Let there be N measurements M_{meas}, with each measured point being approximated well by a fitted value M_{fit} dependant on L fitting or/and input variables φ:

$$\begin{bmatrix} M_{1,fit}(\varphi_1, \dots, \varphi_L) \\ \vdots \\ M_{N,fit}(\varphi_1, \dots, \varphi_L) \end{bmatrix} \cong \begin{bmatrix} M_{1,meas} \\ \vdots \\ M_{N,meas} \end{bmatrix}$$

In the case of a small error in a measured input variable $\varphi_{j,meas}$, the change in a fitted value of M_i can be approximated by a Taylor expansion:

$$M_{i,fit}\left(\varphi_{j,meas} + \Delta\varphi_{j,meas}\right) = M_{i,fit}\left(\varphi_j, meas\right) + \sum_{k=1}^{L} \frac{\partial M_{i,fit}}{\partial \varphi_k} \Delta\varphi_k + \cdots$$

As we always fit the model to the original fitted measurement values, and we assume a good fit:

$$M_{i,fit}\left(\varphi_{j,meas} + \Delta\varphi_{j,meas}\right) \cong M_{i,meas} \equiv M_i$$

and

$$M_{i,fit}\left(\varphi_{j,meas}\right) \cong M_{i,meas} \equiv M_i$$

such that

$$M_i \cong M_i + \sum_{k=1}^{L} \frac{\partial M_i}{\partial \varphi_k} \Delta\varphi_k$$

Thus, any error caused by $\Delta\varphi_{j,meas}$ is compensated by variations in the other fitted parameters $\Delta\varphi_k$:

$$\sum_{\substack{k=1 \\ k\neq j}}^{L} \frac{\partial M_i}{\partial \varphi_k} \Delta\varphi_k + \frac{\partial M_i}{\partial \varphi_{j,meas}} \Delta\varphi_{j,meas} = 0$$

$$\sum_{\substack{k=1 \\ k\neq j}}^{L} \frac{\partial M_i}{\partial \varphi_k} \Delta\varphi_k = -\frac{\partial M_i}{\partial \varphi_{j,meas}} \Delta\varphi_{j,meas}$$

For qMT, the measurement we are interested in as a possible source of error is B_1, and the fitted parameters are F, kf, T2f and T2r ($T_{1,f}$ is omitted, it's simply a function of $T_{1,meas}$, F, and k_f)

$$\frac{\partial M_i}{\partial F} \Delta F + \frac{\partial M_i}{\partial k_f} \Delta k_f + \frac{\partial M_i}{\partial T_{2,f}} \Delta T_{2,f} + \frac{\partial M_i}{\partial T_{2,r}} \Delta T_{2,r} = -\frac{\partial M_i}{\partial B_1} \Delta B_1$$

The sensitivity function is defined as:

$$S_{\varphi_k,i} \equiv \frac{\partial M_i}{\partial \varphi_k}$$

Such that the previous equation can be writing compactly as:

$$\begin{bmatrix} S_{F,i} & S_{k_f,i} & S_{T_{2,f},i} & S_{T_{2,r},i} \end{bmatrix} \begin{bmatrix} \Delta F \\ \Delta k_f \\ \Delta T_{2,f} \\ \Delta T_{2,r} \end{bmatrix} = -S_{B_1,i} \Delta B_1$$

From this equation, we can provide the following general observations:

1. The larger the sensitivity functions $S_{\varphi_k,i}$, the less variation $\Delta\varphi_k$ (e.g. $\Delta F, \Delta k_f, \ldots$) is required for φ_k (e.g. $F, k_f, \ldots$) to compensate the error caused by $\Delta\varphi_{j,meas}$ (e.g. ΔB_1) .
2. The smaller the sensitivity function for the measurement $S_{\varphi_k,i}$ (e.g. $S_{B_1,i} = \frac{\partial M_i}{\partial B_1}$) , the less overall compensation is required by all the φ_k (e.g. $F, k_f, \ldots$).
3. $S_{B_1,i}$ will depend on the T_1 mapping method (e.g $S_{B_1,i}^{VFA}$, $S_{B_1,i}^{IR}$), due to the B_1 dependence of VFA and B_1 independence of IR T_1 maps.

For a set of N measurements (for our qMT protocols, N=10), this equation expands to:

$$\begin{bmatrix} S_{F,1} & S_{k_f,1} & S_{T_{2,f},1} & S_{T_{2,r},1} \\ \vdots & \vdots & \vdots & \vdots \\ S_{F,N} & S_{k_f,N} & S_{T_{2,f},N} & S_{T_{2,r},N} \end{bmatrix} \begin{bmatrix} \Delta F \\ \Delta k_f \\ \Delta T_{2,f} \\ \Delta T_{2,r} \end{bmatrix} = \begin{bmatrix} -S_{B_1,1}\, \Delta B_1 \\ \vdots \\ -S_{B_1,N} \Delta B_1 \end{bmatrix}$$

Which can be written in matrix form:

$$S\psi = \Psi$$

where S is the matrix of sensitivity functions, ψ is the vector of changes in fitted parameters, and Ψ is the vector of $-S_{B_1,i}\Delta B_1$.

As the equation is an overdetermined set of linear equations of the form:

$$Ax = b$$

The optimal changes in parameters ψ can be solved through the following minimization:

$$\min \|S\psi - \Psi\|_2^2$$

for different protocols, B_1 mapping methods, etc. S and Ψ can be calculated analytically (possible, but cumbersome to do so), or estimated through simulations.

From this analysis, and from my qMT B1-insensitivity measurements, we can make the following general predictions:

- $\left|S_{F,i}\right| \gg \left|S_{k_f,i}\right|$ for most measurements (if not all), since $\Delta F \cong 0$ and Δk_f
- $\left|S_{k_f,i}\right| \ll \left|S_{F,i}\right|, \left|S_{T_{2,f},i}\right|, \left|S_{T_{2,r},i}\right|$ for most measurements (if not all) (most of the error is dumped into kf)
- $\left|S_{B_1,i}^{VFA}\right| < \left|S_{B_1,i}^{IR}\right|$ since less ΔF is required for VFA and IR. (This one is less strong, since Δk_f is smaller for IR and VFA, but for ΔF it's the opposite. Possibly a more complicated relationship to the solution of minimization instead of this simple relation)