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Bachelor thesis in Physics
submitted by

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Functional Renormalization and Quantum Gravity

This bachelor thesis has been carried out by

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at the

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at

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under the supervision of

Prof. Dr. Jan M. Pawlowski

Functional Renormalization and Quantum Gravity

Mathieu Kaltschmidt

Abstract

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Zusammenfassung

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Contents

Chapter 1.

Introduction

Throughout this thesis we use units such that $\hbar = c = G \equiv 1$.

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Chapter 2.

Functional methods in Quantum Field Theory

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2.1. Generating Functionals and Correlation Functions

We consider a theory setting of N scalar fields $\varphi_a(x), a \in \{1, \dots, N\}$ in d-dimensional Euclidean space. The corresponding partition sum in presence of sources $J_a(x)$ reads

$$Z[J] = \int \mathcal{D}\varphi \, \mathrm{e}^{-\mathcal{S}[\varphi]+J\cdot\varphi} \,. \tag{2.1}$$

The information content of the partition sum results mainly from the classical action functional $\mathcal{S}[\varphi]$, which determines the classical field equations

$$\frac{\delta \mathcal{S}}{\delta \varphi(x)} = 0. \tag{2.2}$$

Notation: The scalar product sums over field components and integrates over all space …

$$J \cdot \varphi = \int_x J_a(x) \, \varphi_a(x) = \int_p \tilde{J}_a(p) \, \tilde{\varphi}_a(p) \tag{2.3}$$

with

$$\int_x = \int_{\mathbb{R}^d} \mathrm{d}^d x \qquad \text{and} \qquad \int_p = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \tag{2.4}$$

Mean field description:

$$\phi := \langle \varphi \rangle = \frac{1}{Z} \frac{\delta Z}{\delta J}\bigg|_{J=0} = \int \mathcal{D}\varphi \, \varphi \, \mathrm{e}^{-\mathcal{S}[\varphi]+J\cdot\varphi} \tag{2.5}$$

Higher correlations:

$$\langle \varphi_1 \cdots \varphi_n \rangle := \langle \varphi^n \rangle = \frac{1}{Z} \frac{\delta^n Z}{\delta^n J} = \int \mathcal{D}\varphi \; \overbrace{\varphi_1 \cdots \varphi_n}^{:= \varphi^n} \; \mathrm{e}^{-\mathcal{S}[\varphi] + J \cdot \varphi} \tag{2.6}$$

We obtain the Schwinger functional by taking the logarithm:

$$W[J] = \ln Z[J] \tag{2.7}$$

For the special case of $n = 2$ the correlation function yields the connected 2-point function which is also known as the propagator $G_{ab}(x, y)$ correlating the field φ_a at spacetime point x with the field φ_b at y.

$$\begin{aligned} G_{ab}(x, y) &= \frac{\delta^2 W[J]}{\delta J_a(x) \delta J_b(y)} = \frac{\delta}{\delta J_a(x)} \left(\frac{1}{Z} \frac{\delta Z}{\delta J_b(y)} \right) \\ &= \frac{1}{Z} \left(\frac{\delta^2 Z}{\delta J_a(x) \delta J_b(y)} \right) - \frac{1}{Z^2} \left(\frac{\delta Z}{\delta J_a(x)} \right) \left(\frac{\delta Z}{\delta J_b(y)} \right) \\ &= \langle \varphi_a(x) \varphi_b(y) \rangle - \phi_a(x) \phi_b(y) = \langle \varphi_a(x) \varphi_b(y) \rangle_{\mathrm{c}} \end{aligned} \tag{2.8}$$

The Effective Action:

The effective action can be obtained by performing a Legendre transform of the Schwinger funtional, i. e.:

$$\Gamma[\phi] = \sup_J \left\{ \int_x J(x) \phi(x) - \mathcal{W}[J] \right\} = \int_x J_{\mathrm{sub}}(x) \phi(x) - \mathcal{W}[J_{\mathrm{sub}}] \tag{2.9}$$

Quantum equation of motion:

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = J(x) \tag{2.10}$$

Dyson-Schwinger equation:

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = \frac{\delta \mathcal{S}}{\delta \varphi(x)} \left[\varphi = G \cdot \frac{\delta}{\delta \phi} + \phi \right] \tag{2.11}$$

2.2. The Functional Renormalization Group

- Kadanoff Block-Spin model

- maybe visualization of Ising model + phase transitions

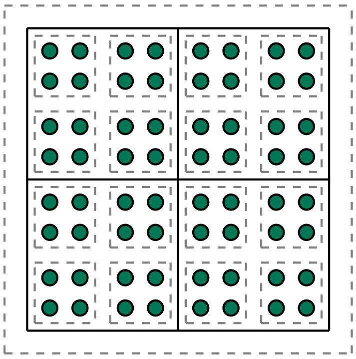

Figure 2.1.: Visualization of the Kadanoff Block-Spin model.[1]

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2.3. Renormalization Group Consistency

This section is mainly based on [2].

Cutoff independence of the full quantum effective action:

$$\Lambda \frac{\mathrm{d}\Gamma}{\mathrm{d}\Lambda} = 0 \tag{2.12}$$

Full effective action in a generic representation:

$$\Gamma[\phi] = \mathcal{D}_\Lambda[\phi] + \Gamma_\Lambda[\phi] \tag{2.13}$$

Formal discussion:

$$\Gamma_k[\phi] = \Gamma_\Lambda[\phi] + \int_\Lambda^k \frac{\mathrm{d}k'}{k'} \mathcal{F}_{k'}[\phi] \tag{2.14}$$

1. This visualization is inspired by an image provided in the PhD thesis of J.R. Laguna.

2.4. Flow Equations for Generating Functionals

We introduce the RG time scale t:

$$\partial_t = \frac{\partial}{\partial \ln(k/\Lambda)} = \frac{k}{\Lambda}\frac{\partial}{\partial(k/\Lambda)} = k\partial_k \tag{2.15}$$

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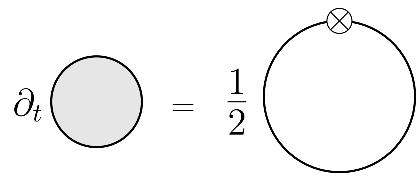
$$\begin{aligned}\partial_t\Gamma_k[\phi] &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{\Gamma_k^{(2)}[\phi]+R_k}\partial_t R_k\right]\\ &= \frac{1}{2}\int_p \frac{1}{\Gamma_k^{(2)}[\phi]+R_k}(p,-p)\;\partial_t R_k(p^2)\end{aligned} \tag{2.16}$$

This translates directly into the following diagrammic representation:



(2.17)

where $\otimes = \partial_t R_k$ represents the insertion of the respective regulator.

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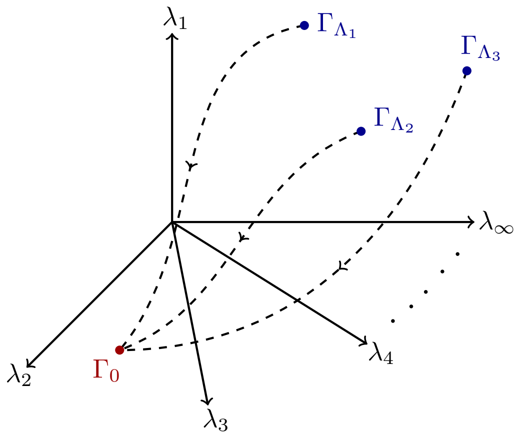

Figure 2.2.: Flow of Γ_k through infinite-dimensional theory space for different regulators, inspired by [5]

Chapter 3.

Fundamentals of General Relativity

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3.1. The Einstein Equations

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The Einstein-Hilbert action:

$$\mathcal{S}_{\mathrm{EH}}[g_{\mu\nu}] = \frac{1}{16\pi G} \int_x \sqrt{-\det g_{\mu\nu}} (\mathcal{R} - 2\Lambda) \tag{3.1}$$

Varying this action as usual yields the Einstein equations in absence of matter:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3.2}$$

where we used $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}$.

Diffeomorphism invariance, Lie derivatives:

$$\mathcal{L}_\omega \phi = \omega^\mu \partial^\mu \phi = \omega^\mu \nabla^\mu \phi \tag{3.3}$$

Now we include matter.

Energy-Momentum Tensor:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-\det g_{\mu\nu}}} \frac{\delta \mathcal{S}_{\text{matter}}}{\delta g^{\mu\nu}} \tag{3.4}$$

Matter part of the action for a minimally coupled scalar field ϕ:

$$\mathcal{S}_{\text{matter}}[g_{\mu\nu}, \phi] = -\frac{1}{2} \int\limits_x \sqrt{-\det g_{\mu\nu}} \left(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} V(\phi) \right) \tag{3.5}$$

From this, we get the Einstein equations including matter by demanding the variation $\sqrt{-\det g_{\mu\nu}} \frac{\delta \mathcal{S}}{\delta g^{\mu\nu}}$ to vanish. This yields:

$$\frac{1}{8\pi G} \left[\mathcal{R}_{\mu\nu} - \frac{1}{2} (\mathcal{R} - 2\Lambda) g_{\mu\nu} \right] = T_{\mu\nu} \tag{3.6}$$

3.2. Perturbative Non-Renormalizability of Gravity

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Chapter 4.

Quantum Gravity in the Einstein-Hilbert Truncation

4.1. RG approach to Quantum Gravity

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4.2. Truncations of the theory space

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4.3. The Effective Action for Quantum Gravity

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4.4. Non-Gaussian Fixed Points

Chapter 5.

Conclusions and Outlook

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Appendix A.

Mathematical Appendix

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A.1. Heat Kernel techniques

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A.2. York decomposition

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Appendix B.

Numerical Implementation

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B.1. Determination of the Fixed Points

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List of Figures

Acknowledgements

First and foremost i would like to thank my supervisor Jan Pawlowski for giving me the opportunity to work on such interesting topic and for his excellent guidance throughout the last months. I learned a lot about theoretical physics …

Group, proofreaders, Heidelberger dudes..

Not to forget, i have to thank all my friends from home, especially Bastian, Chiara, Helena, Jakob, Jana and Lea for all the amazing time we spent together during the last years.

Lastly, i want to thank my parents Marie-Paule and Bernd Kaltschmidt and my sister Céline for their constant support and love and for always allowing me to pursue my dreams. I love you!

Declaration of Authorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise.

Heidelberg, ________________ ____________________________