be a meridian of Q in $\mathbb{P}^2\backslash(C\cup P\cup Q)$. If we assume that α and β are chosen properly in a neighborhood of the point $O = P\cup Q$, then an obvious application of Fujita's lemma (Lemma 3.3.2) shows that α is a meridian of $\widetilde{P}$, and $(\alpha\beta)^n\beta$ is a meridian of $\widetilde{Q}$. Setting $\alpha = 1$ gives

$$\pi_1(\mathbb{P}^2\backslash\widetilde{C}) \simeq \pi_1(\mathbb{P}^2\backslash(C\cup Q))/\ll \beta^{n+1} \gg .$$

Finally, being both meridians of Q, the loops μ and β are conjugate elements in $\pi_1(\mathbb{P}^2\backslash(C\cup Q))$, so that one can replace β by μ in the above isomorphism. This proves the claim. $\square$.

Now suppose that the line Q intersects C transversally. Then by Theorem 5.2.2, the meridian μ of Q is central in $\pi_1(\mathbb{P}^2\backslash(C\cup Q))$, and the subgroup of $\pi_1(\mathbb{P}^2\backslash(C\cup Q))$ generated by μ is isomorphic to $\mathbb{Z}$, so that one has the exact sequences

$$0\to\mathbb{Z}\to\pi_1(\mathbb{P}^2\backslash(C\cup Q))\to\pi_1(\mathbb{P}^2\backslash C)\to 0$$

and

$$0\to(n+1)\mathbb{Z}\to\pi_1(\mathbb{P}^2\backslash(C\cup Q))\to\pi_1(\mathbb{P}^2\backslash\widetilde{C})\to 0.$$

This yields the exact sequence

$$0\to\mathbb{Z}/(n+1)\mathbb{Z}\to\pi_1(\mathbb{P}^2\backslash\widetilde{C})\to\pi_1(\mathbb{P}^2\backslash C)\to 0.$$

It follows that, if $\pi_1(\mathbb{P}^2\backslash C)$ is a finite non-abelian group, then so is $\pi_1(\mathbb{P}^2\backslash\widetilde{C})$. Hence one has the following corollary.

Corollary 4.3.1 *Let G be the group of a curve $C\subset\mathbb{P}^2$, where C is of degree d. Then, for any $n\in\mathbb{N}$, there is a central extension H of G by the cyclic group of order n, such that H is also a curve group. Hence, if G is a finite non-abelian group of order $|G|$, then H is a finite non-abelian group of order $|H| = n|G|$. If G is almost solvable, then so is H.*

One can apply Theorem 4.3.1 recursively to get examples of finite non-abelian groups of curves with many singularities. A similar way of producing finite non-abelian groups can be described as follows. Let $C\subset\mathbb{P}^2$ be an irreducible curve. Even if the group $\pi_1(\mathbb{P}^2\backslash C)$ is abelian, sometimes the group

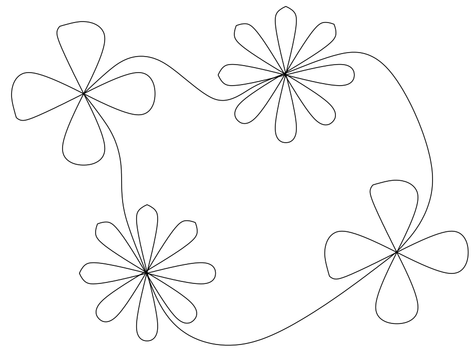


Figure 4.1

$\pi_1(\mathbb{P}^2\backslash(C\cup Q))$ happens to be "small" (e.g. almost solvable) for a line Q non-generic with respect to C. Applying Theorem 4.3.1 to this situation, one can obtain a curve with a finite non-abelian group. As an example, consider the two-cuspidal quartic C with its flex line Q, discussed in the previous section. Degtyarev found that the group of $C\cup Q$ is $\mathbb{Z}\times G$, where

$$G := \langle x, y, z \,|\, x^2 = y^3 = z^5 = xyz\rangle$$

is a finite perfect group of order 120[1]. Now a meridian of Q should be of the form $\mu := (1, g) \in \mathbb{Z}\times G$, where g is a normal generator for G, and for some values $n \in \mathbb{N}$, $\mathbb{Z}\times G/ \ll \mu^n \gg$ will be a finite non-abelian curve group.

Notice that there is another way of producing curves with finite groups: Namely, one may apply an automorphism ϕ of $\mathbb{C}^2 = \mathbb{P}^2\backslash Q$ to a curve whose affine part has a "small" group as discussed above. This is a variation of the method employed by Oka in [76]. It can be applied to the affine quartic with its flex Q discussed above, or to a curve C with a finite non-abelian group with Q intersecting C transversally.

Finally, the Cremona transformations in the proof of Theorem 4.3.1 can be used to obtain new Zariski pairs from the known ones as follows: Suppose that (C_1, C_2) is a Zariski pair, with $\pi_1(\mathbb{P}^2\backslash C_1)$ abelian, and $\pi_1(\mathbb{P}^2\backslash C_2)$ non-abelian (as a concrete example one can consider C_1, C_2 to be the six-cuspidal sextics discussed by Zariski). Then an application of the Cremona

[1] This is an example of an affine curve group with elements of finite order, see also [101].

transformations as in the proof of Theorem 4.3.1 produces two curves $\widetilde{C_1}$, $\widetilde{C_2}$. If one chooses the lines P, Q generically, then these curves will have the same singularities. The pair $(\widetilde{C_1}, \widetilde{C_2})$ will then be a Zariski pair, since the group of $\widetilde{C_1}$ is abelian, whereas the group of $\widetilde{C_2}$ is not.

Remark. If, in Theorem 4.3.1, the line Q meets C transversally, the Alexander polynomials of C and of $\widetilde{C}$ are the same. Also, the commutator subgroup of $\pi_1(\mathbb{P}^2 \backslash C)$ is isomorphic to the commutator subgroup of $\pi_1(\mathbb{P}^2 \backslash \widetilde{C})$ by Lemma 5.2.1 of the next chapter.

By the way, the curves with finite groups constructed above provide many examples of affine curves $\widetilde{C} \backslash \widetilde{Q} \subset \mathbb{P}^2 \backslash \widetilde{Q} \simeq \mathbb{C}^2$, (which do not intersect transversally the line at infinity) and have non-abelian, "small" groups.

Chapter 5

Group theoretical miscellany

5.1 Introduction

Our aim in this chapter is to study the group $\pi_1(\mathbb{P}^2\backslash C)$ from the group theoretical point of view, mostly for the case of an irreducible curve C. The class of groups of irreducible curves is specific enough to allow a study from the abstract group-theoretical point of view. To this end, we introduce a larger class of groups, the class of *irreducible groups*, to which the groups of irreducible curves belong (see Definition 5.2.1). Given a "nice" presentation of such a group, it is possible to give a "nice" set of generators for its commutator subgroup. This leads to a presentation of the group G/G'' for an irreducible group G. The group G/G'' is an invariant of G which is stronger than the Alexander polynomial. In some cases, this group is easier to find than the Alexander polynomial itself. For example, it can easily be shown that the commutator subgroup of the braid group on $n \geq 5$ strands is perfect. Another importance of the group G/G'' is that in some cases it allows one to find finite index subgroups of the group G, which is essential for the study of the Galois coverings of $\mathbb{P}^2$ branched along an irreducible curve. We discuss the problem of finding finite index subgroups in 5.3, this problem will be considered in more detail in the next chapter.

The commutator subgroup of the group of an irreducible curve is finitely presented by well-known results from the group theory. Moreover, the Reidemeister -Schreier algorithm gives an effective way of finding a finite presentation of the commutator subgroup. An explicit application of this algorithm to a curve group is given in 5.4, where the presentation of the commutator subgroup turned out to be quite simple, even though this presentation is obtained by long and messy calculations.

In 5.5 we collect some facts about the generalized triangle groups that we use frequently in this work.

5.2 Irreducible groups

Definition 5.2.1 A group G is said to be *irreducible* if there exists an element $\mu \in G$ with $G = \ll \mu \gg$, i.e. if G is the normal closure of one of its elements. This element μ is said to *normally generate* G.

Clearly, if μ normally generates G, then any conjugate $\mu' = a\mu a^{-1}$ of μ normally generates G, too. The most elementary properties of irreducible groups are the following:

Proposition 5.2.1 *(i) An abelian group is irreducible if and only if it is cyclic.*

(ii) A homomorphic image of an irreducible group is irreducible.

(iii) Abelianization of an irreducible group is a cyclic group.

Remark. A subgroup of an irreducible group need not be irreducible. For example, the braid group $\mathbb{B}_3$ on 3 strands is irreducible, but the commutator subgroup $\mathbb{B}_3'$ is isomorphic to $\mathbb{F}_2$, the free group on two generators, which is not irreducible (e.g. its abelianization is not cyclic). For a presentation of the commutator subgroups of the braid groups, see [33].

Examples of irreducible groups. Besides the cyclic groups, the most next examples of irreducible groups that come into mind are the simple groups. Indeed, a simple group is normally generated by any one of its elements, except the identity element. Symmetric groups are also irreducible, for they are generated by transpositions, and any two transpositions are conjugate. Moreover, many of the groups arising in geometry are irreducible. These include the braid groups (since the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ can be written as $(\sigma_1\sigma_2)\sigma_1(\sigma_1\sigma_2)^{-1} = \sigma_2$), the mapping class groups, the knot groups (by the Wirtinger presentation), groups of irreducible curve complements in $\mathbb{C}^2$ or in $\mathbb{P}^2$.

Presentations of irreducible groups

Let G be a finitely presented irreducible group, with $\mu \in G$ a normal generator. Let X be the conjugacy class of μ. Then, by the following theorem (see [81] for a proof), G admits a presentation, with finitely many generators among the elements of X.

Theorem 5.2.1 *(B.H. Neumann) If X is any set of generators of a finitely presented group G, then this group has a finite presentation of the form $\langle X_0 \mid \mathcal{R}_1 = \mathcal{R}_2 = \cdots = \mathcal{R}_t = 1\rangle$, where $X_0 \subset X$.*

Many examples of finitely presented groups can be obtained as follows: consider the presentation

$$G := \langle a_1, a_2, \ldots, a_n \mid \mathcal{R}_1 = \mathcal{R}_2 = \cdots = \mathcal{R}_{n-1} = \mathcal{R}_n = \mathcal{R}_{n+1} \cdots = 1\rangle \quad \text{with} \quad \mathcal{R}_i = x_i a_i x_i^{-1} a_{i+1}^{-1} \quad \text{for} \quad 1 \le i \le n-1, \tag{5.1}$$

where x_i are some words in $a_1, a_2, \ldots, a_n$. Then G is clearly an irreducible group.

One has the following facts concerning the commutator subgroups of the groups given by the presentation 5.1.

Proposition 5.2.2 *Let G' be the commutator subgroup of an irreducible group given by the presentation 5.1, such that $G/G' = \mathbb{Z}/d\mathbb{Z}$, where $0 < d \le \infty$. Then*

(i) G' is normally generated by the set $X := \{a_i a_{i+1}^{-1} \,:\, 1 \le i \le n-1\}$.

(ii) Furthermore, G' is generated by the set $Y := \{a_i^k a_{i+1}^{-k} \,:\, 1 \le i \le n-1, \quad 1 \le k \le d-1\} \cup \{a_i^d \,:\, 1 \le i \le n-1\}$.

Proof. (i) It is clear that, setting $a_i a_{i+1}^{-1} = 1$, (that is, setting $a_1 = a_2 = \cdots$), one obtains the abelianization of G, so that X should normally generate G'.

(ii) By the relation $\mathcal{R}_i$, one has $x a_i x^{-1} = a_{i+1} \Rightarrow x a_i^k x^{-1} = a_{i+1}^k$. Hence, $a_{i+1}^k a_i^{-k} = x a_i^k x^{-1} a_i^{-k} = [x, a_i^k] \in G'$. On the other hand, if $w = a_{m_1}^{n_1} a_{m_2}^{n_2} \cdots a_{m_k}^{n_k} \in G'$, then one has $n_1 + n_2 + \cdots n_k = 0 \pmod{d}$. Hence, one can write

$$w = (a_{m_1}^{n_1} a_{m_2}^{-n_1}) \cdot (a_{m_2}^{n_2+n_1} a_{m_3}^{-n_2-n_1}) \cdots (a_{m_{k-1}}^{-n_k} a_{m_k}^{n_k}) \cdot a_{m_k}^{rd}$$

for some $r \in \mathbb{Z}$. To finish the argument, observe that if $i < j$, then

$$a_i^n a_j^{-n} = a_i^n a_{i+1}^{-n} \cdot a_{i+1}^n a_{i+2}^{-n} \cdots a_{j-1}^n a_j^{-n}.$$

(If $i > j$, then one applies this to $(a_i^n a_j^{-n})^{-1}$). $\square$

A class of "universal irreducible groups" can be defined as follows: Let us first recall the definition of a C-group, introduced by Kulikov [50], [49]. A group G is said to be a *C-group* (C for conjugate) if it has a presentation in which all the relations are conjugation relations between the generators. That is,

$$G = \langle a_1, a_2, \dots \mid \mathcal{R}_1 = \mathcal{R}_2 = \dots = 1 \rangle, \tag{5.2}$$

where each $\mathcal{R}_i$ is of the form $a_i a_j a_i^{-1} a_k^{-1}$. By Theorem 5.2.1, if G is finitely presented, then the above presentation has finitely many generators.

The first examples of C-groups are the free groups and the free abelian groups. By the Wirtinger presentation, the link groups are C-groups, and the knot groups are irreducible C-groups. The following proposition shows that the braid groups and the groups of affine curve complements are irreducible C-groups.

Proposition 5.2.3 *Consider the presentation*

$$G = \langle a_1, a_2, \dots \mid \mathcal{R}_1 = \mathcal{R}_2 = \dots = 1 \rangle.$$

(i) If all the relations $\mathcal{R}_i$ are of the form $w a_i w^{-1} a_j^{-1}$ then G is a C-group. If G is finitely presented, then G is also finitely presented as a C-group.

(ii) Any irreducible group G is a quotient of an irreducible C-group $\widetilde{G}$ (in many ways).

Proof. (i) We only illustrate the proof for the braid group $\mathbb{B}_3$, the generalization is then obvious. This group has a unique defining relation $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1} = \sigma_2$. Put $x := \sigma_1$, $y := \sigma_2$, $z := \sigma_2\sigma_1\sigma_2^{-1} = yxy^{-1}$. This gives the C-group presentation

$$\mathbb{B}_3 = \langle x, y, z \mid z = yxy^{-1}, \quad y = xzx^{-1} \rangle.$$

Thus, introducing some new generators, any relation $w a_i w^{-1} a_j^{-1}$ can be written as a set of conjugation relations between these generators, applying the procedure illustrated for the braid group. Clearly, if the initial presentation is finite, then this procedure gives a finite C-group presentation.
(ii) Let μ be a normal generator of an irreducible group G, and $X := \{a_1, a_2, \dots\}$ be a set of conjugates of μ generating G. Then for each $i \in \mathbb{N}$ one has a relation $(\mathcal{R}_i) \quad w a_i w^{-1} a_j^{-1}$ in G. Consider the irreducible group H

given by the presentation $\langle a_1, a_2, \ldots \mid \mathcal{R}_1 = \mathcal{R}_2 = \cdots = 1 \rangle$. Then clearly H surjects onto G. $\square$

It follows that the fundamental groups of affine curve complements are C-groups by the Zariski-Van Kampen presentation, since any relation in this presentation is of the form $wa_iw^{-1}a_j^{-1} = 1$. Fundamental groups of projective curve complements are obtained from this presentation by adding a "projective relation", so that they can be viewed as "one relator quotients of C-groups". More precisely, let $C \subset \mathbb{C}^2$ be a curve of degree d, and let L be a line intersecting C generically. Then there is a surjection $\pi_1(L \backslash C) \twoheadrightarrow \pi_1(\mathbb{C}^2 \backslash C)$, that is, if we define a set of "good" generators $a_1, a_2, \ldots a_d$ for $\pi_1(L \backslash C)$, then $a_1, a_2, \ldots, a_d$ generate $\pi_1(\mathbb{C}^2 \backslash C)$, and the Zariski-van Kampen presentation gives some relations of the form $wa_iw^{-1}a_j^{-1} = 1$. Application of the procedure in the proof of Proposition 5.2.3 introduces new generators $a_{d+1}, a_{d+2}, \ldots, a_k$ and gives a C-group presentation of $\pi_1(\mathbb{C}^2 \backslash C)$ with generators $a_1, a_2, \ldots a_d, a_{d+1}, \ldots, a_k$. The projective relation can be written as $\delta := a_d a_{d-1} \cdots a_1 = 1$. Here, δ is a meridian of L_∞, where we assume $\mathbb{C}^2 = \mathbb{P}^2 \backslash L_\infty$. At this point, the following result should be recalled:

Theorem 5.2.2 *Let $C \subset \mathbb{P}^2$ be a curve. Then the pairs $(\mathbb{P}^2, C \cup L)$, $(\mathbb{P}^2, C \cup L')$ are diffeomorphic for the lines L, L' intersecting C generically. Hence, $\pi_1(\mathbb{C}^2 \backslash C) = \pi_1(\mathbb{P}^2 \backslash (C \cup L_\infty))$ is the same for any line L_∞ intersecting C generically. Moreover, under this assumption one has*

(i) The meridian δ of L_∞ is a central element of the group $\pi_1(\mathbb{C}^2 \backslash C)$.

(ii) The commutator subgroups of the groups of the affine and projective curves are isomorpic, i.e.

$$\pi_1'(\mathbb{C}^2 \backslash C) \simeq \pi_1'(\mathbb{P}^2 \backslash C).$$

Proof. The former claim is a direct consequence of the fact that $\mathbb{P}^{2*} \backslash C^*$ is connected, where $\mathbb{P}^{2*}$ is the dual projective space, and C^* is the "complete" dual curve of C, including the dual lines of singular points of C . The part (i) is well known, and can easily be proved by an application of the Zariski-Van Kampen algorithm to a projection with center lying on L_∞, see e.g. [83]. The last part follows from the trivial lemma given below (see [15]). $\square$

Lemma 5.2.1 *Let G be a group, and Z be a central subgroup of G. If $Z \to G/G'$ is an injection, then the commutator subgroup of G/Z is isomorphic to G'.*

Remark. Similarly, it can be shown that the pairs $(\mathbb{P}^2, C \cup L)$ with C an irreducible curve, are diffeomorphic for $L^* \in C^*$, provided that the tangent line L intersects C at exactly $d-1$ smooth points of C, d being the degree of C. Also, the dual of Theorem 5.2.2 holds, that is, if a point $p \in \mathbb{P}^2$ is generic with respect to C, then the pairs $(\mathbb{P}^2, C \cup D)$ are diffeomorphic, where D is the set of lines passing through p and non-generic with respect to C.

The Alexander polynomial

Consider a finitely generated irreducible C-group G given by the presentation 5.2. Clearly, for its abelianization one has an isomorphism $\phi : G/G' \overset{\simeq}{\to} \mathbb{Z}$; and the group G/G' is generated by $\tau := \phi(a_1)$. Then τ acts on G' by conjugation by a_1. This action can be described as follows:

Let $T := \{a_1^n \;:\; n \in \mathbb{Z}\}$ be a set of Schreier transversals for G'. (For a description of the Reidemeister-Schreier algorithm, see e.g. [11]). For $j \neq 1$, put $x_{i,j} := a_1^i a_j a_1^{-i-1}$. Then the set $\{x_{i,j} \;:\; i_i n\mathbb{Z},\; j \in \mathbb{N}\}$ generates G'. It follows that $\tau(x_{i,j}) = a_1 x_{i,j} a_1^{-1} = x_{i+1,j}$. Consider the induced action $\hat{\tau}$ on the torsion-free part A of G'/G''. If A is finitely generated of rank $r > 0$, then the characteristic polynomial $\Delta_G := \det(\hat{\tau} - t \cdot \mathrm{I})$ of $\hat{\tau} : A \simeq \mathbb{Z}^r \to \mathbb{Z}^r$ is called the *Alexander polynomial* of G. Clearly, this definition makes sense for an arbitrary irreducible group. For alternative, equivalent definitions, see [58], [5], [83]. Other relevant references concerning the Alexander polynomials of plane curves are [54], [20], [59], [50], [16].

Theorem 5.2.3 *Let K be the smallest subgroup of G containing both the commutator subgroup G' and the center $\mathcal{Z}(G)$ of G. If G/K is a finite cyclic group of order d, then the commutator subgroup G' is finitely generated, and the order of $\hat{\tau}$ is finite and divides d. Hence, $|\Delta_G(0)| = 1$, and Δ_G is a cyclotomic polynomial.*

Proof. It follows from the hypothesis that there exists an element $z \in \mathcal{Z}(G)$ such that $\phi(z) = \tau^d$. Hence, one can write $z = a_1^d g$, where $g \in G'$. So, the action of $a_1^d = zg^{-1}$ on A is trivial, since z is central and g is a commutator. Thus, $\hat{\tau}$ is of finite order, and this order divides d. The linear map $\hat{\tau} : \mathbb{Z}^r \to \mathbb{Z}^r$ being of finite order, the other assertions follows immediately. $\square$

Let us call a group G *rigid of degree d* if G is irreducible, finitely generated, and if G/K is finite cyclic of order d, where K is the smallest subgroup of

G containing both the center and the commutator subgroup of G. Rigid C-groups are the best candidates for the groups of affine curves intersecting the line at infinity generically. We do not know how to decide from an irreducible C-group presentation whether the group is rigid, i.e. G/K is finite cyclic of order d. A natural way to impose the rigidity on a group is to suppose that the presentation relations comes from a "braid monodromy factorization" (see [65]), but such an approach looks too cumbersome.

A presentation for the group G/G''

Let $G = \pi_1(\mathbb{P}^2 \backslash C)$ be the group of an irreducible curve C of degree d. Then, since the abelianization of G is a finite cyclic group of order d, the commutator subgroup of G is a finitely presented group by the following theorem (see [81] for a proof).

Theorem 5.2.4 *(P. Hall) Let $N \lhd G$ be a normal subgroup and suppose that N and G/N are both finitely presented groups. Then G is finitely presented.*

By Theorem 5.2.2, it follows that if C is an irreducible curve intersecting the line at infinity transversally, then $\pi_1'(\mathbb{C}^2 \backslash C)$ is a finitely presented group, too. Kulikov [48] has shown this by different methods. Moreover, he shows that $\pi_1'(\mathbb{C}^2 \backslash C)$ is a finitely generated group, without any assumptions on the intersection of the irreducible curve C with the line at infinity. Note that $\mathbb{F}_2$ is the group of a reducible curve composed of three lines passing through a commun point. Thus, its commutator subgroup $\mathbb{F}_\infty$ is not finitely generated. On the other hand, $\mathbb{Z}^n$ is the fundamental group of a generic line arrangement, with a trivial commutator subgroup.

Similarly, the commutator subgroup of a rigid group G is finitely presented. If the abelianization of G is finite cyclic, then this directly follows from Theorem 5.2.4. Otherwise, there is a central element z in G, sent to a τ^d (in the notations of Theorem 5.2.3). Let Z be the central subgroup of G generated by z. Then $Z \to G/G'$ is an injection, so the commutator subgroups of G and G/Z are isomorphic by Lemma 5.2.1. But the abelianization of G/Z is a finite cyclic group. Thus, the commutator subgroup of G/Z is finitely presented by Theorem 5.2.4, which implies that the commutator subgroup of G is also finitely presented.

An explicit finite presentation for G' can in fact be found by the Reidemeister-Schreier algorithm, however, the calculations are imposing. We shall give an example of a commutator subgroup of a curve group in Section 5.3, where the presentation of G' turned out to be quite simple. Other known examples of commutator subgroups of curve groups are those of $\mathbb{Z}_p * \mathbb{Z}_q$, where $gcd\,(p,q) = 1$. The commutator subgroup of this group is the free group of rank $(p-1)(q-1)$. For some other examples of commutator subgroups, see [15].

A natural, simpler invariant of a rigid group G is the group G/G''. Obviously, the Alexander polynomial of G can be derived from G/G'', so that this invariant is stronger then the Alexander polynomial. Given a concrete presentation for G, the group G/G'' is in some cases easier to find then the Alexander polynomial itself.

Lemma 5.2.2 *Let (G,μ) be a pair with G a group of an irreducible curve of degree d, and μ a meridian of C. Assume that G is given by the presentation*

$$G = \langle a_1, a_2, \dots a_k \mid \mathcal{R}_1 = \mathcal{R}_2 = \cdots = 1 \rangle,$$

where the generators a_i are conjugate to μ. Then

(i) One has: $\mu^d \in G'$. Thus, $a_i^d \in G'$ for $1 \le i \le k$, since a_i^d is conjugate to μ^d.

(ii) The commutator subgroup G' is generated by the set

$$X := \{x_{i,n} := a_i^n a_{i+1}^{-n} \,:\, 1 \le i \le d-1,\ 1 \le n \le d \cup \{y_i := a_i^d \,:\, 1 \le j \le k\}.$$

Thus, the number of generators is d^2.

(iii) The group G'' is normally generated by the commutators Hence, G/G'' can be obtained from the presentation of G by setting $[X,X] = 1$, that is, by adding the relations $[z,t] = 1$ for $z, t \in X$.

Proof. This is almost a repetition of Proposition 5.2.2.$\square$.

Remarks.

1. Observe that even if G is a curve group, G'' need not be finitely generated. Indeed, $G := \mathbb{Z}_2 * \mathbb{Z}_3$ is an irreducible curve group, with $G' = \mathbb{F}_2$, the free group of rank 2. Hence, $G'' = \mathbb{F}_\infty$. However, we shall prove below that for an affine curve group G the presentation given above for the group G/G'' is finite presentation (see Lemma 5.2.2). In addition, we shall discuss another set of generators for G', obtained by the Reidemeister-Schreier algorithm.

2. The commutator subgroup of an irreducible C-group is not necessarily finitely generated. Many counterexamples are supplied by knot groups. It is well known that the commutator subgroup of a knot group is finitely generated if and only if the knot is fibered, and there are examples of non-fibered knots [11]. Observe that, even if the commutator subgroup is not finitely generated, one can define "truncated Alexander polynomials" of a pair (G, μ), where G is irreducible and μ is a normal generator of G, as follows: Let $a_1, a_2, \ldots, a_k$ be conjugates of the element μ generating the group, and let $n \in \mathbb{N}$. Imposing the relations $[a_i^n, a_j] = 1$ for $1 \le i, j \le k$ (or, equivalently, $a_i^n = a_j^n$), we obtain a quotient group H_n for which the corresponding action on H_n'/H_n'' is of finite order, moreover, H_n' is finitely generated. Hence, one can define the "nth truncated Alexander polynomial" Δ_G^n of G to be the Alexander polynomial of H_n.

Examples. Consider the 3-strand braid group $\mathbb{B}_3$. Then, by a standard application of the Reidemeister-Schreier algorithm, it can be shown that $\mathbb{B}_3'$ is freely generated by the elements $x := \sigma_1^{-1}\sigma_2$ and $y := \sigma_2\sigma_1^{-1}$. (For a presentation of $\mathbb{B}_n'$, see [33].) Setting $[x, y] = 1$ gives the relation

$$[\sigma_1^{-1}\sigma_2, \sigma_2\sigma_1^{-1}] = 1 \Rightarrow \sigma_1^{-1}\sigma_2^2\sigma_1^{-1} = \sigma_2\sigma_1^{-2}\sigma_2 \rightarrow$$

$$\sigma_2^2 = \sigma_1\sigma_2\sigma_1^{-1} \cdot \sigma_1^{-1}\sigma_2\sigma_1 \Rightarrow \sigma_2^2 = \sigma_2^{-1}\sigma_1\sigma_2 \cdot \sigma_2\sigma_1\sigma_2^{-1} \Rightarrow$$

$$\sigma_2^4 = \sigma_1\sigma_2^2\sigma_1 \Rightarrow \sigma_2^6 = \sigma_2\sigma_1\sigma_2 \cdot \sigma_2\sigma_1\sigma_2 \Rightarrow \sigma_2^6 = (\sigma_1\sigma_2)^3 = \sigma_1^6.$$

Hence,

$$\mathbb{B}_3/\mathbb{B}_3'' = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \quad \sigma_1^6 = (\sigma_1\sigma_2)^3 = \sigma_2^6 \rangle.$$

Now consider the braid group $\mathbb{B}_4$ with its standard presentation. Then, from the generators given for $\mathbb{B}_3$ above it is easy to see that the commutator subgroup $\mathbb{B}_4'$ is generated by $x := \sigma_1^{-1}\sigma_2$, $y := \sigma_2\sigma_1^{-1}$, $z := \sigma_2^{-1}\sigma_3$, $t := \sigma_3\sigma_2^{-1}$ and $u := \sigma_1\sigma_3^{-1}$. (note that $(\sigma_1\sigma_3)^n = \sigma_1^n\sigma_3^{-n} = \sigma_3^{-n}\sigma_1^n$ since $[\sigma_1, \sigma_3] = 1$). So, in the group $\mathbb{B}_4/\mathbb{B}_4''$ one has the relations

$$\sigma_1^6 = (\sigma_1\sigma_2)^3 = \sigma_2^6 = (\sigma_2\sigma_3)^3 = \sigma_3^6,$$

induced by the relations $[x, y] = [z, t] = 1$. On the other hand, the relation $[x, z] = 1$ gives,

$$\sigma_1^{-1}\sigma_2\sigma_2^{-1}\sigma_3 = \sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2 \Rightarrow$$

$$\sigma_1^{-1}\sigma_3 = \sigma_2^{-1}\sigma_3\sigma_2 \cdot \sigma_2^{-1}\sigma_1^{-1}\sigma_2 \Rightarrow$$

$$\sigma_1^{-1}\sigma_3 = \sigma_3\sigma_2\sigma_3^{-1} \cdot \sigma_1\sigma_2^{-1}\sigma_1^{-1} \Rightarrow$$

$$\sigma_1 = \sigma_3. \tag{5.3}$$

This proves that $\mathbb{B}_4/\mathbb{B}_4'' \simeq \mathbb{B}_3/\mathbb{B}_3''$. However, for $n \geq 5$, it is easy to see that relations of the type 5.3 forces $\sigma_1 = \sigma_2 = \cdots$, and one obtains $\mathbb{B}_n/\mathbb{B}_n'' \simeq \mathbb{Z}$. Hence, the group $\mathbb{B}_n'$ is perfect for $n \geq 5$, that is, it coincides with its commutator subgroup $\mathbb{B}_n''$. In [33], the authors give an explicit presentation for $\mathbb{B}_n'$ and obtain the same result. So, we have proved the following.

Corollary 5.2.1 *For $n \geq 5$, the groups $\mathbb{B}_n^p$ of Theorem 2.2.1, the groups $\mathbb{B}_n^s$ of Theorem 2.2.2, and the groups $\widetilde{\mathbb{B}_n}$ of Theorem 2.4.1 have perfect commutator subgroups.*

This result is also obtained by Libgober in [58], however, our line of reasoning seems to be more direct. Observe that, as explained in 2.0.4, the group $\widetilde{\mathbb{B}_n}$ is infinite almost solvable, but its commutator subgroup is infinite perfect.

As another example let us consider the group G/G'', where G is the curve group given in Theorem 3.2.2:

$$G := \langle c, b \mid cbc = bcb, \quad b^n c^{n+2} = c^{n+2} b^n, \quad (b^{-n} cb^2)^{n+1} c^{n^2} = 1 \rangle.$$

To find G/G'', it suffices to impose the relations $c^6 = (cb)^3 = b^6$ on the above presentation. Suppose that $n = 6p - q$

If $q = 0$, then $b^n = c^n$, so that the last relation becomes

$$(cb^2)^{6p+1} c^{-6p} = 1 \Rightarrow cb^2 (cb^2)^{3p} c^{-6p} = cb^2 (cb)^{9p} c^{-6p} = 1 \Rightarrow$$

$$cb^2 c^{12p} = cb^2 b^{12p} = 1 \Rightarrow c^{-1} = b^{12p+2}.$$

Thus, the group G/G'' is abelian in this case. One gets easily the same conclusion if $q = 1$ or $q = 2$, so that G' is a perfect group for $q = 0, 1, 2$.

5.3 Residual Groups

Consider the pair (G, μ), where G is an irreducible group, and μ is a normal generator of G. Then we introduce the following invariants of the pair (G, μ).

Definition-Notation. If $a \in G$ is an element of an arbitrary group G, the group $G/ \ll a \gg$ will be denoted by $G(a)$. If (G, μ) is a pair with G an irreducible group, and μ a normal generator of G then for $k \in \mathbb{N}$, we will call the group $G(\mu^k)$ a *residual group* of G and denote it by $G(k)$. It is clear that if μ and ν are conjugate, then $G(\mu^k) = G(\nu^k)$, so that the groups $G(k)$ depends only on the conjugacy class of μ.

Question. Consider the pair (G, μ), where G is a finitely presented irreducible group, and μ a normal generator of G. Is it true that $\cap_{n\in\mathbb{N}} \ll \mu^n \gg = 1$? In particular, is this true if G is the group of an irreducible curve with μ a meridian of C?

Remark. Let (G, μ) be a pair with G an infinite simple group, and $\mu \in G$ an element of infinite order. Then G is normally generated by μ, but $\cap_{n\in\mathbb{N}} \ll \mu^n \gg = G$. Finitely generated infinite simple groups with elements of infinite order exist. Below we give a sketch of a slight modification of Higman's construction of an infinite simple group. For details see [37].

First, observe that by Zorn's Lemma, any non-trivial group has a proper, maximal normal subgroup. Hence any non-trivial group has a non-trivial simple quotient. Now consider the group given by the presentation

$$H := \langle x, y, z, t \,|\, xyx^{-1} = y^2, \quad yzy^{-1} = z^2, \quad ztz^{-1} = t^2, \quad txt^{-1} = x^2 \rangle.$$

Then any quotient of H in which all four generators x, y, z, t are of finite order is trivial. It can be shown that H is a non-trivial group, since it is an amalgamated free product of two non-trivial groups. It follows that H has a quotient G, which is a finitely generated infinite simple group in which at least one of x, y, z, t is of infinite order.

Examples of residual groups. Here, we shall discuss the residual groups of the braid groups. See the next chapter for a discussion of the residual groups of the groups $\mathbb{Z}_p * \mathbb{Z}_q$, where $gcd\,(p, q) = 1$.

The basic example of an irreducible group is the pair $(\mathbb{Z}, 1)$. One has $\mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$. Note that $\mathbb{Z}$ is not the only irreducible curve group G whose abelianization is infinite cyclic, and all residual groups are finite. Degtyarev [15] has shown that the group $H := \mathbb{Z} \times G$ is an affine curve group C, where G is a finite perfect group of order 120 given by the presentation $\langle x, y, z \,|\, x^2 = y^3 = z^5 = xyz \rangle$. It is easy to see that all the residual groups of this groups are finite. Indeed, a meridian of C should be of the form $\mu := (1, g) \in H$. Hence, for the residual groups of H one has $H(\mu^n) = \mathbb{Z}_n \times G(g^n)$, which are finite since G is a finite group.

Now consider the pair $(\mathbb{B}_m, \sigma_1)$, where σ_1 is a half-twist, and $\mathbb{B}_m$ is the braid group on m strands given by the Artin presentation

$$\mathbb{B}_m = \langle \sigma_1, \sigma_2, \ldots, \sigma_{m-1} \,|\, \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad |i-j| = 1, \quad [\sigma_i, \sigma_j] = 1 \quad |i-j| > 1\rangle.$$

It is well known that $\mathbb{B}_m(2) \simeq \Sigma_m$, the mth symmetric group. Coxeter has shown that $\mathbb{B}_m(n)$ is a finite group if and only if $\frac{1}{n} + \frac{1}{m} < \frac{1}{2}$ [12]. The residual groups of $\mathbb{B}_3$ can be described more explicitly as follows: The groups $\mathbb{B}_3(n)$ has the presentation

$$\mathbb{B}_3(n) = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_1^n = 1\rangle.$$

Now put $x := \sigma_1\sigma_2\sigma_1$ and $y := \sigma_2\sigma_1$. Then $\sigma_1 = xy^{-1}$, $\sigma_2 = y^2x^{-1}$, and so one obtains the presentation

$$\mathbb{B}_3(n) = \langle x, y \,|\, x^2 = y^3, \quad (xy^{-1})^n = 1\rangle.$$

Passing to the quotient by the central element x^2, we get the triangle group $T_{2,3,n}$, which is finite for $n \leq 5$, solvable for $n = 6$, and big for $n \geq 7$. The same claim still holds for $\mathbb{B}_3$: For $n = 2, 3, 4, 5$, the residual group $\mathbb{B}_3(n)$ is of order 6, 24, 96 and 600 respectively, and it is solvable when $n = 6$. Another way to see the solvability of $\mathbb{B}_3(6)$ is as follows: In Section 5.1 we have obtained the presentation

$$\mathbb{B}_3/\mathbb{B}_3'' = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_1^6 = (\sigma_1\sigma_2)^3 = \sigma_2^6\rangle.$$

The group $\mathbb{B}_3/\mathbb{B}_3''$ is solvable group since for any group G, the group G/G'' is solvable. Passing to the quotient by $\sigma_1^6 = 1$ one obtains the group

$$G := \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_1^6 = (\sigma_1\sigma_2)^3 = \sigma_2^6 = 1\rangle.$$

Now, note that $z := (\sigma_1\sigma_2)^3$ is central in $\mathbb{B}_3$, thus it is central in $\mathbb{B}_3(6)$. Setting $z = 1$ in $\mathbb{B}_3(6)$, we obtain the same group G. Hence, $B_3(6)$ is solvable.

The spherical braid group $\mathbb{B}_n^s$ is obtained from $\mathbb{B}_n$ by imposing the relation $(\mathcal{R})$: $\quad \sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_2\sigma_1 = 1$. It is clear that for the pair $(\mathbb{B}_n^s, \sigma_1)$ one has $\mathbb{B}_n^s(2) \simeq \mathbb{B}_n(2) \simeq \Sigma_n$, since $(\mathcal{R})$ becomes trivial after passing to the quotient by the relation $\sigma_1^2 = 1$ $(\Rightarrow \sigma_i^2 = 1, \quad i = 1, 2, \ldots, n-1)$. We believe that $\mathbb{B}_n^s(k)$ is finite for $k \leq 5$ and solvable for $k = 6$. MAPLE calculations give

$|\mathbb{B}_4^s(3)| = 12$, $|\mathbb{B}_4^s(4)| = 192$, $|\mathbb{B}_4^s(5)| = 60$,

$|\mathbb{B}_5^s(3)| = 1$, $|\mathbb{B}_5^s(4)| = 7680$, $|\mathbb{B}_5^s(5)| = 62400$.

By Theorem 2.2.1, another example of a curve group close to the braid group is the quotient of $\mathbb{B}_m$ by its center, let us denote this group by $\mathbb{B}_m^z$. Then $\mathbb{B}_m^z(2)$ is the quotient of the symmetric group $\mathbb{B}_m(2) \simeq \Sigma_n$ by its center, i.e. $\mathbb{B}_m^z(2)$ is the alternating group $\mathbb{A}_m$.

Remark. These examples show that there are pairs (G, μ) with G an irreducible infinite group and μ a normal generator of finite order. We do not know if there are such curve groups.

As noticed in Definition 3.3.1, the fundamental groups of curve complements come equipped with some certain equivalence classes, namely meridians and singular meridians. Let μ_p be a singular meridian of a curve $C \subset \mathbb{P}^2$. Then, since μ_p is a meridian of the exceptional curve E of the blow-up at the point p, we have the following.

Lemma 5.3.1 *Let $C \in \mathbb{P}^2$ be a curve, and let $p \in C$ be a singular point, with the singular meridian $\mu_p \in \pi_1(\mathbb{P}^2 \backslash C)$. Let $\sigma_p := X \to \mathbb{P}^2$ be the blow-up at p. Put $\sigma_p^{-1}(C) = E \cup \widetilde{C}$, where $\sigma_p(E) = p$. Then $\pi_1(X \backslash \widetilde{C})$ is the quotient of $\pi_1(\mathbb{P}^2 \backslash C)$ by the relation $\mu_p = 1$, or, in terms of Definition 3.3.2, $\pi_1(X \backslash \widetilde{C}) = G_C^p(1)$.*

Note that the above lemma is still correct if p is a smooth point, and μ_p is a meridian of C. If the curve C is irreducible, one obtains the conclusion that $\pi_1(X \backslash \widetilde{C}) = 1$.

For example, if $p \in C$ is a simple cusp, then the local fundamental group of $\mathbb{P}^2 \backslash C$ at the point p is $\mathbb{B}_3$, with σ_1 (or σ_2) as a meridian, and one can take $\mu_p := \sigma_2 \sigma_1$ as a singular meridian. If one puts $\mu_p = 1$, one gets the relations $\sigma_1 = \sigma_2$ and $\sigma_1^2 = 1$, i.e. one obtains the group $\mathbb{Z}_2$. If the curve C is irreducible, this implies that there is a surjection $G_C(2) \twoheadrightarrow G_C^p(1)$, because of the relation $\sigma_1^2 = 1$. One obtains the same conclusion for a cusp p given by $x^2 = y^{2n+1}$. Note that the element μ_p^{2n+1} is central in the local fundamental group.

If $p \in C$ is an ordinary multiple point of multiplicity m, i.e. transversal intersection of m smooth branches of C, then the local fundamental group at p has the presentation

$$\langle a_1, a_2, \ldots, a_m, \delta \mid [\delta, a_i] = 1 \quad 1 \le i \le m, \quad \delta = a_m a_{m-1} \cdots a_1 \rangle,$$

and one can take $\mu_p = \delta$ as a singular meridian. Obviously, μ_p is central in the local fundamental group. Passing to the quotient by μ_p gives the free group $\mathbb{F}_{m-1}$. This proves the following lemma, which will be used later in the text.

Lemma 5.3.2 *The fundamental group of an arrangement of $m \geq 3$ lines passing through a common point is big.*

Lemma 5.3.1 suggests that the fundamental group admits a "resolution" by passing to quotients by singular meridians. It is interesting to know whether one always ends up with an abelian group after iterating this procedure until a smooth curve in a surface is found. Observe that if C is composed of d lines passing through a common point p, then the singular meridian of C at p is already vanishing, so that $\pi_1(X\backslash\widetilde{C}) \simeq \pi_1(\mathbb{P}^2\backslash C) \simeq \mathbb{F}_{d-1}$, where X is the blow-up of $\mathbb{P}^2$ at p, and $\widetilde{C}$ is the proper transform of C under this blow-up. Hence, the "resolution" of $\pi_1(\mathbb{P}^2\backslash C)$ is not abelian. This case should be considered to be pathological in the context the above question. Indeed, it is easy to see that if one introduce to C a line L not passing through the point p, then the "resolution" of $\pi_1(\mathbb{P}^2\backslash(C \cup L))$ becomes trivial. The above question is interesting from the point of view of Galois coverings: A covering dominated by such a "resolution" can be said to be "non-ramified at the singular points of C".

5.4 A commutator subgroup

In this section, we shall apply the Reidemeister-Schreier algorithm to find a presentation of the commutator subgroup of a curve group. This requires lenghty calculations, however, the final presentation of the commutator subgroup turns out to be very simple, which suggests a more detailed study of these subgroups. For an exposition of the Reidemeister-Schreier algorithm, we refer the reader to [11].

Let G be the group

$$G := \langle a, b \mid (ab)^3 a = b(ab)^3 \;\; (\mathcal{R}_1), \quad (ab)^7 = a^6 \;\; (\mathcal{R}_2)\rangle.$$

This is the group in Theorem 3.2.1 for $k = 3$ and $d = 8$. As shown in [2], this group is big. We repeat this proof for the sake of completeness. By change of generators $c := (ab)^3 a$ and $d := ab$ (with inverse $a = d^{-3}c$, $b = c^{-1}d^4$) we get the presentation

$$G = \langle c, d, \mid c^2 = d^7,\ (d^{-3}c)^6 = d^7 \rangle.$$

Let H be the quotient of G by the relation $c^2 = 1$. Applying the transformation $e = c$, $f = d^{-3}$, (with inverse $c = e$, $d = f^2$) we obtain

$$H = \langle e, f \mid e^2 = f^7 = (fe)^6 \rangle.$$

This is the hyperbolic triangle group $T_{2,7,6}$, which is big by Theorem 5.5.2 in the next section. Hence, G is also a big group.

Now let us find a presentation of the commutator subgroup G' of G. Clearly, one has

$$G/G' \simeq \mathbb{Z}_8,$$

and a Schreier transversal to G' can be given as

$$T = \{1, a, a^2, a^3, a^4, a^5, a^6, a^7\}$$

We will apply the Reidemeister -Schreier rewriting algorithm to find a presentation of G'. By the Reidemeister -Schreier theorem, G' is generated by the elements

$$\begin{aligned}
x_1 &:= ba^{-1},\\
x_2 &:= aba^{-2},\\
x_3 &:= a^2ba^{-3},\\
x_4 &:= a^3ba^{-4},\\
x_5 &:= a^4ba^{-5},\\
x_6 &:= a^5ba^{-6},\\
x_7 &:= a^6ba^{-7},\\
x_8 &:= a^7b,\\
z &:= a^8.
\end{aligned}$$

In order to find the relations among $z, x_1 \dots, x_8$, one has to re-write the presentation relations of G in terms of these letters. So write the relation $\mathcal{R}_1$ as $u = 1$, where

$$u := abababab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}.$$

Then

$$u = aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6} \cdot (bababa ba^{-7})^{-1}$$

$$= aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6}(ba^{-1} \cdot a^2ba^{-3} \cdot a^4ba^{-5} \cdot a^6ba^{-7})^{-1}$$
$$\Rightarrow x_2x_4x_6 = x_1x_3x_5x_7$$

Similarly,

$$aua^{-1} = a^2ba^{-3} \cdot a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot (aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6} \cdot a^7b)^{-1}$$
$$\Rightarrow x_3x_5x_7z = x_2x_4x_6x_8$$
$$a^2ua^{-2} = a^3ba^{-4} \cdot a^5ba^{-6} \cdot a^7b \cdot (a^2ba^{-3} \cdot a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot ba^{-1})^{-1}$$
$$\Rightarrow x_4x_6x_8 = x_3x_5x_7zx_1$$
$$a^3ua^{-3} = a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot (a^3ba^{-4} \cdot a^5ba^{-6} \cdot a^7b \cdot aba^{-2})^{-1}$$
$$\Rightarrow x_5x_7zx_1 = x_4x_6x_8x_2$$
$$a^4ua^{-4} = a^5ba^{-6} \cdot a^7b \cdot aba^{-2} \cdot (a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^2ba^{-3})^{-1}$$
$$\Rightarrow x_6x_8x_2 = x_5x_7zx_1x_3$$
$$a^5ua^{-5} = a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^2ba^{-3} \cdot (a^5ba^{-6} \cdot a^7b \cdot aba^{-2} \cdot a^3ba^{-4})^{-1}$$
$$\Rightarrow x_7zx_1x_3 = x_6x_8x_2x_4$$
$$a^6ua^{-6} = a^7b \cdot aba^{-2} \cdot a^3ba^{-4} \cdot (a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^2ba^{-3} \cdot a^4ba^{-5})^{-1}$$
$$\Rightarrow x_8x_2x_4 = x_7zx_1x_3x_5$$
$$a^7ua^{-7} = a^8 \cdot ba^{-1} \cdot a^2ba^{-3} \cdot a^4ba^{-5} \cdot (a^7b \cdot aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6})^{-1}$$
$$\Rightarrow zx_1x_3x_5 = x_8x_2x_4x_6$$

Now write $\mathcal{R}_2$ as $v = 1$ where

$$v := bababababababa^{-5}.$$

Then one has

$$v = ba^{-1} \cdot a^2ba^{-3} \cdot a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^2ba^{-3} \cdot a^4ba^{-5}$$
$$\Rightarrow x_1x_3x_5x_7zx_1x_3x_5 = 1$$
$$ava^{-1} = aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6} \cdot a^7b \cdot aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6}$$
$$\Rightarrow x_2x_4x_6x_8x_2x_4x_6 = 1$$
$$a^2va^{-2} = a^2ba^{-3} \cdot a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^2ba^{-3} \cdot a^4ba^{-5} \cdot a^6ba^{-7}$$

$$\Rightarrow x_3x_5x_7zx_1x_3x_5x_7 = 1$$

$$a^3va^{-3} = a^3ba^{-4} \cdot a^5ba^{-6} \cdot a^7b \cdot aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6} \cdot a^7b \cdot a^{-8}$$

$$\Rightarrow x_4x_6x_8x_2x_4x_6x_8 = z$$

$$a^4va^{-4} = a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^2ba^{-3} \cdot a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^{-8}$$

$$\Rightarrow x_5x_7zx_1x_3x_5x_7zx_1 = z$$

$$a^5va^{-5} = a^5ba^{-6} \cdot a^7b \cdot aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6} \cdot a^7b \cdot aba^{-2} \cdot a^{-8}$$

$$\Rightarrow x_6x_8x_2x_4x_6x_8x_2 = z$$

$$a^6va^{-6} = a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^2ba^{-3} \cdot a^4ba^{-5} \cdot a^6ba^{-7} \cdot a^8 \cdot ba^{-1} \cdot a^2ba^{-3} \cdot a^{-8}$$

$$\Rightarrow x_7zx_1x_3x_5x_7zx_1x_3 = z$$

$$a^7va^{-7} = a^7b \cdot aba^{-2} \cdot a^3ba^{-4} \cdot a^5ba^{-6} \cdot a^7b \cdot aba^{-2} \cdot a^3ba^{-4} \cdot a^{-8}$$

$$\Rightarrow x_8x_2x_4x_6x_8x_2x_4 = z$$

To sum up, one has the relations

$$\begin{array}{ll}
(\mathcal{A}_1)\ x_2x_4x_6 = x_1x_3x_5x_7 & (\mathcal{B}_1)\ x_1x_3x_5x_7 \cdot zx_1x_3x_5 = 1 \\
(\mathcal{A}_2)\ x_3x_5x_7z = x_2x_4x_6x_8 & (\mathcal{B}_2)\ x_2x_4x_6x_8 \cdot x_2x_4x_6 = 1 \\
(\mathcal{A}_3)\ x_4x_6x_8 = x_3x_5x_7zx_1 & (\mathcal{B}_3)\ x_3x_5x_7zx_1 \cdot x_3x_5x_7 = 1 \\
(\mathcal{A}_4)\ x_5x_7zx_1 = x_4x_6x_8x_2 & (\mathcal{B}_4)\ x_4x_6x_8x_2 \cdot x_4x_6x_8 = z \\
(\mathcal{A}_5)\ x_6x_8x_2 = x_5x_7zx_1x_3 & (\mathcal{B}_5)\ x_5x_7zx_1x_3 \cdot x_5x_7zx_1 = z \\
(\mathcal{A}_6)\ x_7zx_1x_3 = x_6x_8x_2x_4 & (\mathcal{B}_6)\ x_6x_8x_2x_4 \cdot x_6x_8x_2 = z \\
(\mathcal{A}_7)\ x_8x_2x_4 = x_7zx_1x_3x_5 & (\mathcal{B}_7)\ x_7zx_1x_3x_5 \cdot x_7zx_1x_3 = z \\
(\mathcal{A}_8)\ zx_1x_3x_5 = x_8x_2x_4x_6 & (\mathcal{B}_8)\ x_8x_2x_4x_6 \cdot x_8x_2x_4 = z.
\end{array}$$

Substituting the value of $x_1x_3x_5x_7$ given by $(\mathcal{A}_1)$ in $(\mathcal{B}_1)$, we get another relation

$$(\mathcal{C}_1)\ x_2x_4x_6 \cdot zx_1x_3x_5 = 1,$$

such that the pair of relations $(\mathcal{A}_1, \mathcal{B}_1)$ is equivalent to the pair $(\mathcal{A}_1, \mathcal{C}_1)$, so that one can forget the relation $(\mathcal{B}_1)$. By applying the same procedure one can replace the relations $\mathcal{B}_1 - \mathcal{B}_8$ by the following ones

$$\begin{array}{l}
(\mathcal{C}_1)\ x_2x_4x_6 \cdot z \cdot x_1x_3x_5 = 1, \\
(\mathcal{C}_2)\ x_3x_5x_7 \cdot z \cdot x_2x_4x_6 = 1, \\
(\mathcal{C}_3)\ x_4x_6x_8 \cdot x_3x_5x_7 = 1, \\
(\mathcal{C}_4)\ x_5x_7zx_1 \cdot x_4x_6x_8 = z, \\
(\mathcal{C}_5)\ x_6x_8x_2 \cdot x_5x_7zx_1 = z, \\
(\mathcal{C}_6)\ x_7zx_1x_3 \cdot x_6x_8x_2 = z, \\
(\mathcal{C}_7)\ x_8x_2x_4 \cdot x_7zx_1x_3 = z, \\
(\mathcal{C}_8)\ x_1x_3x_5 \cdot x_8x_2x_4 = 1.
\end{array}$$

Put $\alpha := x_1x_3x_5$. Then from relations $(\mathcal{C}_1)-(\mathcal{C}_8)$ one obtains

$$\begin{array}{ll}
(\mathcal{C}_1) \Rightarrow x_2x_4x_6 = (zx_1x_3x_5)^{-1} = \alpha^{-1}z^{-1} & (\mathcal{D}_1)\\
(\mathcal{C}_2) \Rightarrow x_3x_5x_7 = (zx_2x_4x_6)^{-1} = z\alpha z^{-1} & (\mathcal{D}_2)\\
(\mathcal{C}_3) \Rightarrow x_4x_6x_8 = (x_3x_5x_7)^{-1} = z\alpha^{-1}z^{-1} & (\mathcal{D}_3)\\
(\mathcal{C}_4) \Rightarrow x_5x_7zx_1 = z(x_4x_6x_8)^{-1} = z^2\alpha z^{-1} & (\mathcal{D}_4)\\
(\mathcal{C}_8) \Rightarrow x_8x_2x_4 = (x_1x_3x_5)^{-1} = \alpha^{-1} & (\mathcal{D}_8)\\
(\mathcal{C}_7) \Rightarrow x_7zx_1x_3 = (x_8x_2x_4)^{-1}z = \alpha z & (\mathcal{D}_7)\\
(\mathcal{C}_6) \Rightarrow x_6x_8x_2 = (x_7zx_1x_3)^{-1}z = z^{-1}\alpha^{-1}z & (\mathcal{D}_6)\\
(\mathcal{C}_5) \Rightarrow z^{-1}\alpha^{-1}z\cdot z^2\alpha z^{-1} = z \Rightarrow [z^3,\alpha] = 1 & (\mathcal{D}_5)
\end{array}$$

On the other hand, by $(\mathcal{A}_1)-(\mathcal{A}_8)$, one has

$$\begin{array}{l}
(\mathcal{A}_1) \Rightarrow x_1 = x_2x_4x_6\cdot(x_3x_5x_7)^{-1} = \alpha^{-1}z^{-1}\cdot(z\alpha z^{-1})^{-1} = \alpha^{-2}z^{-1}\\
(\mathcal{A}_2) \Rightarrow x_2 = x_3x_5x_7z\cdot(x_4x_6x_8)^{-1} = z\alpha z^{-1}z\cdot(z\alpha^{-1}z^{-1})^{-1} = z\alpha z\alpha z^{-1}\\
(\mathcal{A}_3) \Rightarrow x_3 = x_4x_6x_8\cdot(x_5x_7zx_1)^{-1} = z\alpha^{-1}z^{-1}\cdot(z^2\alpha z^{-1})^{-1} = z\alpha^{-2}z^{-2}\\
(\mathcal{A}_4) \Rightarrow x_4 = x_5x_7zx_1\cdot(x_6x_8x_2)^{-1} = z^2\alpha z^{-1}\cdot(z^{-1}\alpha^{-1}z)^{-1} = z^2\alpha z^{-2}\alpha z\\
(\mathcal{A}_5) \Rightarrow x_5 = x_6x_8x_2\cdot(x_7zx_1x_3)^{-1} = z^{-1}\alpha^{-1}z\cdot(\alpha z)^{-1} = z^{-1}\alpha^{-2}\\
(\mathcal{A}_6) \Rightarrow x_6 = x_7zx_1x_3\cdot(x_8x_2x_4)^{-1} = \alpha z\cdot(\alpha^{-1})^{-1} = \alpha z\alpha\\
(\mathcal{A}_7) \Rightarrow x_7 = x_8x_2x_4\cdot(zx_1x_3x_5)^{-1} = \alpha^{-1}\cdot(z\alpha)^{-1} = \alpha^{-2}z^{-1}\\
(\mathcal{A}_8) \Rightarrow x_8 = zx_1x_3x_5\cdot(x_2x_4x_6)^{-1} = z\alpha\cdot(z^{-1}\alpha^{-1})^{-1} = z\alpha z\alpha
\end{array}$$

It follows that

$$\alpha = x_1x_3x_5 \Rightarrow \alpha = \alpha^{-2}z^{-1}\cdot z\alpha^{-2}z^{-2}\cdot z^{-1}\alpha^{-2}$$

$$\Rightarrow z^{-3} = \alpha^7, \qquad (\mathcal{E}_1)$$

so that one can omit the relation $(\mathcal{D}_5)$. Substituting these expressions for $x_1\cdots x_8$ in the other relations $(\mathcal{D}_1)-(\mathcal{D}_8)$, one obtains

$$\begin{array}{ll}
(\mathcal{D}_1) \Rightarrow z\alpha z\alpha z^{-1}\cdot z^2\alpha z^{-2}\alpha z\cdot\alpha z\alpha = \alpha^{-1}z^{-1} \Rightarrow (\alpha z)^7 = z^3 & (\mathcal{E}_2)\\
(\mathcal{D}_2) \Rightarrow z\alpha^{-2}z^{-2}\cdot z^{-1}\alpha^{-2}\cdot\alpha^{-2}z^{-1} = z\alpha z^{-1} \Rightarrow z^{-3} = \alpha^7, & (\mathcal{E}_1)\\
(\mathcal{D}_3) \Rightarrow z^2\alpha z^{-2}\alpha z\cdot\alpha z\alpha\cdot z\alpha z\alpha = z\alpha^{-1}z^{-1} \Rightarrow (\alpha z)^7 = z^3 & (\mathcal{E}_2)\\
(\mathcal{D}_4) \Rightarrow z^{-1}\alpha^{-2}\cdot\alpha^{-2}z^{-1}\cdot z\cdot\alpha^{-2}z^{-1} = z^2\alpha z^{-1} \Rightarrow z^{-3} = \alpha^7 & (\mathcal{E}_1)\\
(\mathcal{D}_8) \Rightarrow z\alpha z\alpha\cdot z\alpha z\alpha z^{-1}\cdot z^2\alpha z^{-2}\alpha z = \alpha^{-1} \Rightarrow (\alpha z)^7 = z^3 & (\mathcal{E}_2)\\
(\mathcal{D}_7) \Rightarrow \alpha^{-2}z^{-1}\cdot z\cdot\alpha^{-2}z^{-1}\cdot z\alpha^{-2}z^{-2} = \alpha z \Rightarrow z^{-3} = \alpha^7 & (\mathcal{E}_1)\\
(\mathcal{D}_6) \Rightarrow \alpha z\alpha\cdot z\alpha z\alpha\cdot z\alpha z\alpha z^{-1} = z^{-1}\alpha^{-1}z \Rightarrow (\alpha z)^7 = z^3 & (\mathcal{E}_2)
\end{array}$$

Hence,

$$G' = \langle \alpha, z \mid (\alpha z)^7 = z^3, \quad z^{-3} = \alpha^7\rangle,$$

and it is easily seen that G' is not a free group, e.g. one has

$$G'/G'' = \mathbb{Z}_7.$$

Note that $z \in G''$, since the abelianization homomorphism $G' \to \mathbb{Z}_7$ sends z to the unit element. Moreover, $G'' = \ll z \gg$ since imposing the relation $z = 1$ in the above presentation for G' yields the abelianization $\mathbb{Z}_7$ of G'. This implies that G'' is a subgroup of index 56 in G.

On the other hand, since in our example G' is normally generated by z, it is interesting to know the residual groups $G'/ \ll z^k \gg$. We now proceed to investigate them.

Suppose first that $k = 3n + 1$. Then the relation $z^{-3} = \alpha^{-7}$ implies $z^{-1} = z^{3n} = \alpha^{-7n} \Rightarrow z = \alpha^{7n}$. Substituting this in the presentation for $G' \ll z^{3k+1} \gg$, we get

$$G'/ \ll z^{3k+1} \gg = \langle \alpha, \mid \alpha^{7(7n+1)} = \alpha^{21n},\ \alpha^{21n} = \alpha^{-7} \rangle = \langle \alpha, \mid \alpha^{7(4n+1)} = \alpha^{7(3n+1)} = 1 \rangle$$

$$\Rightarrow G'/ \ll z^{3k+1} \gg = \mathbb{Z}_7 = G'/G''.$$

Let us proceed further to find a presentation of the group $G'' = \ll z = \mu^8 \gg$. The set

$$T := \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6\}$$

is clearly a Schreier transversal to G'', so that the corresponding set of generators for G' is given by

$$\begin{aligned} Y := \{\theta &:= \alpha^7, \\ \omega_0 &:= z, \\ \omega_1 &:= \alpha z \alpha^{-1}, \\ \omega_2 &:= \alpha^2 z \alpha^{-2}, \\ \omega_3 &:= \alpha^3 z \alpha^{-3}, \\ \omega_4 &:= \alpha^4 z \alpha^{-4}, \\ \omega_5 &:= \alpha^5 z \alpha^{-5}, \\ \omega_6 &:= \alpha^6 z \alpha^{-6}\} \end{aligned}$$

The relation $z^3\alpha^7 = 1$ is rewritten as, for $0 \leq k \leq 6$,

$$\alpha^k \cdot z^3\alpha^7 \cdot \alpha^{-k} = (\alpha^k z \alpha^{-k})^3 \alpha^7$$

$$\Rightarrow \omega_k^3 \theta = 1, \quad 0 \leq k \leq 6 \quad (\mathcal{F}_k)$$

As for the relation $z^{-3}(\alpha z)^7 = 1$, one has

$$z^{-3}(\alpha z)^7 = z^{-3} \cdot \alpha z \alpha^{-1} \cdot \alpha^2 z\ \alpha^{-2} \cdots \alpha^6 z \alpha^{-6} \cdot \alpha^7 \cdot z$$

$$\Rightarrow \omega_0^3 = \omega_1\omega_2\omega_3\omega_4\omega_5\omega_6\theta\omega_0 \quad (\mathcal{G}_1)$$

Similarly,

$$\alpha \cdot z^{-3}(\alpha_z)^7\alpha^{-3} = (\alpha z \alpha^{-1})^{-3} \cdot \alpha^2 z \alpha^{-3} \cdots \alpha^6 z \alpha^{-3} \cdot \theta \cdot z \cdot \alpha z \alpha^{-1}$$

$$\Rightarrow \omega_1^3 = \omega_2\omega_3\omega_4\omega_5\omega_6\theta\omega_0\omega_1 \quad (\mathcal{G}_2)$$

Repeating this for the relations $\alpha^k z^{-1}(\alpha z)^7\alpha^{-k} = 1$ for $2 \leq k \leq 6$, we get the relations

$$\omega_2^3 = \omega_3\omega_4\omega_5\omega_6\theta\omega_0\omega_1\omega_2 \quad (\mathcal{G}_3)$$

$$\omega_3^3 = \omega_4\omega_5\omega_6\theta\omega_0\omega_1\omega_2\omega_3, \quad (\mathcal{G}_4)$$

$$\omega_4^3 = \omega_5\omega_6\theta\omega_0\omega_1\omega_2\omega_3\omega_4, \quad (\mathcal{G}_5)$$

$$\omega_5^3 = \omega_6\theta\omega_0\omega_1\omega_2\omega_3\omega_4\omega_5, \quad (\mathcal{G}_6)$$

$$\omega_6^3 = \theta\omega_0\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6. \quad (\mathcal{G}_7)$$

The relations $(\mathcal{G}_2) - (\mathcal{G}_7)$ can be derived from the relations $(\mathcal{G}_1)$ and $(\mathcal{F}_1) - (\mathcal{F}_6)$. Indeed, the relations $(\mathcal{F}_k)$ implies $\omega_0^3 = \omega_1^3 = \cdots = \omega_6^3$. Substituting $\omega_0^3 = \omega_1^3$ in $(\mathcal{G}_1)$ gives the relation $(\mathcal{G}_2)$. Repeating this process, one easily obtains the relations $(\mathcal{G}_3 - \mathcal{G}_7)$.

On the other hand, $\theta = \omega_0^3$ by $(\mathcal{F}_1)$. Substituting this in $(\mathcal{G}_1)$ gives the relation

$$\omega_0^5 = \omega_1\omega_2\omega_3\omega_4\omega_5\omega_6.$$

To sum up, we have obtained the presentation

$$G'' = \langle \omega_0, \omega_1, \ldots, \omega_6 \,|\, \omega_1^3 = \omega_2^3 = \cdots = \omega_6^3, \quad \omega_0^5 = \omega_1\omega_2\omega_3\omega_4\omega_5\omega_6 \rangle$$

Therefore, for the abelianization of this group we have

$$G''/G''' = \mathbb{Z} \times \mathbb{Z}_3^5.$$

5.5 Generalized triangle groups and the Tits alternative

Here, we collect some facts about the triangle groups and their generalizations which are used frequently in this work. Their importance lies in the fact that they appear as the residual groups of the free product $\mathbb{Z}_p * \mathbb{Z}_q$, which is a curve group by Theorem 2.2.3. To give an idea how they come out, consider the local fundamental group of the singularity $x^2 = y^{2n+1}$, which can be presented as

$$(G, \mu) := (\langle a, b \mid (ab)^n a = b(ab)^n \rangle,\ a),$$

the generators a, b belonging to the conjugacy class of a meridian μ. Now applying the change of generators $x := (ab)^n a$, $y := ab$, with the inverse $a = y^{-n}x$, $b = x^{-1}y^{n+1}$ gives the presentation

$$(G, \mu) = (\langle x, y \mid x^2 = y^{2n+1} \rangle,\ y^{-n}x).$$

Observe that passing to the quotient by the central element $x^2 = y^{2n+1}$ gives the free product $\mathbb{Z}_2 * \mathbb{Z}_{2n+1}$. For the residual groups of G, one has

$$G(k) = \langle x, y \mid x^2 = y^{2n+1}, \quad (y^{-n}x)^k = 1 \rangle.$$

Setting $x^2 = y^{2n+1} = 1$ as above we get

$$G(k) \twoheadrightarrow \langle x, y \mid x^2 = y^{2n+1} = (y^{-n}x)^k = 1 \rangle,$$

and this latter group is exactly the generalized triangle group as we shall see below.

Recall that a family of finitely generated groups is said to satisfy the *Tits alternative* if any group of this family is either almost solvable (that is, it contains a solvable subgroup of finite index) or big (that is, it has a subgroup isomorphic to $\mathbb{F}_2$, the free group of rank 2).

Theorem 5.5.1 ***(Fine, Howie, Rosenberger [25])*** *Let G be a one-relator product of cyclic groups, that is, a group given by the presentation*

$$G := \langle a_1, a_2, \ldots, a_n \mid a_1^{k_1} = a_2^{k_2} = \cdots = a_n^{k_n} = w(a_1, a_2, \cdots a_n)^m = 1 \rangle,$$

with $n \geq 2$, $m \geq 2$, $2 \leq k_i \leq \infty$ for $i = 1, 2, \ldots, n$, and $w(a_1, a_2, \ldots, a_n)$ is a cyclically reduced word involving every one of $a_1, a_2, \ldots, a_n$. Suppose that one of the following conditions holds *(i) $n \geq 3$, or*

(ii) $n = 2$ and $k_i = 0$ for $i = 1$ or $i = 2$, or

(iii) $n = 2$ and $m \geq 3$, or

(iv) $n = 2$ and $w(a_1, a_2)$ is conjugate in the free group on a_1, a_2 conjugate to a word of the form $a_1^{r_1} a_2^{r_2}$ with $1 \leq r_i < k_i$, $i = 1, 2$.

Then the Tits alternative holds for G.

If $n = 2$ and $k_1, k_2 < \infty$, then G is called a generalized triangle group and can be written in the form

$$G = \langle a, b \mid a^p = b^q = w(a,b)^m = 1 \rangle$$

with $2 \leq p \leq q$, $2 \leq m$ and $w(a,b) = a^{p_1} b^{q_1} a^{p_2} b^{q_2} \cdots a^{p_s} b^{q_s}$ with $1 \leq s$ and $1 \leq p_j < p$, $1 \leq q_j < q$ for $j = 1, 2, \ldots, s$. Theorem 5.5.1 implies that Tits' alternative holds for generalized triangle groups if $m \geq 3$ or $s = 1$. Rosenberger [82] has shown that this is still true for $m = s = 2$, and conjectured that the Tits alternative holds for the generalized triangle groups in general. A classification of finite generalized triangle groups can be found in [40]. Concerning the bigness of the generalized triangle groups, one has the following fact.

Theorem 5.5.2 ***(Baumslag, Morgan, Shalen [7])*** *The generalized triangle group G has a free subgroup of rank 2 if*

$$\kappa := \frac{1}{p} + \frac{1}{q} + \frac{1}{m} < 1.$$

and is infinite if $\kappa = 1$. Any generalized triangle group has an essential representation $\rho : G \to PSL(2, \mathbb{C})$, that is, a representation ρ such that the orders of $\rho(a)$, $\rho(b)$ and $\rho(w)$ are p, q, and m respectively.

Specializing to triangle groups, that is the groups that can be given by the presentation

$$T_{p,q,r} := \langle a, b \mid a^p = b^q = (ab)^r = 1 \rangle,$$

the following trichotomy is well known [13]:

If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, then $T_{p,q,r}$ is called elliptic and is a finite group.

If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then $T_{p,q,r}$ is called Euclidean and it is an abelian-by-finite group.

If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, then $T_{p,q,r}$ is called hyperbolic, and is a big group.

Also, the triangle groups has faithful representations $T_{p,q,r} \to PSL(2,\mathbb{C})$. A famous theorem of Tits [92] states that any subgroup of a matrix group over a field of characteristic 0 satisfies the Tits alternative, hence triangle groups have this property too. This can in fact be seen directly from the trichotomy above: If $T_{p,q,r}$ is elliptic or Euclidean, then it has an abelian subgroup of finite index, and in case it is hyperbolic, it is big. Recall in this connection Zaidenberg's question as to the validity of the Tits alternative for curve groups.

Chapter 6

Fenchel's Problem

6.1 Generalities

In this chapter, we discuss some problems concerning the branched coverings of the projective plane. For the definition of branched coverings and some elementary facts concerning them, we refer the reader to Namba's book [69]. All the facts concerning the generalized triangle groups that we use frequently are collected in 5.5.

Let $\phi : X \to \mathbb{P}^2$ be a finite branched covering, and $C = C_1 \cup C_2 \cdots \cup C_n$ be the branching curve with irreducible components $C_1, C_2, \ldots, C_n$. The *branching divisor* of ϕ is the divisor $D_\phi = m_1C_1 + m_2C_2 + \cdots + m_nC_n$, where m_i is the ramification index of ϕ at C_i. Fenchel's problem can be formulated as follows: Given a divisor D on $\mathbb{P}^2$, is there a finite branched Galois covering $\phi : X \to \mathbb{P}^2$ with $D_\phi = D$? Obviously, this is equivalent to the following (see [69]): Given a divisor $D = m_1C_1 + m_2C_2 + \cdots + m_nC_n$ on $\mathbb{P}^2$, let μ_i be a meridian of C_i in $\mathbb{P}^2 \backslash C$. Is there a normal subgroup N of finite index in $\pi_1(\mathbb{P}^2 \backslash C)$ such that (i) $\mu_i^{m_i} \in N$ and (ii) if $\mu_i^k \in N$, then $m_i | k$?. Alternatively, is there a finite quotient of $\pi_1(\mathbb{P}^2 \backslash C)$ in which the image of μ_i is of order m_i? (Here, we don't assume that X is non-singular.)

For a divisor $D = m_1C_1 + m_2C_2 + \cdots + m_nC_n$ on $\mathbb{P}^2$, let us define the *group of the divisor* D to be the group $Gr(D) := \pi_1(\mathbb{P}^2 \backslash C) / \ll \mu_1^{m_1}, \mu_2^{m_2}, \cdots, \mu_n^{m_n} \gg$, where $C = C_1 \cup C_2 \cdots \cup C_n$ is the support of D (this is a generalization of residual groups introduced in Definition 3.3.2). Let us call an image $\eta : Gr(D) \twoheadrightarrow K$ *essential* if K is a finite group with order of $\eta(\mu_i)$ equals m_i for $1 \leq i \leq n$. Obviously, Fenchel's problem for a divisor D amounts to the study of the essential images of the group $Gr(D)$. As we shall soon see, $Gr(D)$ need not to possess essential images, far from that, this group can be trivial.

We recall that Fenchel's problem was originally posed for divisors on a Riemann surface S. By a theorem of Bundgaard and Nielsen [8], if S is of genus ≥ 1, then there exists a finite Galois covering of S branched at a given divisor on S. The solution for $S = \mathbb{P}^1$ is due to Fox [28]: If D is a divisor on $\mathbb{P}^2$, then there exists a finite covering of $\mathbb{P}^1$ branched at D, except the cases (1) $D = np$ and (2) $D = np + mq$, $n \neq m$, where $p, q \in \mathbb{P}^1$ and $n, m \in \mathbb{N}$.

Namba and Kato has considered the problem for the higher-dimensional case, and in particular in $\mathbb{P}^2$. Relevant references can be found in Namba's book [69], or in his survey article [71]. To begin with, notice that the corresponding affine problem in $\mathbb{C}^2$ always has a positive solution given by an abelian covering, as in the 1-dimensional case. If the support of a divisor $D := n_1C_1 + n_2C_2 + \cdots + n_kC_k$ on $\mathbb{C}^2$ has k irreducible components, and C is the support of D, then $H_1(\mathbb{C}^2 \backslash C, \mathbb{Z}) = \mathbb{Z}^k$. Let μ_i be a meridian of C_i. Then $H_1(\mathbb{C}^2 \backslash C)$ is the free abelian group generated by $\mu_1, \mu_2, \ldots, \mu_k$, and the abelian group given by the presentation

$$\langle \mu_1, \mu_2, \ldots, \mu_k \mid n_1\mu_1 = n_2\mu_2 = \cdots = n_k\mu_k = 1 \rangle$$

is a quotient of $H_1(\mathbb{C}^2 \backslash C)$ and determines abelian Galois covering of $\mathbb{C}^2$ branched at D.

On the other hand, considering Fenchel's problem for a divisor on an arbitrary surface seems like a too general approach. Blowing-up a smooth point on a surface X, denote by D the exceptional divisor on the resulting surface Y. If X is simply connected, then so is $Y \backslash D$, so that there is no covering of Y ramified at D.

For $D = mC$, where $C \subset \mathbb{P}^2$ is a smooth curve of degree d, one has $\pi_1(\mathbb{P}^2 \backslash C) = \mathbb{Z}/d\mathbb{Z}$, so that $Gr(D) = \mathbb{Z}/k\mathbb{Z}$, where $k := gcd\,(m, d)$. Thus, Fenchel's problem for D has a positive solution if and only if $m|d$, and the solution is given by an abelian covering. Obviously, this still gives a solution if $\pi_1(\mathbb{P}^2 \backslash C)$ is non-abelian, since the abelianization of $\pi_1(\mathbb{P}^2 \backslash C)$ is $\mathbb{Z}_d$. Similarly, if $D = m_1C_1 + m_2C_2 + \cdots + m_nC_n$ is a divisor with the support $C = C_1 \cup C_2 \cdots \cup C_n$, where C_i is an irreducible curve of degree d_i with a meridian μ_i, then the abelianization $H_1(\mathbb{P}^2 \backslash C, \mathbb{Z})$ of $\pi_1(\mathbb{P}^2 \backslash C)$ is the abelian group

$$H_1(\mathbb{P}^2 \backslash C, \mathbb{Z}) = \langle \mu_1, \mu_2, \ldots, \mu_n \mid d_1\mu_1 + d_2\mu_2 + \cdots + d_n\mu_n = 0 \rangle.$$

Hence, the abelianization of $Gr(D)$ has the presentation

$$\langle \mu_1, \mu_2, \dots, \mu_n \mid d_1\mu_1 + d_2\mu_2 + \cdots + d_n\mu_n = 0, \quad m_1\mu_1 = m_2\mu_2 = \cdots = m_n\mu_n = 1 \rangle.$$

Put $\kappa_i := m_i / gcd\,(m_i, d_i)$, and let ρ_i be the smallest common multiple of $\{\kappa_j \,|\, i \neq j\}$. Then an abelian covering solves the Fenchel's problem provided that κ_i divides ρ_i for $1 \leq i \leq n$. For details see [69].

However, abelian coverings give a solution to Fenchel's problem only for very restricted cases. For example, let $D := p_1L_1 + p_2L_2 + \cdots + p_dL_d$ be a divisor, whose support $C = L_1 \cup L_2 \cup \cdots \cup L_d$ consists of lines, the coefficients p_i being prime numbers. Then the above condition is never satisfied. But it is easy to see that the group $\pi_1(\mathbb{P}^2 \backslash C)$ is big if it is not abelian (see Remark 1 below). Hence, some non-abelian covers must give a solution to Fenchel's problem. Indeed, Kato proved the following theorem (for a generalization to arrangements of lines and conics, see [69]):

Theorem 6.1.1 ***(Kato [43])*** *Let $D := m_1L_1 + m_2L_2 + \cdots + m_dL_d$ be a divisor with support $C = L_1 \cup L_2 \cup \cdots \cup L_d$ being a line arrangement, such that any line L_i contains a point of multiplicity ≥ 3 of C. Then there is a finite Galois covering of $\mathbb{P}^2$ branched at D.*

Remarks.

1. Let $C \subset \mathbb{P}^2$ be a line arrangement. It is well-known that if C is generic (that is, if all the singularities of C are double points), then the group $\pi_1(\mathbb{P}^2 \backslash C)$ is abelian. Otherwise, C has a point p of multiplicity $m \geq 3$. Let $B \subset C$ be the set of lines passing through p. Then there is a surjection $\pi_1(\mathbb{P}^2 \backslash C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \backslash B)$. But, by Lemma 5.3.2, the group $\pi_1(\mathbb{P}^2 \backslash B)$ is big. Thus, $\pi_1(\mathbb{P}^2 \backslash C)$ is also a big group.

2. Let $D = m_1L_1 + m_2L_2 + \cdots + m_dL_d$ be a divisor, whose support is a line arrangement $C = L_1 \cup L_2 \cup \cdots \cup L_d$. Then it is not true that "$Gr(D)$ is either abelian or big": Indeed, let $d = 3$, and let L_1, L_2, L_3 be three lines passing through a common locus p. Put $D := pL_1 + qL_2 + rL_3$. Then it is easy to see that

$$Gr(D) = \langle a, b, c \,|\, a^p = b^q = c^r = cba = 1 \rangle = T_{p,q,r}.$$

Hence, $Gr(D)$ is finite for $\rho := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, infinite solvable for $\rho = 1$, and big if $\rho < 1$.

Let us further discuss this latter example. Take a line L_4 not passing through the triple point p of $L_1 \cup L_2 \cup L_3$, and consider the divisor $D_1 :=$

$pL_1 + qL_2 + rL_3 + sL_4$. Then one has

$$Gr(D_1) = \langle a, b, c, d \mid a^p = b^q = c^r = d^s = dcba = 1, \quad [a, d] = [b, d] = [c, d] = 1 \rangle.$$

Passing to the quotient by the central element d gives the triangle group $T_{p,q,r}$ again, so that the group $Gr(D_1)$ is solvable for $\rho \geq 1$. Now, if we define $D_2, D_3, \cdots D_k$ similarly, such that the supports of D_i, D_j intersect generically, then by the Oka-Sakamoto-Kaliman theorem $Gr(D_1 + D_2 + \cdots + D_k)$ is still a solvable group (See Lemma 6.1.1 below).

Question. Let $C = L_1 \cup L_2 \cup \cdots \cup L_d$ be a line arrangement whose singularities are at most triple points. Is it true that $Gr(D)$ is solvable, where $D = 2L_1 + 2L_2 + \cdots + 2L_d$?

(For a computation of the group $Gr(D)$ when all the triple points are aligned, and a generalization of this question, see the last chapter.) If this question has a positive answer, then the Galois coverings of $\mathbb{P}^2$, ramified at such divisors can be effectively classified. To our knowledge, this classification has not been carried out, except for the case of abelian coverings. Here, we open a paranthesis and briefly discuss the case where the support C of $D := n_1L_1 + n_2L_2 + \cdots + n_dL_d$ consists of d lines passing through a common point p. The blow-up of the point p gives the Hirzebruch surface X_1, with the ruling $\phi : X_1 \to \mathbb{P}^1$ and the exceptional section E with $E^2 = -1$, E is the blow-up of p. We keep the same notation for the proper transforms of L_i, C, and D in X_1. Now, the singular meridian μ_p at the point p of C is vanishing. This can be seen by observing that a line $Q \not\subseteq C$ passing through the point p intersects C only at p. Let $* \in Q\backslash\{p\}$ be the base point for $\pi_1(\mathbb{P}^2\backslash C)$, and define the singular meridian μ_p by joining $*$ by a path in Q to the boundary of a small disc $\Delta \subset Q$ centered at p. The complement $Q\backslash\{p\} \simeq \mathbb{C}$ being simply connected, μ_p is trivial in $\pi_1(Q\backslash\{p\})$, so it is trivial in $\pi_1(\mathbb{P}^2\backslash C)$ since $Q\backslash\{p\} \subset \mathbb{P}^2\backslash C$.

This shows that a branched covering of $\mathbb{P}^2$ at D can be considered as a branched covering of X_1 at D, since the meridian μ_p of E is vanishing in $X_1\backslash(C \cup E)$. Now, L_i are fibers of the ruling $\phi : X_1 \to \mathbb{P}^1$, and D defines a divisor

$$D|_{\mathbb{P}^1} := n_1\phi(L_1) + n_2\phi(L_2) \cdots + n_d\phi(L_d)$$

on $\mathbb{P}^1$. Let $\psi : S \to \mathbb{P}^1$ be a branched covering of $\mathbb{P}^1$ at $D|_{\mathbb{P}^1}$. Then by the base change

$$\begin{array}{ccc} Y & \xrightarrow[\tilde{\psi}]{} & X_1 \\ \Big\downarrow{\scriptstyle \tilde{\phi}} & & \Big\downarrow{\scriptstyle \phi} \\ S & \xrightarrow[\psi]{} & \mathbb{P}^1 \end{array}$$

one obtains a surface Y with a ruling $\tilde{\phi} : Y \to S$ over the Riemann surface S, and the morphism $\tilde{\psi} : Y \to X$ is a Galois covering of X_1 branched at D. The most interesting case is when $S \simeq \mathbb{P}^1$, i.e. ψ is a branched Galois covering of $\mathbb{P}^1$ by itself. In this case Y becomes a rational ruled surface. This occurs only when $Gr(D)$ is finite, i.e. if $D = nL_1 + nL_2$ with $n \in \mathbb{N}$ or if $D = nL_1 + mL_2 + kL_3$ with $\frac{1}{n} + \frac{1}{m} + \frac{1}{k} > 1$ and $n, m, k \in \mathbb{N}$. The pull-back of the exceptional section of X_1 is a rational curve with self-intersection $-d$, where d is the degree of ψ (or $d = |Gr(D)|$). It turns out that Y is the d^{th} Hirzebruch surface X_d.

Returning to our problem of finding the essential images of $Gr(D)$ for divisors D whose supports C are line arrangements with at most triple points, note that one can alternatively consider the problem in the plane blown-up at the triple points of C. Denote the resulting surface by Z. Let $p_1, p_2, \cdots, p_k$ be the triple points of C, $E_i \subset Z$ be the exceptional line of the blow-up of p_i, and let μ_i be a singular meridian of C at p_i. Then μ_i is a meridian of E in X, so that one can consider the smaller groups

$$Gr(D + n_1E_1 + n_2E_2 + \cdots + n_kE_k) := Gr(D)/ \ll \mu_1^{n_1}, \mu_2^{n_2}, \cdots, \mu_k^{n_k} \gg .$$

Put $\vec{n} := (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$ and $\vec{E} := (E_1, E_2, \ldots, E_k)$.

Question. Is it true that the groups $Gr(D + \vec{n} \cdot \vec{E})$ are finite, where $D := 2L_1 + 2L_2 + \cdots + 2L_d$ is a divisor whose support is a line arrangement with at most triple points?

Note that, if $D = nL_1 + mL_2 + kL_3 + lL_4$, where L_i are four lines passing through a common point p, then the group $Gr(D)$ is big, even if $n = m = k = l = 2$. (For an interesting discussion about this group due to Higman, see [11].) Since, in this case, $Gr(D) = Gr(D + nE)$, where E is the blow-up of p, the group $Gr(D+nE)$ is big, too (because the singular meridian at p is already vanishing.)

The following theorem helps to deduce the solvability of Fenchel's problem for a divisor D from the solvability of the problem for the components of D.

Theorem 6.1.2 *Let D_1, D_2 be two divisors in $\mathbb{P}^2$ with solvable Fenchel's problem, such that the supports of D_1 and D_2 do not have any common component. Then Fenchel's problem is solvable for $D_1 + D_2$.*

Proof. Let $A = A_1 \cup A_2 \cup \cdots \cup A_n$ be the support of D_1 with irreducible components A_i, and $B = B_1 \cup B_2 \cup \cdots \cup B_n$ be the support of D_2 with irreducible components B_i. Let L_∞ be a line intersecting $A \cup B$ transversally. Take meridians δ of L_∞, μ_i of A_i and ν_i of B_i in $\mathbb{P}^2 \backslash (A \cup B \cup L_\infty)$. Then by Theorem 5.2.2, the element δ is central in the group $\pi_1(\mathbb{P}^2 \backslash (A \cup B \cup L_\infty))$, and $\pi_1(\mathbb{P}^2 \backslash (A \cup B))$ is the quotient of the group $\pi_1(\mathbb{P}^2 \backslash (A \cup B \cup L_\infty))$ by the relation $\delta = 1$.

Assume first that A, B intersect transversally. Then, by the Oka-Sakamoto-Kaliman theorem, $\pi_1(\mathbb{P}^2 \backslash (C \cup L_\infty)) = \pi_1(\mathbb{P}^2 \backslash (A \cup L_\infty)) \times \pi_1(\mathbb{P}^2 \backslash (B \cup L_\infty))$. Moreover, since L_∞ intersect A and B transversally, $\pi_1(\mathbb{P}^2 \backslash A)$ is the quotient of $\pi_1(\mathbb{P}^2 \backslash (A \cup L_\infty))$ by a central element δ_1, and $\pi_1(\mathbb{P}^2 \backslash B)$ is the quotient of $\pi_1(\mathbb{P}^2 \backslash (B \cup L_\infty))$ by a central element δ_2. Hence, one can write $\delta = \delta_1 \delta_2$. Passing to the quotient by the relation $\delta_1 = \delta_2 = 1$ in $\pi_1(\mathbb{P}^2 \backslash (A \cup B))$, we obtain a surjection

$$\pi_1(\mathbb{P}^2 \backslash (A \cup B)) \twoheadrightarrow \pi_1(\mathbb{P}^2 \backslash A) \times \pi_1(\mathbb{P}^2 \backslash B).$$

This surjection exists even if A and B do not intersect generically. Indeed, applying projective transformations to A, we can write C as a limit of the equisingular family $A_t \cup B$, where A_t intersects B transversally. By the Zariski semicontinuity theorem of the fundamental group (Theorem 2.3.2), one has a surjection

$$\pi_1(\mathbb{P}^2 \backslash (A \cup B)) \twoheadrightarrow \pi_1(\mathbb{P}^2 \backslash A_t) \times \pi_1(\mathbb{P}^2 \backslash B) \simeq \pi_1(\mathbb{P}^2 \backslash A) \times \pi_1(\mathbb{P}^2 \backslash B).$$

This implies that there is a surjection

$$Gr(D_1 + D_2) \twoheadrightarrow Gr(D_1) \times Gr(D_2).$$

By the hypothesis, there exists essential images $Gr(D_1) \twoheadrightarrow K_1$ and $Gr(D_2) \twoheadrightarrow K_2$. We conclude that there is an essential image

$$Gr(D_1 + D_2) \twoheadrightarrow Gr(D_1) \times Gr(D_2) \twoheadrightarrow K_1 \times K_2. \qquad \square$$

Observe that it is not obvious to derive Theorem 6.1.1 from Theorem 6.1.2, since there is no covering of $\mathbb{P}^2$ ramified only at a line.

The following ideas appeared in the proof of Theorem 6.1.2 merits stating separately.

Lemma 6.1.1 *(i) Let $D = m_1C_1 + m_2C_2 + \cdots + m_nC_n$ be a divisor with support C, and let $C \to C'$ be an equisingular degeneration with a reduced limit. Let C_i' be the induced degeneration of C_i (C_i' can be reducible), and put $D_0 = m_1C_1' + m_2C_2' + \cdots + m_nC_n'$. Then there is a surjection $Gr(D_0) \twoheadrightarrow Gr(D)$. Hence, if Fenchel's problem is solvable for D, then so it is for D_0. Conversely if the group $Gr(D_0)$ is trivial (abelian, finite, solvable), then so is $Gr(D)$.*

(ii) If D_1, D_2 are divisors without common components, then there is a surjection $Gr(D_1 + D_2) \twoheadrightarrow Gr(D_1) \times Gr(D_2)$. If moreover the supports of D_1 and D_2 intersect transversally, then in addition one has an exact sequence

$$1 \to \mathbb{Z}_n \to Gr(D_1 + D_2) \to Gr(D_1) \times Gr(D_2) \to 1.$$

for some n with $1 \leq n \leq \infty$, the kernel $\mathbb{Z}_n$ being a central subgroup of the group $Gr(D_1 + D_2)$.

Proof. The part (i) follows from Theorem 2.3.2. If the supports of D_1 and D_2 intersect transversally, then the part (ii) follows from Theorem 2.4.2. Otherwise one can write $D_1 + D_2$ as a limit of an equisingular degeneration of a transversal intersection as in the proof of Theorem 6.1.2, and the assertion follows from the part (i). □

In view of Theorem 6.1.2, the most interesting case in Fenchel's problem is that of divisors with irreducible supports. Unfortunately, not much is known about the groups $Gr(D)$ with $D = nC$, where C is irreducible. Under a serious restriction on the irreducible curve C the next theorem guarantees the finiteness of $Gr(2C)$, but not its non-triviality.

Theorem 6.1.3 *Let (G, μ) a pair with G being an irreducible group, and μ being a normal generator,[1] such that some two conjugates a, b of μ generate G in the usual sense. Then $G(2)$ is a finite group.*

[1] Recall that if G is the group of an irreducible curve $C \subset \mathbb{P}^2$, and μ is a meridian of C, then (G, μ) is such a pair.

Proof. Let

$$G = \langle a, b \,|\, \mathcal{R}_1 = \mathcal{R}_2 = \cdots = 1 \rangle$$

be a presentation of our group, where $\mathcal{R}_1 = xax^{-1}b$. Since $a^2 = b^2 = 1$ in the group $G(2)$, the relation $\mathcal{R}_1$ can be written in the form $(ab)^n a (ab)^{-n} = b$. Hence, $G(2)$ is a quotient of the group

$$H_n := \langle a, b \,|\, (ab)^n a (ab)^{-n} = b, \quad a^2 = 1 \rangle,$$

and it suffices to show that H_n is finite. First, note that the above presentation of H_n is equivalent to the presentation

$$H_n = \langle a, b \,|\, (ab)^{2n+1} = 1,\ a^2 = b^2 = 1 \rangle = T_{2,2,2n+1}.$$

But this is the triangle group $T_{2,2,2n+1}$, which is known to be finite. Anyway, we shall find its commutator subgroup to understand better its structure.

Evidently, $H_n/H_n' = \mathbb{Z}_2$. So it suffices to show that the commutator subgroup H_n' is finite. We apply the Reidemeister -Schreier method to find a presentation of the group H_n'. Clearly, $T := \{1, a\}$ is a Schreier transversal to H_n', and the corresponding generators are given as $Y := \{x := ba^{-1}, y := ab, z := a^2\}$. The relation $a^2 = 1$ implies $z = 1$, and the relation $b^2 = 1$ is re-written as $xy = 1$. Finally, the relation $(ab)^{2n+1} = 1$ is translated as $x^{2n+1} = y^{2n+1} = 1$. We conclude that

$$H_n' = \mathbb{Z}_{2n+1} \Rightarrow H_n = \mathbb{Z}_{2n+1} \ltimes \mathbb{Z}_2$$

that is, H_n is a non-abelian group of order $4n + 2$. $\square$

Examples. It is easy to see that, if an irreducible curve C of degree d has a flex or a singular point of order $(d-2)$, then the group $G = \pi_1(\mathbb{P}^2 \backslash C)$ is generated by two meridians. All of the groups of rational cuspidal curves with a deep singularity calculated in Chapter 3 are generated by two meridians.

Remark. In the case where there are more then two conjugates a, b, c, d, etc. of μ generating G, one can get a quotient of G generated by only two of them by e.g. setting $b = c = d = \cdots$, and then apply Theorem 6.1.3 to find a finite quotient of G.

The conclusion of Theorem 6.1.3 does not hold for curve groups in general. We now proceed to show the existence of curve groups G with big residual groups $G(n)$ for each $n \geq 2$.

Claim. There exists an irreducible curve $C \subset \mathbb{P}^2$ such that the group $Gr(nC)$ is big for each $n \geq 2$ (or, in the notations of Definition 3.3.2, with the group $G_C(n)$ being big for each $n \geq 2$).

Proof of the claim. According to Theorem 2.2.3, for $p, q \in \mathbb{N}$, g.c.d.$(p,q) = 1$, the free product $\mathbb{Z}_p * \mathbb{Z}_q :=< a, b \mid a^p = b^q = 1 >$ is the fundamental group $G := \pi_1(\mathbb{P}^2 \backslash C_{p,q})$ of an irreducible curve $C_{p,q}$ of degree pq, where $\mu := a^{p_1}b$ is a meridian, p_1 being an integer such that $pp_1 + qq_1 = 1$ for some $q_1 \in \mathbb{Z}$. (The expression $\mu = a^{p_1}b$ for the meridian can be found at Oka's paper [79]; however, the precise form of μ does not play an essential rôle in what follows.)

Then one has

$$Gr(nC_{p,q}) = \langle a, b \mid a^p = b^q = (a^{p_1}b)^n = 1 \rangle,$$

which is a generalized triangle group. Hence, if $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, then $Gr(nC_{p,q})$ is a big group, regardless of $n > 1$.

Let us consider the particular case where $q = p + 1$. Then $p_1 = -1$, and one has

$$Gr(nC_{p,q}) = \langle a, b, \mid a^{p+1} = b^p = (a^{-1}b)^n = 1 \rangle.$$

Put $x := a$, $y := b^{-1}$, $z := a^{-1}b$. Then $xyz = 1$, and so

$$Gr(nC_{p,q}) = \langle x, y, z \mid x^{p+1} = y^p = z^n = xyz = 1 \rangle = T_{p,p+1,n},$$

where $T_{p,p+1,n}$ is the triangle group, which is big for $1/p + 1/(p+1) + 1/n < 1$. This is clearly the case for $n \geq 2$ and $p \geq 4$. Observe that for $(p,q,n) = (2,3,6)$, the group $G(6)$ is infinite solvable. $\square$

Remark. Dimca [18] gives an equisingular deformation of the sextic $C_{2,3}$ (whose equation is given in Theorem 2.2.3) to an irreducible sextic with a singularity of multiplicity 4, thereby giving an example of a unisingular curve of type $(d, d-2)$ with a non-abelian fundamental group. Using Theorem 6.1.3 and Lemma 6.1.1-(ii), one can see that the curve $C_{p,q}$ cannot be deformed to an irreducible curve C with a singularity of multiplicity $pq - 2$ if $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. Indeed, by Lemma 6.1.1-(ii), such a deformation would induce a surjection $Gr(2C) \twoheadrightarrow Gr(2C_{p,q})$. But $Gr(2C)$ should be finite by Theorem 6.1.3, and we have shown above that $Gr(2C_{p,q})$ is a big group.

Now, following an idea due to Namba [67], we exploit the fact that the generalized triangle groups have essential representations. By the following theorem, the groups $Gr(nC_{p,q})$ has essential images for any $p, q, n \in \mathbb{N}$.

Theorem 6.1.4 ***(Selberg [85])*** *Let R be a non-trivial, finitely generated subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then there exists a torsion free normal subgroup N of R of finite index.*

Corollary 6.1.1 *For any Oka curve $C_{p,q}$ as in Theorem 2.2.3, there is a finite Galois covering of $\mathbb{P}^2$ ramified at $nC_{p,q}$ for each $n \in \mathbb{N}$.*

Proof. We have observed above that $Gr(nC_{p,q})$ is a generalized triangle group. By Theorem 5.5.2, there is an essential linear representation $\rho : Gr(nC_{p,q}) \twoheadrightarrow R \subset \mathrm{GL}_n(\mathbb{C})$. By Selberg's theorem, R has a torsion free subgroup N of finite index. The quotient R/N gives the desired essential image of $Gr(nC_{p,q})$. $\square$

For a divisor $D = m_1C_1 + m_2C_2 + \cdots + m_kC_k$ with irreducible components $C_1, C_2, \ldots, C_k$, and with meridians μ_i of C_i, let us call a representation $\phi : Gr(D) \to \mathrm{GL}_n(\mathbb{C})$ *essential* if the order of $\phi(\mu_i)$ equals m_i for $1 \leq i \leq k$. The method we have used for the proof of Corollary 6.1.1 can be summarized as

Theorem 6.1.5 ***(Namba [67])*** *If $Gr(D)$ has an essential representation, then $Gr(D)$ has an essential image, and the Fenchel's problem is solvable for D, in other words, there is a finite Galois covering of $\mathbb{P}^2$ ramified at D.*

The following theorem can be proved easily by imitating the proof of Corollary 6.1.1.

Theorem 6.1.6 *Let $C \subset \mathbb{P}^2$ be an irreducible curve. If there is a surjection $\pi_1(\mathbb{P}^2 \backslash C) \twoheadrightarrow \mathbb{Z}_p * \mathbb{Z}_q$ for some $p \geq 2$, $q \geq 2$, then there is a finite Galois covering of $\mathbb{P}^2$ ramified at nC for any $n \in \mathbb{N}$.*

We say that a divisor D is *strictly greater then 1* (or $D > 1$) if $D = n_1C_1 + n_2C_2 + \cdots + n_kC_k$ with $n_i > 1$ for $1 \leq i \leq k$, the curves C_i being the irreducible components of D.

Definition 6.1.1 Let $C \subset \mathbb{P}^2$ be a curve. Then the group that $\pi_1(\mathbb{P}^2 \backslash C)$ is said to be *abundant* if for any divisor $D > 1$ with support C, there is a Galois covering of $\mathbb{P}^2$ ramified at D.Theorem 6.1.6 asserts that if C is irreducible, then $\pi_1(\mathbb{P}^2 \backslash C)$ is abundant if there is a surjection $\pi_1(\mathbb{P}^2 \backslash C) \twoheadrightarrow \mathbb{Z}_p * \mathbb{Z}_q$.

Hence, the Oka curves of Theorem 2.2.3 are examples of irreducible curves with abundant groups, and the curves of Theorem 6.1.1 are examples of line arrangements with abundant groups.

Remarks

1. A big curve group need not be abundant, an example will be discussed in the next section. It is interesting to know if the converse is true, i.e. if the abundance of a group implies its bigness.

2. An effective way to find essential images of the group $G := \mathbb{Z}_p * \mathbb{Z}_q$ is to consider the group G/G'', a presentation of this group is given in the preceding chapter. In particular, since G' is free of rank $(p-1)(q-1)$ if $gcd\,(p,q) = 1$, one has $G/G'' = \mathbb{Z}^{(p-1)(q-1)} \rtimes \mathbb{Z}_{pq}$.

3. In connection with the problem of finding finite index subgroups of $\pi_1(\mathbb{P}^2 \backslash C)$, one should take Higman's conjecture and the developments in this direction into consideration. This conjecture states that "each triangle group $T_{p,q,r}$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{p} < 1$ contains among its quotients all but finitely many of the alternating or symmetric groups". Higman's conjecture has been verified for many cases, see [21] for details.

6.2 Solution of Fenchel's problem for rational cuspidal curves

It is easy to see that, in view of Theorem 6.1.6, Fenchel's problem is completely solved for the groups in Theorem 3.2.3, except for the groups (4). Indeed, we have shown that these groups are either abelian or admit a surjection to the group $\mathbb{Z}_j * \mathbb{Z}_k$ for some (j,k). Hence, these groups are abundant if they are non abelian. On the other hand, our results for the groups (4), or for the groups of rational three-cuspidal curves of type $(d, d-2)$, $(d, d-3)$ and $(d, d-4)$ are incomplete, and largely negative. Here we will only discuss the group

$$G := \langle c, b \mid cbc = bcb, \quad b^n c^{n+2} = c^{n+2} b^n, \quad (b^{-n} c b^2)^{n+1} c^{n^2} = 1 \rangle,$$

which is by Theorem 3.2.2 the group of a rational three-cuspidal curve C of type $(2n+3, 2n)$. In Proposition 3.3.1, it is shown that if n is odd and $k|n$, there is a surjection $G(k) \twoheadrightarrow T_{2,3,k}$. Hence, under these assumptions, there exists a solution to Fenchel's problem for kC. Note that e.g. if $n = 7$, then the degree of the curve is 17, which is a prime number. So there is a unique abelian covering ramified at the curve. Setting $k = 7$ gives a surjection $G \twoheadrightarrow G(7) \twoheadrightarrow T_{2,3,7}$, showing that G is big in this case. However, there are not too many Galois coverings of $\mathbb{P}^2$ ramified at kC for $k \in \mathbb{N}$, as we proceed to show now.

Assume that $n = 2m$ is even, and $k|n$. Then $x^{n+1} = x^{2m+1} = y^{3m}x$ by the relation $x^2 = y^3$. Hence, we obtain

$$G(k) = \langle x, y \mid x^2 = y^3, \quad y^{3m}x = 1 \quad , (y^{-1}x)^k = 1 \quad \rangle.$$

Thus, $x = y^{-3m}$ in the group $G(k)$, and so

$$G(k) = \mathbb{Z}/gcd\,(6m+3, (3m+1)k)\mathbb{Z} = \mathbb{Z}/gcd\,(k,3)\mathbb{Z},$$

so that if $3 \nmid 3$, then there is no Galois covering of $\mathbb{P}^2$ ramified at kC.

Now suppose that $k \in \mathbb{N}$ is such that $gcd\,(k,n) = i$, $gcd\,(k, n+2) = j$. Then it is easy to see that, the relation $[b^n, c^{n+2}] = 1$ becomes $[b^i, c^j] = 1$ in the group $G(k)$. Hence, if $i = j = 1$, then $[b,c] = 1 \Rightarrow b = c$ by the cusp relation $cbc = bcb$. Thus,

$$G(k) = \mathbb{Z}/gcd\,(k, 2n+3)\mathbb{Z}$$

At this point, we see that the group G, although being big, is not abundant. In fact, G is quite "scarce", i.e. if we suppose that $i = j = gcd\,(k, 2n+3) = 1$, then $G(k) = 1$, i.e. there is no Galois covering of $\mathbb{P}^2$ ramified at kC.

Question. Is it true that, if the group $\pi_1(\mathbb{P}^2 \backslash C)$ is infinite, then there are infinitely many $k \in \mathbb{N}$ such that there is a (finite) Galois covering over $\mathbb{P}^2$ ramified at kC? Is this true if the group $\pi_1(\mathbb{P}^2 \backslash C)$ is big?

Chapter 7

Perspectives

Among the ideas exploited in the preceding pages, perhaps the most promising one for the future research is that of the *group of a divisor*. It is possible to introduce the corresponding local concept: Given a divisor D on $\mathbb{P}^2$, let C be its support. Then, for $p \in C$ a singular point of C, there is the corresponding *local divisor* D_p that can be defined in a neighborhood B of p in $\mathbb{P}^2$ by $D_p := D \cap B$, where the coefficients of the irreducible branches of D_p are inherited from those of D in the obvious way. One can accordingly define the *group of a divisor germ* as follows: Let $(D,0) := n_1(C_1,0)+n_2(C_2,0)+\cdots+n_k(C_k,0)$ be a divisor germ with support $(C,0) := (C_1,0) \cup (C_2,0) \cup \cdots \cup (C_k,0)$. Let $\pi_1^0(B\backslash C)$ be the fundamental group in a sufficiently small neighborhood B of the origin, and let μ_i be a meridian of C_i. Then the group of $(D,0)$ is defined to be the quotient

$$Gr^0(D) := \pi_1^0(B\backslash C)/ \ll \mu_1^{n_1}, \mu_2^{n_2}, \cdots, \mu_k^{n_k} \gg .$$

For example, let $(D,0) := n(C,0)$, where $(C,0)$ is a simple cusp germ. Then $\pi_1^0(B\backslash C) = \mathbb{B}_3$, so that $Gr^0(D)$ is finite if $n \leq 5$, finite solvable for $n = 6$, and big otherwise.

If $(D,0) := n(C_1,0) + m(C_2,0)$, where $(C_1,0)$, $(C_2,0)$ are two smooth germs intersecting with multiplicity k, then $\pi_1^0(C) = \langle a,b \mid (ab)^k = (ba)^k \rangle$, where $(C,0) := (C_1,0) \cup (C_2,0)$. Hence, one has

$$Gr^0(D) = \langle a,b \mid (ab)^k = (ba)^k, \quad a^n = b^m = 1 \rangle.$$

Thus, if $k = 1$, then $Gr^0(D) \simeq \mathbb{Z}_n \times \mathbb{Z}_m$. Otherwise, put $x := ab$, $y := b$, and write the above presentation as

$$Gr^0(D) = \langle [x^k, y] = (xy^{-1})^n = y^m = 1 \rangle.$$

Passing to the quotient by the central element x^k gives the triangle group $T_{n,m,k}$, so that $Gr^0(D)$ is solvable for $\frac{1}{n}+\frac{1}{m}+\frac{1}{k}\geq 1$.

If $(D,0):=n(C_1,0)+m(C_2,0)+k(C_3,0)$, where $(C_1,0)$, $(C_2,0)$ $(C_3,0)$ are three smooth germs intersecting transversally, then one has

$$Gr^0(D)=\langle a,b,c,\delta\,|\,a^n=b^m=c^k=[\delta,a]=[\delta,b]=[\delta,c]=1,\quad \delta=cba\rangle.$$

Passing to the quotient by the central element δ again gives the triangle group $T_{n,m,k}$.

The following question is tempting:

Question. Let D be a divisor on $\mathbb{P}^2$, with the support C. Is it true that, the group $Gr(D)$ is finite (solvable), if for any singular point p of C, the group $Gr^p(D)$ is finite (solvable)?

For example, let us consider the case of a divisor D with an irreducible support. By Theorem 2.2.1, the group $\mathbb{B}_n^p$ is the group of a curve possessing only ordinary nodes and simple cusps as singularities. (Recall that $\mathbb{B}_n^p$ is the quotient of $\mathbb{B}_n$ by its center.) A positive answer to the above question would imply that $\mathbb{B}_n^p(m)$ is finite for $m\leq 5$ and solvable for $m=6$. This seems like too much to expect for the braid group. However, even if the answer to the above question is negative, we think that one can give some general conditions under which the answer is positive.

For certain divisors whose supports are line arrangements with at most triple points, it can be verified that the group is indeed solvable. For example, this is true if all triple points of the arrangement are aligned. As a more concrete example, let $D:=2R+\sum_{i=1}^n 2P_i+2Q_i$ be a divisor, R,P_i,Q_i being lines in $\mathbb{P}^2$ such that P_i and Q_i intersect on R, and such that there are no triple points other then those lying on R. Then by the Oka-Sakamoto-Kaliman theorem, $Gr(D)$ is the group generated by elements c, $a_1,a_2\dots,a_n$, $b_1,b_2,\dots,n$ with the relations

$a_i^2=b_i^2=c^2=1,\quad 1\leq i\leq n$
$[a_i,a_j]=[a_i,b_j]=[b_i,b_j]=1\quad \text{if}\quad i\neq j,$
$c\Pi_{i=1}^n a_ib_i=1.\quad (*)$

Now consider the infinite dihedral group $\mathbb{D}_\infty=\langle a,b\,|\,a^2=b^2=1\rangle$, whose commutator subgroup $\mathbb{Z}$ is generated by $[a,b]=(ab)^2$. It follows that the commutator subgroup $Gr(D)'$ of $Gr(D)$ is generated by the elements $x_i:=$

$[a_i, b_i] = (a_i b_i)^2$ for $1 \leq i \leq n$, since it is easily seen that $[x_i, x_j] = 1$ if $i \neq j$. Finally, by $(*)$ one has $c^{-1} = \Pi_{i=1}^n a_i b_i$, and the relation $c^2 = 1$ implies $\Pi_{i=1}^n (a_i b_i)^2 = \Pi_{i=1}^n x_i = 1$. It follows that $Gr(D)' \simeq \mathbb{Z}^{n-1}$. Hence, one has the exact sequence

$$0 \to \mathbb{Z}^{n-1} \to Gr(D) \to \mathbb{Z}_2^{2n} \to 0.$$

The next interesting case is that of arrangements of lines and conics. For some special classes of such arrangements, the braid monodromy has been studied in detail by Moishezon and Teicher in [61]-[65]. As an example, consider the divisor $D := nP + mQ + kR$, where Q is a smooth conic and P, Q are two distinct tangent lines to Q. Then it is easy to see that

$$Gr(D) = \langle a, b, c \,|\, (ab)^2 = (ba)^2, \quad a^n = b^m = c^k = ab^2c = 1 \rangle.$$

Put $x := ab$, $y := b$. Then $a = xy^{-1}$ and $c^{-1} = xy$. Re-writing the above presentatation in terms of x, y we get

$$Gr(D) = \langle x, y \,|\, [x^2, y] = (xy^{-1})^n = y^m = (xy)^k = 1 \rangle.$$

Passing to the quotient be the central element x^2, one obtains the surjection

$$Gr(D) \twoheadrightarrow \langle x, y \,|\, x^2 = y^m = (xy)^r \rangle \simeq T_{2,m,r},$$

where $T_{2,m,r}$ is the triangle group with $r := gcd\,(n, k)$.

As a final remark, observe the following reformulation of the question above. Note first that, by a theorem of Grothendieck-Raynaud [35], there exists a maximal (possibly infinite) covering $X \to \mathbb{P}^2$ branched at D under the assumption that for any singular point p of D, the group $Gr^p(D)$ is finite. (X is then simply connected.) The question above asks if $X \to \mathbb{P}^2$ is a finite covering.

Abelian branched coverings of $\mathbb{P}^2$ has been studied extensively by several authors, to cite but just a few of them, by Zariski [99], Gaffney and Lazarsfeld [32], Hirzebruch [39], and by Hironaka [38]. Non-abelian coverings of $\mathbb{P}^2$ ramified at divisors with "small" groups, even for the simplest cases discussed above, await study.

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