Department of Physics and Astronomy
Heidelberg University

Bachelor thesis in Physics
submitted by

Mathieu Kaltschmidt

from Kappel-Grafenhausen

Functional Renormalization and Quantum Gravity

This bachelor thesis has been carried out by

Mathieu Kaltschmidt

at the

Institute for Theoretical Physics

at

Heidelberg University

under the supervision of

Prof. Dr. Jan M. Pawlowski

Functional Renormalization and Quantum Gravity

Mathieu Kaltschmidt

Abstract

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Zusammenfassung

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Contents

Chapter 1.

Introduction

Throughout this thesis we use units such that $\hbar = c = G \equiv 1$.

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Chapter 2.

Functional methods in Quantum Field Theory

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2.1. Generating Functionals and Correlation Functions

We consider a theory setting of N scalar fields $\varphi_a(x), a \in \{1, \ldots, N\}$ in d-dimensional Euclidean space. The corresponding partition sum in presence of sources $J_a(x)$ reads

$$Z[J] = \int \mathcal{D}\varphi \, \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \,. \tag{2.1}$$

The information content of the partition sum results mainly from the classical action functional $\mathcal{S}[\varphi]$, which determines the classical field equations

$$\frac{\delta \mathcal{S}}{\delta \varphi(x)} = 0. \tag{2.2}$$

Notation: The scalar product sums over field components and integrates over all space …

$$J \cdot \varphi = \int_x J_a(x) \, \varphi_a(x) = \int_p \tilde{J}_a(p) \, \tilde{\varphi}_a(p) \tag{2.3}$$

with

$$\int_x = \int_{\mathbb{R}^d} \mathrm{d}^d x \qquad \text{and} \qquad \int_p = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \tag{2.4}$$

Mean field description:

$$\phi := \langle \varphi \rangle = \frac{1}{Z} \frac{\delta Z}{\delta J}\bigg|_{J=0} = \int \mathcal{D}\varphi \, \varphi \; \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \tag{2.5}$$

Higher correlations:

$$\langle\varphi_1\cdots\varphi_n\rangle := \langle\varphi^n\rangle = \frac{1}{Z}\frac{\delta^n Z}{\delta^n J} = \int\mathcal{D}\varphi\ \overbrace{\varphi_1\cdots\varphi_n}^{:=\varphi^n}\ \mathrm{e}^{-\mathcal{S}[\varphi]+J\cdot\varphi} \tag{2.6}$$

We obtain the Schwinger functional by taking the logarithm:

$$W[J] = \ln Z[J] \tag{2.7}$$

For the special case of $n = 2$ the correlation function yields the connected 2-point function which is also known as the propagator $G_{ab}(x, y)$ correlating the field φ_a at spacetime point x with the field φ_b at y.

$$\begin{aligned} G_{ab}(x,y) &= \frac{\delta^2 W[J]}{\delta J_a(x)\delta J_b(y)} = \frac{\delta}{\delta J_a(x)}\left(\frac{1}{Z}\frac{\delta Z}{\delta J_b(y)}\right) \\ &= \frac{1}{Z}\left(\frac{\delta^2 Z}{\delta J_a(x)\delta J_b(y)}\right) - \frac{1}{Z^2}\left(\frac{\delta Z}{\delta J_a(x)}\right)\left(\frac{\delta Z}{\delta J_b(y)}\right) \\ &= \langle\varphi_a(x)\varphi_b(y)\rangle - \phi_a(x)\phi_b(y) = \langle\varphi_a(x)\varphi_b(y)\rangle_{\mathrm{c}} \end{aligned} \tag{2.8}$$

The Effective Action:

The effective action can be obtained by performing a Legendre transform of the Schwinger funtional, i. e.:

$$\Gamma[\phi] = \sup_J\left\{\int_x J(x)\phi(x) - \mathcal{W}[J]\right\} = \int_x J_{\mathrm{sub}}(x)\phi(x) - \mathcal{W}[J_{\mathrm{sub}}] \tag{2.9}$$

Quantum equation of motion:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = J(x) \tag{2.10}$$

Dyson-Schwinger equation:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \frac{\delta\mathcal{S}}{\delta\varphi(x)}\left[\varphi = G\cdot\frac{\delta}{\delta\phi} + \phi\right] \tag{2.11}$$

2.2. The Functional Renormalization Group

- Kadanoff Block-Spin model

- maybe visualization of Ising model + phase transitions

2.2.1. Renormalization group consistency

Cutoff independence of the full quantum effective action:

$$\Lambda \frac{\mathrm{d}\Gamma}{\mathrm{d}\Lambda} = 0 \tag{2.12}$$

Full effective action in a generic representation:

$$\Gamma[\phi] = \mathcal{D}_\Lambda[\phi] + \Gamma_\Lambda[\phi] \tag{2.13}$$

Formal discussion:

$$\Gamma_k[\phi] = \Gamma_\Lambda[\phi] + \int\limits_\Lambda^k \frac{\mathrm{d}k'}{k'} \mathcal{F}_{k'}[\phi] \tag{2.14}$$

2.3. The Renormalisation Group Flow for the Effective Action

We introduce the RG time scale t:

$$\partial_t = \frac{\partial}{\partial \ln(k/\Lambda)} = \frac{k}{\Lambda} \frac{\partial}{\partial(k/\Lambda)} = k\partial_k \tag{2.15}$$

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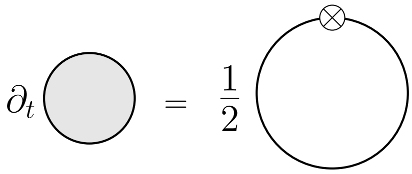
$$\begin{aligned} \partial_t \Gamma_k[\phi] &= \frac{1}{2} \operatorname{Tr} \left[\frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \partial_t R_k \right] \\ &= \frac{1}{2} \int\limits_p \frac{1}{\Gamma_k^{(2)}[\phi] + R_k} (p, -p)\, \partial_t R_k(p^2) \end{aligned} \tag{2.16}$$

This translates directly into the following diagrammatic representation:



where $\otimes = \partial_t R_k$ represents the insertion of the respective regulator.

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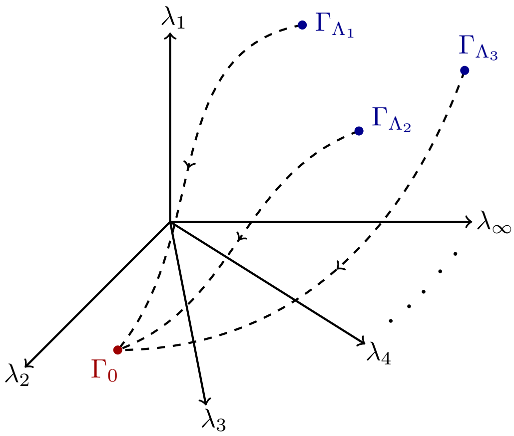

Figure 2.1.: Flow of Γ_k through infinite-dimensional theory space for different regulators, inspired by [5]

Chapter 3.

Fundamentals of General Relativity

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3.1. The Einstein Equations

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The Einstein-Hilbert action:

$$\mathcal{S}_{\mathrm{EH}}[g_{\mu\nu}] = \frac{1}{16\pi G}\int\limits_x \sqrt{-\det g_{\mu\nu}}(\mathcal{R} - 2\Lambda) \tag{3.1}$$

Varying this action as usual yields the Einstein equations in absence of matter:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3.2}$$

where we used $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}$.

Diffeomorphism invariance, Lie derivatives:

$$\mathcal{L}_\omega \phi = \omega^\mu \partial^\mu \phi = \omega^\mu \nabla^\mu \phi \tag{3.3}$$

Now we include matter.

Energy-Momentum Tensor:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-\det g_{\mu\nu}}} \frac{\delta \mathcal{S}_{\text{matter}}}{\delta g^{\mu\nu}} \tag{3.4}$$

Matter part of the action for a minimally coupled scalar field ϕ:

$$\mathcal{S}_{\text{matter}}[g_{\mu\nu}, \phi] = -\frac{1}{2} \int_x \sqrt{-\det g_{\mu\nu}} \left(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} V(\phi)\right) \tag{3.5}$$

From this, we get the Einstein equations including matter by demanding the variation $\sqrt{-\det g_{\mu\nu}} \frac{\delta \mathcal{S}}{\delta g^{\mu\nu}}$ to vanish. This yields:

$$\frac{1}{8\pi G} \left[\mathcal{R}_{\mu\nu} - \frac{1}{2}(\mathcal{R} - 2\Lambda) g_{\mu\nu}\right] = T_{\mu\nu} \tag{3.6}$$

3.2. Perturbative Non-Renormalizability of Gravity

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Chapter 4.

Quantum Gravity in the Einstein-Hilbert Truncation

4.1. RG approach to Quantum Gravity

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4.2. Truncations of the theory space

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4.3. The effective action for Quantum Gravity

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4.4. Non-Gaussian Fixed Points

Chapter 5.

Conclusions and Outlook

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Appendix A.

Mathematical Appendix

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A.1. Heat Kernel techniques

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A.2. York decomposition

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Appendix B.

Numerical Implementation

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B.1. Determination of the Fixed Points

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List of Figures

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Declaration of Authorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise.

Heidelberg, ________________ ____________________________