
Decentralized Optimization with Heterogeneous Delays: a Continuous-Time Approach

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Abstract

In decentralized optimization, nodes of a communication network privately possess a local objective function, and communicate using gossip-based methods in order to minimize the average of these per-node objectives. While synchronous algorithms can be heavily slowed down by a few nodes and edges in the graph (the *straggler problem*), their asynchronous counterparts lack from a sharp analysis taking into account heterogeneous delays in the communication network. In this paper, we propose a novel continuous-time framework to analyze asynchronous algorithms, which does not require to define a global ordering of the events, and allows to finely characterize the time complexity in the presence of (heterogeneous) delays. Using this framework, we describe a fully asynchronous decentralized algorithm to minimize the sum of smooth and strongly convex functions. Our algorithm (*DCDM*, Delayed Coordinate Dual Method), based on delayed randomized gossip communications and local computational updates, achieves an asynchronous speed-up: the rate of convergence is tightly characterized in terms of the eigengap of the graph weighted by local delays only, instead of the global worst-case delays as in previous analyses.

1 Introduction

In this paper, we consider *asynchronous* and *decentralized* optimization methods that do not rely on a central coordinator. This is particularly relevant in large-scale systems in which centralized approaches suffer from a communication bottleneck at the server. These decentralized optimization methods cover classical settings of supervised learning of models in large datacenters, but also more recent federated learning applications where data and computations are distributed among agents who do not wish to share their local data. We focus on asynchronous operations due to their scalability in the number of agents in the system, and their robustness to node failures and to *stragglers*. More precisely, we consider the following optimization problem:

$$\min_{x \in \mathbb{R}^d} \left\{ f(x) := \sum_{i=1}^n f_i(x) \right\}, \quad (1)$$

where each partial objective function $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ for $i \in [n]$ is held privately by an agent i . In the case of empirical risk minimization, f_i represents the empirical risk for the local dataset of node i .

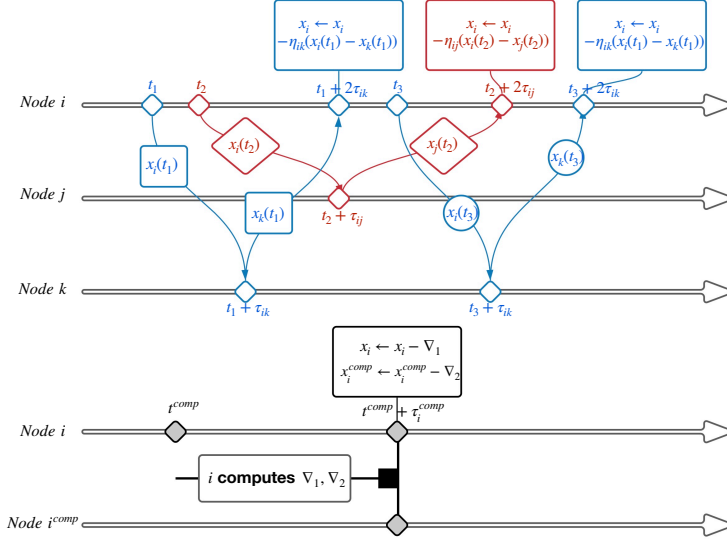


Figure 1: Algorithm structure, viewpoint of Node i . Communications (upper part) and computations (lower part) take place in parallel, but we separate them for clarity.

Another important example is that of network averaging, corresponding to $f_i(x) = \|x - c_i\|^2$ where c_i is a vector attached to node i . In this case, the solution of Problem (1) reads $\bar{c} = \frac{1}{n} \sum_{i=1}^n c_i$.

We classically assume that agents are located at the nodes of a connected graph $G = (V, E)$ on the set of nodes $V = [n]$. An agent $i \in V$ can compute first-order quantities related to its local objective function f_i , and can communicate with any adjacent agent in the graph. Our model of asynchrony is built on the historical randomized gossip model [5], in which we add local computations, and consider different communication and computation delays. Time is indexed continuously by $t \in \mathbb{R}^+$. Every agent in the network keeps a local estimate - $x_i(t) \in \mathbb{R}^d$ for node $i \in V$ at time $t \in \mathbb{R}^+$, performs local computation updates using its partial objective function f_i , and randomly communicates with its neighbors following a communication/computation scheme as described in Figure 1.

1.1 Related Works

Decentralized Optimization and Gossip Algorithms: Gossip algorithms [5, 9] were initially introduced to compute the global average of local vectors with local pairwise communications only (no central coordinator). Yet, these algorithms can be generalized to address our distributed optimization problem with local functions f_i beyond the special case of averaging ($f_i(x) = \|x - c_i\|^2$). Two types of gossip algorithms appear in the literature: synchronous ones, where all nodes communicate with each other simultaneously [9, 35, 20], and randomized ones [5, 28], where at a defined time $t \geq 0$, only a pair of adjacent nodes can communicate. A third category considers directed (non-symmetric) communication graphs [41, 2] which are much easier to implement asynchronously. In the synchronous framework, the communication speed is limited by the slowest node (*straggler* problem), whereas in the historical randomized gossip framework [5], although one could hope for a better dependence, communications are assumed to happen instantly and the question of dealing with delays remains largely unanswered. We refer the reader to Nedich et al. [29] for a more exhaustive survey on gossip algorithms.

Handling Asynchrony: The dynamics of asynchronous optimization algorithms is significantly more complex than their synchronous counterparts. Their study goes back to the monograph of Bertsekas [4], where asynchrony is modelled through a global ordering of events, providing the necessary formalism classically used. Most of the recent literature is then derived from a distributed asynchronous variant of SGD called *HOGWILD!* [33]. In the *after-write* ordering, denoting by $(t_k)_k$ the times at which updates commit, with $X_k = x(t_k)$, these updates write:

$$X_{k+1} = X_k - \eta_k g_{i_k}(X_{\ell_k}), \quad (2)$$

where g_{i_k} is a stochastic (through i_k) estimate of the gradient and $\ell_k \in \mathbb{N}$ is such that $t_{\ell_k} \leq t_k - \tau_{t_k, i_k} < t_{\ell_k+1}$, so that X_{ℓ_k} is the state in which the parameter was when the k -th update started computing at time $t_k - \tau_{t_k, i_k}$, assuming consistent reads. Leblond et al. [22] introduce alternative orderings of the iterates (before-read and after-read) that correspond to different views of the same sequence of updates, and which simplify the analysis. Asynchrony in the distributed shared-memory model (distributed coordinate gradient descent or distributed SGD) and the impact of delays over convergence rates is well understood [24, 43, 13], even though reader-friendly proofs and convergence guarantees under realistic assumptions on the intricacies between iterates, delays and choices of coordinates, are a challenging problem [39, 7]. However, very few works study asynchronous models without shared-memory (or a centralized architecture), and no sharp analysis quantifies the impact of heterogeneous delays over a decentralized communication network: analyses are worst-case in terms of delays in the whole network [2, 38]. We refer the interested reader to Assran et al. [1] for a more exhaustive survey on advances in asynchrony, both in shared-memory and decentralized models.

1.2 Main Contributions

We propose and analyze a fully decentralized and asynchronous algorithm described in Figure 1. This algorithm, based on delayed randomized gossip, is to our knowledge the first to benefit from an asynchronous speed-up: it achieves a linear rate of convergence that depends on the eigengap of the Laplacian of the graph weighted by local communication and computation constraints instead of global worst-case ones. Our model encompasses delayed updates and both communication and computation capacity limitations.

To obtain sharp dependence on the delays, we develop a novel general framework for the analysis of asynchronous algorithms. Carefully avoiding discrete iterates and a tedious global ordering of the updates, that we believe to be a bottleneck to analysis and implementation of asynchronous algorithms, we analyze our algorithm in continuous time, although the updates themselves are discrete. More precisely, our asynchronous descent (continuous-time analog of (2)) on some function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ (Appendix E), writes as, for any $t \in \mathbb{R}^+$:

$$dx(t) = - \sum_i \delta_{t \in \mathcal{P}_i} \eta_{t,i} g_i(x(t - \tau_{t,i})), \quad (3)$$

where g_i is a partial estimate of the gradient, and $\delta_{t \in \mathcal{P}_i} = 1$ only if $t \in \mathcal{P}_i \subset \mathbb{R}^+$, the instants at which updates are made using partial estimate g_i , for some stepsizes and delays $\eta_{t,i}, \tau_{t,i}$. In contrast to discrete iterates such as in (2), our continuous-time approach allows for much more natural assumptions on the delays. Delays are naturally represented as continuous-time parameters corresponding to the time required to perform a computation or a communication. Thus, no work is needed in order to have their discrete analog: boundedness of these physical delays is therefore realistic, and capturing delay heterogeneity becomes possible, while dependence intricacies between delays, choice of coordinates, and iterates are much more natural. We believe this continuous-time framework to be particularly well-suited to study asynchrony, especially in decentralized algorithms. Finally, nodes of the network can share the continuous-time parameter $t \in \mathbb{R}^+$ by simply synchronizing clocks once, while a discrete global counter $k \in \mathbb{N}$ is not locally available if we assume an asynchronous and decentralized network.

We identify from our sharp stability conditions and convergence guarantees a Braess's paradox-like phenomenon: deleting some carefully-chosen edges can increase the global connectivity and thus lead to faster convergence. Taking then into consideration communication costs, we use our results to minimize the energy used for communications by sparsifying our communication graph, while keeping the same rate of convergence.

This paper articulates as follows: we first present the model and the algorithm in Section 2, then detail the convergence results in Section 3. We refine the communication model in Section 4 and finally investigate the Braess's Paradox-type phenomenon and graph sparsification in Section 5.

2 Model Description and Algorithm

2.1 Assumptions and Notations

The communication network is represented by an undirected graph $G = (V, E)$ on the set of nodes $V = [n]$, and is assumed to be connected. Two nodes are said to be neighbors or adjacent in the

graph, and we write $i \sim j$, if $(ij) \in E$ or $i = j$. Two edges $(ij), (kl) \in E$ are adjacent in the graph if $(ij) = (kl)$ or if they share a node. Each node $i \in V$ has access to a local function f_i defined on \mathbb{R}^d , assumed to be L_i -smooth and σ_i -strongly convex [6], i.e. $\forall x, y \in \mathbb{R}^d$:

$$\frac{\sigma_i}{2} \|x - y\|^2 \leq f_i(x) - f_i(y) - \langle \nabla f_i(y), x - y \rangle \leq \frac{L_i}{2} \|x - y\|^2. \quad (4)$$

Denote $f(z) = \sum_{i \in [n]} f_i(z)$ for $z \in \mathbb{R}^d$ and $F(x) = \sum_{i \in [n]} f_i(x_i)$ for $x = (x_1^\top, \dots, x_n^\top) \in \mathbb{R}^{n \times d}$ where $x_i \in \mathbb{R}^d$ is attached to node $i \in [n]$. Let $L = \max_i L_i$ and $\sigma = \min_i \sigma_i$ denote the global regularity constants. For any function $g : \mathbb{R}^p \rightarrow \mathbb{R}$, g^* denotes its *Fenchel conjugate* on \mathbb{R}^p defined as

$$\forall y \in \mathbb{R}^p, g^*(y) = \sup_{x \in \mathbb{R}^p} \langle x, y \rangle - g(x) \in \mathbb{R} \cup \{+\infty\}.$$

The rate of convergence of our algorithm depends on the smallest positive eigenvalue γ of the Laplacian of graph G [26], weighted by some constants K_{ij} that depend on the local communication and computation delays.

Definition 1 (Graph Laplacian). *Let $(K_{ij})_{(ij) \in E}$ be a set of non-negative real numbers. The Laplacian of the graph G weighted by the K_{ij} 's is the matrix $\Delta_G(K)$ with (i, j) entry equal to $-K_{ij}$ if $(ij) \in E$, $\sum_{k \sim i} K_{ik}$ if $j = i$, and 0 otherwise. In the sequel K_{ij} always refers to the weights of the Laplacian, and $\lambda_2(\Delta_G(K))$ denotes this Laplacian's second smallest eigenvalue.*

2.2 Formulation of the Problem

We now introduce and formalize our approach and model in order to solve Problem (1). Classically [35], we consider that all nodes have a local parameter $x_i \in \mathbb{R}^d$, but that they should all be equal in order to have the minimizer of the sum. The problem thus writes:

$$\min_{x \in \mathbb{R}^{n \times d}} \sum_{i=1}^n f_i(x_i) \text{ such that } x_i = x_j \text{ for } (ij) \in E. \quad (5)$$

Problem (5) can be reformulated using Lagrangian duality, so that constraints become dual variables (the associated Lagrangian multipliers), and thus performing an update on one of these dual variables consists in performing an update along an edge of the network. Thus, decentralized algorithms can be obtained by applying accelerated gradient methods [35, 40] or coordinate gradient descent [14, 16] to the dual version of Problem (5).

Yet, in the basic dual formulation, nodes need to compute dual gradients each time they communicate. To alleviate this issue, we follow Hendrikx et al. [14] and consider an augmented graph approach with virtual nodes and edges for computations, in order to handle computation and communication delays separately. For each node $i \in V$ in the initial network, we add a computational node i^{comp} and an edge between i and i^{comp} . The augmented graph, denoted $E^+ = (V^+, E^+)$ has $2n$ nodes and $\#E + n$ edges. Each function f_i for $i \in V$ is then divided between nodes i and i^{comp} , by setting $g_i(x) = \frac{\sigma}{4} \|x\|^2$ and $g_{i^{\text{comp}}}(x) = f_i(x) - \frac{\sigma}{4} \|x\|^2$, so that both g_i and $g_{i^{\text{comp}}}$ are $\sigma/2$ -strongly convex¹. This yields the following minimization problem:

$$\min_{x \in \mathbb{R}^{2n \times d}} \sum_{j \in V^+} g_j(x_j) \text{ such that } x_i = x_j \text{ for } (ij) \in E, \text{ and } x_i = x_{i^{\text{comp}}} \text{ for } i \in V. \quad (6)$$

Our approach then consists in applying coordinate gradient steps on the dual formulation of Problem 6 [23], where coordinates become edges (either real ones $(ij) \in E$, or virtual ones (ii^{comp}) for $i \in V$). Since $\nabla g_i^*(x) = 2x/\sigma$, a dual coordinate step alongside $(ij) \in E$ consists in a (weighted) local averaging of variables x_i, x_j , involving communications between i and j . On the other hand, a step alongside a virtual edge (ii^{comp}) involves local computation of a conjugate gradient $\nabla g_{i^{\text{comp}}}^*$ but no communications. In the rest of the paper, x^* is the solution to (1), while $\bar{x}^* \in \mathbb{R}^{V^+ \times d}$ is the vector with x^* at every node i.e. the solution to (6).

¹We split the strong convexity uniformly for simplicity, but other choices are possible (see Appendix F).

2.3 Poisson Point Process Model with Delays

The times at which updates take place are ruled by independent *Poisson point processes* [19] (presented in Appendix A), noted as *P.p.p.* in the rest of the paper. This is a design choice of the algorithm: these point processes are easily simulated with exponential random variables, while making the analysis possible with the study of infinitesimal increments of time. Updates alongside an edge $(ij) \in E$ (resp. a virtual edge (ii^{comp}) for $i \in V$) take place at the instants of \mathcal{P}_{ij} a *P.p.p.* of intensity $p_{ij} > 0$ (resp. $\mathcal{P}_i^{\text{comp}}$ a *P.p.p.* of intensity $p_i^{\text{comp}} > 0$). The processes $\mathcal{P}^{\text{comm}} = \bigcup_{(ij) \in E} \mathcal{P}_{ij}$ and $\mathcal{P}^{\text{comp}} = \bigcup_{i \in V} \mathcal{P}_i^{\text{comp}}$ are again *P.p.p.* of respective communication and computation intensity:

$$I^{\text{comm}} = \sum_{(ij) \in E} p_{ij} \text{ and } I^{\text{comp}} = \sum_{i \in V} p_i^{\text{comp}}.$$

Local variable on node $i \in V$ (resp. on virtual node i^{comp}) at time $t \geq 0$ is denoted $x_i(t)$ (resp. $x_{i^{\text{comp}}}(t)$). Throughout the paper, \mathcal{F}_t for $t \in \mathbb{R}^+$ denotes the filtration of the point process $\mathcal{P} = \mathcal{P}^{\text{comm}} \cup \mathcal{P}^{\text{comp}}$ up to time $t \in \mathbb{R}$. Intensities p_{ij} and p_i^{comp} correspond to unnormalized probabilities: at a defined time t , the probability that $(ij) \in E$ is the next edge alongside which an update is made, is p_{ij}/I^{comm} , while the probability that node i is the next one to perform a computational update is $p_i^{\text{comp}}/I^{\text{comp}}$. Global intensities I^{comm} and I^{comp} correspond to the frequency of these updates: the number of communication (resp. computational) updates made in an interval of time of length δ is a discrete Poisson random variable of mean δI^{comm} (resp. δI^{comp}). The algorithm we describe below (communication and computation update rules) is called *DCDM*, for Delayed Coordinate Dual Method. We stress the fact that instants at which updates are made are distributed according to a *P.p.p.* by design of the algorithm. In particular, this is not an assumption on the statistics of the updates, but this is enforced by the implementation of *DCDM*, which we discuss further in Section 4 and Appendix C.

Communications: For a communication edge $(ij) \in E$, we denote by $\tau_{ij} > 0$ an upper-bound on the time it takes to send a full vector between nodes i and j . An activation time $t \in \mathcal{P}_{ij}$ of edge (ij) corresponds to a communication update on x_i and x_j at this time t , consecutive to variables sent at a time $t - \tau_{ij}$. The update resulting from the coordinate step on the dual variable associated to edge (ij) is, for some $K_{ij} > 0$:

$$\begin{cases} x_i(t) \stackrel{t}{\leftarrow} x_i(t) - \frac{K_{ij}}{2p_{ij}}(x_i(t - \tau_{ij}) - x_j(t - \tau_{ij})), \\ x_j(t) \stackrel{t}{\leftarrow} x_j(t) + \frac{K_{ij}}{2p_{ij}}(x_i(t - \tau_{ij}) - x_j(t - \tau_{ij})). \end{cases} \quad (7)$$

Parameters $0 < K_{ij} \leq 1$ counterbalance the variance induced by the delays. Setting $K_{ij} = p_{ij}$ yields a (delayed) local averaging.

Local Computations: For computation edge (ii^{comp}) between node $i \in V$ and its virtual node i^{comp} , we denote by τ_i^{comp} an upper-bound on the time it takes to make a computational update. A computation update at a time $t \in \mathcal{P}_i^{\text{comp}}$ corresponds to a computation started at $t - \tau_i^{\text{comp}}$, and for such a $t \in \mathcal{P}_i^{\text{comp}}$:

$$\begin{cases} x_i(t) \stackrel{t}{\leftarrow} x_i(t) - \frac{K_i^{\text{comp}}}{2p_i^{\text{comp}}}(x_i(t - \tau_i^{\text{comp}}) - \nabla g_{i^{\text{comp}}}^*(y_{i^{\text{comp}}}(t - \tau_i^{\text{comp}}))) \\ y_{i^{\text{comp}}}(t) \stackrel{t}{\leftarrow} y_{i^{\text{comp}}}(t) - \frac{\sigma K_i^{\text{comp}}}{p_i^{\text{comp}}}(\nabla g_{i^{\text{comp}}}^*(y_{i^{\text{comp}}}(t - \tau_i^{\text{comp}})) - x_i(t - \tau_i^{\text{comp}})) \end{cases}, \quad (8)$$

where $K_i^{\text{comp}} > 0$, $y_{i^{\text{comp}}} := \nabla g_{i^{\text{comp}}}(x_{i^{\text{comp}}})$ and equivalently $x_{i^{\text{comp}}} := \nabla g_{i^{\text{comp}}}^*(y_{i^{\text{comp}}})$. Equation (15) in Appendix A is a more formal writing of these updates, using stochastic calculus formalism. To sum-up the communication and computation time constants, we have: (i) *Poisson intensities* p_{ij} and p_i^{comp} that correspond to the frequency of respectively local communication and computation updates. (ii) Constants τ_{ij} (resp. τ_i^{comp}), upper-bounds on the time a communication alongside edge (ij) (resp. a computation at node i) takes. While Poisson intensities p are parameters of the algorithm, delays τ are physical constraints.

3 Stability Conditions and Convergence Guarantees

We now present our main results: stability conditions for our *DCDM* algorithm and Delayed Randomized Gossip, and convergence guarantees.

Assumption 1 (Stability Condition). *Assume that for all $(ij) \in E$ and $i \in V$, we have (note that $(ij) \sim (ji)$ and $i^{\text{comp}} \sim i$, constant e is $\exp(1)$):*

$$K_{ij} \leq \frac{p_{ij}}{1 + \sum_{(kl) \sim (ij)} p_{kl}(\tau_{ij} + e\tau_{kl})}, \quad K_i^{\text{comp}} \leq \frac{p_i^{\text{comp}}}{1 + \sum_{j \sim i} p_{ij}(\tau_i^{\text{comp}} + e\tau_{ij})}. \quad (9)$$

Theorem 1. *Let $\Delta_G(K)$ be the Laplacian matrix of graph G with weights K_{ij} on its edges, which satisfy Assumption 1. Denote by $\lambda_2(K)$ its second smallest eigenvalue. Let $\gamma > 0$ such that $\gamma < 1/\tau_{\max}$ and:*

$$\gamma \leq \frac{\sigma}{2} \min \left(\frac{\lambda_2(\Delta_G(K))}{L + \sigma}, \min_{i \in V} \frac{K_i^{\text{comp}}}{L_i} \right). \quad (10)$$

For any $T \geq 0$, we have:

$$\frac{\int_0^T e^{\gamma t} \mathbb{E} [\|x(t) - \bar{x}^*\|^2] dt}{\int_0^T e^{\gamma t} dt} \leq e^{-\gamma T} \frac{L}{\sigma} \frac{1 + \frac{\tau_{\max}}{T}}{1 - \gamma \tau_{\max}} \|x(0) - \bar{x}^*\|^2. \quad (11)$$

A direct consequence is the case where each f_i is of the form $f_i(x) = \|x - c_i\|^2$ for some $c_i \in \mathbb{R}^d$, which reduces to computing the mean $\bar{c} = \frac{1}{n} \sum_i c_i$ in a decentralized way. In this case, computations are trivial and so $g_{i^{\text{comp}}} = 0$ for all i . We call the algorithm resulting from communication updates as in Equation (7) the Delayed Randomized Gossip algorithm.

Corollary 1. *Delayed Randomized Gossip for the averaging problem for K_{ij} verifying Equation (9), has linear rate of convergence $\gamma = \lambda_2(\Delta_G(K)) \wedge 1/\tau_{\max}$.*

For large graphs ($n \rightarrow \infty$) and delays independent from the size of the graph, we have $\lambda_2(\Delta_G) \rightarrow 0$ and so $\lambda_2(\Delta_G) \wedge 1/\tau_{\max} \wedge K_{\min}^{\text{comp}} = \lambda_2(\Delta_G)$. The *asynchronous speed-up* consists in having a rate of convergence as the eigengap of the Laplacian of the graph, weighted by local communication and computation constraints: the term $\lambda_2(\Delta_G(K))$, where each K_{ij} is impacted only by local quantities. Assumption 1 suggests a scaling of $p_{ij} \approx 1/\tau_{ij}$ and $p_i^{\text{comp}} \approx 1/\tau_i^{\text{comp}}$, giving local weights K_{ij} in the Laplacian of order $1/(\text{degree}_{ij} \tau_{ij})$ where degree_{ij} is the degree of edge (ij) in the edge-edge graph. On the other hand, synchronous algorithms are slowed-down by the slowest node: the equivalent term would be of order $\lambda_2(\Delta_G(1/(\text{degree}_{ij} \tau_{\max})))$, as explained in Appendix B.4. For some $\varepsilon > 0$, we compare the continuous time $T_\varepsilon^{\text{DCDM}}$ it takes for *DCDM* to reach precision ε to the same quantity for a standard optimal synchronous algorithm such as *MSDA* [35], where $\kappa = L/\sigma$ and $K_{\min}^{\text{comp}} = \min_{i \in V} K_i^{\text{comp}} L/L_i$:

$$T_\varepsilon^{\text{DCDM}} = \mathcal{O} \left(\kappa \left[\lambda_2 \left(\Delta_G \left(\frac{1}{\tau_{ij} \text{degree}_{ij}} \right) \right)^{-1} + K_{\min}^{\text{comp}-1} \right] \log(\varepsilon^{-1}) \right).$$

$$T_\varepsilon^{\text{Synch, Opt}} = \mathcal{O} \left(\sqrt{\kappa} \left[\tau_{\max}^{\text{comm}} \lambda_2 \left(\Delta_G \left(\frac{1}{\text{degree}_{ij}} \right) \right)^{-\frac{1}{2}} + \tau_{\max}^{\text{comp}} \right] \log(\varepsilon^{-1}) \right).$$

We clearly see the differences with our approach here: *MSDA* uses acceleration and thus obtains $\sqrt{\kappa}$ and $\sqrt{\lambda_2(\Delta_G(1/\text{degree}))}$ factors. Yet, this is a purely synchronous algorithm and so it depends on $\tau_{\max}^{\text{comm}}$ and $\tau_{\max}^{\text{comp}}$. Our asynchronous algorithm *DCDM* depends on the mixing time of a random walk on the graph where heuristically, jumping from node i to j takes a time τ_{ij} instead of τ_{\max} : this is the *asynchronous speed-up*. Similarly, computation times are weighted by the *local* Lipschitz constant, so that slow computing nodes do not slow the system down if their local L_i is small enough. Finally, applying a Jensen inequality in Equation (11) yields Corollary 2, and min-max characterization of Perron-Frobenius eigenvalues gives Corollary 3 (Appendix F.5).

Corollary 2. *For $\hat{x}(T)$ the exponentially weighted average on the trajectory of $x(t)$ defined as:*

$$\hat{x}(T) = \frac{\int_0^T e^{\gamma t} x_t dt}{\int_0^T e^{\gamma t} dt}, \quad \text{we have} \quad \mathbb{E} [\|\hat{x}(T) - \bar{x}^*\|^2] \leq C e^{-\gamma T}$$

for some constant $C > 0$ that depends on the initialization, σ , L and the graph.

Corollary 3. For non-negative $(K_{ij})_{(ij) \in E^+}$, let $M(K) \in \mathbb{R}^{E^+ \times E^+}$ the matrix indexed by the edges of the augmented graph, with $M(K)_{(ij),(kl)} = \mathbb{I}_{(ij) \sim (kl)} K_{ij}(\tau_{ij} + e\tau_{kl})$. Then, there exists Poisson intensities $(p_e)_{e \in E^+} \in \mathbb{R}^{+*}$ such that (9) is verified, if and only if $\rho(M(K)) < 1$. The latter condition can be easily satisfied if the row-sums of $M(K)$ are strictly less than 1, or equivalently if:

$$\forall (ij) \in E^+, K_{ij} < \frac{1}{\sum_{(kl) \sim (ij)} \tau_{ij} + e\tau_{kl}}.$$

Finally, delayed randomized gossip can achieve a rate $\gamma = \sup_{K: \rho(M(K)) < 1} \lambda_2(\Delta_G(K)) \wedge 1/\tau_{\max}$.

3.1 Proof Overview of Theorem 1

A complete proof can be found in Appendix F. However, we present here a quick overview of the techniques used, as we believe them to be of independent interest. As explained in Section 2.2, each update (communication or computation) is derived from a coordinate-gradient step on a dual formulation of Problem (6). Our main work is then to prove a continuous-time coordinate gradient descent lemma (Theorem 3 in Appendix E). Consider a function $F: \mathbb{R}^p \rightarrow \mathbb{R}$ minimized with an algorithm such as Equation (3) for partial estimates of the gradient $g_i(x) = \nabla_i F(x)$, where \mathcal{P}_i are Poisson point processes. The first step of the proof consists in studying infinitesimal increments:

$$\mathbb{E}[F(x(t+dt)) - F(x(t))] \leq dt \sum_i \mathbb{E} \left[-A_i \|\nabla_i F(x(t - \tau_i))\|^2 + B_i \int_{t-\tau_i}^t \|\nabla_i F(x(s - \tau_i))\|^2 ds \right],$$

an inequality true up to a $o(dt)$, in order to upper-bound $\frac{d\mathbb{E}[F(x(t)) - F(x^*)]}{dt}$ and thus $\frac{de^{\gamma t} \mathbb{E}[F(x(t)) - F(x^*)]}{dt}$. Stability conditions (Assumption 1) ensure that $A_i \geq 2\tau_i B_i$. That being done, a potential function inspired by the analysis of time-delayed ordinary differential equations (ODE) [32] and Lyapunov-Krasovskii function techniques for stability of such systems [12] is introduced:

$$\mathcal{L}_T^\gamma = \int_0^T e^{\gamma t} \mathbb{E}[F(x(t)) - F(x^*)] dt.$$

Integrating our upper-bound on $\frac{de^{\gamma t} \mathbb{E}[F(x(t)) - F(x^*)]}{dt}$ for $0 \leq t \leq T$ then yields:

$$\frac{d\mathcal{L}_T^\gamma}{dt} \leq F(0) - F(x^*) + \gamma(\mathcal{L}_T^\gamma - \mathcal{L}_{T-\tau_{\max}}^\gamma).$$

This delayed differential inequality then ensures the sub-linear growth of \mathcal{L}_T^γ by Lemma 2 (Appendix D) and concludes the proof of Theorem 1.

3.2 Sharpness

We now focus on the gossip averaging problem (Corollary 1), and study to what extent the stability conditions on each K_{ij} as well as the rate of convergence are sharp. Consider (7) without any delays ($\tau_{ij} = 0$). In this case (Appendix B.2), if $K_{ij} < p_{ij}$:

$$\mathbb{E}[\|x(t) - \bar{x}(0)\|^2] \leq e^{-\lambda_2(\Delta(K_{ij}))t} \|x(0) - \bar{x}(0)\|,$$

and we know that this rate of convergence is sharp. For a given choice of K_{ij}, p_{ij} such that $K_{ij}/p_{ij} \leq 1$, the exponential rate is not impacted by the delays. Yet, delays impact the stability of the system, and so the stepsizes K_{ij} need to be tuned down accordingly. We now explain why (9) actually corresponds (up to constant factors) to the highest K_{ij} that ensure stability, so that our rate of convergence is sharp indeed. First, ensuring that $K_{ij}/p_{ij} \leq 1$ is the standard condition that relates the step-size with the directional smoothness (which is equal to 1 in this case) and the sampling probability. It is a consequence of the discrete updates and it does not come from the delayed framework.

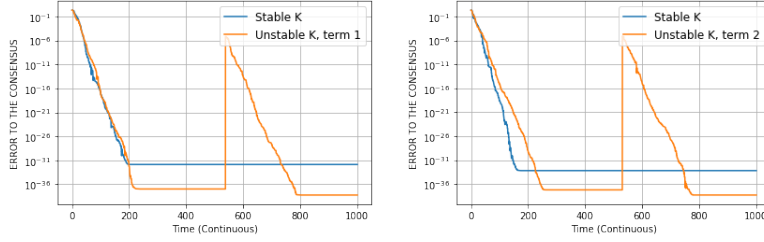
Inspired by the theory of time-delayed systems and delayed ordinary differential equations [32, 30], we study stability conditions on parameters K_{ij} of the stochastic process $(x(t))$ first by studying the stability of its mean. Writing $\tilde{x}_i(t) = \mathbb{E}[x_i(t)]$, we have:

$$\frac{d\tilde{x}_i(t)}{dt} = - \sum_{i \sim j} \frac{K_{ij}}{2} (\tilde{x}_i(t - \tau_{ij}) - \tilde{x}_j(t - \tau_{ij})). \quad (12)$$

By linearizing this delayed ODE, we obtain that it is stable whenever $\rho(\Delta_G(K_{ij}\tau_{ij})) < 1$ [25]. The only way to ensure locally that this global condition is satisfied is to have that $\sum_{j \sim i} K_{ij}\tau_{ij} < 1$ for all nodes i . Two sufficient conditions for this to be true are to set:

$$\left(\forall (ij) \in E, K_{ij} < \frac{p_{ij}}{\sum_{kl \sim ij} p_{kl}\tau_{kl}} \right) \text{ or } \left(\forall (ij) \in E, K_{ij} < \frac{p_{ij}}{\sum_{kl \sim ij} p_{kl}\tau_{ij}} \right). \quad (13)$$

Our expression of K_{ij} (9) combines both these conditions, together with $K_{ij} < p_{ij}$. Our sufficient conditions (13) thus naturally imply $\rho(\Delta_G(K_{ij}\tau_{ij})) < 1$, and we conjecture that they are tight. We verify this experimentally in Figure 2, which shows that if either one of the conditions (13) is not satisfied, delayed randomized gossip is unstable.



(a) Without the first condition

(b) Without the second condition

Figure 2: Cyclic graph with 3 nodes with pathological delays: 2 edges with $\tau = 1, p = 1$, an edge with $\tau' = 500, p' = 1/500$. Figure 2(a) (resp. 2(b)) compares delayed randomized gossip with K as in (9) to delayed randomized gossip where the term $\sum_{kl \sim ij} p_{kl}\tau_{ij}$ (resp. $\sum_{kl \sim ij} p_{ij}\tau_{ij}$) was removed from the denominator of K_{ij} . For safety, the unstable K is divided by 2, to ensure that explosions are not due to globally bigger stepsizes.

4 Handling Communication and Computation Capacity Limitations

As presented, *DCDM* and Delayed Randomized Gossip cannot be implemented in the presence of communication and/or computation limited capacities. In this section, we introduce a more realistic model, under which our algorithm keeps the same convergence guarantees.

Definition 2 (Communication and Computation Capacities). *We model communication and computation limitations in the following way:*

1. Computation Capacity: Node i can compute only q_i^{comp} gradients in an interval of time of length τ_i^{comp} ;
2. Communication Capacity, edge-wise limitations: Only q_{ij} messages can be exchanged simultaneously between adjacent nodes $i \sim j$ in an interval of time of length τ_{ij} ;
3. Communication Capacity, node-wise limitations: Node i can only send q_i^{comm} messages in any interval of time of length $\tau_i^{\text{comm}} = \min_{j \sim i} \tau_{ij}$.

Under these constraints, times of updates can not be *Poisson point processes*. We thus use another point process: $\tilde{\mathcal{P}}$ is a *truncated P.p.p.* of intensity p and parameters $\tau > 0, q \in \mathbb{N}$ on \mathbb{R}^+ if there exists a *P.p.p.* \mathcal{P} of intensity p on \mathbb{R} such that:

$$\tilde{\mathcal{P}} = \{t \in \mathcal{P} \cap \mathbb{R}^+ : |\mathcal{P} \cap [t - \tau, t]| < q\}.$$

In other words, $\tilde{\mathcal{P}}$ is a *P.p.p.* of intensity p conditioned to be bounded by q on every interval of length τ . Let $\tilde{\mathcal{P}}_{ij}$ (resp. $\tilde{\mathcal{P}}_i^{\text{comp}}$) a *truncated P.p.p.* of intensity p_{ij} and parameters τ_{ij}, q_{ij} (resp. intensity p_i^{comp} and parameters $\tau_i^{\text{comp}}, q_i^{\text{comp}}$). Furthermore, we condition as before the point process $\cap_{j \sim i} \tilde{\mathcal{P}}_{ij}$ to be bounded by q_i^{comm} on every interval of length τ_i^{comm} . These processes are easily generated, as explained in Appendix C. We have the following result:

Theorem 2. For DCDM as described in Section 2.3, with update times ruled by \tilde{P}_{ij} and $\tilde{P}_i^{\text{comp}}$, if:

$$2p_i^{\text{comp}}\tau_i^{\text{comp}} \leq q_i^{\text{comp}}, \quad 2p_{ij}\tau_{ij} \leq q_{ij}, \quad 2\sum_{j \sim i} p_{ij}\tau_i^{\text{comm}} \leq q_i^{\text{comm}}.$$

then for K_{ij} and γ as in Theorem 1 the same convergence guarantee holds, replacing γ by $\gamma/2$.

The two first conditions are usually verified, since $p_{ij}, p_i^{\text{comp}}$ are usually chosen of respective order $1/\tau_{ij}, 1/\tau_i^{\text{comp}}$. The last condition is more subtle and suggests that a node's degree should be of order q_i^{comm} , thus enforcing sparsification of the graph. For instance, having $q_i^{\text{comm}} = 1$ would lead to blocking nodes while they are communicating, like it is done in loss-networks [18].

5 Braess's Paradox and Graph Sparsification to Improve Communications

Braess's paradox. In this section, we present a well-known phenomenon that usually arises in road-traffic: removing one or more roads in a road network can speed up the overall traffic flow. This phenomenon, called *Braess's paradox* [10], also arises in loss networks [3]. In our problem, this translates as adding an edge (ij) with a non-negligible Poisson intensity p_{ij} . Consider the line graph with constant delays $\tau_{i,i+1} = \tau$. Add edge $(1, n)$ in order to close the line, with a delay $\tau_{1n} = \tau'$ such that τ' is arbitrarily large. If the added Poisson intensity p_{1n} satisfies $\tau_{1n}p_{1n} \rightarrow \infty$, then $K_{12}, K_{n-1,n} \rightarrow 0$ and $\gamma \rightarrow 0$ (nodes 1 and n isolated), decreasing the overall connectivity. In order to bypass the phenomenon, we would need to virtually delete the edge, by setting $p_{1n} = 0$. Figures 3(a) and 3(d) show that by simply deleting some edges in a more complex graph, the spectral gap can increase. In all simulations below, N is the number of edges, λ_2 is the un-normalized second smallest eigenvalue of the Laplacian of the graph with weights K as in Theorem 1.

Sparsification and Improved Communications. Corollary 3 writes the optimal rate of convergence as an optimization problem over K . Reversely, the optimal choice of Poisson weights p_{ij} solve:

$$\sup_{p_{ij} \geq 0, (ij) \in E} \inf_{\|x\|=1, \sum_i x_i = 0} \sum_{(ij) \in E} \frac{p_{ij}}{1 + \sum_{kl \sim ij} p_{kl}(\tau_{ij} + e\tau_{kl})} (x_i - x_j)^2.$$

As explained above, we expect to obtain a sparser graph when solving this optimization problem. Due to physical constraints (Theorem 2), sparsification of the graph can also lead to huge improvements in terms of communication power required. In wireless communications, the energy spent by a node scales as the number of messages sent, while in wired communications, it scales as the number of messages transiting in the network. The respective associated energies thus are $\sum_{(ij)} p_{ij}$ and $\sum_{(ij)} p_{ij}\tau_{ij}$. We focus on wired communications. Adding a regularization term to the problem to enforce sparsification and limit the energy spent in communications, we solve:

$$\sup_{p_{ij} \geq 0, (ij) \in E} \lambda_2 \left(\Delta_G \left(\frac{p_{ij}}{1 + \sum_{kl \sim ij} p_{kl}(\tau_{ij} + e\tau_{kl})} \right) \right) - \omega \sum_{(ij) \in E} p_{ij}\tau_{ij}, \quad (14)$$

for a constant $\omega > 0$ small enough. We take $\omega = \lambda_2 / (2 \sum_{ij} p_{ij}\tau_{ij})$ for (p_{ij}) the initialization of the algorithm. Our results are shown in Figures 3: starting with $p_{ij} = 1/\tau_{ij}$ on the complete graph (with τ_{ij} being squared random variables with uniform distribution in $[0, 1]$, or proportional to a random geometry), we run our algorithm (simulated annealing, Appendix G) to approximately solve (14). As shown in Figure 3, in both scenarios, we drastically decreased the number of edges in the graph, at little or no cost in convergence speed (λ_2) and energy (J).

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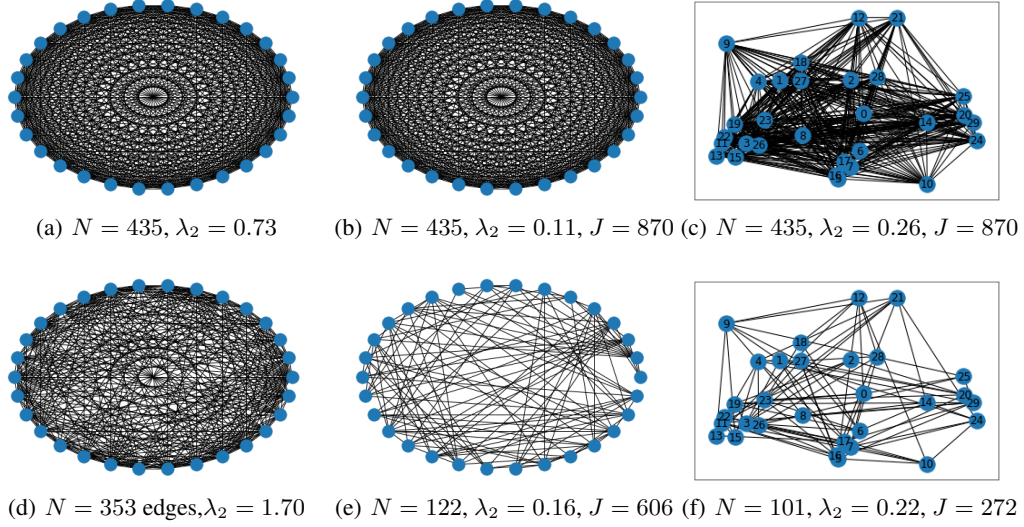


Figure 3: **Braess's paradox:** Figure 3(a) is the complete graph with 30 nodes, delays τ_{ij} such that $\sqrt{\tau_{ij}}$ random uniform in $[0, 1]$, and intensities $p_{ij} = 1/\tau_{ij}$. Figure 3(d) obtained simply by deleting edges (setting some p_{ij} equal to 0), without changing the value of the intensities of the edges kept. **Sparsification:** on a complete graph with random *i.i.d.* uniform delays squared (Figures 3(b), 3(e)) or with delays proportional to random geometry (Figures 3(c), 3(f)). We denote $J = \sum_{ij} p_{ij} \tau_{ij}$.

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The appendix articulates as follows:

1. In Appendix A, we define and present preliminary results on Poisson point processes and truncated Poisson point processes;
2. In Appendix B, we present known results about standard synchronous and randomized gossip algorithms, mentioned in the article, and discuss the intuition behind the asynchronous speed-up we bring.
3. In Appendix C, we describe in details the *DCDM* and randomized gossip communication-computation schemes, under communication and computation capacity constraints.
4. In Appendix D, we define and analyze a continuous-time gradient descent with delays, under constant and *i.i.d.* delays;
5. In Appendix E, we introduce and analyse a continuous-time delayed coordinate descent, crucial in the analysis of our algorithms;
6. In Appendix F, we derive *DCDM* and delayed randomized gossip from a dual formulation of the problem, and prove Theorem 1 and its related corollaries;
7. In Appendix G, we explain the algorithm used (a simple simulated annealing) to obtain Figure 3.

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A Poisson Point Measures and Processes

We here define and present *Poisson point processes*, and a crucial property for our analysis. For a more formal and exhaustive study of these processes, we refer the reader to [19] or [21].

Definition 3. A Poisson point measure on \mathbb{R}^+ with intensity $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a random measure ν on \mathbb{R}^+ such that:

1. for all measurable and disjoint subsets A, B of \mathbb{R}^+ , $\nu(A)$ and $\nu(B)$ are independent random variables;
2. for any measurable $A \subset \mathbb{R}^+$, $\nu(A)$ follows a discrete Poisson law of mean $\int_A \phi(t)dt$.

Homogeneous Poisson point measures are obtained for constant functions ϕ .

Proposition 1. Let ν be a homogeneous Poisson point measure of constant intensity $p > 0$. There exists a decomposition $d\nu(t) = \sum_{k \geq 0} \delta_{T_k}(dt)$ such that:

1. $(T_k)_{k \geq 0}$ is an increasing sequence;
2. $(T_{k+1} - T_k)_{k \geq 0}$ are i.i.d. of lax exponential with rate $p > 0$.

The Poisson point process \mathcal{P} of constant intensity $p > 0$ is then defined as $\mathcal{P} = \{T_k, k \geq 0\}$.

The following simple property is fundamental in our proofs, and allows for the study of infinitesimal increments.

Proposition 2. Let \mathcal{P} be a Poisson point process of intensity $p > 0$. Then, for any $t \geq 0$ and $\varepsilon > 0$, we have when $\varepsilon \rightarrow 0$:

$$\begin{aligned} \mathbb{P}(\#\mathcal{P} \cap [t, t + \varepsilon] = 0) &= 1 - p\varepsilon + o(\varepsilon) \\ \mathbb{P}(\#\mathcal{P} \cap [t, t + \varepsilon] = 1) &= p\varepsilon + o(\varepsilon) \\ \mathbb{P}(\#\mathcal{P} \cap [t, t + \varepsilon] > 1) &= o(\varepsilon). \end{aligned}$$

Next proposition justifies the fact that modelling the update times or the times at which computations/communications start as P.p.p. is the same.

Proposition 3 (Translation). Let $\mathcal{P} = \{t_k, k \in \mathbb{N}\}$ be a P.p.p. of intensity p on \mathbb{R} . Let $\{\tau_k, k \in \mathbb{N}\}$ be a sequence of i.i.d. real random variables. Then, the point process $\{t_k + \tau_k, k \in \mathbb{N}\}$ is still a P.p.p. of intensity p .

Finally, using stochastic processes formalism, updates (7) and (8) writes as:

$$\begin{cases} dx_i(t) = - \sum_{j \sim i} N_{ij}(dt) \frac{K_{ij}}{2p_{ij}} (x_i(t - \tau_{ij}) - x_j(t - \tau_{ij})) \\ dy_{i\text{comp}}(t) = -N_i^{\text{comp}}(dt) \frac{K_i^{\text{comp}} \sigma}{2p_{i\text{comp}}} (\nabla g_{i\text{comp}}^*(y_{i\text{comp}})(t - \tau_i^{\text{comp}}) - x_i(t - \tau_i^{\text{comp}})), \end{cases} \quad (15)$$

where we denote $N_{ij}(t) = \#\mathcal{P}_{ij}([0, t])$ and $N_i^{\text{comp}}(t) = \#\mathcal{P}_i^{\text{comp}}([0, t])$ the *Poisson point measures* associated to the communication and computation *P.p.p.*, so that $N_{ij}(dt) = \delta_{t \in \mathcal{P}_{ij}}$ is a Dirac at every point in \mathcal{P}_{ij} (and similarly for $\mathcal{P}_i^{\text{comp}}$).

In the presence of capacity limitations, *Poisson point processes* above are replaced by *truncated P.p.p.*, defined as follows.

Definition 4 (Truncated P.p.p.). *Let \mathcal{P} a P.p.p. of intensity $p > 0$ on \mathbb{R} , and $\tau \in \mathbb{R}_+, q \in \mathbb{N}^*$. Let N be the Poisson point measure associated to \mathcal{P} . We define the truncated Poisson point measure \tilde{N} of intensity p on \mathbb{R}^+ and parameters τ, q as:*

$$d\tilde{N} = \sum_{t \in \mathcal{P} \cap \mathbb{R}^+} \mathbb{I}_{\{N([t-\tau, t]) \leq q\}} \delta_t,$$

and we let $\tilde{\mathcal{P}}$ be the point process associated to this point measure. In other words: $\tilde{\mathcal{P}}$ is a P.p.p. of intensity restricted to be of cardinality less than q on any interval of length τ .

B Gossip Algorithms: General Considerations on the Averaging Problem and Asynchronous Speed-Up

B.1 Synchronous Gossip

In the synchronous setting, all nodes are allowed to share a common clock, which enables them to perform operations synchronously. Formally, a *gossip matrix* is defined as follows:

Definition 5 (Gossip Matrix). *A gossip matrix is a matrix $W \in \mathbb{R}^{n \times n}$ such that:*

- $\forall (i, j) \in [n]^2, W_{i,j} > 0 \implies i \sim j \text{ or } i = j$ (supported by G),
- $\forall i \in [n], \sum_{j \sim i} W_{i,j} = 1$ (stochastic),
- $\forall (i, j) \in [n]^2, W_{i,j} = W_{j,i}$ (symmetric).

Iteratively, at times $t = 0, 1, 2, \dots$, if $x(t) = (x_i(t))_i \in \mathbb{R}^{n \times d}$ describes the information stacked locally at each node ($x_i(t)$ being the vector at node i), we perform the operation $x(t+1) = Wx(t)$. It is to be noted that, thanks to the sparsity of the gossip matrix, this operation is local: for all node i ,

$$x_i(t+1) = \sum_{j \sim i} W_{ij} x_j(t), \quad (16)$$

where $i \sim j$ if they are neighbors or if $i = j$. The convergence bound will be stated below. Intuitively, at each iteration, each node i sends a proportion of its mass to each one of its neighbour, the condition $\sum_{j \sim i} W_{ij} = 1$ being the mass conservation.

Proposition 4 (Synchronous Gossip). *Let γ_W be the eigengap of the laplacian of G weighted by W_{ij} at each edge. Then, for all $k = 0, 1, 2, \dots$:*

$$\|x(k) - \bar{c}\| \leq (1 - \gamma_W)^k \|c - \bar{c}\|, \quad (17)$$

where $x(0) = c$, and \bar{c} is when consensus is reached

Proof. For $k \geq 0$,

$$\begin{aligned} x(k+1) - \bar{c} &= W(x(k) - \bar{c}) \\ \implies \|x(k+1) - \bar{c}\| &\leq \lambda_2(W) \|x(k) - \bar{c}\|, \end{aligned}$$

where λ_2 is the second largest eigenvalue of W , 1 being the largest (W is stochastic symmetric), and \bar{c} being in the corresponding eigenspace. We conclude by saying that $\lambda_2(W) = 1 - \gamma_W$ where γ_W is the smallest non null eigenvalue of $Id - W$. Notice that $Id - W$ is the laplacian of the graph weighted by $\nu_{ij} = W_{ij}$. \square

Then, since every iteration takes a time τ_{max} , denoting time in a continuous way by $t \in \mathbb{R}^+$, we have:

$$\|x(t) - \bar{c}\| \leq (1 - \gamma_W)^{t/\tau_{max}-1} \leq \exp\left(-\frac{\gamma_W}{\tau_{max}}(t - \tau_{max})\right), \quad (18)$$

and $\gamma_W/\tau_{max} = \gamma_{synch}$ where γ_{synch} is the smallest non-null eigenvalue of the laplacian of the graph with weights $\nu_{ij} = W_{ij}/\tau_{max}$.

B.2 Randomized Gossip

Time is indexed in a continuous way, by \mathbb{R}^+ . For every edge $e = (ij) \in E$, let \mathcal{P}_{ij} be a Poisson point process (P.p.p.) of constant intensity $p_{ij} > 0$ that we will call "clocks", all independent from each other. Updates will be ruled by these processes: at every clock ticking of \mathcal{P}_{ij} , nodes i and j update the value they stack by the mean $\frac{x_i + x_j}{2}$. For more generality and in order to mimick (7), we write the updates of the form:

$$\begin{cases} x_i \leftarrow x_i - \frac{K_{ij}}{2p_{ij}}(x_i - x_j), \\ x_j \leftarrow x_j + \frac{K_{ij}}{2p_{ij}}(x_i - x_j), \end{cases}$$

recovering local averagings for $K_{ij} = p_{ij}$. If we write $\mathcal{P} = \bigcup_{ij \in E} \mathcal{P}_{ij}$, \mathcal{P} is a P.p.p. of global intensity $I := \sum_{ij \in E} p_{ij}$.

Proposition 5 (Asynchronous Continuous Time Bound). *Assume that $K_{ij}/p_{ij} \leq 1$. Let $(x_t(i))_i$ be the vector stacked on the graph, and $\bar{c} = (\frac{1}{n} \sum_i c_i, \dots, \frac{1}{n} \sum_i c_i)^\top$ the consensus, where $c_i = x_i(0)$. Let σ_{asynch} be the smallest non null eigenvalue of the laplacian of the graph, weighted by the K_{ij} 's. For $t \geq 0$, we have:*

$$\mathbb{E}[\|x(t) - \bar{c}\|^2] \leq \exp(-t\sigma_{asynch})\|c - \bar{c}\|^2.$$

Proof. First, it is to be noted that, if \mathcal{P} is a P.p.p. of intensity $\lambda > 0$, for all $t \in \mathbb{R}$ and $dt \rightarrow 0$:

$$\mathbb{P}([t, t + dt] \cap \mathcal{P} \neq \emptyset) = \lambda dt + o(dt). \quad (19)$$

When ij activated at time t , we multiply $x(t)$ by $W_{ij} = I_n - \frac{K_{ij}}{p_{ij}}$. Compute W_{ij}^2 :

$$W_{ij}^2 = I_n - \frac{K_{ij}}{p_{ij}} \left(2 - \frac{K_{ij}}{p_{ij}}\right) \frac{(e_i - e_j)(e_i - e_j)^\top}{2}.$$

Notice then that $\sum_{ij} p_{ij} W_{ij}^2 = II_n - L$, where L is the laplacian of the graph weighted by the $\frac{K_{ij}}{p_{ij}}(2 - \frac{K_{ij}}{p_{ij}})$. We then get that, with $R_t^2 = \|x(t) - \bar{c}\|^2$ the squared error to the consensus at time t , up to a $o(dt)$:

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_t}[R_{t+dt}^2] &= (1 - Idt) \mathbb{E}^{\mathcal{F}_t}[R_{t+dt}^2 | \text{no activations in } [t, t + dt]] \\ &\quad + dt \sum_{ij} p_{ij} \mathbb{E}^{\mathcal{F}_t}[R_{t+dt}^2 | ij \text{ activated in } [t, t + dt]] + o(dt) \\ &= (1 - Idt) R_t^2 + dt(x(t) - \bar{c})^\top \left(\sum_{ij} p_{ij} W_{ij}^2 \right) (x(t) - \bar{c}) \\ &= R_t^2 - dt(x(t) - \bar{c})^\top \left(\sum_{ij} p_{ij} \frac{K_{ij}}{p_{ij}} \left(2 - \frac{K_{ij}}{p_{ij}}\right) \frac{(e_i - e_j)(e_i - e_j)^\top}{2} \right) (x(t) - \bar{c}) \\ &= R_t^2 - dt(x(t) - \bar{c})^\top L(x(t) - \bar{c}) \\ &\leq R_t^2 - \sigma_{asynch} R_t^2. \end{aligned}$$

Then, taking the mean, dividing by $dt \rightarrow 0$ and integrating concludes the proof. \square

For randomized gossip and local averagings ($K_{ij} = p_{ij}$), we thus recover the eigengap of the Laplacian weighted by p_{ij} . In the general case, due to Laplacian monotonicity and $K_{ij} \leq p_{ij}$, we get a rate of convergence between $\lambda_2(\Delta_G(K))$ and $2\lambda_2(\Delta_G(K))$. This is sharp: assume that $x(0) - \bar{c}$ is equal to the eigenvector of L associated to the eigenvalue σ_{asynch} . The inequality above becomes an equality.

B.3 Laplacian Monotonicity

We finish by proving the following intuitive result:

Proposition 6 (Monotonicity of the Laplacian). *Let $\Lambda(\lambda_{ij}, (ij) \in E)$ be the laplacian of the graph weighted by λ_{ij} . Then, its second smallest eigenvalue σ is a non decreasing function of each weight λ_{ij} .*

Proof. First compute $\langle \Lambda u, u \rangle$, the weights λ_{ij} being fixed:

$$\begin{aligned} \langle \Lambda u, u \rangle &= \sum_i \sum_{j \sim i} u_i(u_i - u_j) \lambda_{ij} \\ &= \frac{1}{2} \sum_i \sum_{j \sim i} (u_i - u_j)^2 \lambda_{ij}. \end{aligned}$$

It appears that for any $u \in \mathbb{R}^n$, these are non decreasing quantities in each λ_{ij} . If we take Λ and Λ' two laplacians with weights $\lambda_{ij} \leq \lambda'_{ij}$, we get, for all $u \in \mathbb{R}^n$, $\langle \Lambda u, u \rangle \leq \langle \Lambda' u, u \rangle$. Then, using that $\sigma = \min_{\|u\|=1, \langle u, \mathbb{I} \rangle=0} \langle \Lambda u, u \rangle$ (as \mathbb{I} is a eigenvector associated to the eigenvalue 0), we have $\sigma' \leq \sigma$ the desired result. \square

B.4 Asynchronous Speed-up

The rate of convergence of our algorithms are controlled by the smallest positive eigenvalue γ of the Laplacian of graph G [26], weighted by some constants K_{ij} that depend on the local communication and computation delays.

Definition 6 (Graph Laplacian). *Let $(K_{ij})_{(ij) \in E}$ be a set of non-negative real numbers. The Laplacian of the graph G weighted by the K_{ij} 's is the matrix $\Delta_G(K)$ with (i, j) entry equal to $-K_{ij}$ if $(ij) \in E$, $\sum_{k \sim i} K_{ik}$ if $j = i$, and 0 otherwise. In the sequel K_{ij} always refers to the weights of the Laplacian, and $\lambda_2(K)$ denotes this Laplacian's second smallest eigenvalue.*

We refer the reader to Figure 4 for some orders of magnitude for classical graphs. For synchronous Gossip, note that the Laplacian has weights W_{ij}/τ_{\max} . In these regular graphs, W_{ij} is often taken equal to the degree, hence the third column.

Graph	$\lambda_2(K)$ with $K_{ij} = 1$	$\lambda_2(K)$ with $K_{ij} = \frac{1}{\text{degree}}$
Line with n nodes	$\mathcal{O}(1/n^2)$	$\mathcal{O}(1/n^2)$
Cyclic graph with n nodes	$\mathcal{O}(1/n^2)$	$\mathcal{O}(1/n^2)$
2D-grid with $n = m^2$ nodes	$\mathcal{O}(1/n)$	$\mathcal{O}(1/n)$
Complete graph with n nodes	$\mathcal{O}(n)$	$\mathcal{O}(1)$

Figure 4: Order of magnitude of λ_2 for some classical graphs.

Proposition 7 (Gossip Algorithms). *For any $t \in \mathbb{R}^+$, let $x_{\text{synch}}(t)$ and $x_{\text{asynch}}(t)$ be respectively the iterates obtained with synchronous gossip and randomized gossip. We have:*

$$\begin{aligned} \|x_{\text{synch}}(t) - \bar{c}\| &\leq \|x_{\text{synch}}(0) - \bar{c}\| e^{-t\lambda_2(W/\tau_{\max})}, \\ \mathbb{E} \left[\|x_{\text{asynch}}(t) - \bar{c}\|^2 \right] &\leq \|x_{\text{asynch}}(0) - \bar{c}\|^2 e^{-t\lambda_2(p)}, \end{aligned}$$

for any $t \in \mathbb{R}^+$.

The *asynchronous speed-up* thus corresponds to achieving as exponential rate of convergence the eigengap weighted by local quantities: here intensities of local *P.p.p.* in the randomized gossip model. We therefore introduced delays and computations in the randomized gossip original algorithm, in order to obtain both a realistic asynchronous framework, and local stability conditions and convergence guarantees: the exponential rate of convergence we obtained in Theorems 1 and corollary 1 are of the form $\lambda_2(K)$, where K_{ij} for $(ij) \in E$ only depends on p_{ij} and on the p_{kl}, τ_{kl} for $(kl) \sim (ij)$.

We now compare synchronous rate of convergence $\lambda_2(\Delta_G(I - W))$ with our asynchronous one $\lambda_2(\Delta_G(K))$ in Figure 5. Proposition 6 ensures us that having local delays instead of τ_{\max} in the Laplacian is an improvement. However, no result quantifies clearly this improvement, hence the following numerical examples. We compare these eigengaps on 3 different graphs: 2D-Grid, Erdos-Renyi (parameter $p = 1/10$) and complete, graphs with 100 nodes each, with heterogeneous delays: 10% of edges have delays equal to 100 units of time, while the rest have delays equal to 1 unit of time. γ_{synch} and γ_{asynch} respectively denote in each configuration the Laplacian of the graph with weights $1/\tau_{\max}$, and K_{ij} as in Corollary 1 with Poisson intensities $p_{ij} = 1/\tau_{ij}$.

Graph (100 nodes)	Synchronous	Asynchronous
2D-grid	$\gamma_{\text{synch}} = 9.5 \times 10^{-4}$	$\gamma_{\text{asynch}} = 2.0 \times 10^{-2}$
Complete graph	$\gamma_{\text{synch}} = 1.0 \times 10^{-2}$	$\gamma_{\text{asynch}} = 1.3 \times 10^{-1}$
Erdos-Renyi (100,0.1)	$\gamma_{\text{synch}} = 1.6 \times 10^{-3}$	$\gamma_{\text{asynch}} = 3.9 \times 10^{-2}$

Figure 5: Synchronous VS Asynchronous eigengaps on some classical graphs in a heterogeneous setting in terms of delays.

C Discussion on our algorithm *DCDM*: realistic implementation, limitations

C.1 Communication and Computation Overloads

Our versions of *DCDM* and Delayed Randomized Gossip presented in Section 2.3 can suffer from communication and/or computation overloads, leading to unrealistic assumptions. Indeed, delays are assumed to be bounded, but modelling times of updates as *P.p.p.* leads to an unbounded number of updates in each interval of time of positive length. To alleviate that issue, we introduced in Section 4 the notion of *truncated Poisson point processes* of intensity p , and parameters $\tau \in \mathbb{R}_+, q \in \mathbb{N}^*$, which we recall here, and develop further in our case.

Definition 7 (Truncated *P.p.p.*). *Let \mathcal{P} a P.p.p. of intensity $p > 0$, and $\tau \in \mathbb{R}_+, q \in \mathbb{N}^*$. Let N be the Poisson point measure associated to \mathcal{P} . We define the truncated Poisson point measure \tilde{N} of intensity p and parameters τ, q as:*

$$d\tilde{N} = \sum_{t \in \mathcal{P}} \mathbb{I}_{\{N([t-\tau, t]) \leq q\}} \delta_t,$$

and we let $\tilde{\mathcal{P}}$ be the point process associated to this point measure. In other words: $\tilde{\mathcal{P}}$ is a P.p.p. of intensity restricted to be of cardinality less than q on any interval of length τ .

Computation capacity limitations: assume now that node i cannot compute more than q_i^{comp} gradients in an interval of time of length τ_i^{comp} . In order for our model not to fail, instead of considering $\mathcal{P}_i^{\text{comp}}$ a *P.p.p.* of intensity p_i^{comp} for the times at which computational updates are made, we now consider $\tilde{\mathcal{P}}_i^{\text{comp}}$ a *truncated P.p.p.* of intensity p_i^{comp} , and parameters $\tau_i^{\text{comp}}, q_i^{\text{comp}}$.

Communication capacity limitations: dealing with communication limitations is a bit trickier, since depending on the structure, limitations can happen node-wise (a node can only send a limited number of messages in an interval of time of fixed length) or edge-wise (a limited number of messages can flow between nodes $i \sim j$ in an interval of time of fixed length). Both models find to be of interest depending the type of communications considered (wired or wire-less).

1. *Edge-wise limitations:* if only q_{ij} messages can be sent between adjacent nodes i and j in any interval of time of length τ_{ij} , the times at which communication updates between nodes i and j are made are $\tilde{\mathcal{P}}_{ij}$ *truncated P.p.p.* of intensity p_{ij} and parameters τ_{ij}, q_{ij} .

2. *Node-wise limitations:* assume that every node i in the network can send at most q_i^{comm} messages in an interval of time of length τ_i^{comm} . For every node i , consider $\tilde{\mathcal{P}}_i^{\text{comm}}$ a *truncated P.p.p.* of intensity $\sum_{j \sim i} p_{ij}/2$ and parameters $\tau_i^{\text{comm}}, q_i^{\text{comm}}$. Communication-update times between nodes $i \sim j$ are then $\tilde{\mathcal{P}}_{ij}$, point process associated to the point measure \tilde{N}_{ij} built as follows:

$$d\tilde{N}_{ij} = \sum_{t \in \tilde{\mathcal{P}}_i^{\text{comm}}} \varepsilon_{i,t}(j) \delta_t + \sum_{t \in \tilde{\mathcal{P}}_j^{\text{comm}}} \varepsilon_{j,t}(i) \delta_t,$$

where $(\varepsilon_{i,t}(j))_{j \sim i}$ for $t \in \tilde{\mathcal{P}}_i^{\text{comm}}$ is a random vector with all entries equal to zero, except one which is 1, chosen at random with probability $\frac{p_{ij}}{\sum_{k \sim i} p_{ik}}$ amongst all neighbors of i . In

other words, $(\tilde{\mathcal{P}}_{ij})_{ij \in E}$ are *P.p.p.* restricted to the events $\left| \bigcap_{j \sim i} \tilde{\mathcal{P}}_{ij} \right| \leq q_i^{\text{comm}}$ for every node i .

The time-continuous coordinate gradient descent upon which our analysis is built (Theorem 3) encompasses all these scenarios. However, in order for the rates of convergence to be the same as in Theorem 1 (up to a factor 1/2), the following assumption is needed.

Assumption 2 (Communication and computation overloads). *In the presence of computation and communication overloads, Poisson intensities of the truncated P.p.p. are restricted to be upper-bounded:*

- Computation Limitations:

$$2p_i^{\text{comp}} \tau_i^{\text{comp}} \leq q_i^{\text{comp}}.$$

- Edge-wise Communication Limitations:

$$2p_{ij} \tau_{ij} \leq q_{ij}.$$

- Computation Limitations:

$$2 \sum_{j \sim i} p_{ij} \tau_i^{\text{comm}} \leq q_i^{\text{comm}}.$$

These conditions are derived from the following probability bound, where Z_μ is a discrete Poisson variable of mean μ , for any $x \geq 0$:

$$\mathbb{P}(Z_\mu \geq \mu + x) \leq e^{\frac{-x^2}{\mu+x}}.$$

and

$$\mathbb{P}(Z_\mu \geq 1) = 1 - e^{-\mu}.$$

C.2 Realistic Implementation and Limitations of our Algorithm

In order for *DCDM* (or Delayed Randomized Gossip) to be implemented (with or without *truncated P.p.p.*), some conditions are required, and are the main limitations of our work. First, as highlighted in Figure 1 and Section 2.3, communication steps at a time t are of the form $-\eta_{ij}(x_i(t - \tau_{ij}) - x_j(t - \tau_{ij}))$ (on node i): the (delayed) iterates are required to have the same delay. This comes from our dual formulation (Appendix F). Secondly, it is not clear who has the initiative of communications in our algorithm, since we treat delays and communications edge-wise. Finally, delays in the analysis are assumed to be constant or *i.i.d.* random variables. However, we propose here a realistic implementation of *DCDM*, taking into account all these limitations: nodes have the initiative of communications and delayed variables are enforced to have the same constant delays (equal to an upper-bound of these delays).

Computation Updates at node i : let $\mathcal{N}_i^{\text{comp}}(t)$ be the number of computations launched at node i between times t and $t - \tau_i^{\text{comp}}$. Initialize $\mathcal{P}_i^{\text{comp}}$ as $\{t_0\}$ where t_0 is a random time of exponential law of parameter p_i^{comp} . At any time $t \in \mathcal{P}_i^{\text{comp}}$:

1. Compute a random time T of exponential law of parameter p_i^{comp} , and add $t + T$ to $\mathcal{P}_i^{\text{comp}}$;

2. *Computation:* at time t , if $\mathcal{N}_i^{\text{comp}}(t) < q_i^{\text{comp}}$, i computes $\nabla g_{i^{\text{comp}}}^*(y_i^{\text{comp}}(t))$ and saves into memory node-variable $x_i(t)$;
3. *Update:* at time $t + \tau_i^{\text{comp}}$, the computation is finished, and the computation update (8) can be performed.

Communication Updates at node i : let $\mathcal{N}_i^{\text{comm}}(t)$ and let $\mathcal{N}_{ij}(t)$ for $j \sim i$ be respectively the number of communications started by node i between times t and $t - \tau_i^{\text{comm}}$ and the number of communications started between nodes i and j in $[t - \tau_{ij}, t]$. Initialize $\mathcal{P}_i^{\text{comm}}$ as $\{t_0\}$ where t_0 is a random time of exponential law of parameter $\sum_{j \sim i} p_{ij}/2$. At any time $t \in \mathcal{P}_i^{\text{comm}}$:

1. Compute a random time T of exponential law of parameter $\sum_{j \sim i} p_{ij}/2$, and add $t + T$ to $\mathcal{P}_i^{\text{comm}}$;
2. *Local synchronization:* at time t , if $\mathcal{N}_i^{\text{comm}}(t) < q_i^{\text{comm}}$, i chooses an adjacent node j with probability $p_{ij}/\sum_{k \sim i} p_{ik}$, and sends a *ping* (smallest message possible). We assume that sending a *ping* takes a time upper-bounded by τ_{ij}^{ping} . Upon reception of this *ping*, if $\mathcal{N}_j^{\text{comm}}(t) < q_j^{\text{comm}}$, j returns the same *ping* to i . i and j are thus synchronized at time $t + 2\tau_{ij}^{\text{ping}}$.
3. *Communication:* if $\mathcal{N}_{ij}(t + 2\tau_{ij}^{\text{ping}}) < q_{ij}$, i sends $x_i(t)$ to j and j sends $x_j(t)$ to i , while each agent keeps in memory the vector sent. For this to be possible, at a time s , node i needs to keep in memory its local values at times between $s - \min_{j \sim i} \tau_{ij}^{\text{ping}}$ and s (a small number of values usually, so that is not too restrictive).
4. *Update:* at time $t + 2\tau_{ij}^{\text{ping}} + \tau_{ij}$, update (7) can thus be performed.

Proposition 3 then justifies the fact that the induced process has the same law as the one studied and analyzed. The communication/computation scheme above emphasizes the fact that quantities $\tau_{ij}, \tau_i^{\text{comp}}$ are upper-bounds on the delays of local communication/computations. The delay is here $\tau_{ij} + 2\tau_{ij}^{\text{ping}}$, as we pay the price of local synchronization.

D Time Continuous Delayed Gradient Descent

In this section, we present simple results on basic gradient descent for the analysis of time-continuous delay, in order to introduce proof techniques in this framework.

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ a σ -strongly convex and L -smooth function. For a stepsize $\eta > 0$, consider the following time-continuous algorithm:

$$dx(t) = -\delta_{t \in \mathcal{P}} \varepsilon(t) \eta \nabla f(x(t - \tau_t)), \quad (20)$$

where $\mathcal{P} \subset \mathbb{R}^+$ denotes the times at which updates can be made, variable $\varepsilon(t) \in \{0, 1\}$ for $t \in \mathcal{P}$ denotes if this update can actually be made, and τ_t is the delay for update made at time $t \in \mathcal{P}$.

If $\tau > 0$ is a constant time such that in an interval of time of length τ , $q \in \mathbb{N}^*$ gradients can be computed, we can take $\varepsilon(t) = \mathbb{I}_{\#\mathcal{P} \cap [t - \tau, t] < q}$, and $\tau_t = \tau$ constant. We thus artificially restrict the point process of updates to be bounded by q locally on each interval of length τ . More generally, we assume that $\varepsilon(t)$ is a $\sigma(\mathcal{P} \cap [t - \tau_t, t])$ -measurable random variable, non-increasing function of $\#\mathcal{P} \cap [t - \tau_t, t]$. This assumption encompasses *truncated Poisson point processes*.

Proposition 8 (Constant Delays). *Assume delays τ_t are constant equal to $\tau > 0$. If $\eta = \frac{K}{pL}$ with:*

$$K \leq \frac{p}{1 + (1 + e)p\tau}, \quad (21)$$

then

$$\frac{1}{T} \int_0^T \mathbb{E} [f(x(t)) - f(x^*)] dt \leq f(x(0)) - f(x^*). \quad (22)$$

Moreover, for $0 < \gamma < \tau^{-1}$ such that :

$$\gamma \leq \varepsilon \frac{\sigma K}{L}, \quad (23)$$

we have, for any $T \geq 0$, where x^* minimizes f :

$$\frac{1}{T} \int_0^T e^{\gamma t} \mathbb{E} [f(x(t)) - f(x^*)] dt \leq \frac{1 + \frac{\tau}{T}}{1 - \gamma\tau} (f(x(0)) - f(x^*)). \quad (24)$$

Proposition 9 (I.I.D. delays). Assume that $(\tau_t)_{t \in \mathcal{P}}$ is a sequence of i.i.d. random variables, almost-surely upper-bounded by some constant τ_{\max} . Let $0 < \gamma < \tau_{\max}^{-1}$ and $\eta = \frac{K}{pL}$ with:

$$\begin{aligned} K &\leq \frac{p}{1 + p\mathbb{E}[\tau] + p\tau_{\max}}, \\ \gamma &< \varepsilon \frac{\sigma K}{L}. \end{aligned} \quad (25)$$

Then, for any $T > 0$, we have:

$$\frac{1}{T} \int_0^T e^{\gamma t} \mathbb{E} [f(x(t)) - f(x^*)] dt \leq \frac{1 + \frac{\tau}{T}}{1 - \gamma\tau} (f(x(0)) - f(x^*)). \quad (26)$$

Proof of Proposition 8. Let $t \geq 0$ and some $dt > 0$. We have at a $o(dt)$ precision (basic property of P.p.p., Proposition 2), using smoothness of function f :

$$\mathbb{E} [f(x(t + dt)) - f(x(t))] \leq p dt \mathbb{E} [-\varepsilon_t \eta \langle \nabla f(x(t)), \nabla f(x(t - \tau)) \rangle] + p dt \frac{L\eta^2}{2} \mathbb{E} [\varepsilon_{t-\tau} \|\nabla f(x(t - \tau))\|^2].$$

From this point, we write $\nabla f(x(t)) = \nabla f(x(t - \tau)) + (\nabla f(x(t)) - \nabla f(x(t - \tau)))$, in order to have:

$$\begin{aligned} \mathbb{E} [f(x(t + dt)) - f(x(t))] &\leq -p dt \eta \mathbb{E} [\|\varepsilon_t \nabla f(x(t - \tau))\|^2] + dt \frac{L\eta^2}{2} \mathbb{E} [\|\nabla f(x(t - \tau))\|^2] \\ &\quad + p dt \eta \mathbb{E} [\varepsilon_t \langle \nabla f(x(t - \tau)), \nabla f(x(t - \tau)) - \nabla f(x(t)) \rangle]. \end{aligned}$$

In the equation above, having classical condition on gradient descent $\eta < 1/L$ leads to, if we forget about the bold term:

$$\mathbb{E} [f(x(t + dt)) - f(x(t))] \leq -p dt \frac{\eta}{2} \mathbb{E} [\|\varepsilon_t \nabla f(x(t - \tau))\|^2].$$

From this point, we could quite easily obtain linear convergence. However, we need to treat both this bold term and the ε_t , leading to a smaller η . For the bold term:

$$\begin{aligned} &\eta \mathbb{E} [\varepsilon_t \langle \nabla f(x(t - \tau)), \nabla f(x(t - \tau)) - \nabla f(x(t)) \rangle] \\ &\leq \eta \mathbb{E} [\varepsilon_t \|\nabla f(x(t - \tau))\| \|\nabla f(x(t - \tau)) - \nabla f(x(t))\|] \end{aligned}$$

using a Cauchy-Schwarz inequality. Then, using smoothness properties of function f to measure the difference between the delayed and the actual gradient:

$$\begin{aligned} &\eta \mathbb{E} [\varepsilon_t \langle \nabla f(x(t - \tau)), \nabla f(x(t - \tau)) - \nabla f(x(t)) \rangle] \\ &\leq \eta \mathbb{E} [\varepsilon_t \|\nabla f(x(t - \tau))\| L \|x(t) - x(t - \tau)\|] \\ &= L \eta \mathbb{E} \left[\varepsilon_t \|\nabla f(x(t - \tau))\| \left\| \int_{(t-\tau)^+}^t \varepsilon_s \eta \nabla f(x(s - \tau)) N(ds) \right\| \right], \end{aligned}$$

where we used and integrated the identity $dx(s) = -N(ds) \varepsilon_s \eta \nabla f(x(s - \tau))$. Using a triangular inequality:

$$\begin{aligned} &\eta \mathbb{E} [\varepsilon_t \langle \nabla f(x(t - \tau)), \nabla f(x(t - \tau)) - \nabla f(x(t)) \rangle] \\ &\leq L \eta \mathbb{E} \left[\varepsilon_t \|\nabla f(x(t - \tau))\| \int_{(t-\tau)^+}^t \varepsilon_s \eta \|\nabla f(x(s - \tau))\| N(ds) \right] \\ &L \eta^2 \int_{(t-\tau)^+}^t \mathbb{E} [\varepsilon_t \|\nabla f(x(t - \tau))\| \varepsilon_s \|\nabla f(x(s - \tau))\| N(ds)]. \end{aligned}$$

For any $s \in (t - \tau, t)$, using $ab \leq \frac{a^2+b^2}{2}$, since ε_s is $\sigma(\mathcal{P} \cap [s - \tau, s])$ -measurable (and similarly for ε_t):

$$\begin{aligned} & \mathbb{E} [\varepsilon_t \|\nabla f(x(t - \tau))\| \varepsilon_s \|\nabla f(x(s - \tau))\| N(ds)] \\ & \leq \mathbb{E} [\varepsilon_t \|\nabla f(x(t - \tau))\|^2 N(ds)] / 2 + \mathbb{E} [\varepsilon_s \|\nabla f(x(s - \tau))\|^2 N(ds)] / 2 \\ & = \mathbb{E} [\varepsilon_t N(ds)] \mathbb{E} [\|\nabla f(x(t - \tau))\|^2] / 2 + \mathbb{E} [\varepsilon_s N(ds)] \mathbb{E} [\|\nabla f(x(s - \tau))\|^2] / 2 \\ & = \mathbb{E} [\varepsilon_t N(ds)] \mathbb{E} [\|\nabla f(x(t - \tau))\|^2] / 2 + p ds \mathbb{E} [\varepsilon_s] \mathbb{E} [\|\nabla f(x(s - \tau))\|^2] / 2. \end{aligned}$$

since $\mathbb{E} [\varepsilon_s N(ds)] = \mathbb{E} [\varepsilon_s] \mathbb{E} [N(ds)] = \mathbb{E} [\varepsilon_s] p ds$. Integrating for $t - \tau \leq s \leq t$ gives, using again the independence properties of $\varepsilon_s, \varepsilon_t$, and the monotoneous property of ε_t with respect to $N(t - \tau, t)$ (namely, $\mathbb{E} [\varepsilon_t N([t - \tau, t])] \leq \mathbb{E} [\varepsilon_t] \mathbb{E} [N([t - \tau, t])]$):

$$\begin{aligned} & \eta \mathbb{E} [\varepsilon_t \langle \nabla f(x(t - \tau)), \nabla f(x(t - \tau)) - \nabla f(x(t)) \rangle] \\ & \leq \frac{\eta^2 L}{2} \mathbb{E} [\varepsilon_t N([t - \tau, t])] \mathbb{E} [\|\nabla f(x(t - \tau))\|^2] / 2 + \frac{p \eta^2 L}{2} \int_{(t-\tau)^+}^t \mathbb{E} [\varepsilon_s \|\nabla f(x(s - \tau))\|^2] ds \\ & \leq \frac{p \tau \eta^2 L}{2} \mathbb{E} [\varepsilon_t \|\nabla f(x(t - \tau))\|^2] + \frac{p \eta^2 L}{2} \int_{(t-\tau)^+}^t \mathbb{E} [\varepsilon_s \|\nabla f(x(s - \tau))\|^2] ds, \end{aligned}$$

since $\mathbb{E} [N([t - \tau, t])] = p \tau$. All in one, we have:

$$\begin{aligned} \mathbb{E} [f(x(t + dt)) - f(x(t))] & \leq -p dt \eta \mathbb{E} [\|\varepsilon_t \nabla f(x(t - \tau))\|^2] + p dt \frac{L \eta^2}{2} \mathbb{E} [\|\nabla f(x(t - \tau))\|^2] \\ & \quad + p dt \frac{p \tau \eta^2 L}{2} \mathbb{E} [\varepsilon_t \|\nabla f(x(t - \tau))\|^2] \\ & \quad + p dt \frac{p \eta^2 L}{2} \int_{(t-\tau)^+}^t \mathbb{E} [\varepsilon_s \|\nabla f(x(s - \tau))\|^2] ds. \end{aligned}$$

Dividing by dt and making $dt \rightarrow 0$:

$$\begin{aligned} \mathbb{E} \left[\frac{df(x(t))}{dt} \right] & \leq -p \eta \mathbb{E} [\|\varepsilon_t \nabla f(x(t - \tau))\|^2] + p \frac{L \eta^2}{2} \mathbb{E} [\|\nabla f(x(t - \tau))\|^2] \\ & \quad + \frac{p^2 \tau \eta^2 L}{2} \mathbb{E} [\varepsilon_t \|\nabla f(x(t - \tau))\|^2] \\ & \quad + \frac{p^2 \eta^2 L}{2} \int_{(t-\tau)^+}^t \mathbb{E} [\varepsilon_s \|\nabla f(x(s - \tau))\|^2] ds. \end{aligned}$$

All these terms are of the form $\mathbb{E} [\varepsilon_s \|\nabla f(x(s - \tau))\|^2]$ for $s = t$, or $t - \tau \leq s \leq t$. In order for these terms to compensate themselves, we thus need to integrate the above inequality for $0 \leq t \leq T$. However, we first fix some $\gamma > 0$ in order to consider the evolution of $\mathbb{E} \left[\frac{de^{\gamma t} f(x(t))}{dt} \right]$:

$$\begin{aligned} \mathbb{E} \left[\frac{de^{\gamma t} f(x(t))}{dt} \right] & \leq \gamma \mathbb{E} [e^{\gamma t} f(x(t))] \\ & \quad - p \eta \mathbb{E} [e^{\gamma t} \|\varepsilon_t \nabla f(x(t - \tau))\|^2] + \frac{p L \eta^2}{2} \mathbb{E} [e^{\gamma t} \|\nabla f(x(t - \tau))\|^2] \\ & \quad + \frac{p^2 \tau \eta^2 L}{2} \mathbb{E} [e^{\gamma t} \varepsilon_t \|\nabla f(x(t - \tau))\|^2] \\ & \quad + \frac{p^2 \eta^2 L e^{\gamma \tau}}{2} \int_{(t-\tau)^+}^t \mathbb{E} [e^{\gamma s} \varepsilon_s \|\nabla f(x(s - \tau))\|^2] ds. \end{aligned}$$

For the integral term, remark that if we integrate it between 0 and T :

$$\int_0^T \int_{(t-\tau)^+}^t \mathbb{E} [e^{\gamma s} \varepsilon_s \|\nabla f(x(s - \tau))\|^2] ds dt \leq \tau \int_0^{t-\tau} \mathbb{E} [e^{\gamma t} \varepsilon_t \|\nabla f(x(t - \tau))\|^2] dt.$$

We now use the following very simple lemma, in order to integrate our upper-bound on $\frac{de^{\gamma t} \mathbb{E}[f(x(t))]}{dt}$:

Lemma 1. Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a differentiable function such that g and g' are locally bounded. Define $G(T) = \int_0^T g(t)dt$ for $T \geq 0$. We have for any $T \geq 0$:

$$G'(T) = g(0) + \int_0^T g'(t)dt. \quad (27)$$

Using this with $L_T^\gamma = \int_0^T e^{\gamma t} \mathbb{E}[f(x(t)) - f(x^*)]dt$:

$$\begin{aligned} \frac{dL_T^\gamma}{dT} &\leq f(x(0)) - f(x^*)\gamma L_T^\gamma \\ &\quad - p\eta \left(1 - \frac{\eta L}{2}(1 + (1 + e^{\gamma\tau})p\tau)\right) \int_0^{T-\tau} \mathbb{E}[\varepsilon_t \|\nabla f(x(t))\|^2] dt \\ &\leq \gamma L_T^\gamma - p\eta \left(1 - \frac{\eta L}{2}(1 + (1 + e^{\gamma\tau})p\tau)\right) 2\sigma e^{\gamma\tau} L_{T-\tau}^\gamma, \end{aligned}$$

where we used the fact that for all x , $F(x) - F(x^*) \leq \frac{1}{2\sigma} \|\nabla F(x)\|^2$ if F is σ -strongly convex. Finally, assume that $\gamma < 1/\tau$, and:

$$\eta \leq \frac{1}{pL} \frac{p}{1 + (1 + e)p\tau},$$

and:

$$\gamma < p\sigma\eta.$$

Under these assumptions, we have:

$$\frac{dL_T^\gamma}{dT} \leq f(x(0)) - f(x^*) + \gamma(L_T^\gamma - L_{T-\tau}^\gamma). \quad (28)$$

This time-delayed differential inequation gives us the desired result through the following Lemma:

Lemma 2. Let $h : \mathbb{R} \rightarrow \mathbb{R}^+$ a differentiable function such that:

$$\begin{aligned} \forall t \leq 0, h(t) &= 0, \\ \forall t \geq 0, h'(t) &\leq a + b(h(t) - h(t - \tau)), \end{aligned}$$

for some positive constants a, b, τ verifying $\tau b < 1$. Then:

$$\forall t \in \mathbb{R}, h(t) \leq \frac{a(t + \tau)}{1 - \tau b}. \quad (29)$$

Proof of Lemma 2. Let $\delta(t) = h(t) - h(t - \tau)$. For any $t \geq 0$, we have:

$$\begin{aligned} \delta(t) &= \int_{t-\tau}^t h'(s)ds \\ &\leq \int_{t-\tau}^t (a + b\delta(s))ds \\ &\leq \tau(a + b \sup_{s \leq t} \delta(s)). \end{aligned}$$

Let $c = \frac{\tau a}{1 - \tau b}$ (solution of $x = \tau(a + bx)$) and $t_0 = \inf\{t > 0 | \delta(t) \geq c\} \in \mathbb{R} \cup \{\infty\}$. Assume that t_0 is finite. Then, by continuity, $\delta(t_0) = c$ and:

$$c \leq \tau(a + b \sup_{s \leq t_0} \delta(s)) < \tau(a + bc) < c,$$

as for all $s < t_0$, $\delta(s) < c$. This is absurd, and thus t_0 is not finite: $\forall t > 0, \delta(t) < c$, giving us $h(t) \leq c(t + \tau)/\tau$ for all $t \geq 0$. \square

We hence have, for all $T > 0$:

$$L_T^\gamma \leq \frac{T + \tau}{1 - \gamma\tau} (f(x(0)) - f(x^*)).$$

\square

Proof with random delays. We now present the proof with *i.i.d.* random delays: $(\tau_t)_{t \in \mathcal{P}}$ is a sequence of *i.i.d.* random variables, almost surely upper-bounded by some constant τ_{\max} . The proof is based on the following lemma:

Lemma 3. *Let $T \geq 0$ and $g, h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous functions. We have:*

$$\int_0^T \mathbb{E} \left[g(\tau_t) h(t - \tau_t) \|\nabla f(x(t - \tau_t))\|^2 \right] dt \geq \int_0^{T - \tau_{\max}} \mathbb{E} [g(\tau)] h(t) \mathbb{E} \left[\|\nabla f(x(t))\|^2 \right] dt.$$

This proposition is proved as follows, where μ_τ is the law of τ :

$$\begin{aligned} \int_0^T \mathbb{E} \left[g(\tau_t) h(t - \tau_t) \|\nabla f(x(t - \tau_t))\|^2 \right] dt &= \int_{-\tau_{\max}}^T \mathbb{E} \left[h(t) \|\nabla f(x(t))\|^2 \right] \int_0^{T-t} g(s) \mu_\tau(ds) dt \\ &\geq \int_0^{T - \tau_{\max}} h(t) \mathbb{E} \left[\|\nabla f(x(t))\|^2 \right] dt \int_0^{\tau_{\max}} g(s) \mu_\tau(ds) \\ &= \mathbb{E} [g(\tau)] \int_0^{T - \tau_{\max}} h(t) \mathbb{E} \left[\|\nabla f(x(t - \tau_t))\|^2 \right] dt. \end{aligned}$$

The goal is now to find ourselves in a position where Lemma 3 can be used. Following the proof with constant delays leads to:

$$\begin{aligned} \mathbb{E} \left[\frac{de^{\gamma t} f(x(t))}{dt} \right] &\leq \gamma \mathbb{E} [e^{\gamma t} f(x(t))] \\ &\quad - p\eta \mathbb{E} \left[e^{\gamma t} \|\varepsilon_t \nabla f(x(t - \tau_t))\|^2 \right] + \frac{pL\eta^2}{2} \mathbb{E} \left[e^{\gamma t} \|\nabla f(x(t - \tau_t))\|^2 \right] \\ &\quad + \frac{p^2\eta^2 L}{2} \mathbb{E} \left[e^{\gamma t} \tau_t \varepsilon_t \|\nabla f(x(t - \tau_t))\|^2 \right] \\ &\quad + \frac{p^2\eta^2 L}{2} \mathbb{E} \left[\int_{(t-\tau_t)^+}^t e^{\gamma t} \varepsilon_s \|\nabla f(x(s - \tau_s))\|^2 ds \right]. \end{aligned}$$

Integrating that last term for $0 \leq t \leq T$ gives:

$$\begin{aligned} \int_0^T \mathbb{E} \left[\int_{(t-\tau_t)^+}^t e^{\gamma t} \varepsilon_s \|\nabla f(x(s - \tau_s))\|^2 ds dt \right] &\leq \mathbb{E} \left[\int_0^T \varepsilon_s \|\nabla f(x(s - \tau_s))\|^2 \int_s^{s+\tau'_s} e^{\gamma t} dt ds \right] \\ &\leq \int_0^T \mathbb{E} \left[\varepsilon_s e^{\gamma s} \|\nabla f(x(s - \tau_s))\|^2 \right] \mathbb{E} \left[\frac{e^{\gamma \tau} - 1}{\gamma} \right] ds, \end{aligned}$$

where τ'_s is an independent copy of τ_s (plays the role of τ_t but indexed by s). Finally, integrating $\mathbb{E} \left[\frac{de^{\gamma t} f(x(t))}{dt} \right]$ for $0 \leq t \leq T$ yields:

$$\begin{aligned} \int_0^T \mathbb{E} \left[\frac{de^{\gamma t} f(x(t))}{dt} \right] dt &\leq \gamma L_T^\gamma \\ &\quad - p\eta \int_0^T \mathbb{E} \left[g(\tau_t) h(t - \tau_t) \|\nabla f(x(t - \tau_t))\|^2 \right] dt, \end{aligned}$$

where $h(t) = e^{\gamma t}$ and $g(\tau) = 1 - \frac{L\eta e^{\gamma \tau}}{2} (1 + p\tau + ep\mathbb{E}[\tau])$. Assume then that $\gamma < 1/\tau_{\max}$ and:

$$\eta \leq \frac{2}{Le} \frac{1}{1 + p\tau_{\max} + ep\mathbb{E}[\tau]}.$$

Then $g \geq 0$, and we can use Lemma 3:

$$\begin{aligned} \int_0^T \mathbb{E} \left[\frac{de^{\gamma t} f(x(t))}{dt} \right] dt &\leq \gamma L_T^\gamma \\ &\quad - p\eta \int_0^{T - \tau_{\max}} \mathbb{E} [g(\tau)] e^{\gamma t} \mathbb{E} \left[\|\nabla f(x(t))\|^2 \right] dt. \end{aligned}$$

We then proceed as in Proposition 8, and obtain:

$$\gamma < \sigma p \eta \mathbb{E} [g(\tau)] \leq \sigma p \eta,$$

concluding the proof. \square

E Time-Continuous Analysis of a Delayed Coordinate Gradient Descent

Let F be a σ -strongly convex and L -smooth function on \mathbb{R}^d . For $i = 1, \dots, n$, let E_i be a subspace of \mathbb{R}^d , and assume that:

$$\mathbb{R}^d = \bigoplus_{i=1}^n E_i. \quad (30)$$

For $x \in \mathbb{R}^d$, denote x_i its orthogonal projection on E_i and $\nabla_i F = (\nabla F)_i$, and assume that the subspaces E_1, \dots, E_n are orthogonal. For $i, j \in [n]$, we say that i and j are adjacent and we write $i \sim j$ iff $\nabla_i \nabla_j F$ is not constant equal to 0. This forms a graph structure on the coordinates, that we hope as sparse as possible.

Algorithm Let \mathcal{P}_i be independent *P.p.p.* of intensity p_i denoting the times at which an update can be performed on subspace E_i . For $t \in \mathcal{P}_i$ let $\varepsilon_i(t) \in \{0, 1\}$ denote if the update is performed. Consider the following time-continuous process $X(t)$, where $X_i(t)$ evolves according to:

$$dX_i(t) = -\delta_{t \in \mathcal{P}_i} \varepsilon_i(t) \eta_{t,i} \nabla_i F(X(t - \tau_{t,i})),$$

Assumption 3. F is σ -strongly convex, and L_i -smooth on E_i for $i \in [n]$. Furthermore, there exists non-negative real numbers $M_{i,j}$ (and $M_{j,i}$) for $i \sim j$ such that for all $i = 1, \dots, n$ and $x, y \in \mathbb{R}^d$, we have:

$$\|\nabla_i F(x) - \nabla_i F(y)\| \leq \sum_{j \sim i} M_{i,j} \|x_j - y_j\|. \quad (31)$$

If F is L_i smooth on E_i (true under Assumption 3), then $M_{i,j} = L_j$ for any $i \sim j$ works. If $\nabla_i F$ is M_i -Lipschitz, then $M_{i,j} = L_i$ works. More generally, this assumption in Equation (31) allows for more freedom, and is particularly well suited for our analysis. We will apply this in Appendix F to prove Theorem 1, where M_{ij} will have the form $\sqrt{L_i L_j}$.

Assumption 4. (i) Continuous-time delays $\tau_{t,i} = \tau_i$ are time-independent.

(ii) For $t \in \mathcal{P}_i$, random variable $\varepsilon_i(t)$ is $\sigma(\mathcal{P}_j \cap [t - \tau_i, t], j \in [n])$ -measurable, there exists a constant $\varepsilon_i > 0$ such that:

$$\mathbb{E}[\varepsilon_i(t)] \geq \varepsilon_i,$$

and we assume that $\varepsilon_i(t)$ is monotonous with each $\mathcal{P}_j \cap [t - \tau_i, t]$ in the sense that: $\mathbb{E}[\varepsilon_i(t) \# \mathcal{P}_j \cap [t - \tau_i, t]] \leq \mathbb{E}[\varepsilon_i(t)] \mathbb{E}[\# \mathcal{P}_j \cap [t - \tau_i, t]]$.

Theorem 3. Assume that Assumptions 3 and 4 are verified. Then, for stepsizes $\eta_i = \frac{K_i}{p_i L_i}$ such that for all $i = 1, \dots, n$:

$$K_i \leq \frac{p_i}{1 + \sum_{j \sim i} p_j \left(\frac{\tau_i M_{i,j} + e \tau_j M_{j,i}}{\sqrt{L_i L_j}} \right)} \quad (32)$$

and for $0 \leq \gamma < 1/\tau_{\max}$ such that:

$$\gamma < \sigma \min_i \frac{\varepsilon_i K_i}{L_i}, \quad (33)$$

then:

$$\frac{1}{T} \int_0^T e^{\gamma t} \mathbb{E}[F(X(t)) - F(x^*)] dt \leq \frac{1 + \frac{\gamma \tau_{\max}}{T}}{1 - \gamma \tau_{\max}} (F(X(0)) - F(x^*)). \quad (34)$$

Remark 1. Random variables $\varepsilon_i(t)$ and constants ε_i let us take into account many scenarios in terms of communication and computational constraints. We here describe a classical scenario. For all $i = 1, \dots, n$, coordinate i can only have up to $q_i \in \mathbb{N}^*$ computations in parallel. This leads to: $\varepsilon_i(t) = 1$ if $\sum_{s \in \mathcal{P}_i \cap [t, t + \tau_i)} \varepsilon_i(s) < q_i$ and $\mathbb{E}[\varepsilon_i(t)] \geq 1 - e^{-(q_i^{\text{comp}} - p_i \tau_i)^2 / q_i}$ using that for Z_μ a discrete Poisson variable of mean μ , we have:

$$\mathbb{P}(Z_\mu \geq \mu + x) \leq e^{-\frac{x^2}{x + \mu}}. \quad (35)$$

For $q_i = 1$, we have $\mathbb{E}[\varepsilon_i(t)] \geq 1 - e^{-p_i \tau_i}$. In the decentralized optimization problem, subspaces E_k correspond to communication edges $e \in E$ and to computational ones (i, i^{comp}) : communication and computation overloads can thus be handled.

Proof Theorem 3. Let $t \geq 0$ and $dt > 0$. We first write the infinitesimal increment between times t and $t + dt$, with a $o(dt)$ precision that we omit to write in the following equations:

$$\begin{aligned}\mathbb{E}[F(X(t + dt)) - F(X(t)) | \mathcal{F}_t] &= dt \sum_{i=1}^n p_i \left(F \left(X(t) - \varepsilon_i(t) \frac{K_i}{p_i L_i} U_i \nabla_i F(X(t - \tau_i)) \right) - F(X(t)) \right) \\ &\leq dt \sum_{i=1}^n p_i \left(-\frac{K_i}{p_i L_i} \langle \varepsilon_i(t) \nabla_i F(X(t - \tau_i)), \nabla_i F(X(t)) \rangle \right. \\ &\quad \left. + \frac{L_i}{2} \left\| \varepsilon_i(t) \frac{K_i}{p_i L_i} \nabla_i F(X(t - \tau_i)) \right\|^2 \right).\end{aligned}$$

First,

$$\begin{aligned}-\varepsilon_i(t) \frac{K_i}{p_i L_i} \langle \nabla_i F(X(t - \tau_i)), \nabla_i F(X(t)) \rangle &= -\frac{K_i}{p_i L_i} \|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\|^2 \\ &\quad - \frac{K_i}{p_i L_i} \langle \varepsilon_i(t) \nabla_i F(X(t - \tau_i)), \nabla_i F(X(t)) - \nabla_i F(X(t - \tau_i)) \rangle,\end{aligned}$$

and we bound this last term as follows:

$$\begin{aligned}& -\frac{K_i}{p_i L_i} \langle \varepsilon_i(t) \nabla_i F(X(t - \tau_i)), \nabla_i F(X(t)) - \nabla_i F(X(t - \tau_i)) \rangle \\ & \leq \frac{K_i}{p_i L_i} \|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\| \|\nabla_i F(X(t)) - \nabla_i F(X(t - \tau_i))\| \\ & \leq \frac{K_i}{p_i L_i} \|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\| \sum_{j \sim i} M_{i,j} \|X_j(t) - X_j(t - \tau_i)\|,\end{aligned}$$

where we used local gradient-Lipschitz properties and gradient sparsity. As:

$$\|X_j(t) - X_j(t - \tau_i)\| = \left\| \int_{(t-\tau_i)^+}^t \varepsilon_j(s) \frac{K_j}{p_j L_j} \nabla_j F(X(s - \tau_j)) N_j(ds) \right\|,$$

thanks to a triangular inequality we have:

$$\begin{aligned}& \mathbb{E} \left[-\frac{K_i}{p_i L_i} \langle \varepsilon_i(t) \nabla_i F(X(t - \tau_i)), \nabla_i F(X(t)) - \nabla_i F(X(t - \tau_i)) \rangle \right] \\ & \leq \mathbb{E} \left[\sum_{j \sim i} \int_{(t-\tau_i)^+}^t M_{i,j} \frac{K_i}{L_i p_i} \|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\| \left\| \varepsilon_j(s) \frac{K_j}{p_j L_j} \nabla_j F(X(s - \tau_j)) \right\| N_j(ds) \right] \\ & \leq \mathbb{E} \left[\sum_{j \sim i} \int_{(t-\tau_i)^+}^t \frac{1}{2} \left(\frac{K_i^2 t M_{i,j}}{p_i^2 L_i \sqrt{L_i L_j}} \|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\|^2 + \frac{K_j^2 t M_{i,j}}{p_j^2 L_j \sqrt{L_i L_j}} \|\varepsilon_j(s) \nabla_j F(X(s - \tau_j))\|^2 \right) N_j(ds) \right] \\ & \leq \sum_{j \sim i} \frac{1}{2} \left(\frac{K_i^2 t M_{i,j}}{p_i^2 \sqrt{L_i L_j}} \mathbb{E}[N_j(t - \tau_i, t) \varepsilon_i(t)] \mathbb{E}[\|\nabla_i F(X(t - \tau_i))\|^2] + \int_{(t-\tau_i)^+}^t \frac{K_j^2 M_{i,j}}{p_j L_j \sqrt{L_i L_j}} \mathbb{E}[\|\varepsilon_j(s) \nabla_j F(X(s - \tau_j))\|^2] ds \right)\end{aligned}$$

where we used the fact that $\varepsilon_j(s)$ is independent from $X(s - \tau_j)$ and $N_j(ds)$ in the integral. Furthermore, $\mathbb{E}[N_j(t - \tau_i, t) \varepsilon_i(t)] \leq \mathbb{E}[N_j(t - \tau_i, t)] \mathbb{E}[\varepsilon_i(t)] = p_j \tau_i \mathbb{E}[\varepsilon_i(t)]$. We then recover:

$$\begin{aligned}& \mathbb{E} \left[-\frac{K_i}{p_i L_i} \langle \varepsilon_i(t) \nabla_i F(X(t - \tau_i)), \nabla_i F(X(t)) - \nabla_i F(X(t - \tau_i)) \rangle \right] \\ & \leq \sum_{j \sim i} \frac{1}{2} \left(\frac{K_i^2 M_{i,j} p_j \tau_i}{p_i^2 L_i \sqrt{L_i L_j}} \mathbb{E}[\varepsilon_i(t) \|\nabla_i F(X(t - \tau_i))\|^2] + \int_{(t-\tau_i)^+}^t \frac{K_j^2 M_{i,j}}{p_j L_j \sqrt{L_i L_j}} \mathbb{E}[\|\varepsilon_j(s) \nabla_j F(X(s - \tau_j))\|^2] ds \right),\end{aligned}$$

All in one, we have, with a $o(dt)$ precision:

$$\begin{aligned}\mathbb{E}[F(X(t+dt)) - F(X(t)) | \mathcal{F}_t] &\leq -dt \sum_{i=1}^n \frac{K_i}{L_i} \mathbb{E}[\|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\|^2] \\ &\quad + dt \sum_{i=1}^n \frac{K_i^2}{2p_i L_i} \left(1 + \tau_i \sum_{j \sim i} p_j \frac{M_{i,j}}{\sqrt{L_i L_j}}\right) \mathbb{E}[\|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\|^2] \\ &\quad + dt \sum_{i=1}^n p_i \sum_{j \sim i} \int_{(t-\tau_i)^+}^t \frac{K_j^2}{2p_j L_j} \frac{M_{i,j}}{\sqrt{L_i L_j}} \mathbb{E}[\|\varepsilon_j(s) \nabla_j F(X(s - \tau_j))\|^2] ds.\end{aligned}$$

Dividing by dt and making $dt \rightarrow 0$:

$$\begin{aligned}\frac{d\mathbb{E}[F(X(t))]}{dt} &\leq - \sum_{i=1}^n \frac{K_i}{L_i} \mathbb{E}[\|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\|^2] \\ &\quad + \sum_{i=1}^n \frac{K_i^2}{2p_i L_i} \left(1 + \tau_i \sum_{j \sim i} p_j \frac{M_{i,j}}{\sqrt{L_i L_j}}\right) \mathbb{E}[\|\varepsilon_i(t) \nabla_i F(X(t - \tau_i))\|^2] \\ &\quad + \sum_{i=1}^n p_i \sum_{j \sim i} \int_{(t-\tau_i)^+}^t \frac{K_j^2}{2p_j L_j} \frac{M_{i,j}}{\sqrt{L_i L_j}} \mathbb{E}[\|\varepsilon_j(s) \nabla_j F(X(s - \tau_j))\|^2] ds.\end{aligned}$$

Notice that integrating that last term (multiplied by $e^{\gamma t}$ for $0 \leq t \leq T$ yields:

$$\begin{aligned}&\int_0^T \sum_{i=1}^n e^{\gamma t} p_i \sum_{j \sim i} \int_{(t-\tau_i)^+}^t \frac{K_j^2}{2p_j L_j} \frac{M_{i,j}}{\sqrt{L_i L_j}} \mathbb{E}[\|\varepsilon_j(s) \nabla_j F(X(s - \tau_j))\|^2] ds dt \\ &\leq \sum_{i=1}^n \left(\sum_{j \sim i} p_j e^{\gamma(\tau_i + \tau_j)} \frac{K_i^2}{2p_i L_i} \frac{M_{j,i} \tau_j}{\sqrt{L_i L_j}} \right) \int_0^{T-\tau_i} \mathbb{E}[e^{\gamma t} \|\varepsilon_i(t) \nabla_i F(X(t))\|^2] dt,\end{aligned}$$

by interchanging time variables s, t and coordinate indices i, j . Let us now denote $L_T^\gamma = \int_0^T e^{\gamma t} \mathbb{E}[F(X(t)) - F(X(x^*))] dt$ and $\ell_t^\gamma = e^{\gamma t} \mathbb{E}[F(X(t)) - F(X(x^*))]$ for some $\gamma > 0$, such that $L_T^\gamma = \int_0^T \ell_t^\gamma dt$. Since $\frac{d\ell_t^\gamma}{dt} = e^{\gamma t} \left(\frac{d\mathbb{E}[F(X(t))]}{dt} + \gamma \mathbb{E}[F(X(t))] \right)$, we have:

$$\frac{dL_T^\gamma}{dt} = F(0) - F(x^*) + \int_0^T e^{\gamma t} \frac{d\mathbb{E}[F(X(t))]}{dt} dt + \gamma L_T^\gamma \quad (36)$$

We now use Lemma 1, in order to integrate our upper-bound on $\frac{d\mathbb{E}[F(X(t))]}{dt}$:

$$\begin{aligned}\frac{dL_T^\gamma}{dt} &\leq (F(X(0)) - F(x^*)) \\ &\quad - \sum_{i=1}^n \int_0^{T-\tau_i} \frac{K_i}{L_i} \left(1 - \frac{K_i}{2p_i} \left(1 + \sum_{j \sim i} p_j \left(\frac{M_{i,j}}{\sqrt{L_i L_j}} \tau_i + \frac{M_{j,i}}{\sqrt{L_i L_j}} e^{\gamma \tau_j} \tau_j\right)\right)\right) e^{\gamma(t+\tau_i)} \mathbb{E}[\|\varepsilon_i(t + \tau_i) \nabla_i F(X(t))\|^2] dt \\ &\quad + \gamma \int_0^T e^{\gamma t} \mathbb{E}[\|\nabla F(X(t))\|^2] dt.\end{aligned}$$

Thus, if for all $i = 1, \dots, n$:

$$K_i \leq \frac{p_i}{1 + \sum_{j \sim i} p_j \left(\frac{M_{i,j}}{\sqrt{L_i L_j}} \tau_i + \frac{M_{j,i}}{\sqrt{L_i L_j}} e^{\gamma \tau_j} \tau_j\right)}, \quad (37)$$

we have $\frac{dL_T^\gamma}{dt} \leq 0$ for $\gamma = 0$. For $\gamma > 0$, and with such an assumption on each K_i :

$$\begin{aligned} \frac{dL_T^\gamma}{dt} &\leq (F(X(0)) - F(x^*)) \\ &\quad - \sum_{i=1}^n \int_0^{T-\tau_i} \frac{K_i}{2L_i} e^{\gamma(t+\tau_i)} \mathbb{E} \left[\|\varepsilon_i(t+\tau_i) \nabla_i F(X(t))\|^2 \right] dt \\ &\quad + \gamma \int_0^T e^{\gamma t} \mathbb{E} \left[\|\nabla F(X(t))\|^2 \right] dt. \end{aligned}$$

Using that $\mathbb{E} \left[\|\varepsilon_i(t+\tau_i) \nabla_i F(X(t))\|^2 \right] = \mathbb{E} [\varepsilon_i(t+\tau_i)] \mathbb{E} \left[\|\nabla_i F(X(t))\|^2 \right] \geq \varepsilon_i \mathbb{E} \left[\|\nabla_i F(X(t))\|^2 \right]$ ($\varepsilon_i(t+\tau_i)$ is $\sigma(\mathcal{P}([t, t+\tau_i]))$ -measurable):

$$\begin{aligned} \frac{dL_T^\gamma}{dt} &\leq (F(X(0)) - F(x^*)) - \sum_{i=1}^n \int_{-\tau_i}^{T-\tau_i} \frac{\varepsilon_i K_i}{2L_i} e^{\gamma(t+\tau_i)} \mathbb{E} \left[\|\nabla_i F(X(t))\|^2 \right] dt \\ &\quad + \gamma \int_0^T e^{\gamma t} \mathbb{E} [F(X(t)) - F(X(x^*))] dt \\ &\leq (F(X(0)) - F(x^*)) - \sum_{i=1}^n \int_{-\tau_i}^{T-\tau_i} \frac{\varepsilon_i K_i}{2L_i} e^{\gamma(t+\tau_i)} \mathbb{E} \left[\|\nabla_i F(X(t))\|^2 \right] dt \\ &\quad + \gamma \int_0^{T-\tau_{\max}} e^{\gamma t} \frac{1}{2\sigma} \mathbb{E} \left[\|\nabla F(X(t))\|^2 \right] dt \\ &\quad + \gamma (L_T^\gamma - L_{T-\tau_{\max}}^\gamma), \end{aligned}$$

where we used the fact that for all x , $F(x) - F(x^*) \leq \frac{1}{2\sigma} \|\nabla F(x)\|^2$ if F is σ -strongly convex. Finally, if for all $i = 1, \dots, n$, we have:

$$\frac{\varepsilon_i K_i e^{\gamma \tau_i}}{L_i} > \frac{\gamma}{\sigma}, \quad (38)$$

then:

$$\frac{dL_T^\gamma}{dT} \leq F(X(0)) - F(x^*) + \gamma (L_T^\gamma - L_{T-\tau_{\max}}^\gamma). \quad (39)$$

We then use Lemma 2 in order to obtain a sublinear growth of L_T^γ : if $\gamma \tau_{\max} < 1$, we have:

$$\forall T \geq 0, L_T^\gamma \leq \frac{(T + \tau_{\max})}{1 - \gamma \tau_{\max}} (F(X(0)) - F(x^*)). \quad (40)$$

Finally, since $1 + \gamma \tau_i \leq e^{\gamma \tau_i}$ and $K_i \tau_i < 1$, a sufficient condition for $\frac{\varepsilon_i K_i e^{\gamma \tau_i}}{L_i} > \frac{\gamma}{\sigma}$ to be true, is:

$$\gamma \leq \frac{\sigma \varepsilon_i K_i}{L_i}, \quad (41)$$

concluding the proof. \square

F Proof of Theorem 1

Proof of Theorem 1 goes as follows: we first introduce our dual formulation of the problem, where edges (both non-virtual and virtual ones) become variables. We then recover the updates described in Section 2.3 as coordinate gradient steps on the dual problem. Finally, we compute the quantities necessary for Theorem 3 to be used: strong convexity parameter, local smoothness, etc.

N.B: In this section, we treat a non-uniform splitting (and thus more general) of the strong convexity of local objectives. This writes: $g_i(x) = \frac{\sigma_i}{4} \|x\|^2$ and $g_{i\text{comp}}(x) = f_i(x) - \frac{\sigma_i}{4} \|x\|^2$, instead of $g_i(x) = \frac{\sigma}{4} \|x\|^2$ and $g_{i\text{comp}}(x) = f_i(x) - \frac{\sigma}{4} \|x\|^2$.

F.1 Dual Formulation of Problem (6)

$\mathbb{R}^{V^+ \times n}$ corresponds to the space of vectors $(x_i)_{i \in V^+}$ where $x_i \in \mathbb{R}^d$ for any $i \in V^+$. Similarly, $\mathbb{R}^{E^+ \times n}$ corresponds to the space of vectors $(x_{ij})_{ij \in E^+}$ where $x_{ij} \in \mathbb{R}^d$ for any $ij \in E^+$. We denote by $(e_i)_{i \in V^+}$ the canonical basis of \mathbb{R}^{V^+} and by $(e_{ij})_{ij \in E^+}$ the canonical basis of \mathbb{R}^{E^+} .

A standard way to deal with the constraint $x_i = x_j$ for all $i \sim j$, is to use a dual formulation, by introducing a dual variable λ indexed by the edges (both virtual and non-virtual ones). We first introduce a matrix $A \in \mathbb{R}^{V^+ \times E^+}$ such that $\text{Ker}(A^\top) = \text{Vect}(\mathbb{I})$ where \mathbb{I} is the constant vector $(1, \dots, 1)^\top$ of dimension $2n$. A is chosen such that:

$$\forall (ij) \in E^+, Ae_{ij} = \mu_{ij}(e_i - e_j). \quad (42)$$

for some non-null constants μ_{ij} . We define $\mu_{ij} = -\mu_{ji}$ for this writing to be consistent. This matrix A is a square root of the laplacian of the augmented graph G^+ weighted by $\nu_{ij} = \mu_{ij}^2$. The constraint $x_i = x_j$ if $i \sim j$ can then be written $A^\top x = 0$. The dual problem reads as follows:

$$\min_{x \in \mathbb{R}^{n \times d}, A^\top x = 0} \sum_{i \in V^+} g_i(x_i) = \min_{x \in \mathbb{R}^{n \times d}} \max_{\lambda \in \mathbb{R}^{E^+}} \sum_{i \in V^+} g_i(x_i) - \langle A^\top x, \lambda \rangle.$$

Let $F_A^*(\lambda) := F^*(A\lambda)$ for $\lambda \in \mathbb{R}^{E^+ \times d}$ where F^* is the Fenchel conjugate of F . The dual problem reads

$$\min_{x \in \mathbb{R}^{V^+ \times d}, x_i = x_j, (ij) \in E^+} F(x) = \max_{\lambda \in \mathbb{R}^{E^+ \times d}} -F_A^*(\lambda).$$

Thus $F_A^*(\lambda) = \sum_{i \in V^+} g_i^*((A\lambda)_i)$ is to be minimized over the dual variable $\lambda \in \mathbb{R}^{E^+ \times d}$.

We now make a parallel between pairwise operations between adjacent nodes in the (augmented) network and coordinate gradient steps on F_A^* . As $F_A^*(\lambda) = \max_{x \in \mathbb{R}^{V^+ \times d}} -F(x) + \langle A\lambda, x \rangle$, to any $\lambda \in \mathbb{R}^{E^+ \times d}$ a primal variable $x \in \mathbb{R}^{V^+ \times d}$ is uniquely associated through the formula $\nabla F(x) = A\lambda$. The partial derivative of F_A^* with respect to coordinate $(ij) \in E^+$ of λ reads :

$$\nabla_{ij} F_A^*(\lambda) = (Ae_{ij})^\top \nabla F^*(A\lambda) = \mu_{ij}(\nabla g_i^*((A\lambda)_i) - \nabla g_j^*((A\lambda)_j)).$$

Consider then the following step of coordinate gradient descent for F_A^* on coordinate $(ij) \in E^+$ of λ , performed when edge (ij) is activated at iteration k (corresponding to time t_k), and where $U_{ij} = e_{ij}e_{ij}^\top$:

$$\lambda_{t_{k+1}} = \lambda_{t_k} - \frac{1}{(\sigma_i^{-1} + \sigma_j^{-1})\mu_{ij}^2} U_{ij} \nabla_{ij} F_A^*(\lambda_{t_k}). \quad (43)$$

Denoting $v_k = A\lambda_{t_k} \in \mathbb{R}^{V^+ \times d}$, we obtain the following formula for updating coordinates i, j of v when ij activated (when there is no delays):

$$v_{k+1,i} = v_{k,i} - \frac{\nabla g_i^*(v_{k,i}) - \nabla g_j^*(v_{k,j})}{\sigma_i^{-1} + \sigma_j^{-1}}, \quad (44)$$

$$v_{k+1,j} = v_{k,j} + \frac{\nabla g_i^*(v_{k,i}) - \nabla g_j^*(v_{k,j})}{\sigma_i^{-1} + \sigma_j^{-1}}. \quad (45)$$

Such updates can be performed locally at nodes i and j after communication between the two nodes. We refer in the sequel to this scheme as the Coordinate Descent Method (CDM). While $\lambda \in \mathbb{R}^{E^+ \times d}$ is a dual variable defined on the edges, $v \in \mathbb{R}^{n \times d}$ is also a dual variable, but defined on the nodes. The *primal surrogate* of v is defined as $x = \nabla F^*(v)$ i.e. $x_i = \nabla f_i^*(v_i)$ at node i . It can hence be computed with local updates on v ((44) and (45)).

F.2 Recovering our Decentralized Updates

We here translate the updates of our algorithm described in Section 2.3 into edge-dual variable updates on $\lambda \in \mathbb{R}^{E^+ \times d}$.

Note that for $i \in V$ a non-virtual node and $x \in \mathbb{R}^d$, we have:

$$\nabla g_i(x) = \frac{\sigma_i}{2} x \text{ and } \nabla g_i^*(x) = \frac{2}{\sigma_i} x,$$

while for a virtual node i^{comp} :

$$\nabla g_{i^{\text{comp}}}(x) = \nabla f_i(x) - \frac{\sigma_i}{2}x \text{ and } \nabla g_{i^{\text{comp}}}^*(x) = \nabla f_i^*(x) - \frac{2}{\sigma_i}x.$$

For $(ij) \in E^+$ a communication or computation edge and $\lambda \in \mathbb{R}^{E^+ \times d}$, we have where $x_i = g_i^*((A\lambda)_i)$ and $x_j = g_j^*((A\lambda)_j)$ are the node-primal surrogates at i and j of λ :

$$\nabla_{ij} F_A^*(\lambda) = \mu_{ij}(x_i - x_j).$$

Hence, in light of Equations (44) and (45), updates:

$$\begin{aligned} \nabla g_i(x_i) &\leftarrow \nabla g_i(x_i) - \eta_{ij}(x_i - x_j) \\ \nabla g_j(x_j) &\leftarrow \nabla g_j(x_j) + \eta_{ij}(x_i - x_j) \end{aligned}$$

are equivalent to, on λ edge-dual variable:

$$\lambda \leftarrow \lambda - \eta_{ij} \mu_{ij}^2 \nabla_{ij} F_A^*(\lambda).$$

Setting $\eta_{ij} = \frac{K_{ij}}{p_{ij}(\sigma_i^{-1} + \sigma_j^{-1})}$ for a communication edge $(ij) \in E$ thus yields communication update (7), while setting $\eta_{i,i^{\text{comp}}} = \frac{K_{i,i^{\text{comp}}}}{2p_{i,i^{\text{comp}}}}$ yields computational update (8). We study function F_A^* in the next subsection in order to apply delayed coordinate gradient steps, thus making Theorem 1 appear as a consequence of Theorem 3.

F.3 Some properties of F_A^*

Lemma 4. For any $(ij) \in E^+$, F_A^* is $L_{ij} := 4\mu_{ij}^2(\sigma_i^{-1} + \sigma_j^{-1})$ -smooth on E_{ij} the subspace of coordinates (ij) .

Proof. Let $h_{ij} \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^{E^+ \times d}$. Using the $2\sigma_i^{-1}$ and $2\sigma_j^{-1}$ -smoothness of g_i^* and g_j^* :

$$\begin{aligned} F_A^*(\lambda + e_{ij}h_{ij}^\top) - F_A^*(\lambda) &= g_i^*((A(\lambda + e_{ij}h_{ij}^\top))_i) - g_i^*((A\lambda)_i) \\ &\quad + g_j^*((A(\lambda + e_{ij}h_{ij}^\top))_j) - g_j^*((A\lambda)_j) \\ &\leq \langle \nabla_{ij} F_A^*(\lambda), e_{ij}h_{ij}^\top \rangle \\ &\quad + \frac{2\sigma_i^{-1}}{2} \|(Ae_{ij}h_{ij}^\top)_i\|^2 + \frac{2\sigma_j^{-1}}{2} \|(Ae_{ij}h_{ij}^\top)_j\|^2, \end{aligned}$$

concluding the proof, as $\|(Ae_{ij}h_{ij}^\top)_i\|^2 = 2\mu_{ij}^2 \|h_{ij}\|^2$. □

Lemma 5. For any $(ij) \in E$, any $\lambda, \lambda' \in \mathbb{R}^{E^+ \times d}$:

$$\|\nabla_{ij} F_A^*(\lambda) - \nabla_{ij} F_A^*(\lambda')\| \leq \sum_{(kl) \sim (ij)} M_{(ij),(kl)} \|\lambda_{kl} - \lambda'_{kl}\|, \quad (46)$$

where $M_{(ij),(kl)} = \sqrt{L_{ij}L_{kl}}$.

Proof. As, $\nabla_{ij} F_A^*(\lambda) = (Ae_{ij})^\top ((\nabla g_i^*((A\lambda)_i) - \nabla g_j^*((A\lambda)_j)))$, we have:

$$\begin{aligned}
\|\nabla_{ij} F_A^*(\lambda) - \nabla_{ij} F_A^*(\lambda')\| &= \|(Ae_{ij})^\top ((\nabla g_i^*((A\lambda)_i) - \nabla g_i^*((A\lambda')_i) - \nabla g_j^*((A\lambda)_j) + \nabla g_j^*((A\lambda')_j)))\| \\
&\leq \|Ae_{ij}\| (\|\nabla g_i^*((A\lambda)_i) - \nabla g_i^*((A\lambda')_i)\| + \|\nabla g_j^*((A\lambda)_j) - \nabla g_j^*((A\lambda')_j)\|) \\
&\leq \sqrt{2} |\mu_{ij}| (2\sigma_i^{-1} \|(A\lambda)_i - (A\lambda')_i\| + 2\sigma_j^{-1} \|(A\lambda)_j - (A\lambda')_j\|) \\
&= \sqrt{2} |\mu_{ij}| \left(2\sigma_i^{-1} \left\| \sum_{k \sim i} \mu_{ik} (\lambda_{ik} - \lambda'_{ik}) \right\| + 2\sigma_j^{-1} \left\| \sum_{l \sim j} \mu_{jl} (\lambda_{jl} - \lambda'_{jl}) \right\| \right) \\
&\leq \sqrt{2} |\mu_{ij}| \left(2\sigma_i^{-1} \sum_{k \sim i} |\mu_{ik}| \|\lambda_{ik} - \lambda'_{ik}\| + 2\sigma_j^{-1} \sum_{l \sim j} |\mu_{jl}| \|\lambda_{jl} - \lambda'_{jl}\| \right) \\
&\leq \sqrt{2} |\mu_{ij}| \left(2\sqrt{\sigma_i^{-1} + \sigma_j^{-1}} \sqrt{\sigma_i^{-1} + \sigma_k^{-1}} \sum_{k \sim i} |\mu_{ik}| \|\lambda_{ik} - \lambda'_{ik}\| \right. \\
&\quad \left. + 2\sqrt{\sigma_i^{-1} + \sigma_j^{-1}} \sqrt{\sigma_l^{-1} + \sigma_j^{-1}} \sum_{l \sim j} |\mu_{jl}| \|\lambda_{jl} - \lambda'_{jl}\| \right) \\
&\leq \sum_{(kl) \sim (ij)} \sqrt{L_{ij} L_{kl}} \|\lambda_{kl} - \lambda'_{kl}\|,
\end{aligned}$$

where L_{ij}, L_{kl} as in Lemma 4. □

Lemma 6. σ_A the strong convexity parameter of F_A^* on the orthogonal of $\text{Ker}(A)$ is lower bounded by $\lambda_{\min}^+(A^T D A)/2$, where $\lambda_{\min}^+(M)$ is the smallest non null eigenvalue of M , $D \in \mathbb{R}^{V^+ \times V^+}$ is a diagonal matrix with entries $D_{i,i} = 1/\sigma_{\max}$ for $i \in V$, and $D_{i^{\text{comp}}, i^{\text{comp}}} = 1/L_i$ for virtual nodes.

Proof. Let $\lambda, \lambda' \in \mathbb{R}^{E \times d}$. For $i \in V$, by $(2\sigma_i)^{-1}$ and thus $(2\sigma_{\max})^{-1}$ -strong convexity of g_i^* :

$$g_i^*((A\lambda)_i) - g_i^*((A\lambda')_i) \geq \langle \nabla g_i^*((A\lambda')_i), (A(\lambda - \lambda'))_i \rangle + \frac{1}{4\sigma_{\max}} \|(A(\lambda - \lambda'))_i\|^2.$$

Then, for a computational node i^{comp} , by $(2L_i)^{-1}$ and thus $(2L_{\max})^{-1}$ strong convexity of $g_{i^{\text{comp}}}^*$:

$$\begin{aligned}
g_{i^{\text{comp}}}^*((A\lambda)_{i^{\text{comp}}}) - g_{i^{\text{comp}}}^*((A\lambda')_{i^{\text{comp}}}) &\geq \langle \nabla g_{i^{\text{comp}}}^*((A\lambda')_{i^{\text{comp}}}), (A(\lambda - \lambda'))_{i^{\text{comp}}} \rangle \\
&\quad + \frac{1}{4L_i} \|(A(\lambda - \lambda'))_{i^{\text{comp}}}\|^2.
\end{aligned}$$

Summing over all $i \in V^+$ and using $\nabla F_A^*(\lambda') = A^\top (\nabla_i f_i^*((A\lambda')_i))_{i \in V^+}$ leads to:

$$\begin{aligned}
F_A^*(\lambda) - F_A^*(\lambda') &\geq \langle \nabla F_A^*(\lambda'), \lambda - \lambda' \rangle + \frac{1}{4} \|\sqrt{D} A(\lambda' - \lambda)\|^2 \\
&\geq \langle \nabla F_A^*(\lambda'), \lambda - \lambda' \rangle + \frac{\lambda_{\min}^+(A^T D A)}{4} \|\lambda - \lambda'\|^{*2}.
\end{aligned}$$

where $\|\cdot\|^*$ is the euclidean norm on the orthogonal of $\text{Ker}(A)$. □

Lemma 7. For any $u \in \mathbb{R}^{V^+}$, we have:

$$u^\top \sqrt{D} A A^\top \sqrt{D} u = \sum_{(ij) \in E} \frac{\mu_{ij}^2}{\sigma_{\max}} (u_i - u_j)^2 + \sum_{i \in V} \mu_{i, i^{\text{comp}}}^2 \left(\frac{u_i}{\sqrt{\sigma_{\max}}} - \frac{u_{i^{\text{comp}}}}{\sqrt{L_i}} \right)^2,$$

and thus:

$$\begin{aligned}
\lambda_{\min}^+(A^T D A) &\geq \frac{\lambda_2(\Delta_G(\mu_{ij}^2))}{\lambda_2(\Delta_G(\mu_{ij}^2)) \max_{i \in V} L_i \mu_{i, i^{\text{comp}}}^{-2} + (L_{\max} + \sigma_{\max})} \\
&\geq \frac{1}{2} \min \left(\frac{\lambda_2(\Delta_G(\mu_{ij}^2))}{L_{\max} + \sigma_{\max}}, \min_{i \in V} \frac{\mu_{i, i^{\text{comp}}}^2}{L_i} \right)
\end{aligned}$$

Proof. For the first part, one just needs to notice that $u^\top \sqrt{D} A^\top A \sqrt{D}$ is the quadratic form associated to the Laplacian of the augmented graph, applied to vector $\sqrt{D}u$.

Let us now write $u = (x, y) \in \mathbb{R}^{V^+}$ where x corresponds to non-virtual nodes, and y to virtual ones. Let $Q = \Delta_G(\mu_{ij}^2)$ Laplacian of the non-augmented graph with weights μ_{ij}^2 , and let $W = \min_i \mu_{i,i\text{comp}}^2$. First, we have, where \bar{x} is the projection of x onto $(1, \dots, 1)$:

$$\begin{aligned} \sum_{(ij) \in E} \frac{\mu_{ij}^2}{\sigma_{\max}} (u_i - u_j)^2 &= x^\top Q x / \sigma_{\max} \\ &\geq \lambda_2(Q) / \sigma_{\max} \|x - \bar{x}\|^2. \end{aligned}$$

Hence:

$$u^\top \sqrt{D} A A^\top \sqrt{D} u \geq \sum_{i \in V} \lambda_2(Q) / \sigma_{\max} (x_i - \bar{x})^2 + \mu_{i,i\text{comp}}^2 \left(\frac{x_i - \bar{x}}{\sqrt{\sigma_{\max}}} - \frac{y_i - \sqrt{\frac{L_i}{\sigma_{\max}}} \bar{x}}{\sqrt{L_i}} \right)^2.$$

Now, for any real numbers x, y , any $\lambda, W > 0$ and any $\sigma, L > 0$

$$\lambda \left(\frac{x}{\sqrt{\sigma}} \right)^2 + W \left(\frac{x}{\sqrt{\sigma}} - \frac{y}{\sqrt{L}} \right)^2 \geq \frac{\lambda W}{\lambda L + W(L + \sigma)} (x^2 + y^2).$$

Indeed, for any c such that $0 < c \leq W/L$

$$\begin{aligned} &\lambda \left(\frac{x}{\sqrt{\sigma}} \right)^2 + W \left(\frac{x}{\sqrt{\sigma}} - \frac{y}{\sqrt{L}} \right)^2 - c(x^2 + y^2) \\ &= \left(\frac{\lambda + W}{\sigma} - c \right) x^2 + \left(\frac{W}{L} - c \right) y^2 - 2 \frac{W}{\sqrt{\sigma L}} xy \\ &= \left(\sqrt{\frac{W}{L}} - cy - \frac{\frac{W}{\sqrt{\sigma L}} x}{\sqrt{\frac{W}{L} - c}} \right)^2 + \left(\frac{\lambda + W}{\sigma} - c - \frac{W^2}{\sigma L (\frac{W}{L} - c)} \right) x^2. \end{aligned} \quad (47)$$

This quantity is non-negative for any $x, y \in \mathbb{R}$ as long as $\frac{\lambda + W}{\sigma} - c - \frac{W^2}{\sigma L (\frac{W}{L} - c)} \geq 0$. This is verified if and only if:

$$L\sigma c^2 - (\lambda L + W(L + \sigma))c + \lambda W \geq 0.$$

Thus, we would like to find the roots of a polynomial (in c) of the form $ac^2 - bc + d$, with $a, b, d \geq 0$. We know that the smallest one is given by:

$$\frac{b - \sqrt{b^2 - 4ad}}{2a} = \frac{b}{2a} \left(1 - \sqrt{1 - \frac{4ad}{b^2}} \right) \geq \frac{b}{2a} \times \frac{4ad}{2b^2} = \frac{d}{b},$$

where we used that $1 - \sqrt{1 - x} \geq \frac{x}{2}$. In particular, the quadratic from Equation (47) is always non-negative if

$$c \leq \frac{\lambda W}{\lambda L + W(L + \sigma)}. \quad (48)$$

Note that this c also respects the condition that $c \leq W/L$. For such a $c > 0$, going back to our matrix $\sqrt{D} A A^\top \sqrt{D}$, we have:

$$\begin{aligned} u^\top \sqrt{D} A A^\top \sqrt{D} u &\geq \sum_{i \in V} \lambda_2(Q) / \sigma_{\max} (x_i - \bar{x})^2 + \mu_{i,i\text{comp}}^2 \left(\frac{x_i - \bar{x}}{\sqrt{\sigma_{\max}}} - \frac{y_i - \sqrt{\frac{L_i}{\sigma_{\max}}} \bar{x}}{\sqrt{L_i}} \right)^2 \\ &\geq \sum_{i \in V} \frac{\lambda \mu_{i,i\text{comp}}^2}{\lambda L_i + \mu_{i,i\text{comp}}^2 (L_i + \sigma)} \left[(x_i - \bar{x})^2 + \left(y_i - \sqrt{\frac{L_{\max}}{\sigma_{\max}}} \bar{x} \right)^2 \right] \\ &\geq \frac{\lambda}{\lambda \max_i (L_i / \mu_{i,i\text{comp}}^2) + (L_{\max} + \sigma)} \left\| u - \frac{\bar{x}}{\sqrt{\sigma_{\max}}} D^{-\frac{1}{2}} \mathbf{1} \right\|^2. \end{aligned}$$

Since $\text{Span}(D^{-\frac{1}{2}}\mathbf{1}) = \ker(\sqrt{D}A^\top A\sqrt{D})$, denoting u^+ the orthogonal projection of u on $\ker(\sqrt{D}A^\top A\sqrt{D})$, we have that $\|u - \alpha D^{-\frac{1}{2}}\mathbf{1}\|^2 \geq \|u - u^+\|^2$ for all $\alpha \in \mathbb{R}$, and so:

$$\begin{aligned} (u - u^+)^\top \sqrt{D}A^\top A\sqrt{D}(u - u^+) &= u^\top \sqrt{D}A^\top A\sqrt{D}u \\ &\geq \frac{\lambda}{\lambda \max_i (L_i/\mu_{i,i^{\text{comp}}}^2) + (L_{\max} + \sigma)} \|u - u^+\|^2, \end{aligned}$$

proving our lower bound on $\lambda_2(\sqrt{D}A^\top A\sqrt{D})$ and thus on $\lambda_2(A^\top DA)$ and on σ_A the strong-convexity parameter of F_A^* on the orthogonal of $\text{Ker}(A)$. \square

F.4 Applying Theorem 3 to recover Theorem 1

The coordinate blocks for our coordinate gradient descent are E_{ij} for $(ij) \in E$ and $E_{i,i^{\text{comp}}}$ coordinates of $\lambda \in \mathbb{R}^{E^+}$ associated to edges and virtual edges.

We have:

1. $L_{ij} = 2\mu_{ij}^2(\sigma_i^{-1} + \sigma_j^{-1})$ local smoothness;
2. $M_{(ij),(kl)} = \sqrt{L_{ij}L_{kl}}$ for Assumption 3;
3. Strong convexity parameter of F_A^* :

$$\sigma_A \geq \frac{1}{2} \min \left(\frac{\lambda_2(\Delta_G(\mu_{ij}^2))}{L_{\max} + \sigma_{\max}}, \min_{i \in V} \frac{\mu_{i,i^{\text{comp}}}^2}{L_i} \right);$$

4. Delays associated to coordinates are time-independent, and for $ij \in E$, are equal to τ_{ij} , while for virtual edge i, i^{comp} we have τ_i^{comp} ;
5. Local Poisson intensities are respectively $p_{ij}, p_i^{\text{comp}}$ for non-virtual edges and virtual ones.
6. In the presence of capacity limitations, the assumptions made in Theorem 2 ensure that for all $(ij) \in E^+$, $\varepsilon_{ij} \geq 1/2$ when applying Theorem 3.

Therefore, one can apply Theorem 3 to our function F_A^* on the orthogonal of $\text{Ker}(A)$ to obtain:

1. **Stability condition:** for all $(ij) \in E^+$,

$$K_{ij} \leq \frac{p_{ij}}{1 + \sum_{kl \sim ij} p_{kl}(\tau_{ij} + e\tau_{kl})};$$

2. **Rate of convergence:**

$$\gamma < \sigma_A \min_{e \in E^+} \frac{\varepsilon_{ij} K_{ij}}{\mu_{ij}^2(\sigma_i^{-1} + \sigma_j^{-1})}.$$

As weights μ_{ij}^2 of matrix A can be tuned as desired, we choose for $(ij) \in E^+$:

$$\mu_{ij}^2 = \frac{\varepsilon_{ij} K_{ij}}{\mu_{ij}^2(\sigma_i^{-1} + \sigma_j^{-1})} \geq \sigma_{\min} \frac{K_{ij}}{2\varepsilon_{ij}}.$$

Since $\varepsilon_{ij} = 1$ in Theorem 1 and $1/2$ in Theorem 2, we recover:

$$\gamma < \frac{\sigma_{\min}}{2} \min \left(\frac{\lambda_2(\Delta_G(K_{ij}))}{L_{\max} + \sigma_{\max}}, \min_{i \in V} \frac{K_{i,i^{\text{comp}}}}{L_i} \right)$$

for Theorem 1, and multiply this by $\varepsilon = 1/2$ for Theorem 2.

F.5 Proof of Corollary 3

We recall some notions on Perron-Frobenius eigenvectors and eigenvalues [34]. We say that a matrix or a vector is elementwise non-negative (*resp.* elementwise positive) if all its entries are non-negative (*resp.* positive). A matrix $M \in \mathbb{R}^{m \times m}$ is said regular if there exists some $k \in \mathbb{N}^*$ such that A^k is elementwise positive.

Theorem 4 (Perron-Frobenius Theorem for regular matrices). *For a regular and elementwise non-negative matrix $M \in \mathbb{R}^{m \times m}$:*

1. *There is a real and positive eigenvalue $\lambda_{\text{PF}}(M)$ of M , with elementwise positive left and right eigenvectors;*
2. *For any other eigenvalue (possibly complex) λ of M , $|\lambda| < \lambda_{\text{PF}}$, and thus $\rho(M) = \lambda_{\text{PF}}(A)$;*
3. *The eigenvalue $\lambda_{\text{PF}}(M)$ is simple: its multiplicity is 1 and corresponds to a 1×1 Jordan block.*

$\lambda_{\text{PF}}(M)$ is called the Perron-Frobenius eigenvalue of M . The associated left and right elementwise eigenvectors are Perron-Frobenius eigenvectors (unique up to scaling).

We use the following min-max ratio characterization of Perron-Frobenius eigenvalues, corollary 8.1.31 p. 493 in Horn [17].

Proposition 10. *For a non-negative element wise and regular matrix $M \in \mathbb{R}^{m \times m}$, its Perron-Frobenius eigenvalue satisfies:*

$$\lambda_{\text{PF}}(A) = \min_{x=(x_1, \dots, x_m) \in \mathbb{R}^+, * } \max_{i=1, \dots, m} \frac{(Ax)_i}{x_i}.$$

Let us now prove Corollary 3. Theorem 4 and Proposition 10 can be applied to matrix $M(K)$: it is non-negative elementwise, and since graph G^+ is connected, $M(K)$ is regular.

Assume first that $\rho(M(K)) < 1$. Using Theorem 4, $\lambda_{\text{PF}}(M(K)) < 1$. Let p^0 a Perron-Frobenius right eigenvector of $M(K)$: for all $(ij) \in E^+$, we have:

$$p_{ij}^0 > 0 \quad \text{and} \quad K_{ij} \sum_{(kl) \sim (ij)} p_{kl}^0 (\tau_{ij} + e\tau_{kl}) = \rho(M(K)) p_{ij}^0.$$

Let $C = \max_{(ij) \in E^+} \frac{K_{ij}}{(1 - \rho(M(K))) p_{ij}^0}$, such that for all $(ij) \in E^+$, denoting $p = Cp^0$:

$$\begin{aligned} K_{ij} \left(1 + \sum_{kl \sim ij} p_{kl} (\tau_{ij} + e\tau_{kl}) \right) &= K_{ij} + \rho(M(K)) p \\ &= K_{ij} + \rho(M(K)) C p_{ij}^0 \\ &\leq C(1 - \rho(M(K))) p_{ij}^0 + C \rho(M(K)) p_{ij}^0 \\ &= p_{ij}. \end{aligned}$$

Hence, for this choice of $p_{ij} > 0$, $(ij) \in E^+$, Assumption 1 is verified.

Assume now that there exist some positive Poisson intensities p_{ij} , $(ij) \in E^+$ such that Assumption 1 is verified. For any $(ij) \in E^+$, we have:

$$\begin{aligned} (M(K)p)_{ij} &= K_{ij} \sum_{kl \sim ij} p_{kl} (\tau_{ij} + e\tau_{kl}) \\ &< K_{ij} \left(1 + \sum_{kl \sim ij} p_{kl} (\tau_{ij} + e\tau_{kl}) \right) \\ &\leq p_{ij}. \end{aligned}$$

so that:

$$\max_{(ij) \in E^+} \frac{(M(K)p)_{ij}}{p_{ij}} < 1.$$

Using Proposition 10 and the fact that $M(K)$ is regular and non-negative entrywise, we get:

$$\rho(M(K)) = \lambda_{\text{PF}}(M(K)) \leq \max_{(ij) \in E^+} \frac{(M(K)p)_{ij}}{p_{ij}} < 1,$$

which concludes the proof.

G Experiments

G.1 Simulating Delayed Randomized Gossip

We here explain a simple way to simulate delayed randomized gossip (and similarly *DCDM*) with the following pseudo-code.

Assume graph G , initialization vector x , Poisson intensities p_{ij} , stepsizes K_{ij} and delays τ_{ij} are given. Let $T > 0$ be the stopping (continuous-)time.

Let U be a list, that will stack the updates to be made later. An entry of U is of the form $(t_k, (i_k j_k), (y_{i_k}, y_{j_k}))$: t_k is the time at which this update will take place, $(i_k j_k)$ is the edge alongside this update is made, and (y_{i_k}, y_{j_k}) are the vectors that will be used in this update.

Draw $\mathcal{T} = (T_1, \dots, T_r)$ such that $(T_{k+1} - T_k)_k$ are *i.i.d.* exponential random variables of parameter $I = \sum_{ij} p_{ij}$, with $T_{r-1} < T < T_r$.

For k in $\{1, \dots, r\}$:

1. Sort U increasingly (with respect to the first argument of its entries). For any $(t, (ij), (y_i, y_j))$ in U such that $t \leq T_k$, perform:

$$\begin{aligned} x_i &\leftarrow x_i - \frac{K_{ij}}{p_{ij}}(y_i - y_j), \\ x_j &\leftarrow x_j + \frac{K_{ij}}{p_{ij}}(y_i - y_j), \end{aligned}$$

and delete this entry from U . These updates are performed with the order from which the entries are in U .

2. Draw an edge (ij) with probability p_{ij}/I ;
3. Add $(t + \tau_{ij}, (ij), (x_i, x_j))$ to U .

The obtained vector is then x_T .

G.2 Simulated Annealing

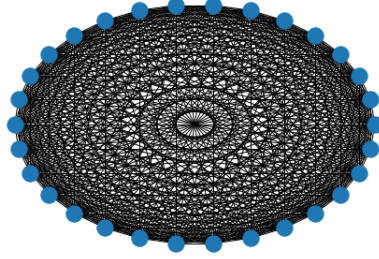
Figures 3(d) 3(e) and 3(f) are obtained by minimizing (14) using simulated annealing and bootstrapping 5 times:

1. Run a simulated annealing (described below) with $\omega = \lambda_2/(2 \sum_{ij} p_{ij} \tau_{ij})$ for T epochs;
2. Initialize with the result obtained;
3. Restart.

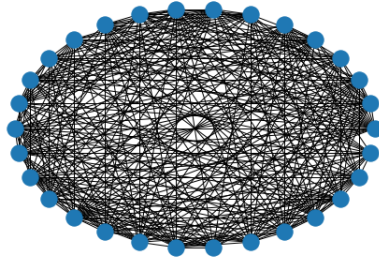
Simulated annealing in our case goes as follows, where for a configuration of intensity p , $J(p)$ is the energy (14), and $0 \leq t \leq T$ are the iterates:

1. Temperature $\beta^{-1} = 1/\sqrt{1+t}$;
2. Draw an edge (ij) uniformly at random;
3. Set $p_{ij}^{t+1} = 0$ if $p_{ij}^t > 0$, or $p_{ij}^{t+1} = p_{ij}^0$ if $p_{ij}^t = 0$;
4. If $J(p^{t+1}) > J(p^t)$, keep the modification. Else, keep it with probability $\exp(-\beta(J(p^t) - J(p^{t+1})))$.
5. Return p_T .

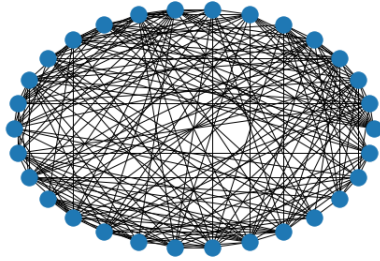
Figures below show the graphs obtained after each iteration of a simulated annealing algorithm for $T = 1000$ epochs.



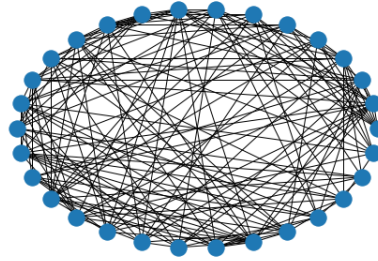
(a) Original Graph, $N = 435$, $\lambda_2 = 0.11$



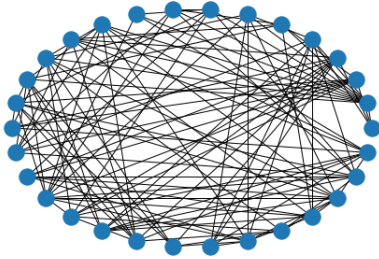
(b) Sparsified 1, $N = 301$, $\lambda_2 = 0.86$



(c) Sparsified 2, $N = 208$, $\lambda_2 = 0.47$

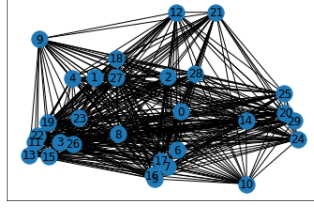


(d) Sparsified 3, $N = 161$, $\lambda_2 = 0.53$

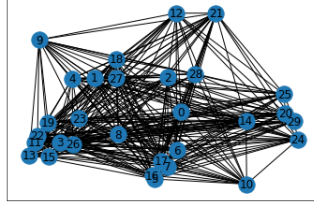


(e) Sparsified 4, $N = 122$, $\lambda_2 = 0.16$

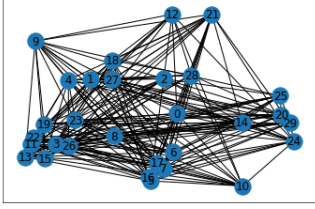
Figure 6: Illustration of our sparsification method on complete graph with random *i.i.d.* uniform delays squared: N is the number of edges obtained, λ_2 represents the spectral gap of the graph weighted by K as in Theorem 1.



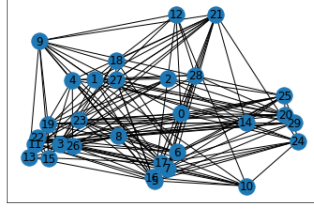
(a) Original Graph, $N = 435$, $\lambda_2 = 0.65$



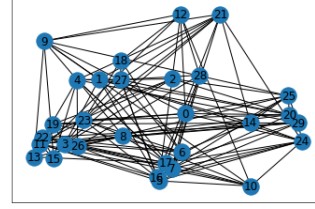
(b) Sparsified 1, $N = 311$, $\lambda_2 = 0.61$



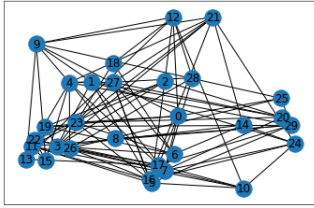
(c) Sparsified 2, $N = 212$, $\lambda_2 = 0.59$



(d) Sparsified 3, $N = 157$, $\lambda_2 = 0.60$



(e) Sparsified 4, $N = 124$, $\lambda_2 = 0.56$



(f) Sparsified 5, $N = 101$, $\lambda_2 = 0.53$

Figure 7: Illustration of our sparsification method on a random geometrical graph, with delays proportional to distance: N is the number of edges obtained, λ_2 represents the spectral gap of the graph weighted by K as in Theorem 1.