Journal of
Mathematical Physics

Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial-value problem of general relativity

Cite as: Journal of Mathematical Physics **14**, 456 (1973); https://doi.org/10.1063/1.1666338
Submitted: 25 July 1972 . Published Online: 03 November 2003

James W. York

Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial-value problem of general relativity*

James W. York Jr.

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 25 July 1972)

It is shown that an arbitrary symmetric tensor ψ^{ab} (or ψ_{ab}) of any weight can be covariantly decomposed on a Riemannian manifold (M,g) into a unique sum of transverse-traceless, longitudinal, and pure trace parts. The summands involve only linear operators and are mutually orthogonal in the global scalar product on (M,g). Each summand transforms separately into itself if the decomposition is carried out properly in a conformally related space $(M,\bar{g})$. The decomposition is therefore determined by a conformal equivalence class of Riemannian manifolds. This property makes the decomposition ideally suited to the initial-value problem of general relativity, which becomes, as a result, a well-defined system of elliptic equations. Three of the four initial-value equations are linear and determine the decomposition of a symmetric tensor. The fourth equation is quasilinear and determines the conformal factor. The decomposition applied to the space of symmetric tensors on (M,g) can be written in terms of a direct sum of orthogonal linear spaces and gives a framework for treating and classifying deformations of Riemannian manifolds pertinent to the theory of gravitation and to pure geometry.

1. INTRODUCTION AND SUMMARY

The purpose of this paper is essentially twofold. Firstly, we give a conformally invariant, orthogonal, covariant decomposition of symmetric tensors on positive-definite Riemannian manifolds into transverse-traceless, longitudinal, and pure trace parts. Secondly, we show that this decomposition enables one to set the initial-value problem of general relativity as a system of four second-order elliptic equations for four unknown functions. Three of these equations are linear with a conformally-invariant vector field as the solution. The other is quasilinear and determines the conformal factor. Physically, the *unconstrained* fields correspond to pure spin-two transverse traceless dynamical variables. Geometrically, they describe the *anisotropy of space*. This description is of course equivalent to the conventional treatment of space-time in terms of a hyperbolic four-metric ${}^{(4)}g_{\mu\nu}$ satisfying Einstein's field equations, but is more useful for a number of purposes.

It is well known[1] that in flat or in curved Riemannian spaces one can decompose an arbitrary vector or one-form into its transverse and longitudinal parts. Physically, this procedure leads to the identification of the true canonical degrees of freedom of the electromagnetic field and to the identification of the gauge, or non-dynamical, variables. Thus, if $\mathbf{A}$ is the magnetic vector potential and $\mathbf{E}$ the electric field, then the transverse fields $\mathbf{A}_t$ and $\mathbf{E}_t (\nabla \cdot \mathbf{A}_t = \nabla \cdot \mathbf{E}_t = 0)$ are the dynamical or pure spin-one variables, while the longitudinal (l) part of $\mathbf{A}$ is determined by a choice of gauge. Moreover, this decomposition is not only covariant with respect to arbitrary coordinate transformations, it is also orthogonal in the natural global scalar product. That is, for any two vectors $\mathbf{v}$ and $\mathbf{w}$, we have

$$\int_M v_g V_t{}^a W_l{}^b g_{ab} \equiv (V_t, W_l) = 0, \tag{1}$$

where v_g denotes the volume element which makes the integral invariant and the integration extends over the entire manifold M. Geometrically, the decomposition of 1-forms, and more generally p-forms, leads via de Rham's theorem to a characterization of topological invariants of M (i.e., Betti Numbers).[2]

In this work, we shall consider *three*-dimensional Riemannian spaces because this case is of the greatest interest for physical applications. However, the results are easily generalized to n-spaces, provided they are Riemannian, i.e., have positive-definite metric.[3] Also, it is assumed that the three-spaces are *closed* (compact manifolds without boundary) and complete. The choice of closed spaces is made for mathematical convenience, and because closed three-spaces are of greatest interest in recent studies of the classical and quantum dynamics of general relativity.[4] However, the decomposition is also valid for open, asymptotically flat 3-spaces, certain assumptions being made as to the decay of the metric and other tensor fields as one approaches infinity.[5] In the pioneering work of Deser[6] on the covariant decomposition of tensors, only the case of asymptotic flatness was treated. Deser's procedure is also satisfactory for *closed* spaces; however, his decomposition is neither conformally invariant nor completely orthogonal in general.

There are two compelling physical motivations for decomposing symmetric tensors, both having to do with gravitation. One is the desire to separate gravitational variables into irreducible (spin) parts, so as to distinguish the dynamical (spin-2) variables from gauge variables and constrained variables. This was done in general relativity by Arnowitt, Deser, and Misner[1] using a *non*covariant decomposition and was treated covariantly by Deser[6] in later works. Another strong reason is that a covariant procedure should lead in the initial-value problem to a well-defined system of equations determining the constrained variables (Sec. 4).

From a mathematical point of view, the decomposition of tensor fields is also significant. A certain "T-decomposition"[6] (T = transverse) of symmetric tensors characterizes possible deformations of Riemannian manifolds, as shown by Ebin.[7] This is also important in general relativity, for the dynamics of gravity may be viewed as a continuous (time-dependent) deformation of spacelike hypersurfaces in enveloping spacetimes satisfying Einstein's field equations. This deformation process may also be viewed in superspace $\mathcal{S}$, which is the collection of Riemannian metrics *modulo* diffeomorphisms of M. Each point of $\mathcal{S}$ is a *3-geometry* ${}^3\mathcal{G} = (M,g)$. The structure of superspace was examined in detail by Fischer[8] and was discussed in a helpful article by B. DeWitt.[9] In the Appendix, we show that for metrics possessing symmetries, one can readily construct from one transverse tensor others which will also be transverse. Similar results hold for the TT decomposition.

In a certain sense the superspace picture and the associated T-decomposition of symmetric tensors are not sufficiently "fine" for all purposes in treating gravitation. This is the case because, using Wheeler's phrase,[10] "3-geometry is the carrier of information about time". Thus, $^3\mathcal{G}$ may be regarded as being specified by three functions (g_{ab} *modulo* diffeomorphisms of M) and its associated momentum, a transverse tensor density, is also specified by three functions. If these six functions were subject to no further constraints on a spacelike hypersurface, we would conclude that the gravitational field has three, not two, canonical degrees of freedom at each point of space. However, the metric and momentum *are* subject to an additional constraint. (The transversality of the momentum already comprises three of the four equations of constraint). We may regard, therefore, one of the metric variables and one of the momentum variables as describing a nondynamical pair of variables subject to a constraint. To what physical quantities could they correspond? Not surprisingly, they correspond to "time" and the "Hamiltonian."[11] Qualitatively similar arguments to this effect have been given in analyses of the dynamics of gravity by Dirac,[12] by Misner,[4] by Arnowitt, Deser, and Misner,[1] by Wheeler,[10] by Kuchař,[13] by the author,[11] and by others. Therefore, because there is *one* constraint on these *two* variables, one of them may be specified and the other must be determined by satisfying the remaining initial-value equation.

Details of the procedure followed differ among different students of the problem. Some prefer simply to carry along the extra pair, letting them, in effect, be governed implicitly by the fourth constraint written as a "Hamilton-Jacobi" type functional differential equation. This is what may be called roughly the "superspace approach."[14] Others make a choice of space and time coordinates based on some "preferred" background metric and select dynamical and nondynamical variables according to the criterion of convenience in treating the problem at hand. The essence of the latter formulation is found in the "mini-superspace" approach,[15,16] where only metrics with certain specified symmetries are considered. Thus, all degrees of freedom of the field are assumed to be frozen except those with the given symmetry. This procedure involves an element of risk because the gravitational field is nonlinear: degrees of freedom with different symmetry, or no symmetry, *do* interact; thus, ignoring some of them could lead to misleading conclusions. Moreover, in the quantum version of this approach, freezing certain degrees of freedom manifestly violates the uncertainty principle. These points are well known and have been discussed in the literature.[15,16] Of course, there are a number of other schemes for dealing with the problems engendered by the constraints, but I will not discuss them here.

One notices the "dual" nature of this problem: (1) picking variables in order to satisfy ultimately the initial-value equations on a spacelike hypersurface; (2) treating the dynamics of gravity based on this choice. In reality, however, there is only *one* problem because if the constraints are satisfied at one moment, they automatically continue to hold at succeeding moments by virtue of the field equations at that moment and the contracted Bianchi identities ${}^{(4)}\nabla_\nu {}^{(4)}G^{\mu\nu} \equiv 0$. [We use a prefix (4) and Greek indices when referring to space-time, as opposed to space.] Moreover, it has been shown that from the initial-value equations one can recover the entire dynamical content of general relativity.[17]

There is, however, a specific choice of variables which lead to a refinement of the superspace approach.[11,18] To proceed further, I make the only apparent simple three-dimensionally covariant refinement of variables, namely, the scale or conformal factor of the metric and the trace of the momentum tensor (actually the *scalar* part of this trace) are identified as separate entities. These two variables are selected to comprise the "extra" pair. There are compelling physical reasons for this choice. Firstly, I have previously shown that the 3-geometry *modulo* conformal factor specifies the pure spin-2 (TT) representation of space geometry.[18] Secondly, the scalar trace of the momentum is just the "volume Hubble parameter", i.e., the specific rate of volume expansion of the 3-space as it evolves in space-time.[11] Therefore, this scalar is naturally identifiable with "time," in that it identifies the expansion or contraction epoch of the 3-space in its history. Thirdly, if we use this scalar as $x^0 = t =$ time, then on the surfaces $t =$ const., the trace-free part of the momentum tensor is transverse. Hence, the dynamical part of the momentum tensor is TT, and a TT tensor is determined only by the underlying *conformal* geometry.[18] (See Sec. 3). Mathematically, this procedure is fruitful in the initial-value problem (Sec. 4). Earlier, Dirac[12] made a related choice of variables, in that he treated the scale factor of the geometry as a nondynamical variable. However, he did not show that the conformal geometry has the properties we have mentioned above, nor did he use the scalar trace of the momentum as time. Rather, he set this trace equal to *zero* for all time as an implicit condition on the time coordinate. It is by now well known that time variables compatible with his procedure do not exist in closed universes, i.e., for closed 3-spaces, in general.[14] This defect is not present in our case; in fact, the scalar trace need not necessarily be a constant (Sec. 4). In any case, our time variable is local and identifies the surface in spacetime to which one is referring.

Because the 3-geometry contains as an "extra" variable the scale factor, it does in a sense "carry information about time," though somewhat indirectly. The fact that the 3-geometry must carry extra information of some kind led Wheeler[19] to ask, in effect, "What is two-thirds of superspace?" An answer to this question is given in Sec. 6, namely, "conformal superspace."

The "TT-decomposition" described in this paper characterizes deformations of *conformal* Riemannian manifolds $(M,\bar{g}) \equiv {}^3\mathfrak{C}$, where $\bar{g}_{ab} \equiv (\det g)^{-1/3} g_{ab}$. Thus ${}^3\mathfrak{C}$ is determined by a Riemannian metric g_{ab} *modulo* diffeomorphisms of M and *modulo* conformal mappings $\bar{g}_{ab} = \phi^4 g_{ab}$, where $\phi(x)$ is an arbitrary, real, nonvanishing scalar function on M. (We may say ${}^3\mathfrak{C}$ is determined by g_{ab} *modulo* "conformeomorphisms"[20] for short.) The collection of conformal 3-geometries may be called "conformal superspace" $\tilde{\mathcal{S}}$. Each point of $\tilde{\mathcal{S}}$ is a ${}^3\mathfrak{C}$. We sometimes find it convenient to regard ${}^3\mathfrak{C}$ also as a set of conformally equivalent Riemannian 3-geometries. In all respects, the TT-decomposition bears the same relation to $\tilde{\mathcal{S}}$ and to the dynamics of ${}^3\mathfrak{C}$ in space-time that the T-decomposition bears to $\mathcal{S}$ and to the dynamics of $^3\mathcal{G}$ in space-time.

However, the TT-decomposition has features that the T-decomposition does not possess. Notably, it is a "finer" splitting of a symmetric tensor than the T-decomposition in that it has more independent pieces. This fineness leads to its conformal invariance, which, in turn, makes it quite useful in analysis of the initial-value equations. Moreover, since conformally invariant objects described the pure spin-2 aspects of gravity.[18] the TT-decomposition is ideally suited to the construction of such objects.

In Sec. 2, the TT-decomposition of a symmetric tensor ψ^{ab} is defined by

$$\psi^{ab} = \psi^{ab}_{\mathrm{TT}} + \psi^{ab}_{\mathrm{L}} + \psi^{ab}_{\mathrm{Tr}}, \tag{2}$$

where the longitudinal part is

$$\psi^{ab}_{\mathrm{L}} \equiv \nabla^a W^b + \nabla^b W^a - \tfrac{2}{3} g^{ab} \nabla_c W^c \equiv (LW)^{ab} \tag{3}$$

and the trace part is

$$\psi^{ab}_{\mathrm{Tr}} \equiv \tfrac{1}{3}\,\psi g^{ab}, \qquad \psi \equiv g_{cd}\psi^{cd}. \tag{4}$$

We use ∇_a to denote covariant differentiation and ∂_a ordinary differentiation. Our conventions are such that

$$\nabla_{[b}\nabla_{c]}\, V_a = \tfrac{1}{2}\, V_d R^d{}_{acb}, \tag{5}$$

$$R_{ab} = R^c{}_{acb}. \tag{6}$$

The transversality requirement leads to the only equations that need be solved in this procedure. This can be done, uniquely, with W^a the solution. The TT, L, and TR parts are mutually orthogonal. Upon conformal mapping of g_{ab} and ψ^{ab}, (2) maintains the same form and the equations guaranteeing transversality have the *same* solution W^a after the mapping as they did before the mapping. These conformal properties are demonstrated in Sec. 3.

In Sec. 4, we discuss application of the TT-decomposition to the initial-value problem. In Sec. 5, the decomposition is written in terms of orthogonal projection operators on the space of symmetric tensors. Deformations of conformal Riemannian manifolds are treated in the final section. In the Appendix, we show that when symmetries are present, one may readily construct from a given TT tensor others that are automatically TT.

2. TRANSVERSE–TRACELESS DECOMPOSITION

We define ψ^{ab}_{TT} in accordance with (2) by

$$\psi^{ab}_{\mathrm{TT}} \equiv \psi^{ab} - \tfrac{1}{3}\,\psi g^{ab} - (LW)^{ab}. \tag{7}$$

Let us suppose that both ψ^{ab} and g_{ab} are C^∞ tensor fields. For concreteness, we work here with tensors rather than with tensor densities; one need only multiply through by an appropriate power of $g^{1/2}$ for the densities. We note that the trace condition

$$g_{ab}\,\psi^{ab}_{\mathrm{TT}} = 0 \tag{8}$$

is satisfied by construction. The transversality requirement

$$\nabla_b\, \psi^{ab}_{\mathrm{TT}} = 0 \tag{9}$$

leads to covariant equations for the vector field W^a:

$$(DW)^a \equiv -\nabla_b (LW)^{ab} = -\nabla_b(\psi^{ab} - \tfrac{1}{3}\,\psi g^{ab}). \tag{10}$$

Notice that only the divergence of the *trace-free* part of ψ^{ab} enters (10). As a result, it is helpful to introduce an abbreviated notation. Define the algebraic operator Λ that projects any symmetric tensor into its trace-free part:

$$\Lambda^{ab}_{cd}\psi^{cd} \equiv (\delta_c^{(a}\delta_d^{b)} - \tfrac{1}{3}\, g^{ab} g_{cd})\psi^{cd} \equiv \psi^{ab} - \tfrac{1}{3}\,\psi g^{ab}. \tag{11}$$

Also, let the divergence of ψ^{ab} be denoted by $\nabla \cdot \psi$. Then (10) assumes the abbreviated form

$$DW \equiv -\nabla \cdot LW = -\nabla \cdot \Lambda\psi. \tag{10'}$$

Let us now discuss the basic properties of (10). The operator D is linear and second order as we see by inspection. Moreover, as we show below, this operator is positive-definite, Hermitian,[21] and its "harmonic"[22] functions are always orthogonal to the source (right-hand side) in (10). Thus (10) will always possess solutions[2] W unique up to conformal Killing vectors (see below). These solutions can be obtained by the eigenfunction method, assuming that D possesses a complete set of orthogonal eigenfunctions. We expect the spectrum of D to be discrete for closed spaces. If not, we assume its eigenvalues do not have zero as an accumulation point.[21] The operator inverse to D, D^{-1}, therefore exists and we may write

$$W = D^{-1}[-\nabla \cdot \Lambda\psi]. \tag{12}$$

To show that D is positive definite, we multiply $(DW)^a$ by W_a, form the global scalar product, and integrate by parts to show

$$(W, DW) = \tfrac{1}{2}(LW, LW), \tag{13}$$

where

$$(LW, LW) \equiv \int_M v_g\, (LW)_{ab}(LW)^{ab} \geq 0. \tag{14}$$

Thus D is positive unless $LW = 0$, a case discussed below. That D is Hermitian follows from a similar argument in which one integrates by parts twice to find

$$(V, DW) = (DV, W) \tag{15}$$

for any vectors V and W.

The right-hand side of (13) can vanish only if $LW = 0$. This means either $W = 0$ or $W =$ conformal Killing vector (CKV) of the metric. The condition for a CKV is, of course, not satisfied for an arbitrary ("conformally wild") metric. The condition for a CKV is given by $\pounds_W \tilde{g}_{ab} = 0$ or

$$\pounds_W g_{ab} = \lambda g_{ab} \tag{16}$$

for some scalar function λ, where $\pounds_W$ denotes the Lie derivative along W. Equation (16) is just

$$\nabla_a W_b + \nabla_b W_a = \lambda g_{ab}. \tag{17}$$

Taking the trace of both sides, we find

$$\lambda = \tfrac{2}{3}\,\nabla_c W^c. \tag{18}$$

Therefore, W is a CKV if and only if

$$\nabla^a W^b + \nabla^b W^a - \tfrac{2}{3}\, g^{ab}\nabla_c W^c \equiv (LW)^{ab} = 0. \tag{19}$$

It follows that the only nontrivial solutions of $DW = 0$ are CKV's, if they exist. Hence the nontrivial "harmonic" functions of D are CKV's. We shall now show that even if these "harmonic" solutions exist, they are always orthogonal to the right-hand side of (10) and, hence, can cause no difficulties in solving equation (10) by an eigenfunction expansion.

Denote the CKV's by $W^a = C^a$, where by definition

$LC = 0$. Form the scalar product of the right-hand side of (10) with C and integrate by parts to find

$$(-\nabla\cdot\Lambda\psi, C) = \tfrac{1}{2}(\Lambda\psi, LC) = 0. \tag{20}$$

Hence the source is in the domain of D^{-1} and D^{-1} gives the solution of (10) even in the presence of conformal symmetries.

The above results also show that the solution of (10) must be unique up to CKV's. However, since only $(LW)^{ab}$ enters in the definition (7) of ψ^{ab}_{TT}, CKV's cannot affect ψ^{ab}_{TT}.

The orthogonality of ψ^{ab}_{TT}, $(LW)^{ab}$, and $\frac{1}{3}\psi g^{ab}$ is easily demonstrated. We see readily that $\frac{1}{3}\psi g^{ab}$ is pointwise orthogonal to $(LW)^{ab}$ and to ψ^{ab}_{TT}, as $(LW)^{ab}$ and ψ^{ab}_{TT} are both trace-free. To show that ψ^{ab}_{TT} and $(LV)^{ab}$ are orthogonal for any vector V and any TT tensor, we have only to show that

$$(LV, \psi_{TT}) = 0, \tag{21}$$

which follows readily using integration by parts, Gauss's theorem, and $\nabla\cdot\psi_{TT} = 0$. We conclude, therefore, that the decomposition defined by (7) exists, is unique, and is orthogonal. The properties of (7) under conformal mappings of g_{ab} and ψ^{ab} are the subject of the next section.

One can further decompose the vector W^a uniquely into *its* transverse (t) and longitudinal (l) parts with respect to the metric g_{ab}. This splitting is orthogonal, as in (1). But W_t and W_l themselves "mix" under the conformal transformations defined in the next section. In any event, since this further splitting is well defined, we see that an arbitrary symmetric tensor field can be split into a sum of pure spin-two (TT), pure spin-one (W_t) and spin-zero ($g_{ab}\psi^{ab}$ and $\nabla\cdot W_l$) parts.

Also, it is easily verified in this procedure that a given tensor that is already TT has no L or Tr parts; a pure L tensor has no TT or Tr parts; and a pure Tr tensor has no TT or L parts.

3. CONFORMAL TRANSFORMATIONS

Understanding the conformal properties of (7) is of great interest in itself and is essential in the application of these results to the gravity initial-value problem (Sec. 4). A space conformally related to (M,g) is $(M,\bar{g})$, where

$$\bar{g}_{ab} = \phi^4 g_{ab}. \tag{22}$$

Therefore, we have for the connection coefficients

$$\bar{\Gamma}_{bc}{}^{a} = \Gamma_{bc}{}^{a} + 2(\delta^a_b\nabla_c \ln\phi + \delta^a_c\nabla_b \ln\phi - g_{bc}\nabla^a \ln\phi), \tag{23}$$

with $\phi(x)$ an arbitrary real positive scalar function. The freely given tensor ψ^{ab}, which is to be decomposed, will *also* be mapped conformally by the transformation

$$\bar{\psi}^{ab} = \phi^{-10}\psi^{ab}. \tag{24}$$

Thus, on $(M,\bar{g})$ we will decompose $\phi^{-10}\psi^{ab}$, not ψ^{ab} itself. We shall prove that decomposing $\phi^{-10}\psi^{ab}$ on $(M,\bar{g})$ is completely equivalent to decomposing ψ^{ab} on (M,g), for an *arbitrary* choice of $\phi(x)$. This fact is of essential importance in the application of (7) to the gravitational initial-value problem. The choice of (24) is not arbitrary, but is dictated by the form of (7), as we shall see below. If ψ^{ab} were a tensor density of weight one, we would use in place of (24), $\bar{\psi}^{ab} = \phi^{-4}\psi^{ab}$. If it were of weight 5/3, we would have $\bar{\psi}^{ab} = \psi^{ab}$. This follows from (24) and $\bar{g}^{1/2} = \phi^6 g^{1/2}$. These cases are also important in the initial-value problem.[18]

Returning to the tensor case, we rewrite (7), using Eqs. (22), (23), (24), to obtain

$$\psi^{ab}_{TT} = \phi^{10}(\bar{\psi}^{ab} - \tfrac{1}{3}\bar{\psi}\bar{g}^{ab}) - (LW)^{ab}, \tag{25}$$

where $\bar{\psi} \equiv \bar{g}_{cd}\bar{\psi}^{cd}$. We now wish to transform the longitudinal part, $(LW)^{ab}$. Substitution of (22) and (23) into (3) gives

$$(LW)^{ab} = \phi^4(\bar{\nabla}^a W^b + \bar{\nabla}^b W^a - \tfrac{2}{3}\bar{g}^{ab}\bar{\nabla}_c W^c) \equiv \phi^4(\bar{L}W)^{ab}, \tag{26}$$

or

$$(\bar{L}W)^{ab} = \phi^{-4}(LW)^{ab}. \tag{27}$$

By using (27) and multiplying through by ϕ^{-10}, (25) becomes

$$\phi^{-10}\psi^{ab}_{TT} = (\bar{\psi}^{ab} - \tfrac{1}{3}\bar{g}^{ab}\bar{\psi}) - \phi^{-6}(\bar{L}W)^{ab}. \tag{28}$$

Let us put $\bar{T}^{ab} \equiv \phi^{-10}\psi^{ab}_{TT}$. We note that

$$\bar{g}_{ab}\bar{T}^{ab} = 0 \tag{29}$$

by construction. Furthermore, $\bar{T}^{ab}$ will also be transverse on $(M,\bar{g})$ provided $\bar{\nabla}_b\bar{T}^{ab} = 0$, or

$$-\bar{\nabla}_b[\phi^{-6}(\bar{L}W)^{ab}] = -\bar{\nabla}_b(\bar{\psi}^{ab} - \tfrac{1}{3}\bar{\psi}\bar{g}^{ab}). \tag{30}$$

However, for any choice of ϕ, this equation for the "vector potential" W^a is already satisfied by the *same* vector W^a that satisfied (10)! This statement is proved as follows: The left-hand side of (30) is simply

$$\phi^{-6}[-\bar{\nabla}_b(\bar{L}W)^{ab} + 6(\bar{L}W)^{ab}\bar{\nabla}_b \ln\phi]. \tag{31}$$

Since $\ln\phi$ is a scalar, we have $\bar{\nabla}_b \ln\phi = \nabla_b \ln\phi = \partial_b \ln\phi$. Moreover, using (22), (23), and (27), we have

$$-\bar{\nabla}_b(\bar{L}W)^{ab} = -\phi^{-4}[\nabla_b(LW)^{ab} + 6(LW)^{ab}\nabla_b \ln\phi]. \tag{32}$$

By using (31) and (32), the left-hand side of (30) becomes

$$-\phi^{-10}\nabla_b(LW)^{ab}. \tag{33}$$

From (22), (23), and (24), the right-hand side of (30) becomes

$$-\phi^{-10}\nabla_b(\psi^{ab} - \tfrac{1}{3}\psi g^{ab}). \tag{34}$$

It is just this last result that uniquely dictates the conformal transformation (24) on ψ^{ab}. From (33) and (34) we see that (30) implies

$$-\nabla_b(LW)^{ab} = -\nabla_b(\psi^{ab} - \tfrac{1}{3}\psi g^{ab}), \tag{10}$$

which is just Eq. (10). Conversely, Eq. (10) implies (30), so that (10) and (30) are completely equivalent for any choice of ϕ and, therefore, have precisely the same solution $W^a(x)$. The "harmonic" functions of the operator $-\bar{\nabla}_b[\phi^{-6}(\bar{L}W)^{ab}]$ are precisely the same as those of $-\nabla_b(LW)^{ab} \equiv (DW)^a$, because $-\bar{\nabla}_b[\phi^{-6}(\bar{L}W)^{ab}]$ is a positive-definite operator which can only vanish if $W^a = 0$ or if $W^a =$ CKV of $\bar{g}_{ab}$. Since $(\bar{L}W)^{ab} = \phi^{-4}(LW)^{ab}$, we see that $(\bar{L}W)^{ab} = 0$ if and only if $(LW)^{ab} = 0$. This is not surprising, as it only says that if W^a is a CKV of g_{ab}, then it is also a CKV of any conformally related metric, as one might expect. That is, the condition for a CKV is conformally invariant.

Hence, we identify $\bar{T}^{ab}$ with $\bar{\psi}_{\rm TT}^{ab}$, yielding the result

$$\bar{\psi}_{\rm TT}^{ab} = \phi^{-10}\psi_{\rm TT}^{ab}. \tag{35}$$

Equation (21) may now be written

$$\bar{\psi}_{\rm TT}^{ab} = (\bar{\psi}^{ab} - \tfrac{1}{3}\bar{g}^{ab}\bar{\psi}) - \phi^{-6}(\bar{L}W)^{ab}. \tag{36}$$

We see that the longitudinal part of $\bar{\psi}^{ab}$ is simply

$$\bar{\psi}_L^{ab} = (\bar{L}\bar{W})^{ab} = \phi^{-6}(\bar{L}W)^{ab} = \phi^{-10}(LW)^{ab}. \tag{37}$$

In summary, given ψ^{ab} on (M,g), we decompose it by means of

$$\psi^{ab} = \psi_{\rm TT}^{ab} + \psi_{\rm L}^{ab} + \psi_{\rm Tr}^{ab}, \tag{2}$$

with $\psi_{\rm L}^{ab}$ and $\psi_{\rm Tr}^{ab}$ given by (3), (4), and the solution of (10). On a conformally related manifold $(M,\bar{g})$, the tensor $\bar{\psi}^{ab} = \phi^{-10}\psi^{ab}$ decomposes in the same way:

$$\bar{\psi}^{ab} = \bar{\psi}_{\rm TT}^{ab} + \bar{\psi}_{\rm L}^{ab} + \bar{\psi}_{\rm Tr}^{ab}, \tag{38}$$

where

$$\bar{\psi}_{\rm TT}^{ab} = \phi^{-10}\psi_{\rm TT}^{ab}, \tag{35}$$

$$\bar{\psi}_{\rm L}^{ab} = \phi^{-10}\psi_{\rm L}^{ab}, \tag{37'}$$

$$\bar{\psi}_{\rm Tr}^{ab} = \phi^{-10}\psi_{\rm Tr}^{ab} \tag{39}$$

with the vector W^a determining the longitudinal part being the same for $\bar{\psi}_L^{ab}$ as for ψ_L^{ab}.

One can see from the above that the form of the decomposition may be viewed as being *determined* by conformal invariance, for we know *without* performing a decomposition that if a tensor is TT with respect to a given metric g_{ab}, that ϕ^{-10} times the tensor will also be TT with respect to the conformally transformed metric $\phi^4 g_{ab}$.[18] It is actually this latter observation whose significance led to the present method of decomposition. In particular, one can see that the form for the longitudinal part is crucial because of (27).

4. INITIAL-VALUE PROBLEM OF GENERAL RELATIVITY

Because of its conformal properties, the decomposition (7) is ideally suited for use in the gravitational initial-value problem. For simplicity, we shall first describe principally the case involving vacuum gravity fields ${}^{(4)}R_{\mu\nu} = 0$.

The initial-value problem is to construct a spacelike Riemannian three-manifold (M,g) and a symmetric tensor density of weight one, π^{ab}, such that

$$\nabla_b \pi^{ab} = 0, \tag{40}$$

$$g^{-1/2}(\pi_{ab}\pi^{ab} - \tfrac{1}{2}\pi^2) - g^{1/2}R = 0, \tag{41}$$

where R is the scalar curvature of (M,g). The conformal approach to this problem is to solve (40) in a conformally invariant manner, then to choose the conformal factor ϕ in such a way as to satisfy (41). Equations (40) and (41) are the Gauss–Codazzi equations giving necessary and sufficient conditions for the embedding of (M,g) with second fundamental tensor

$$K_{ab} = g^{-1/2}(\tfrac{1}{2}\pi g_{ab} - \pi_{ab}) \tag{42}$$

in a spacetime satisfying Einstein's equations ${}^{(4)}R_{\mu\nu} = 0$. For global analysis of this problem, it is convenient to convert (40) and (41) into second-order elliptic partial differential equations.

Here let us treat the case where (M,g) is to be embedded in such a way that its volume "Hubble parameter" is constant on the surface. The specific variable[11] which has proven to be of fundamental significance for this and other purposes is

$$\tau \equiv \tfrac{2}{3} g^{-1/2}\pi = \tfrac{4}{3}K, \tag{43}$$

which measures the rate of change of the local volume elements of (M,g) per unit volume, per unit proper time, i.e., per unit proper distance orthogonal to the surface. The "maximal" case is $\tau = 0$, but in general we simply assume that $\tau = $ const., i.e., $\nabla_a\tau = \partial_a\tau = 0$ on (M,g).

In this case (40) can be written as

$$\nabla_b(\pi^{ab} - \tfrac{1}{3}\pi g^{ab}) = 0 \tag{40'}$$

since by hypothesis $\nabla^a\pi = 0$. Thus (36) requires that we construct a TT tensor density $\sigma^{ab} \equiv \pi^{ab} - \frac{1}{3}\pi g^{ab}$. For this purpose, we give arbitrarily a tensor ψ^{ab} and construct its TT part as above by solving (10) for W^a. We then set $\sigma^{ab} = g^{1/2}\psi_{\rm TT}^{ab}$.

Of course, the variables g_{ab}, σ^{ab} will not in general satisfy (41). However, we now map them conformally onto new variables which do satisfy (41), i.e., such that

$$\bar{g}^{-1/2}(\bar{\pi}_{ab}\bar{\pi}^{ab} - \tfrac{1}{2}\bar{\pi}^2) - \bar{g}^{1/2}\bar{R} = 0. \tag{44}$$

We first note that

$$\pi^{ab} = \sigma^{ab} + \tfrac{1}{2}g^{1/2}g^{ab}\tau, \tag{45}$$

which follows from the definition of σ^{ab} and τ. We know that the transformation $\bar{\sigma}^{ab} = \phi^{-4}\sigma^{ab}$ preserves the TT character of σ^{ab}, so that

$$\bar{\pi}^{ab} = \bar{\sigma}^{ab} + \tfrac{1}{2}\bar{g}^{1/2}\bar{g}^{ab}\tau = \phi^{-4}\sigma^{ab} + \tfrac{1}{2}\phi^2 g^{1/2}g^{ab}\tau \tag{46}$$

will satisfy (40) in the form $\bar{\nabla}_b\bar{\pi}^{ab} = 0$. Substituting (39) into (37) and using the well-known formula

$$\bar{R} = \phi^{-4}R + 8\phi^{-5}\Delta\phi, \tag{47}$$

with $\Delta\phi \equiv -g^{ab}\nabla_a\nabla_b\phi$, we obtain the equation determining ϕ:

$$(8\Delta + R)\phi = \mathfrak{M}\phi^{-7} - \tfrac{3}{8}\tau^2\phi^5, \tag{48}$$

where $\mathfrak{M} \equiv g^{-1}g_{ac}g_{bd}\sigma^{ab}\sigma^{cd}$. Equation (48) is a quasilinear elliptic equation determining $\phi(x)$. All of its coefficients involve only known or given functions, so it is not coupled back to the momentum constraint $\bar{\nabla}_b\bar{\pi}^{ab} = 0$, the solution of which, as has been pointed out, is a conformally invariant problem, when $\tau = $ const.

Quasilinear elliptic equations such as (48) are discussed, for example, in the treatise of Ladyzhenskaya and Ural'tseva.[23] The existence of solutions, particularly real positive[24] solutions $0 < \phi < \infty$, depends to a large extent on the detailed nature of the nonlinear terms and the possible values of their coefficients. For the particular form exhibited by (48), N. O'Murchadha and the author[25] have classified all cases of physical interest. For example, we have shown that solutions ϕ, such that $0 < \phi < \infty$ on M, exist for any $\mathfrak{M} > 0$, $\tau \neq 0$, and for any choice of the initial metric g_{ab} on a closed C^∞ manifold.

The case $\tau = 0$ is discussed below. It is exceptional only for *closed* manifolds. For asymptotically flat (open) spaces, existence of solutions has also been demonstrated. Moreover, for (48), we have shown that the solution is *unique* for closed *or* open spaces. (In the asymptotically flat case, the value of ϕ at infinity must also be specified, of course.) Together with the TT-decomposition, these results give a highly satisfactory description of the initial-value problem for the vacuum gravitational field. The same conclusion holds for the gravitational field with sources, described below.

The case $\tau = 0$ is exceptional for closed spaces and it is worthwhile to see why this is true. We shall see that if one chooses an initial metric with $R < 0$ everywhere on M, no solution of (48) exists if $\tau = 0$. Of course, the set of configurations with $\tau = 0$ is of "measure zero" in the full configuration space $\tilde{S} \times \{\text{possible choices of } \tau\}$. Nevertheless, the discussion of this case sheds light on the meaning of the conformal method in general. Let us first, then, examine the conformal properties of (48) itself.

Had we started not from g_{ab}, but from any other metric $g'_{ab} = \nu^4 g_{ab}$ in the same conformal equivalence class as g_{ab}, we would of course have $\sigma'^{ab}_{\mathrm{TT}} = \nu^{-4}\sigma^{ab}_{\mathrm{TT}}$. The analog of (48) would be of the same form:

$$(8\Delta' + R')\phi' = \mathfrak{M}'(\phi')^{-7} - \tfrac{3}{8}\tau^2(\phi')^5, \tag{49}$$

where $\mathfrak{M}' = \nu^{-12}\mathfrak{M}$, $\tau' = \tau$, and $\phi' = \phi\nu^{-1}$. Note that $(8\Delta + R)$ is just the conformally invariant scalar Laplacian, so the fact that (48) is itself conformally form-invariant is not surprising. From this we see that the solution of (48) admits the conformal "gauge" transformation $\phi' = \phi\nu^{-1}$. Therefore, the uniqueness of solutions to (48) for given *conformal equivalence classes* of initial data is only uniqueness *modulo* this gauge behavior. It is clear, however, that the final metric $\bar{g}_{ab}$ and momentum $\bar{\sigma}^{ab}_{\mathrm{TT}}$ are themselves unique with respect to the given conformal class of initial data. Thus,

$$\bar{g}_{ab} = \phi^4 g_{ab} = (\phi')^4 g'_{ab}, \tag{50}$$

$$\bar{\sigma}^{ab}_{\mathrm{TT}} = \phi^{-4}\sigma^{ab}_{\mathrm{TT}} = (\phi')^{-4}\sigma'^{ab}_{\mathrm{TT}}. \tag{51}$$

The conclusion is that *the complete set of initial value equations for W^a and ϕ is conformally covariant. Their solutions transform by the rules $W^{a'} = W^a$, $\phi' = \phi\nu^{-1}$. From these solutions one obtains a unique initial-data set $\bar{g}_{ab}$, $\bar{\sigma}^{ab}_{\mathrm{TT}}$, τ.*

We now return to the case $\tau = 0$, $\mathfrak{M} > 0$. We see from (44) that $\bar{R}$ must be positive on all of M. If we choose an initial metric such that $R(g) < 0$ everywhere on M, then if a positive ϕ satisfying (48) exists, it must map from a space with $R < 0$ everywhere to one with $\bar{R} > 0$ everywhere. However, it is easy to show that no such mapping can exist for M *closed*. From (47), we have

$$8\Delta\phi = -R\phi + \bar{R}\phi^5. \tag{52}$$

Integrating (47) over a closed M gives

$$0 = \int_M v_g(-R\phi + \bar{R}\phi^5), \tag{53}$$

which cannot be satisfied in the present case.

However, the argument in the case $\tau = 0$ does not apply to asymptotically flat (open) spaces, as one cannot there *discard* the gradient of ϕ at infinity. In fact, this very boundary integral determines the mass-at-infinity of the gravitational configuration.[26]

What this simple argument also demonstrates is that, whereas the scalar curvature is not a conformally invariant object in a local sense, *globally* a *uniform sign* of R over a closed M is conformally significant. That is, if R has one sign only on M and if $\bar{R}$ also has only one sign (which is not true, in general, if $\tau \neq 0$), these signs must be the same.

This argument is closely related to *Yamabe's theorem*:[27] Every compact C^∞ Riemannian manifold of dimension ≥ 3 can be conformally deformed to a C^∞ Riemannian structure of *constant* scalar curvature. Let the constant be called k. Then $\mathrm{sgn}(k)$ provides a convenient conformally invariant "index" for conformal equivalence classes of (closed) Riemannian manifolds. For the exceptional case $\tau = 0$, the conformal classes with $\mathrm{sgn}(k) = 0$ and $\mathrm{sgn}(k) = -1$ are here ruled out; whereas, if $\tau \neq 0$, they are not. For asymptotically flat spaces, no classification such as that provided by Yamabe's theorem is possible.

For a closed universe, $\tau = 0$ corresponds to the moment of maximum expansion. According to the argument above, we have the result that a closed vacuum gravitational configuration with $\mathrm{sgn}(k) = -1$ or $\mathrm{sgn}(k) = 0$ cannot correspond to a moment of maximum expansion. Examples of closed vacuum universes with no moment of maximum expansion are known.[28] These universes expand for all time at an ever-slowing rate which only approaches $\tau = 0$ asymptotically as the volume approaches infinity, which means the closed space is becoming effectively "open".

Conversely, at a moment of maximum expansion, $\mathrm{sgn}(k) = +1$ if $\mathfrak{M} > 0$. If $\tau = 0$ *and* $\mathfrak{M} = 0$ (vanishing shear), we have a "moment of time symmetry",[29,30] for which $\mathrm{sgn}(k) = 0$ is the only permissible case if M is closed.

To conclude this part of the discussion, we repeat that for $\mathfrak{M} > 0$, only cases involving $\tau = 0$ everywhere on M lead to any restrictions on the choice of conformal equivalence classes of initial data in the construction of solutions to the constraint equations.

In general, we see that when $\tau = \text{const.}$, (40) and (41) split into two separate problems. We give freely (M, g) and ψ^{ab}. We solve the linear elliptic Eqs. (10) for W^a, and thereby construct $\sigma^{ab}_{\mathrm{TT}}$. Substituting g_{ab}, $\sigma^{ab}_{\mathrm{TT}}$, and τ into (41), we find ϕ. The final initial data set satisfying the complete set of constraints is therefore $(M, \bar{g})$ with $\bar{g}_{ab} = \phi^4 g_{ab}$, and $\bar{\pi}^{ab}$ given by (39). This means that the initial-value problem on surfaces $\tau = \text{const.}$ is an uncoupled elliptic second-order system of four Eqs. (10) and (48) for four functions W^a and ϕ. Of these, the three Eqs. (10) are linear and (48) is quasilinear.

The conformal treatment of the initial-value equations can be generalized to cases where $\tau(x) \neq \text{const.}$ is a prescribed function. In place of the momentum conditions $\nabla_b\sigma^{ab} = 0$ for $\tau = \text{const.}$, one has

$$\nabla_b\sigma^{ab} = -\tfrac{1}{2}g^{1/2}\nabla^a\tau, \tag{54}$$

where σ^{ab} is still trace-free. We can view this problem as requiring the construction of a traceless tensor in purely *longitudinal* form $(LZ)^{ab}$. Equation (54) determines $(LZ)^{ab}$; however, *this* procedure is not independent of ϕ. Equations (54) and (48) are now coupled. On such surfaces the gravitational initial variables are not pure spin-two objects. The presence of $\nabla^a\tau$ introduces effectively a vector part Z to the complete set of initial variables. Of course, to the solutions σ^{ab} of (54) itself, may always be added a free (unconstrained) field in the form of some TT-variable. So, even in this case, TT momenta describe free gravitational fields ("waves"). Moreover, regardless of the presence of $\nabla^a\tau \neq 0$, the *intrinsic* conformal geometry of the surface still corresponds to the pure spin-two part of the field. This is true because the three-dimensional conformal curvature tensor is *identically* TT regardless of the value of τ on the surface. Thus, the vacuum gravitational field behaves somewhat like an electromagnetic field with sources when $\tau \neq$ const.

We now wish to point out how matter or other field sources enter (48). The form of (44) is[32,33]

$$\bar{g}^{-1/2}(\bar{\pi}_{ab}\bar{\pi}^{ab} - \tfrac{1}{2}\bar{\pi}^2) - \bar{g}^{1/2}\bar{R} = -16\pi\bar{g}^{1/2}\bar{T}^*_*, \tag{55}$$

where $\bar{T}^*_* = \bar{T}^\mu{}_\nu u^\nu u_\mu$, $T^\mu{}_\nu$ = matter tensor, and u^μ is the unit timelike four-vector normal to the surface. Physically, $\bar{T}^*_*$ is the positive-definite scalar measuring the mass or energy per unit proper three-volume on the surface. The key question here is: Can $\bar{T}^*_*$ itself be freely specified, or only specified up to some conformal factor? Both dimensional arguments and arguments based on the free electromagnetic field[32] as source point to the fact that one can only give $\bar{T}^*_*$ *up to* a conformal factor. Thus, for example, the conformal properties of the electromagnetic field coupled to gravity using a "3 + 1" formalism identical to that of the present paper show that one may freely give T^*_*, where $\bar{T}^*_* = \phi^{-8}T^*_*$. This choice is consistent with the decoupling of the momentum and energy constraints (see below) *and* with the fact that the *electromagnetic*-initial-value problem on the surface must simultaneously be satisfied.

In place of (48), one finds[32,33]

$$(8\Delta + R)\phi = \mathfrak{M}\phi^{-7} - \tfrac{3}{8}\tau^2\phi^5 + 16\pi T^*_*\phi^{-3}, \tag{56}$$

where all coefficients are known (see below), and where the sign of each coefficient on the right-hand-side is known. Again, one can show[25,32] the existence of a unique solution for all conformal equivalence classes of initial data, except those with $\tau = 0$ have to be treated separately, as above. Choquet-Bruhat[34] treated the case with $\tau = 0$ and $\bar{T}^*_* = T^*_*$, i.e., the energy density scalar *completely* specified in advance. However, this is not consistent with massless, integral spin sources nor with dimensional analysis. It is an attempt to specify *more* data about the sources that is consistent with the physics of gravity coupled to other fields. Not surprisingly, she found a number of serious restrictions on the existence and uniqueness of solutions for certain gravitational-matter configurations. Understanding these restrictions more deeply could be important in further elucidating the physical content of initial-value problems with sources.

We have already indicated how matter sources enter (48). Likewise, sources may be inserted into the momentum constraints in the form[11,32,33]

$$\nabla_b\sigma^{ab} = 8\pi g^{1/2}g^{ab}T^*_b \tag{57}$$

when τ = const. Here $G = c = 1$, T^μ_ν is the matter tensor, and $T^*_b = B^\nu_b u_\mu T^\mu_\nu$. The factor B^ν_b projects onto the surface, for which u^μ is the unit normal four-vector. The construction of a trace-free σ^{ab} satisfying (57) is again a conformally invariant problem *not* coupled to (48) (just as before) if we conformally map T^*_b by the transformation[32,35]

$$T^*_b \to \bar{T}^*_b = \phi^{-6}T^*_b. \tag{58}$$

This means that here $g^{1/2}T^*_b$ is freely specifiable, not T^*_b itself. Effectively, one may only prescribe the directional properties of the matter current T^*_b but not its absolute magnitude. Observe that T^*_b does *not* enter (56); only the solution σ^{ab} of (54) enters (56) and, of course, T^*_*. So none of the discussion of (56) has to be altered in this generalized procedure. Therefore, the inclusion of matter currents occasions no difficulties. Again, we may allow $\tau(x) \neq$ const., but the equations then become coupled, introducing, however, no difficulties in principle.

Lastly, we mention that there is an elliptic "sandwich" version of the initial value problem.[36] Here, one gives freely, on two nearby slices τ_1 = const., τ_2 = const., and two infinitesimally different conformal metrics. One finds ϕ and W^a by a method similar to our discussion above. The proper orthogonal distance N between the two surfaces is determined by an elliptic equation[11] for N, resulting from the demand that the τ's be two infinitesimally differing constants on the two surfaces. The "sandwich" problem becomes five elliptic equations for the five variables ϕ, W^a, and N. The W^a equations are linear and the ϕ and N equations are quasilinear. These five equations are coupled. Coupled elliptic systems are very hard to analyse, no sufficiently powerful mathematical theorems being readily available. However, this *particular* set has simplifying features that permit this "conformal thin-sandwich" result to be analyzed.

5. PROJECTION OPERATORS FOR THE TT-DECOMPOSITION

In this section, following a notation close to that of Berger and Ebin,[7] I shall denote the operators relevant to the TT-decomposition by

$$(\tilde{\delta}\psi)^a \equiv -\nabla_b(\psi^{ab} - \tfrac{1}{3}\psi g^{ab}) \equiv -[\nabla\cdot(\Lambda\psi)]^a, \tag{59}$$

where ψ^{ab} is a symmetric tensor. Omitting indices, we write the solution of (10) as

$$W = D^{-1}[\tilde{\delta}\psi]. \tag{60}$$

The longitudinal part of the tensor ψ is written

$$\psi_L = LW = LD^{-1}[\tilde{\delta}\psi]. \tag{61}$$

The trace part is given by

$$\psi_{Tr} = (I - \Lambda)\psi, \tag{62}$$

where I denotes the identity operator. Therefore,

$$\psi_{TT} = (\Lambda - LD^{-1}\tilde{\delta})\psi. \tag{63}$$

The appropriate projection operators acting on ψ are thus

$$P_{TT} = \Lambda - LD^{-1}\tilde{\delta}, \tag{64}$$

$$P_L = LD^{-1}\tilde{\delta}, \tag{65}$$

$$P_{Tr} = I - \Lambda, \tag{66}$$

where $I = P_{TT} + P_L + P_{Tr}$.

The space of pure trace tensors may be written[37] $\Lambda^{-1}(0)$. The space of longitudinal tensors may be written $L(V^1)$, where V^1 denotes the space of vectors on M. We now seek a further characterization of the spaces of TT-tensors as defined in this paper.

First we note that the operator L^+ adjoint to L is $L^+ = \frac{1}{2}\tilde{\delta}$. The vector "Laplacian" D of equation (10) is given by

$$D = \tilde{\delta} L. \tag{67}$$

The dimension of the kernel $D^{-1}(0)$ for this operator equals the number of linearly independent conformal Killing vectors admitted by (M,g). This dimension is conformally invariant. Its maximum value in an N-space is $\frac{1}{2}(N+1)(N+2)$. The maximum is achieved if and only if the space is conformally flat.

Now we define the *tensor* "Laplacian" $\tilde{\tilde{\Delta}}$ by

$$\tilde{\tilde{\Delta}} = L\tilde{\delta}. \tag{68}$$

This is a linear, second-order, elliptic, positive-definite operator which vanishes if and only if the tensor it acts on is TT as defined in this paper. To see this, note that

$$(\psi, \tilde{\tilde{\Delta}}\psi) = 2(\tilde{\delta}\psi, \tilde{\delta}\psi), \tag{69}$$

which vanishes if and only if $\tilde{\delta}\psi = 0$. We can have $\tilde{\delta}\psi = 0$ if ψ is TT or if the tracefree part of ψ is transverse. However, according to our definition of TT tensors, if the tracefree part is transverse, then the longitudinal part must vanish and we have $\psi_{\mathrm{TT}} = \Lambda\psi$, i.e., the tracefree part *is* the TT part in this case. Hence, it follows that the space of TT tensors can be written as $\tilde{\delta}^{-1}(0)$ or $\tilde{\tilde{\Delta}}^{-1}(0)$. It follows also that these kernels are conformally invariant because the TT-property is preserved under the conformal mappings we have defined. Therefore, one sees by virtue of their conformal properties an interesting "duality" between the vector operator $D = \tilde{\delta}L$ and the tensor operator $\tilde{\tilde{\Delta}} = L\tilde{\delta}$.[38]

Again following the notation of Berger and Ebin,[7] we write the present splitting of the space of C^∞ symmetric tensor fields on (M,g) as

$$C^\infty(S^2) = \tilde{\tilde{\Delta}}^{-1}(0) \oplus L(V^1) \oplus \Lambda^{-1}(0), \tag{70}$$

where the summands are orthogonal.

Using different operators than those we have employed here, Berger and Ebin[7] achieved an orthogonal TT decomposition in the special case that $R = \text{const}$. Yamabe's theorem[27] should provide a link between their decomposition and the one defined in this paper.

6. DEFORMATIONS OF CONFORMAL RIEMANNIAN MANIFOLDS

Any infinitesimal deformation of a Riemannian geometry may be represented by a symmetric tensor δg_{ab}, where δ now stands for "variation," not for the negative divergence as in Sec. 5. Since any symmetric tensor may be decomposed by our procedure, we may write

$$\delta g_{ab} = \delta g_{ab}^{\mathrm{TT}} + (LW)_{ab} + \tfrac{1}{3} g_{ab} g^{cd} \delta g_{cd}. \tag{71}$$

On the other hand, by definition, any small variation of a conformal metric $\tilde{g}_{ab} = (\det g)^{-1/3} g_{ab}$ must be tracefree. Hence, the last term of (71) represents an infinitesimal conformal transformation which cannot affect the underlying conformal geometry.

Consider an infinitesimal shift of coordinates $x^a \to x^a - W^a$. The change induced in $\tilde{g}_{ab}$ is just

$$\pounds_W \tilde{g}_{ab} = (\det g)^{-1/3} (LW)_{ab}. \tag{72}$$

Hence, the second term on the right of (71) denotes a relabeling of coordinates in the underlying conformal geometry and thus to no change in the intrinsic conformal geometry itself. Thus, the only term of (71) corresponding to a variation of the intrinsic conformal geometry is $\delta g_{ab}^{\mathrm{TT}}$. Therefore, every deformation in conformal superspace $\tilde{\mathcal{S}}$ is represented by a TT tensor only, and conversely. Thus, in passing from one point of $\tilde{\mathcal{S}}$ to a neighboring point, only TT-tensors need be considered.

There is a method of constructing TT tensors that does not rely on any TT-decomposition as such. In three dimensions, the conformal curvature tensor density[18] is defined by

$$\tilde{\beta}^{ab} \equiv \tfrac{1}{2} (\det g)^{1/3} (\epsilon^{efa} g^{bm} + \epsilon^{efb} g^{am}) \nabla_e R_{fm}, \tag{73}$$

where ϵ^{efa} is the unit alternating tensor density. If $\delta g_{ab} = \lambda g_{ab}$, then one can readily show that $\delta\tilde{\beta}^{ab} = 0$. Let $\beta^{ab} = (\det g)^{-1/3} \tilde{\beta}^{ab}$. Then we also have $\delta\beta^a_b = 0$. $\tilde{\beta}^{ab}$ is identically symmetric and TT, giving rise to the conclusion that the conformal geometry represents the pure spin-2 part of the full Riemannian 3-geometry, as we described earlier.

If we construct from $\tilde{\beta}^{ab}$ a conformally invariant scalar density of weight one and integrate it over M, then the functional derivative of this integral with respect to g_{ab} will be automatically a TT density of a weight 1. The weight of this tensor can be changed at will, of course. As an example,

$$\mu^{ab} = \delta/\delta g_{ab} \int_M (\beta^c_d \beta^d_c)^{1/2} d^3x \tag{74}$$

is a TT tensor density of weight 1.[39] It is worthwhile to note that the functional derivation of a TT tensor density of coordinate weight 1, as in (74), may be readily used to prove that such an object *must* transform by the rule $\bar{\mu}^{ab} = \phi^{-4}\mu^{ab}$ under a conformal mapping $\bar{g}_{ab} = \phi^4 g_{ab}$. Thus the *tensor* form $\psi^{ab}_{\mathrm{TT}} = g^{-1/2}\mu^{ab}$ must transform as $\overline{\psi}^{ab}_{\mathrm{TT}} = \phi^{-10}\psi^{ab}_{\mathrm{TT}}$, as shown in Sec. 2.

ACKNOWLEDGMENTS

I thank V. Bargmann, S. Deser, R. Gowdy, D. Markiewicz, and especially N. O'Murchadha and R. Wald for discussions and helpful remarks.

APPENDIX: TT TENSORS AND SYMMETRIES

Let X^a be a CKV of g_{ab} and consider the tensor ψ^{TT}_{ab}. By using the well-known rules for interchanging $\pounds_X$ and ∇_b, one can easily verify that

$$\psi'^{\mathrm{TT}}_{ab} = \psi^{\mathrm{TT}}_{ab} + \pounds_X \psi^{\mathrm{TT}}_{ab} + \tfrac{1}{3}\psi^{\mathrm{TT}}_{ab} \nabla_c X^c \tag{A1}$$

is also TT with respect to g_{ab}. Similar results hold for ψ^{ab}_{TT} and for different weights. Therefore, although CKV's do not show up directly in the construction of ψ^{TT}_{ab}, as we explained, they do give automatically other TT tensors.

The *transverse* decomposition[6,7] of a tensor T_{ab} is defined by

$$T_{ab} = T^T_{ab} + \nabla_a V_b + \nabla_b V_a, \tag{A2}$$

for some unique V_a. If the metric g_{ab} possesses Killing vectors $Y(\pounds_Y g_{ab} = 0)$, then, similarly to the above,

$$T'^T_{ab} = T^T_{ab} + \pounds_Y T^T_{ab} \tag{A3}$$

is also automatically transverse with respect to g_{ab}. These results relate, respectively, to the "stratification" of $\tilde{\mathcal{S}}$, and to the "stratification" of $\mathcal{S}$.[8]

*Supported in part by National Science Foundation Grant GP30799X to Princeton University.

[1]See, for example, the article by R. Arnowitt, S. Deser, and C. W. Misner in *Gravitation*, edited by L. Witten (Wiley, New York, 1962).

[2]G. deRham, *Varietés Differentiables* (Hermann, Paris, 1955).

[3]For manifolds with indefinite metrics, such as spacetime, the decomposition treated in this paper may also be used. However, the equations to be solved become hyperbolic (for spacetime) instead of elliptic, necessitating a different choice of boundary conditions. The hyperbolic case will be treated elsewhere.

[4]See, for example, C. W. Misner, Phys. Rev. **186**, 1319 (1969); R. Gowdy, Phys. Rev. Lett. **27**, 826 (1971).

[5]The behavior of tensors at spacelike infinity may also be treated as done by R. Geroch, J. Math. Phys. **13**, 956 (1972), by adding a conformal "point at infinity", thereby compactifying the asymptotically flat space. This "conformal compactification" appears to be quite compatible with the present techniques.

[6]S. Deser, Ann. Inst. Henri Poincaré 7, 149 (1967). The present decomposition is similar to one of Deser, but has a different trace term. Our procedure is similar to one of Deser's transverse (not TT) decompositions [his equation (10)], but ours is applied to the *trace-free* part of a tensor T_{ab} whereas his is applied to T_{ab} itself. I thank Professor Deser for correspondence concerning this point.

[7]Cf. M. Berger and D. Ebin, J. Diff. Geom. 3, 379 (1969).

[8]A. E. Fischer, in *Relativity*, edited by M. Carmeli, S. I. Ficklin, and L. Witten (Plenum, New York, 1970).

[9]B. DeWitt, article in Ref. 8.

[10]R. F. Baierlein, D. Sharp, and J. A. Wheeler, Phys. Rev. **126**, 1864 (1962).

[11]J. W. York, Phys. Rev. Lett. **28**, 1082 (1972).

[12]P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).

[13]K. Kuchař, J. Math. Phys. **13**, 768 (1972).

[14]Cf. B. DeWitt, Phys. Rev. **160**, 1113 (1967).

[15]Cf. C. W. Misner, article in *Magic without Magic*, edited by J. Klauder (Freeman, San Francisco, 1972).

[16]Cf. K. Kuchař, Phys. Rev. D **4**, 955 (1971).

[17]U. Gerlach, Phys. Rev. **177**, 1929 (1969).

[18]J. W. York, Phys. Rev. Lett. **26**, 1656 (1971).

[19]J. A. Wheeler, Rev. Mod. Phys. **34**, 873 (1962).

[20]I am grateful to R. Geroch for suggesting this term.

[21]Strictly speaking, one needs "self-adjointness." However, on closed C^∞ Riemannian manifolds, there is no necessary distinction for our purposes. See the results on elliptic operators in Ref. 7. For the relation in general, cf. F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Ungar, New York, 1955).

[22]By "harmonic," we mean here simply the kernel of D, $D^{-1}(0)$, consisting of the linearly independent conformal Killing vectors (if any) on $(M,\bar{g})$.

[23]O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasi-Linear Elliptic Equations* (Academic, New York, 1968).

[24]Vanishing ϕ means the Riemannian structure is singular, e.g., "pinched off." Here we are considering only nonsingular manifolds without boundary (or nonsingular manifolds with boundary only "at infinity").

[25]N. O'Murchadha and J. W. York, "Existence and Uniqueness of Solutions of the Hamiltonian Constraint," paper presented by J. W. Y. at the Boston Conference on Gravitation and Quantization, 2 November 1972 (to be published); N. O'Murchadha, thesis, January, 1973.

[26]D. R. Brill and S. Deser, Ann. Phys. (N.Y.) **50**, 548 (1968); and N. O'Murchadha and J. W. York, "Positive Energy in General Relativity" (to be published).

[27]H. Yamabe, Osaka Math. J. 12, 21 (1960). That Yamabe's original proof is incomplete was noticed by N. S. Trudinger, Annal. Scuola di Pisa, Ser. 3, 22, 265 (1968), who gave a partial proof of the theorem, and by T. Aubin, C.R. Acad. Sci. A **266**, 69 (1968), who gave a new proof of Yamabe's theorem. This theorem is quite interesting, but is not assumed in the present considerations. There is no analogous theorem for nonclosed manifolds.

[28]R. Gowdy, "Vacuum Spacetimes with Two-Parameter Spacelike Isometries: Topologies and Boundary Conditions", to be published (1972).

[29]Cf. J. A. Wheeler's discussion of D. R. Brill's treatment of the time-symmetric initial-value problem in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[30]Y. Choquet-Bruhat discussed mathematical aspects of the case $\tau = 0$ in C.R. Acad. Sci. A **274**, 682 (1972).

[31]J. W. York, "General Decompositions of Symmetric Tensors" (to be published).

[32]N. O'Murchadha and J. W. York, "Coupled Gravitational-Electromagnetic Initial-Value Problem" (to be published).

[33]In this reference, matter tensor "currents" were not included, but they can be included without difficulty, as pointed out in Ref. 11. See the discussion of (57) below.

[34]See Ref. 30.

[35]The transformation (58) follows from a dimensional analysis or from consideration of electromagnetism as a source for gravity (Ref. 32).

[36]J. W. York, "Conformal Thin-Sandwich Theorem" (to be published).

[37]If O is an operator, $O^{-1}(0)$ denotes the kernel of the operator O.

[38]An analogous duality exists whenever one decomposes a symmetric tensor of any valence r in terms of a transverse part (or TT part) and a longitudinal part, where the longitudinal part is defined in terms of a symmetric tensor of valence $(r-1)$. For example, in decomposing a symmetric tensor T^{abc}, the elliptic operator defining the "tensor potential" P^{ab} [analogous to W^a in (10)] has as its kernel only "Killing tensors" C^{ab}, satisfying $\nabla^{(a}C^{bc)}=0$, if any exist on the given (M,g).

[39]It is probably well known that the functional derivative with respect to the metric of a conformally invariant functional is TT. For example, one may obtain the "contracted Bianchi identities" in this way for a sourceless Maxwell field coupled to the Einstein gravitational field.