

# Arithmetic and dynamics around the outer automorphism of $\mathrm{PGL}(2, \mathbb{Z})$

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# Foreword

In a paper of his on binary quadratic forms, Poincaré states:



**“it is not possible, for the  
indefinite quadratic forms to  
find invariants, in the sense that  
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Several attempts have been made  
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- 1 Definition of Jimm and functional equations
- 2 Dynamics
- 3 Tree automorphisms and Lebesgue's measure

# PART I

## Definition of Jimm and functional equations

There are two fundamental involutions of the real line  $\mathbf{R}$ :

$$V : x \in \mathbf{R} \rightarrow -x \in \mathbf{R}$$

$$K : x \in \mathbf{R} \rightarrow 1 - x \in \mathbf{R}$$

and a third one if we add the point at infinity:

$$U : x \in \mathbf{R} \rightarrow 1/x \in \mathbf{R}$$

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$$U : x \in \mathbf{R} \rightarrow 1/x \in \mathbf{R}$$

$$V(x) = -x, \quad K(x) = 1 - x, \quad U(x) = 1/x$$

... together they generate the group

$$\mathrm{PGL}_2(\mathbf{Z}) = \left\{ \frac{px + q}{rx + s} \mid ps - qr = \pm 1, p, q, r, s \in \mathbf{Z} \right\}$$

$$\simeq \langle U, V, K \mid U^2 = V^2 = K^2 = (UV)^2 = (KU)^3 = 1 \rangle$$

Our aim here is to introduce a fourth involution, which we call Jimm

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## Notation

Every  $x \in \mathbf{R}$  can be written as a continued fraction

$$[n_0, n_1, n_2, \dots] = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{\dots}}}$$

( $n_0 \in \mathbf{Z}$ ,  $n_i \in \mathbf{Z}_{>0}$  for  $i > 0$ ), uniquely if  $x$  is irrational.

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By  $1_k$  we denote the sequence  $1, 1, \dots, 1$  of length  $k$ .

We introduce a 'singular' function  $\mathbf{R} \rightarrow \mathbf{R}$ :

### Definition

$$\zeta([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

This is a kind of 'real' modular function, as we shall see.

But let us consider some examples first...

## Definition (Recall)

$$\zeta([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

## Examples

$$\zeta([3, 3, 3, \dots]) = [1_{3-1}, 2, 1_{3-2}, 2, 1_{3-2}, 2, \dots] = [1, 1, 2, 1, 2, 1, 2, \dots]$$

$$\zeta([5, 5, 5, \dots]) = [1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, \dots]$$



## Definition (Recall)

$$\zeta([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

This definition works only if  $n_k \geq 2$ . To make it work for  $n_k = 2$ , use

## RULE I

$$\dots, n, 1_0, m, \dots = \dots, n, m, \dots$$

## Examples

$$\zeta([2, 2, 2, \dots]) = [1, 2, 1_0, 2, 1_0, 2, \dots] = [1, 2, 2, 2, \dots]$$

$$\zeta([2, 3, 2, 3, \dots]) = [1, 2, 1, 2, 2, 1, 2, 2, 1, \dots]$$

## Definition (Recall)

$$\zeta([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

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$$\zeta([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

To make it work also when  $n_k = 1$ , use

## RULE II

$$\dots, n, 1_{-1}, m, \dots = \dots, n + m - 1, \dots$$

## Examples

$$\begin{aligned} \zeta([1, 1, 2, 1, 2, 1, 2, \dots]) &= \\ [1_0, \underbrace{2, 1_{-1}, 2}_3, 1_0, \underbrace{2, 1_{-1}, 2}_3, 1_0, \underbrace{2, 1_{-1}, 2}_3, \dots] &= \\ &= [3, 3, 3, \dots] \end{aligned}$$

remember?

## Definition (Recall)

$$\zeta([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

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remember?

## Definition (Recall)

$$\zeta([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

## Example

$$\begin{aligned} \zeta([\dots, 7, \textcolor{blue}{1}, \textcolor{red}{1}, \textcolor{green}{1}, \textcolor{magenta}{1}, 13, \dots]) &= \\ [\dots 1_5, 2, \underbrace{\textcolor{blue}{1}_{-1}, 2}_{}, \textcolor{red}{1}_{-1}, 2, \textcolor{green}{1}_{-1}, 2, \textcolor{magenta}{1}_{-1}, 2, 1_{11}, \dots] &= \\ [\dots 1_5, 3, \underbrace{\textcolor{red}{1}_{-1}, 2}_{}, \textcolor{green}{1}_{-1}, 2, \textcolor{magenta}{1}_{-1}, 2, 1_{11}, \dots] &= \\ [\dots 1_5, 4, \underbrace{\textcolor{green}{1}_{-1}, 2}_{}, \textcolor{magenta}{1}_{-1}, 2, 1_{11}, \dots] &= \\ [\dots 1_5, 5, \underbrace{\textcolor{magenta}{1}_{-1}, 2}_{}, 1_{11}, \dots] &= \\ [\dots 1_5, 6, 1_{11}, \dots] \end{aligned}$$

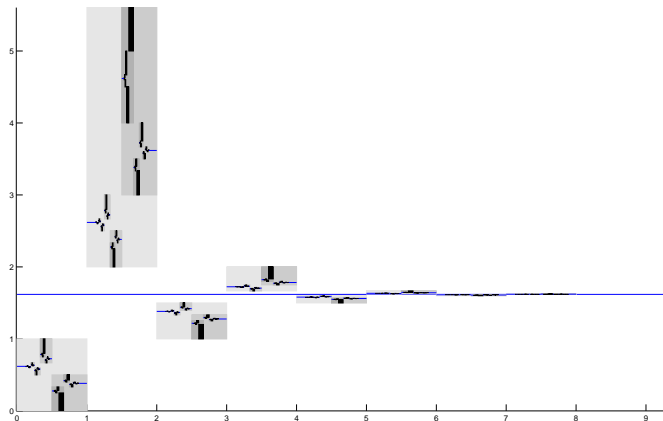
## Definition (Recall)

$$\zeta([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

With these two rules,  $\zeta$  becomes well-defined on  $\mathbf{R} \setminus \mathbf{Q}$  and it is involutive:

$$\zeta(\zeta(x)) = x$$

Here is the plot of  $\zeta$  (the graph lies inside the darker boxes)



# Some continuity properties of jimm

It can be shown that..

- $\zeta$  is continuous on  $\mathbf{R} \setminus \mathbf{Q}$
- have jump discontinuities on  $\mathbf{Q}$
- $\zeta$  is differentiable almost everywhere
- its derivative vanish almost everywhere
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Now consider....

### Example

$$\begin{aligned}\zeta(1 + [3, 3, 3 \dots]) &= \zeta([4, 3, 3 \dots]) = [1, 1, 1, 2, 1, 2, 1, \dots] \\ &= 1 + \frac{1}{\underbrace{[1, 1, 2, 1, 2, 1, \dots]}_{=\zeta([3, 3, 3, \dots])}}\end{aligned}$$

We have, in general

### FUNCTIONAL EQUATION

$$\zeta(1 + x) = 1 + \frac{1}{\zeta(x)}$$

This functional equation can be derived from the following fundamental set of functional equations

$$\zeta(\zeta(x)) = x \quad (\text{involutivity})$$

$$\zeta\left(\frac{1}{x}\right) = \frac{1}{\zeta(x)} \quad \text{equivariance}$$

$$\zeta(-x) = -\frac{1}{\zeta(x)} \quad \text{“twisted” equivariance}$$

$$\zeta(1-x) = 1 - \zeta(x) \quad \text{equivariance}$$

Now notice

$$xy = 1 \iff y = 1/x \iff$$

$$\zeta(y) = \zeta\left(\frac{1}{x}\right) = \frac{1}{\zeta(x)}$$

Hence

$$xy = 1 \iff \zeta(y)\zeta(x) = 1$$

We may do the same for the other equations, which gives

## Two-variable form of functional equations

$$\zeta(x) = y \iff \zeta(y) = x$$

$$xy = 1 \iff \zeta(x)\zeta(y) = 1$$

$$x + y = 0 \iff \zeta(x)\zeta(y) = -1$$

$$x + y = 1 \iff \zeta(x) + \zeta(y) = 1$$

$$\frac{1}{x} + \frac{1}{y} = 1 \iff \frac{1}{\zeta(x)} + \frac{1}{\zeta(y)} = 1$$

$\implies \zeta$  preserves harmonic pairs of numbers.



Recall that

$$Ux := \frac{1}{x}, \quad Vx := -x, \quad Kx := 1 - x$$

The functional equations say

$$\zeta U = U \zeta, \quad \zeta K = K \zeta, \quad \zeta V = UV \zeta$$

$\implies \zeta$  is **Dyer's outer automorphism of  $\mathrm{PGL}_2(\mathbf{Z})$** .

This is the only non-trivial outer automorphism:  $\mathrm{Out}(\mathrm{PGL}_2(\mathbf{Z})) \simeq \mathbf{Z}/2\mathbf{Z}$ .

(In fact we worked out the continued fraction-definition of  $\zeta$  from the above functional equations)

The most general functional equation has the form

$$\zeta(Mx) = \zeta(M)\zeta(x), \quad M \in \mathrm{PGL}_2(\mathbf{Z})$$

(where  $\zeta(M)$  is the image of  $M$  under Dyer's automorphism).

Hence  $\zeta$  is a “twisted” **equivariant** function.

$f$  is said to be  **$\mathrm{PSL}_2(\mathbf{Z})$ -equivariant** if  $f(Mx) = Mf(x)$ ,  $\forall M \in \mathrm{PSL}_2(\mathbf{Z})$ .

If  $G$  is weight- $k$  modular (i.e.  $G(Mz) = j_M(z)^k G(z)$ ) then

$$H(z) = z + k \frac{G(z)}{G'(z)}$$

is  $\mathrm{PSL}_2(\mathbf{Z})$ -equivariant, i.e. it satisfies the functional equations

$$H(Tz) = TH(z), \quad H(Sz) = SH(z),$$

where  $Tz = KUVz = z + 1$  and  $Sz = UVz = -1/z$  generate  $\mathrm{PSL}_2(\mathbf{Z})$ .

**Question.** Are there analytic analogues of  $\zeta$ ? i.e. are there analytic functions with  $H(Mx) = \zeta(M)H(x)$ ,  $\forall M \in \mathrm{PGL}_2(\mathbf{Z})$ ? (needs to be properly formulated)

# Action on quadratic irrationals

## Fact I

$\zeta$  sends ultimately periodic continued fractions to ultimately periodic continued fractions.

$\implies$

$\zeta$  sends **quadratic irrationals** to **quadratic irrationals**  
i.e.  $\zeta$  preserves the "real multiplication-set."

(it does not preserve nor respect the trace, norm, signature, etc)

$$\zeta(\sqrt{2}) = \zeta([1, 2, 2, \dots]) = 1 + \sqrt{2}$$

Not so simple in general:

$$\zeta(\sqrt{11}) = \frac{15 + \sqrt{901}}{26}, \quad \zeta(-\sqrt{11}) = \frac{15 - \sqrt{901}}{26}$$

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## Fact II

$\zeta$  respects ends of continued fractions (i.e. if  $x, y$  has continued fractions that eventually coincide, then so does  $\zeta(x)$  and  $\zeta(y)$ ).



$\zeta$  respects the  $\mathrm{PGL}_2(\mathbb{Z})$ -action (i.e. if  $x$  and  $y$  are in the same  $\mathrm{PGL}_2(\mathbb{Z})$ -orbit, then so are  $\zeta(x)$  and  $\zeta(y)$ .)

$$\zeta(Mx) = \zeta(M)\zeta(x) \quad M \in \mathrm{PGL}_2(\mathbb{Z}), x \in \mathbb{R}$$

so that

$$x = My \implies \zeta(x) = \zeta(M)\zeta(y), \quad \zeta(M) \in \mathrm{PGL}_2(\mathbb{Z})$$



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## More precisely

$$\zeta(Mx) = \zeta(M)\zeta(x) \quad M \in \mathrm{PGL}_2(\mathbf{Z}), x \in \mathbf{R}$$

so that

$$x = My \implies \zeta(x) = \zeta(M)\zeta(y), \quad \zeta(M) \in \mathrm{PGL}_2(\mathbf{Z})$$

Facts I&II together imply:

### Fact III

$\zeta$  induces an involution of the “moduli space of degenerate rank-2 lattices” inside  $\mathbf{R}$ , preserving setwise the “real-multiplication” locus.

$$\zeta \circ \zeta = \text{id} \text{ on } \mathbf{R}/\text{PGL}_2(\mathbf{Z})$$

The facts imply...

$\zeta$  is really a modular function.

Furthermore, one has ....

## Fact IV

$\zeta$  commutes with the Galois conjugation on quadratic irrationals, i.e.

$$\zeta(a + \sqrt{b}) = A + \sqrt{B}$$

$$\iff$$

$$\zeta(a - \sqrt{b}) = A - \sqrt{B}$$

Now go back to the two-variable functional equations....

$$xy = 1 \iff \zeta(x)\zeta(y) = 1$$

$$x + y = 0 \iff \zeta(x)\zeta(y) = -1$$

$$x + y = 1 \iff \zeta(x) + \zeta(y) = 1$$

$$\frac{1}{x} + \frac{1}{y} = 1 \iff \frac{1}{\zeta(x)} + \frac{1}{\zeta(y)} = 1$$

...and set  $y = \bar{x}$ , where  $x = a + \sqrt{b}$  is a quadratic irrational:

$$x\bar{x} = 1 \iff \zeta(x)\zeta(\bar{x}) = 1$$

$$x + \bar{x} = 0 \iff \zeta(x)\zeta(\bar{x}) = -1$$

$$x + \bar{x} = 1 \iff \zeta(x) + \zeta(\bar{x}) = 1$$

$$\frac{1}{x} + \frac{1}{\bar{x}} = 1 \iff \frac{1}{\zeta(x)} + \frac{1}{\zeta(\bar{x})} = 1$$

## Recall from number theory

If  $x = a + \sqrt{b}$  ( $a, b \in \mathbf{Q}$ ,  $b > 0$ ), then

**norm** of  $x$  is  $N(x) := x\bar{x} \iff N(a + \sqrt{b}) = a^2 - b$

**trace** of  $x$  is  $T(x) := x + \bar{x} \iff T(a + \sqrt{b}) = 2a$

### Example

$$N(1 + \sqrt{2}) = -1, \quad T(1 + \sqrt{2}) = 2$$

The functional equations means

$$N(x) = x\bar{x} = 1 \iff \zeta(x)\zeta(\bar{x}) = 1 = N(\zeta x)$$

$$Tr(x) = x + \bar{x} = 0 \iff \zeta(x)\zeta(\bar{x}) = -1 = N(\zeta x)$$

$$Tr(x) = x + \bar{x} = 1 \iff \zeta(x) + \zeta(\bar{x}) = 1 = Tr(\zeta x)$$

$$\frac{Tr(x)}{N(x)} = \frac{1}{x} + \frac{1}{\bar{x}} = 1 \iff \frac{1}{\zeta(x)} + \frac{1}{\zeta(\bar{x})} = 1 = \frac{Tr(\zeta x)}{N(\zeta x)}$$

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We get...

### Correspondence I

$$x\bar{x} = 1 \iff \zeta(x)\zeta(\bar{x}) = 1; \text{ i.e. } N(x) = 1 \iff N(\zeta(x)) = 1$$

$$\implies$$

$\zeta$  restricts to an involution of the set of **units of norm +1** of the rings of integers in quadratic number fields.

$$\zeta \circ \{a + \sqrt{a^2 - 1} \mid 1 < a \in \mathbf{Q}\}$$

We get...

## Correspondence II

$$x + \bar{x} = 0 \iff \zeta(x)\zeta(\bar{x}) = -1; \text{ i.e. } T(x) = 0 \iff N(\zeta(x)) = -1.$$

$\implies \zeta$  establishes a bijection between the set of **square roots of positive rationals** and the set of **units of norm -1** of the rings of integers of quadratic number fields.

$$\zeta : \{\sqrt{q} \mid q \in \mathbf{Q}\} \rightarrow \{a + \sqrt{a^2 + 1} \mid a \in \mathbf{Q}\}$$

.... and these correspondences are far from being trivial:

### Correspondence II-Example

$$\begin{aligned}\sqrt{\frac{39}{17}} &= [1, \overline{1, 1, 16, 1, 1, 2}] \implies \\ \zeta\left(\sqrt{\frac{39}{17}}\right) &= [4, \overline{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 4, 4}] = A \implies \\ N(A) &= N\left(\frac{7663 + \sqrt{70845893}}{3482}\right) = -1.\end{aligned}$$

## Correspondence II-More Examples

$$\begin{aligned}
 \sqrt{N} &\rightarrow \zeta(\sqrt{N}) \\
 \sqrt{3} &\rightarrow \frac{1}{2}(\sqrt{13} + 3) \\
 \sqrt{5} &\rightarrow \frac{1}{3}(\sqrt{10} + 1) \\
 \sqrt{6} &\rightarrow \frac{1}{14}(\sqrt{221} + 5) \\
 \sqrt{7} &\rightarrow \frac{1}{6}(\sqrt{37} + 1) \\
 \sqrt{8} &\rightarrow \frac{1}{4}(\sqrt{17} + 1) \\
 \sqrt{10} &\rightarrow \frac{1}{7}(\sqrt{65} + 4) \\
 \sqrt{11} &\rightarrow \frac{1}{26}(\sqrt{901} + 15) \\
 \sqrt{12} &\rightarrow 134(\sqrt{1517} + 19) \\
 \sqrt{13} &\rightarrow \frac{1}{3}(\sqrt{13} + 2) \\
 \sqrt{14} &\rightarrow \frac{1}{5}(\sqrt{34} + 3) \\
 \sqrt{15} &\rightarrow \frac{1}{18}(\sqrt{445} + 11) \\
 \sqrt{17} &\rightarrow \frac{1}{19}(\sqrt{442} + 9)
 \end{aligned}$$

We get...

### Correspondence III

$$x + y = 1 \iff \zeta(x) + \zeta(\bar{x}) = 1; \text{ i.e. } T(x) = 1 \iff T(\zeta(x)) = 1$$

$$\zeta \circ \left\{ \frac{1}{2} + \sqrt{a} \mid 0 < a \in \mathbf{Q} \right\}$$

We get...

### Correspondence IV

$$\frac{1}{x} + \frac{1}{\bar{x}} = 1 \iff \frac{1}{\mathfrak{C}(x)} + \frac{1}{\mathfrak{C}(\bar{x})} = 1; \text{ i.e. } T\left(\frac{1}{x}\right) = 1 \iff T\left(\frac{1}{\mathfrak{C}(x)}\right) = 1$$

$$T(x) = N(x) \iff T(\mathfrak{C}x) = N(\mathfrak{C}x)$$

Equivalently,

$$\mathfrak{C} \circ \{a + \sqrt{a^2 - 2a} \mid 1 < a \in \mathbf{Q}\}$$

... and there are more correspondences of this type



What about algebraic numbers of higher degree?

### Conjecture

If  $x$  is algebraic of degree  $> 2$ , then  $\zeta(x)$  is transcendental<sup>a</sup>

---

<sup>a</sup>Testing the transcendence conjecture of Jimm and its continued fraction statistics (joint with H. Ayral, to appear)

**Why?** Because if  $x$  algebraic of degree  $> 2$ , then it is widely believed that  $x$  obeys the Gauss-Kuzmin statistics.

- $\implies$  the frequency of  $1$ 's in the continued fraction of  $\zeta(x)$  is  $1$ .
- $\implies \zeta(x)$  does not obey the GK statistics
- $\implies \zeta(x)$  is can not be algebraic.

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There is also a much bolder version of this transcendence conjecture:

**Strong transcendence conjecture:** In addition to the transcendence conjecture, any set of algebraically related numbers in the set

$$S := \{\zeta(x) \mid x \in \overline{\mathbf{Q}}, \deg(x) > 2\}.$$

are in the same  $\mathrm{PGL}_2(\mathbf{Z})$ -orbit.

For example,  $\zeta(x)$  and  $\zeta(\frac{ax+b}{cx+d})$  are (provably) algebraically dependent for any  $x \in \mathbf{R}$ , whereas if  $x \in S$ , then  $\zeta(x)$ ,  $\zeta(x^2)$  and  $\zeta(2x)$  are (conjecturally) not.

## A few examples...

$$\begin{aligned}\zeta(\sqrt[3]{2}) &= \zeta([1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, \dots]) \\ &= [2, 1, 3, 1, 1, 1, 4, 1, 1, 4, 1_6, 3, 1_{12}, 3, 1_8, 2, 3, 1, 1, 2, \dots] \\ &= 2.784731558662723 \dots\end{aligned}$$

$$\begin{aligned}\zeta(\pi) &= \zeta([3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \dots]) = \\ &= [1_2, 2, 1_5, 2, 1_{13}, 3, 1_{290}, 5, 3, \dots] \\ &= 1.7237707925480276079699326494931025145558144289232 \dots\end{aligned}$$

$$\begin{aligned}\zeta(e) &= \zeta([2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]) = \\ &= [1, 3, 4, 1, 1, 4, 1, 1, 1, 1, \dots, \overline{4, 1_{2n}}] \\ &= 1.3105752928466255215822495496939143349712038085627 \dots\end{aligned}$$

(We tried to recognize these numbers by the PSLQ-algorithm with various sets of constants—we couldn't get any results)

# PART II

## Dynamics



# Dynamics

## Fact

$\zeta$  conjugates the Gauss map to the “Fibonacci map”


$$T_{Gauss} : [0, n_1, n_2, n_3, \dots] \in [0, 1] \longrightarrow [0, n_2, n_3, n_4, \dots] \in [0, 1]$$

$$\implies$$

$$T_{Fibonacci} = \zeta T_{Gauss} \zeta : [0, 1_k, n_{k+1}, n_{k+2}, \dots] \rightarrow [0, n_{k+1} - 1, n_{k+2}, \dots]$$


# Dynamics

The Gauss map

[ 1, 1, 1, 6, 13, 2, 2, 7, ...]

# Dynamics

The Gauss map

  $[1, 1, 6, 13, 2, 2, 7, \dots]$




# Dynamics

The Gauss map

$$[\text{Pac-Man}, 1, 6, 13, 2, 2, 7, \dots]$$


# Dynamics

The Gauss map

[ 6, 13, 2, 2, 7, ...]

# Dynamics

The Gauss map

[ 13, 2, 2, 7, ...]

# Dynamics

The Gauss map

$$[\text{Pac-Man}, 2, 2, 7 \dots]$$

# Dynamics

The Gauss map

$$[\text{Pac-Man}, 2, 7 \dots]$$


# Dynamics

The Gauss map

$$[\text{Pac-Man}, 7 \dots]$$


# Dynamics

The Fibonacci map

[ 1, 1, 1, 6, 13, 2, 2, 7, ...]

# Dynamics


The Fibonacci map

[ 5, 13, 2, 2, 7, ...]




# Dynamics

The Fibonacci map

[ 4, 13, 2, 2, 7, ...]


# Dynamics

The Fibonacci map

[ 3, 13, 2, 2, 7, ...]


# Dynamics

The Fibonacci map

[ 2, 13, 2, 2, 7, ...]


# Dynamics

The Fibonacci map

[1, 13, 2, 2, 7, ...]


# Dynamics

The Fibonacci map

[ 12, 2, 2, 7, ...]


# Dynamics

The Fibonacci map

[ 11, 2, 2, 7, ...]


# Dynamics

The Fibonacci map

[ 10, 2, 2, 7, ...]

# Dynamics


The Fibonacci map

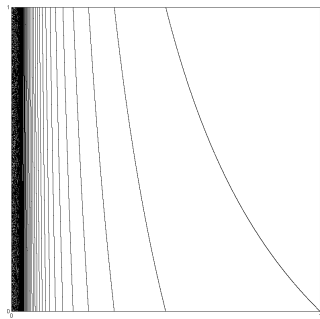
[ 9, 2, 2, 7, ...]



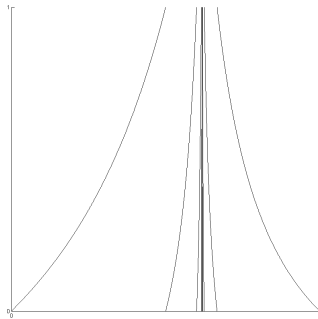
# Dynamics

The Fibonacci map

[ 8, 2, 2, 7, ...]



**Gauss map**



**Fibonacci map**

Dynamics of these two maps are closely related (Isola et al).

The transfer operator of the Fibonacci map is

$$(\mathcal{L}_s^{Fib}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(F_{k+1}y + F_k)^{2s}} \psi\left(\frac{F_k y + F_{k-1}}{F_{k+1}y + F_k}\right) \quad (1)$$

The transfer operator of the Gauss map is

$$(\mathcal{L}_s^{Gauss}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^{2s}} \psi\left(\frac{1}{k+x}\right) \quad (2)$$

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## A.C. invariant measures

$$T_{Fibonacci} \leftrightarrow \frac{1}{x(x+1)} \text{ (infinite)}, \quad T_{Gauss} \leftrightarrow \frac{1}{x+1}$$

**Zeta functions** (the transfer operator evaluated at Lebesgue's measure)

$$T_{Fibonacci} \leftrightarrow (\mathcal{L}_s^{Fib} \psi)(\mathbf{1}) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad (\text{"Fibonacci zeta"})$$

$$T_{Gauss} \leftrightarrow (\mathcal{L}_s^{Gauss} \psi)(\mathbf{1}) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\text{"Riemann zeta"})$$

Eigenfunctions of the Fibonacci transfer operator satisfies the three-term functional equation

$$\psi(y) = \frac{1}{y^{2s}} \psi\left(\frac{y+1}{y}\right) + \frac{1}{\lambda} \frac{1}{(y+1)^{2s}} \psi\left(\frac{y}{y+1}\right) \quad (3)$$

(Equivalent to three-term functional equation studied by Lewis and Zagier)

# Dynamics

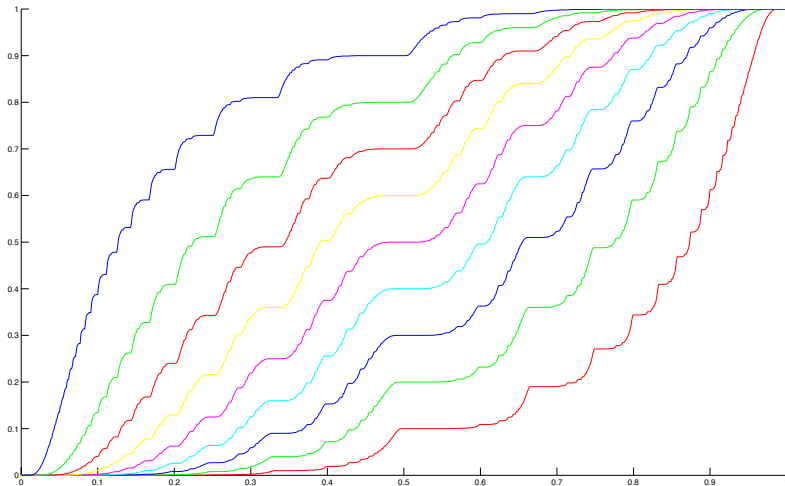
The Denjoy-Minkowski measure is the measure whose cumulative distribution function is

$$\mu([0, n_1, n_2, \dots, n_{k-1}, n_k]) = \sum_{k=1}^{\infty} (-1)^{1+k} 2^{-n_1 - n_2 \cdots - n_k}. \quad (4)$$

$\mu$  is a common invariant measure for the Gauss and the Fibonacci maps (however, it is not absolutely continuous w.r.t Lebesgue's measure).

Actually,  $\mu$  is the common invariant measure of a much wider class of maps.

# Dynamics



Plots of the cumulative laws of Denjoy-Minkowski measures  $F_p$   
( $p = 0.1, 0.2, \dots 0.9$ )



There is a common generalization of the Gauss and Fibonacci maps:

$$\mathbb{T}_\alpha(x) = \begin{cases} [0, m_{k+1}, m_{k+2}, m_{k+3}, \dots] & n_k > m_k (*) \\ [0, m_k - n_k, m_{k+1}, m_{k+2}, \dots] & n_k < m_k (**) \end{cases} \quad (5)$$

where  $\alpha = [0, n_1, n_2, \dots]$  and  $x = [0, m_1, m_2, \dots]$ .

One has

$$\mathbb{T}_0 = \mathbb{T}_{Gauss}, \quad \mathbb{T}_{\Phi^*} = \mathbb{T}_{Fibonacci}$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{Pac-Man}, 1, 1, 1, 6, 13, 2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{C}1, 1, 6, 13, 2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{C}, 1, 6, 13, 2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{◐}, 6, 13, 2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{C}, 4, 13, 2, 2, 7, \dots]$$



# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{C}11, 2, 2, 7, \dots]$$



# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2}-1 = [0, 2, 2, 2, \dots]$ .

$$[\text{C}, 9, 2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{C}, 7, 2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2}-1 = [0, 2, 2, 2, \dots]$ .

$$[\text{⦿} 5, 2, 2, 7, \dots]$$

# Dynamics

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{◐}, 3, 2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{C}1, 2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{⦿}2, 2, 7, \dots]$$

# Dynamics

## Example

The map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2} - 1 = [0, 2, 2, 2, \dots]$ .

$$[\text{⦿}5, \dots]$$

# Dynamics

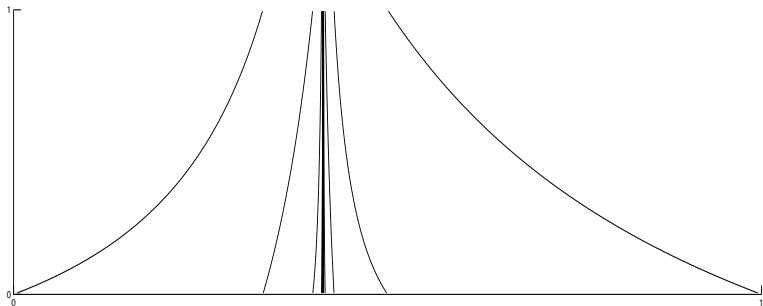


Figure: Plot of  $\mathbb{T}_{\sqrt{2}-1}$



# Dynamics

The following functional equation is satisfied:

$$\mathbb{T}_{\zeta(\alpha)}(\zeta x) = \zeta \mathbb{T}_\alpha(x).$$

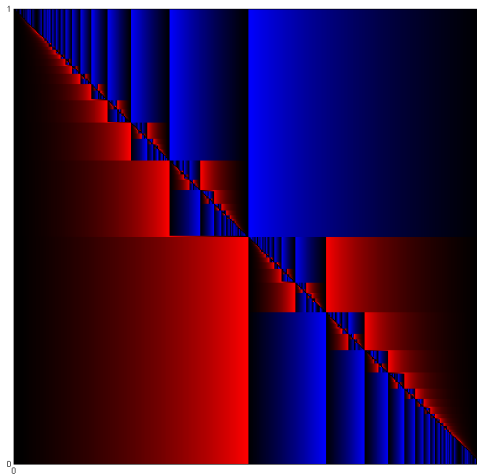
For  $\alpha = \Phi^*$ , this specialises to

$$\mathbb{T}_0(\zeta x) = \zeta \mathbb{T}_{\Phi^*}(x) \iff \zeta \mathbb{T}_0(\zeta x) = \mathbb{T}_{\Phi^*}(x)$$

i.e. the fact that  $\zeta$  conjugates the Gauss and the Fibonacci maps.

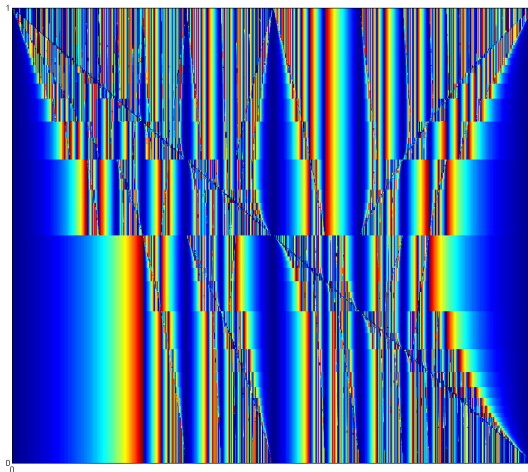
$$(\text{recall that } \mathbb{T}_0 = \mathbb{T}_{\text{Gauss}}, \quad \mathbb{T}_{\Phi^*} = \mathbb{T}_{\text{Fibonacci}})$$

# Dynamics



Plot of  $T_\alpha(x)$  as a function of  $\alpha$  and  $x$ .  
 The intensity is proportional to the value of  $T_\alpha(x)$ .  
 The symmetry is due to  $T_{1-\alpha}(1-x) = T_\alpha(x)$ .

# Dynamics



Third iteration of  $\mathbb{T}_\alpha(x)$ . The intensity is proportional to the value of  $T_\alpha^3(x)$ .

# Dynamics

The transfer operator is  $(\mathcal{L}_{s,\alpha}\psi)(y) =$

$$-\frac{1}{y^{2s}}\psi\left(\frac{1}{y}\right) + \sum_{k=1}^{\infty} \sum_{i=0}^{n_k-1} \left| \frac{d}{dy}[0, n_1, \dots, n_{k-1}, i+y] \right|^s \psi[0, n_1, \dots, n_{k-1}, i+y]$$

eigenfunctions of which satisfy the functional equation

$$\psi(y) - \psi(1+y) + \frac{1}{y^{2s}} \left\{ \psi\left(\frac{1}{y}\right) - \psi\left(1 + \frac{1}{y}\right) \right\} =$$

$$\frac{1}{\lambda(1+y)^{2s}} \left\{ \psi\left(\frac{y}{1+y}\right) + \psi\left(\frac{1}{1+y}\right) \right\}$$

Observe that the LHS=0 is precisely Lewis' three-term functional equation, and the RHS is Isola's transfer operator of the Farey map.

# Dynamics

## Example

For the map  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2}-1 = [0, 2, 2, 2, \dots]$  we have

$$\mathcal{L}_{s,\alpha}\psi(y) = \sum_{i=1}^{\infty} \frac{1}{(P_{i+1}y + P_i)^s} \psi\left(\frac{P_i y + P_{i-1}}{P_{i+1}y + P_i}\right) +$$

$$\sum_{j=1}^{\infty} \frac{1}{(P_{j+1}y + P_{j+1} + P_j)^s} \psi\left(\frac{P_j y + P_j + P_{j-1}}{P_{j+1}y + P_{j+1} + P_j}\right),$$

where  $0, 1, 2, 5, 12, 29, 70, 169, 408, \dots$  is the Pell sequence defined by  $P_0 = 0$ ,  $P_1 = 1$  and  $P_k = 2P_{k-1} + P_{k-2}$ .

# Dynamics

## Example

An a.c. invariant measure for  $\mathbb{T}_{\sqrt{2}-1}$  with  $\sqrt{2}-1 = [0, 2, 2, 2, \dots]$

$$\psi(y) = \sum_{i=0}^{\infty} \frac{1}{(1+2iy)(1+2y+2iy)} - \frac{1}{(y+2i+3)(y+2i+2)}.$$

## Questions.

- What are the a.c. invariant measures for  $\mathbb{T}_{\alpha}$  in general?
- How are the dynamics of  $\mathbb{C}$ -conjugate maps related?
- Same questions for the continued fraction maps defined below

# Dynamics

**Fact:** Denjoy-Minkowski measure is a common invariant measure of all  $\mathbb{T}_\alpha$ 's.

In fact, this is true for an even wider class of maps (called **continued fraction maps**)  $T : [0, 1] \mapsto [0, 1]$ , whose inverse branches are all  $\text{PGL}_2(\mathbf{Z})$  on  $[0, 1]$ . These are generalized Pacman maps (i.e. pacman with powers equal to several  $\mathbb{T}_\alpha$ -pacmen combined)

(There is a systematic way to define these maps as topological covering maps of the boundary of the Farey tree)

# Dynamics

Indeed, suppose the inverse branches of  $T$  are  $\{\varphi_\beta\}_{\beta=1,2,\dots}$ . Then each  $\varphi_\beta$  can be written as

$$\varphi_\beta(y) = [0, n_1, n_2, \dots, n_{k-1}, i + y],$$

where  $0 < k, n_1, n_2, \dots$  and  $0 \leq i$  depends on  $\beta$ . Suppose  $X$  is a random variable on  $[0, 1]$  with law  $\mathbf{?}$  and set  $Y := T(X)$ . The law  $\mathbf{F}_Y$  of  $Y$  is

$$\begin{aligned} \mathbf{F}_Y(y) &= \text{Prob}\{Y \leq y\} = \text{Prob}\{T(X) \leq y\} = \sum_{\beta} |\mathbf{?}(\varphi_\beta(y)) - \mathbf{?}(\varphi_\beta(0))| \\ &= \sum_{\beta} |\mathbf{?}[0, n_1, n_2, \dots, n_{k-1}, i + y] - \mathbf{?}[0, n_1, n_2, \dots, n_{k-1}, i]| \\ &= \sum_{\beta} \mathbf{?}(y) 2^{-(n_1 + \dots + n_{k-1} + i)} \implies \mathbf{F}_Y(y) = \mathbf{?}(y) \sum_{\beta} 2^{-(n_1 + \dots + n_{k-1} + i)}, \end{aligned}$$

and the series of the last line *must* sum up to 1, because  $\mathbf{F}_Y$  and  $\mathbf{?}(y)$  are both probability laws.

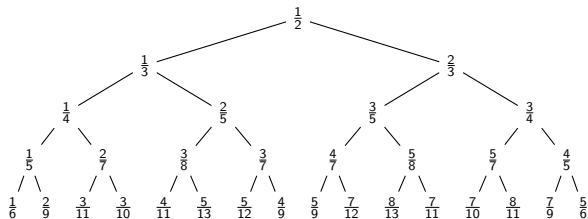


## PART III

# Tree automorphisms and Lebesgue's measure

# $\mathcal{T}$ as a tree automorphism

## The Farey tree

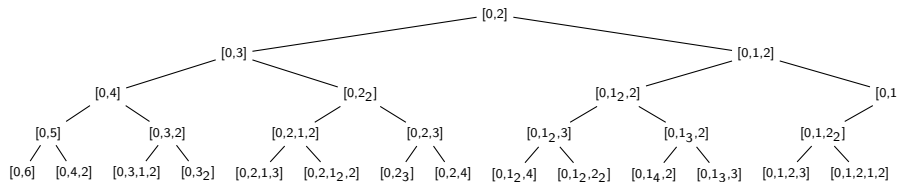


Produced by the Farey sum rule:

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$$

# $\tau$ as a tree automorphism

## The Farey tree by continued fractions



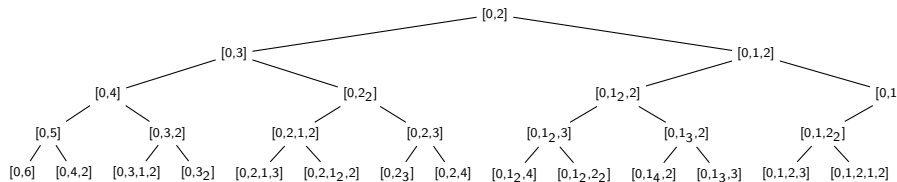
The boundary  $\partial\mathcal{F}$  is the set of all infinite paths based at the root.

### Fact

The map  $\partial\mathcal{F} \rightarrow [0, 1]$  sending path to its continued fraction, parametrize irrationals in  $[0, 1]$  (and is 2-to-1 over the rationals).

# $\mathcal{T}$ as a tree automorphism

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## $\mathcal{T}$ as a tree automorphism

The automorphism group  $\mathbf{Aut}(\mathcal{F})$  naturally acts on  $\partial\mathcal{F}$ .

$\implies \mathbf{Aut}(\mathcal{F})$  acts on continued fractions via the above identification.  
(ignoring a countable set of numbers for each automorphism).

## $\tau$ as a tree automorphism

Shuffle description of **Aut**( $\mathcal{F}$ ).

$\implies \tau$  is the automorphism which shuffles every other vertex.

## $\mathcal{T}$ as a tree automorphism

Shuffle description of **Aut**( $\mathcal{F}$ ).

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## $\mathcal{T}$ as a tree automorphism

# Twist description of $\mathbf{Aut}(\mathcal{F})$

$\Rightarrow \mathcal{T}$  is the automorphism which twists every vertex.



## $\mathcal{T}$ as a tree automorphism

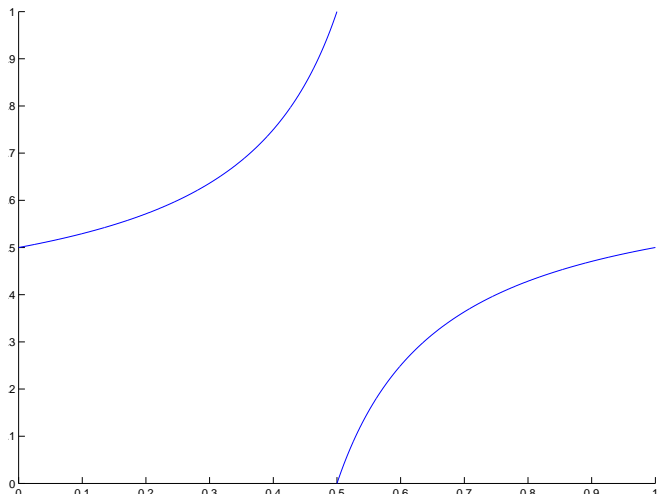
Twist description of **Aut**( $\mathcal{F}$ )

$\Rightarrow \mathcal{T}$  is the automorphism which twists every vertex.

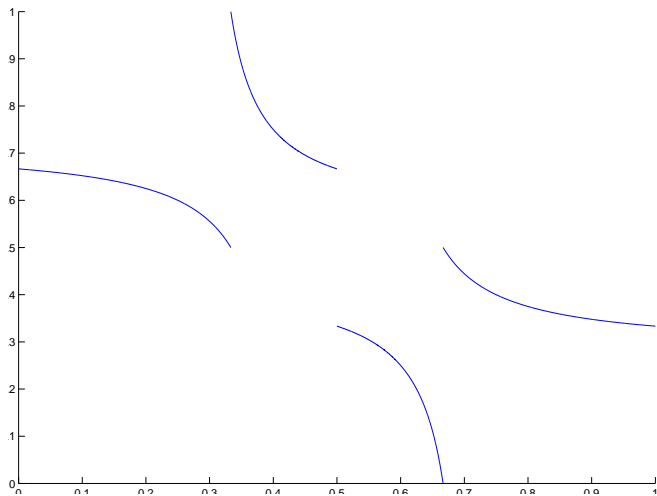
$\zeta$  sends zig-zag segments on a path to straight segments and vice versa

Looking at the boundary actions of shuffles (or twists), yields a presentation of  $\zeta$  as a limit of piecewise- $\mathrm{PGL}_2(\mathbf{Z})$  maps....

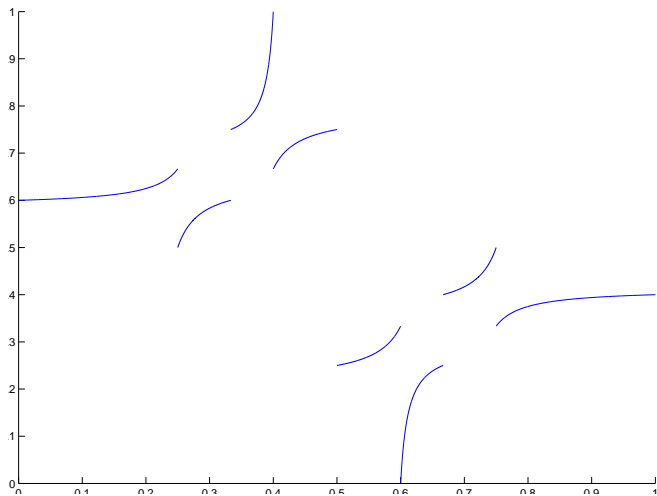
# Jimm as a limit of piecewise-PGL<sub>2</sub>( $\mathbb{Z}$ ) maps



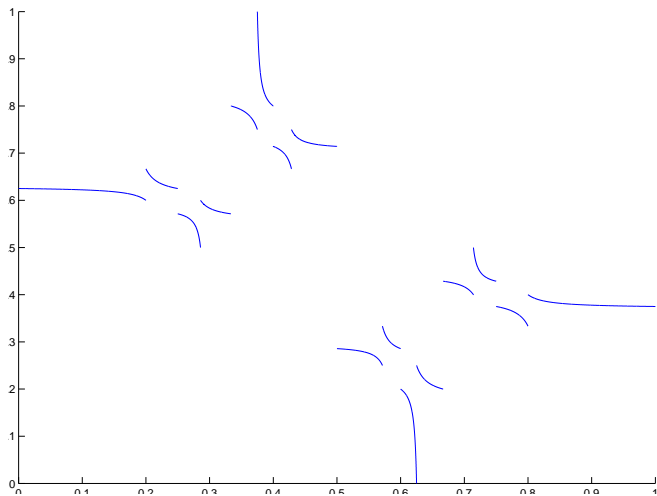
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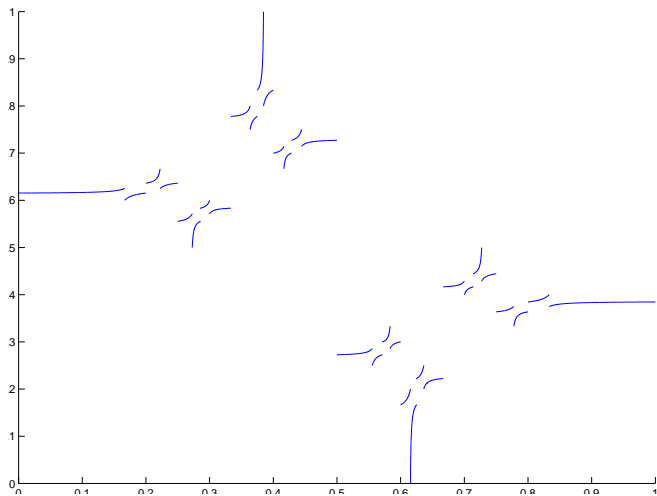
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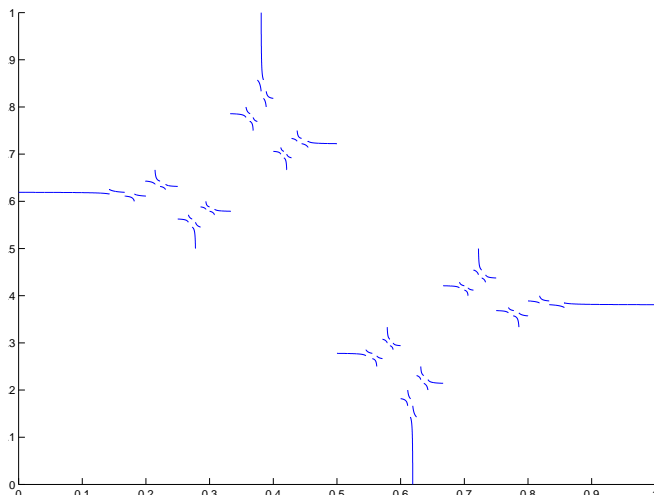
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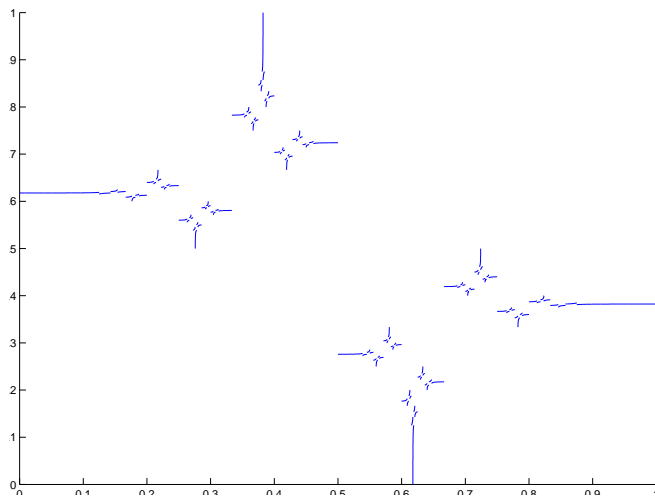


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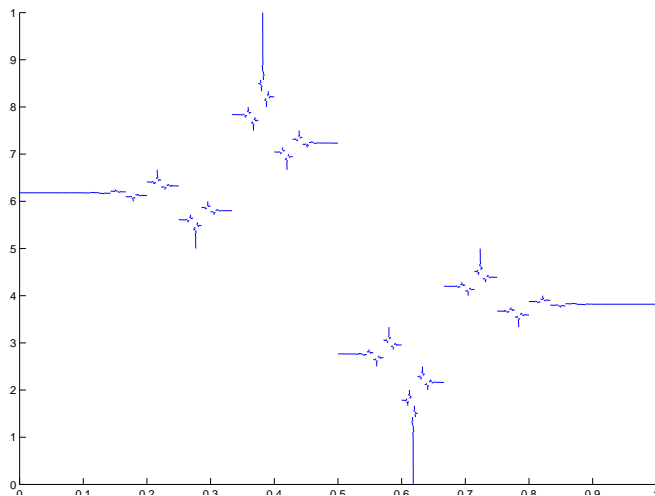




# Jimm as a limit of piecewise-PGL<sub>2</sub>( $\mathbb{Z}$ ) maps

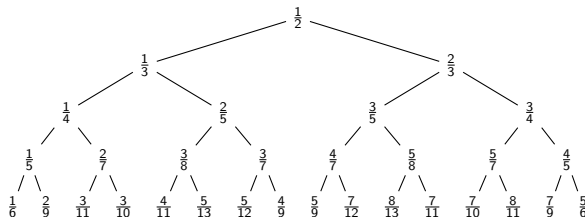


# Jimm as a limit of piecewise-PGL<sub>2</sub>( $\mathbb{Z}$ ) maps



# $\tau$ as a symmetry of Lebesgue's measure

Let's turn back to the Farey tree...

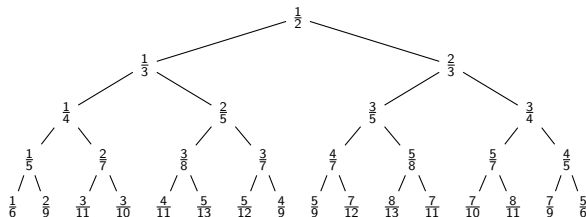


A random walker starts from the root vertex. For each vertex  $x$ , we are given the probability  $\pi(x)$  of **arriving** to that vertex from its parent.

This induces a measure on the set of continued fractions, i.e. on  $[0, 1]$ .

# $\tau$ as a symmetry of Lebesgue's measure

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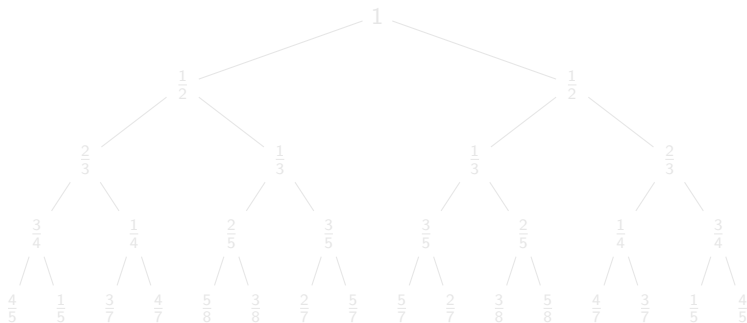
## $\tau$ as a symmetry of Lebesgue's measure

If we set  $\pi(x) \equiv 1/2$ , then the c.d.f. of the induced measure on  $[0, 1]$  is the Minkowski-Denjoy measure.  
(which by the way is the unique  $\text{Aut}(\mathcal{F})$ -invariant measure on  $\partial\mathcal{F}$ .)

# $\tau$ as a symmetry of Lebesgue's measure

## Question

Which 'arrival' probability function  $\pi_{\text{Leb}}(x)$  induce the Lebesgue measure?

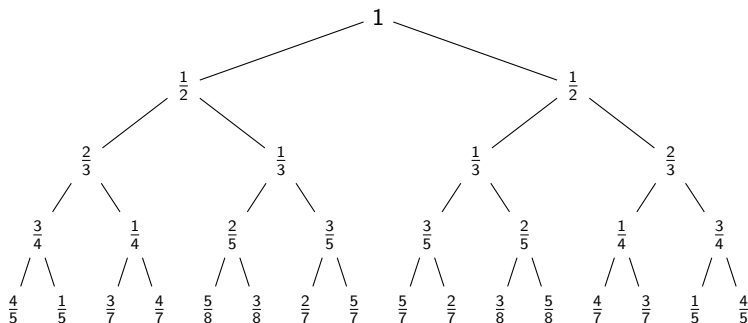


The "Lebesgue tree"  $\mathcal{L}$ .

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The "Lebesgue tree"  $\mathcal{L}$ .

## Answer

Assume  $n_k > 1$ . Then the arrival probabilities

$$\pi_{Leb}([0, n_1, n_2, \dots, n_{k-1}, n_k]) = 1 - [0, n_k - 1, n_{k-1}, \dots, n_2, n_1]$$

induces the Lebesgue measure on  $[0, 1]$ .



A subtle symmetry of Lebesgue's measure:

$$\pi_{Leb} \mathfrak{I}(x) = \mathfrak{I} \pi_{Leb}(x)$$

(On the left hand side  $\mathfrak{I}$  acts on the tree whereas on the right it acts on the rationals)

# Three singular measures

How does this symmetry manifests itself on the superficial level?

There are many questions pertaining to the measures induced by the transition functions

- $\pi(x) := K\pi_\lambda(x)$
- $\pi(x) := \mathcal{C}\pi_\lambda(x) = \pi_\lambda\mathcal{C}(x)$
- $\pi(x) := K\mathcal{C}\pi_\lambda(x) = \mathcal{C}K\pi_\lambda(x).$

These are, in a sense, basic deformations of Lebesgue's measure.

# Three singular measures

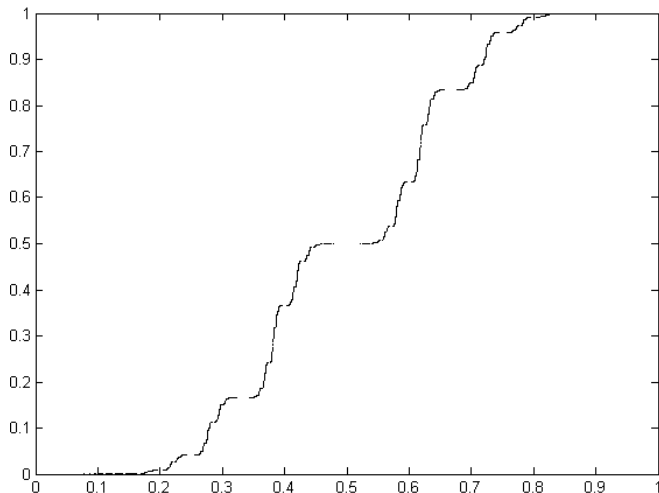


Figure: c.d.f. of  $K\pi_\lambda$

# Three singular measures

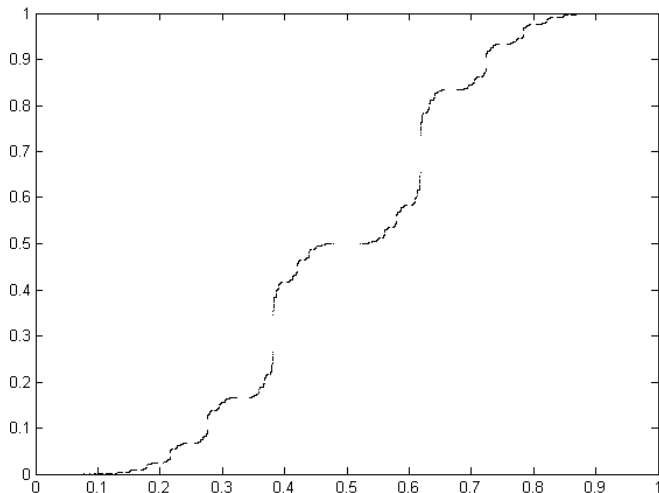


Figure: c.d.f. of  $\zeta\pi_\lambda = \pi_\lambda\zeta$

# Three singular measures

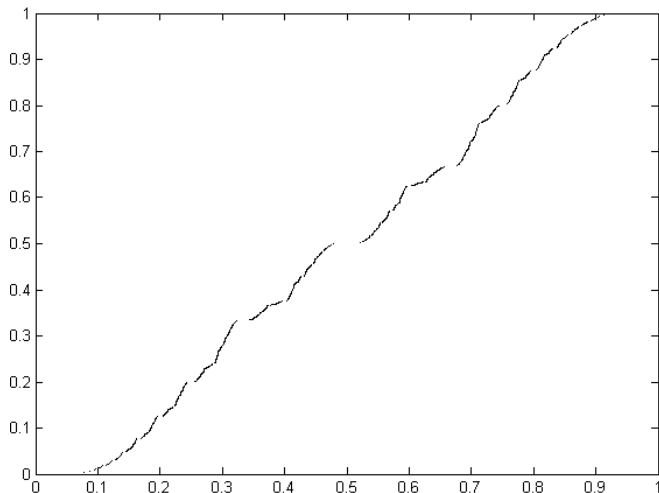


Figure: c.d.f. of  $K\mathfrak{C}\pi_\lambda = \mathfrak{C}K\pi_\lambda$

# Three singular measures

## Some Questions

- Are the measures  $\pi(x) := K\pi_\lambda(x)$  and  $\pi(x) := \zeta\pi_\lambda(x) = \pi_\lambda\zeta(x)$  singular with respect to Lebesgue's measure? Denjoy-Minkowski measure?
- How do these measure behave under the continued fraction maps?

# More measures

Recall  $Kx := 1 - x$  and define the flip operation on  $\mathbf{Q} \cap (0, 1)$  as

$$\varphi([0, n_1, n_2, \dots, n_k]) = [0, n_k - 1, n_{k-1}, \dots, n_2, n_1 + 1]$$

where it is assumed that  $n_k > 1$ .

Let  $T_F$  be the *Farey map*

$$T_F : (n_1, n_2, \dots, n_{k-1}, n_k) \in X \rightarrow (n_1 - 1, n_2, \dots, n_{k-1}, n_k) \in X, \quad (6)$$

Then

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$$\pi_{Leb}(r) = K\varphi T_F(r)$$

# More measures

## Lemma

- (i)  $(K\varphi)^4 = Id$ .
- (ii) Both  $K$  and  $\mathfrak{C}$  preserves the relations  $x + y = 1$  and the relation of being sibling (this latter is preserved with any automorphism)
- (iii) If  $\pi$  is any measure, then so are  $K\pi$ ,  $\varphi K\varphi\pi$ ,  $K\varphi K\varphi\pi$
- (iv)  $x, y$  are siblings if and only if  $\varphi(x) + \varphi(y) = 1$ .

This lemma permits us to construct a limited number of deformations of the Lebesgue measure.

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- *On the involution of the real line induced by Dyer's outer automorphism of  $PGL(2, \mathbb{Z})$ .* (with H. Ayrál) arXiv:1605.03717
- *A subtle symmetry of Lebesgue's measure.* (with H. Ayrál) arXiv:1605.07330
- *Testing the transcendence conjecture of Jimm and its continued fraction statistics.* (with H. Ayrál, to appear)
- *An involution of reals, discontinuous on rationals and whose derivative vanish almost everywhere.* (with H. Ayrál, to appear)
- Some deformations of Lebesgue's measure on the boundary of the Farey tree (with H. Ayrál, in progress)
- Dynamics of a family of continued fraction maps (with H. Ayrál, in progress)
- Conumerator and the conominator, in progress.

*M E R C I*

# BONUS MATERIAL

$\mathcal{C}$  acts on..

- Binary quadratic forms (tears apart class groups)
- Beatty partitions of  $\mathbf{N}$ .

$$r \in \mathbf{R} \setminus \mathbf{Q} \rightsquigarrow \mathcal{B}_r = \lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, \dots = (\lfloor nr \rfloor)_{n \geq 1}$$

If  $r > 1$  and  $\frac{1}{r} + \frac{1}{s} = 1$  then  $\mathcal{B}_r \cup \mathcal{B}_s = \mathbf{N}$ .

(  $\implies \mathcal{C}$  induce a duality of Beatty partitions of  $\mathbf{N}$ ).

- Sturmian words  $a_n := \lfloor r(n+1) \rfloor - \lfloor rn \rfloor$ .
- Trivalent ribbon graphs  $\simeq$  dessins  $\simeq$  decorated TM spaces.  
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**Example.**

$$\mathfrak{Z}([0; \overline{1_{n-1}, a}]) = [0; n, \overline{1_{a-2}, n+1}] \implies$$

$$\begin{aligned} \mathfrak{Z}\left(\frac{a}{2} \left[ \sqrt{1 + 4 \frac{aF_{n-1} + F_{n-2}}{a^2 F_n}} - 1 \right] \right) \\ = \frac{1}{n + \frac{n+1}{2} \left( \sqrt{1 + 4 \frac{(n+1)F_{a-2} + F_{a-3}}{(n+1)^2 F_{a-1}}} - 1 \right)} \end{aligned}$$

(notice the exchange  $(a, F_n) \leftrightarrow (F_a, n)$ )

## Functional equations on the upper half plane

One must consider the  $\mathrm{PGL}_2(\mathbf{Z})$ -action on  $\{\mathrm{Im}z > 0\}$  given by

$$M \cdot z := \begin{cases} Mz, & \det(M) = +1 \\ M\bar{z}, & \det(M) = -1 \end{cases}$$

The generators of  $\mathrm{PGL}_2(\mathbf{Z})$  in this representation are

$$\bar{U} : z \rightarrow \frac{1}{\bar{z}}, \quad \bar{V} : z \rightarrow -\bar{z}, \quad \bar{K} : z \rightarrow 1 - \bar{z},$$

and the functional equations become

$$f(\bar{U}) = \bar{U}f, \quad f(\bar{V}) = \bar{U}\bar{V}f, \quad f(\bar{K}) = \bar{K}f,$$

in other words

$$f\left(\frac{1}{\bar{z}}\right) = \frac{1}{f(z)}, \quad f(-\bar{z}) = -\frac{1}{f(z)}, \quad f(1 - \bar{z}) = 1 - \overline{f(z)},$$

## Functional equations on the upper half plane

If  $f$  satisfies the functional equations, i.e.

$$f(M \cdot z) = \zeta(M) \cdot f(z) \implies$$

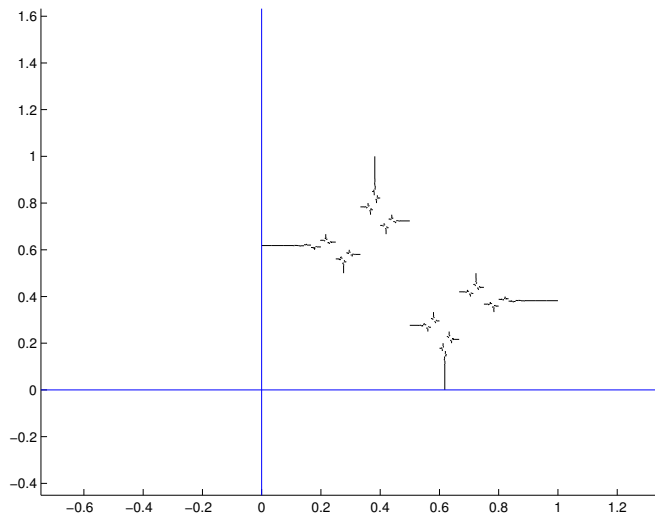
$$f \circ f(M \cdot z) = f(\zeta(M) \cdot f(z)) = M \cdot f(z),$$

in other words,  $f \circ f$  is  $\mathrm{PGL}_2(\mathbf{Z})$ -equivariant. Moreover, if  $g$  is a modular function, then

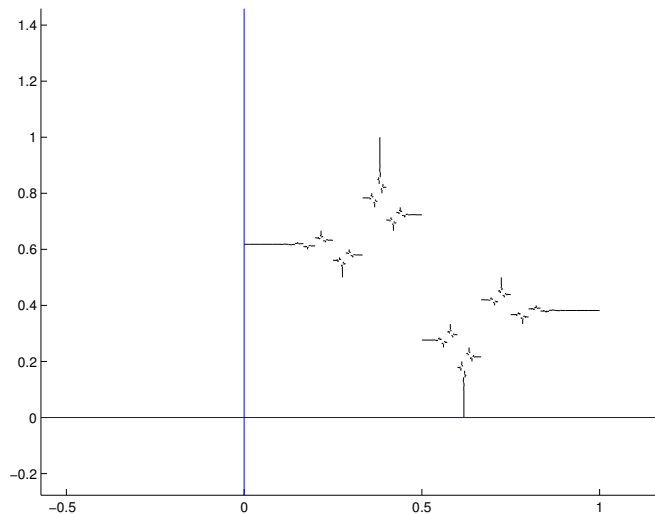
$$g \circ f(M \cdot z) = g(\zeta(M) \cdot f(z)) = f(z),$$

i.e.  $g \circ f$  is also modular.

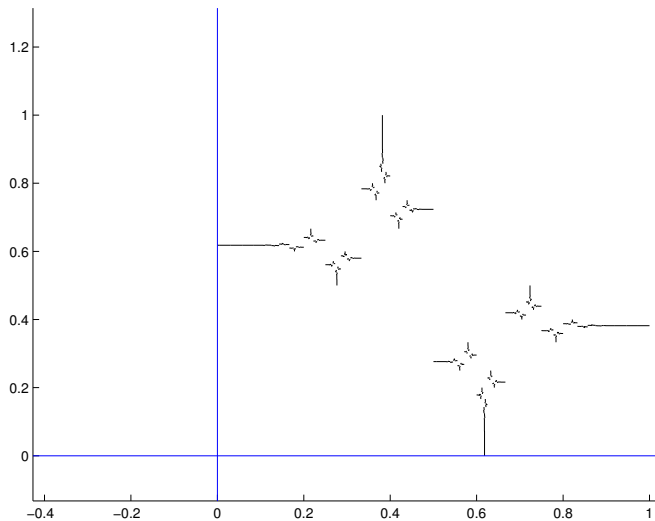
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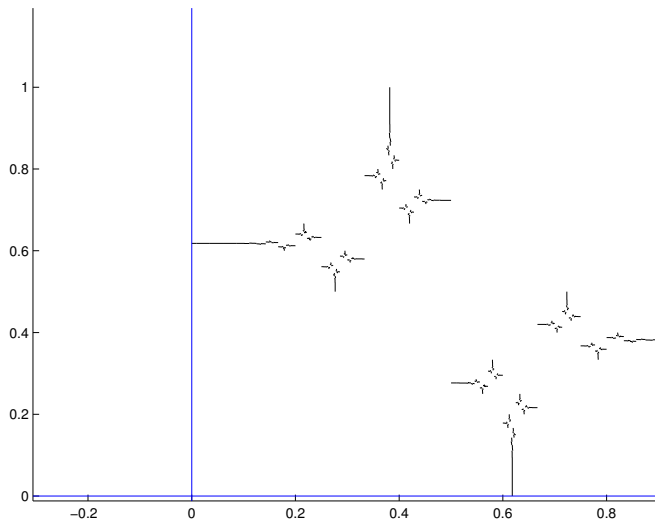


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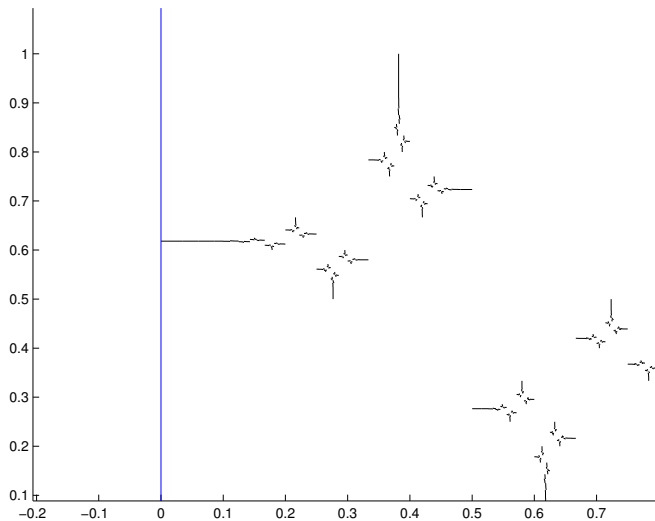




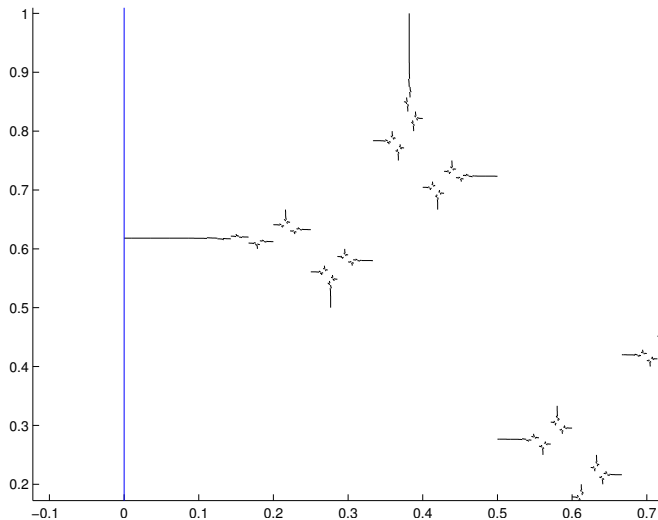
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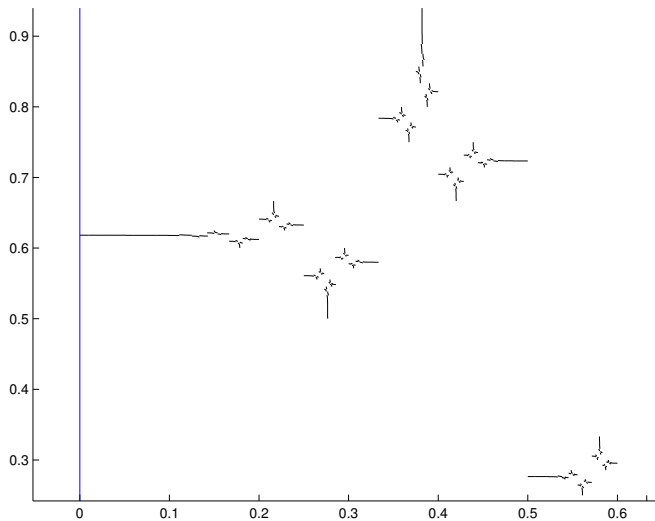
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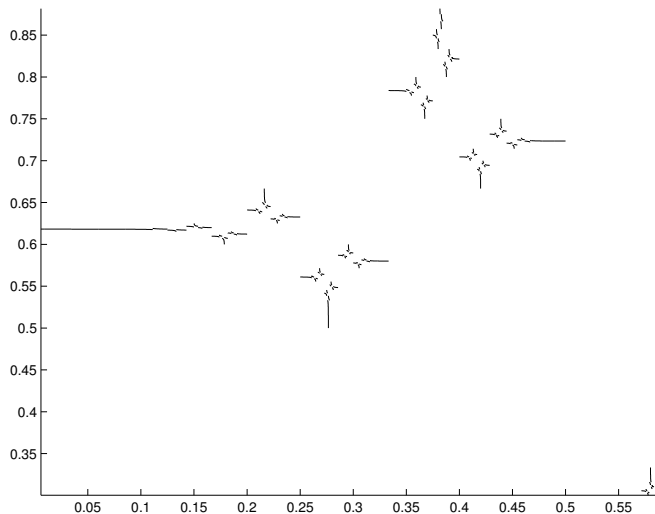
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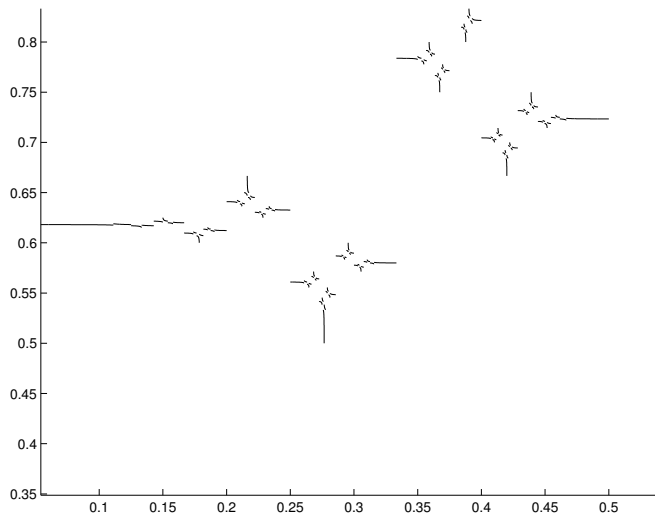
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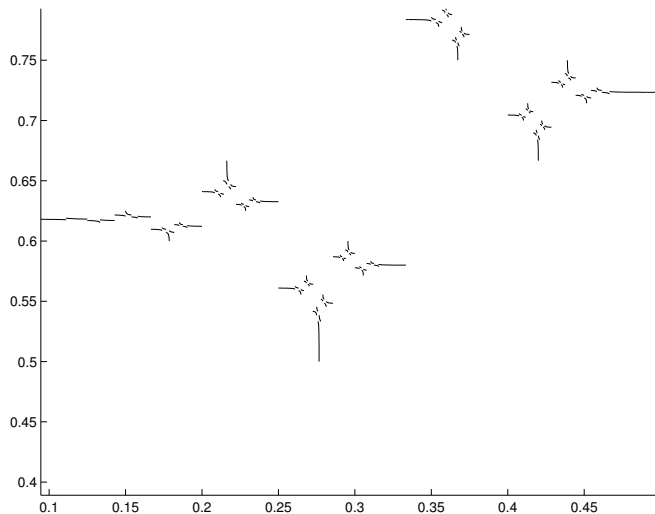
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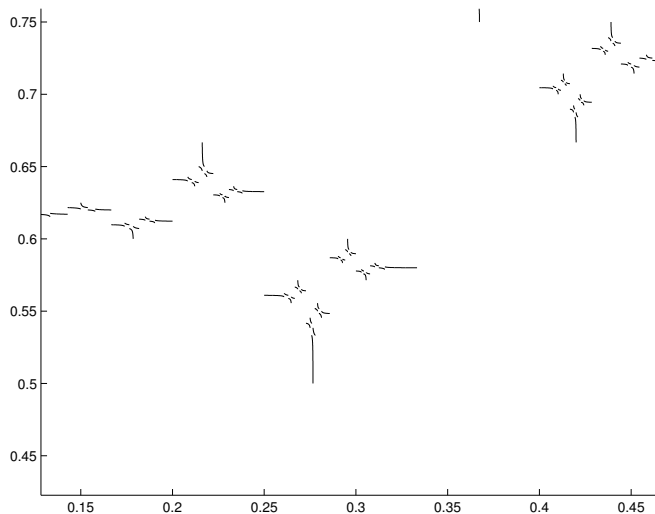
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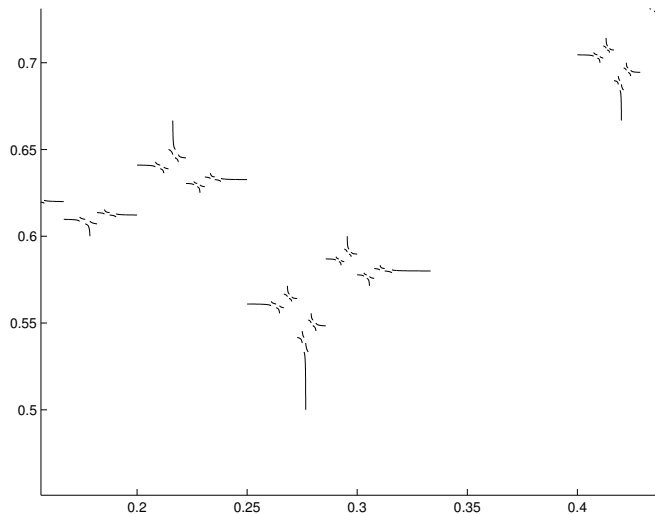


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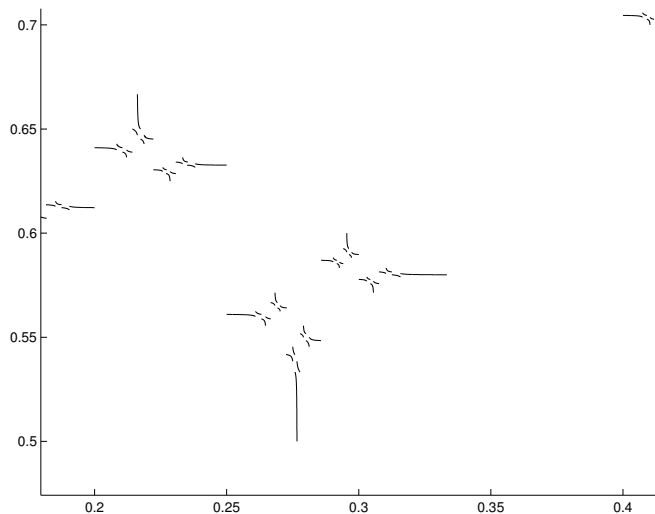




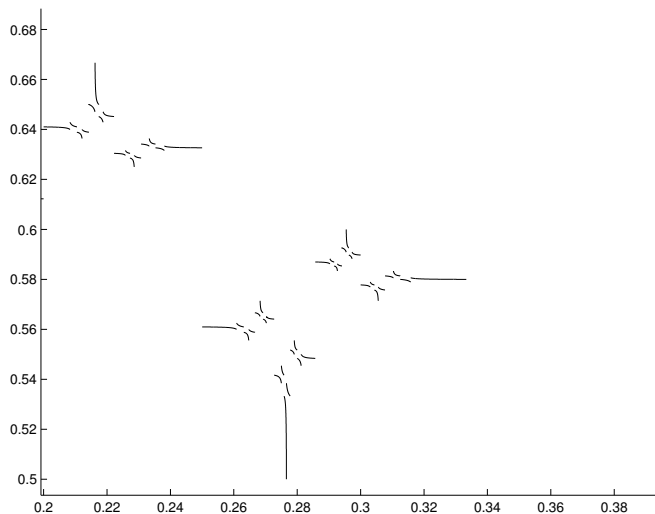
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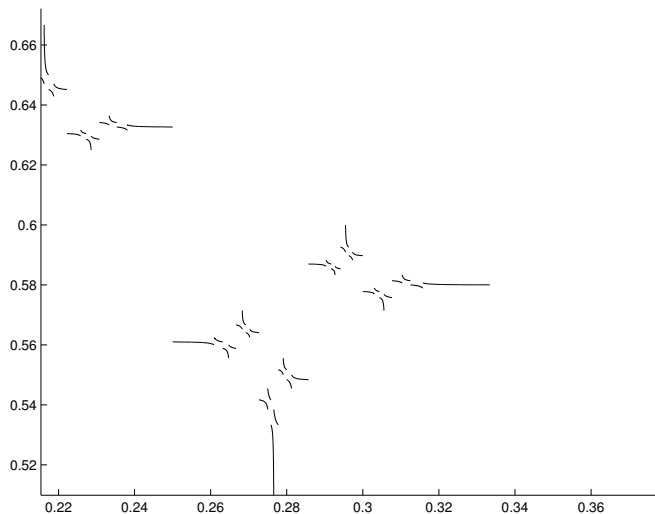
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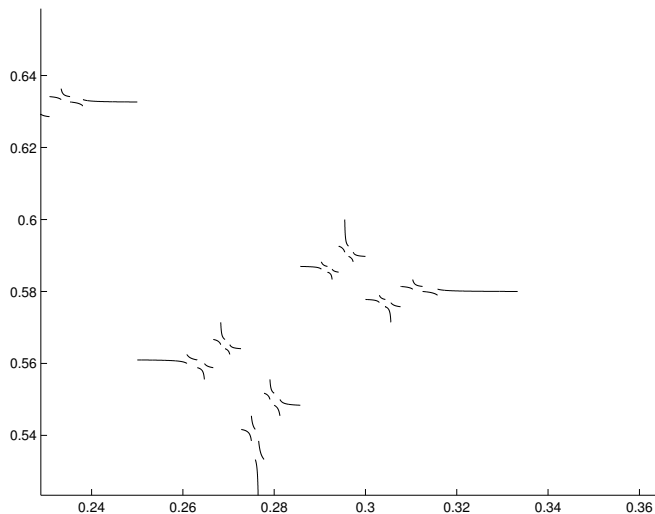
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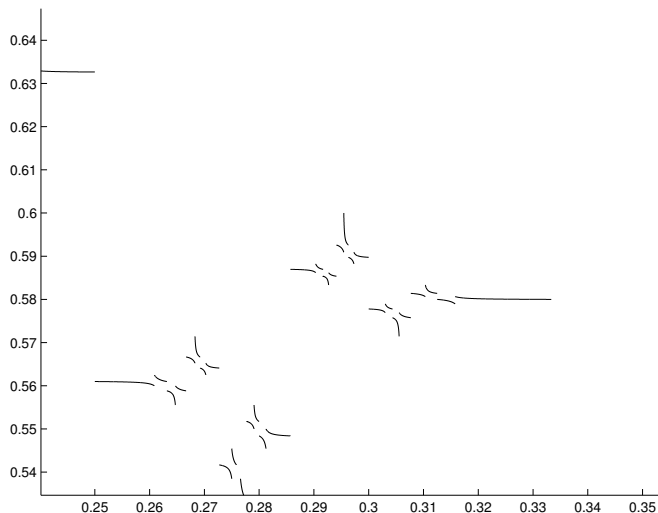
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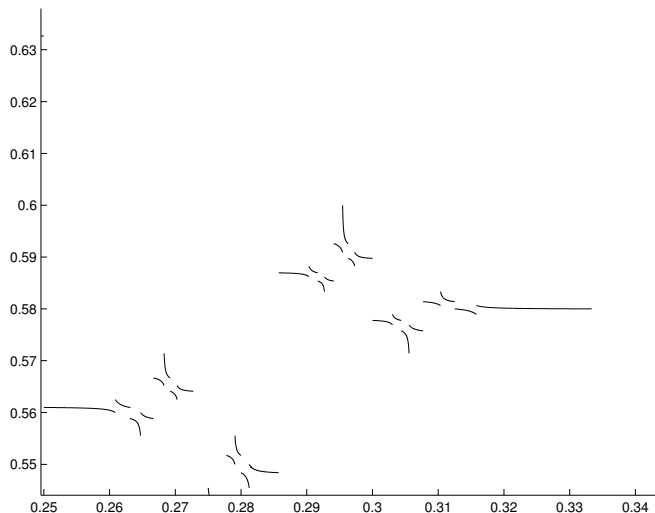
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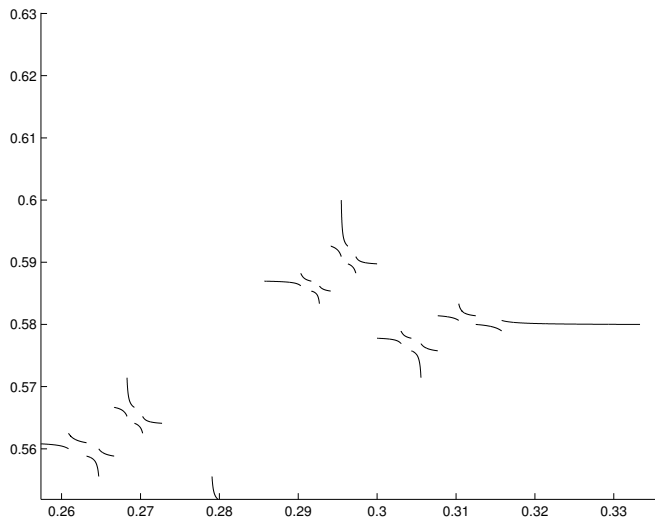
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