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Sensitivity of Linear Control Systems to Large Parameter Variations*

Sensibilité des systèmes de commande linéaires aux grandes variations des paramètres

Empfindlichkeit linearer Kontrollsysteme gegenüber grossen Parametervariationen

Чувствительность линейных систем управления к широким изменениям параметров

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A measure of sensitivity of linear systems to large parameter variations is established. Results are presented which relate these measures to the corresponding parameter variations.

Summary—This paper is concerned with the sensitivity of a linear system-quadratic type cost functional to large parameter variations in the description of the linear system. For some class of parameter variations ε and some real number ρ, the concept of ρ-sensitivity is introduced: ρ-sensitivity occurs if the value of the cost functional does not increase by more than a factor of ρ for any change in the class ε in comparison with a nominal or errorless system. For fixed ρ several methods are developed which allow determination of certain error classes ε for which ρ-sensitivity occurs. Analogous results are obtained for both the finite time problem and the infinite time problem. The special case where the linear system is optimally designed for the given cost functional is also considered. Three examples illustrating the calculations and application of the methods are presented.

INTRODUCTION

MANY authors have recently been concerned with determining the effects of variations in system parameters on the performance of an optimally designed system. This problem is of considerable importance since in any physical system uncertainties of various kinds inevitably occur. In particular, uncertainty may occur in the state measurement process if the control is of a feedback form. Uncertainty may occur in the plant due to environmental and aging effects. Also, there are always inherent uncertainties in the choice of a mathematical model both for the controlled system and the controller. Thus, for several reasons there will be discrepancies between any physical process and the mathematical model which is chosen as its representation.

There have been various attempts in the literature to develop theoretical methods with which to study this problem. The so-called "performance index sensitivity vector", as proposed by DORATO [1], has been used by PAGUREK [2, 3], WITSENHAUSEN [4] and DUNN [5] to investigate the change in a performance index due to sufficiently small changes in the system parameters. In particular, for a certain class of problems it has been shown that various implementations of the optimal control, e.g. feedback or open loop, lead to the same performance index sensitivity vector. However, this approach has certain inherent disadvantages due to the fact that the sensitivity vector is defined as the gradient of the performance index with respect to a parameter vector. Thus, this approach can yield information only of a local nature. Several other approaches based on the consideration of essentially finite changes in the system characteristics have been considered. HOWARD and REKASIUS [6] have considered the worst possible parameter variations within some class in the sense

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that the performance index is maximized. In addition, RISSANEN [7] and McCLAMROCH [8] have considered the problem of specifying an upper bound on the change in the performance index and determining admissible variations in the parameters. Similar results have appeared in papers by RISSANEN and DURBECK [9] and SARMA and DEEKSHATULU [10].

The related problem of determining the admissible parameter variations so that the value of the performance index does not change has also been studied. Various results have been reported in papers by BARNET and STOREY [11], by McCLAMROCH and AGGARWAL [12, 13], and by McCLAMROCH, AGGARWAL, and CLARK [14].

The approach towards sensitivity taken in this paper is in the spirit of the approaches taken in [7–10]. Basically the results presented here are clarifications and extensions of the results in [7, 8].

OPTIMAL CONTROL PROBLEM

In optimal control theory one of the most commonly studied problems is the state-regulator problem. This problem can be stated as follows. For the linear differential system

$$\dot{x}=F(t)x+B(t)u\,,\qquad x(t_0)=x_0\,, \tag{1}$$

choose the control u from the set of all bounded piecewise continuous functions defined on the time interval $[t_0, t_1]$, so that the value of the performance index given by

$$J=\tfrac{1}{2}x'(t_1)\Lambda x(t_1)+\tfrac{1}{2}\int_{t_0}^{t_1}\{x'(t)H(t)x(t)+u'(t)R(t)u(t)\}\mathrm{d}t \tag{2}$$

is a minimum with respect to all admissible controllers. We make the following assumptions:

(a) x is an n-vector, u is an r-vector
(b) $F(t)$, $B(t)$, $H(t)$, $R(t)$ are matrices of appropriate size, all with elements continuous in t, Λ is a constant matrix of the appropriate size,
(c) the time interval $[t_0, t_1]$ is specified, but the final state $x(t_1)$ is not specified,
(d) the square matrices Λ, $H(t)$, and $R(t)$ are symmetric; Λ and $H(t)$ are non-negative definite, and $R(t)$ is positive definite.

This state-regulator problem has been extensively investigated, and it is well known that the optimal control, in feedback form, is given by

$$u^*(t)=-R^{-1}(t)B'(t)M(t)x(t) \tag{3}$$

and the optimal value of the performance index is given by

$$J^*=\tfrac{1}{2}x_0'M(t_0)x_0 \tag{4}$$

where $M(t)$ is a symmetric $n\times n$ matrix which is the unique solution to

$$\dot{M}+MF+F'M-MBR^{-1}B'M+H=0$$
$$M(t_1)=\Lambda\,. \tag{5}$$

THE SENSITIVITY PROBLEM

First consider the relation between changes in the value of the performance index (2) and corresponding changes in the differential system (1) when the control is optimal, i.e. given by (3). It is assumed that the differential system (1) changes as

$$\dot{x}=F(t)x+B(t)u^*+E(t)x \tag{6}$$

where the $n\times n$ matrix $E(t)$ represents the change or error in the system (1). The matrix $E(t)$ could correspond to a variation in either the matrix $F(t)$ or $B(t)$. For physical reasons there is no need to assume that the matrices Λ, $H(t)$, or $R(t)$ change. Thus the value of the performance index (1) changes because the trajectory changes. Note also that since the optimal control u^* is given in feedback form by (3), it obviously changes, since the trajectory changes.

Now consider a slightly more general problem in which the control does not appear explicitly. The linear differential system is given by

$$\dot{x}=A(t)x+E(t)x\,,\qquad x(t_0)=x_0 \tag{7}$$

and the performance index is

$$J=x'(t_1)\Lambda x(t_1)+\int_{t_0}^{t_1}x'(t)Q(t)x(t)\mathrm{d}t\,. \tag{8}$$

where Λ is non-negative definite and $Q(t)$ is positive definite. The matrix $E(t)$ is again considered to represent the system error, the errorless or nominal system being given by (7) with $E(t)\equiv 0$. If the optimal feedback control (3) is substituted into (2) and (6), then we obtain (7) and (8) with $A=F-BR^{-1}B'M$ and $Q=H+MBR^{-1}B'M$. In fact (7) and (8) can be obtained by using any linear feedback controller for (2) and (6), optimal or not. Thus it suffices to consider only the system (7) with the performance measure (8).

Since the matrix $E(t)$ represents an error its value is not known exactly; in this work it is assumed that the error is a member of some appropriate class of errors, i.e. $E\in\varepsilon$. In order to include the possibility of an errorless system it is also assumed that $0\in\varepsilon$.

With these preliminaries the following definition makes clear the concept of sensitivity.

Definition. For some real number ρ and some class of errors ε the system (7) and the performance measure (8) are said to be ρ-sensitive if for each $x_0 \in R^n$

$$J_E \leq \rho J_0 \text{ for all } E \in \varepsilon. \tag{9}$$

Here J_E denotes the value of the performance index (8), evaluated along the solution of (7), assuming an error matrix $E \in \varepsilon$. J_0 denotes the value of the performance measure for no system error.

Since $0 \in \varepsilon$ by assumption it is sufficient to consider only $\rho \geq 1$. It should be noticed that if $0 \in \varepsilon_1$, $0 \in \varepsilon_2$ and ρ-sensitivity holds for the class ε_1, then $\varepsilon_2 \subset \varepsilon_1$ implies that ρ-sensitivity holds for the class of errors ε_2. Thus these various classes can be ordered by inclusion. It would be desirable to know the maximum class under the above ordering; however for practical reasons the definition is in terms of an arbitrary error class ε.

TIME VARYING LINEAR SYSTEMS

Consider the general time-varying problem for the state equation (7) and performance measure (8). The generic matrix $E(\cdot) \in \varepsilon$ denotes the unknown system error in (7).

Theorem 1. The system (7) and (8) is ρ-sensitive with respect to ε if and only if for each $E(\cdot) \in \varepsilon$,

$$(\rho-1)P_0(t_0) - P_1(t_0) \geq 0 \tag{10}$$

where $P_0(t)$ and $P_1(t)$ satisfy

$$\dot{P}_0 + P_0 A + A' P_0 + Q = 0, \qquad P_0(t_1) = \Lambda \tag{11}$$

$$\dot{P}_1 + P_1(A+E) + (A'+E')P_1 + P_0 E + E' P_0 = 0, \quad P_1(t_1) = 0. \tag{12}$$

[If L is a square matrix then $L \geq 0 (L > 0)$ means that L is non-negative (positive) definite.]

Proof: It follows from (7) and (8) (see [8] or [14]) that $J_E = V(x_0, t_0)$ where $V(x, t)$ satisfies $V_t + V_x(A+E)x + x'Qx = 0$,

$$V(x, t_1) = x' \Lambda x$$

and also $J_0 = W(x_0, t_0)$ where $W(x, t)$ satisfies $W_t + W_x Ax + x'Qx = 0$,

$$W(x, t_1) = x' \Lambda x.$$

Solutions to the above equations are given by

$$V(x, t) = x'[P_0(t) + P_1(t)]x$$

and

$$W(x, t) = x' P_0(t) x$$

where $P_0(t)$ and $P_1(t)$ satisfy (11) and (12). Using the definition of ρ-sensitivity it follows that

$$x'[P_0(t_0) + P_1(t_0)]x \leq \rho x' P_0(t_0) x$$

must hold for all x, which is equivalent to inequality (10).

Q.E.D.

The conditions of Theorem 1 define the maximum class of ρ-sensitive errors. However, from a practical viewpoint Theorem 1 is very difficult to apply. The following theorem is easier to apply, but the conclusion is weaker than that of Theorem 1.

Theorem 2. The system (7) and (8) is ρ-sensitive with respect to ε if for each $E(\cdot) \in \varepsilon$,

$$(\rho-1)Q(t) - \rho P_0(t)E(t) - \rho E'(t)P_0(t) \geq 0 \quad \text{for all } t_0 \leq t \leq t_1 \tag{13}$$

where $P_0(t)$ satisfies (11).

Proof: We need to show that condition (13) guarantees that condition (10) is satisfied. Define the matrix $L(t) = (\rho-1)P_0(t) - P_1(t)$. It follows from (11) and (12) that $L(t)$ satisfies

$$\dot{L} + L(A+E) + (A'+E')L + N = 0$$

$$L(t_1) = (\rho-1)\Lambda$$

where $N(t) = (\rho-1)Q(t) - \rho P_0(t)E(t) - \rho E'(t)P_0(t)$. Solving for $L(t)$, we obtain

$$L(t) = (\rho-1)\Phi(t, t_1)\Lambda\Phi'(t, t_1) + \int_t^{t_1} \Phi(t, \sigma)N(\sigma)\Phi'(t, \sigma)\mathrm{d}\sigma$$

where $\Phi(t, t_1)$ is the state transition matrix for the adjoint system, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, t_1) = -[A'(t) + E'(t)]\Phi(t, t_1)$$

$$\Phi(t_1, t_1) = 1.$$

Since Λ is non-negative definite and (13) is simply that $N(t)$ is non-negative definite, the above expression for $L(t)$ guarantees that $L(t_0)$ is non-negative definite. Thus (10) is satisfied.

Q.E.D.

If a class of errors ε consists only of matrices which satisfy condition (13), then we are guaranteed that the system (7) and (8) is ρ-sensitive with respect to ε. It is straightforward to apply Theorem 2 since, as a result of Sylvester's criterion, (13) represents n inequalities to be satisfied for all $t_0 \leq t \leq t_1$. Further the matrix $P_0(t)$ is determined solely from knowledge of the nominal system. Condition (13) in Theorem 2 is only one condition which guarantees ρ-sensitivity. It is possible to determine other conditions; however, condition (13) represents a simple as well as a relatively strong condition.

The above results can be directly applied to the situation where the nominal system is optimal. Using the optimal feedback control (3) in (6), the state equation is

$$\dot{x}=[F(t)-B(t)R^{-1}(t)B'(t)M(t)+E(t)]x$$
$$x(t_0)=x_0 \tag{14}$$

and the performance measure is given by

$$J=\tfrac{1}{2}x'(t_1)\Lambda x(t_1)+\tfrac{1}{2}\int_{t_0}^{t_1}x'(\sigma)[H(\sigma)+M(\sigma)B(\sigma)R^{-1}(\sigma)B'(\sigma)M(\sigma)]x(\sigma)\mathrm{d}\sigma. \tag{15}$$

Theorem 2 applied to (14) and (15) yields

Theorem 3. The optimal system (14) and (15) is ρ-sensitive with respect to ε if for each $E(\cdot)\epsilon\varepsilon$,

$$(\rho-1)(H+MBR^{-1}B'M)-\rho ME-\rho E'M\geq 0$$

for all $t_0\leq t\leq t_1$

where $M(t)$ satisfies (5).

TIME INVARIANT LINEAR SYSTEMS

Consider the case where the matrices F, B, H, R are time-invariant, $\Lambda=0$, and $t_1\to\infty$. The optimal control problem is to choose the control for the system

$$\dot{x}=Fx+Bu, \qquad x(0)=x_0 \tag{16}$$

which minimizes the performance index

$$J=\tfrac{1}{2}\int_0^\infty[x'(t)Hx(t)+u'(t)Ru(t)]\mathrm{d}t \tag{17}$$

for positive definite matrices R and H. If the pair $[F, B]$ is controllable, then the optimal control is given by

$$u^*=-R^{-1}B'Mx \tag{18}$$

and the optimal value function by

$$J=\tfrac{1}{2}x_0'Mx_0 \tag{19}$$

where M is the symmetric positive definite solution of

$$MF+F'M-MBR^{-1}B'M+H=0. \tag{20}$$

As before, we are interested in determining the effects of system changes on the value of the performance index for the control system (16) using the optimal feedback control (18).

As before the more general formulation is given by the state equation

$$\dot{x}=(A+E)x, \qquad x(0)=x_0 \tag{21}$$

and the performance measure

$$J=\int_0^\infty x'(t)Qx(t)\mathrm{d}t \tag{22}$$

with A a stable matrix and Q a positive definite matrix. The matrix E represents the system error, in this case, time-invariant.

Theorem 4. The system (21) and (22) is ρ-sensitive with respect to ε if for each $E\epsilon\varepsilon$,

$$(\rho-1)Q-\rho P_0E-\rho E'P_0\geq 0 \tag{23}$$

where P_0 is the solution of

$$P_0A+A'P_0=-Q. \tag{24}$$

Proof: In the limit as $t_1\to\infty$, if $A+E$ is a stable matrix, it follows that P_0 and P_1 approach constant matrices, so that (10) reduces to

$$(\rho-1)P_0-P_1\geq 0$$

where P_0 and P_1 satisfy

$$P_0A+A'P_0=-Q$$

$$P_1(A+E)+(A'+E')P_1=-P_0E-E'P_0.$$

Define $L=(\rho-1)P_0-P_1$, so that the matrix L satisfies

$$L(A+E)+(A'+E')L=-N$$

where $N=(\rho-1)Q-\rho P_0E-\rho E'P_0$. The solution for L is given by

$$L=\int_0^\infty\Phi'(\sigma)N\Phi(\sigma)\mathrm{d}\sigma$$

where $\Phi(\sigma)=\mathrm{e}^{(A+E)\sigma}$ is the transition matrix for $\dot{x}=(A+E)x$. Since $\Phi(\sigma)$ is nonsingular, it follows that $L\geq 0$ if $N\geq 0$.

Q.E.D.

Returning to the particular case where the nominal system is optimal, obtain

$$\dot{x}=[A-BR^{-1}B'M+E]x, \qquad x(0)=x_0 \tag{25}$$

with the performance measure

$$J=\tfrac{1}{2}\int_0^\infty x'(Q+MBR^{-1}B'M)x\mathrm{d}t. \tag{26}$$

Theorem 4 applied to (25) and (26) yields

Theorem 5. The optimal system (25) and (26) is ρ-sensitive with respect to ε if for each $E\epsilon\varepsilon$,

$$(\rho-1)(H+MBR^{-1}B'M)-\rho ME-\rho E'M\geq 0 \quad (27)$$

where M is the positive definite solution of (20).

EXAMPLES

For the first example consider the linear time-varying system

$$\dot{x}_1=x_2$$
$$\dot{x}_2=-\alpha_1(t)x_1-\alpha_2(t)x_2$$

with the nominal system given by

$$\alpha_1(t)=\alpha_2(t)=0.$$

The performance measure is given by

$$J=\int_0^{1\cdot 0}(x_1^2+x_2^2)\mathrm{d}t.$$

In order to apply Theorem 2, $P_0(t)$ is first obtained from (11) as

$$P_0(t)=\begin{bmatrix} 1-t & \frac{1}{2}(1-t)^2 \\ \frac{1}{2}(1-t)^2 & (1-t)^2+\frac{1}{3}(1-t)^3\end{bmatrix}.$$

Applying Sylvester's conditions for (13) obtain

$$\alpha_1\geq\frac{-(\rho-1)}{2\rho P_{12}}$$

$$(\alpha_2P_{12}-\alpha_1P_{22})^2\leq\frac{(\rho-1)^2}{\rho^2}+\frac{2(\rho-1)}{\rho}\alpha_2P_{22}$$
$$+\frac{2(\rho-1)}{\rho}\alpha_1P_{12}$$

which if satisfied for all $0\leq t\leq 1\cdot 0$ guarantees that the system is ρ-sensitive, where

$$P_{12}(t)=\tfrac{1}{2}(1-t)^2$$
$$P_{22}(t)=(1-t)^2+\tfrac{1}{3}(1-t)^3.$$

The two inequalities can be thought of as representing two geometrical constraints in a three-dimensional $\alpha_1-\alpha_2-t$ space. In Figs. 1(a) and 1(b) the two cases $\alpha_1\equiv 0$, and $\alpha_2\equiv 0$ are considered. The indicated curves are the lower bounds for the corresponding errors. Note that an error for which the conditions of Theorem 2 are not satisfied but for which the system is ρ-sensitive is also indicated.

Next consider a simple example illustrating the use of Theorem 5. A simple problem in the theory

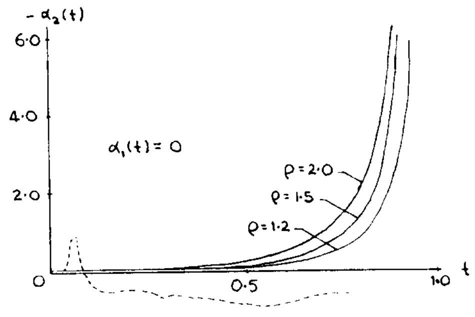

FIG. 1(a). Region for ρ-sensitivity, $\alpha_1=0$.

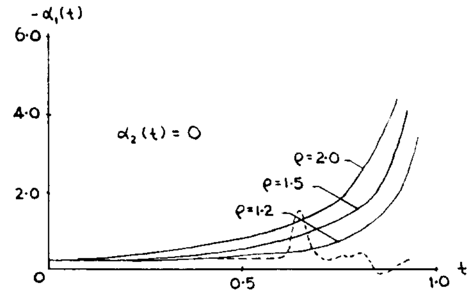

FIG. 1(b). Region for ρ-sensitivity, $\alpha_2=0$.

of optimal control is to determine the feedback control $u(x)$ which stabilizes the system

$$\dot{x}_1=x_2,\qquad x_1(0)=x_{10}$$
$$\dot{x}_2=u,\qquad x_2(0)=x_{20}$$

and minimizes the cost

$$J=\tfrac{1}{2}\int_0^\infty(x_1^2+2x_2^2+u^2)\mathrm{d}t.$$

The optimal feedback control is easily calculated as

$$u^*=-x_1-2x_2,$$

and the optimal cost is

$$J=x_{10}^2+x_{10}x_{20}+x_{20}^2.$$

For some control u, the system changes are given by

$$\dot{x}_1=x_2$$
$$\dot{x}_2=\alpha_1x_1+\alpha_2x_2+u$$

where the nominal system is given for $\alpha_1=\alpha_2=0$. If the control is optimal, then the controlled state equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = (\alpha_1 - 1)x_1 + (\alpha_2 - 2)x_2 .$$

In order to determine the effects of changes in the system indicated by α_1 and α_2, make use of Theorem 5. Condition (26) reduces for this example to the satisfaction of the two inequalities

$$\alpha_1 \leq \frac{(\rho - 1)}{\rho}$$
$$8(\rho-1)^2 - 4\rho(\rho-1)(\alpha_1+\alpha_2) - \rho^2(2\alpha_1 - \alpha_2)^2 \geq 0 .$$

The boundaries of various ρ = constant curves in the α_1–α_2 parameter plane are shown in Fig. 2. For a given value of ρ, the inside of the region defined

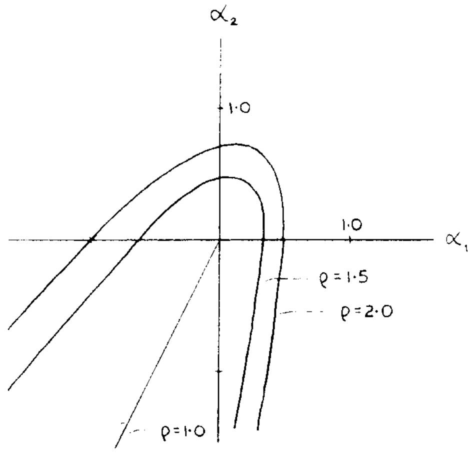

FIG. 2. Region for ρ-sensitivity in parameter plane.

by the ρ = constant curve together with the boundary define a set of parameters for which the system is ρ-sensitive.

As the third example consider a linear process for which a linear feedback control has been chosen. The controlled system is given by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 - \alpha_1(x_1 - x_3)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -x_3 - x_4 - \alpha_2(x_3 - x_1)$$

with the nominal system being given by $\alpha_1 = \alpha_2 = 0$. Thus the nominal system represents two uncoupled second-order oscillators, and the parameters α_1 and α_2 determine the amount of coupling between these two oscillators. It is assumed that a measure of the performance of the above system is

$$J = \int_0^\infty (x_1^2 + x_2^2 + x_3^2 + x_4^2)\mathrm{d}t .$$

Since the controlled system is not necessarily optimal, use is made of Theorem 4 to determine the effects of variations in α_1 and α_2. Solving for P_0 from (24) and substituting into (23) it follows that the system is ρ-sensitive for those values of α_1 and α_2 for which the matrix

$$\begin{bmatrix} (\rho-1)+\rho\alpha_1 & \rho\alpha_1 & -\frac{1}{2}\rho(\alpha_1+\alpha_2) & -\rho\alpha_2 \\ \rho\alpha_1 & (\rho-1) & -\rho\alpha_1 & 0 \\ -\frac{1}{2}\rho(\alpha_1+\alpha_2) & -\rho\alpha_1 & (\rho-1)+\rho\alpha_2 & \rho\alpha_2 \\ -\rho\alpha_2 & 0 & \rho\alpha_2 & (\rho-1) \end{bmatrix}$$

is non-negative definite. The boundaries of various ρ = constant curves in the $\alpha_1 - \alpha_2$ parameter plane are shown in Fig. 3. For a given value of ρ, the interior of the ρ = constant curve together with its boundary defines a set of parameters for which the system is ρ-sensitive.

It is interesting to note that the ρ = constant curves for the second example (Fig. 2) define unbounded regions in the parameter plane, while for the third example (Fig. 3) they define bounded regions in the parameter plane. By using the invariance results in [14] it is easy to determine general conditions guaranteeing that these regions in the parameter space are unbounded.

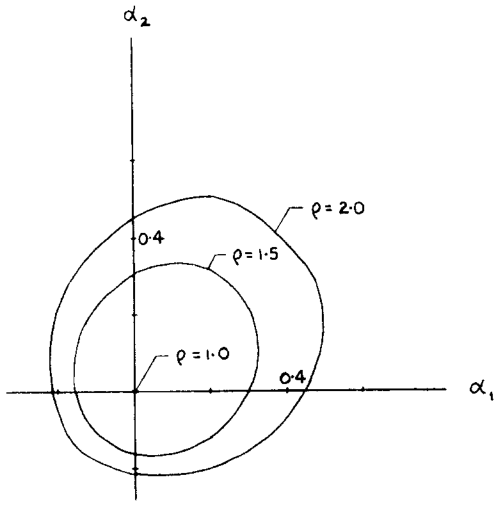

FIG. 3. Region for ρ-sensitivity in parameter plane.

CONCLUSIONS

In this work there has been established a viewpoint towards sensitivity that takes into consideration parameter variations which are not necessarily small. It is felt that this consideration is of major importance. The basic results allow one to determine the effects of finite parameter variations on the performance of linear feedback control systems, either optimal or suboptimal. The approach has been rather natural, since we have restricted ourselves to linear systems only. It appears that the definition for ρ-sensitivity is both meaningful and useful for linear systems. Regretfully, the definition does not seem to be so useful for non-linear-nonquadratic problems.

There are, however, some points which need to be considered in greater depth. There are two cases for which it becomes very difficult to verify Sylvester's conditions: (a) if the dimension of the state space is large, and (b) if the number of errors (parameters) is large. Hopefully, new computational schemes can be developed so that Sylvester's conditions can be applied more efficiently. If there are more than two errors, the parameter plane cannot be used to visualize the regions for ρ-sensitivity. In such cases it might be feasible to determine spheres in the parameter plane for which the system is ρ-sensitive for errors inside the sphere. Such questions are currently being studied.

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Résumé—Cet article se rapporte à la sensibilité d'une fonctionnelle de coût du type quadratique d'un système linéaire à de grandes variations des paramètres dans la déscription du système linéaire. Pour une certaine classe de variations de paramètres ε et un certain nombre réel ρ, on introduit le concept de sensibilité ρ: la sensibilité ρ a lieu si la valeur de la fonctionnelle de coût n'augmente pas davantage que d'un facteur de ρ pour toute variation de la classe ε en comparaison avec un système nominal ou depourvu d'erreur. Pour des ρ donnés on trouve plusieurs méthodes permettant le determination de certaines classes d'erreurs ε pour lesquelles la sensibilité ρ a lieu. Des résultats analogues sont obtenus aussi bien pour le problème à temps fini que pour le problème à temps infini. Le cas particulier dans lequel le système linéaire est optimalement calculé pour la fonctionnelle de coût donnée est également examiné. Trois exemples illustrant les calculs et les applications des methodes sont présentés.

Zusammenfassung—Die Arbeit befaßt sich mit der Empfindlichkeit eines linearen Systems mit quadratischen Kostenfunktional im Hinblick auf große Parametervariationen bei der Beschreibung des linearen Systems. Für eine Klasse von Parametervariationen ε und eine reelle Zahl ρ wird der Begriff der ρ -Empfindlichkeit eingeführt: ρ -Empfindlichkeit liegt vor, wenn der Wert des Kostenfunktionals um nicht mehr als den Faktor ρ anwächst und zwar für jede Änderung in der Klasse ε im Vergleich mit einem nominellen oder fehlerfreien System. Für fixierte ρ werden verschiedene Methoden entwickelt, die die Bestimmung gewisser Fehlerklassenerlauben, für die die ρ -Empfindlichkeit vorkommt. Analoge Resultate werden für das Endlichzeitproblem und für das Unendlichzeitproblem erhalten. Der Spezialfall, bei dem das lineare System für das gegebene Kostenfunktional optimal entworfen ist, wird ebenfalls betrachtet. Drei Beispiele veranschaulichen die Rechnungen und die Anwendung der Methoden.

Резюме—Эта статья относится к чувствительности функционала стоимости квадратичного типа линейной системы к широким изменениям параметров в описании линейной системы. Для некоторого класса изменении параметров ε и для некоторого реального значения ρ, вводится концепт ρ-чувствительности: ρ-чувствительность имеет место если значение функционала стоимости не увеличивается больше чем на фрактор ρ для всякого изменения класса ε по сравнению с номинальной или с безошибочной системой. Для данных ρ, выводится несколько методов позволяющих определить некоторые классы ошибок ε для которых ρ-чувствительность имеет место. Аналогичные результаты получены как для проблемы с конечным временем, так и для проблемы с бесконечным временем. Также рассматривается частный случай когда линейная система оптимально вычислена для данного функционала стоимости. Приведены три примера иллюстрирующие вычисления и применения методов.