Chapter 2

Basic Sensitivity and Stability Results

2.1. INTRODUCTION

Comprehensive treatments of linear programming sensitivity analysis and parametric linear programming (i.e., obtaining a solution as a function of problem parameters) have been forthcoming (Dinkelbach, 1969; Nožička, Guddat, Hollatz, and Bank, 1974; Gal, 1979) along with basic sensitivity results (parameter derivatives) in quadratic programming (Boot, 1963). Unified treatments of nonlinear parametric programming methodology have only very recently begun to emerge (Bank, Guddat, Klatte, Kummer, and Tammer, 1982; Brosowski, 1982).

Activity in the study of sensitivity and stability of solutions of general mathematical programs to problem perturbations has been sporadic and results are scattered in the literature. This chapter endeavors to give a flavor of the state of the art in this area. Many important contributions are necessarily omitted in this overview, which is confined to parametric perturbations of nonlinear programming problems. This and the following section, and most of Sections 2.3 and 2.4, are based on Fiacco and Hutzler (1979b), though numerous definitions, additional results, and refinements have been introduced.

As the next two examples demonstrate, the solution of very simple mathematical programs may vary smoothly or change drastically for arbitrarily small perturbations of the problem parameters.

Example 2.1.1

Consider the nonlinear program

$$\min_x (x_1 - \varepsilon)^2 + (x_2 + 1)^2$$

$$\text{s.t. } x_2 \geq x_1,$$

$$x_2 \geq -x_1.$$

The problem may be viewed as seeking $x(\varepsilon) \in R(\varepsilon)$ (the feasible region) that minimizes the distance from $(\varepsilon, -1)$ to $R(\varepsilon)$. A depiction and three solutions are illustrated in Fig. 2.1.1.

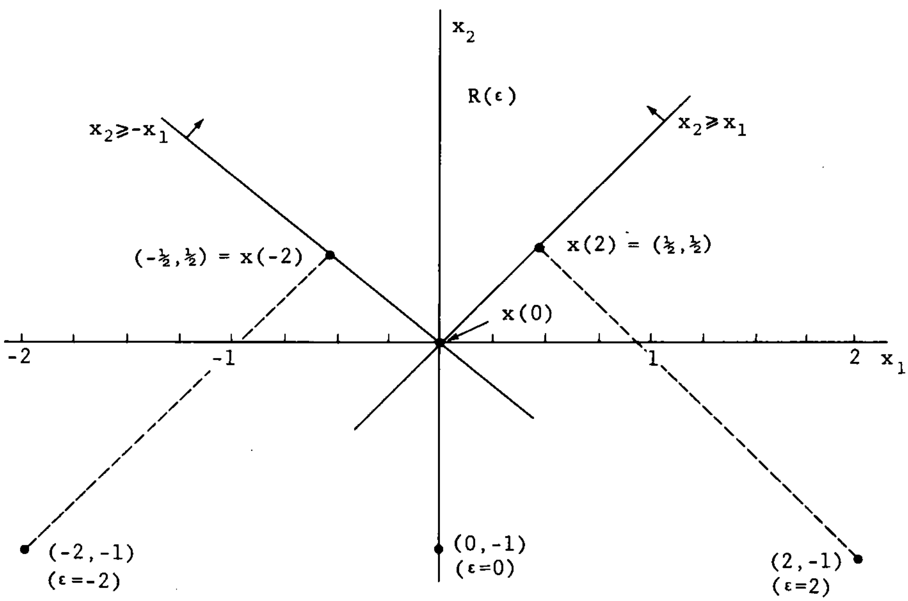

Fig. 2.1.1. Depiction of Example 2.1.1 and solutions for $\varepsilon = -2, 0, 2$.

The analytical solution of this problem is easily seen to be

$$x(\varepsilon)^T = [x_1(\varepsilon), x_2(\varepsilon)] = \begin{cases} ((1+\varepsilon)/2, -(1+\varepsilon)/2), & \varepsilon < -1 \\ (0, 0), & -1 \leq \varepsilon \leq 1 \\ ((\varepsilon - 1)/2, (\varepsilon - 1)/2), & \varepsilon > 1. \end{cases}$$

It is clear that $x(\varepsilon)$ is piecewise linear, continuous, and differentiable everywhere except for $\varepsilon = \pm 1$. It is readily shown that the optimal value function of this problem, $f^*(\varepsilon) = f[x(\varepsilon), \varepsilon]$, is convex (defined just after Theorem 2.2.4), and twice differentiable everywhere except for $\varepsilon = \pm 1$, where it is only once differentiable.

Unfortunately, as the next example illustrates, the solutions of mathematical programs do not always behave so nicely.

Example 2.1.2

$\min_x \varepsilon x$

s.t. $x \geq -1$.

The solution of this problem is given by $x(\varepsilon) = -1$ if $\varepsilon > 0$; $x(\varepsilon)$ can be chosen as any value in $[-1, \infty)$ if $\varepsilon = 0$; and if $\varepsilon < 0$ there is no finite solution of this problem. Thus, $f^*(\varepsilon) = -\varepsilon$, 0 and has no lower bound, for the respective values of ε. Thus, as ε varies in a small neighborhood of the origin in E^1 the solution may be finite and unique, it may be unbounded, or there may be infinitely many solutions.

It should be clear from these two simple examples that very small perturbations of the parameters of a mathematical program can cause a wide variety of results. The purpose of this chapter is to summarize and illustrate the work that has been done to date in providing conditions under which the solutions of nonlinear programs are locally well behaved and in estimating solution properties as a function of problem parameters.

The reader not familiar with other spaces and norms may substitute E^n for all more general spaces and the usual Euclidean norm for other norms in the following results.

2.2 OBJECTIVE FUNCTION AND SOLUTION SET CONTINUITY

Some of the earliest work in stability analysis for nonlinear programming was concerned with the variation of the optimal value function with changes in a parameter appearing in the right hand side of the constraints, i.e., involving problems of the form:

$$\min_x f(x), \quad \text{s.t. } g(x) \geq \varepsilon, \qquad P_1(\varepsilon)$$

where $f: E^n \to E^1$, $g: E^n \to E^m$ and ε is in E^m. The theory of point-to-set maps (Berge, 1963) has been used for much of the analysis of this problem. Hogan (1973d) has provided an excellent development of those properties of point-to-set maps that are especially useful in deriving such results.

Next we present several definitions and properties relating to point-to-set maps that are needed in a number of important results. The terms "map," "mapping," and "point-to-set map" are used interchangeably in the sequel.

Given two topological spaces T and X, a point-to-set mapping Γ from T to X is a function which associates with every point in T a subset of X. Following Berge (1963), we say that the point-to-set mapping Γ is continuous at $t_0 \in T$ if it is both upper semicontinuous and lower semicontinuous at

t_0. These last two notions are established by the following definition, which we include for completeness.

Definition 2.2.1. Let Γ be a point-to-set mapping from T to subsets of X.

(i) Γ is lower semicontinuous at $t_0 \in T$ if, for each open set $S \subset X$ satisfying $S \cap \Gamma(t_0) \neq \emptyset$ there exists a neighborhood N of t_0, $N(t_0)$, such that for each t in $N(t_0)$, $\Gamma(t) \cap S \neq \emptyset$.

(ii) Γ is upper semicontinuous at $t_0 \in T$ if, for each open set $S \subset X$ containing $\Gamma(t_0)$ there exists a neighborhood N of t_0, $N(t_0)$, such that for each t in $N(t_0)$, $\Gamma(t) \subset S$.

Furthermore, if Γ is lower semicontinuous at each point of T, then it is said to be lower semicontinuous in T; and if Γ is upper semicontinuous at each point of T with $\Gamma(t)$ compact for each t, then Γ is said to be upper semicontinuous in T. The mapping Γ is said to be continuous in T if it is both upper and lower semicontinuous in T.

Definitions, simpler and generally easier to apply, that are closely related to these semicontinuity properties for point-to-set maps are based on the properties of their graphs. Let Γ be a point-to-set mapping from T to subsets of X. Then the graph G of Γ is $G \equiv \bigcup_{t\in T}\bigcup_{x\in\Gamma(t)}(t, x)$, i.e., $G = \{(t, x) \in T \times X \mid x \in \Gamma(t)\}$.

Definition 2.2.2. Let $t_n \in T$ be such that $t_n \to t_0$. Then

(i) Γ is open at $t_0 \in T$ if, for each $x_0 \in \Gamma(t_0)$ there exists a value m and a sequence $\{x_n\} \subset X$ such that $x_n \in \Gamma(t_n)$ for each $n > m$ and $x_n \to x_0$.

(ii) Γ is closed at $t_0 \in T$ if $x_n \in \Gamma(t_n)$ for each n and $x_n \to x_0$ together imply that $x_0 \in \Gamma(t_0)$.

The map Γ is said to be open (closed) in T if it is open (closed) at each point of T. It may also be noted that Γ is closed in T if and only if its graph is a closed set in $T \times X$.

An alternative to the Berge definition of continuity of the map Γ is given in the setting of open and closed maps by Hogan (1973d). The map Γ is said to be continuous at $t_0 \in T$ if it is both open and closed at t_0, and it is said to be continuous in T if it is both open and closed in T. The definition that is applicable in the following results will be clear from the context, i.e., the framework for a given result will involve either semicontinuity conditions or open-closed conditions, but not both. In the former case, the Berge definitions will apply and in the latter case the Hogan definitions will apply.

The relationship between these definitions may be found in Hogan (1973d), and is summarized as follows: (i) the definitions of lower semicontinuous and open at a point are equivalent, and (ii) if Γ is uniformly compact near t_0, i.e., if there is a neighborhood N of t_0 such that the closure of the set $\bigcup_{t \in N} \Gamma(t)$ is compact, then Γ is closed at t_0 if and only if $\Gamma(t_0)$ is compact and $\Gamma(t)$ is upper semicontinuous at t_0.

It is clear that if $\Gamma(t_0)$ is a closed set and $\Gamma(t)$ is upper semicontinuous at t_0, then Γ must be closed at t_0. However, the converse does not hold. The following very simple example exhibits this distinction between upper semicontinuous and closed, in the absence of the compactness assumptions. Consider $f\colon E^1 \to E^1$ defined by $f(x) = 1/x$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$. It may readily be verified that f is closed but not upper semicontinuous at $x = 0$.

Following Berge (1963), for real-valued functions we use the notation lsc and usc for lower and upper semicontinuity, respectively. These are defined, according to the customary convention, as follows. Let ϕ be a real-valued function defined on the topological space X. Then

(i) ϕ is said to be lsc at a point $x_0 \in X$ if
$\liminf_{x \to x_0} \phi(x) \geq \phi(x_0)$,

(ii) ϕ is said to be usc at a point $x_0 \in X$ if
$\limsup_{x \to x_0} \phi(x) \leq \phi(x_0)$.

The function ϕ is said to be continuous at x_0 if it is both lsc and usc at x_0 and lsc (usc) on X if ϕ is lsc (usc) at each point of X. If ϕ is continuous at x_0, then

$$\liminf_{x \to x_0} \phi(x) = \limsup_{x \to x_0} \phi(x) = \lim_{x \to x_0} \phi(x).$$

Using these definitions, the following results are easily established for real-valued functions (Berge, 1963). Assume X and T are topological spaces in the next two results.

Theorem 2.2.1. If f is a real-valued usc (lsc) function defined on $X \times T$ and if R is a lower semicontinuous (upper semicontinuous) mapping from T into X such that for each ε in T, $R(\varepsilon) \neq \emptyset$, then the (real-valued) function f^*, defined by

$$f^*(\varepsilon) = \inf_x \{f(x, \varepsilon) \mid x \in R(\varepsilon)\}$$

is usc (lsc).

Theorem 2.2.2. If f is a continuous real-valued function defined on the space $X \times T$ and R is a continuous mapping of T into X such that $R(\varepsilon) \neq \emptyset$ for each ε in T, then the (real-valued) function f^*, defined by

$$f^*(\varepsilon) = \inf_x\{f(x, \varepsilon) \mid x \in R(\varepsilon)\}$$

is continuous in T. Furthermore, the mapping S, defined by

$$S(\varepsilon) = \{x \in R(\varepsilon) \mid f(x, \varepsilon) = f^*(\varepsilon)\},$$

is an upper semicontinuous mapping of T into X.

Conditions that imply the continuity of the solution of a mathematical program have been given by Dantzig, Folkman, and Shapiro (1967) and by Robinson and Day (1974). Letting f be a function from a metric space X to E^1, with $R(\varepsilon) \subset X$, and defining $S(\varepsilon) = \{x \in R(\varepsilon) \mid f(x) = \inf_z\{f(z) \mid z \in R(\varepsilon)\}\}$ Dantzig *et al.* (1967) obtain conditions for S to vary in a closed manner. When R is defined by linear inequalities, they obtain under appropriate conditions that S is a closed mapping. Under this same hypothesis, letting S^* denote the mapping S when it is a singleton, they obtain conditions which yield the continuity of S^* as a function of the parameter ε, in its domain. The domain D of a point-to-set map $\Gamma: T \to X$ is defined as $D = \{t \in T \mid \Gamma(t) \neq \emptyset\}$.

The following classical concept of pointwise topological limits of a sequence $\{R_n\}$ of subsets of a metric space X is utilized. Define the inner limit and the outer limit of $\{R_n\}$, respectively, as

$$\underline{\lim}_{n\to\infty} R_n \equiv \left\{x \in X \mid \exists\ x_n \in R_n \quad \text{for } n \geq n^* \text{ and } x_n \to x\right\}$$

and

$$\overline{\lim}_{n\to\infty} R_n \equiv \left\{x \in X \mid \exists\ x_{n_j} \in R_{n_j} \quad \text{for } j \geq j^* \text{ and } x_{n_j} \to x\right\}.$$

If $\underline{\lim}_{n\to\infty} R_n = \overline{\lim}_{n\to\infty} R_n = R$, then the limit R of $\{R_n\}$ is said to exist and we write

$$\lim_{n\to\infty} R_n = R \quad \text{or} \quad R_n \to R.$$

A connected set M is such that there do not exist disjoint open sets A_1 and A_2 such that $M \subset A_1 \cup A_2$, $M \cap A_1 \neq \emptyset$, and $M \cap A_2 \neq \emptyset$.

Theorem 2.2.3. Let R be a point-to-set mapping from the metric space T to the set of subsets of E^n, with $R(\varepsilon)$ a closed set for each ε in T. Let D be the domain of R. Suppose $R(\varepsilon)$ is a connected set for each ε in D, and for some ε^* in D, $R(\varepsilon^*)$ is compact. Furthermore, assume that for every sequence $\{\varepsilon_n\} \subset D$, $\varepsilon_n \to \varepsilon^*$ implies $R(\varepsilon_n) \to R(\varepsilon^*)$. Then, if $f \in C(E^n)$, the mapping S^* is continuous at ε^*, if ε^* is in its domain.

If g is an affine function from E^n to E^m, i.e., if $g(x) = Ax + b$, where A is an $m \times n$ constant matrix and $b \in E^m$ is a constant vector, and if $R_M(g) = \{x \in M | g(x) \geq 0\}$ where $M \subset E^n$, the function g is said to be nondegenerate with respect to the set M if $R_M(g)$ has a nonempty interior and no component of g is identically zero. The continuity of the map defined by $S_M(g) = \{x \in R_M(g) | f(x) = \inf_z\{f(z) | z \in R_M(g)\}\}$ and S_M^* (the mapping S_M when it is a singleton) as functions of g is given by the following theorems. The set M is said to be convex if $x, y \in M$ imply that $\alpha x + (1 - \alpha)y \in M$ for any $\alpha \in [0, 1]$.

Theorem 2.2.4. If f is continuous, g is affine, and M is closed and convex, then S_M is closed at every nondegenerate point g.

A function ϕ is said to be convex on a convex set M if $\phi(\alpha x + (1 - \alpha)y) \leq \alpha\phi(x) + (1 - \alpha)\phi(y)$ for any $x, y \in M$ and any $\alpha \in [0, 1]$. A function ϕ is quasi-convex on a convex set M if and only if $\{x \in M | \phi(x) \leq k\}$ is convex for any real number

k. Equivalently, $\phi(x)$ is quasi-convex on M if $\max\{\phi(x), \phi(y)\} \geq \phi(\lambda x + (1 - \lambda)y)$ for every $x, y \in M$ and every $\lambda \in [0, 1]$. It follows that a convex function is quasi-convex.

Theorem 2.2.5. If the assumptions of Theorem 2.2.4 hold and f is quasi-convex or $R_M(g)$ is bounded, then S_M^* is continuous at a nondegenerate point g of its domain.

Motivated by applications to the theory of economic choice, Robinson and Day (1974), considering a general constraint set $R(\varepsilon)$, provide conditions that guarantee the continuity of the point-to-set mapping whose value at ε is the set of solutions of the mathematical program

$$\begin{aligned} &\min_x f(x, \varepsilon) \\ &\text{s.t. } x \in R(\varepsilon) \\ &\qquad \varepsilon \in T, \end{aligned}$$

where $R(\varepsilon)$ represents a constraint set as a function of the parameter ε, and X and T are topological vector spaces. To that end, let $f^*(\varepsilon) = \inf_x\{f(x, \varepsilon) \mid x \in R(\varepsilon)\}$, and define the mapping $S: T \to X$ by $S(\varepsilon) = \{x \in R(\varepsilon) \mid f(x, \varepsilon) = f^*(\varepsilon)\}$.

A function ϕ is said to be strictly quasi-convex on a convex set M if, for all $x_1, x_2 \in M$ such that $\phi(x_1) < \phi(x_2)$ and all $\lambda \in (0, 1)$, $\phi(\lambda x_1 + (1 - \lambda)x_2) < \phi(x_2)$. A convex function is also strictly quasi-convex.

Theorem 2.2.6. Assume that the space X is locally convex and that $R(\varepsilon) \neq \emptyset$ for each $\varepsilon \in T$ and that R is continuous and convex-valued on T (i.e., $R(\varepsilon)$ is convex for each ε in T). If $f(x, \varepsilon) = \min\{f_1(x, \varepsilon), f_2(\varepsilon)\}$, where f_1 is continuous on $X \times T$ and strictly quasi-convex in x for each fixed ε, and f_2 is continuous on T, then S is continuous and convex-valued on T.

As might be expected, stronger stability characterizations were initially found for the ostensibly simpler right hand side (rhs) problems $P_1(\varepsilon)$ and $P_2(\varepsilon)$. (However, it is noted in Section 2.6 that the general problem $P_3(\varepsilon)$ may be reformulated as an *equivalent* rhs problem of the form $P_2(\varepsilon)$.)

Recall that the inequality-constrained rhs problem is

$$\min_x f(x) \quad \text{s.t. } g(x) \geq \varepsilon \qquad P_1(\varepsilon)$$

where now $f: E^n \to E^1$ and $g: E^n \to E^m$. Associated with problem $P_1(\varepsilon)$ are: (i) the feasible region, $R(\varepsilon) = \{x \in E^n | g(x) \geq \varepsilon\}$; (ii) the set $D = \{\varepsilon \in E^m | R(\varepsilon) \neq \emptyset\}$, i.e., the domain of R and the effective domain of f^*; (iii) a set $I(\varepsilon) = \{x \in E^n | g(x) > \varepsilon\}$ associated with the interior of the feasible region; and (iv) the optimal value function $f^*(\varepsilon) = \inf_x\{f(x) | x \in R(\varepsilon)\}$, also called the "marginal function" when the perturbations are rhs perturbations.

The following three theorems, due to Evans and Gould (1970), provide conditions for the stability of the constraint set, i.e., the feasible region $R(\varepsilon)$, as well as for the continuity of the optimal value function. In the statements of the following results in this section we will denote by R a point-to-set mapping from the set $D \subset E^m$ to the set of all subsets of E^n, with the value at ε in D given by $R(\varepsilon)$. The interior of the set D is denoted by D^0, and the closure of $I(\varepsilon)$ is denoted by $\overline{I(\varepsilon)}$. Assume g is continuous.

Theorem 2.2.7.

(i) The mapping R is upper semicontinuous at ε if and only if there exists a vector $\varepsilon' < \varepsilon$ such that $R(\varepsilon')$ is compact.

(ii) If $R(\varepsilon)$ is compact and $I(\varepsilon) \neq \emptyset$, then R is lower semicontinuous at ε if and only if $\overline{I(\varepsilon)} = R(\varepsilon)$.

Theorem 2.2.8.

(i) If f is lsc and R is upper semicontinuous at ε, then f^* is lsc at ε.

(ii) If $I(\varepsilon) \neq \emptyset$, f is usc and R is lower semicontinuous at ε, then f^* is usc at ε.

The following theorem is an immediate consequence of these two results, which give important realizations of Theorem 2.2.1.

Theorem 2.2.9. Suppose ε is in D^0 and that f is continuous in E^n. Also assume that there exists a vector $\varepsilon' < \varepsilon$ such that $R(\varepsilon')$ is compact, and that $\overline{I(\varepsilon)} = R(\varepsilon)$. Then, R is a continuous mapping and f^* a continuous function at ε.

Theorem 2.2.9 is related to Theorem 2.2.2, giving conditions that imply the satisfaction of the hypotheses of Berge's theorem. The question of the stability of the set of optimal solutions and the stability of the optimal value function has been addressed by a number of authors. Greenberg and Pierskalla (1972), referring to problem P_1, have shown that the solution set point-to-set mapping S is upper semicontinuous at ε if R is upper semicontinuous at ε and if f^* is continuous at ε. This result is very similar to one given by Dantzig *et al.* (1967) and by Berge (1963). The essential differences among these results lie in the use of semicontinuity in Greenberg and Pierskalla (1972) and the closedness of maps in Dantzig *et al.* (1967), while the conclusion drawn in Berge (1963) is based on the continuity of R. Greenberg and Pierskalla (1972) also extend the Evans-Gould results to allow for general constraint perturbations.

Considering the marginal function for convex programming problems, which is central to the construction of decomposition algorithms for large-scale nonlinear programs, Hogan (1973c) established conditions for its continuity. These results, including the case in which the parameter of the problem appears in the objective function as well as in the constraints, are given in the next two theorems. The function ϕ was said to be convex on a convex set M if $\phi(\alpha x + (1 - \alpha)y) \leq \alpha\phi(x) + (1 - \alpha)\phi(y)$ for any $x, y \in M$ and any $\alpha \in [0, 1]$. If for $x \neq y$ and $\alpha \in (0, 1)$ the inequality is strict, then ϕ is said to be strictly convex. A function ϕ is said to be (strictly) concave if $-\phi$ is (strictly) convex.

Theorem 2.2.10. Let $f^*(\varepsilon) = \inf_x\{f(x) \mid x \in M, g(x) \geq \varepsilon\}$. If

(i) M is a compact convex set in E^n,

(ii) f is continuous on M,

(iii) g_i is usc on M, and

(iv) each g_i is strictly concave on M,

then f^* is continuous on its effective domain in E^m.

Theorem 2.2.11. Let $f^*(\varepsilon) = \inf_x\{f(x, \varepsilon) \mid x \in M, g(x, \varepsilon) \geq 0\}$. If

(i) M is a compact convex set in E^n,

(ii) f and g are both continuous on $M \times E^k$, and

(iii) each component of g is strictly concave on M for each ε,

then f^* is continuous on its effective domain.

Referring back to problem $P_1(\varepsilon)$, let $S_\delta(\varepsilon) = \{x \in R(\varepsilon) \mid f(x) \leq f^*(\varepsilon) + \delta \text{ for } \delta \geq 0\}$ define a point-to-set map S_δ from E^m to E^n.

Stern and Topkis (1976), defining a notion of linear continuity, establish conditions under which f^* is Lipschitz continuous. Under convexity assumptions on the problem functions they also show that the map S_δ whose value is $S_\delta(\varepsilon)$, the set of δ-optimal solutions, is continuous. The notation introduced above for $P_1(\varepsilon)$ is used. It is assumed that f and g are continuous.

Definition 2.2.3. The real-valued function ϕ is said to be Lipschitz continuous on a set $M \subset X$, where X is a normed space, if there exists a value $\lambda > 0$ such that $|\phi(x) - \phi(y)| \leq \lambda\|x - y\|$ for all $x, y \in M$.

Definition 2.2.4. Suppose Γ is a point-to-set mapping from $T \subset E^m$ to subsets of E^n. Then Γ is said to be uniformly linearly continuous on $T_0 \subset T$ if there exists a value $\lambda > 0$ such that $\inf_{x \in \Gamma(\bar{t})} \|z - x\| \leq \lambda \cdot \|t - \bar{t}\|$, for all $z \in \Gamma(t)$ and for all $t, \bar{t} \in T_0$.

Theorem 2.2.12. Let $D_0 \subset D$ (the domain of R) and suppose $R(\bar{\varepsilon})$ is bounded for some $\bar{\varepsilon}$ in D. If R is uniformly linearly continuous with constant k on $T_0 = D_0 \cap \{\varepsilon | \varepsilon \geq \bar{\varepsilon}\}$, and if f is Lipschitz continuous with constant c on $R(\bar{\varepsilon})$, then f^* is Lipschitz continuous with constant kc on T_0.

Stern and Topkis show that the following conditions imply that R is uniformly linearly continuous on a subset T_1 of D: (i) D_0 is a closed subset of D, $I(\varepsilon) \neq \emptyset$ for each ε in D_0, $R(\bar{\varepsilon})$ bounded and convex, and $g_j \in C^1$ and convex on $R(\bar{\varepsilon})$ for all j, with $T_1 = T_0$; or (ii) the Cottle Constraint Qualification (see (CQ4) in the next section) holds at some $\hat{\varepsilon}$ in D, there exists $\bar{\varepsilon} < \hat{\varepsilon}$ such that $R(\bar{\varepsilon})$ is bounded and $g_j \in C^2$ for all j, with $T_1 = D \cap \{\varepsilon | \, \|\varepsilon - \hat{\varepsilon}\| \leq \alpha\}$ for some $\alpha > 0$. They also show that the conditions given in (i), along with $f \in C^1$ and convex imply that the map S_δ of the set $S_\delta(\varepsilon)$ of δ-optimal solutions is uniformly linearly continuous on T_0 for each $\delta > 0$ and also obtain the next continuity result for S_δ.

Theorem 2.2.13. If f and $-g_i$, i = 1, ..., m, are strictly quasi-convex, $R(\bar{\varepsilon})$ is bounded, and $I(\bar{\varepsilon}) \neq \emptyset$, then S_δ is continuous at $\bar{\varepsilon}$ for each $\delta > 0$.

Hager (1979) recently obtained Lipschitz results for quadratic programs with unique solutions, assuming that these exist for small perturbations and that the perturbation parameter varies over a convex set. Lipschitz continuity of the solution and multiplier vector of such programs follows under the hypothesis that the gradients of the binding constraints satisfy the linear independence criterion (CQ3, Section 2.3). In both instances an estimate of the Lipschitz constant is provided.

Lipschitz properties of the optimal value function and estimates of the Clarke (1975) generalized gradient were obtained by Gauvin (1979) for problem $P_2(\varepsilon)$ (Section 2.3) and extended by Gauvin and Dubeau (1979) to problem $P_3(\varepsilon)$ (Section 2.3). Extensions in a more general setting have been provided recently by Gollan (1981a,b) and Rockafellar (1982a,b).

2.3. DIFFERENTIAL STABILITY

In this section we concentrate on theory that has been developed to analyze various stability properties of the optimal value function, evolving from an analysis of the rate of change of this function at a solution point. We begin with a brief discussion of several well known constraint qualifications that are typically invoked in obtaining many of the results.

The constraint qualifications used in mathematical programming are regularity conditions which are generally imposed to insure that the set of Karush-Kuhn-Tucker multipliers (Karush, 1939; Kuhn and Tucker, 1951) corresponding to an optimal solution of a mathematical program is nonempty. We present here five qualifications that are frequently applied. These and a number of others are treated in some detail by Mangasarian (1969). Throughout this discussion we shall assume the constraint set $R \equiv \{x \in E^n | \ g(x) \geq 0, \ h(x) = 0\}$, where $g\colon E^n \to E^m$ and $h\colon E^n \to E^p$ are once continuously differentiable vector functions. Define $B(x) \equiv \{i | \ g_i(x) = 0\}$. The operators ∇ and ∇^2, without subscripts, denote differentiation with respect to x.

CQ1. The Mangasarian-Fromovitz constraint qualification (Mangasarian and Fromovitz, 1967) is said to hold at a point $x^* \in R$ if:

(i) there exists a vector $z \in E^n$ such that $\nabla g_i(x^*)z > 0$ for all $i \in B(x^*)$, $\nabla h_j(x^*)z = 0$ for $j = 1, \ldots, p$, and

(ii) the gradients $\{\nabla h_j(x^*), \ j = 1, \ldots, p\}$ are linearly independent.

This condition is equivalent to (CQ1)': $\Sigma_{i \in B(x^*)} u_i \nabla g_i(x^*) + \Sigma_{j=1}^{p} w_j \nabla h_j(x^*) = 0$ has no nonzero solution $u_i \geq 0$, w_j, for $x^* \in R$.

If the g_i are concave (or even pseudo-concave) functions and the h_j are affine, then CQ1 is equivalent to the well known "Slater condition" (Slater, 1950), a general form of which we give as CQ2.

Definition 2.3.1. If the function $\phi: E^n \to E^1$ is differentiable on the convex set M and $\phi(y) \geq \phi(x)$ for all $x, y \in M$, with $\nabla\phi(x)(y - x) \geq 0$, then ϕ is said to be pseudo-convex on M. (The function ϕ is said to be pseudo-concave if $-\phi$ is pseudo-convex.)

A differentiable convex function is pseudo-convex and a pseudo-convex function is both quasi-convex and strictly quasi-convex.

CQ2. The Slater constraint qualification is satisfied at $x^* \in R$ if h_j is affine for each j, g_i is pseudo-concave for each i and there exists a point $\bar{x} \in R$ with $g_i(\bar{x}) > 0$ for each i in $B(x^*)$.

CQ3. The linear independence assumption is said to hold at $x^* \in R$ if the gradients $\{\nabla g_i(x^*),\ i \in B(x^*);\ \nabla h_j(x^*),\ j = 1, \ldots, p\}$ are linearly independent.

CQ4. If there are no equality constraints and $\sum_{i \in B(x^*)} u_i \nabla g_i(x^*) = 0$ has no nonzero solution $u_i \geq 0$, for $x^* \in R$, the Cottle constraint qualification is said to hold at x^*. (In the absence of equality constraints, CQ1 is equivalent to CQ4.)

CQ5. The Kuhn-Tucker constraint qualification (Kuhn and Tucker, 1951) is satisfied at $x^* \in R$ if, for each nonzero vector $z \in E^n$ satisfying $\nabla g_i(x^*)z \geq 0$ for each $i \in B(x^*)$ and $\nabla h_j(x^*)z = 0$, $j = 1, \ldots, p$, z is tangent to a once-differentiable arc originating at x^* and contained in R.

The relationships that hold among these qualifications, in addition to those already mentioned, are that CQ3 implies CQ1, which, in turn, is sufficient for CQ5. Also, CQ2 implies CQ4, which also implies CQ5. For a proof and further discussion of the relationships among these constraint qualifications see Arrow, Hurwicz, and Uzawa (1961), Bazaraa, Goode, and Shetty (1972), Mangasarian (1969), and Peterson (1973).

Robinson (1976b) has shown the equivalence of CQ1 and a form of local stability of the set of solutions of a system of inequalities. Gauvin (1977) has shown that CQ1 is both necessary and sufficient for the set of Lagrange multiplier vectors corresponding to a given local solution of a general NLP problem to be nonempty, compact, and convex. In addition, Gauvin and Tolle (1977) established that CQ1 is preserved under rhs perturbations. If CQ3 holds, it is well known that the Lagrange multiplier vector, corresponding to a given solution, exists and is unique.

One of the earliest characterizations of the differential stability of the optimal value function of a mathematical program was provided by Danskin (1966, 1967). Addressing the problem of minimizing $f(x, \varepsilon)$ subject to $x \in R$, R some topological space, Danskin derived conditions under which the directional derivative of the optimal value function exists, and also determined its representation.

Definition 2.3.2. The (one-sided) directional derivative of the function ϕ at the point t in the direction z is defined to be:

$$D_z\phi(t) = \lim_{\beta \to 0^+} [\phi(t + \beta z) - \phi(t)]/\beta$$

if the limit exists.

Theorem 2.3.1. Let R be nonempty and compact and let f and the partial derivatives $\partial f/\partial \varepsilon_i$ be continuous. Then at any point ε in E^k and for any direction $z \in E^k$ the directional derivative of f^* exists and is given by

$$D_z f^*(\varepsilon) = \min_{x \in S(\varepsilon)} \nabla_\varepsilon f(x, \varepsilon) z,$$

where $S(\varepsilon) = \{x \in R | f(x, \varepsilon) = f^*(\varepsilon)\}$.

This result has wide applicability in the sense that the constraint space R can be any compact topological space. It has been extended by a number of authors, including Dem'yanov and Rubinov (1968), to other spaces and a variety of functional forms. The principal restriction of this result is that the set R does not vary with the parameter ε. However, since inequality and equality constraints can be "absorbed" into the objective function of a program through the use of an appropriate auxiliary function (Lagrangian, penalty function, etc.), Danskin's result is often applicable to auxiliary function methods. It can also be readily applied to the objective function of the dual of a convex program with rhs perturbations. Turning to the problem where R depends on a parameter ε through the inequality $g(x, \varepsilon) \geq 0$, and f and the components $-g_i$ are convex, Hogan (1973b) has shown that $D_z f^*(\varepsilon)$ exists and is finite for all $z \in E^k$. The following theorem presents the details of this result for $f^*(\varepsilon) = \inf_x\{f(x, \varepsilon) | x \in M, g(x, \varepsilon) \geq 0\}$, where M is a subset of E^n. The Lagrangian for this problem is defined as $L(x, u, \varepsilon) \equiv f(x, \varepsilon) - u^T g(x, \varepsilon)$. For convenience and without loss of generality, we shall assume that the parameter value of interest is $\varepsilon = 0$, unless otherwise stated.

Theorem 2.3.2. Let M be a closed and convex set. Suppose f and the $-g_i$ are convex on M for each fixed ε and are continuously differentiable on $M \times N(0)$, where N(0) is a neighborhood of $\varepsilon = 0$ in E^k. If $S(0) \equiv \{x \in M| \ g(x, 0) \geq 0$ and

$f^*(0) = f(x, 0)\}$ is nonempty and bounded, $f^*(0)$ is finite, and there is a point $\bar{x} \in M$ such that $g(\bar{x}, 0) > 0$, then $D_z f^*(0)$ exists and is finite for all $z \in E^k$, and

$$D_z f^*(0) = \min_{x\in S(0)} \max_{u\in K(x,0)} \nabla_\varepsilon L(x, u, 0) z$$

$$= \min_{x\in S(0)} \max_{u\in K(x,0)} \left\{(\nabla_\varepsilon f(x, 0) - u^T \nabla_\varepsilon g(x, 0)) z\right\},$$

where $K(x, 0)$ is the set of optimal Lagrange multipliers for the given $x \in S(0)$.

For convex programs the set $K(x, 0)$ is the same for each $x \in S(0)$. This does *not* allow us to drop the minimization over $x \in S(0)$ in the expression for $D_z f^*(0)$, however, since the quantity $\nabla_\varepsilon L(x, u, 0) z$ being minimized will generally depend on x. However, it does allow for a considerable economy in the calculation of $D_z f^*$ since the constraint set $K(x, 0)$ associated with the inner maximization problem does not depend on x.

Some recent investigations of this sort have focused on the extremal value function inequality-equality constrained optimization problems with rhs perturbations, of the form

$$\min_x f(x) \quad \text{s.t. } g(x) \geq \varepsilon^1,\ h(x) = \varepsilon^2. \qquad P_2(\varepsilon)$$

Let $R(\varepsilon) = \{x \mid g(x) \geq \varepsilon^1,\ h(x) = \varepsilon^2\}$, and let

$$f^*(\varepsilon) = \begin{cases} \inf_x \{f(x) \mid x \in R(\varepsilon)\}, & R(\varepsilon) \neq \emptyset \\ +\infty, & R(\varepsilon) = \emptyset, \end{cases}$$

consistent with our notation for $P_3(\varepsilon)$, where $x \in E^n$, $\varepsilon^1 \in E^m$, $\varepsilon^2 \in E^p$, $\varepsilon = (\varepsilon^1, \varepsilon^2)^T \in E^k (k = m + p)$ in the following results in this section, unless otherwise specified.

We also define, for $R(\varepsilon) \neq \emptyset$, $S(\varepsilon) = \{x \in R(\varepsilon) | f(x) = f^*(\varepsilon)\}$, and the Lagrangian $L(x, u, w, \varepsilon) \equiv f(x) - u^T[g(x) - \varepsilon^1] + w^T[h(x) - \varepsilon^2]$. Given these definitions, Gauvin and Tolle (1977) have proved the following continuity property of f^*.

Theorem 2.3.3. If $R(0) \neq \emptyset$, with R uniformly compact near $\varepsilon = 0$, and if CQ1 holds for some $x^* \in S(0)$, then f^* is continuous at $\varepsilon = 0$.

Fiacco (1980b,c) and Gauvin and Dubeau (1979) showed that this result holds for the general problem

$$\begin{aligned} &\min_x f(x, \varepsilon) \\ &\text{s.t. } g(x, \varepsilon) \geq 0, \\ &\qquad h(x, \varepsilon) = 0, \end{aligned} \qquad P_3(\varepsilon)$$

where the problem functions are C^1 in (x, ε), $R(\varepsilon)$, and $S(\varepsilon)$ are defined to be the feasible region and solution sets of $P_3(\varepsilon)$, respectively, and the parameter ε is a vector in E^k.

In the absence of equality constraints, Rockafellar (1974) has shown that, under certain second-order conditions, the function f^* of $P_2(\varepsilon)$ satisfies a stability of degree two condition, i.e., in a neighborhood of $\varepsilon = 0$ there exists a twice-differentiable function $\phi: E^{m+p} \to E^1$ with $f^*(\varepsilon) \geq \phi(\varepsilon)$ and $f^*(0) = \phi(0)$. Under this stability property, bounds on the directional derivatives of f^* (when they exist) can be derived. For convex programming problems of the form $P_3(\varepsilon)$, Gol'stein (1972) has shown that a saddle-point condition is satisfied by the directional derivative of f^*. Gauvin and Tolle (1977), not assuming convexity, but limiting their analysis to problem $P_2(\varepsilon)$, extend the work of Gol'stein and provide sharp bounds on the Dini upper and lower derivatives of f^*, also without requiring the existence of second-order derivatives. These results were extended by Fiacco and Hutzler (1979a) to the general inequality-constrained problem and by Fiacco (1980c) and Gauvin and Dubeau (1979) to the more general problem $P_3(\varepsilon)$, and are presented next. The Gauvin-Dubeau extension follows the approach of Auslender (1979), while the author's approach is based on eliminating the equality constraint and using elementary arguments to deal with the resulting simpler inequality-constrained problem.

Let $L(x, u, w, \varepsilon) \equiv f(x, \varepsilon) - u^T g(x, \varepsilon) + w^T h(x, \varepsilon)$ denote the usual Lagrangian of problem $P_3(\varepsilon)$ and let $K(x, 0)$ denote the set of Karush-Kuhn-Tucker vectors (u, w) corresponding to a solution x of $P_3(\varepsilon)$ at $\varepsilon = 0$, let $Q(z, \beta) = [f^*(\beta z) - f^*(0)]/\beta$, where $z \in E^k$ is a unit vector. As above, $R(\varepsilon)$ denotes the feasible region, $S(\varepsilon)$ the solution set, and the functions f, g, h are assumed jointly once continuously differentiable in (x, ε). The following results hold for problem $P_3(\varepsilon)$.

The Dini upper and lower derivatives are defined as $\limsup_{\beta \to 0^+} Q(z, \beta)$ and $\liminf_{\beta \to 0^+} Q(z, \beta)$, respectively.

Theorem 2.3.4. If $R(0) \neq \emptyset$, R is uniformly compact near $\varepsilon = 0$, and CQ1 holds for some $x^* \in S(0)$ then, for any $z \in E^k$,

$$\liminf_{\beta \to 0^+} Q(z, \beta) \geq \min_{(u,w) \in K(x^*, 0)} \nabla_\varepsilon L(x^*, u, w, 0) z. \quad (2.3.1)$$

The next corollary, following immediately from the theorem, gives a result that is weaker, but useful in the sequel.

Corollary 2.3.5. If $R(0) \neq \emptyset$, R is uniformly compact near $\varepsilon = 0$, and CQ1 holds at each $x \in S(0)$, then, for any $z \in E^k$,

$$\liminf_{\beta \to 0^+} Q(z, \beta) \geq \inf_{x \in S(0)} \min_{(u,w) \in K(x, 0)} \nabla_\varepsilon L(x, u, w, 0) z. \quad (2.3.2)$$

Theorem 2.3.6. Under the conditions of Corollary 2.3.5, for any $z \in E^k$,

$$\limsup_{\beta \to 0^+} Q(z, \beta) \le \inf_{x \in S(0)} \max_{(u,w) \in K(x,0)} \nabla_\varepsilon L(x, u, w, 0) z. \quad (2.3.3)$$

Corollary 2.3.7. If, in the hypotheses of Corollary 2.3.5, CQ1 is replaced with CQ3, then for each $z \in E^k$, $D_z f^*(0)$ exists and

$$D_z f^*(0) = \inf_{x \in S(0)} \nabla_\varepsilon L[x, u(x), w(x), 0] z, \quad (2.3.4)$$

where $[u(x), w(x)]$ is the unique optimal Lagrange multiplier vector associated with $x \in S(0)$.

Auslender (1979) also obtained these bounds for the rhs perturbation problem $P_2(\varepsilon)$, extending the results of Gauvin and Tolle (1977) by using a weaker form of the Mangasarian-Fromovitz constraint qualification and adapting implicit function theorem results due to Hestenes (1975). This allows him to replace the differentiability assumption on the objective and inequality functions with the weaker requirement that they be locally Lipschitz. Fontanie (1980) used the author's reduction approach in addressing an extension of Auslender's results to allow general perturbations. Utilizing concepts of subdifferential analysis and generalized derivatives, Rockafellar (1982a) has provided general bounds and sharper results under weaker assumptions than those given heretofore. Further significant extensions of these results have been provided by Rockafellar (1982b), utilizing a second-order constraint qualification and saddle-point properties of augmented Lagrangians.

The following theorem (Fiacco and Hutzler, 1979a; Fiacco, 1980b) corresponds, under slightly different assumptions, to results obtained by Gol'stein (1972) and Hogan (1973b) (Theorem 2.3.2) for a general class of problems of the form $P_3(\varepsilon)$ that are convex in x.

Corollary 2.3.8. Let f and $-g_i$, i = 1, ..., m be convex functions in x, and let the functions h_j, j = 1, ..., p be affine in x, with all functions jointly C^1 in (x, ε). If R(0) ≠ ∅, R is uniformly compact near ε = 0, and CQ1 is satisfied for each x ∈ S(0), then $D_z f^*(0)$ exists for each z ∈ E^k, and

$$D_z f^*(0) = \inf_{x \in S(0)} \max_{(u,w) \in K(x,0)} \nabla_\varepsilon L(x, u, w, 0) z. \tag{2.3.5}$$

As noted following Theorem 2.3.2, the expression for (2.3.5) is theoretically and computationally simplified by noting that K(x, 0) ≡ K(0), a set that is the same for each x ∈ S(0).

Although we are focusing attention on programs for which the spaces involved are finite dimensional, we note that most of these sensitivity results have been extended to infinite dimensional programs. For example, Maurer (1977a,b) obtained a characterization of the directional derivative of the extremal value function subgradient for problem $P_3(\varepsilon)$, and has applied his results to a class of optimal control problems. Lempio and Maurer (1980) have obtained similar bounds under analogous assumptions that are required to handle general perturbed infinite-dimensional programs of the form minimize f(x, ε), subject to x ∈ M_1 and g(x, ε) ∈ M_2, where M_1 and M_2 are arbitrary closed convex sets. Other extensions in infinite-dimensional spaces may be found in the works of Dem'yanov and Pevnyi (1972), Gollan (1981a,b), Levitin (1976), Rockafellar (1982a,b), and others.

If the inequalities (2.3.2) and (2.3.3) are applied to problem $P_2(\varepsilon)$, then we obtain the result

$$\inf_{x \in S(0)} \min_{(u,w) \in K(x,0)} \left(u^T z_1 + w^T z_2\right) \le \liminf_{\beta \to 0^+} Q(z, \beta)$$

$$\le \limsup_{\beta \to 0^+} Q(z, \beta) \le \inf_{x \in S(0)} \max_{(u,w) \in K(x,0)} \left(u^T z_1 + w^T z_2\right)$$

obtained by Gauvin and Tolle (1977), where we have defined $z = (z_1, z_2)^T$. The equation (2.3.4) resulting from the additional assumption of linear independence (CQ3) becomes

$$D_z f^*(0) = \inf_{x \in S(0)} \left(u(x)^T z_1 + w(x)^T z_2\right).$$

These relationships explicitly link the rate of change of the optimal value function f^* with the optimal Lagrange multipliers (u, w). In particular, they show that, under the given assumptions, the rate of change of f^* in any direction z is bounded if the union of the set of optimal Lagrange multipliers associated with S(0) is bounded. It is clear that many additional valuable insights might be obtained from this connection, in deducing optimal value stability properties from optimal multiplier properties and conversely.

This line of inquiry has been extremely fertile. We next indicate some results, relating to this connection, that provide an elegant unification of important necessary optimality conditions.

For the problem

$$\min_x f(x) \quad \text{s.t. } g(x) \ge \varepsilon, \qquad P_1'(\varepsilon)$$
$$x \in M$$

where M is a convex subset of E^n, f is convex and the g_i are concave, Geoffrion (1971) gives a clear presentation of some basic results. He first notes that the optimal value function f^* is convex. For the result of interest, the following definitions are needed. Let ϕ be a convex function and assume $\phi(\overline{x})$ is finite. A vector v is said to be a *subgradient* of ϕ at the point $\overline{x}$ if

$$\phi(x) \geq \phi(\overline{x}) + (x - \overline{x})^T v \text{ for all } x.$$

The program $P_1'(\varepsilon)$ is called "stable" if $f^*(\varepsilon)$ is finite and there exists $c > 0$ such that

$$\frac{f^*(\varepsilon') - f^*(\varepsilon)}{\|\varepsilon' - \varepsilon\|} \geq -c \quad \text{for all } \varepsilon' \neq \varepsilon.$$

This condition was apparently first introduced by Gale (1967) to obtain duality results. Assuming $P_1'(\varepsilon)$ has a solution, Geoffrion shows that $P_1'(\varepsilon)$ is stable if and only if an optimal Karush-Kuhn-Tucker multiplier u^* exists, and u^* is an optimal multiplier for $P_1'(\varepsilon)$ if and only if u^* is a subgradient of f^* at ε.

Variants of the last result have apparently been known for some time. For example, Rockafellar (1970) shows that if $P_1'(\varepsilon)$ is a convex program satisfying Slater's condition and has a solution, then the stated result holds.

Rockafellar (1967) used an equivalent notion of stability of a problem of the form

$$\min_x f(x) - g(Ax)$$

$$\text{s.t. } x \in M \quad \text{and} \quad Ax \in T_0.$$

Here, A is a linear transformation from a real (finite or infinite-dimensional) vector space X to another similar space T, f is a finite-valued convex function on $M \subset X$, where $M \neq \emptyset$

and convex, and g is a finite-valued concave function on $T_0 \subset T$ for $T_0 \neq \emptyset$ and convex. The perturbed problem considered will be called $\tilde{P}(\varepsilon)$ and is the above problem with the objective function $f(x) - g(Ax - \varepsilon)$, where ε is in T. Rockafellar shows first that the optimal value f^* of $\tilde{P}(\varepsilon)$ is convex on T. Suppose that $f^*(0)$ is finite. He then defines $\tilde{P}(0)$ to be "stably set" if the directional derivative

$$D_z f^*(0) = \lim_{\alpha \to 0^+} \frac{f^*(\alpha z) - f^*(0)}{\alpha} > -\infty \quad \text{for all } z \in T$$

and the constraints of $\tilde{P}(0)$ are "consistent" (i.e., satisfied by at least one x). (Geoffrion (1971) notes that this condition is equivalent to Gale's condition, given in the last paragraph.) Rockafellar proves that if there exists at least one $x \in X$ where f is finite at x and g is finite and continuous at Ax, then $\tilde{P}(0)$ is stably set and f^* is continuous near $\varepsilon = 0$. He proceeds to prove many strong duality relationships between the given convex program $\tilde{P}(0)$ and a concave program similar in form to $\tilde{P}(0)$ and constructed by means of the conjugate function theory of Fenchel (1949).

Clarke (1976) has shown that if X is a Banach space and f is locally Lipschitz, then programs of the form $P_1(\varepsilon)$, with $x \in M \subset X$ are "normal" in the sense that generalized Karush-Kuhn-Tucker conditions can be shown to hold, even in the absence of differentiability and convexity assumptions. Clarke terms the program

$$\begin{aligned} &\min_x f(x) \\ &\text{s.t. } g(x) \geq \varepsilon, \qquad\qquad P_1'(\varepsilon) \\ &\qquad x \in M, \end{aligned}$$

where M is a closed subset of E^n, normal if Karush-Kuhn-Tucker type multipliers exist for any solution x. The problem $P_1'(\varepsilon)$ is said to be "calm" if $f^*(\varepsilon)$ is finite and

$$\liminf_{\varepsilon' \to \varepsilon} [f^*(\varepsilon') - f^*(\varepsilon)]/\|\varepsilon' - \varepsilon\| > -\infty \qquad (2.3.6)$$

where as usual

$$f^*(\varepsilon) = \begin{cases} \inf_x\{f(x) \mid x \in M,\ g(x) \geq \varepsilon\}, & R(\varepsilon) \neq \emptyset \\ +\infty, & R(\varepsilon) = \emptyset. \end{cases}$$

(We note that the limit quotient in Eq. (2.3.6) is a form of stability also used by Rockafellar (1967).) Using these notions, Clarke showed that if the problem is calm it is also normal, and if f^* is finite in a neighborhood of $\varepsilon = 0$, then the problem is calm and normal for almost all ε in a neighborhood of 0. Conditions sufficient for the calmness (and hence the normality) of the problem are given in the following theorem.

Theorem 2.3.9. If

(i) $-g_i$, $i = 1, \ldots, m$, are convex,

(ii) M is convex and bounded,

(iii) f is bounded and Lipschitz continuous on M, and

(iv) there exists a point $x \in M$ such that $g(x) > \varepsilon$, then

$P_1'(\varepsilon)$ is calm.

Extensions, refinements, and generalizations of many of the results in this section may be found in Hiriart-Urruty (1979) and Gollan (1981a,b), and a further sharpening and more general and unified treatment of these and related results in Rockafellar (1982a,b).

2.4. IMPLICIT FUNCTION THEOREM RESULTS

There are many forms of implicit function theorems that have found extensive application in functional analysis. These theorems treat the general problem of solving an equation of the form

$$\phi(x, y) = 0 \tag{2.4.1}$$

for x in terms of y. The classical results in this area are well known. For completeness we present two forms of the implicit function theorem. A more complete discussion of these and other theorems is contained in Bochner and Martin (1948), Hestenes (1966, 1975), and in most advanced texts in functional analysis. These results extend to more general spaces.

Theorem 2.4.1. [Implicit Function Theorem for C^k Function.] Suppose $\phi: E^{n+m} \to E^n$ is a k times continuously differentiable mapping whose domain is T. Suppose $(\bar{x}, \bar{y}) \in T$, $\phi(\bar{x}, \bar{y}) = 0$, and the Jacobian with respect to x, $\nabla_x\phi(\bar{x}, \bar{y})$, is nonsingular. Then there exists a neighborhood of $\bar{y}$, $N(\bar{y}) \subset E^m$, and a unique function $\gamma \in C^k[N(\bar{y})]$, $\gamma: N(\bar{y}) \to E^n$, with $\gamma(\bar{y}) = \bar{x}$ and $\phi[\gamma(y), y] = 0$ for all $y \in N(\bar{y})$.

The function γ is said to be defined implicitly by the equation $\phi[\gamma(y), y] = 0$.

In the next theorem, the notation ϕ_j is used to denote the j-th component of $\phi: E^{k+\ell} \to E^k$ with respect to its j-th argument, and $\partial(\phi_1, \ldots, \phi_k)/\partial(x_1, \ldots, x_k)$ denotes the Jacobian of $(\phi_1, \ldots, \phi_k)$ with respect to $(x_1, \ldots, x_k)$.

Theorem 2.4.2. [Implicit Function Theorem for Analytic Function.] If $\phi_j(x_1, \ldots, x_k; y_1, \ldots, y_\ell)$ is analytic in a neighborhood of the origin with $\phi_j(0, 0) = 0$ for $j = 1, \ldots, k$, and $[\partial(\phi_1, \ldots, \phi_k)/\partial(x_1, \ldots, x_k)]^{-1}$ exists at $x = y = 0$, then the system of equations $\phi_j(x_1, \ldots, x_k; y_1, \ldots, y_\ell) = 0$, for $j = 1, \ldots, k$, has a unique solution $x_j = x_j(y_1, \ldots, y_\ell)$, which vanishes for $y = 0$ and which is analytic in a neighborhood of the origin.

Results of this type have particular applicability to sensitivity analysis in nonlinear optimization and have only recently been exploited in NLP. Hildebrandt and Graves (1927) have provided results on the existence and differentiability of solutions of Eq. 2.4.1. Cesari (1966) has established conditions under which the equation $\phi(y, y) = 0$ has at least one solution, and discusses the continuous dependence of y on parameters of the equation. Rheinboldt (1969) has given global existence theorems for the solution of (2.4.1), which leads to a "continuation property." This continuation property has been applied to the solution of parametric optimization programs. Recent applications of implicit function theory results due to Hestenes (1975) have been made in differential stability by Auslender (1979) and Gauvin and Dubeau (1979), as noted in the previous section.

Fiacco and McCormick (1968) and Duffin, Peterson, and Zener (1967) (for geometric programming (GP)) provided some of the first applications of an implicit function theorem to obtaining sensitivity information about the solution of a mathematical program. Since then, additional results in this area have been obtained by Bigelow and Shapiro (1974), Armacost and Fiacco (1974, 1975, 1976, 1977, 1978), Armacost (1976a,b), Fiacco (1976), Jittorntrum (1978, 1981), Robinson (1974), and Spingarn (1977).

Robinson (1979) provided an implicit function theorem for a "generalized equation" where we seek a "solution" of

$$0 \in F(y, \varepsilon) + T(y) \tag{2.4.2}$$

where $F: Y \times P \to E^s$ is a vector function, Y is an open set in E^s, P is a topological space, T is a closed point-to-set mapping from E^s into itself, and ε is a parameter vector. Given ε, the problem is to find y such that the point-to-set map on the rhs of (2.4.2) contains 0, or equivalently, such that the vector $-F(y, \varepsilon) \in T(y)$. It is assumed that $F(y, \varepsilon)$ is (jointly) continuous and the partial Fréchet derivative $\nabla_y F$ is continuous on $Y \times P$.

Given $\bar{\varepsilon} \in P$, the linearization of F at $\bar{\varepsilon}$ around $y_0 \in Y$ is given by $LF_{y_0}(y) = F(y_0, \bar{\varepsilon}) + \nabla_y F(y_0, \bar{\varepsilon})(y - y_0)$. It is assumed that Y_0 is a nonempty bounded convex set and that $Y_\gamma = Y_0 + \gamma B \subset Y$ for some $\gamma > 0$, where B is the (open) unit ball in E^s. Conditions are given such that: (i) the set of solutions $\hat{S}(\varepsilon)$ of (2.4.2) in $Y_0 + \delta B$ is upper semicontinuous in some neighborhood $N(\bar{\varepsilon})$ of $\bar{\varepsilon}$ for some $\delta \in (0, \gamma]$ (with $\hat{S}(\varepsilon) = \phi$ if $\varepsilon \notin N(\bar{\varepsilon})$); (ii) $\hat{S}(\bar{\varepsilon}) = Y_0$; and (iii) for each $\alpha > 0$ and some neighborhood $N_\alpha(\bar{\varepsilon})$ and each $\varepsilon \in N_\alpha(\bar{\varepsilon})$,

$$\phi \neq \hat{S}(\varepsilon) \subset \hat{S}(\bar{\varepsilon}) + (\lambda + \alpha) \max_y \{ \| F(y, \varepsilon) - F(y, \bar{\varepsilon}) \| \mid y \in Y_0 \} B \tag{2.4.3}$$

for some $\lambda > 0$.

Nonlinear complementarity and equilibrium problems and the Karush-Kuhn-Tucker conditions for $P_3(\varepsilon)$ can be expressed as realizations of (2.4.2). The latter constitute our interest here and are expressed in the form (2.4.2) when we take $s = n + m + p$, $y = (x, u, w)^T$, $F = \left[\nabla_x L, g^T, -h^T\right]^T$, and

$T = \partial\psi_C$, where ψ_C denotes the indicator function of C defined by

$$\psi_C(y) = \begin{cases} 0\,, & y \in C \\ +\infty, & y \notin C \end{cases}$$

and ∂ denotes the "subdifferential operator" (i.e., $\partial\psi_C(y)$ is the set of all subgradients of ψ_C at y) [Rockafellar (1970)] and $C = E^n \times E^m_+ \times E^p$, where $E^m_+ = \{u \in E^m | u \geq 0\}$.

The additional conditions required for these results are the existence of $\eta > 0$, along with the following assumptions for each $y_0 \in Y_0$:

(i) $Y_\gamma \cap G^{-1}_{y_0}(0) = Y_0$;

(ii) $Y_\gamma \cap G^{-1}_{y_0}(y) \subset Y_\gamma \cap G^{-1}_{y_0}(0) + \lambda\|y\|B$ for each $y \in \eta B$;

(iii) $Y_\gamma \cap G^{-1}_{y_0}(y) \neq \emptyset$ and convex for each $y \in \eta B$,

where $G_{y_0} = LF_{y_0} + T$, and the inverse Γ^{-1} of a point-to-set map Γ: $X \to Z$ at $\bar{z} \in Z$ is defined to be $\Gamma^{-1}(\bar{z}) = \{x \in X | \bar{z} \in \Gamma(x)\}$.

Robinson also shows that assumption (iii) may be replaced by

(iii)' $\nabla_y F(y_0, \bar{\epsilon})$ is positive semidefinite and T is a maximal monotone operator. An operator T is called "monotone" if for each (y_1, v_1), (y_2, v_2) in the graph of T, it follows that $(y_1 - y_2)^T(v_1 - v_2) \geq 0$. It is called "maximal monotone" if it is monotone and its graph is not properly contained in that of any other monotone operator.

Many other results for parametric nonlinear programs are developed by Robinson, making use of the generalized equation framework briefly indicated here.

An illustration of an application of the above results is provided by Robinson to analyze the solution set of the *linear* generalized equation

$$0 \in Ay + a + T(y) \tag{2.4.4}$$

where A is a positive semidefinite $s \times s$ matrix, $a \in E^s$, $T = \partial\psi_C$ and C is a nonempty polyhedral convex set in E^s. It is shown that the solution set of (2.4.4) is nonempty and bounded if and only if there exists $\delta_0 > 0$ such that for each $s \times s$ matrix A' and each $a \in E^s$ with $\delta' \equiv \max\{\|A' - A\|, \|a' - a\|\} < \delta_0$, the set $\hat{S}(A', a') \equiv \{y | 0 \in A'y + a' + \partial\psi_C(y)\} = \emptyset$. Under the assumption that μ be a bound on $\hat{S}(A, a) \neq \emptyset$, it is shown that $\exists\lambda > 0$ such that

$$\phi \neq \hat{S}(A', a') \cap M \subset \hat{S}(A, a) + \lambda\delta'(1 - \lambda\delta')^{-1}(1 + \mu)B \tag{2.4.5}$$

where M is any open bounded set containing $\hat{S}(A, a)$ and $\delta' < \delta_1$ for some $\delta_1 > 0$. Linear and quadratic programming and linear complementarity problems can be posed in the form (2.4.4).

Applied to a quadratic program, for example, (2.4.5) provides an extension of a result due to Daniel (1973). It concerns the solution stability for positive definite quadratic programs, which is given next for comparison with (2.4.5). Daniel considers a program in E^n of the form

$$\min_x (1/2)x^T Kx - k^T x$$

$$\text{s.t. } Cx \leq c$$

$$Dx = d,$$

where K is positive definite and symmetric with $\tilde{\lambda} > 0$ its smallest eigenvalue. Daniel obtained the following special case of (2.4.5).

Theorem 2.4.3. If $\delta = \max\{\|K' - K\|, \|k' - k\|\}$, then for $\delta < \tilde{\lambda}$,

$$\|x_0' - x_0\| \leq \delta(\tilde{\lambda} - \delta)^{-1}(1 + \|x_0\|),$$

where x_0 solves the program above and x_0' solves that program when K' and k' replace K and k, respectively.

Using additional assumptions, a number of stronger results have been obtained which characterize more completely the relationship between a solution set and the optimal value function of a mathematical program to general perturbations appearing simultaneously in the objective function and anywhere in the constraints. These problems generally have the form

$$\begin{aligned} &\min_x \ f(x, \varepsilon) \\ &\text{s.t. } g(x, \varepsilon) \geq 0 \qquad\qquad P_3(\varepsilon) \\ &\qquad h(x, \varepsilon) = 0, \end{aligned}$$

where $f: E^n \times E^k \to E^1$, $g: E^n \times E^k \to E^m$, and $h: E^n \times E^k \to E^p$, unless specifically stated otherwise.

McCormick (Fiacco and McCormick, 1968) obtained conditions that guarantee the existence of a differentiable function of ε that locally solves a particular form (problem $P_3''(\varepsilon)$, Section 3.4) of $P_3(\varepsilon)$. Robinson (1974) and Fiacco (1976) extended this result to programs in which the perturbations appear generally, as in $P_3(\varepsilon)$. These results are established by applying an appropriate implicit function theorem to the first-order necessary conditions that must hold at a local solution of $P_3(\varepsilon)$ to prove the existence of a continuous or differentiable Karush-Kuhn-Tucker triple. The author's extension, given in the next theorem, establishes the existence of a once continuously differentiable (local) solution of $P_3(\varepsilon)$, along with associated unique differentiable optimal Lagrange multipliers. This is the basic theorem for the sensitivity results developed in this book, and is proved (with slightly weaker differentiability assumptions) in Chapter 3 (Theorem 3.2.2).

Theorem 2.4.4. If

(1) f, g, h are C^2 in (x, ε) in a neighborhood of $(x^*, 0)$,

(2) the second order sufficiency conditions hold at $[x^*, u^*, w^*]$ (see Lemma 3.2.1),

(3) the linear independence assumption holds at x^*, and

(4) $u_i^* > 0$ for all i such that $g_i(x^*) = 0$, i.e., strict complementary slackness with respect to u^* holds at x^*,

then

(i) x^* is a local isolated (i.e., locally unique) minimizing point of $P_3(0)$ with unique Lagrange multipliers u^*, w^*,

(ii) for ε near 0, there exists a unique C^1 function $y(\varepsilon) = [x(\varepsilon), u(\varepsilon), w(\varepsilon)]^T$ satisfying the second-order sufficiency conditions for problem $P_3(\varepsilon)$, with $y(0) = [x^*, u^*, w^*]^T$, hence $x(\varepsilon)$ is an isolated local minimizing point of $P_3(\varepsilon)$ with associated unique Lagrange multipliers $u(\varepsilon)$ and $w(\varepsilon)$, and

(iii) for ε near 0, the gradients of the binding constraints are linearly independent, and strict complementary slackness holds for $u(\varepsilon)$ and $g(x, \varepsilon)$.

Fiacco (1976) also provided an explicit formula for the partial derivatives, as well as approximations based on classical penalty functions (Theorem 2.4.9). These results provide the basis for the methodology presented in Part II of this book.

Jittorntrum (1978, 1981), essentially completing the development of results pursued by Bigelow and Shapiro (1974), obtained the following results, which do not require the strict complementarity condition (4) of Theorem 2.4.4. However, the second-order part of the second-order sufficiency conditions (2) of the theorem must be strengthened to: $z^T \nabla^2 L(x^*, u^*, w^*, 0)z > 0$ for all $z \neq 0$ such that $\nabla g_i(x^*, 0)z = 0$, for all i such that $g_i(x^*) = 0$ and $u_i^* > 0$ and such that $\nabla h_j(x^*, 0)z = 0$ for all j. Following Robinson (1980a), we call this condition, taken together with the first-order Karush-Kuhn-Tucker conditions (see Lemma 3.2.1), "the strong second-order sufficient conditions for problem $P_3(0)$." Conclusions (i) - (v) of the next theorem were obtained by Jittorntrum in the cited 1978 doctoral dissertation, Conclusion (vi) in the cited 1981 paper.

Theorem 2.4.5. If f, g, h are C^2 in (x, ε) in a neighborhood of $(x^*, 0)$, if the strong second-order sufficient conditions for problem $P_3(0)$ hold at (x^*, u^*, w^*), and if the linear independence condition holds at x^*, then

(i) x^* is a local isolated minimizing point of $P_3(0)$ and the associated Lagrange multipliers u^* and w^* are unique,

(ii) for ε in a neighborhood of 0, there exists a unique *continuous* vector function $y(\varepsilon) = [x(\varepsilon), u(\varepsilon), w(\varepsilon)]^T$ satisfying the strong second-order sufficiency condition for a local minimum of the problem $P_3(\varepsilon)$ such that $y(0) = (x^*, u^*, w^*)^T$, and hence, $x(\varepsilon)$ is a locally unique minimizer of $P_3(\varepsilon)$ with associated unique Lagrange multipliers $u(\varepsilon)$ and $w(\varepsilon)$,

(iii) linear independence of the binding constraint gradients holds at $x(\varepsilon)$ for ε near 0,

(iv) there exist $0 < \alpha, \beta, \gamma < \infty$ and $\delta > 0$ such that for every ε with $\|\varepsilon\| < \delta$,

$$\|x(\varepsilon) - x^*\| \le \alpha\|\varepsilon\|,$$

$$\|u(\varepsilon) - u^*\| \le \beta\|\varepsilon\|,$$

and

$$\|w(\varepsilon) - w^*\| \le \gamma\|\varepsilon\|,$$

(v) $f^*(\varepsilon) = f[x(\varepsilon), \varepsilon]$ is differentiable with respect to ε at $\varepsilon = 0$, with

$$\nabla_\varepsilon f^*(0) = \nabla_\varepsilon L(x^*, u^*, w^*, 0),$$

and

(vi) in any direction $z \neq 0$, the (uniquely determined, one-sided) directional derivative $D_z y(\varepsilon)$ of (the components of) $[x(\varepsilon), u(\varepsilon), w(\varepsilon)]$ exists at $\varepsilon = 0$.

We note that, under the assumptions of Theorem 2.4.5, it may be further concluded that $f^*(\varepsilon)$ is once continuously differentiable and $[x(\varepsilon), u(\varepsilon), w(\varepsilon)]$ has directional derivatives in any direction *near* $\varepsilon = 0$, thus extending Conclusions v and vi of the theorem. Furthermore, since $\nabla_\varepsilon f^*(\varepsilon) = \nabla_\varepsilon L[y(\varepsilon), \varepsilon]$ near $\varepsilon = 0$ and the problem functions are C^2, we note that the (component by component) directional derivative D_z of $\nabla_\varepsilon f^*$ exists near $\varepsilon = 0$ in any direction z and is given by

$$D_z\left[\nabla_\varepsilon f^*(\varepsilon)^T\right] = \nabla^2_{y\varepsilon} L D_z y + \nabla^2_\varepsilon L\, z \Big|_{[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon]}.$$

Also, the directional second derivative D_z^2 of $f^*(\varepsilon)$ exists, given by

$$\begin{aligned} D_z^2 f^*(\varepsilon) &= z^T D_z\left[\left(\nabla_\varepsilon f^*\right)^T\right] \\ &= z^T \nabla^2_{y\varepsilon} L D_z y + z^T \nabla^2_\varepsilon L\, z \Big|_{[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon]}. \end{aligned}$$

These expressions are consistent with the formula we obtain in Theorem 3.4.1 for $\nabla_{\varepsilon}^{2} f^{*}(\varepsilon)$ when $f^{*} \in C^{2}$.

Jittorntrum also provides an approach for computing the directional derivatives, based on solving a collection of equations and inequalities derived from necessary optimality conditions, that must hold at a solution. Using the strong second-order sufficient conditions for $P_3(\varepsilon)$, Robinson (1980a) subsequently obtained general results that also dispense with strict complementarity and essentially subsume Theorem 2.4.5 (Conclusions (i) - (v)) as particular realizations. Kojima (1980) proved additional stability results for $P_3(\varepsilon)$, without strict complementary slackness, using the degree theory of continuous maps. It is also relevant to note that some efforts have been made to relax the linear independence assumption, as well as the strict complementarity assumption, obtaining similar results. Notably, Kojima (1980) showed that $x(\varepsilon)$ will be continuous and an *isolated* local minimum of $P(\varepsilon)$ near $\varepsilon = 0$ if the Mangasarian-Fromovitz constraint qualification (CQ1) holds at x^{*}, rather than linear independence of the binding constraint gradients, *providing* the strong second-order sufficient conditions hold at x^{*} for *all* optimal multipliers (u, w) associated with x^{*}. This use of the strong second-order sufficient conditions is consistent with Robinson's finding (1980a,b) that CQ1 and the *usual* second-order sufficient conditions (Lemma 3.2.1) holding at x^{*} for *all* optimal (u, w) are not sufficient that $x(\varepsilon)$ be an isolated local minimum of $P(\varepsilon)$ for ε near 0 (although these assumptions suffice to conclude that $x^{*} = x(0)$ *is* an isolated local minimum of $P(0)$, as noted in the remarks following Lemma 3.2.1).

Spingarn (1977) extended Theorem 2.4.4 by considering the problem $P_3(\varepsilon)$ with additional constraints that restrict ε to a C^2 submanifold T in E^k and restrict x to a "cyrtohedron" H of class C^2 in E^n. He has shown a certain set of second-order conditions to be necessary for optimality, and that these conditions also imply the results obtained in Theorem 2.4.4. Before stating this result, we introduce Spingarn's definitions.

Let A and B be finite (possibly empty) index sets, and for $i \in A$ and $j \in B$, let $\{g_i\}$ and $\{h_j\}$ be finite collections of C^1 functions defined on the open set $U \subset E^n$. Also, for $x \in U$ and $A' \subset A$, let

$$V(x, A') = \{\nabla g_i(x) \mid i \in A'\} \cup \{\nabla h_j(x) \mid j \in B\}, \text{ and}$$

$$Z(A') = \{x \in U \mid g_i(x) = 0 \text{ and}$$

$$h_j(x) = 0, \text{ for all } i \in A' \text{ and } j \in B\}.$$

Definition 2.4.1. Let H be a nonempty connected subset of E^n. Then, for $k \geq 1$, H is a cyrtohedron of class C^k if there exist sets of C^k functions $\{g_i\}$ for $i \in A$ and $\{h_j\}$ for $j \in B$, defined on a neighborhood N of $x^* \in E^n$ with

(i) $x^* \in Z(A)$, and for $x \in N$, $x \in H$ if and only if $g_i(x) \geq 0$ for all $i \in A$ and $h_j(x) = 0$ for all $j \in B$,

(ii) if $\Sigma a_i \nabla g_i + \Sigma b_j \nabla h_j = 0$ for some a_i, b_j, with $a_i \geq 0$, then $a_i = b_j = 0$ for all $i \in A$ and $j \in B$,

(iii) $A_0 \subset A_1 \subset A$ and $V\left(x^*, A_1\right) \subset \operatorname{span} V\left(x^*, A_0\right)$ implies that $Z(A_0) = Z(A_1)$.

Consider now the problem

$$\min_x f(x, \varepsilon)$$

$$\text{s.t. } g(x, \varepsilon) \geq 0,\ h(x, \varepsilon) = 0,\ x \in H,\ \varepsilon \in T \qquad P_3'(\varepsilon)$$

which is $P_3(\varepsilon)$ with the additional constraints mentioned earlier. The following definition contains conditions which are sufficient for optimality for $P_3'(\varepsilon)$.

Definition 2.4.2. Let H be a cyrtohedron of class C^2. The point $y^* = (x^*, u^*, w^*)^T$ is said to satisfy the strong second-order conditions for $P_3'(\varepsilon)$ if

(i) $x^* \in \{x \mid g(x, \varepsilon) \geq 0\} \cap \{x \mid h(x, \varepsilon) = 0\}$,

(ii) $-\nabla L(x^*, u^*, w^*, \varepsilon)$ is in the relative interior of the normal cone to H at x^*,

(iii) the gradients of the constraints that are binding at x^* are linearly independent,

(iv) for each $i = 1, \ldots, m$, $u_i^* > 0$ if and only if $g_i(x^*, \varepsilon) = 0$, and

(v) $z^T[\nabla^2 L(x^*, u^*, w^*, \varepsilon) + K[\nabla L(x^*, u^*, w^*, \varepsilon)]]z > 0$ for all nonzero $z \in E^n$ for which

(a) z is in the largest linear subspace contained in the tangent cone to H at x^*,

(b) $\nabla g_i(x^*, \varepsilon)z = 0$ for all $i \in B^*(\varepsilon) \equiv \{i \mid g_i(x^*, \varepsilon) = 0\}$, and

(c) $\nabla h_j(x^*, \varepsilon)z = 0$ for $j = 1, \ldots, p$,

where $K[\cdot]$, the curvature of the facial submanifold of H which contains x^*, is an $n \times n$ matrix.

If the set H is taken to be E^n, then $P_3'(\varepsilon)$ reduces to the program $P_3(\varepsilon)$ and (ii) and (v) above become the familiar conditions

(ii') $\nabla L(x^*, u^*, w^*, \varepsilon) = 0$, and

(v') $z^T\nabla^2 L(x^*, u^*, w^*, \varepsilon)z > 0$ for all nonzero $z \in E^n$ for which (b) and (c) above hold.

With these definitions, we now state Spingarn's result.

Theorem 2.4.6. Consider the problem $P_3'(\varepsilon)$. If the strong second-order conditions hold at $y^* = (x^*, u^*, w^*)^T \in H \times E^m \times E^p$, $\varepsilon^* \in E^k$, then there exist neighborhoods $N_1 \subset E^k$ and $N_2 \subset E^n$ of ε^* and x^*, respectively, and a C^1 function $y(\varepsilon) = [x(\varepsilon), u(\varepsilon), w(\varepsilon)]^T$ defined on N_1 such that:

(i) $y(\varepsilon)$ satisfies the strong second-order conditions for $P_3'(\varepsilon)$,

(ii) for each ε in N_1, $x(\varepsilon) \in N_2$ is an isolated local minimizer for $P_3'(\varepsilon)$, and

(iii) for each ε in N_1, the Lagrange multipliers $u(\varepsilon)$, $w(\varepsilon)$, associated with $x(\varepsilon)$ are uniquely determined.

Under slightly weaker assumptions than those invoked by Fiacco (1976), Robinson (1974) has obtained results for problem $P_3(\varepsilon)$ similar to those stated in Theorem 2.4.4, proving the continuity of the Karush-Kuhn-Tucker triple, and using the results to derive bounds on the variation of $y(\varepsilon)$.

Theorem 2.4.7. Let T be a Banach space, $T_0 \subset T$, $M \subset E^n$, with M and T_0 open sets. Let f, g, and h have second partial derivatives with respect to x which are jointly continuous on $M \times T_0$. For ε^* in T_0, suppose (x^*, u^*, w^*) is a Karush-Kuhn-Tucker triple of $P_3(\varepsilon)$. Also assume that the linear independence, strict complementary slackness, and second-order sufficiency conditions hold at (x^*, u^*, w^*). Then

(i) there exists a continuous function $y(\varepsilon)$ with $y(\varepsilon^*) = (x^*, u^*, w^*)^T$, and for each ε in T_0, $y(\varepsilon)$ is the unique Karush-Kuhn-Tucker triple of $P_3(\varepsilon)$ and the unique zero of $[\nabla L(x, u, w, \varepsilon), u_1 g_1(x, \varepsilon), \ldots, u_m g_m(x, \varepsilon), h_1(x, \varepsilon), \ldots, h_p(x, \varepsilon)]$,

(ii) for ε near ε^*, $x(\varepsilon)$ is an isolated local minimizing point of $P_3(\varepsilon)$, and

(iii) linear independence, strict complementary slackness, and the second-order sufficiency conditions hold for ε near ε^*.

Theorem 2.4.8. Under the hypotheses of the previous theorem, for any $\lambda \in (0, 1)$, there exist neighborhoods N_λ^1 of ε^* and N_λ^2 of (x^*, u^*, w^*) such that for any ε in N_λ^1 and any y in N_λ^2 we have

$$\|y - y(\varepsilon)\| \leq (1 - \lambda)^{-1}\|M[y(\varepsilon^*), \varepsilon^*]^{-1}\| \cdot \|F(y, \varepsilon)\|,$$

where M is the Jacobian of $F = [\nabla L, u_1 g_1, \ldots, u_m g_m, h_1, \ldots, h_p]^T$ with respect to y. Robinson applies these results to determine the rate of convergence of a large family of algorithms for solving $P_3(\varepsilon^*)$.

Extensions of results analogous to Theorem 2.4.4 but in infinite dimensional spaces have been obtained. Ioffe and Tikhomirov (1980; Russian original, 1974) use an implicit function theorem to prove that $[x(\varepsilon), w(\varepsilon)] \in C^1$ is the unique Karush-Kuhn-Tucker triple near a given ε for the problem

$$\begin{aligned} &\min_x \ f(x) \\ &\text{s.t. } F(x) = \varepsilon, \end{aligned} \qquad P_0(\varepsilon)$$

where $x \in X$, $F: X \to T$, and X and T are Banach spaces, and assumptions analogous to those of Theorem 2.4.4 are invoked. Similar results for $P_2'(\varepsilon)$: $\min_x f(x)$ s.t. $g(x) \geq \varepsilon^1$, $F(x) = \varepsilon^2$, $\varepsilon^1 \in E^m$, $\varepsilon^2 \in T$, $\varepsilon = (\varepsilon^1, \varepsilon^2)^T$ were obtained by Wierzbicki and Kurcyusz (1977), again using an implicit function, where X and T are Hilbert spaces.

In the remainder of this section we summarize several additional important results associated with Theorem 2.4.4. These will be treated in detail in Chapters 3-8.

If we consider the logarithmic-quadratic mixed barrier-penalty function (Fiacco and McCormick, 1968) associated with problem $P_3(\varepsilon)$ given by

$$W(x, \varepsilon, r) = f(x, \varepsilon) - r\sum_{i=1}^{m} \ln g_i(x, \varepsilon) + (1/2r)\sum_{j=1}^{p} h_j^2(x, \varepsilon),$$

we have the following theorem due to Fiacco (1976). The proof of this result is given in Chapter 6 (Theorem 6.2.1).

Theorem 2.4.9. [Locally unique C^1-KKT triple associated with a locally unique unconstrained local minimum of the barrier-penalty function $W(x, \varepsilon, r)$.] Assume the hypotheses of Theorem 2.4.4 are satisfied. Then, in a neighborhood of $(\varepsilon, r) = (0, 0)$ there exists a unique, once continuously differentiable function $y(\varepsilon, r) = [x(\varepsilon, r), u(\varepsilon, r), w(\varepsilon, r)]^T$ satisfying:

$$\begin{aligned} &\nabla L(x, u, w, \varepsilon) = 0, \\ &u_i g_i(x, \varepsilon) = r, \quad i = 1, \ldots, m, \text{ and} \\ &h_j(x, \varepsilon) = w_j r, \quad j = 1, \ldots, p, \end{aligned} \tag{2.4.6}$$

with $y(0, 0) = [x^*, u^*, w^*]^T$. Furthermore, for any (ε, r) near $(0, 0)$ with $r > 0$, $x(\varepsilon, r)$ is a locally unique unconstrained local minimizing point of $W(x, \varepsilon, r)$ with $g_i[x(\varepsilon, r), \varepsilon] > 0$ for each $i = 1, \ldots, m$, and $\nabla_x^2 W(x, \varepsilon, r)$ is positive definite at $x = x(\varepsilon, r)$.

The existence of higher-order derivatives of $y(\varepsilon)$ depends on the degree of (continuous) differentiability of the problem functions. This follows directly from an application of the classical implicit function theorem to the first-order necessary conditions for a solution of $P_3(\varepsilon)$. An analogous result holds for $y(\varepsilon, r)$, the solution of (2.4.6). In fact, it follows easily that under the appropriate conditions not only do higher-order derivatives of $y(\varepsilon, r)$ exist, but these derivatives converge to the corresponding derivatives of $y(\varepsilon)$. This result is stated next and later subsumed in Corollary 6.4.1.

Theorem 2.4.10. Let f, g, and h have continuous derivatives of all orders up to p + 1. Assume that the conditions of Theorem 2.4.4 are satisfied. Then, in a neighborhood of $(\varepsilon, r) = (0, 0)$, there exists a unique function $y(\varepsilon, r) \in C^p$, $y(\varepsilon, r) = [x(\varepsilon, r), u(\varepsilon, r), w(\varepsilon, r)]^T$ satisfying (2.4.6), with

$$y(\varepsilon, r) \to y(\varepsilon),$$

and the j-th partial derivative of y with respect to ε at (ε, r) converges to the j-th partial derivative of y at ε as $r \to 0$ for (ε, r) near $(0, 0)$, where $j = 1, \ldots, p$.

An analog of Theorem 2.4.9 was subsequently given by Buys and Gonin (1977) and Armacost and Fiacco (1977), using an augmented Lagrangian previously utilized by Buys (1972). It gives "exact" sensitivity results, i.e., results coinciding with those of Theorem 2.4.4. The augmented Lagrangian associated with problem $P_3(\varepsilon)$ is defined as

$$\hat{L}(x, u, w, \varepsilon, c)$$

$$= f(x, \varepsilon) - \sum_{i \in J(\varepsilon)} (u_i - (1/2)cg_i(x, \varepsilon))g_i(x, \varepsilon)$$

$$+ \sum_{j=1}^{p} (w_j + (1/2)ch_j(x, \varepsilon))h_j(x, \varepsilon) - (1/2c) \sum_{i \in K(\varepsilon)} u_i^2$$

where $J(\varepsilon) = \{i \mid u_i - cg_i(x, \varepsilon) \geq 0\}$ and $K(\varepsilon) = \{i \mid u_i - cg_i(x, \varepsilon) < 0\}$.

Theorem 2.4.11. Under the assumptions of Theorem 2.4.4, there exists c^* such that for ε near 0 and $c > c^*$, there exists a unique C^1 function $y(\varepsilon, c) = [x(\varepsilon, c), u(\varepsilon, c), w(\varepsilon, c)]^T$ satisfying

(i) $\nabla\hat{L}(x, u, w, \varepsilon, c) = 0$,

(ii) $u_i g_i(x, \varepsilon) = 0$, $i = 1, \ldots, m$, and

(iii) $h_j(x, \varepsilon) = 0$, $j = 1, \ldots, p$,

with $y(\varepsilon, c) = y(\varepsilon)$. Furthermore, for any ε near 0 and $c > c^*$ we have that $x(\varepsilon, c)$ is a locally unique unconstrained local minimizing point of $\hat{L}[x, u(\varepsilon, c), w(\varepsilon, c), \varepsilon, c]$ and $\nabla^2\hat{L}$ is positive definite for $[x, u, w]$ near $[x^*, u^*, w^*]$. This result, readily following from the definition of $\hat{L}$ and Theorem 2.4.4, is proved and formulas for $\nabla_\varepsilon y(\varepsilon, c)$ are developed in Chapter 7.

Armacost and Fiacco (1975) also obtained formulas for the first- and second-order derivatives of the optimal value function of the problem $P_3(\varepsilon)$. These follow from the next result, which is an easy consequence of Theorem 2.4.4.

Theorem 2.4.12. Suppose the assumptions of Theorem 2.4.4 hold for $P_3(\varepsilon)$. Then, in a neighborhood of $\varepsilon = 0$, the optimal value function $f^*(\varepsilon)$ is twice continuously differentiable as a function of ε, and

(i) $f^*(\varepsilon) = L[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon]$,

(ii) $\nabla_\varepsilon f^*(\varepsilon) = \nabla_\varepsilon L[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon]$, and

(iii) $\nabla_\varepsilon^2 f^*(\varepsilon) = \nabla_\varepsilon\left[\nabla_\varepsilon L[x(\varepsilon), u(\varepsilon), w(\varepsilon), \varepsilon]^T\right]$.

Consider the problem

$$\begin{aligned} &\min_x \ f(x) \\ &\text{s.t. } g_i(x) \geq \varepsilon_i, \quad i = 1, \ldots, m, \\ &\qquad h_j(x) = \varepsilon_{j+m} \quad j = 1, \ldots, p. \end{aligned} \qquad P_2(\varepsilon)$$

The Lagrangian for $P_2(\varepsilon)$ is given by

$$L(x, u, w, \varepsilon) = f(x) - \sum_{i=1}^{m} u_i [g_i(x) - \varepsilon_i] + \sum_{j=1}^{p} w_j [h_j(x) - \varepsilon_{j+m}].$$

The following result follows immediately from Theorem 2.4.12. It is stated separately because of its importance.

Theorem 2.4.13. Let f, g, and h be twice continuously differentiable in x in a neighborhood of x^*. Suppose the assumptions of Theorem 2.4.4 hold for $P_2(\varepsilon)$. Then, in a neighborhood of $\varepsilon = 0$,

(i) $\nabla_\varepsilon f^*(\varepsilon) = [u(\varepsilon)^T, -w(\varepsilon)^T]^T$

(ii) $\nabla_\varepsilon^2 f^*(\varepsilon) = \begin{bmatrix} \nabla_\varepsilon u(\varepsilon) \\ -\nabla_\varepsilon w(\varepsilon) \end{bmatrix}$

Theorem 2.4.12 and 2.4.13 are proved in Chapter 3 (Theorem 3.4.1 and Corollary 3.4.4). Formulas for $\nabla_\varepsilon f^*$ in terms of the derivatives of the original problem functions are developed in Chapter 3, Section 3.4.

Applications of the sensitivity results given in Theorems 2.4.4, 2.4.9, and 2.4.12 have been implemented via a penalty function algorithm and are reported subsequently. Here, we briefly summarize some of the initial computational experience.

Armacost and Fiacco (1974) illustrated computational aspects of some of the results outlined in this section. Using the SUMT (Sequential Unconstrained Minimization Technique) Version 4 computer code developed by Mylander, Holmes, and McCormick (1971) and a subroutine for sensitivity analysis coded by Armacost and Mylander (1973) that implemented a procedure for calculating the solution parameter derivatives based on the results given in Theorems 2.4.4, 2.4.9, and 2.4.12, they obtained results that computationally corroborated the theory, e.g., the convergence of the first partial derivatives of the optimal solution and the optimal value function of several problems. The resulting code was further developed and refined by Armacost (1976a,b). Subsequently, Armacost and Fiacco (1978) used this computer program, which came to be known as SENSUMT, to analyze the behavior of the solution of an inventory problem relative to changes in several problem parameters. This application is discussed in Chapter 8. Further development of SENSUMT and other applications will be discussed as the theory is developed.

2.5. OPTIMAL VALUE AND SOLUTION BOUNDS

The foregoing results are concerned mainly with the continuity or differentiability properties of the optimal value function $f^*(\varepsilon)$ or a Karush-Kuhn-Tucker triple $[x(\varepsilon), u(\varepsilon), w(\varepsilon)]$. Among the results we have given, only a few, e.g., Robinson's generalized implicit function theorem result (2.4.3) for the general system (2.4.2), the particular realization of this obtained earlier for the positive definite quadratic program by Daniel in Theorem 2.4.3, Robinson's bounds on a Karush-Kuhn-Tucker triple given in Theorem 2.4.8, and Stern and Topkis' Lipschitz condition on the optimal value function of $P_1(\varepsilon)$ given in Theorem 2.2.12 provide parametric solution bounds. All of these results involve valuable existence proofs, providing incisive characterizations of solution properties relative to parametric perturbations. Nonetheless, much remains to be done to develop computable techniques for calculating useful numerical estimates of bounds for general classes of problems.

The question arises whether practically useful computable parametric solution bounds can be calculated for general classes of NLP problems. Practicality demands that the required effort be feasible and the perturbations be finite, while usefulness suggests additionally that a suitable degree of accuracy be achieved. We find that generally applicable bounds results meeting the prescribed properties are rare indeed. In fact, there appear to be two fundamental directions, one due to Kantorovich (1948) and one due to Moore (1966, 1977). Both involve Newton's method, the latter incorporating results from interval arithmetic. Most general approaches for obtaining parametric solution bounds and error bounds seem to be associated with one of these basic results. Examples of intriguing and important recent error bounds results of the Kantorovich type may be found in the work of Miel (1980), and of a mixture of these types in the work of Mancini and McCormick (1976, 1979), and McCormick (1980). Theoretical and numerical comparisons of the two basic techniques for computing error bounds have been reported by Rall (1979).

Another strategy for obtaining solution bounds is the exploitation of structure of particular classes of functions. For example, if the given problem is a linear programming problem, then under usual nondegeneracy assumptions one can essentially construct an exact parametric solution for finite and rather general classes of parametric perturbations. Of course, this may entail an inordinate amount of effort and, if this is prohibitive, questions arise as to whether suitably accurate parametric bounds on a solution might be calculated with tolerable effort. In nonlinear programming an important computationally exploitable property is *separability* of the problem functions, i.e., the representation of a function as the sum of functions of one variable. Since each one-dimensional "component" of a separable function can generally be separately estimated or bounded by simpler functions, this leads to the possibility of estimating or bounding a multi-dimensional separable problem by a simpler problem. Assuming that the one-dimensional estimation errors or error bounds are tractable, then the solution and error bound information associated with the approximation problem can possibly be related to the solution- and error-bound information of the original problem.

The most striking and well known application of the exploitation of separability is undoubtedly the extensively developed and widely used branch and bound methods for non-convex separable programming pioneered by Falk and Soland (1969), based on globally solving the given problem by generating a sequence of separable convex underestimating programs. Closer to the spirit of solution error bounds, however, Geoffrion (1977) noted that the error bounds involved in fitting functions (of one variable) to the respective components of a given separable problem could be very precisely related to the discrepancy between the solution value of the given problem and that of the approximation problem. He gave several theoretical results for bounding the difference of the optimal value and constructive curve-fitting techniques for reducing the resulting deviation. More recently, Meyer (1979, 1980) developed techniques for calculating error bounds on the optimal value of large convex separable programs and has been investigating the exploitation of separability to calculate error bounds on solution points. Thakur (1978, 1980, 1981) computes optimal value function and solution point error-bound information for convex (highly nonlinear) separable programs, based on an analysis of a corresponding piecewise linear approximation problem. An intriguing use of this information is to accelerate convergence by reducing the region known to contain a solution in a scheme based on solving a sequence of progressively tighter linear approximation problems. Again, in the context of separable, but now separable *parametric* programs, Benson (1980) has extended the usual branch and bound approach to allow for the *simultaneous* calculation of a set of solutions of a given separable program corresponding to a finite number of resource levels (right-hand sides).

Dinkel and Kochenberger (1977, 1978) and Dinkel, Kochenberger, and Wong (1978, 1982) have conducted numerous experiments in calculating the parameter derivatives of a Karush-Kuhn-Tucker triple $[x(\varepsilon), u(\varepsilon), w(\varepsilon)]$ and extrapolating for new solutions, in a variety of applied GP problems (essentially under the assumptions of Theorem 2.4.4). A distinctive feature of their work is the use of a technique generally attributed to Davidenko (1953): extrapolation over a small parameter interval using first parameter-derivative information followed by a reevaluation of functions involved and a recalculation of derivatives at the new point, followed by extrapolation, etc. They have obtained much greater accuracy using this incremental approach than is obtained by using one linear extrapolation, the accuracy apparently being proportional to the number of increments – though obviously, there is a trade-off with computational effort that dictates a compromise strategy. They have also developed heuristics for estimating the range over which extrapolation is valid.

Our computational approach to the calculation of optimal value and Karush-Kuhn-Tucker triple parameter derivatives is discussed in detail in Chapters 3 - 8, so we confine ourselves here to a few remarks concerning a central topic of this section, the calculation of optimal solution bounds. Our point of view has been to start with the highly structured problem and to focus our attention explicitly on the calculation of parametric bounds on the optimal value function corresponding to finite parameter changes. We have found the class of problems with convex (or concave) optimal value function to have many computationally exploitable properties, while being general enough to be of wide practical interest, and have thus been naturally led to the jointly convex parametric NLP $P_3(\varepsilon)$ as a logical starting point on which to build a computable bounding methodology. The convexity of the problem in x leads to immediate connections with duality theory and the determination of dual parametric bounds. The convexity of the optimal value function allows for the immediate calculation of a global lower parametric linear bound whenever a solution and a subgradient is calculated at any specified parameter value, and an upper linear bound via linear interpolation between the optimal values whenever solutions for two parameter values have been calculated. The joint convexity of the feasible region also easily yields a feasible parametric vector, whenever solutions are calculated for two parameter values, by linear interpolation between the given solution points. Also, the bounds on $f^*(\varepsilon)$ open the door to computable techniques for bounding a Karush-Kuhn-Tucker triple $[x(\varepsilon), u(\varepsilon), w(\varepsilon)]$. Finally, the possibility of estimating corresponding results for nonconvex parametric programs is realized for those programs that can be estimated by jointly convex programs. In particular, results can be obtained for jointly separable nonconvex programs since, as noted earlier, an important approach for solving standard (i.e., nonparametric) nonconvex separable problems is branch and bound methodology that is based on solving a sequence of convex underestimating problems. It suffices to observe that the same techniques can be used jointly in (x, ε) to generate parametric jointly convex underestimating problems of a parametric jointly separable nonconvex program. Further, since the same ideas involving separability can be extended to a significantly more general structure called "factorability" (McCormick, 1976), encompassing most functions commonly encountered in practice, the applicability of results based on jointly convex programs to nonconvex programs is clear.

The details of the indicated results and their application in developing solution bounds and refined parametric solution estimates will not be pursued further here, but are presented in Chapter 9.

We close by noting that the class of problems that have an optimal value function with convexity or concavity properties is considerably larger than might be imagined from a cursory analysis. Reverse convex programs (e.g., $P_2(\varepsilon)$ with f concave, the g_i convex and the h_j affine in x) may be expected to have a piecewise concave optimal value function. Also, large classes of parametric posynomial GP problems have a convex optimal value function (all the more intriguing since signomial GP problems may be underestimated or overestimated by posynomial GP problems). Finally, it is noted that GP duality theory can be exploited to calculate global parametric bounds on the optimal value function.

2.6. GENERAL RESULTS FROM RHS RESULTS

The general parametric problem $P_3(\varepsilon)$ and the rhs parametric problem $P_2(\varepsilon)$ are more closely associated than their formulations may suggest. In fact, surprisingly, any general problem of the form

$$\min_x f(x, \varepsilon) \quad \text{s.t. } g(x, \varepsilon) \geq 0,\ h(x, \varepsilon) = 0 \qquad P_3(\varepsilon)$$

may be formulated as an *equivalent* rhs problem

$$\min_x f(x) \quad \text{s.t. } g(x) \geq \varepsilon^1,\ h(x) = \varepsilon^2 \qquad P_2(\varepsilon)$$

by simply redefining ε in $P_3(\varepsilon)$ to be a variable and introducing a new parameter α such that $\varepsilon = \alpha$. This results in the problem

$$\min_{(x,\varepsilon)} f(x, \varepsilon) \quad \text{s.t. } g(x, \varepsilon) \geq 0,\ h(x, \varepsilon) = 0,$$
$$\varepsilon = \alpha \qquad P(\alpha)$$

which is clearly equivalent to $P_3(\varepsilon)$ and of the form $P_2(\varepsilon)$.

The relevance and utility of this equivalence is its use in organizing more tightly and relating more precisely the body of results that hold for the various formulations. It is obvious that results for $P_3(\varepsilon)$ can be applied to $P_2(\varepsilon)$. The reformulation $P(\alpha)$ of $P_3(\varepsilon)$ implies that the converse of this fact is true: results for $P_2(\varepsilon)$ obviously apply to $P(\alpha)$ and can be applied to $P_3(\varepsilon)$, when the assumptions on $P_3(\varepsilon)$ can be shown to imply the required $P_2(\varepsilon)$ assumptions via $P(\alpha)$. This has many interesting applications in immediately extending certain results for $P_2(\varepsilon)$ to their natural generalization for $P_3(\varepsilon)$. For example, CQ1 holds for $P_3(\varepsilon)$ at $x = \bar{x}$ with $\varepsilon = \bar{\varepsilon}$, with respect to the vector $\bar{y}$, if and only if $\nabla_x \bar{g}(\bar{x}, \bar{\varepsilon})\bar{y} > 0$, $\nabla_x h(\bar{x}, \bar{\varepsilon})\bar{y} = 0$, and the rows of $\nabla_x h(\bar{x}, \bar{y})$ are linearly independent. But this means that $\nabla_{(x,\varepsilon)}\bar{g}(\bar{x}, \bar{\varepsilon})\binom{\bar{y}}{0} = \nabla_x g(\bar{x}, \bar{\varepsilon})\bar{y} > 0$. Also, defining $H \equiv \binom{h}{\varepsilon-\alpha}$, it follows that

$$\nabla_{(x,\varepsilon)} H\begin{pmatrix}\bar{y}\\ 0\end{pmatrix} = \begin{bmatrix}\nabla_x h(\bar{x}, \bar{\varepsilon})\bar{y}\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

and the rows of

$$\nabla_{(x,\varepsilon)} H(x, \varepsilon) = \begin{bmatrix}\nabla_x h(\bar{x}, \varepsilon) & \nabla_\varepsilon h(\bar{x}, \bar{\varepsilon})\\ 0 & I\end{bmatrix}$$

are linearly independent. This means that CQ1 holds for the problem $P(\alpha)$ at $(x, \varepsilon) = (\bar{x}, \bar{\varepsilon})$ with $\alpha = \bar{\varepsilon}$, with respect to the vector $(\bar{y}, 0)$. Thus, CQ1 in $P(\varepsilon)$ is inherited by a realization of $P_2(\varepsilon)$. As an immediate application, it follows that the Gauvin-Tolle results, the continuity of $f^*(\varepsilon)$ and the differential stability bounds on the Dini derivatives of $P_2(\varepsilon)$ described in Section 2.3, easily extend to $P_3(\varepsilon)$ via $P(\alpha)$, if the natural generalizations of the Gauvin-Tolle assumptions are assumed to hold for $P_3(\varepsilon)$. This particular application was noted by Janin (personal communication, 1981) and Rockafellar (1982a) and independently applied by Gollan (1981a) to prove some general results.

Another important example is based on well known results and involves deducing the convexity of the optimal value function of the general problem $P_3(\varepsilon)$ from the convexity of the optimal value function for $P_2(\varepsilon)$. A classical result is the convexity of the optimal value function of $P_2(\varepsilon)$, if this problem is convex in x. It follows that the optimal value of $P(\alpha)$ is convex if this problem is jointly convex in (x, ε). But clearly, if $P_3(\varepsilon)$ is convex in (x, ε), then so is $P(\alpha)$. Hence, the following result, proved independently of the result for $P_2(\varepsilon)$ by Mangasarian and Rosen (1964), is seen to be immediately implied by the $P_2(\varepsilon)$ result: the optimal value of $P_3(\varepsilon)$ is convex if this problem is jointly convex in (x, ε). We make important use of this result in the bounding procedure described in Chapter 9, as indicated in the last section.

The situation is not always so straightforward, but clearly careful examination of extensions to $P_3(\varepsilon)$ via $P(\alpha)$ is warranted whenever a result for $P_2(\varepsilon)$ has been obtained. Obviously, if a result for $P_3(\varepsilon)$ can be proved directly with little more effort then that required for the corresponding $P_2(\varepsilon)$ result, this would probably be preferable. However, considerable theoretical simplifications and a more constructive proof might make it desirable to develop a result for $P_2(\varepsilon)$ and extend to $P_3(\varepsilon)$ through $P(\alpha)$. It would be interesting to explore existing results from this point of view.

2.7. SUMMARY

This chapter provides a concise survey of a number of basic sensitivity and stability results for general classes of nonlinear parametric problems. It surely does not contain all the important findings, but hopefully gives a good idea of several mainstream directions of research that have evolved and are still being developed. In the past several years, as the references will attest, developments have accelerated and the level of sophistication of results has been appreciably elevated.

We shall endeavor to anticipate some of the future research directions in the final chapter. Otherwise, except for the bounds results in Chapter 9, the rest of this book is devoted to a rather detailed treatment of a fundamental segment of the methodology that has been developed rather thoroughly in the past few years. This body of results is perhaps the most highly structured for the general problem of interest. It is based on one of the strongest collections of assumptions that might be invoked, while still remaining in the province of assumptions that are typically utilized to obtain the strongest consequences in other important mathematical programming applications, e.g., strong characterizations of optimality, and convergence and rate of convergence of algorithms.

The assumptions needed for these ideal results are the assumptions of Theorem 2.4.4: second-order sufficiency, linear independence, and strict complementary slackness conditions. These provide an abundance of theoretical sensitivity and stability results and sufficient structure to readily allow the development of simple formulas and solution-algorithm-based computational approximation techniques. It is perhaps to be expected that this collection of results would be one of the first to be most fully developed and applied. The results are of intrinsic interest and are useful, since practical problems meet the requisite conditions surprisingly often. They also provide an essential interface with more general results and a vital perspective and point of departure, establishing essentially the most that can be expected under rather ideal circumstances.