Arithmetic and dynamics around the outer automorphism of PGL(2,Z)

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(joint work with Hakan Ayral)

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Rencontre autours de la fonction de Minkowski
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Foreword

In a paper of his on binary quadratic forms, Poincaré states:

"it is not possible, for the indefinite quadratic forms to find invariants, in the sense that we gave to this word..."

Several attempts have been made since then...

Our study can be understood as another attempt to see what can be done by modifying the meaning of the word "invariant"....

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PART I

Definition of Jimm and functional equations

There are two fundamental involutions of the real line $\mathbf{R}$:

$$V : x \in \mathbf{R} \to -x \in \mathbf{R}$$

$$K : x \in \mathbf{R} \to 1 - x \in \mathbf{R}$$

and a third one if we add the point at infinity:

$$U : x \in \mathbf{R} \to 1/x \in \mathbf{R}$$

$$V(x) = -x, \quad K(x) = 1 - x, \quad U(x) = 1/x$$

... together they generate the group

$$\mathrm{PGL}_2(\mathbf{Z}) = \left\{ \frac{px + q}{rx + s} \mid ps - qr = \pm 1, p, q, r, s \in \mathbf{Z} \right\}$$

$$\simeq \langle U, V, K \mid U^2 = V^2 = K^2 = (UV)^2 = (KU)^3 = 1 \rangle$$

Our aim here is to introduce a fourth involution, which we call Jimm

Notation

Every $x \in \mathbf{R}$ can be written as a continued fraction

$$[n_0, n_1, n_2, \dots] = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\dots}}}$$

($n_0 \in \mathbf{Z}$, $n_i \in \mathbf{Z}_{>0}$ for $i > 0$), uniquely if x is irrational.

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Notation

By 1_k we denote the sequence $1, 1, \dots, 1$ of length k.

We introduce a 'singular' function $\mathbf{R} \to \mathbf{R}$:

Definition

$$\text{ج}([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

This is a kind of 'real' modular function, as we shall see.

But let us consider some examples first...

Definition (Recall)

$$\mathfrak{Z}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

Examples

$$\mathfrak{Z}([3, 3, 3, \ldots] = [1_{3-1}, 2, 1_{3-2}, 2, 1_{3-2}, 2 \ldots] = [1, 1, 2, 1, 2, 1, 2, \ldots]$$

$$\mathfrak{Z}([5, 5, 5, \ldots] = [1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, \ldots]$$

Definition (Recall)

$$\text{Ğ}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

This definition works only if $n_k \geq 2$. To make it work for $n_k = 2$, use

RULE I

$$\ldots, n, 1_0, m, \cdots = \ldots, n, m, \ldots$$

Examples

$$\text{Ğ}([2, 2, 2, \ldots]) = [1, 2, 1_0, 2, 1_0, 2 \ldots] = [1, 2, 2, 2, \ldots]$$

$$\text{Ğ}([2, 3, 2, 3 \ldots]) = [1, 2, 1, 2, 2, 1, 2, 2, 1, \ldots]$$

Definition (Recall)

$$\text{Ꝣ}([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

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Definition (Recall)

$$\text{ج}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

To make it work also when $n_k = 1$, use

RULE II

$$\ldots, n, 1_{-1}, m, \cdots = \ldots, n+m-1, \ldots$$

Examples

$$\text{ج}([1, 1, 2, 1, 2, 1, 2, \ldots]) =$$

$$[1_0, \underbrace{2, 1_{-1}, 2}_{3}, 1_0, \underbrace{2, 1_{-1}, 2}_{3}, 1_0, \underbrace{2, 1_{-1}, 2}_{3}, \ldots] =$$

$$= [3, 3, 3, \ldots]$$

remember?

Definition (Recall)

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Examples

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$$= [3, 3, 3, \dots]$$

remember?

Definition (Recall)

$$\text{ج}([n_0, n_1, n_2, \dots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \dots]$$

Example

$$\text{ج}([\dots, 7, 1, 1, 1, 1, 13, \dots]) =$$

$$[\dots 1_5, \underbrace{2, 1_{-1}, 2}, 1_{-1}, 2, 1_{-1}, 2, 1_{-1}, 2, 1_{11}, \dots]$$

$$[\dots 1_5, \underbrace{3, 1_{-1}, 2}, 1_{-1}, 2, 1_{-1}, 2, 1_{11}, \dots]$$

$$[\dots 1_5, \underbrace{4, 1_{-1}, 2}, 1_{-1}, 2, 1_{11}, \dots]$$

$$[\dots 1_5, \underbrace{5, 1_{-1}, 2}, 1_{11}, \dots]$$

$$[\dots 1_5, 6, 1_{11}, \dots]$$

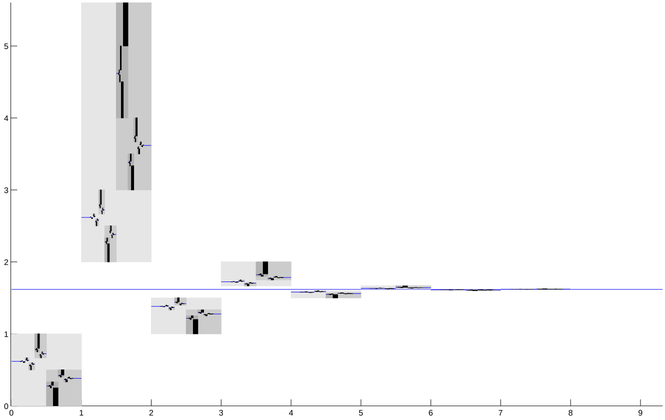
Definition (Recall)

$$\text{ج}([n_0, n_1, n_2, \ldots]) = [1_{n_0-1}, 2, 1_{n_1-2}, 2, 1_{n_2-2}, \ldots]$$

With these two rules, ج becomes well-defined on $\mathbf{R} \backslash \mathbf{Q}$ and it is involutive:

$$\text{ج}(\text{ج}(x)) = x$$

Here is the plot of ج (the graph lies inside the darker boxes)



Some continuity properties of jimm

It can be shown that..

- ج is continuous on $\mathbf{R}\backslash\mathbf{Q}$
- have jump discontinuities on $\mathbf{Q}$
- ج is differentiable almost everywhere
- its derivative vanish almost everywhere
- admits a natural extension to $\mathbf{Q}\backslash 0$.

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Now consider....

Example

$$\mathfrak{Z}(1+[3,3,3\dots]) = \mathfrak{Z}([4,3,3\dots]) = [1,1,1,2,1,2,1,\dots]$$
$$= 1 + \frac{1}{\underbrace{[1,1,2,1,2,1,\dots]}_{=\mathfrak{Z}([3,3,3,\dots])}}$$

We have, in general

FUNCTIONAL EQUATION

$$\mathfrak{Z}(1+x) = 1 + \frac{1}{\mathfrak{Z}(x)}$$

This functional equation can be derived from the following fundamental set of functional equations

$$\text{ج}(\text{ج}(x)) = x \quad \text{(involutivity)}$$

$$\text{ج}\left(\frac{1}{x}\right) = \frac{1}{\text{ج}(x)} \quad \text{equivariance}$$

$$\text{ج}(-x) = -\frac{1}{\text{ج}(x)} \quad \textbf{“twisted”} \text{ equivariance}$$

$$\text{ج}(1-x) = 1 - \text{ج}(x) \quad \text{equivariance}$$

Now notice

$$xy = 1 \iff y = 1/x \iff$$

$$\text{ج}(y) = \text{ج}\left(\frac{1}{x}\right) = \frac{1}{\text{ج}(x)}$$

Hence

$$xy = 1 \iff \text{ج}(y)\text{ج}(x) = 1$$

We may do the same for the other equations, which gives

Two-variable form of functional equations

$$\text{ج}(x) = y \iff \text{ج}(y) = x$$

$$xy = 1 \iff \text{ج}(x)\text{ج}(y) = 1$$

$$x + y = 0 \iff \text{ج}(x)\text{ج}(y) = -1$$

$$x + y = 1 \iff \text{ج}(x) + \text{ج}(y) = 1$$

$$\frac{1}{x} + \frac{1}{y} = 1 \iff \frac{1}{\text{ج}(x)} + \frac{1}{\text{ج}(y)} = 1$$

$\implies$ ج preserves harmonic pairs of numbers.

Recall that

$$Ux := \frac{1}{x}, \quad Vx := -x, \quad Kx := 1 - x$$

The functional equations say

$$\text{ج}U = U\text{ج}, \quad \text{ج}K = K\text{ج}, \quad \text{ج}V = UV\text{ج}$$

$\Longrightarrow$ ج **is Dyer's outer automorphism of $\mathrm{PGL}_2(\mathbf{Z})$.**

This is the only non-trivial outer automorphism: $Out(\mathrm{PGL}_2(\mathbf{Z})) \simeq \mathbf{Z}/2\mathbf{Z}$.

(In fact we worked out the continued fraction-definition of ج from the above functional equations)

The most general functional equation has the form

$$\text{ج}(Mx) = \text{ج}(M)\text{ج}(x), \quad M \in \mathrm{PGL}_2(\mathbf{Z})$$

(where $\text{ج}(M)$ is the image of M under Dyer's automorphism).

Hence ج is a "twisted" **equivariant** function.

f is said to be $\mathrm{PSL}_2(\mathbf{Z})$-equivariant if $f(Mx) = Mf(x)$, $\forall M \in \mathrm{PSL}_2(\mathbf{Z})$.

If G is weight-k modular (i.e. $G(Mz) = j_M(z)^k G(z)$) then

$$H(z) = z + k\frac{G(z)}{G'(z)}$$

is $\mathrm{PSL}_2(\mathbf{Z})$-equivariant, i.e. it satisfies the functional equations

$$H(Tz) = TH(z), \quad H(Sz) = SH(z),$$

where $Tz = KUVz = z + 1$ and $Sz = UVz = -1/z$ generate $\mathrm{PSL}_2(\mathbf{Z})$.

Question. Are there analytic analogues of ℨ? i.e. are there analytic functions with $H(Mx) = \mathfrak{Z}(M)H(x)$, $\forall M \in \mathrm{PGL}_2(\mathbf{Z})$? (needs to be properly formulated)

Action on quadratic irrationals

Fact I

ج sends ultimately periodic continued fractions to ultimately periodic continued fractions.

$$\Longrightarrow$$

ج sends **quadratic irrationals** to **quadratic irrationals**
i.e. ج preserves the "real multiplication-set."

(it does not preserve nor respect the trace, norm, signature, etc)

Examples

$$\text{ج}(\sqrt{2}) = \text{ج}([1, 2, 2, \ldots]) = 1 + \sqrt{2}$$

Not so simple in general:

$$\text{ج}(\sqrt{11}) = \frac{15 + \sqrt{901}}{26}, \quad \text{ج}(-\sqrt{11}) = \frac{15 - \sqrt{901}}{26}$$

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Fact II

$\rJ$ respects ends of continued fractions (i.e. if x, y has continued fractions that eventually coincide, then so does $\rJ(x)$ and $\rJ(y)$).

$$\Longleftrightarrow$$

$\rJ$ **respects the** $\mathrm{PGL}_2(\mathbf{Z})$**-action** (i.e. if x and y are in the same $\mathrm{PGL}_2(\mathbf{Z})$-orbit, then so are $\rJ(x)$ and $\rJ(y)$.)

More precisely

$$\rJ(Mx) = \rJ(M)\rJ(x) \quad M \in \mathrm{PGL}_2(\mathbf{Z}), x \in \mathbf{R}$$

so that

$$x = My \implies \rJ(x) = \rJ(M)\rJ(y), \quad \rJ(M) \in \mathrm{PGL}_2(\mathbf{Z})$$

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$$x = My \implies \mathfrak{J}(x) = \mathfrak{J}(M)\mathfrak{J}(y), \quad \mathfrak{J}(M) \in \mathrm{PGL}_2(\mathbf{Z})$$

Facts I&II together imply:

Fact III

ج induces an involution of the "moduli space of degenerate rank-2 lattices" inside $\mathbf{R}$, preserving setwise the "real-multiplication" locus.

$$\text{ج} \circlearrowright \mathbf{R}/\mathrm{PGL}_2(\mathbf{Z})$$

The facts imply...

ج is really a modular function.

Furthermore, one has ....

Fact IV

ج **commutes with the Galois conjugation** on quadratic irrationals, i.e.

$$ج(a+\sqrt{b}) = A+\sqrt{B}$$

$$\Longleftrightarrow$$

$$ج(a-\sqrt{b}) = A-\sqrt{B}$$

Now go back to the two-variable functional equations....

$$\boxed{\begin{aligned} xy = 1 &\iff \jimm(x)\jimm(y) = 1 \\ x + y = 0 &\iff \jimm(x)\jimm(y) = -1 \\ x + y = 1 &\iff \jimm(x) + \jimm(y) = 1 \\ \frac{1}{x} + \frac{1}{y} = 1 &\iff \frac{1}{\jimm(x)} + \frac{1}{\jimm(y)} = 1 \end{aligned}}$$

...and set $y = \bar{x}$, where $x = a + \sqrt{b}$ is a quadratic irrational:

$$\boxed{\begin{aligned} x\bar{x} = 1 &\iff \mathfrak{J}(x)\mathfrak{J}(\bar{x}) = 1 \\ x + \bar{x} = 0 &\iff \mathfrak{J}(x)\mathfrak{J}(\bar{x}) = -1 \\ x + \bar{x} = 1 &\iff \mathfrak{J}(x) + \mathfrak{J}(\bar{x}) = 1 \\ \frac{1}{x} + \frac{1}{\bar{x}} = 1 &\iff \frac{1}{\mathfrak{J}(x)} + \frac{1}{\mathfrak{J}(\bar{x})} = 1 \end{aligned}}$$

Recall from number theory

If $x = a + \sqrt{b}$ $(a, b \in \mathbf{Q},\ b > 0)$, then

norm of x is $N(x) := x\bar{x} \iff N(a + \sqrt{b}) = a^2 - b$

trace of x is $T(x) := x + \bar{x} \iff T(a + \sqrt{b}) = 2a$

Example

$$N(1 + \sqrt{2}) = -1, \quad T(1 + \sqrt{2}) = 2$$

The functional equations means

$$\boxed{\begin{gathered}
N(x) = x\bar{x} = 1 \iff \text{ج}(x)\text{ج}(\bar{x}) = 1 = N(\text{ج}x) \\
Tr(x) = x + \bar{x} = 0 \iff \text{ج}(x)\text{ج}(\bar{x}) = -1 = N(\text{ج}x) \\
Tr(x) = x + \bar{x} = 1 \iff \text{ج}(x) + \text{ج}(\bar{x}) = 1 = Tr(\text{ج}x) \\
\frac{Tr(x)}{N(x)} = \frac{1}{x} + \frac{1}{\bar{x}} = 1 \iff \frac{1}{\text{ج}(x)} + \frac{1}{\text{ج}(\bar{x})} = 1 = \frac{Tr(\text{ج}x)}{N(\text{ج}x)}
\end{gathered}}$$

We get...

Correspondence I

$$x\bar{x} = 1 \iff \text{ج}(x)\text{ج}(\bar{x}) = 1; \text{ i.e. } N(x) = 1 \iff N(\text{ج}(x)) = 1$$

$$\Longrightarrow$$

ج restricts to an involution of the set of **units of norm +1** of the rings of integers in quadratic number fields.

$$\text{ج} \circlearrowright \{a + \sqrt{a^2 - 1} \,|\, 1 < a \in \mathbf{Q}\}$$

We get...

Correspondence II

$$x+\bar{x}=0 \iff \text{ج}(x)\text{ج}(\bar{x})=-1; \text{ i.e. } T(x)=0 \iff N(\text{ج}(x))=-1.$$

$\Longrightarrow$ ج establishes a bijection between the set of **square roots of positive rationals** and the set of **units of norm -1** of the rings of integers of quadratic number fields.

$$\text{ج}:\{\sqrt{q}\mid q\in\mathbf{Q}\}\to\{a+\sqrt{a^2+1}\mid a\in\mathbf{Q}\}$$

.... and these correspondences are far from being trivial:

Correspondence II-Example

$$\sqrt{\frac{39}{17}} = [1, \overline{1, 1, 16, 1, 1, 2}] \implies$$

$$\text{ج}\left(\sqrt{\frac{39}{17}}\right) = [4, \overline{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 4, 4}] = A \implies$$

$$N(A) = N\left(\frac{7663 + \sqrt{70845893}}{3482}\right) = -1.$$

Correspondence II-More Examples

$$
\begin{array}{lcl}
\sqrt{N} & \rightarrow & \text{ج}(\sqrt{N}) \\
\sqrt{3} & \rightarrow & \frac{1}{2}(\sqrt{13}+3) \\
\sqrt{5} & \rightarrow & \frac{1}{3}(\sqrt{10}+1) \\
\sqrt{6} & \rightarrow & \frac{1}{14}(\sqrt{221}+5) \\
\sqrt{7} & \rightarrow & \frac{1}{6}(\sqrt{37}+1) \\
\sqrt{8} & \rightarrow & \frac{1}{4}(\sqrt{17}+1) \\
\sqrt{10} & \rightarrow & \frac{1}{7}(\sqrt{65}+4) \\
\sqrt{11} & \rightarrow & \frac{1}{26}(\sqrt{901}+15) \\
\sqrt{12} & \rightarrow & 134(\sqrt{1517}+19) \\
\sqrt{13} & \rightarrow & \frac{1}{3}(\sqrt{13}+2) \\
\sqrt{14} & \rightarrow & \frac{1}{5}(\sqrt{34}+3) \\
\sqrt{15} & \rightarrow & \frac{1}{18}(\sqrt{445}+11) \\
\sqrt{17} & \rightarrow & \frac{1}{19}(\sqrt{442}+9)
\end{array}
$$

We get...

Correspondence III

$$x + y = 1 \iff \text{ج}(x) + \text{ج}(\bar{x}) = 1; \text{ i.e. } T(x) = 1 \iff T(\text{ج}(x)) = 1$$

$$\text{ج} \circlearrowright \{\frac{1}{2} + \sqrt{a} \,|\, 0 < a \in \mathbf{Q}\}$$

We get...

Correspondence IV

$$\frac{1}{x}+\frac{1}{\bar{x}}=1 \iff \frac{1}{\text{ج}(x)}+\frac{1}{\text{ج}(\bar{x})}=1; \text{ i.e. } T(\frac{1}{x})=1 \iff T(\frac{1}{\text{ج}(x)})=1$$

$$T(x)=N(x) \iff T(\text{ج}x)=N(\text{ج}x)$$

Equivalently,

$$\text{ج} \circlearrowright \{a+\sqrt{a^2-2a}\,|\,1<a\in\mathbf{Q}\}$$

... and there are more correspondences of this type

What about algebraic numbers of higher degree?

Conjecture

If x is algebraic of degree > 2, then $\text{Ꞩ}(x)$ is transcendental[a]

[a] Testing the transcendence conjecture of Jimm and its continued fraction statistics (joint with H. Ayral, to appear)

Why? Because if x algebraic of degree > 2, then it is widely believed that x obeys the Gauss-Kuzmin statistics.

$\Longrightarrow$ the frequency of $1's$ in the continued fraction of $\text{Ꞩ}(x)$ is 1.

$\Longrightarrow$ $\text{Ꞩ}(x)$ does not obey the GK statistics

$\Longrightarrow$ $\text{Ꞩ}(x)$ is can not be algbraic.

A few examples...

$$\begin{aligned}\text{ج}(\sqrt[3]{2}) &= \text{ج}([1;3,1,5,1,1,4,1,1,8,1,14,1,10,2,1,4,\ldots]) \\ &= [2,1,3,1,1,1,4,1,1,4,1_6,3,1_{12},3,1_8,2,3,1,1,2,\ldots] \\ &= 2.78473155862723\ldots\end{aligned}$$

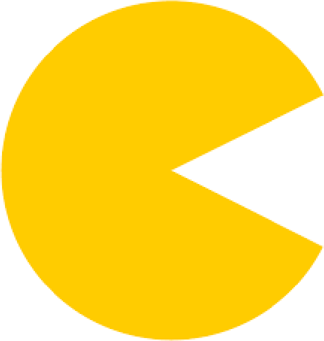
$$\begin{aligned}\text{ج}(\pi) = \text{ج}([3,7,15,1,292,1,1,1,2,1,3,\ldots]) = \\ [1_2,2,1_5,2,1_{13},3,1_{290},5,3,\ldots] \\ = 1.7237707925480276079699326494931025145558144289232\ldots\end{aligned}$$

$$\begin{aligned}\text{ج}(e) = \text{ج}([2,1,2,1,1,4,1,1,6,1,1,8,\ldots]) = \\ [1,3,4,1,1,4,1,1,1,1,\ldots,\overline{4,1_{2n}}] \\ = 1.3105752928466255215822495496939143349712038085627\ldots\end{aligned}$$

(We tried to recognize these numbers by the PSLQ-algorithm with various sets of constants–we couldn't get any results)

PART II

Dynamics



Dynamics

Fact

ج conjugates the Gauss map to the "Fibonacci map"

$$T_{Gauss} : [0, n_1, n_2, n_3, \ldots] \in [0, 1] \longrightarrow [0, n_2, n_3, n_4, \ldots] \in [0, 1]$$

$$\Longrightarrow$$

$$T_{Fibonacci} = \text{ج}\, T_{Gauss}\, \text{ج} : [0, 1_k, n_{k+1}, n_{k+2}, \ldots] \to [0, n_{k+1} - 1, n_{k+2}, \ldots]$$

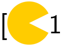
Dynamics

The Gauss map

[ $1, 1, 1, 6, 13, 2, 2, 7, \ldots$]

Dynamics

The Gauss map

$[$  $1, 1, 6, 13, 2, 2, 7, \ldots]$

Dynamics

The Gauss map

$$[\;1, 6, 13, 2, 2, 7, \dots]$$

Dynamics

The Gauss map

$[$  $6, 13, 2, 2, 7, \ldots]$

Dynamics

The Gauss map

[ $13, 2, 2, 7, \ldots]$

Dynamics

The Gauss map

$[$  $2, 2, 7 \ldots]$

Dynamics

The Gauss map

$[$  $2, 7 \ldots]$

Dynamics

The Gauss map

[, 7 . . .]

Dynamics

The Fibonacci map

$$[\;1, 1, 1, 6, 13, 2, 2, 7, \ldots]$$

Dynamics

The Fibonacci map

$[$  $5, 13, 2, 2, 7, \dots]$

Dynamics

The Fibonacci map

$$[\ 4, 13, 2, 2, 7, \ldots]$$

Dynamics

The Fibonacci map

$$[\ \ 3, 13, 2, 2, 7, \dots]$$

Dynamics

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$$[\ 2, 13, 2, 2, 7, \ldots]$$

Dynamics

The Fibonacci map

$$[\ 1, 13, 2, 2, 7, \ldots]$$

Dynamics

The Fibonacci map

$$[\;12, 2, 2, 7, \ldots]$$

Dynamics

The Fibonacci map

[ $11, 2, 2, 7, \dots]$

Dynamics

The Fibonacci map

[$10, 2, 2, 7, \ldots]$

Dynamics

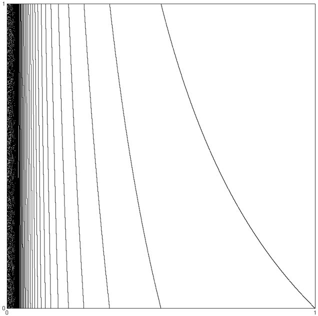
The Fibonacci map

[$9, 2, 2, 7, \ldots$]

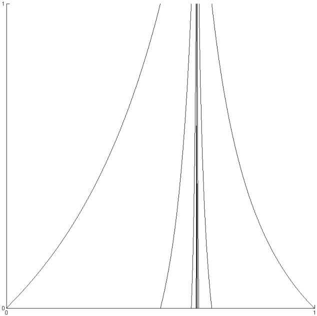
Dynamics

The Fibonacci map

$[$  $8, 2, 2, 7, \ldots]$



$\longrightarrow$



Gauss map $\longrightarrow$ **Fibonacci map**

Dynamics of these two maps are closely related (Isola et al).

The transfer operator of the Fibonacci map is

$$(\mathscr{L}_s^{Fib}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(F_{k+1}y + F_k)^{2s}} \psi\left(\frac{F_k y + F_{k-1}}{F_{k+1}y + F_k}\right) \tag{1}$$

The transfer operator of the Gauss map is

$$(\mathscr{L}_s^{Gauss}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^{2s}} \psi\left(\frac{1}{k+x}\right) \tag{2}$$

Dynamics of these two maps are closely related (Isola et al).

The transfer operator of the Fibonacci map is

$$(\mathscr{L}_s^{Fib}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(F_{k+1}y + F_k)^{2s}} \psi\left(\frac{F_k y + F_{k-1}}{F_{k+1}y + F_k}\right) \tag{1}$$

The transfer operator of the Gauss map is

$$(\mathscr{L}_s^{Gauss}\psi)(y) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^{2s}} \psi\left(\frac{1}{k+x}\right) \tag{2}$$

A.C. invariant measures

$$T_{Fibonacci} \leftrightarrow \frac{1}{x(x+1)} \text{ (infinite)}, \qquad T_{Gauss} \leftrightarrow \frac{1}{x+1}$$

Zeta functions (the transfer operator evaluated at Lebesgue's measure)

$$T_{Fibonacci} \leftrightarrow (\mathscr{L}_s^{Fib}\psi)(\mathbf{1}) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \qquad \text{(“Fibonacci zeta”)}$$

$$T_{Gauss} \leftrightarrow (\mathscr{L}_s^{Gauss}\psi)(\mathbf{1}) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{(“Riemann zeta”)}$$

Eigenfunctions of the Fibonacci transfer operator satisfies the three-term functional equation

$$\psi(y) = \frac{1}{y^{2s}}\psi\left(\frac{y+1}{y}\right) + \frac{1}{\lambda}\frac{1}{(y+1)^{2s}}\psi\left(\frac{y}{y+1}\right) \tag{3}$$

(Equivalent to three-term functional equation studied by Lewis and Zagier)

Dynamics

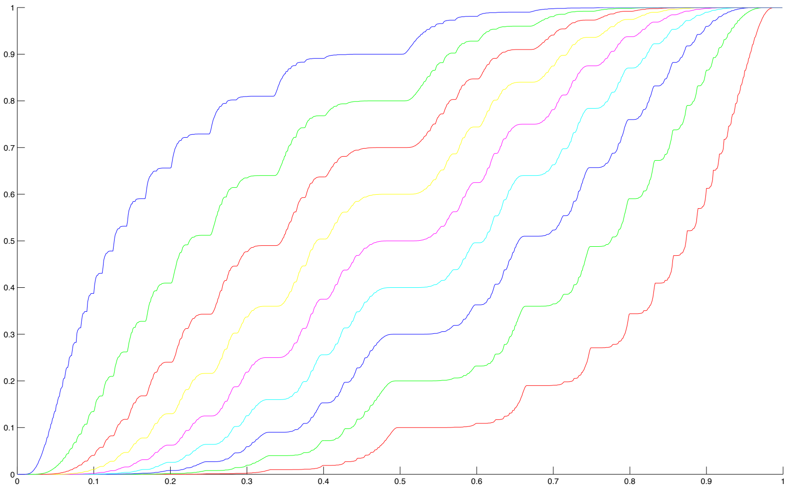
The Denjoy-Minkowski measure is the measure whose cumulative distribution function is

$$\mathbf{?}([0, n_1, n_2, \ldots, n_{k-1}, n_k]) = \sum_{k=1}^{\infty} (-1)^{1+k} 2^{-n_1 - n_2 \cdots - n_k}. \tag{4}$$

? is a common invariant measure for the Gauss and the Fibonacci maps (however, it is not absolutely continuous w.r.t Lebesgue's measure).

Actually, **?** is the common invariant measure of a much wider class of maps.

Dynamics



Plots of the cumulative laws of Denjoy-Minkowski measures F_p $(p = 0.1, 0.2, \ldots 0.9)$

There is a common generalization of the Gauss and Fibonacci maps:

$$\mathbb{T}_\alpha(x) = \begin{cases} [0, m_{k+1}, m_{k+2}, m_{k+3}, \dots] & n_k > m_k \, (*) \\ [0, m_k - n_k, m_{k+1}, m_{k+2}, \dots] & n_k < m_k \, (**) \end{cases} \tag{5}$$

where $\alpha = [0, n_1, n_2, \dots]$ and $x = [0, m_1, m_2, \dots]$.

One has

$$\mathbb{T}_0 = \mathbb{T}_{Gauss}, \quad \mathbb{T}_{\Phi^*} = \mathbb{T}_{Fibonacci}$$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\ldots]$.

$$[\;1,1,1,6,13,2,2,7,\ldots]$$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1 = [0, 2, 2, 2, \dots]$.

[$1, 1, 6, 13, 2, 2, 7, \dots]$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1 = [0, 2, 2, 2, \ldots]$.

[ $1, 6, 13, 2, 2, 7, \ldots]$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1 = [0, 2, 2, 2, \ldots]$.

$$[\quad 6, 13, 2, 2, 7, \ldots]$$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\dots]$.

$$[$$  $$4,13,2,2,7,\dots]$$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\dots]$.

[$2,13,2,2,7,\dots]$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\ldots]$.

$$[\;11,2,2,7,\ldots]$$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\ldots]$.

$$[\;9,2,2,7,\ldots]$$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1 = [0, 2, 2, 2, \ldots]$.

$$[\;7, 2, 2, 7, \ldots]$$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\ldots]$.

$$[\ 5,2,2,7,\ldots]$$

Dynamics

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\ldots]$.

$[$  $3,2,2,7,\ldots]$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1 = [0, 2, 2, 2, \ldots]$.

[ $1, 2, 2, 7, \ldots]$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1 = [0, 2, 2, 2, \dots]$.

[$2, 2, 7, \dots]$

Dynamics

Example

The map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\dots]$.

[$5,\dots$]

Dynamics

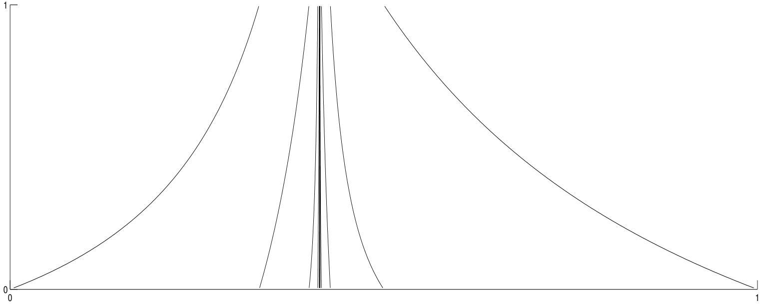


Figure: Plot of $\mathbb{T}_{\sqrt{2}-1}$

Dynamics

The following functional equation is satisfied:

$$\mathbb{T}_{\text{ج}(\alpha)}(\text{ج}x) = \text{ج}\mathbb{T}_\alpha(x).$$

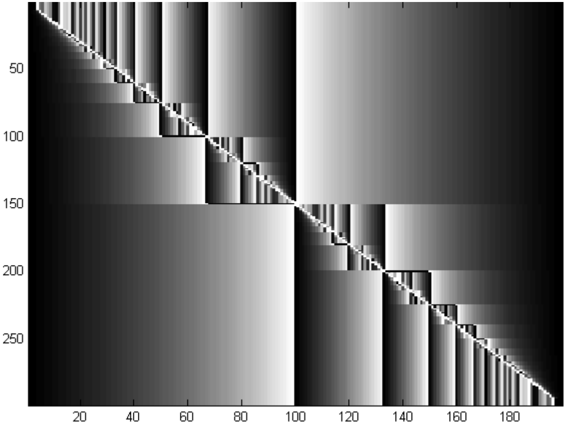
For $\alpha = \Phi^*$, this specialises to

$$\mathbb{T}_0(\text{ج}x) = \text{ج}\mathbb{T}_{\Phi^*}(x) \iff \text{ج}\mathbb{T}_0(\text{ج}x) = \mathbb{T}_{\Phi^*}(x)$$

i.e. the fact that ج conjugates the Gauss and the Fibonacci maps.

$$\left(\text{recall that } \mathbb{T}_0 = \mathbb{T}_{Gauss}, \quad \mathbb{T}_{\Phi^*} = \mathbb{T}_{Fibonacci}\right)$$

Dynamics



Plot of $\mathbb{T}_\alpha(x)$ as a function of α and x.
The intensity is proportional to the value of $T_\alpha(x)$.
The symetry is due to $T_{1-\alpha}(1-x) = T_\alpha(x)$.

Dynamics

The transfer operator is $(\mathscr{L}_{s,\alpha}\psi)(y) =$

$$-\frac{1}{y^{2s}}\psi\left(\frac{1}{y}\right) + \sum_{k=1}^{\infty}\sum_{i=0}^{n_k-1}\left|\frac{\mathrm{d}}{\mathrm{d}y}[0, n_1, \ldots, n_{k-1}, i+y]\right|^s \psi[0, n_1, \ldots, n_{k-1}, i+y]$$

eigenfunctions of which satisfy the functional equation

$$\psi(y) - \psi(1+y) + \frac{1}{y^{2s}}\left\{\psi\left(\frac{1}{y}\right) - \psi\left(1+\frac{1}{y}\right)\right\} = \frac{1}{\lambda(1+y)^{2s}}\left\{\psi\left(\frac{y}{1+y}\right) + \psi\left(\frac{1}{1+y}\right)\right\}$$

Observe that the LHS=0 is precisely Lewis' three-term functional equation, and the RHS is Isola's transfer operator of the Farey map.

Dynamics

Example

For the map $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1 = [0,2,2,2,\dots]$ we have

$$\mathscr{L}_{s,\alpha}\psi(y) = \sum_{i=1}^{\infty} \frac{1}{(P_{i+1}y + P_i)^s}\psi\left(\frac{P_i y + P_{i-1}}{P_{i+1}y + P_i}\right) + \sum_{j=1}^{\infty} \frac{1}{(P_{j+1}y + P_{j+1} + P_j)^s}\psi\left(\frac{P_j y + P_j + P_{j-1}}{P_{j+1}y + P_{j+1} + P_j}\right),$$

where $0, 1, 2, 5, 12, 29, 70, 169, 408, \dots$ is the Pell sequence defined by $P_0 = 0$, $P_1 = 1$ and $P_k = 2P_{k-1} + P_{k-2}$.

Dynamics

Example

An a.c. invariant measure for $\mathbb{T}_{\sqrt{2}-1}$ with $\sqrt{2}-1=[0,2,2,2,\ldots]$

$$\psi(y)=\sum_{i=0}^{\infty}\frac{1}{(1+2iy)(1+2y+2iy)}-\frac{1}{(y+2i+3)(y+2i+2)}.$$

Questions.

- What are the a.c. invariant measures for $\mathbb{T}_\alpha$ in general?
- How are the dynamics of $\mathfrak{J}$-conjugate maps related?
- Same questions for the continued fraction maps defined below

Dynamics

Fact: Denjoy-Minkowski measure is a common invariant measure of all $\mathbb{T}_\alpha$'s.

In fact, this is true for an even wider class of maps (called continued fraction maps) $T : [0,1] \mapsto [0,1]$, whose inverse branches are all $\mathrm{PGL}_2(\mathbf{Z})$ on $[0,1]$. These are generalized Pacman maps (i.e. pacmen with powers equal to several $\mathbb{T}_\alpha$-pacmen combined)

(There is a systematic way to define these maps as topological covering maps of the boundary of the Farey tree)

Dynamics

Indeed, suppose the inverse branches of T are $\{\varphi_\beta\}_{\beta=1,2,\dots}$. Then each φ_β can be written as

$$\varphi_\beta(y) = [0, n_1, n_2, \dots, n_{k-1}, i + y],$$

where $0 < k, n_1, n_2, \dots$ and $0 \le i$ depends on β. Suppose X is a random variable on $[0, 1]$ with law **?** and set $Y := T(X)$. The law $\mathbf{F}_Y$ of Y is

$$\mathbf{F}_Y(y) = \mathrm{Prob}\{Y \le y\} = \mathrm{Prob}\{T(X) \le y\} = \sum_\beta \left|\mathbf{?}\left(\varphi_\beta(y)\right) - \mathbf{?}\left(\varphi_\beta(0)\right)\right|$$

$$= \sum_\beta \left|\mathbf{?}[0, n_1, n_2, \dots, n_{k-1}, i + y] - \mathbf{?}[0, n_1, n_2, \dots, n_{k-1}, i]\right|$$

$$= \sum_\beta \mathbf{?}(y) 2^{-(n_1+\cdots+n_{k-1}+i)} \implies \mathbf{F}_Y(y) = \mathbf{?}(y) \sum_\beta 2^{-(n_1+\cdots+n_{k-1}+i)},$$

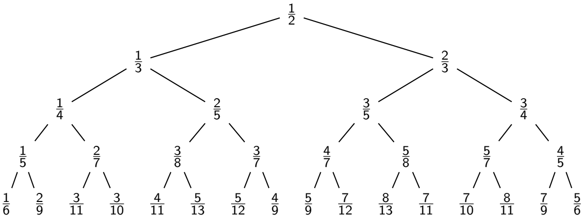
and the series of the last line *must* sum up to 1, because $\mathbf{F}_Y$ and $\mathbf{?}(y)$ are both probability laws.

PART III

Tree automorphisms and Lebesgue's measure

ج as a tree automorphism

The Farey tree

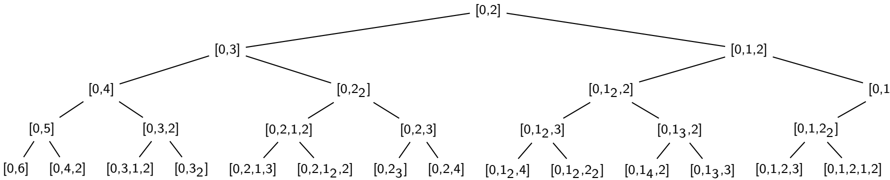


Produced by the Farey sum rule:

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$$

ج as a tree automorphism

The Farey tree by continued fractions



The boundary $\partial\mathcal{F}$ is the set of all infinite paths based at the root.

Fact

The map $\partial\mathcal{F} \to [0,1]$ sending path to its continued fraction, parametrize irrationals in $[0,1]$ (and is 2-to-1 over the rationals).

ج as a tree automorphism

The automorphism group $\mathbf{Aut}(\mathcal{F})$ naturally acts on $\partial\mathcal{F}$.
$\implies$ $\mathbf{Aut}(\mathcal{F})$ acts on continued fractions via the above identification.
(ignoring a countable set of numbers for each automorphism).

ج as a tree automorphism

Shuffle description of $\mathbf{Aut}(\mathcal{F})$.

$\implies$ ج is the automorphism which shuffles every other vertex.

ج as a tree automorphism

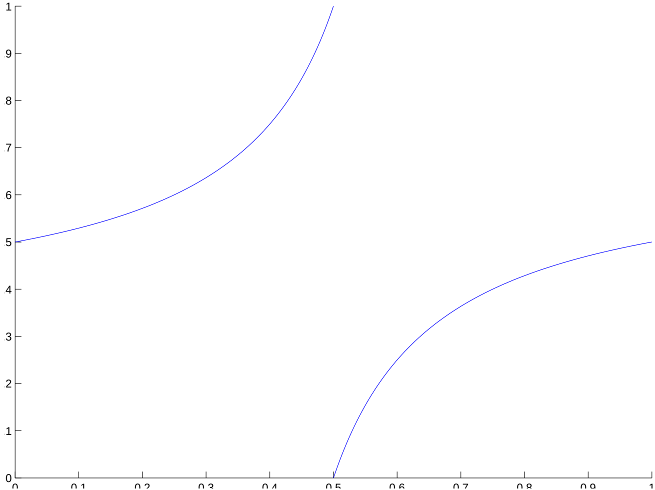
Twist description of $\mathbf{Aut}(\mathcal{F})$

$\Longrightarrow$ ج is the automorphism which twists every vertex.

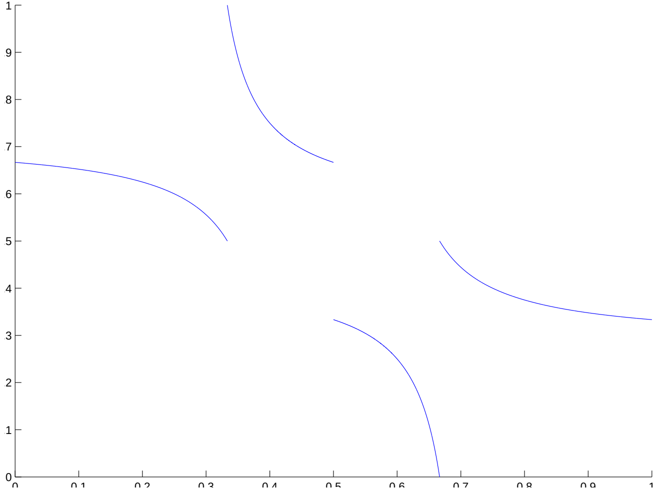
ج sends zig-zag segments on a path to straight segments and vice versa

Looking at the boundary actions of shuffles (or twists), yields a presentation of ج as a limit of piecewise-$\mathrm{PGL}_2(\mathbf{Z})$ maps....

Jimm as a limit of piecewise-$\mathrm{PGL}_2(\mathbf{Z})$ maps



Jimm as a limit of piecewise-$\mathrm{PGL}_2(\mathbf{Z})$ maps



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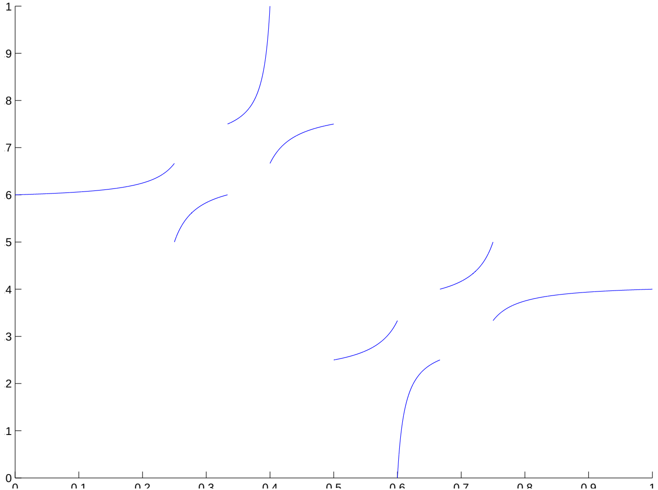

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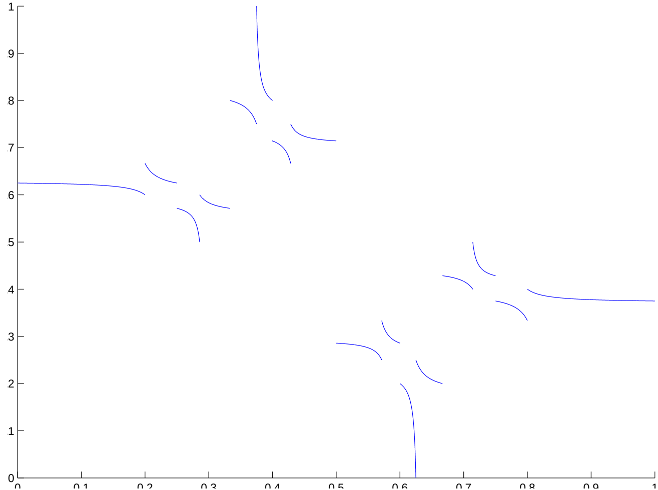

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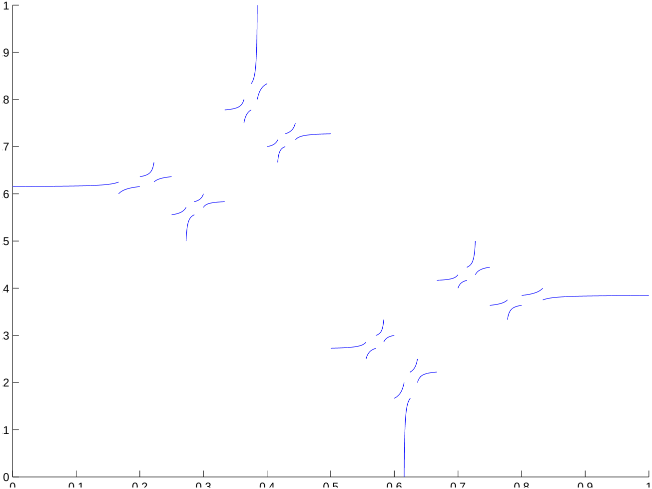

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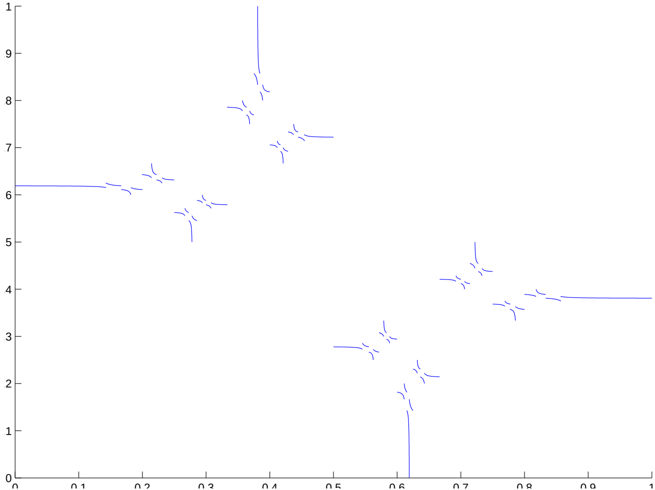

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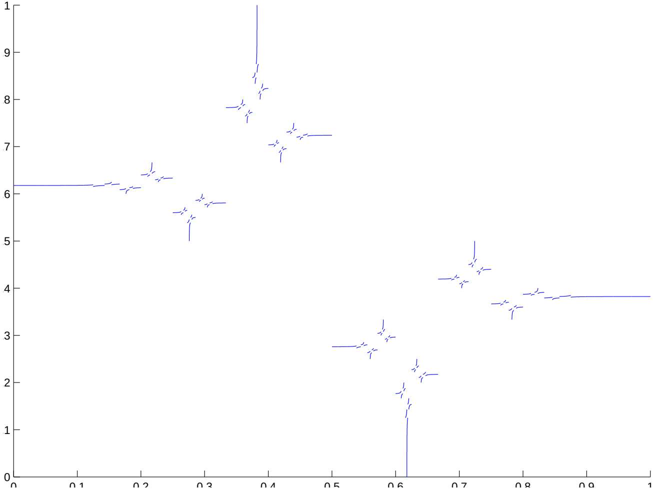

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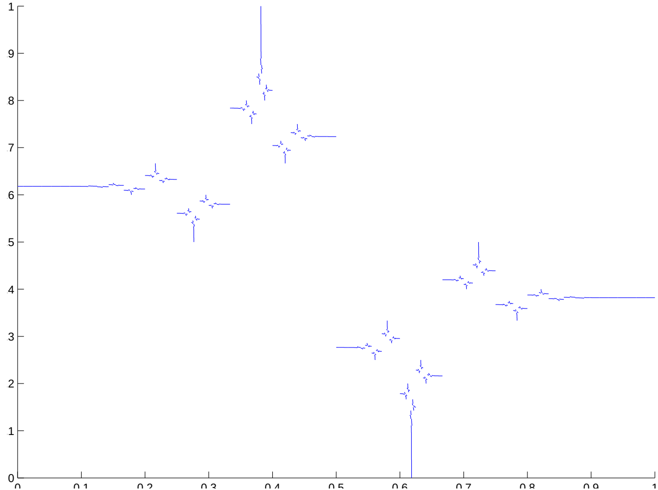
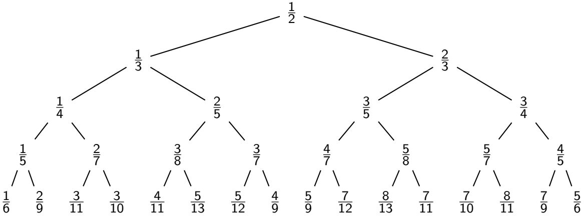


ج as a symmetry of Lebesgue's measure

Let's turn back to the Farey tree...



A random walker starts from the root vertex. For each vertex x, we are given the probability $\pi(x)$ of **arriving** to that vertex from its parent.

This induces a measure on the set of continued fractions, i.e. on $[0, 1]$.

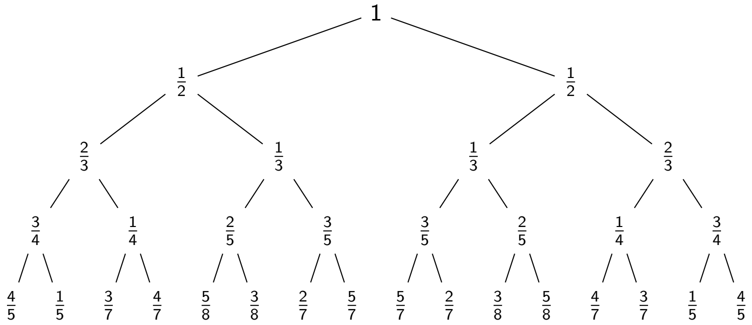
ج as a symmetry of Lebesgue's measure

If we set $\pi(x) \equiv 1/2$, then the c.d.f. of the induced measure on $[0, 1]$ is the Minkowski-Denjoy measure.
(which by the way is the unique $Aut(\mathcal{F})$-invariant measure on $\partial\mathcal{F}$.)

ج as a symmetry of Lebesgue's measure

Question

Which 'arrival' probability function $\pi_{Leb}(x)$ induce the Lebesgue measure?



The "Lebesgue tree" $\mathcal{L}$.

Answer

Assume $n_k > 1$. Then the arrival probabilities

$$\pi_{Leb}([0, n_1, n_2, \ldots, n_{k-1}, n_k]) = 1 - [0, n_k - 1, n_{k-1}, \ldots, n_2, n_1]$$

induces the Lebesgue measure on $[0, 1]$.

A subtle symmetry of Lebesgue's measure:

$$\pi_{Leb}\text{ج}(x) = \text{ج}\pi_{Leb}(x)$$

(On the left hand side ج acts on the tree whereas on the right it acts on the rationals)

Three singular measures

How does this symmetry manifests itself on the superficial level?

There are many questions pertaining to the measures induced by the transition functions

- $\pi(x) := K\pi_\lambda(x)$
- $\pi(x) := \mathfrak{I}\pi_\lambda(x) = \pi_\lambda\mathfrak{I}(x)$
- $\pi(x) := K\mathfrak{I}\pi_\lambda(x) = \mathfrak{I}K\pi_\lambda(x)$.

These are, in a sense, basic deformations of Lebesgue's measure.

Three singular measures

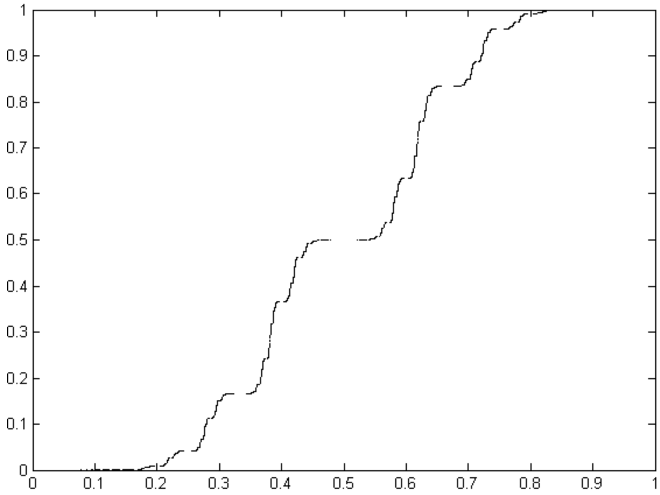


Figure: c.d.f. of $K\pi_\lambda$

Three singular measures

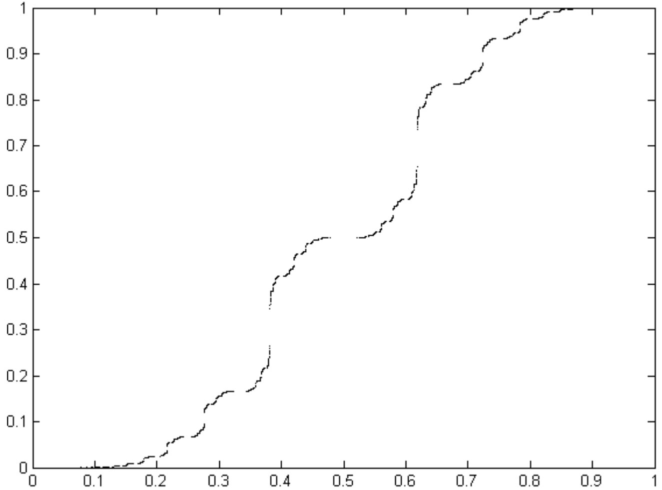


Figure: c.d.f. of $\daleth\pi_\lambda = \pi_\lambda\daleth$

Three singular measures

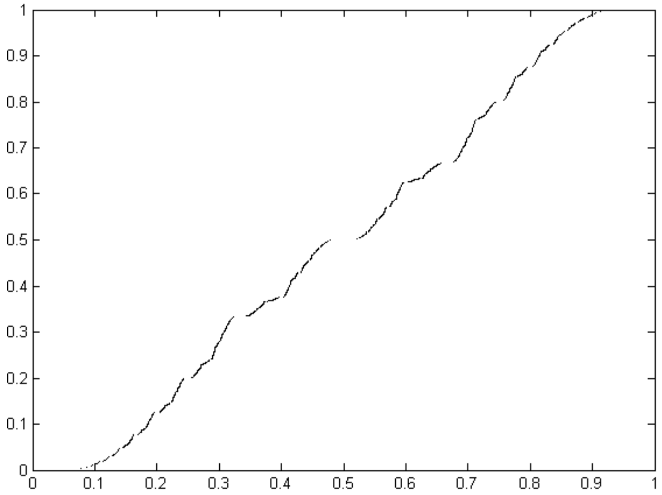


Figure: c.d.f. of $K\zeta\pi_\lambda = \zeta K\pi_\lambda$

Three singular measures

Some Questions

- Are the measures $\pi(x) := K\pi_\lambda(x)$ and $\pi(x) := Ꜿ\pi_\lambda(x) = \pi_\lambda Ꜿ(x)$ singular with respect to Lebesgue's measure? Denjoy-Minkowski measure?
- How do these measure behave under the continued fraction maps?

More measures

Recall $Kx := 1 - x$ and define the flip operation on $\mathbf{Q} \cap (0,1)$ as

$$\varphi([0, n_1, n_2, \ldots, n_k]) = [0, n_k - 1, n_{k-1}, \ldots n_2, n_1 + 1]$$

where it is assumed that $n_k > 1$.

Let T_F be the *Farey map*

$$T_F : (n_1, n_2, \ldots, n_{k-1}, n_k) \in X \to (n_1 - 1, n_2, \ldots, n_{k-1}, n_k) \in X, \qquad (6)$$

Then

$$\pi_{Leb}(r) = K\varphi T_F(r)$$

More measures

Lemma

(i) $(K\varphi)^4 = Id$.
(ii) Both K and ج preserves the relations $x + y = 1$ and the relation of being sibling (this latter is preserved with any automorphism)
(iii) If π is any measure, then so are $K\pi$, $\varphi K\varphi\pi$, $K\varphi K\varphi\pi$
(iv) x, y are siblings if and only if $\varphi(x) + \varphi(y) = 1$.

This lemma permits us to construct a limited number of deformations of the Lebesgue measure.

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- *On the involution of the real line induced by Dyer's outer automorphism of PGL(2,Z).* (with H. Ayral) arXiv:1605.03717
- *A subtle symmetry of Lebesgue's measure.* (with H. Ayral) arXiv:1605.07330
- *Testing the transcendence conjecture of Jimm and its continued fraction statistics.* (with H. Ayral, to appear)
- *An involution of reals, discontinuous on rationals and whose derivative vanish almost everywhere.* (with H. Ayral, to appear)
- Some deformations of Lebesgue's measure on the boundary of the Farey tree (with H. Ayral, in progress)
- Dynamics of a family of continued fraction maps (with H. Ayral, in progress)
- Conumerator and the conominator, in progress.

M E R C I

BONUS MATERIAL

ج acts on..

- Binary quadratic forms (tears apart class groups)
- Beatty partitions of $\mathbf{N}$.

$$r \in \mathbf{R} \setminus \mathbf{Q} \rightsquigarrow \mathcal{B}_r = \lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, \ldots = (\lfloor nr \rfloor)_{n \geq 1}$$

If $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$ then $\mathcal{B}_r \cup \mathcal{B}_s = \mathbf{N}$.
($\implies$ ج induce a duality of Beatty partitions of $\mathbf{N}$).

- Sturmian words $a_n := \lfloor r(n+1) \rfloor - \lfloor rn \rfloor$.
- Trivalent ribbon graphs $\simeq$ dessins $\simeq$ decorated TM spaces.
($\implies$ ج induces a duality of punctured Riemann surfaces.)
- Dynamical continued fraction maps..
- .....

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- Sturmian words $a_n := \lfloor r(n+1) \rfloor - \lfloor rn \rfloor$.
- Trivalent ribbon graphs $\simeq$ dessins $\simeq$ decorated TM spaces.
($\implies$ ج induces a duality of punctured Riemann surfaces.)
- Dynamical continued fraction maps..
- .....

BONUS MATERIAL

ج acts on..

- Binary quadratic forms (tears apart class groups)
- Beatty partitions of **N**.

$$r \in \mathbf{R} \setminus \mathbf{Q} \rightsquigarrow \mathcal{B}_r = \lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, \ldots = (\lfloor nr \rfloor)_{n \geq 1}$$

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Example.

$$\text{ج}([0;\overline{1_{n-1},a}]) = [0;n,\overline{1_{a-2},n+1}] \implies$$

$$\text{ج}\left(\frac{a}{2}\left[\sqrt{1+4\frac{aF_{n-1}+F_{n-2}}{a^2F_n}}-1\right]\right)$$

$$= \frac{1}{n+\frac{n+1}{2}\left(\sqrt{1+4\frac{(n+1)F_{a-2}+F_{a-3}}{(n+1)^2F_{a-1}}}-1\right)}$$

(notice the exchange $(a, F_n) \leftrightarrow (F_a, n)$)

Functional equations on the upper half plane

One must consider the $\mathrm{PGL}_2(\mathbf{Z})$-action on $\{\mathbf{Im}z > 0\}$ given by

$$M \cdot z := \begin{cases} Mz, & \det(M) = +1 \\ M\bar{z}, & \det(M) = -1 \end{cases}$$

The generators of $\mathrm{PGL}_2(\mathbf{Z})$ in this representation are

$$\bar{U} : z \to \frac{1}{\bar{z}}, \quad \bar{V} : z \to -\bar{z}, \quad \bar{K} : z \to 1 - \bar{z},$$

and the functional equations become

$$f(\bar{U}) = \bar{U}f, \quad f(\bar{V}) = \bar{U}\bar{V}f, \quad f(\bar{K}) = \bar{K}f,$$

in other words

$$f(\frac{1}{\bar{z}}) = \frac{1}{\overline{f(z)}}, \quad f(-\bar{z}) = -\frac{1}{f(z)}, \quad f(1-\bar{z}) = 1 - \overline{f(z)},$$

Functional equations on the upper half plane

If f satisfies the functional equations, i.e.

$$f(M \cdot z) = \mathfrak{J}(M) \cdot f(z) \implies$$

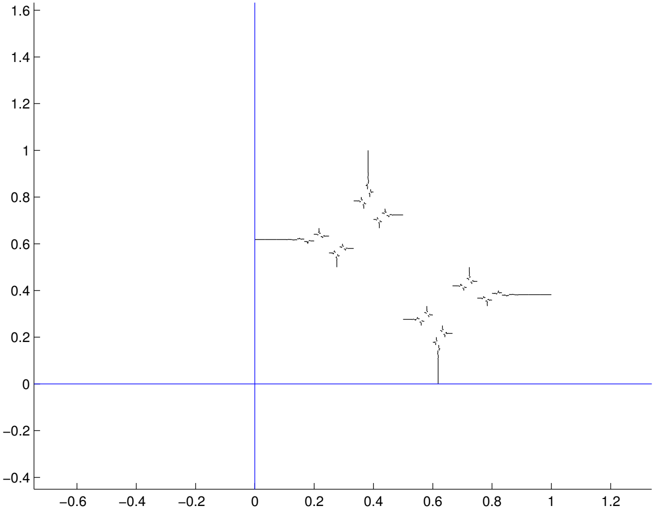
$$f \circ f(M \cdot z) = f(\mathfrak{J}(M) \cdot f(z)) = M \cdot f(z),$$

in other words, $f \circ f$ is $\mathrm{PGL}_2(\mathbf{Z})$-equivariant. Moreover, if g is a modular function, then

$$g \circ f(M \cdot z) = g(\mathfrak{J}(M) \cdot f(z)) = f(z),$$

i.e. $g \circ f$ is also modular.

Zoom



Zoom

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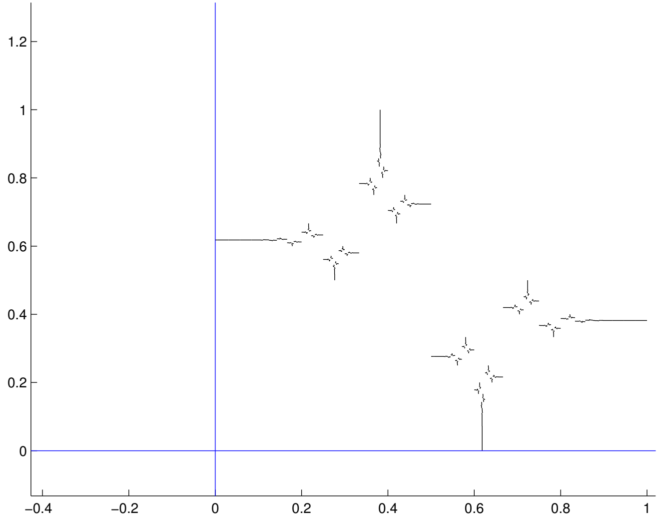

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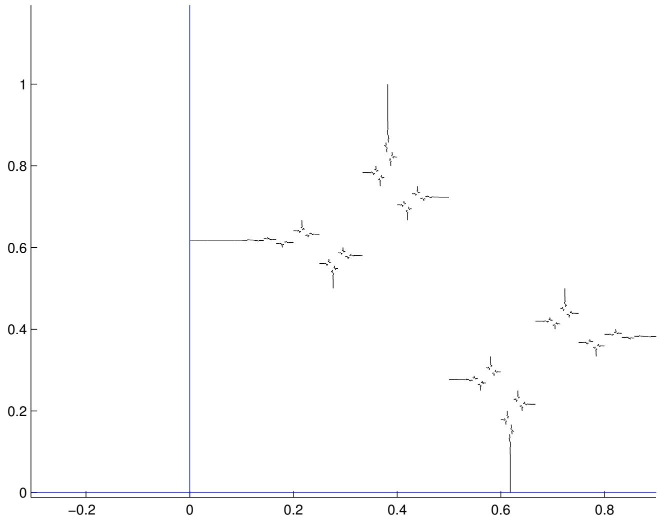

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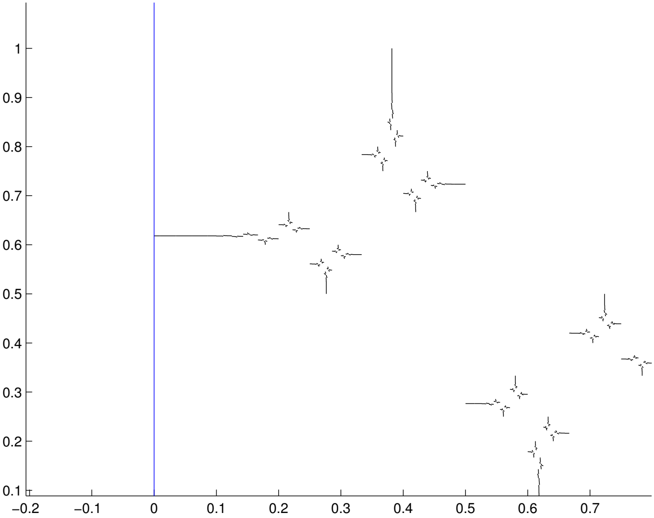

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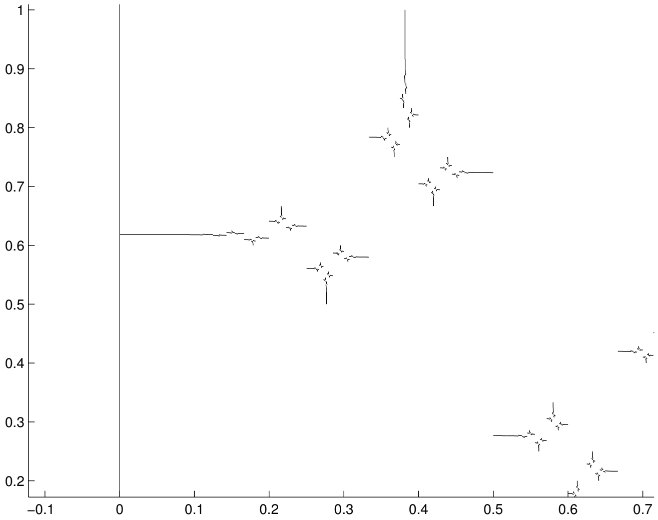

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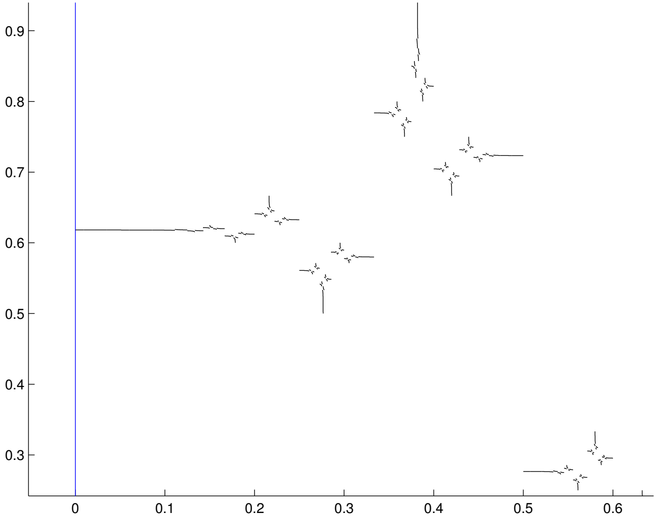

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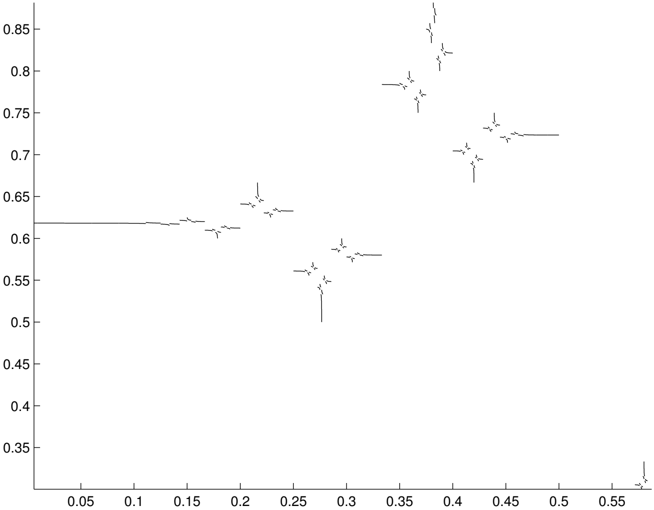
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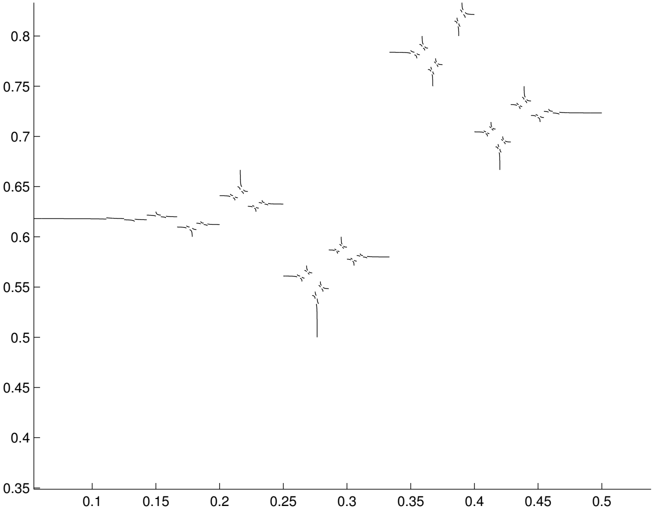
Zoom



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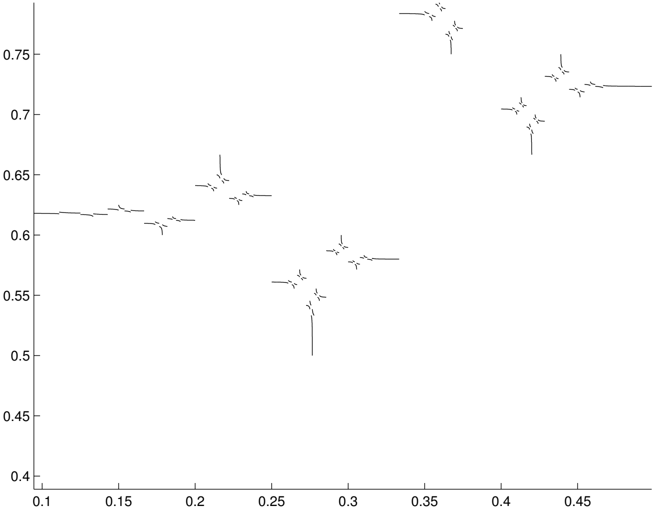

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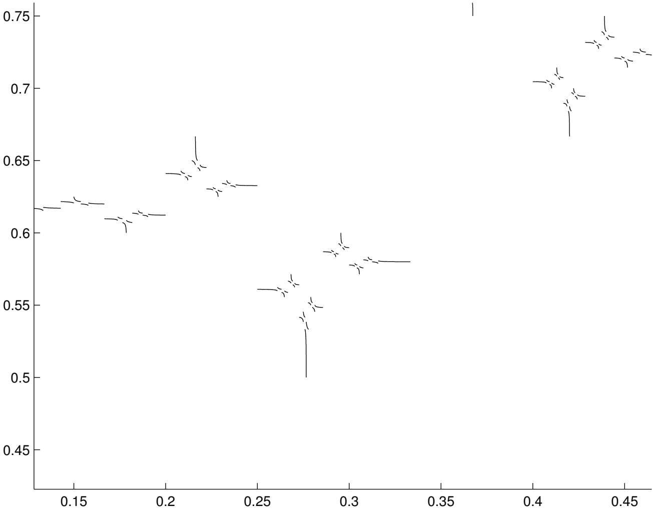

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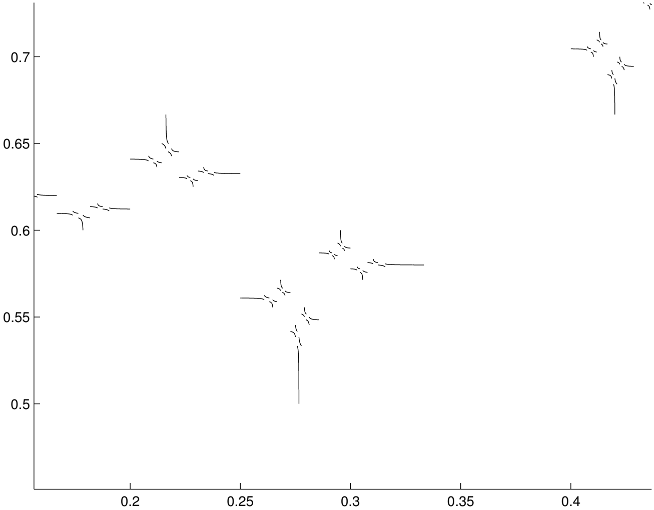

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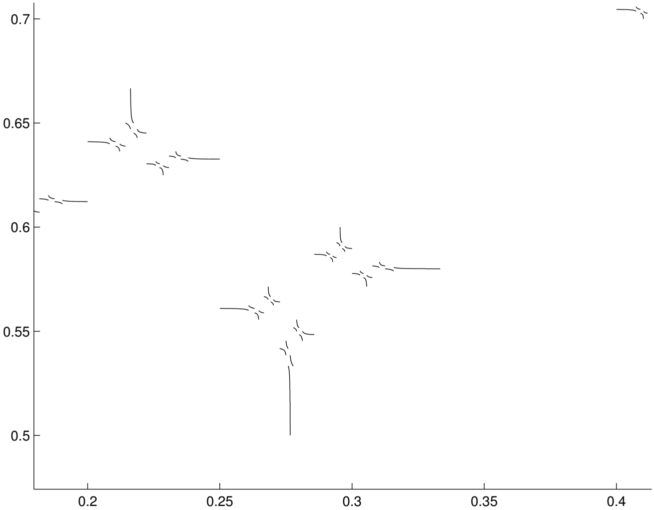

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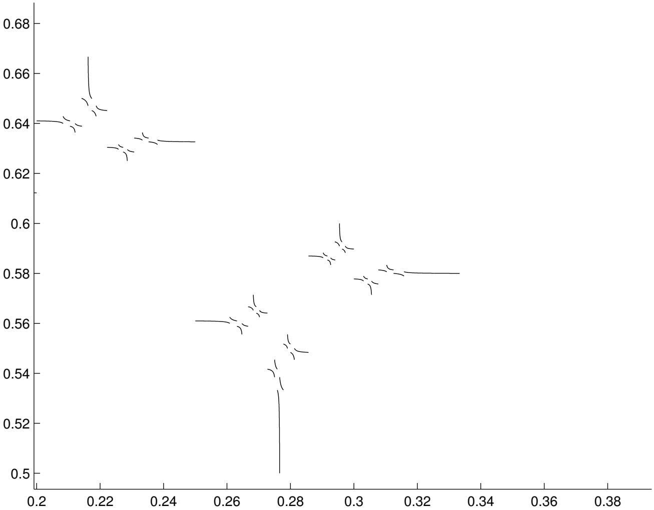
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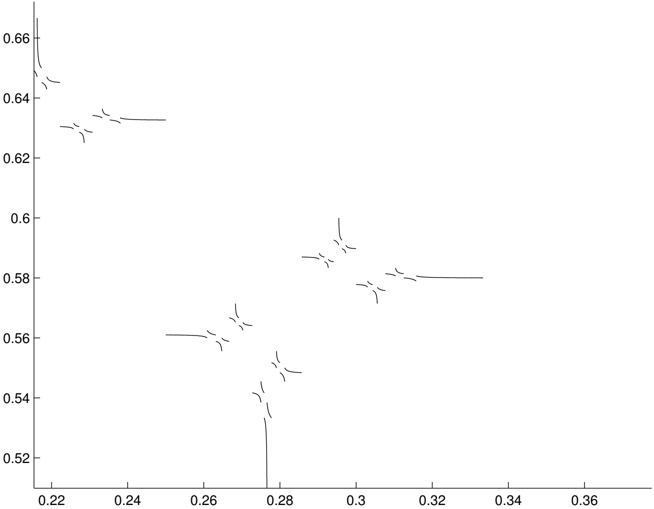
Zoom



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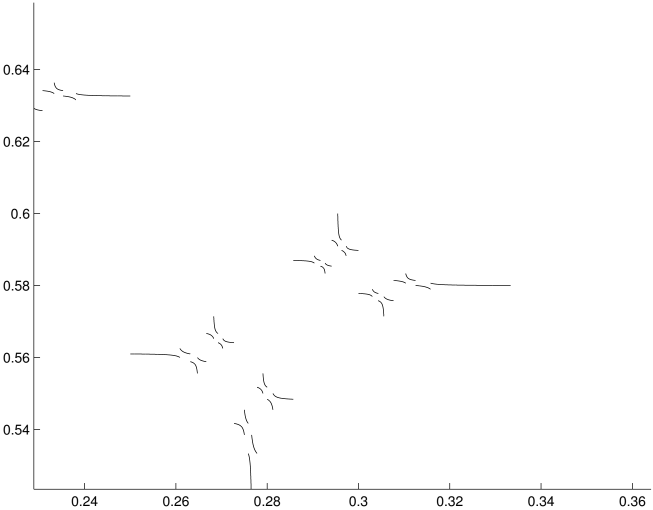

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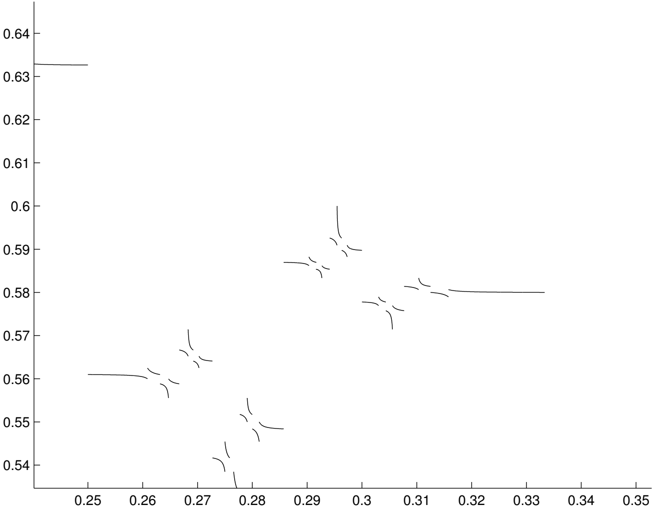
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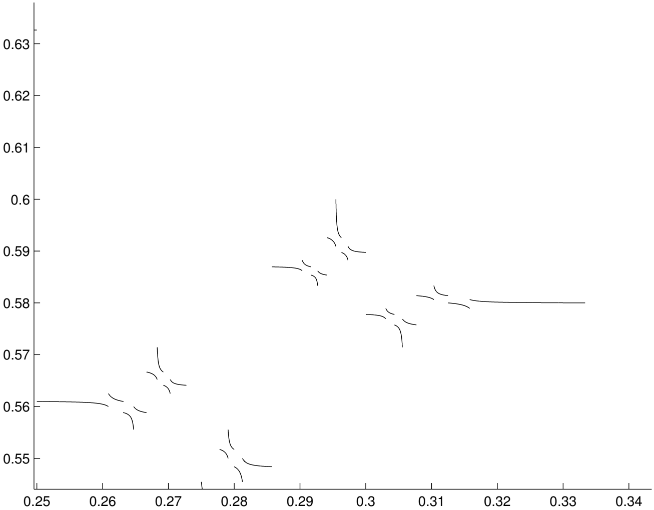
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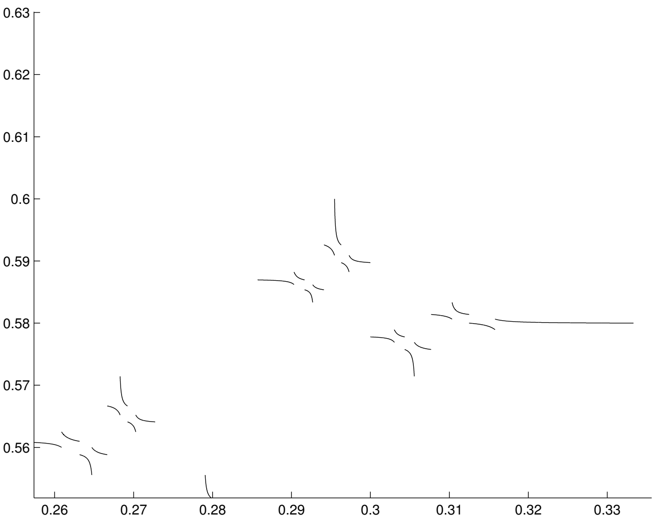

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