Department of Physics and Astronomy
Heidelberg University

Bachelor thesis in Physics
submitted by

Mathieu Kaltschmidt

from Kappel-Grafenhausen

2019

The Asymptotic Safety approach to Quantum Gravity

This Bachelor thesis has been carried out by

Mathieu Kaltschmidt

at the

Institute for Theoretical Physics

at

Heidelberg University

under the supervision of

Prof. Dr. Jan M. Pawlowski

The Asymptotic Safety approach to Quantum Gravity

Mathieu Kaltschmidt

Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Zusammenfassung

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Contents

Chapter 1.

Introduction

Throughout this thesis we use units such that $\hbar = c = G \equiv 1$.

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

This is the second paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

And after the second paragraph follows the third paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

After this fourth paragraph, we start a new paragraph sequence. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Chapter 2.

Functional Methods in Quantum Field Theory

This chapter introduces the the treatment of quantum field theory using functional methods. The main goal is to get familiar with the physical concepts and the notation used throughout this work and to derive the flow equation for the average effective action, introduced by Christof Wetterich in 1993 [14]. For the derivation of the flow equation we are following [5, 9].

2.1. Generating Functionals and Correlation Functions

We consider a theory setting of N real scalar fields $\varphi_a(x), a \in \{1, \ldots, N\}$ in d-dimensional Euclidean space. The corresponding partition sum in presence of sources $J_a(x)$ reads

$$Z[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\varphi \, \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \,. \tag{2.1}$$

The action $\mathcal{S}$ is specified together with an ultraviolet cutoff scale Λ, later being the momentum scale where we initialize the flow equations and some normalization factor $\mathcal{N}$.

In this notation, the scalar product sums over field components and integrates over all space,

$$J \cdot \varphi = \int_x J_a(x) \, \varphi_a(x) = \int_p \tilde{J}_a(p) \, \tilde{\varphi}_a(p), \tag{2.2}$$

with

$$\int_x = \int_{\mathbb{R}^d} \mathrm{d}^d x \qquad \text{and} \qquad \int_p = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d p}{(2\pi)^d}. \tag{2.3}$$

The partition sum $Z[J]$ is called a *generating functional*. It directly allows us to compute field expectation values

$$\phi := \langle \varphi \rangle = \frac{1}{Z} \frac{\delta Z}{\delta J}\bigg|_{J=0} = \int \mathcal{D}\varphi \, \varphi \, \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \tag{2.4}$$

and higher order correlation functions

$$\langle \varphi_1 \cdots \varphi_n \rangle := \langle \varphi^n \rangle = \frac{1}{Z} \frac{\delta^n Z}{\delta^n J}\bigg|_{J=0} = \int \mathcal{D}\varphi \, \overbrace{\varphi_1 \cdots \varphi_n}^{:= \varphi^n} \, \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \tag{2.5}$$

via functional differentiation of $Z[J]$. This means, that we are basically able to compute all contributing Feynman diagrams for our theory setting, if we have knowledge of its corresponding (grand) canonical partition sum.
For a more efficient description of the theory in terms of only the *connected* correlation functions, we define the Schwinger functional $W[J]$ as the logarithm of $Z[J]$,

$$W[J] = \ln Z[J]. \tag{2.6}$$

It is the generating functional for the connected correlation functions. The normalization factor $\mathcal{N}$, introduced in (2.1) enters here as an additive constant, which drops out for all higher order correlation functions, except the zero-point function. This term is connected to the thermodynamic quantities of our system and becomes important, when external parameters such as temperature, volume or the chemical potential are varied. For the case of quantum gravity, it is linked to the cosmological constant Λ. Nevertheless, in general we are only interested in correlation functions with $n \geq 1$ and therefore we drop this term.
Consider for example the connected two-point function $G_{ab}(x, y) = G_{\alpha\beta}$[1], known as the propagator, correlating the field φ_a at spacetime point x with the field φ_b at y,

$$\begin{aligned} G_{\alpha\beta} &= \frac{\delta^2 W[J]}{\delta J_\alpha \delta J_\beta} = \frac{\delta}{\delta J_\alpha}\left(\frac{1}{Z}\frac{\delta Z}{\delta J_\beta}\right) \\ &= \frac{1}{Z}\left(\frac{\delta^2 Z}{\delta J_\alpha \delta J_\beta}\right) - \frac{1}{Z^2}\left(\frac{\delta Z}{\delta J_\alpha}\right)\left(\frac{\delta Z}{\delta J_\beta}\right) \\ &= \langle \varphi_\alpha \varphi_\beta \rangle - \phi_\alpha \phi_\beta = \langle \varphi_\alpha \varphi_\beta \rangle_{\text{c}}. \end{aligned} \tag{2.7}$$

The propagator is the key object in functional approaches to quantum field theory. It depends on the chosen background via J.
It is still possible to make our computations even more efficient, because $W[J]$ still contains some redundant information. Connected correlation functions can be separated into so-called one-particle irreducible (1PI) and one-particle reducible ones. The 1PI correlation functions are those, whose corresponding Feynman diagrams can *not* be separated into two disconnected ones by cutting a single internal line. As an example, the connected four-point function for Yukawa theory, which has 1PI and and reducible parts is depicted in figure (2.1).

The generating functional for the 1PI correlation functions, the *effective action* Γ, is obtained from the Schwinger functional via a Legendre transform,

$$\Gamma[\phi] = \sup_J \left\{ \int_x J(x)\phi(x) - W[J] \right\} = \int_x J_{\text{sup}}(x)\phi(x) - W\left[J_{\text{sup}}\right], \tag{2.8}$$

where J_{sup} has to be understood as a field-dependent current $J_{\text{sup}}[\phi]$. In the following,

1. To save on notation, we introduce collective indices $\alpha = (x, a)$ respectively (q, a) in momentum space.

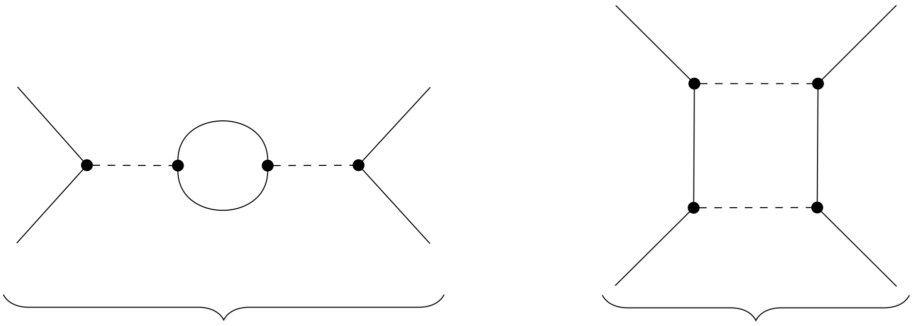


Figure 2.1.: Contributing one-particle reducible and 1PI diagrams for the four-point-function for Yukawa theory.

we will drop the subscript, its meaning is implicitly understood. From a physical point of view, the effective action Γ is the quantum analogue of the classical action $\mathcal{S}$. The performed Legendre transform leads us to a mean field description of our theory with $\phi = \langle\varphi\rangle$ on a given background, as introduced before. The symmetries of the classical action are in general still present in the effective action.
In terms of the effective action, correlation functions are again obtained by performing functional derivatives, but now w. r. t. to the mean field ϕ,

$$\Gamma^{(n)}\left(x_1,\ldots,x_n\right) = \frac{\delta^n\Gamma}{\delta\phi\left(x_1\right)\cdots\delta\phi\left(x_n\right)}. \tag{2.9}$$

For the transition from connected to 1PI correlation functions we have to convert J- derivatives into ϕ-derivatives, i. e.

$$\frac{\delta}{J(x)} = \int_y \frac{\delta\phi(y)}{\delta J(x)}\frac{\delta}{\phi(y)} = \int_y G(x,y)\frac{\delta}{\phi(y)} \tag{2.10}$$

where we used, that $^{\delta\phi}/_{\delta J} = {}^{\delta W^{(1)}}/_{\delta J} = G$. Evaluating the product of the two two-point functions obtained from W and Γ respectively, gives us another important result:

$$\begin{aligned}\int_y \frac{\delta^2 W}{\delta J\left(x_1\right)\delta J(y)}\frac{\delta^2\Gamma}{\delta\phi(y)\phi\left(x_2\right)} &= \int_y \frac{\delta}{\delta J\left(x_1\right)}\left[\frac{\delta W}{\delta J(y)}\right]\frac{\delta}{\delta\phi(y)}\left[\frac{\delta\Gamma}{\delta\phi\left(x_2\right)}\right]\\ &= \int_y \frac{\delta\phi(y)}{\delta J\left(x_1\right)}\frac{\delta}{\delta\phi(y)}J\left(x_2\right)\\ &= \delta\left(x_1 - x_2\right).\end{aligned} \tag{2.11}$$

The full propagator G is the inverse of the 1PI two point function:

$$W^{(2)}(x_1),(x_2) = G(x_1,x_2) = \frac{1}{\Gamma^{(2)}}(x_1,x_2). \tag{2.12}$$

In the next section, we introduce the functional renormalization group (FRG) based on the concepts and results discussed in this section.

2.2. Functional Renormalization Group

The functional renormalization group is a mathematical tool, allowing us to investigate the dynamics of physical systems on different scales, i. e. energy or momentum scales. This idea is based on a continuous version of Kadanoffs block spin model on the lattice and was developed by Kenneth G. Wilson in 1971. It aims at solving the theory by integrating successively momentum shell by momentum shell, being the reason why the path integral approach to quantum field theory, as introduced before, provides a suitable framework. The main advantage of the FRG approach is, that no regularization or renormalization procedure has to be applied. The latter one is already implemented systematically, which secures the self-consistency of the approach.

As a first step towards a FRG equation we need to introduce an infrared cutoff scale k in our theory, below which the modes are not integrated out. A common way to introduce such a scale is by adding a scale-dependent cutoff term $\Delta\mathcal{S}_k$ in the definition of the partition sum (2.1) and therefore automatically also in the definition of the Schwinger functional (2.6)

$$W_k[J] = \ln Z_k[J] = \ln \int \mathcal{D}\varphi \, \mathrm{e}^{-\mathcal{S}[\varphi]+J\cdot\varphi-\Delta\mathcal{S}_k[\varphi]} \,. \tag{2.13}$$

The physical scale k we introduced here is known as *renormalization scale* and has units of inverse length, meaning large k correspond to small distances and vice versa. The cutoff term $\Delta\mathcal{S}_k$ is a quadratic functional depending on the field φ,

$$\Delta\mathcal{S}_k[\varphi] = \frac{1}{2}\varphi \cdot R_k \cdot \varphi = \frac{1}{2}\int_{x,y} \varphi_\alpha \, R_{k,\alpha\beta} \, \varphi_\beta . \tag{2.14}$$

The function R_k is called regulator. It plays an important role for this formulation of quantum field theory. The regulator is chosen such that only the propagation for momentum modes with $p^2 \lesssim k^2$ is suppressed. The most important physical limits are summarized in the following:

$$R_k(p^2) \to \begin{cases} k^2 & \text{for } p \to 0 \\ 0 & \text{for } p \to \infty \\ 0 & \text{for } k \to 0 \\ \infty & \text{for } k \to \Lambda \end{cases} \tag{2.15}$$

We will come back to these limits after deriving the FRG equation, to get a deeper insight into the physical interpretation of the regulator. A convenient choice of the regulator is given by

$$R_k(p^2) = p^2 \cdot r_k(y), \tag{2.16}$$

with $y := \frac{p^2}{k^2}$, and a dimensionless regulator shape function r_k, only depending on the dimensionless momentum ratio p^2/k^2. There is a plethora of different types of shape functions. For the computations performed in this work, we restrict ourselves to a class of rather simple, so-called Litim-type regulators with shape functions

$$r_k(y) = \left(\frac{1}{y} - 1\right) \theta(1-y), \tag{2.17}$$

where θ is the Heaviside step function. This class of (sharp) regulators is a good choice for finding analytic FRG equations in simple approximations. For numerical approaches, exponential regulators, which are in general more complicated, are well suited. In this setting, (2.13) provides a well defined starting point for solving the theory by successively lowering the cutoff scale k infinitesimally and integrating out all momentum modes $\varphi_{p\approx k}$. This procedure can be formalized by taking a scale derivative of our scale-dependent functional (2.13)

$$k\partial_k W_k[J] = -\langle k\partial_k \Delta\mathcal{S}_k[\varphi]\rangle. \tag{2.18}$$

At this point it is quite convenient to introduce the RG time t as

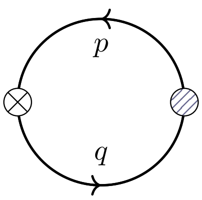
$$\partial_t = \frac{\partial}{\partial \ln(k/\Lambda)} = \frac{k}{\Lambda}\frac{\partial}{\partial(k/\Lambda)} = k\partial_k, \tag{2.19}$$

where Λ is a fixed reference scale. Usually one chooses the ultraviolet cutoff scale, where the flow is initialized.

With the definition of the propagator, we finally arrive at the FRG equation, also called Wetterich equation or flow equation for the effective action:

$$\begin{aligned}\partial_t\Gamma_k[\phi] &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{\Gamma_k^{(2)}[\phi] + R_k}\partial_t R_k\right] \\ &= \frac{1}{2}\int_p \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(p,-p)\,\partial_t R_k(p^2).\end{aligned} \tag{2.20a}$$

It has a rather simple diagrammic representation as one-loop equation:

$$\partial_t\Gamma_k[\phi] = \frac{1}{2}\sum_{i,j=1}^{N}\int_{p,q} \partial_t R_{k,ij}\,(p,q)$$

$$\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}_{ji}(q,p), \tag{2.20b}$$

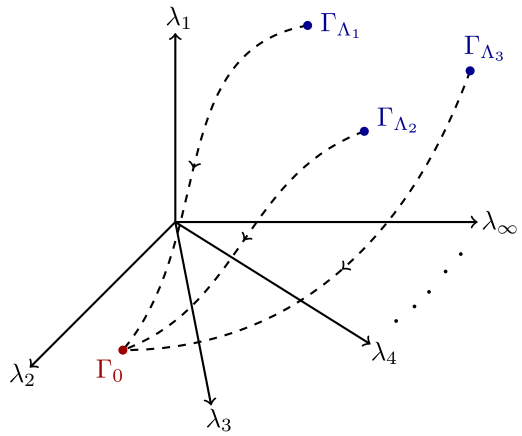


Figure 2.2.: Flow of Γ_k through infinite-dimensional theory space for different regulators.

where $\partial_t R_{k,ij}(p,q) = \partial_t R_k(p^2)(2\pi)^d\ \delta_{ij}\ \delta(p-q)$ and therefore the trace in on the r. h. s. effectively sums over just one index i and integrates over one loop momentum p.

2.3. Systematic expansion schemes

Chapter 3.

Curved Spacetimes

This section is based on [2].

3.1. An Introduction to Spacetime Geometry

inner product:

$$g(X,Y) = g_{\mu\nu} X^\mu Y^\nu = X^\mu Y_\mu = g^{\mu\nu} X_\mu Y_\nu = X_\mu Y^\mu \tag{3.1}$$

we can use the metric tensor to raise and lower spacetime indices,
Christoffel symbols (connection):

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) \tag{3.2}$$

Geodesic equation:

$$\int ds = \int d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\mu}{d\tau}} \tag{3.3}$$

$$\ddot{x}^\mu + \Gamma^\mu{}_{\sigma\rho} \dot{x}^\sigma \dot{x}^\rho = 0 \tag{3.4}$$

Riemann/Curvature tensor:

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\epsilon{}_{\beta\delta} \Gamma^\alpha{}_{\epsilon\gamma} - \Gamma^\epsilon{}_{\beta\gamma} \Gamma^\alpha{}_{\epsilon\delta} \tag{3.5}$$

Definition using the commutator of covariant derivatives

$$\left[\nabla_\mu, \nabla_\nu \right] A^\sigma = R^\sigma{}_{\rho\mu\nu} A^\rho \tag{3.6}$$

Contractions of the Curvature tensor:

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = g^{\alpha\beta} R^\beta{}_{\mu\alpha\nu} \tag{3.7}$$

Curvature Scalar:

$$\mathcal{R} = g_{\mu\nu} R^{\mu\nu} = R^\mu{}_\mu \tag{3.8}$$

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

3.2. From Geometry to the Einstein Equations

This is the second paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

And after the second paragraph follows the third paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

After this fourth paragraph, we start a new paragraph sequence. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

This is the second paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

The Einstein-Hilbert action:

$$\mathcal{S}_{\text{EH}}[g_{\mu\nu}] = \frac{1}{16\pi G} \int\limits_x \sqrt{-\det g_{\mu\nu}} (\mathcal{R} - 2\Lambda) \tag{3.9}$$

Varying this action as usual yields the Einstein equations in absence of matter:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3.10}$$

where we used $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}$.

Diffeomorphism invariance, Lie derivatives:

$$\mathcal{L}_\omega \phi = \omega^\mu \partial^\mu \phi = \omega^\mu \nabla^\mu \phi \tag{3.11}$$

3.3. Gravity with Matter

Energy-Momentum Tensor:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-\det g_{\mu\nu}}} \frac{\delta \mathcal{S}_{\text{matter}}}{\delta g^{\mu\nu}} \tag{3.12}$$

Matter part of the action for a minimally coupled scalar field ϕ:

$$\mathcal{S}_{\text{matter}}[g_{\mu\nu}, \phi] = -\frac{1}{2} \int\limits_x \sqrt{-\det g_{\mu\nu}} \left(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} V(\phi) \right) \tag{3.13}$$

From this, we get the Einstein equations including matter by demanding the variation $\sqrt{-\det g_{\mu\nu}} \frac{\delta \mathcal{S}}{\delta g^{\mu\nu}}$ to vanish. This yields:

$$\frac{1}{8\pi G}\left[\mathcal{R}_{\mu\nu} - \frac{1}{2}(\mathcal{R} - 2\Lambda)g_{\mu\nu}\right] = T_{\mu\nu} \tag{3.14}$$

3.4. Perturbative Non-Renormalizability of Gravity

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Chapter 4.

Functional Renormalization and Quantum Gravity

4.1. RG approach to Quantum Gravity

Flow equation for QG:

$$\partial_t \Gamma_k[\bar{g}, \Phi] = \frac{1}{2} \operatorname{Tr}\, G_{\text{hh}}[\Phi]\, \partial_t R_k - \operatorname{Tr}\, G_{\text{c}\bar{\text{c}}}[\Phi]\, \partial_t R_k \tag{4.1}$$

4.2. Einstein-Hilbert Truncation

We want to solve the Flow equation (4.2) approximately. All terms that are invariant under the imposed symmetry, i. e. invariant under diffeomorphism transformations need to be taken into account.

Easiest truncation takes only the scalar curvature $\mathcal{R}$ and the cosmological constant Λ into account (No higher order terms ...) and was performed by Martin Reuter in 1993 [11].

This truncation reads

$$\Gamma_k = 2\kappa^2 Z_k \int_x \sqrt{\det g}\, [-\mathcal{R} + 2\Lambda_k] + \mathcal{S}_{\text{gf}} + \mathcal{S}_{\text{gh}} \tag{4.2}$$

with

$$\kappa^2 = \frac{1}{32\pi G}, \qquad G_k = G Z_k^{-1} \tag{4.3}$$

Linear gauge fixing F_μ and corresponding ghost term induced by the Faddeev-Popov procedure:

$$\begin{aligned} \mathcal{S}_{\text{gf}} &= \frac{1}{2\alpha} \int_x \sqrt{\det \bar{g}}\, \bar{g}^{\mu\nu} F_\mu F_\nu \\ \mathcal{S}_{\text{gh}} &= \int_x \sqrt{\det \bar{g}}\, \bar{g}^{\mu\mu'} \bar{g}^{\nu\nu'} \bar{c}_{\mu'} \mathcal{M}_{\mu\nu} c_{\nu'} \end{aligned} \tag{4.4}$$

with the Faddeev-Popov operator $\mathcal{M}_{\mu\nu}(\bar{g}, h)$ for the gauge fixing $F_\mu(\bar{g}, h)$.
A linear, de-Donder type gauge fixing with

$$
\begin{aligned}
F_\mu &= \bar{\nabla}^\nu h_{\mu\nu} - \frac{1+\beta}{4} \bar{\nabla}_\mu h^\nu{}_\nu \\
\mathcal{M}_{\mu\nu} &= \bar{\nabla}^\rho (g_{\mu\nu} \nabla_\rho + g_{\rho\nu} \nabla_\mu) - \bar{\nabla}_\mu \nabla_\nu ,
\end{aligned}
\tag{4.5}
$$

is employed, where $\beta = 1$ and $\alpha \to 0$ represents a fixed point of the RG flow. Note, that the limit $\alpha \to 0$ is performed after the gauge fixing process.

anomalous dimension:

$$
\eta_g = -\frac{\partial_t Z_k}{Z_k} = -\partial_t \ln Z_k
$$

dimensionless renormalized cosmological constant:

$$
\lambda_k = \Lambda_k k^{-2}
$$

dimensionless renormalized cosmological constant:

$$
g_k = G_k k^{d-2} = \frac{G k^{d-2}}{Z_k}
$$

corresponding beta function:

$$
\beta_g = \partial_t g_k = \left(d - 2 + \eta_g\right) g_k \tag{4.6}
$$

maximally symmetric space:

$$
\bar{\mathcal{R}}_{\mu\nu} = \frac{1}{d}\, \bar{g}_{\mu\nu} \bar{\mathcal{R}} \tag{4.7}
$$

$$
\bar{\mathcal{R}}_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)}\, (\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}) \bar{\mathcal{R}} \tag{4.8}
$$

suitable tensor basis:

$$
h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{d}\, \bar{g}_{\mu\nu} \bar{\Delta}\right) \sigma + \frac{1}{d}\, \bar{g}_{\mu\nu} h \tag{4.9}
$$

As a first approximation, we only take the contribution from the spin-two graviton mode $h^{\mathrm{TT}}_{\mu\nu}$ into account. This is motivated by the fact, that this mode carries the the most degrees

of freedom.
In this setting, we want to solve the Wetterich equation (2.20b) by computing the left hand side and the right hand side separately and extract the β-functions for the Newton coupling g_k and the cosmological constant λ_k by a comparison of all terms of order $\sim \sqrt{\det g}$ and $\sim \sqrt{\det g}\,\mathcal{R}$. Here, only the most important steps of the calculation are presented. For the complete calculation have a look at the Appendix A.
In our spin-two graviton mode approximation, we don't have to deal with the gauge-fixing and ghost parts ocuring in the effective action. The simplified version of equation (4.2) reads

$$\Gamma_{k,h^{\mathrm{TT}}} = 2\kappa^2 Z_k \int_x \sqrt{\det g}\,[-\mathcal{R} + 2\Lambda_k]. \tag{4.10}$$

We start by computing the transverse-traceless graviton two-point function

$$\Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}} = \frac{Z_k}{32\pi}\left(\bar{\Delta} - 2\Lambda_k + \frac{2}{3}\bar{\mathcal{R}}\right). \tag{4.11}$$

Using a regulator of the form

$$R_k = \Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}}\Big|_{\Lambda_k=\bar{\mathcal{R}}=0} \cdot r_k\left(\frac{\bar{\Delta}}{k^2}\right) = \frac{Z_k}{32\pi}\bar{\Delta}\left(\frac{k^2}{\bar{\Delta}} - 1\right)\Theta\left(1 - \frac{\bar{\Delta}}{k^2}\right),$$

with a Litim-type cutoff

$$r_k(x) = \left(\frac{1}{x} - 1\right)\Theta(1-x), \tag{4.12}$$

we are directly able to compute the l.h.s. of the Wetterich equation, i. e. the scale derivative of the effective average action:

$$\partial_t \Gamma_{k,h^{\mathrm{TT}}} = 2\kappa^2 Z_k \int_x \sqrt{\det g}\left\{\eta_g \mathcal{R} + 2\left(k^2(\partial_t \lambda_k) + \Lambda_k(2-\eta_g)\right)\right\} \tag{4.13}$$

One finds the β-function for the Newton coupling without performing the analysis of the Wetterich equation, i. e.

$$\beta_g = \partial_t g_k = \partial_t\left(\frac{G\cdot k^2}{Z_k}\right) = g_k\,(2+\eta_g). \tag{4.14}$$

The computation of the r.h.s. of the flow equation is more complicated because it involves the computation of a trace of a function depending on the Laplacian. We can use heat-kernel techniques to solve such equations. Heat-kernel computations are based on a curvature expansion in powers of the curvature scaler $\mathcal{R}$. For more details, have a look at the appendix (A.2). As a first step, we simplify the trace expression as much as possible.

$$
\begin{aligned}
\operatorname{Tr}\left[\frac{1}{\Gamma_k^{(2)}+R_k}\partial_t R_k\right] &= \operatorname{Tr}\left[\frac{\partial_t\left(\frac{Z_k}{32\pi}\bar{\Delta}\right)r_k}{\left(\frac{Z_k}{32\pi}\right)\left(\bar{\Delta}+2\Lambda_k+\frac{2}{3}\bar{\mathcal{R}}\right)+\left(\frac{Z_k}{32\pi}\bar{\Delta}\right)r_k}\right] \\
&= \operatorname{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_g r_k\right)}{\bar{\Delta}(1+r_k)-2\Lambda_k+\frac{2}{3}\bar{\mathcal{R}}}\right]
\end{aligned}
\tag{4.15}
$$

We expand this expression around vanishing curvature and get

$$
\operatorname{Tr}\left[\frac{1}{\Gamma_k^{(2)}+R_k}\partial_t R_k\right] = \operatorname{Tr}\left[\frac{\bar{\Delta}(\partial_t r_k-\eta_g r_k)}{\bar{\Delta}(1+r_k)-2\Lambda_k}\right] - \frac{2}{3}\bar{\mathcal{R}}\operatorname{Tr}\left[\frac{\bar{\Delta}(\partial_t r_k-\eta_g r_k)}{\left(\bar{\Delta}(1+r_k)-2\Lambda_k\right)^2}\right] + \mathcal{O}(\mathcal{R}^2) \tag{4.16}
$$

Now we are able to evaluate these two terms separately using heat-kernel techniques. One finds for the first term

$$
\operatorname{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_g r_k\right)}{\bar{\Delta}(1+r_k)-2\Lambda_k}\right] = \frac{1}{(4\pi)^2}\int_x\sqrt{\det\bar{g}}\left[5\Phi_2^1(-2\Lambda_k)-\frac{5}{6}\bar{\mathcal{R}}\Phi_1^1(-2\Lambda_k)\right], \tag{4.17}
$$

with the threshold functions

$$
\Phi_n^p(\omega) = \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\, z^{n-1}\frac{z(-2zr_k(z)-\eta_g r_k(z))}{(z(1+r_k(z))+\omega)^p}. \tag{4.18}
$$

Analogously, the second term in our expansion reads

$$
-\frac{2}{3}\bar{\mathcal{R}}\operatorname{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_g r_k\right)}{\left(\bar{\Delta}(1+r_k)-2\Lambda_k\right)^2}\right] = -\frac{10}{3}\frac{\bar{\mathcal{R}}}{(4\pi)^2}\int_x\sqrt{\det\bar{g}}\frac{1-\frac{\eta_g}{6}}{(1-2\lambda_k)^2}. \tag{4.19}
$$

For the cosmological constant, comparing the $\int\sqrt{\det g}$ terms yields

$$
\beta_\lambda = \partial_t\lambda_k = -4\lambda_k + \frac{\lambda_k}{g_k}\partial_t g_k + \frac{5}{4\pi}g_k\frac{1-\frac{\eta_g}{6}}{1-2\lambda_k}. \tag{4.20}
$$

where the anomalous dimension η_g is determined by comparing the $\int\sqrt{\det g}\mathcal{R}$ terms:

$$
\eta_g = -\frac{5}{3\pi}\left(\frac{1-\frac{\eta_g}{4}}{1-2\lambda_k}+2\frac{1-\frac{\eta_g}{6}}{(1-2\lambda_k)^2}\right). \tag{4.21}
$$

The solution of this system of coupled differential equations is evaluated using Python. We arrive at the following fixed point values for the Newton coupling and the cosmolog-

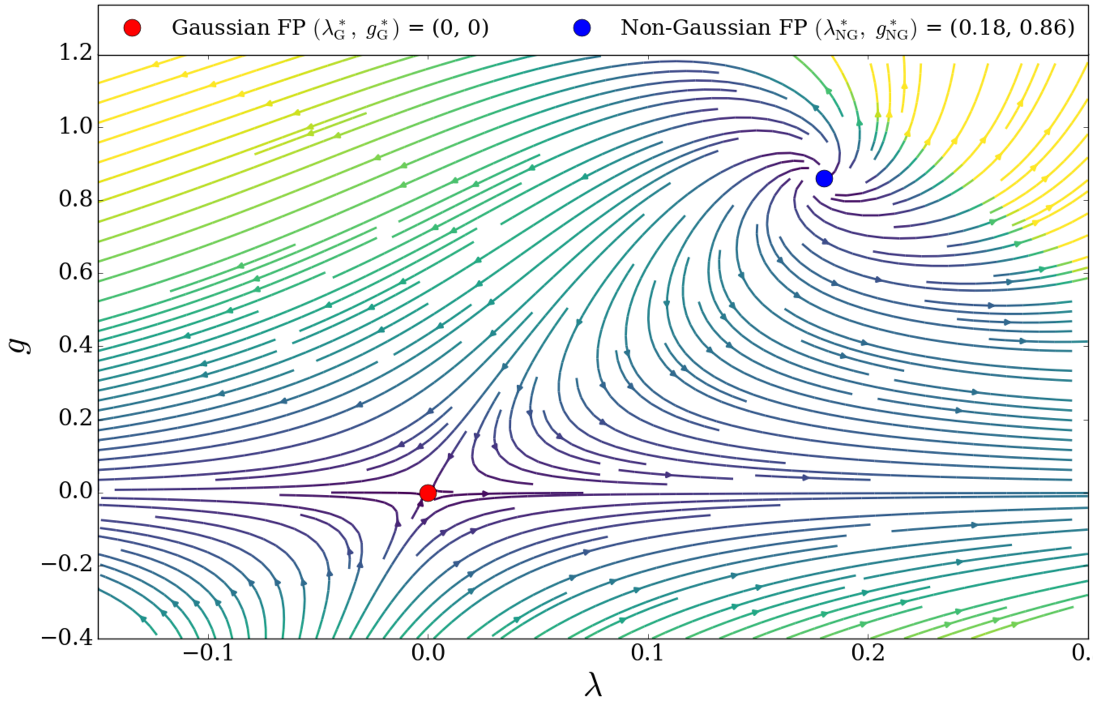


Figure 4.1.: RG flow diagram for the Einstein-Hilbert truncation in TT approximation as computed in this work. The flow points towards the infrared.

ical constant:

$$(g_k^*, \lambda_k^*) = (0.86, 0.18). \tag{4.22}$$

The corresponding critical exponents, i. e. minus the eigenvalues of the stability matrix evaluated at the fixed point, are given by

$$\theta_{1,2} = 2.9 \pm 2.6i \tag{4.23}$$

Chapter 5.

Asymptotic Safety of Gravity-Matter Systems

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

5.1. Matter contributions

$$\Gamma_k = \Gamma_{\text{EH}} + \mathcal{S}_{\text{gf}} + \mathcal{S}_{\text{gh}} + \Gamma_{\text{matter}} \tag{5.1}$$

where the different matter contributions come from

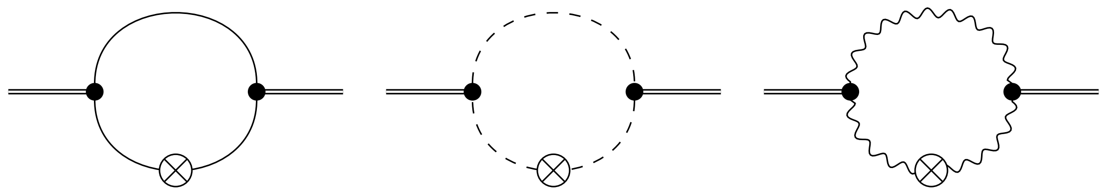
$$\Gamma_{\text{matter}} = \mathcal{S}_{\text{S}} + \mathcal{S}_{\text{D}} + \mathcal{S}_{\text{V}} \tag{5.2}$$

with

$$\mathcal{S}_{\text{S}} = \frac{Z_{\text{S}}}{2} \int_x \sqrt{\det g}\, g^{\mu\nu} \sum_{i=1}^{N_{\text{S}}} \partial_\mu \phi^i \partial_\nu \phi^i \tag{5.3}$$

$$\mathcal{S}_{\text{D}} = i Z_{\text{D}} \int_x \sqrt{\det g} \sum_{i=1}^{N_{\text{D}}} \bar{\psi}^i \not{\nabla} \psi^i \tag{5.4}$$

$$\begin{aligned} \mathcal{S}_{\text{V}} = \frac{Z_{\text{V}}}{4} \int_x \sqrt{\det g} \sum_{i=1}^{N_{\text{V}}} g^{\mu\nu} g^{\kappa\lambda} F^i_{\mu\kappa} F^i_{\nu\lambda} \\ + \frac{Z_{\text{V}}}{2\xi} \int_x \sqrt{\det \bar{g}} \sum_{i=1}^{N_{\text{V}}} \left(\bar{g}^{\mu\nu} \bar{D}_\mu A^i_\nu \right)^2 \\ + \frac{1}{2} \int_x \sqrt{\det \bar{g}} \sum_{i=1}^{N_{\text{V}}} \bar{c}_i (-\bar{D}^2) c_i \end{aligned} \tag{5.5}$$



(a) Fermions. **(b)** Scalars. **(c)** Gauge Fields.

Figure 5.1.: Different matter contributions to the graviton anomalous dimension η_h.

5.2. Fermionic fields

This section is mainly based on [6] and [7] where the spin-base invariant formalism for treating fermions in curved spacetimes has been developed. The goal of this part of the thesis is to get a rough idea on how to perform calculations involving Dirac fermions, especially in the context of Asymptotic Safety of gravity-matter systems.

Covariant derivative:

$$\nabla_\mu = \partial_\mu + \frac{1}{8}\left[\gamma^a, \gamma^b\right]\omega_\mu^{ab} \tag{5.6}$$

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

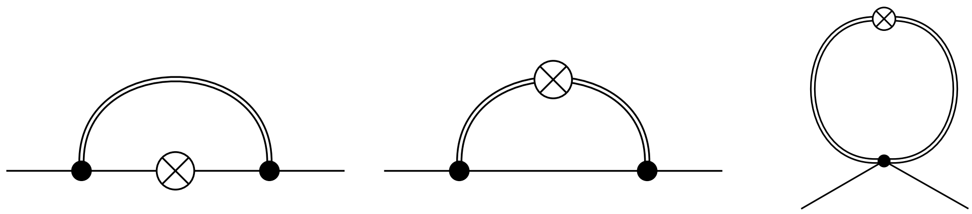


Figure 5.2.: Contributing diagrams to the fermion anomalous dimension η_D. Analogous contributions arise for external scalars and gauge fields to η_S and η_V.

Chapter 6.

Summary and Outlook

Appendix A.

Mathematical Appendix

In this part of the appendix we want to present some of the mathematical tools we used during the calculations presented in the scope of this thesis in a more formal manner.

A.1. York Decomposition

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

A.2. Heat-Kernel Techniques

We use heat-kernel techniques to evaluate the r.h.s. of the flow equation (2.20b), where we need to compute traces over functions depending on the Laplacian on a curved background. In general, the method can be understood as a curvature expansion about a flat background.

The formula to compute such traces is given by

$$\mathrm{Tr}\, f(\Delta) = N \sum\!\!\!\!\!\!\int_{\ell} \rho(\ell) f(\lambda(\ell)), \tag{A.1}$$

with some normalization N, the spectral values $\lambda(\ell)$ and their corresponding multiplicities $\rho(\ell)$.

On flat backgrounds, the computation of (A.1) is simply a standard momentum integral, whereas on curved backgrounds, consider for example a four-sphere with constant background curvature $r = \frac{\bar{\mathcal{R}}}{k^2} > 0$, the spectrum of the Laplacian is discrete and we need to sum over all spectral values in (A.1).

For our example of the four-sphere, we have

$$\lambda(\ell) = \frac{\ell(3+\ell)}{12} r \qquad \text{and} \qquad \rho(\ell) = \frac{(2\ell+3)(l+2)!}{6\ell!}. \tag{A.2}$$

and the normalization is given by the inverse of the four-sphere-volume $N = V_{S^4}^{-1} = \frac{k^4 r^2}{384\pi^2}$. This leads us to the formula for our computation of the r.h.s. of the flow equation on a background with constant positive curvature

$$\operatorname{Tr} f(\Delta) = \frac{k^4 r^2}{384\pi^2} \sum_{\ell=0}^{\infty} \frac{(2\ell+3)(\ell+2)!}{6\ell!} f\left(\frac{\ell(3+\ell)}{12} r\right). \tag{A.3}$$

This is called the spectral sum. For large curvatures r the convergence of the series is rather fast, whereas in the limit $r \to 0$ one finds exponentially slow convergence.
The master equation for heat kernel computations reads

$$\operatorname{Tr} f(\Delta) = \frac{1}{(4\pi)^{\frac{d}{2}}} \left[B_0(\Delta) Q_2[f(\Delta)] + B_2(\Delta) Q_1[f(\Delta)]\right] + O\left(\bar{\mathcal{R}}^2\right) \tag{A.4}$$

with the heat-kernel coefficients

$$B_n(\bar{\Delta}) = \int \mathrm{d}^d x \sqrt{\det \bar{g}}\ \operatorname{Tr} b_n(\bar{\Delta}) \tag{A.5}$$

and

$$Q_n[f(x)] = \frac{1}{\Gamma(n)} \int \mathrm{d}x x^{n-1} f(x). \tag{A.6}$$

For our computation on S^4, the traces over the coefficients $b_n(\bar{\Delta})$ are presented in the following.

	TT	TV	S
$\operatorname{Tr} b_0$	5	3	1
$\operatorname{Tr} b_2$	$-\frac{5}{6}\mathcal{R}$	$\frac{1}{4}\mathcal{R}$	$\frac{1}{6}\mathcal{R}$

Table A.1.: Heat-kernel coefficients for transverse-traceless tensors (TT), transverse vectors (TV) and scalars (S) for computations on the four-sphere S^4.

The basic idea of the proof of equation (A.1) is based on the Laplace transform

$$f(\Delta) = \int_0^{\infty} \mathrm{d}s\, \mathrm{e}^{-s\Delta}\, \tilde{f}(s). \tag{A.7}$$

References

[1] Jens Braun, Marc Leonhardt, and Jan M. Pawlowski. "Renormalization group consistency and low-energy effective theories". In: (2018). arXiv: 1806.04432 [hep-ph].

[2] Sean M. Carroll. "Lecture notes on general relativity". In: (1997). arXiv: gr-qc/9712019 [gr-qc] (cit. on p. 9).

[3] Nicolai Christiansen et al. "Asymptotic safety of gravity with matter". In: *Phys. Rev.* D97.10 (2018), p. 106012. arXiv: 1710.04669 [hep-th].

[4] Pietro Donà, Astrid Eichhorn, and Roberto Percacci. "Matter matters in asymptotically safe quantum gravity". In: *Phys. Rev.* D89.8 (2014), p. 084035. arXiv: 1311.2898 [hep-th].

[5] Stefan Floerchinger and Christof Wetterich. *Lectures on Quantum Field Theory*. Lecture Notes (Access currently restricted to students). Heidelberg University. 2019 (cit. on p. 3).

[6] Stefan Lippoldt. "Fermions in curved spacetimes". (Link). PhD thesis. Friedrich-Schiller-University Jena, 2016 (cit. on p. 20).

[7] Stefan Lippoldt. "Spin-base invariance of Fermions in arbitrary dimensions". In: *Phys. Rev.* D91.10 (2015), p. 104006. arXiv: 1502.05607 [hep-th] (cit. on p. 20).

[8] Jan Meibohm, Jan M. Pawlowski, and Manuel Reichert. "Asymptotic safety of gravity-matter systems". In: *Phys. Rev.* D93.8 (2016), p. 084035. arXiv: 1510.07018 [hep-th].

[9] Jan. M. Pawlowski et al. *The Functional Renormalization Group & applications to gauge theories and gravity*. Lecture Notes (Access currently restricted to students). Heidelberg University. 2019 (cit. on p. 3).

[10] Martin Reuter and Frank Saueressig. *Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2019.

[11] Martin Reuter and Frank Saueressig. "Renormalization group flow of quantum gravity in the Einstein-Hilbert truncation". In: *Phys. Rev.* D65 (2002), p. 065016. arXiv: hep-th/0110054 [hep-th] (cit. on p. 13).

[12] Timo Weigand. *Quantum Field Theory I+II*. Lecture Notes (Link). Heidelberg University. 2014.

[13] Christof Wetterich. "Effective average action in statistical physics and quantum field theory". In: *Int. J. Mod. Phys.* A16 (2001), pp. 1951–1982. arXiv: hep-ph/0101178 [hep-ph].

[14] Christof Wetterich. "Exact evolution equation for the effective potential". In: *Phys. Lett.* B301 (1993), pp. 90–94. arXiv: 1710.05815 [hep-th] (cit. on p. 3).

List of Figures

Acknowledgements

First and foremost i would like to thank my supervisor Prof. Jan Pawlowski for giving me the opportunity to work on such interesting topic and for always finding the time to discuss about ... . I learned a lot about theoretical physics …

I would like the thank the whole Quantum Gravity group at ITP for the nice atmosphere and many interesting and helpful discussions. Especially i would like to thank Gustavo Brito for all the interesting blackboard sessions, which improved my understanding of the subject a lot.

Group, proofreaders, Heidelberger dudes..

Not to forget, i have to thank all my friends for the amazing time we spent together during the last years.

Lastly, i want to thank my parents Marie-Paule and Bernd Kaltschmidt and my sister Céline for their constant support and love and for always allowing me to pursue my dreams.

Declaration of Authorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise.

Heidelberg, ________________ ____________________________