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Gravity-Matter Systems
in
Asymptotically Safe Quantum Gravity

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Gravity-Matter Systems in Asymptotically Safe Quantum Gravity

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Abstract

In this work we discuss the Asymptotic Safety approach as a possible realization of a theory of quantum gravity based on path integral quantization using functional renormalization group methods. First, the exact renormalization group equation is solved in a spin-2 graviton approximation using the background field formalism and the respective fixed point structure is analyzed. In the second part, we investigate minimally coupled scalar, fermion and gauge fields and their impact on the system. Finally, we discuss the validity of our computations in the background field method and present the fluctuation field formalism as a modern, alternative approach.

Zusammenfassung

In dieser Arbeit wird der Asymptotic-Safety Zugang als möglicher Ansatz zur Realisierung einer Theorie der Quantengravitation im Rahmen der Pfadintegral-Quantisierung mit Methoden der funktionalen Renormierungsgruppe untersucht. Die exakte Renormierungsgruppengleichung wird zunächst in einer Spin-2-Graviton Näherung im Hintergrundformalismus gelöst und die resultierende Fixpunkt-Struktur analysiert. Im zweiten Teil der Arbeit wird der Einfluss von minimal gekoppelten Skalar-, Fermion- und Eichfeldern auf das System überprüft. Abschließend wird die Hintergrundfeld-Methode kritisch hinterfragt und mit der Fluktuationsfeld-Methode ein moderner, alternativer Zugang präsentiert.

Contents

Chapter 1.

Introduction

- Current understanding of Gravity
- Need of NP approach due to failure of perturbative quantization
- Some words on Wilson, Weinberg etc.
- Different approaches to Quantum Gravity (Strings, Loop QG, Causal Sets etc.)
- Throughout this thesis we use natural units such that $\hbar = c \equiv 1$.
- Einsteins sum convention is implicitly understood.
- Difference between roman and greek indices..

The structure of this work is the following. In chapter 2 the field theoretical language and the Functional Renormalization Group (FRG) are introduced. A derivation of Wetterich's exact renormalization group equation, a. k. a. the flow equation, completes our discussion of non-perturbative approaches to quantum field theory. Chapter 3 provides the background knowledge on gravity and curved spacetimes. In chapter 4, as a first step towards quantum gravity, the Asymptotic Safety approach is motivated and the flow equation is solved within the Einstein-Hilbert truncation in a transverse-traceless spin-2 graviton approximation. Our calculation is extended in chapter 5, by taking minimally coupled scalar, fermion and gauge fields into account. In chapter 6 we critically review the background field approximation. The results are summarized and discussed in chapter 7. To conclude this work, an outlook on current progress and open questions in Asymptotic Safety research is presented.

Chapter 2.

Functional Methods in Quantum Field Theory

This chapter introduces a treatment of quantum field theory using functional methods. The main goal is to get familiar with the physical concepts and the notation used throughout this work and to derive the flow equation for the average effective action, introduced by Christof Wetterich in 1993 [21]. For the derivation of the flow equation we are following [7, 13].

2.1. Generating Functionals and Correlation Functions

Consider a theory setting of N real scalar fields $\varphi_a(x), a \in \{1, \dots, N\}$ in d-dimensional Euclidean space. The corresponding partition sum in presence of sources $J_a(x)$ reads

$$Z[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\varphi \, \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \,. \tag{2.1}$$

The action $\mathcal{S}$ is specified together with an ultraviolet cutoff scale Λ, later being the momentum scale where we initialize the flow equations and some normalization factor $\mathcal{N}$. In this notation, the scalar product sums over field components and integrates over all space,

$$J \cdot \varphi = \int_x J_a(x)\, \varphi_a(x) = \int_p \tilde{J}_a(p)\, \tilde{\varphi}_a(p), \tag{2.2}$$

with

$$\int_x = \int_{\mathbb{R}^d} \mathrm{d}^d x \qquad \text{and} \qquad \int_p = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d p}{(2\pi)^d}. \tag{2.3}$$

The partition sum $Z[J]$ is called a *generating functional*. It directly allows us to compute field expectation values

$$\phi := \langle \varphi \rangle = \frac{1}{Z} \frac{\delta Z}{\delta J}\bigg|_{J=0} = \int \mathcal{D}\varphi \, \varphi \; \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \tag{2.4}$$

and higher order correlation functions

$$\langle \varphi_1 \cdots \varphi_n \rangle := \langle \varphi^n \rangle = \frac{1}{Z} \frac{\delta^n Z}{\delta^n J}\bigg|_{J=0} = \int \mathcal{D}\varphi \, \overbrace{\varphi_1 \cdots \varphi_n}^{:= \varphi^n} \; \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \tag{2.5}$$

via functional differentiation. This means, we are basically able to compute all contributing Feynman diagrams for our theory setting, if we have knowledge of its corresponding (grand) canonical partition sum.

For a more efficient description of the theory in terms of only the *connected* correlation functions, we define the Schwinger functional $W[J]$ as the logarithm of $Z[J]$,

$$W[J] = \ln Z[J]. \tag{2.6}$$

It is the generating functional for the connected correlation functions. The normalization factor $\mathcal{N}$, introduced in (2.1) enters here as an additive constant, which drops out for all higher order correlation functions, except for the zero-point function. This term is connected to the thermodynamic quantities of our system and becomes important, when external parameters such as temperature, volume or the chemical potential are varied. For the case of quantum gravity, it is linked to the cosmological constant Λ. Nevertheless, in general we are only interested in correlation functions with $n \geq 1$ and therefore we drop this term.

Consider for example the connected two-point function $G_{ab}(x,y) = G_{\alpha\beta}$[1], known as the propagator, correlating the field φ_a at spacetime point x with the field φ_b at y,

$$\begin{aligned} G_{\alpha\beta} &= \frac{\delta^2 W[J]}{\delta J_\alpha \delta J_\beta} = \frac{\delta}{\delta J_\alpha}\left(\frac{1}{Z}\frac{\delta Z}{\delta J_\beta}\right) \\ &= \frac{1}{Z}\left(\frac{\delta^2 Z}{\delta J_\alpha \delta J_\beta}\right) - \frac{1}{Z^2}\left(\frac{\delta Z}{\delta J_\alpha}\right)\left(\frac{\delta Z}{\delta J_\beta}\right) \\ &= \langle \varphi_\alpha \varphi_\beta \rangle - \phi_\alpha \phi_\beta = \langle \varphi_\alpha \varphi_\beta \rangle_{\mathrm{c}}. \end{aligned} \tag{2.7}$$

The propagator is the key object in functional approaches to quantum field theory. It depends on the chosen background via J.

It is still possible to make our computations even more efficient, because $W[J]$ still contains some redundant information. Connected correlation functions can be separated into so-called one-particle irreducible (1PI) and one-particle reducible ones. The 1PI correlation functions are those, whose corresponding Feynman diagrams can *not* be separated into two disconnected ones by cutting a single internal line. As an example, contributing 1PI and reducible diagrams to the connected four-point function for Yukawa theory, are depicted in figure (2.1).

The generating functional for the 1PI correlation functions, the *effective action* Γ, is obtained from the Schwinger functional via a Legendre transform,

$$\Gamma[\phi] = \sup_J \left\{ \int_x J(x)\phi(x) - W[J] \right\} = \int_x J_{\sup}(x)\phi(x) - W\left[J_{\sup}\right], \tag{2.8}$$

1. To save on notation, we introduce collective indices $\alpha = (x, a)$ respectively (q, a) in momentum space.

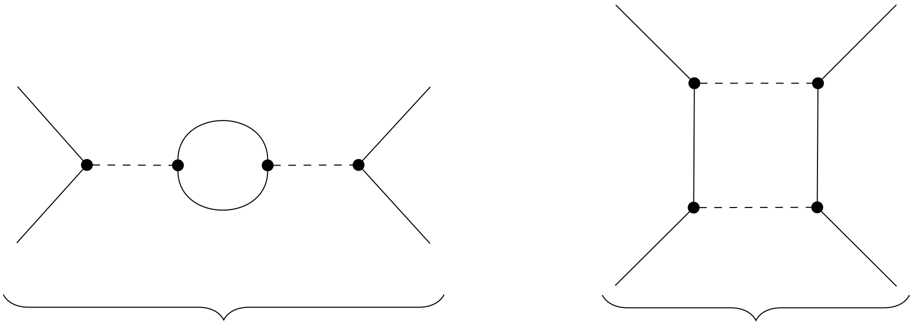


Figure 2.1.: Contributing one-particle reducible and 1PI diagrams to the four-point-function in Yukawa theory.

where J_{sup} has to be understood as a field-dependent current $J_{\text{sup}}[\phi]$. In the following, we will drop the subscript, its meaning is implicitly understood. From a physical point of view, the effective action Γ is the quantum analogue of the classical action $\mathcal{S}$. The performed Legendre transform leads us to a mean field description of our theory with $\phi = \langle\varphi\rangle$ on a given background, as introduced before. The symmetries of the classical action are in general still present in the effective action.
In terms of the effective action, correlation functions are again obtained by performing functional derivatives, but now w. r. t. the mean field ϕ,

$$\Gamma^{(n)}\left(x_1, \ldots, x_n\right) = \frac{\delta^n \Gamma}{\delta\phi\left(x_1\right)\cdots\delta\phi\left(x_n\right)}. \tag{2.9}$$

For the transition from connected to 1PI correlation functions we have to convert J- derivatives into ϕ-derivatives, i. e.

$$\frac{\delta}{\delta J(x)} = \int_y \frac{\delta\phi(y)}{\delta J(x)}\frac{\delta}{\delta\phi(y)} = \int_y G(x,y)\frac{\delta}{\delta\phi(y)}, \tag{2.10}$$

where we used, that ${}^{\delta\phi}/_{\delta J} = {}^{\delta W^{(1)}}/_{\delta J} = G$. Evaluating the product of the two two-point functions obtained from W and Γ respectively, gives us another important result:

$$\begin{aligned}\int_y \frac{\delta^2 W}{\delta J\left(x_1\right)\delta J(y)}\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi\left(x_2\right)} &= \int_y \frac{\delta}{\delta J\left(x_1\right)}\left[\frac{\delta W}{\delta J(y)}\right]\frac{\delta}{\delta\phi(y)}\left[\frac{\delta\Gamma}{\delta\phi\left(x_2\right)}\right] \\ &= \int_y \frac{\delta\phi(y)}{\delta J\left(x_1\right)}\frac{\delta}{\delta\phi(y)}J\left(x_2\right) \\ &= \delta\left(x_1 - x_2\right).\end{aligned} \tag{2.11}$$

The full propagator G is the inverse of the 1PI two-point function:

$$W^{(2)}(x_1,x_2) = G(x_1,x_2) = \frac{1}{\Gamma^{(2)}}(x_1,x_2). \tag{2.12}$$

In the next section, we want to use these concepts to introduce the functional renormalization group (FRG)...

2.2. Functional Renormalization Group

The functional renormalization group is a mathematical tool, allowing us to investigate the dynamics of physical systems on different scales, i. e. energy or momentum scales. This idea is based on a continuous version of Kadanoffs block spin model on the lattice and was developed by Kenneth G. Wilson in 1971. It aims at solving the theory by integrating successively momentum shell by momentum shell, being the reason why the path integral approach to quantum field theory provides a suitable framework. The main advantage of the FRG approach is, that no regularization or renormalization procedure has to be applied. The latter one is already implemented systematically, which secures the self-consistency of the approach.

As a first step towards a FRG equation we need to introduce an infrared cutoff scale k in our theory, below which the modes are not integrated out. A common way to introduce such a scale is by adding a scale-dependent cutoff term $\Delta\mathcal{S}_k$ in the definition of the partition sum (2.1) and therefore automatically also in the definition of the Schwinger functional (2.6)

$$W_k[J] = \ln Z_k[J] = \ln \int \mathcal{D}\varphi \, \mathrm{e}^{-\mathcal{S}[\varphi]+J\cdot\varphi-\Delta\mathcal{S}_k[\varphi]} \,. \tag{2.13}$$

The physical scale k we introduced here is known as *renormalization scale* and has units of inverse length, meaning large k correspond to small distances and vice versa. The cutoff term $\Delta\mathcal{S}_k$ is a quadratic functional depending on the field φ,

$$\Delta\mathcal{S}_k[\varphi] = \frac{1}{2}\varphi\cdot R_k\cdot\varphi = \frac{1}{2}\int_{x,y}\varphi_\alpha \, R_{k,\alpha\beta} \, \varphi_\beta. \tag{2.14}$$

The function R_k is called regulator. It plays an important role for this formulation of quantum field theory. The regulator is chosen such that only the propagation for momentum modes with $p^2 \lesssim k^2$ is suppressed. The most important physical limits are summarized in the following:

$$R_k(p^2) \to \begin{cases} k^2 & \text{for } p \to 0 \\ 0 & \text{for } p \to \infty \\ 0 & \text{for } k \to 0 \\ \infty & \text{for } k \to \Lambda \end{cases} \tag{2.15}$$

We will come back to these limits after deriving the FRG equation, to get a deeper insight into the physical interpretation of the regulator. A convenient choice of the regulator is given by

$$R_k(p^2) = p^2 \cdot r_k(y), \tag{2.16}$$

with $y := \frac{p^2}{k^2}$, and a dimensionless regulator shape function r_k, only depending on the dimensionless momentum ratio p^2/k^2. There is a plethora of different types of shape functions. For the computations performed in this work, we restrict ourselves to a class of rather simple, so-called Litim-type regulators with shape functions

$$r_k(y) = \left(\frac{1}{y} - 1\right) \theta(1 - y), \tag{2.17}$$

where θ is the Heaviside step function. This class of *sharp* regulators is a good choice for finding analytic FRG equations in simple approximations. For numerical approaches, exponential regulators, which are in general more complicated, are well suited. In this setting, (2.13) provides a good starting point for solving the theory by successively lowering the cutoff scale k infinitesimally and integrating out all momentum modes $\varphi_{p\approx k}$. This procedure can be formalized by taking a scale derivative of our scale-dependent functional (2.13)

$$k\partial_k W_k[J] = -\langle k\partial_k \Delta\mathcal{S}_k[\varphi]\rangle. \tag{2.18}$$

At this point it is quite convenient to introduce derivatives w. r. t. the *RG time t* as

$$\partial_t = \frac{\partial}{\partial \ln(k/\Lambda)} = \frac{k}{\Lambda}\frac{\partial}{\partial(k/\Lambda)} = k\partial_k, \tag{2.19}$$

where Λ is a fixed reference scale. Usually one chooses the ultraviolet cutoff scale, where the flow is initialized.

With the definition of the propagator, we finally arrive at the FRG equation, also called Wetterich equation or flow equation for the effective action:

$$\begin{aligned}\partial_t \Gamma_k[\phi] &= \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1} \partial_t R_k\right] \\ &= \frac{1}{2}\int_p \left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}(p,-p)\, \partial_t R_k(p^2).\end{aligned} \tag{2.20a}$$

It has a rather simple diagrammic representation as one-loop equation:

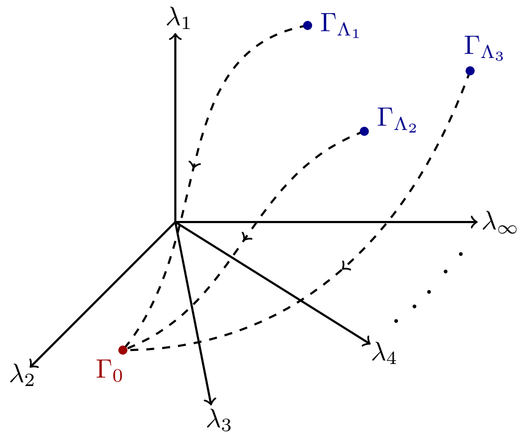
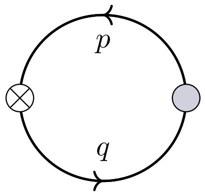


Figure 2.2.: Flow of Γ_k through infinite-dimensional theory space for different regulators.

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \sum_{i,j=1}^{N} \int_{p,q} \partial_t R_{k,ij}\,(p,q)$$



$$\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}_{ji}(q,p), \tag{2.20b}$$

where $\partial_t R_{k,ij}(p,q) = \partial_t R_k(p^2)(2\pi)^d\,\delta_{ij}\,\delta_{\mathrm{D}}(p-q)$ and therefore the trace on the r. h. s. effectively sums over just one index i and integrates over one loop momentum p.

$\mathbb{Z}$

2.3. Systematic expansion schemes

Chapter 3.

Curved Spacetimes and Gravity

Our current understanding of gravity is manifested in Einsteins theory of General Relativity. Different to the treatment of the other fundamental forces, all described by gauge theories and summarized in the Standard Model of Particle Physics, gravity is based on the concept of curved spacetime. This chapter summarizes some of the general concepts and notions of General Relativity, needed for a basic understanding of the subject. For most of the concepts we present here, we are following Sean Carrolls notes [1]. At the end of this chapter, we show why gravity can not be quantized in a perturbative manner, opposite to the other three fundamental forces. For this part, we follow [13].

3.1. An Introduction to Spacetime Geometry

When talking about the concept of curved spacetimes, one first needs a mathematical framework to quantify curvature and to understand how mathematical concepts such as differentiation and integration are generalized to curved spaces. The central objects in our discussion of curved spaces are *differentiable manifolds*, i.e. topological spaces, that are locally diffeomorphic to $\mathbb{R}^n$. Locally in this sense means, that we can find coordinate maps $\phi_i : M \underset{\text{open}}{\supset} U_i \to \mathbb{R}^n$, such that the image $\phi_i(U_i)$ is open in $\mathbb{R}^n$, for every point on M, whereas globally the manifold may have a very complicated topology. A set of such coordinate maps $\{(U_\alpha, \phi_\alpha)\}$ that covers the entire manifold and where the charts are smoothly sewed together is called an *atlas*. For overlapping charts $U_\alpha \cap U_\beta \neq \emptyset$, the maps $(\phi_\alpha \circ \phi_\beta^{-1})$, a.k.a. coordinate transformations, must be smooth and differentiable. They are directly connected to the coordinates x^μ we'll work with later on.

Further, we need to introduce additional structures, such as vectors and tensors on manifolds, since they are the objects we are interested in when it comes to the discussion of physical models. To be able to talk about vectors, one needs to associate a *tangent space* T_p to every point p of the manifold. The tangent space is the set of all vectors at p and has the structure of a vector space with the same dimension as M. The disjoint union of all tangent spaces on M is called the *tangent bundle*. To specify the concept of the tangent space we claim, that it can be identified with the space of directional derivative operators along curves $\gamma : \mathbb{R} \to M$ through p. In this case, we find a basis of T_p as the set $\{\hat{\partial}_\mu\}$ of directional derivatives at p. It can be shown, that the directional derivatives can be decomposed into a sum of real numbers times partial derivatives, i.e. $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$, where λ is the parameter of the curve γ. This allows us to represent a vector $V = V^\mu \partial_\mu$ independent of the chosen coordinates. The basis vectors in some different coordinate system $x^{\mu'}$ are then simply related to the initial basis via $\partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\mu}$ which yields the transformation law for vector components under general coordinate transformations,

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}. \tag{3.1}$$

Components obeying this transformation law are called *contravariant*. At this point it follows quite naturally to define the *cotangent space* T_p^* as the set of linear maps $\omega : T_p \to \mathbb{R}$. Elements of the cotangent space are called one-forms or dual vectors and similarly to the discussion of the tangent space, we find a suitable basis for T_p^* as the gradients $\{\mathrm{d}\hat{x}^{\mu}\}$, allowing us to represent arbitrary one-forms as $\omega = \omega_{\mu}\mathrm{d}x^{\mu}$. As before, we are interested in the transformation behavior of our basis one-forms, i. e. $\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\mathrm{d}x^{\mu}$ and the dual vector components

$$\omega_{\mu'} = \frac{\partial x_{\mu}}{\partial x^{\mu'}}\omega_{\mu}. \tag{3.2}$$

This transformation behavior differs from the one found for vectors. We call components transforming as in equation (3.2) *covariant*.

Now we are able to generalize these concepts by introducing tensors T of type (k, l) as

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_l}. \tag{3.3}$$

The general transformation law for tensors follows naturally as expected from equations (3.1) and (3.2),

$$T^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}. \tag{3.4}$$

Having understood the basic structures and their respective behavior under coordinate transformations, we are now able to present some of the most important tensors in general relativity.
Maybe the most important object to quantify curved space is the *metric tensor* $g_{\mu\nu}$[1] and its inverse $g^{\mu\nu}$, related via $g^{\mu\nu}g_{\nu\sigma} = \delta^{\mu}{}_{\sigma}$. The metric and its inverse can be used to raise and lower indices, e. g. $x^{\mu} = g^{\mu\nu}x_{\nu}$. Additionally we can compute path lengths and proper time via the definition of the line element

$$ds^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}. \tag{3.5}$$

For arbitrary vector fields V and W the scalar product induced by the metric tensor reads

$$g(V, W) = g_{\mu\nu}V^{\mu}W^{\nu} = V^{\mu}W_{\mu} = g^{\mu\nu}V_{\mu}W_{\nu} = V_{\mu}W^{\mu}. \tag{3.6}$$

1. It is convenient to write the components $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$ when speaking about tensors T.

We will see, that the metric tensor already contains all the information on the geometrical structure of the respective manifold we need to quantify curvature. Nevertheless, we first have to think about differentiation of general tensors again.
In flat space, the partial derivative is a map from (k, l) to $(k, l + 1)$ tensor fields satisfying linearity and the Leibniz product rule. We want to generalize this concept to curved space by introducing the *covariant derivative* ∇[2]. Different to the usual partial derivative, the covariant derivative is independent on the chosen set of coordinates. Consider for example the covariant derivative of a vector field V, which can be written as a partial derivative plus some correction term due to its property to obey the Leibniz rule:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda. \tag{3.7}$$

Here, the correction term is specified by the so-called *Christoffel symbols* a. k. a. *connection coefficients*. They are determined by derivatives of the metric tensor:

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) {}^{3}. \tag{3.8}$$

It can be shown, that the connection coefficients themselves do *not* transform like tensor components, but are constructed in a way such that the combination (3.7) does. Note, that the covariant derivative reduces to the partial when applied to scalars. With this definition of the connection, we are now finally able to introduce the remaining tensor structures needed for the understanding of the calculations presented later on in this work.
The central object in our discussion of curvature is the *Riemann tensor* $R^\alpha{}_{\beta\gamma\delta}$. It is a $(1, 3)$-tensor given by

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} + \Gamma^\epsilon{}_{\beta\delta} \Gamma^\alpha{}_{\epsilon\gamma} - \Gamma^\epsilon{}_{\beta\gamma} \Gamma^\alpha{}_{\epsilon\delta}. \tag{3.9}$$

It contains all the information about the curvature of the respective manifold. Another useful definition of the Riemann tensor is related to the commutator of two covariant derivatives, acting on a vector field:

$$\left[\nabla_\mu, \nabla_\nu \right] A^\sigma = R^\sigma{}_{\rho\mu\nu} A^\rho. \tag{3.10}$$

We are also interested in contractions of the Riemann tensor, especially the *Ricci tensor*

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = g_{\alpha\beta} R^\beta{}_{\mu\alpha\nu} \tag{3.11}$$

2. In the context of quantum field theory, the gauge covariant derivative is often written as D. Nevertheless, throughout this thesis we will use ∇ to indicate any kind of covariant derivative.
3. This holds only true, if the connection is *torsion free* i. e. $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = 2\Gamma^\lambda{}_{[\mu\nu]} = 0$ and fullfils *metric compatibility* i. e.$\nabla_\rho g_{\mu\nu} = 0$. For the most important connection in the context of General Relativity, the *Levi-Civita connection*, these properties are fullfilled. The fundamental theorem of Riemannian geometry states, that for every Riemannian manifold there exists a unique Levi-Civita connection. It is determined by the Koszul formula.

and the *Ricci scalar*

$$\mathcal{R} = g_{\mu\nu} R^{\mu\nu} = R^{\mu}{}_{\mu}. \tag{3.12}$$

At this point, we also want to introduce the *Einstein tensor*, defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}. \tag{3.13}$$

Having introduced the setup for the calculations performed in this work, we are now ready to introduce the *Einstein-Hilbert action*, providing the starting point for an investigation of quantum gravity within the Functional Renormalization Group approach.

3.2. From Geometry to Einsteins Equations

The Einstein-Hilbert action, given by

$$\mathcal{S}_{\text{EH}} = \frac{1}{16\pi G} \int_x \sqrt{g}\, (\mathcal{R} - 2\Lambda), \tag{3.14}$$

where G is Newtons coupling and Λ is the cosmological constant, describes a minimally coupled theory of gravity, leading to a $^1/_r$ gravitational potential in the non-relativistic limit. Note, that compared to the usual spacetime measure a factor of $\sqrt{g} := \sqrt{-\det g_{\mu\nu}}$ is included to preserve diffeomorphism invariance.[4]
Varying the Einstein-Hilbert action w. r. t. the inverse metric $g^{\mu\nu}$ yields Einsteins equations in absence of matter:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{3.15}$$

The non-vacuum Einstein equations are obtained the same way, after the inclusion of matter in this setting by adding a matter part to the Einstein-Hilbert action:

$$\mathcal{S} = \frac{1}{8\pi G} \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{matter}}. \tag{3.16}$$

With the definition of the Energy-Momentum tensor $T_{\mu\nu}$, given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{g}} \frac{\delta \mathcal{S}_{\text{matter}}}{\delta g^{\mu\nu}}, \tag{3.17}$$

4. Diffeomorphism invariance, i. e. the freedom of choosing an appropriate coordinate system, is the central symmetry in the context of General Relativity, based on the assumption, that coordinates do not exist a priori in nature, but are rather a mathematical tool used to describe it, that should not change the fundamental laws of physics.

we arrive at

$$\frac{1}{8\pi G}\left[G_{\mu\nu}+\Lambda g_{\mu\nu}\right]=T_{\mu\nu}. \tag{3.18}$$

In this form, Einsteins equations perfectly embody the direct correlation between curvature (l. h. s.) and the dynamics of the matter content of the theory (r. h. s.).
At the end of this chapter we want to emphasize the problem of perturbative non-renormalizability in the context of finding a quantum field theoretical description of gravity.

3.3. Perturbative Non-Renormalizability of Gravity

Naively, one could try to quantize gravity via the path integral formalism with a generating functional, given by $\int_{g_{\mu\nu}} \mathrm{e}^{-\mathcal{S}_{\mathrm{EH}}}$, as usual. The main problem in this approach is the lack of positivity of $\mathcal{S}_{\mathrm{EH}}$ causing problems with unitarity of the theory. In quantum gravity one usually introduces a linear split of the *full* metric $g_{\mu\nu}$, to perform expansions about a given background $\bar{g}_{\mu\nu}$, comparable to classical perturbation theory, which is based on coupling or amplitude expansions about the free Gaussian theory. The linear split reads

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+\sqrt{G}h_{\mu\nu}, \tag{3.19}$$

with the metric fluctuation $h_{\mu\nu}$ defined as $h_{\mu\nu}=1/\sqrt{G}\left(g_{\mu\nu}-\bar{g}_{\mu\nu}\right)$. This allows us to write the path integral in terms of the fluctuation field as

$$Z\left[J^{\mu\nu};\bar{g}_{\mu\nu}\right]\propto\int_{h_{\mu\nu}}\mathrm{e}^{-S_{\mathrm{EH}}\left[\bar{g}_{\mu\nu}+\sqrt{G}h_{\mu\nu}\right]+\int_x\sqrt{\bar{g}}\,J^{\mu\nu}h_{\mu\nu}}. \tag{3.20}$$

Note, that the source term depends on the determinant of the background metric, otherwise the usual $J^{\mu\nu}$ derivatives would not generate the n-point functions of the fluctuation field $h_{\mu\nu}$. We will come back to this problem, which is often referred to as *background independence*, at the end of this thesis in chapter 6.
After a suitable tensor decomposition of the fluctuation field and a gauge fixing procedure à la Faddeev-Popov[5], we are left with the gauge fixed Einstein-Hilbert action

$$\mathcal{S}_{\mathrm{grav}}[\bar{g},\Phi]=\mathcal{S}_{\mathrm{EH}}[g]+\mathcal{S}_{\mathrm{gf}}[\bar{g},h]+\mathcal{S}_{\mathrm{gh}}[\bar{g},\Phi]. \tag{3.21}$$

5. The functional quantization of gauge theories requires a gauge fixing procedure due to redundancies in the path integral measure. The idea of Faddeev and Popov is to represent the gauge fixing condition, implemented in the functional integral, as an additional functional integral over a set of Grassmann fields c and $\bar{c}$, known as *Faddeev-Popov ghosts*. Even though they are anticommuting Grassmann fields, they transform as scalars under Lorentz transformations. They also violate spin statistics. Nevertheless, they can be treated as additional particles in the computation of Feynman diagrams. For a detailed discussion, see e. g. ch. 16 in [15] or sec. 5.2 in [13].

Here the pure gravity multi-field $\Phi = (h_{\mu\nu}, c_\mu, \bar{c}_\mu)$ was introduced. All together, this yields the gauge-fixed path integral representation of quantum gravity:

$$Z[J;\bar{g}] = \int_\Phi \mathrm{e}^{-\mathcal{S}_{\text{grav}}\left[\bar{g}_{\mu\nu},\Phi\right]+\int_x \sqrt{\bar{g}}\; J\cdot\Phi}\,. \tag{3.22}$$

An analysis of the canonical momentum dimensions of the essential couplings of this theory, G and Λ results in:

$$[G] = \left[\mathrm{d}^d x\sqrt{g}\,\mathcal{R}\right] = 2-d, \qquad\qquad [\Lambda] = 2. \tag{3.23}$$

This implies, that the Newton coupling has a negative mass dimension in $d = 4$ spacetime dimensions. To investigate the consequences of this, one can consider the grade of divergence $\Lambda^{\delta(\gamma)}$ for a general graph γ with E external lines, I internal propagators and L loops. Here, Λ is a UV cutoff for the momentum integrals and $\delta(\gamma)$ is the index of the graph,

$$\delta(\gamma) = dL - 2\left(I - \sum_{n=3}^{\infty}\nu_n\right), \tag{3.24}$$

where the ν_n represent n-graviton vertices. After expressing the number of loops in terms of the internal lines and the n-graviton vertices and restricting ourselves to graphs satisfying $E + 2I = \sum_{n=3}^{\infty}\nu_n$, we find

$$\delta(\gamma) = d - \frac{d-2}{2}E + \sum_{n=3}^{\infty} v_n\delta\left(v_n\right), \tag{3.25}$$

where $\delta\left(v_n\right) = \frac{1}{2}(n-2)(d-2)$. After fixing the number of external lines, e. g. to $E = 2$, representing the case of vacuum polarization, one is now able to investigate the grade of divergence for diagrams of different loop orders. t'Hooft and Veltman proved that the theory is renormalizable up to 1-loop order [18], but already at 2-loop order, Goroff and Sagnotti showed, that non-vanishing counterterms are generated [9]. In general, this is interpreted as the failure of perturbative quantization of gravity due to the negative mass dimension of the Newton coupling. This leads us to our discussion of Asymptotic Safety as a non-perturbative approach based on the functional renormalization group methods we presented in the last chapter.

Chapter 4.

Functional Renormalization and Quantum Gravity

4.1. Asymptotic Safety

4.2. Einstein-Hilbert Truncation

We want to solve the Flow equation (4.1) approximately. All terms that are invariant under the imposed symmetry, i. e. invariant under diffeomorphism transformations need to be taken into account.

Easiest truncation takes only the scalar curvature $\mathcal{R}$ and the cosmological constant Λ into account (No higher order terms ...) and was performed by Martin Reuter in 1993 [17].

This truncation reads

$$\Gamma_k = 2\kappa^2 Z_k \int_x \sqrt{g}\,[-\mathcal{R} + 2\Lambda_k] + \mathcal{S}_{\text{gf}} + \mathcal{S}_{\text{gh}} \tag{4.1}$$

with

$$\kappa^2 = \frac{1}{32\pi G}, \qquad G_k = G Z_k^{-1} \tag{4.2}$$

anomalous dimension:

$$\eta_g = -\frac{\partial_t Z_k}{Z_k} = -\partial_t \ln Z_k$$

dimensionless renormalized cosmological constant:

$$\lambda_k = \Lambda_k k^{-2}$$

dimensionless renormalized cosmological constant:

$$g_k = G_k k^{d-2} = \frac{G k^{d-2}}{Z_k}$$

corresponding beta function:

$$\beta_g = \partial_t g_k = \left(d - 2 + \eta_g\right) g_k \tag{4.3}$$

maximally symmetric space:

$$\bar{\mathcal{R}}_{\mu\nu} = \frac{1}{d}\,\bar{g}_{\mu\nu}\bar{\mathcal{R}} \tag{4.4}$$

$$\bar{\mathcal{R}}_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)}\,(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})\bar{\mathcal{R}} \tag{4.5}$$

suitable tensor basis:

As a first approximation, we only take the contribution from the spin-two graviton mode $h^{\mathrm{TT}}_{\mu\nu}$ into account. This is motivated by the fact, that this mode carries the the most degrees of freedom.
In this setting, we want to solve the Wetterich equation (2.20b) by computing the left hand side and the right hand side separately and extract the β-functions for the Newton coupling g_k and the cosmological constant λ_k by a comparison of all terms of order $\sim \sqrt{g}$ and $\sim \sqrt{g}\,\mathcal{R}$. Here, only the most important steps of the calculation are presented. For the complete calculation have a look at Appendix A.
In our spin-two graviton mode approximation, we don't have to deal with the gauge-fixing and ghost parts ocuring in the effective action. The simplified version of equation (4.1) reads

$$\Gamma_{k,h^{\mathrm{TT}}} = 2\kappa^2 Z_k \int_x \sqrt{g}\,[-\mathcal{R} + 2\Lambda_k]. \tag{4.6}$$

We start by computing the transverse-traceless graviton two-point function

$$\Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}} = \frac{Z_k}{32\pi}\left(\bar{\Delta} - 2\Lambda_k + \frac{2}{3}\bar{\mathcal{R}}\right). \tag{4.7}$$

Using a regulator of the form

$$R_k = \Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}}\Big|_{\Lambda_k=\bar{\mathcal{R}}=0} \cdot r_k\left(\frac{\bar{\Delta}}{k^2}\right) = \frac{Z_k}{32\pi}\bar{\Delta}\left(\frac{k^2}{\bar{\Delta}} - 1\right)\Theta\left(1 - \frac{\bar{\Delta}}{k^2}\right),$$

with a Litim-type cutoff

$$r_k(y) = \left(\frac{1}{y} - 1\right)\Theta(1-y), \tag{4.8}$$

as discussed in chapter (2), we are directly able to compute the l. h. s. of the Wetterich equation, i. e. the scale derivative of the effective average action:

$$\partial_t \Gamma_{k,h^{\mathrm{TT}}} = 2\kappa^2 Z_k \int_x \sqrt{g} \left\{ \eta_g \mathcal{R} + 2\left(k^2 (\partial_t \lambda_k) + \Lambda_k (2-\eta_g) \right) \right\} \tag{4.9}$$

One can extract the β-function for the Newton coupling without performing the analysis of the Wetterich equation, i. e.

$$\beta_g = \partial_t g_k = \partial_t \left(\frac{G \cdot k^2}{Z_k} \right) = g_k \left(2 + \eta_g \right). \tag{4.10}$$

The computation of the r. h. s. of the flow equation is more complicated because it involves the computation of a trace of a function depending on the Laplacian on a curved background. We can use heat-kernel techniques to solve such equations. Heat-kernel computations are based on a curvature expansion in powers of the curvature scaler $\mathcal{R}$. For more details, have a look at the appendix (A.2). As a first step, we simplify the trace expression as much as possible.

$$\begin{aligned} \mathrm{Tr}\left[\frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k \right] &= \mathrm{Tr}\left[\frac{\partial_t \left(\frac{Z_k}{32\pi} \bar{\Delta} \right) r_k}{\left(\frac{Z_k}{32\pi} \right) \left(\bar{\Delta} - 2\Lambda_k + \frac{2}{3}\bar{\mathcal{R}} \right) + \left(\frac{Z_k}{32\pi} \bar{\Delta} \right) r_k} \right] \\ &= \mathrm{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k - \eta_g r_k \right)}{\bar{\Delta}(1+r_k) - 2\Lambda_k + \frac{2}{3}\bar{\mathcal{R}}} \right] \end{aligned} \tag{4.11}$$

We expand this expression around vanishing curvature and get

$$\mathrm{Tr}\left[\tfrac{1}{\Gamma_k^{(2)}+R_k} \partial_t R_k \right] = \mathrm{Tr}\left[\tfrac{\bar{\Delta}\left(\partial_t r_k - \eta_g r_k\right)}{\bar{\Delta}(1+r_k) - 2\Lambda_k} \right] - \tfrac{2}{3}\bar{\mathcal{R}}\, \mathrm{Tr}\left[\tfrac{\bar{\Delta}\left(\partial_t r_k - \eta_g r_k\right)}{\left(\bar{\Delta}(1+r_k) - 2\Lambda_k\right)^2} \right] + \mathcal{O}(\mathcal{R}^2) \tag{4.12}$$

Now we are able to evaluate these two terms separately using heat-kernel techniques. One finds for the first term

$$\mathrm{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k - \eta_g r_k \right)}{\bar{\Delta}(1+r_k) - 2\Lambda_k} \right] = \frac{1}{(4\pi)^2} \int_x \sqrt{g} \left[5\Phi_2^1(-2\Lambda_k) - \frac{5}{6}\bar{\mathcal{R}} \Phi_1^1(-2\Lambda_k) \right], \tag{4.13}$$

with the threshold functions

$$\Phi_n^p(\omega) = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{z(-2zr_k(z) - \eta_g r_k(z))}{(z(1+r_k(z)) + \omega)^p}. \tag{4.14}$$

Analogously, the second term in our expansion reads

$$-\frac{2}{3}\bar{\mathcal{R}}\,\mathrm{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_g r_k\right)}{\left(\bar{\Delta}(1+r_k)-2\Lambda_k\right)^2}\right]=-\frac{10}{3}\frac{\bar{\mathcal{R}}}{(4\pi)^2}\int_x\sqrt{g}\frac{1-\frac{\eta_g}{6}}{(1-2\lambda_k)^2}. \tag{4.15}$$

For the cosmological constant, comparing the $\int\sqrt{g}$ terms yields

$$\beta_\lambda=\partial_t\lambda_k=-4\lambda_k+\frac{\lambda_k}{g_k}\partial_t g_k+\frac{5}{4\pi}g_k\frac{1-\frac{\eta_g}{6}}{1-2\lambda_k}, \tag{4.16}$$

where the anomalous dimension η_g is determined by comparing the $\int\sqrt{g}\mathcal{R}$ terms:

$$\eta_g=-\frac{5}{3\pi}\left(\frac{1-\frac{\eta_g}{4}}{1-2\lambda_k}+2\frac{1-\frac{\eta_g}{6}}{(1-2\lambda_k)^2}\right). \tag{4.17}$$

The solution of this system of coupled differential equations is evaluated using `Python3` and `Wolfram Mathematica`. We arrive at the following fixed point values for the Newton coupling and the cosmological constant:

$$(g_k^*,\lambda_k^*)=(0.86,0.18). \tag{4.18}$$

The corresponding critical exponents, i. e. minus the eigenvalues of the stability matrix evaluated at the fixed point, are given by the complex conjugated pair

$$\theta_{1,2}=2.9\pm 2.6i. \tag{4.19}$$

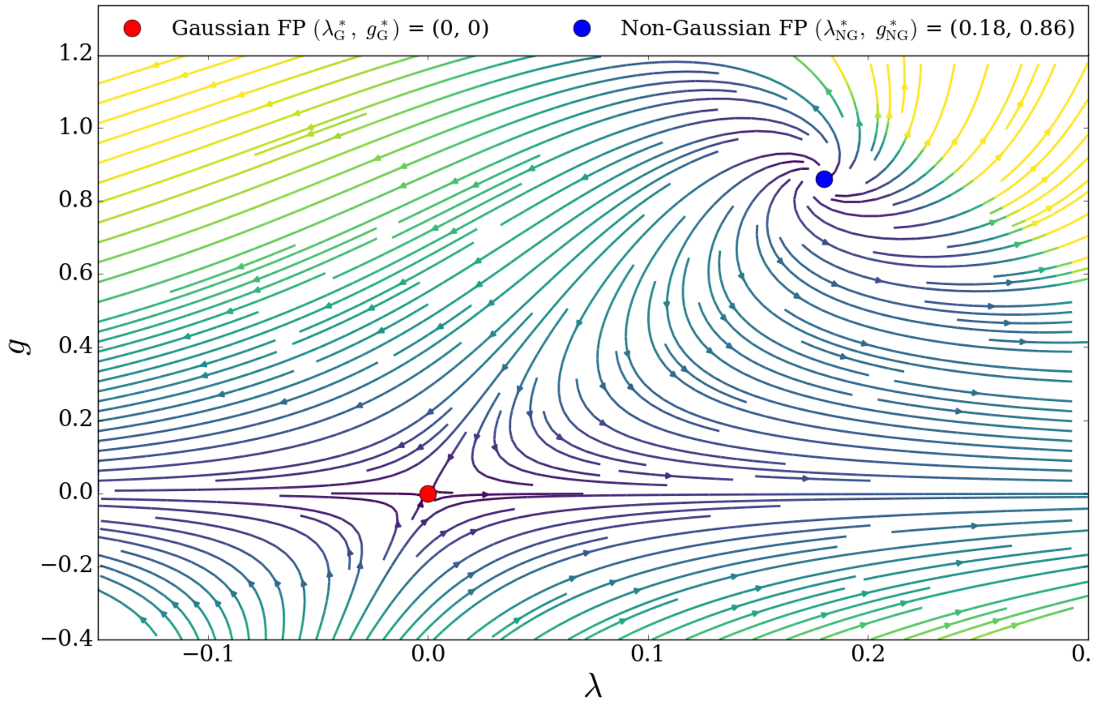


Figure 4.1.: RG flow diagram for the Einstein-Hilbert truncation in TT approximation as computed in this work. The flow points towards the infrared.

Asymptotic Safety of Gravity-Matter Systems

The calculation in h^{TT} approximation in the last chapter already allowed us to investigate the characteristic fixed point structure of the Einstein-Hilbert truncation. Nevertheless, in this part of the thesis, where the impact of minimally coupled matter fields is investigated, we want to work with the full result, including also the vector and scalar modes arising after the York decomposition (**??**) of the fluctuation field. This also means, that we have to take care of additional gauge fixing and ghost terms, given by

$$\mathcal{S}_{\mathrm{gf}} = \frac{1}{2\alpha}\int_x \sqrt{\bar{g}}\,\bar{g}^{\mu\nu}F_\mu F_\nu$$
$$\mathcal{S}_{\mathrm{gh}} = \int_x \sqrt{\bar{g}}\,\bar{g}^{\mu\mu'}\bar{g}^{\nu\nu'}\bar{c}_{\mu'}\mathcal{M}_{\mu\nu}c_{\nu'} \tag{5.1}$$

with the Faddeev-Popov operator $\mathcal{M}_{\mu\nu}(\bar{g},h)$ for the gauge fixing $F_\mu(\bar{g},h)$.

$$F_\mu = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1+\beta}{4}\bar{\nabla}_\mu h^\nu{}_\nu$$
$$\mathcal{M}_{\mu\nu} = \bar{\nabla}^\rho(g_{\mu\nu}\nabla_\rho + g_{\rho\nu}\nabla_\mu) - \bar{\nabla}_\mu\nabla_\nu, \tag{5.2}$$

The inclusion of matter in this theory setting is in principle straightforward. We extend our truncation (4.1) by including an additional matter term:

$$\Gamma_k = \Gamma_{\mathrm{EH}} + \mathcal{S}_{\mathrm{gf}} + \mathcal{S}_{\mathrm{gh}} + \Gamma_{\mathrm{matter}}, \tag{5.3}$$

where Γ_{matter} consists of scalar, fermion and gauge field contributions, denoted with $\mathcal{S}_S, \mathcal{S}_D$ and $\mathcal{S}_V$ respectively:

$$\Gamma_{\mathrm{matter}} = \mathcal{S}_S + \mathcal{S}_D + \mathcal{S}_V. \tag{5.4}$$

The different actions will be specified later on, every matter type will be treated separately. For conventions regarding the choice of the respective regulators and the general structure of this calculation, we are following [5].

In this truncation we have two essential couplings, G and Λ and five inessential[1] wave function renormalizations Z_Ψ with $\Psi = (h, c, S, D, V)$. As before, the wave function renormalizations Z_Ψ do not enter the beta functions for G and Λ directly, but are still present

1. Inessential in this sense means, that they can be eliminated by field rescalings.

in a non-trivial way via the anomalous dimension η_Ψ, defined as

$$\eta_\Psi = -\partial_t \ln Z_\Psi . \tag{5.5}$$

For the scalar and gauge field regulators we choose

$$R_{k,S/V}(z) = Z_{S/V} \cdot \mathbb{1}_{S/V} \cdot \tilde{\Delta} \cdot r_k\left(\frac{\tilde{\Delta}}{k^2}\right) , \tag{5.6}$$

where $\tilde{\Delta} = -\nabla^2 + \mathbf{E}_\Psi$ is a modified Laplacian[2], occurring as kinetic operator in the different matter field actions. The regulator choice for the Dirac fermions is slightly different, details are discussed in the respective subsection. Nevertheless, we already present the values of $\mathbf{E}_\Psi$ for all three kinetic operators:

$$\mathbf{E}_\Psi = \begin{cases} 0 & \text{for } \Psi = S \\ \frac{\mathcal{R}}{4} & \text{for } \Psi = D \\ R^\mu{}_\nu & \text{for } \Psi = V. \end{cases} \tag{5.7}$$

The Litim-type shape function r_k is in this case the same as the one defined in equation (4.8), now as a function of the modified Laplacian $\tilde{\Delta}$.

5.1. Matter Contributions in Background Field Approximation

After having introduced the setup for the following calculation, we are now able to determine the different contributions from the matter fields step by step, by evaluating the functional traces ocurring on the r. h. s. of the flow equation separately. For the matter configuration in our setting the flow equation reads

$$\begin{aligned} \partial_t \Gamma_k = &\frac{1}{2} \operatorname{Tr}\left[\left(\Gamma^{(2)} + R_k\right)^{-1} \partial_t R_k\right]_{hh} - \operatorname{Tr}\left[\left(\Gamma^{(2)} + R_k\right)^{-1} \partial_t R_k\right]_{\bar{c}c} \\ &+ \frac{1}{2} \operatorname{Tr}\left[\left(\Gamma^{(2)} + R_k\right)^{-1} \partial_t R_k\right]_{\phi\phi} - \operatorname{Tr}\left[\left(\Gamma^{(2)} + R_k\right)^{-1} \partial_t R_k\right]_{\bar{\psi}\psi} \\ &+ \frac{1}{2} \operatorname{Tr}\left[\left(\Gamma^{(2)} + R_k\right)^{-1} \partial_t R_k\right]_{AA} . \end{aligned} \tag{5.8}$$

In figure (5.1), a digrammatical representation of the flow equation (5.8) is depicted.

2. A more detailed discussion on how these modified Laplacians effect the values of the heat-kernel coefficients is presented in appendix A.

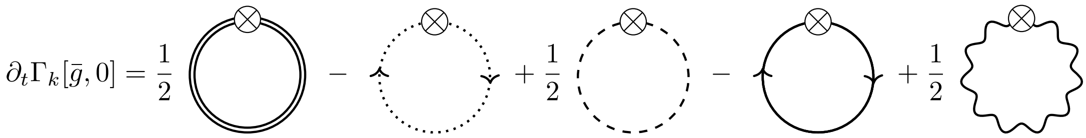


Figure 5.1.: Flow equation (5.8) for the average effective action Γ_k including different matter contributions in diagrammatic representation. The double, dotted,dashed, solid and wiggly lines correspond to the graviton, ghost, scalar, fermion and gauge field propagators, respectively. The crossed circles denote the insertion of the respective regulator.

5.1.1. Scalar fields

The action for N_S scalar fields, minimally coupled to gravity reads

$$\begin{aligned} \mathcal{S}_S &= \frac{Z_S}{2} \int_x \sqrt{g}\, g^{\mu\nu} \sum_{i=1}^{N_\mathrm{S}} \partial_\mu \phi^i \partial_\nu \phi^i \\ &= \frac{Z_S}{2} \int_x \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \sum_{i=1}^{N_\mathrm{S}} \partial_\mu \phi^i \partial_\nu \phi^i + \mathcal{O}(h) \\ &= \frac{Z_S}{2} \int_x \sqrt{\bar{g}} \sum_{i=1}^{N_\mathrm{S}} \phi^i \left(-\bar{\nabla}^2\right) \phi^i + \mathcal{O}(h). \end{aligned} \tag{5.9}$$

For our computation, we expand the action on some background $\bar{g}_{\mu\nu}$ and drop all contributions of $\mathcal{O}(h)$. In the last step, we use integration by parts and assume vanishing boundary terms. Since $\mathbf{E} = 0$ for scalars, we use the initial definition of the Laplacian $\bar{\Delta} = -\bar{\nabla}^2$ for further calculations. These simple manipulations directly allow us to read off the corresponding two-point function

$$\Gamma^{(2)}_{\phi\phi} = \frac{\delta^2 \mathcal{S}_S}{\delta\phi^i\, \delta\phi^j} = Z_S \cdot \bar{\Delta} \cdot \mathbb{1}_S + \mathcal{O}(h), \tag{5.10}$$

where $\mathbb{1}_S$ has to be understood as the identity in field space. Using the regulator defined in (5.6), we find the regularized two-point-function as

$$\Gamma^{(2)}_{k,\phi\phi} = \left[\Gamma^{(2)}_{\phi\phi} + R_{k,S}\right] = Z_S \cdot \bar{\Delta} \cdot \mathbb{1}_S \left(1 + r_k\left(\frac{\bar{\Delta}}{k^2}\right)\right). \tag{5.11}$$

This expression is already diagonal in field space, meaning we are directly able to invert it to obtain the propagator. Together with the scale derivative of the regulator

$$\partial_t R_{k,S} = Z_S \cdot \mathbb{1}_S \cdot \bar{\Delta} \left(\partial_t r_k - \eta_S r_k\right), \tag{5.12}$$

we can start to evaluate the r. h. s. of the flow equation:

$$
\begin{aligned}
\frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma^{(2)}_{k,\phi\phi}\right)^{-1}\partial_t R_{k,S}\right] &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{Z_S\bar{\Delta}\left(\partial_t r_k-\eta_s r_k\right)}{Z_S\bar{\Delta}\left(1+r_k\right)}\mathbb{1}_S\right] \\
&= \frac{N_S}{2}\,\mathrm{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_s r_k\right)}{\bar{\Delta}\left(1+r_k\right)}\right].
\end{aligned}
\tag{5.13}
$$

Here, we already performed the trace operation on the internal indices, leading to an overall factor of N_S. The functional trace is again evaluated using heat-kernel techniques.

$$
\begin{aligned}
\frac{N_S}{2}\,\mathrm{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_s r_k\right)}{\bar{\Delta}\left(1+r_k\right)}\right] &= \frac{N_S}{2}\frac{1}{(4\pi^2)}\left[\int_x\sqrt{\bar{g}}\;\Phi^1_2(0)+\frac{1}{6}\int_x\sqrt{\bar{g}}\;\bar{\mathcal{R}}\;\Phi^1_1(0)\right] \\
&= \frac{N_S}{2}\frac{1}{(4\pi)^2}\int_x\sqrt{\bar{g}}\left[\left(1-\frac{\eta_S}{6}\right)+\frac{\bar{\mathcal{R}}}{3}\left(1-\frac{\eta_S}{6}\right)\right].
\end{aligned}
\tag{5.14}
$$

5.1.2. Fermionic fields

For the fermionic contribution, we proceed slightly different. First, we present the action for N_D minimally coupled Dirac fermions:

$$
\begin{aligned}
\mathcal{S}_D &= iZ_D\int_x\sqrt{g}\;\sum_{i=1}^{N_D}\bar{\psi}^i\not{\nabla}\psi^i \\
&= iZ_D\int_x\sqrt{\bar{g}}\;\sum_{i=1}^{N_D}\bar{\psi}^i\bar{\not{\nabla}}\psi^i+\mathcal{O}(h).
\end{aligned}
\tag{5.15}
$$

The Dirac operator $\not{\nabla}$ satisfies $-\not{\nabla}^2=-\nabla^2+\frac{\bar{\mathcal{R}}}{4}:=\Delta_{\left(1/2\right)}$. The notation for the conjugated field $\bar{\psi}=\psi^\dagger\mathfrak{h}$[3] should not be confused with the bar referring to the background field. As usual, the slashed notation implies contraction with gamma matrices[4], i. e. $\not{\nabla}=\gamma^\mu\nabla_\mu$. In principle, this allows us to read off the fermion two-point function

$$
\Gamma^{(2)}_{\bar{\psi}\psi}=\frac{\delta^2\mathcal{S}_D}{\delta\psi^i\;\delta\bar{\psi}^j}=iZ_D\cdot\not{\nabla}\cdot\mathbb{1}_D+\mathcal{O}(h).
\tag{5.16}
$$

3. The spin metric $\mathfrak{h}$ satisfies $|\det\mathfrak{h}|=1$ and $\mathfrak{h}^{-1}=-\mathfrak{h}$.

4. In the discussion of Dirac fermions, the gamma matrices $\left\{\gamma^0,\gamma^1,\gamma^2,\gamma^3\right\}$ are a set of complex valued matrices, that constitute an irreducible representation of the Clifford algebra, defined by the anticommutation relation $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}\mathbb{1}_{d_\gamma\times d_\gamma}$, with $d_\gamma=2^{\left[d/2\right]}$. A more formal treatment of fermions in curved spacetimes is presented in [10].

In general one chooses the regulator, such that the symmetries of the kinetic term are conserved. As before, the general form of such a regulator is given by

$$R_{k,D} = Z_D \cdot \tilde{\Delta} \cdot \mathbb{1}_D \cdot r_{k,D}\left(\frac{\tilde{\Delta}}{k^2}\right). \tag{5.17}$$

When computing fermion propagators in other theory settings, it follows quite naturally to consider the Dirac dispersion as the "square root" of the scalar Klein-Gordon dispersion. For a more detailed discussion of this idea in the context of Fermi-Bose mixtures, we refer to chapter 2 of [13] or in the context of gravity-matter systems in quantum gravity to the appendix of [4].
This assumption allows us to express r_k for the fermions as a function of the scalar shape function:

$$\Big[1 + r_{k,D}\Big]^2 = 1 + r_{k,S} \qquad \longrightarrow \qquad r_{k,D} = \sqrt{1 + r_{k,S}} - 1 \tag{5.18}$$

In total, this yields the final expression for regularized two-point function for the fermions:

$$\Gamma^{(2)}_{\bar{\psi}\psi,k} = \tag{5.19}$$

5.1.3. Gauge fields

The structure of the gauge field contribution is more complex than for the other fields. This is due to the fact, that we have to employ a gauge fixing procedure w. r. t. the background field $\bar{g}_{\mu\nu}$. This ensures gauge invariance w. r. t. background gauge transformations. The action for N_V gauge fields, minimally coupled to gravity reads

$$\begin{aligned} \mathcal{S}_V =& \frac{Z_\text{V}}{4}\int_x \sqrt{g}\sum_{i=1}^{N_\text{V}} g^{\mu\nu} g^{\kappa\lambda} F^i_{\mu\kappa} F^i_{\nu\lambda} + \frac{Z_\text{V}}{2\xi}\int_x \sqrt{\bar{g}}\sum_{i=1}^{N_\text{V}}\left(\bar{g}^{\mu\nu}\bar{\nabla}_\mu A^i_\nu\right)^2 \\ &+ \int_x \sqrt{\bar{g}}\sum_{i=1}^{N_\text{V}} \bar{C}_i(-\bar{\nabla}^2)C_i, \end{aligned} \tag{5.20}$$

where the second term is the gauge fixing term with gauge parameter ξ and the third term is the Abelian ghost term. Since the two-point function is obtained from a functional derivative w. r. t. the fields A^i, we have to evaluate the ghost-term separately. We start by manipulating the first term:

$$\begin{aligned} \frac{Z_V}{4}\int_x \sqrt{g}\sum_{i=1}^{N_\text{V}} g^{\mu\nu} g^{\kappa\lambda} F^i_{\mu\kappa} F^i_{\nu\lambda} &= \frac{Z_V}{4}\int_x \sqrt{\bar{g}}\sum_{i=1}^{N_\text{V}} \bar{g}^{\mu\nu}\bar{g}^{\kappa\lambda}\bar{F}^i_{\mu\kappa}\bar{F}^i_{\nu\lambda} + \mathcal{O}(h) \\ &\overset{(B.2)}{=} \frac{Z_V}{2}\int_x \sqrt{\bar{g}}\sum_{i=1}^{N_\text{V}} A^i_\lambda\left[\bar{\nabla}^\mu\bar{\nabla}^\lambda - \bar{g}^{\mu\lambda}\bar{\nabla}^2\right]A^i_\mu + \mathcal{O}(h). \end{aligned} \tag{5.21}$$

The steps we skipped can be found in appendix B. For the gauge fixing term we find:

$$\begin{aligned}\frac{Z_{\mathrm{V}}}{2\xi}\int_x\sqrt{\bar{g}}\sum_{i=1}^{N_{\mathrm{V}}}\left(\bar{g}^{\mu\nu}\bar{\nabla}_\mu A^i_\nu\right)^2 &= \frac{Z_{\mathrm{V}}}{2\xi}\int_x\sqrt{\bar{g}}\sum_{i=1}^{N_{\mathrm{V}}}\bar{g}^{\mu\nu}\bar{\nabla}_\mu A^i_\nu g^{\kappa\lambda}\bar{\nabla}_\kappa A^i_\lambda \\ &= \frac{Z_{\mathrm{V}}}{2\xi}\int_x\sqrt{\bar{g}}\sum_{i=1}^{N_{\mathrm{V}}}A^i_\lambda\left[-\bar{\nabla}^\lambda\bar{\nabla}^\mu\right]A^i_\mu.\end{aligned} \tag{5.22}$$

In the last step, we integrated by parts and assumed vanishing boundary terms. This allows us to write

$$\mathcal{S}_V = \frac{Z_V}{2}\int_x\sqrt{\bar{g}}\sum_{i=1}^{N_{\mathrm{V}}}A^i_\lambda\left[-\bar{g}^{\mu\lambda}\bar{\nabla}^2+\bar{\nabla}^\mu\bar{\nabla}^\lambda-\frac{1}{\xi}\bar{\nabla}^\lambda\bar{\nabla}^\mu\right]A^i_\mu \;+\; \text{ghost term} \tag{5.23}$$

In Feynman gauge, where we set $\xi \equiv 1$, this simplifies to

$$\begin{aligned}\mathcal{S}_V &= \frac{Z_V}{2}\int_x\sqrt{\bar{g}}\sum_{i=1}^{N_{\mathrm{V}}}A^i_\lambda\left[-\bar{g}^{\mu\lambda}\bar{\nabla}^2+\left[\bar{\nabla}^\mu,\bar{\nabla}^\lambda\right]\right]A^i_\mu \;+\; \text{ghost term} \\ &\overset{(\mathrm{B.3})}{=} \frac{Z_V}{2}\int_x\sqrt{\bar{g}}\sum_{i=1}^{N_{\mathrm{V}}}A^i_\lambda\left[-\bar{g}^{\mu\lambda}\bar{\nabla}^2+\bar{R}^{\mu\lambda}\right]A^i_\mu \;+\; \text{ghost term}\end{aligned} \tag{5.24}$$

In this form, we are again directly able to read off the two-point-function:

$$\Gamma^{(2)}_{AA} = \frac{\delta^2\mathcal{S}_V}{\delta A^i\,\delta A^j} = Z_V\underbrace{\left[-\bar{g}^{\mu\lambda}\bar{\nabla}^2+\bar{R}^{\mu\lambda}\right]}_{=:\,\bar{\Delta}^{\mu\nu}_{(1)}}\mathbb{1}_V+\mathcal{O}(h), \tag{5.25}$$

where $\bar{\Delta}^{\mu\nu}_{(1)}$ is a modified spin-one Laplacian. With the respective regulator we find

$$\Gamma^{(2)}_{k,AA} = \left[\Gamma^{(2)}_{AA}+R_{k,V}\right] = Z_V\cdot\bar{\Delta}^{\mu\nu}_{(1)}\cdot\mathbb{1}_V\left(1+r_k\left(\frac{\bar{\Delta}^{\mu\nu}_{(1)}}{k^2}\right)\right) \tag{5.26}$$

and

$$\partial_t R_{k,V} = Z_V\cdot\mathbb{1}_V\cdot\bar{\Delta}^{\mu\nu}_{(1)}\cdot\left(\partial_t r_k-\eta_V r_k\right). \tag{5.27}$$

As for the fermions, we have to take care of the heat-kernel coefficients for $\bar{\Delta}^{\mu\nu}_{(1)}$. With equation (A.28), we find

$$\begin{aligned}\mathrm{Tr}\,\mathbf{b}_0 &= 4 \\ \mathrm{Tr}\,\mathbf{b}_2 &= -\frac{\bar{\mathcal{R}}}{3},\end{aligned} \tag{5.28}$$

and therefore, the result for the heat-kernel expansion for the gauge fields is given by:

$$
\begin{aligned}
\frac{1}{2}\,\mathrm{Tr}\left[\frac{Z_V\bar{\Delta}^{\mu\nu}_{(1)}\left(\partial_t r_k-\eta_V r_k\right)}{Z_V\bar{\Delta}^{\mu\nu}_{(1)}(1+r_k)}\mathbb{1}_V\right] &= \frac{N_V}{2}\,\mathrm{Tr}\left[\frac{\bar{\Delta}^{\mu\nu}_{(1)}\left(\partial_t r_k-\eta_V r_k\right)}{\bar{\Delta}^{\mu\nu}_{(1)}(1+r_k)}\right]\\
&= \frac{N_V}{2}\frac{1}{(4\pi)^2}\left[\int_x\sqrt{\bar{g}}\;\Phi^1_2(0)-\frac{1}{3}\int_x\sqrt{\bar{g}}\;\bar{\mathcal{R}}\;\Phi^1_1(0)\right] \qquad (5.29)\\
&= \frac{N_V}{2}\frac{1}{(4\pi)^2}\int_x\sqrt{\bar{g}}\left[\left(1-\frac{\eta_V}{6}\right)-\frac{2}{3}\bar{\mathcal{R}}\left(1-\frac{\eta_V}{6}\right)\right]
\end{aligned}
$$

To finish the calculation of the gauge field contribution, we need to take the ghost term into account. Fortunately, it has already the desired form, where we can directly read off the two-point function:

$$
\Gamma^{(2)}_{\bar{C}C}=\frac{\delta^2\mathcal{S}_V}{\delta C^i\,\delta\bar{C}^j}=\mathbb{1}_V\cdot\bar{\Delta}. \tag{5.30}
$$

Note, that we have the usual Laplacian $\bar{\Delta}=-\bar{\nabla}^2$ as kinetic operator and that no wave function renormalization was introduced for the Abelian ghosts. The ghost regulator $R_{k,\mathrm{gh}}$ is the same as for the scalar fields and therefore the regularized two-point function reads

$$
\Gamma^{(2)}_{k,\bar{C}C}=\left[\Gamma^{(2)}_{\bar{C}C}+R_{k,\mathrm{gh}}\right]=\bar{\Delta}\cdot\mathbb{1}_V\left(1+r_k\left(\frac{\bar{\Delta}}{k^2}\right)\right). \tag{5.31}
$$

In absence of a wave function renormalization, the scale derivative only acts on the shape function r_k and therefore the final contribution is given by

$$
\begin{aligned}
-\,\mathrm{Tr}\left[\frac{\bar{\Delta}\partial_t r_k}{\bar{\Delta}\,(1+r_k)}\mathbb{1}_V\right] &= -N_V\,\mathrm{Tr}\left[\frac{\bar{\Delta}\partial_t r_k}{\bar{\Delta}\,(1+r_k)}\right]\\
&= -N_V\frac{1}{(4\pi)^2}\left[\int_x\sqrt{\bar{g}}\;\Phi^1_2(0)+\frac{1}{6}\int_x\sqrt{\bar{g}}\bar{\mathcal{R}}\;\Phi^1_1(0)\right] \qquad (5.32)\\
&= -N_V\frac{1}{(4\pi)^2}\int_x\sqrt{\bar{g}}\left[1+\frac{1}{3}\bar{\mathcal{R}}\right].
\end{aligned}
$$

In the next section, we combine the obtained results and give the final expressions for the beta functions.

5.2. Beta-Functions and Perturbative Approximation

We investigate the impact of the different matter fields in a qualitative analysis.

Chapter 6.

Background independence in Quantum Gravity

- Nielsen Identity
- Vertex Expansion in powers of fluctuation field h on flat background
- breaking of the split symmetry due to regulator depending only on background

Chapter 7.

Summary and Outlook

Appendix A.

Mathematical Background

In this part of the appendix we want to discuss some of the mathematical tools we used during the calculations presented in the scope of this thesis in a more formal manner. The part on the York decomposition is mainly inspired by [14], whereas the conventions for the heat-kernel computations are taken from [13] and extended for the matter part, using the conventions from [3].

A.1. York Decomposition

In the discussion of gauge theories, it is often very useful to decompose the gauge field A_μ into transversal and longitudinal parts:

$$A_\mu = A_\mu^{\mathrm{T}} + \nabla_\mu \phi. \tag{A.1}$$

The transversal part is characterized by the fact, that $\nabla^\mu A_\mu^{\mathrm{T}} = 0$. Using this decomposition, we are able to separate the pure gauge spin-0 degrees of freedom from the physical ones, contained in the spin-1 part A_μ^{T}.
Assuming vanishing boundary terms, integration by parts allows us to change the integration variables in the functional integral, i. e.

$$\int_x \sqrt{g}\, A_\mu A^\mu = \int_x \sqrt{g}\, A_\mu^{\mathrm{T}} A^{\mathrm{T},\mu} + \int_x \sqrt{g}\, \phi \left(-\nabla^2\right) \phi. \tag{A.2}$$

Note, that we have to take care of the Jacobian J of this variable transformation:

$$(\mathrm{d}A_\mu) \longrightarrow J \left(\mathrm{d}A_\mu^{\mathrm{T}}\right) (\mathrm{d}\phi)\,. \tag{A.3}$$

To be able to determine the Jacobian for our transformation, the integration measure needs to be normalized. A quite convenient choice is to evaluate the Gaussian integral over the different fields ψ and set the result to one:

$$\int (\mathrm{d}\psi) \exp\left\{-\int \mathrm{d}x\, \sqrt{g}\, \psi^2\right\} = 1, \tag{A.4}$$

where we are assuming an Euclidean signature and a curved background metric. With this condition we find:

$$1 = J \int \left(\mathrm{d}A_\mu^{\mathrm{T}}\right) \mathrm{e}^{-\int \mathrm{d}x \sqrt{g}\, A_\mu^{\mathrm{T}} A^{\mathrm{T},\mu}} \int (\mathrm{d}\phi)\, \mathrm{e}^{-\int \mathrm{d}x \sqrt{g}\, \phi\left(-\nabla^2\right)\phi} = J \left(\mathrm{det}'_\phi \left(-\nabla^2\right)\right)^{-1/2}. \tag{A.5}$$

This allows us to determine the Jacobian J as follows:

$$J = \left(\det'_{\phi}\left(-\nabla^2\right)\right)^{1/2}. \tag{A.6}$$

The prime denotes the fact, that the zero mode has to be removed, when computing the determinant to obtain a consistent result. Physically this is in accordance with the fact, that a constant ϕ does not contribute to A_μ.
Four our computation in chapters 4 and 5, we were using the background field method, where we assume a linear split of the *full* metric $g_{\mu\nu}$ into a background metric $\bar{g}_{\mu\nu}$ and a fluctuation field $h_{\mu\nu}$. There is an analogous way of decomposing the fluctuation field in the background field formalism. First, we split $h_{\mu\nu}$ into

$$h_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu} + \frac{1}{d}\,\bar{g}_{\mu\nu} h, \tag{A.7}$$

where $h^{\mathrm{T}}_{\mu\nu}$ is traceless, i. e. $\bar{g}^{\mu\nu} h^{\mathrm{T}}_{\mu\nu} = 0$ and $h = \bar{g}^{\mu\nu} h_{\mu\nu}$. The traceless part can be further decomposed in flat space using the irreducible representations of the Lorentz group with spins 0, 1 and 2 respectively, but in our case a more sophisticated approach, the so-called *York decomposition* is chosen:

$$h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{d}\,\bar{g}_{\mu\nu}\bar{\nabla}^2\right)\sigma + \frac{1}{d}\,\bar{g}_{\mu\nu} h. \tag{A.8}$$

Here, $h^{\mathrm{TT}}_{\mu\nu}$ is a transverse-traceless, spin-2 degree of freedom, ξ_μ is transverse and carries a spin-1 d. o. f. and σ and h have spin-0. As before, we want to find the Jacobian J for this variable transformation:

$$(\mathrm{d}h_{\mu\nu}) \longrightarrow J\left(\mathrm{d}h^{\mathrm{TT}}_{\mu\nu}\right)(\mathrm{d}\xi_\mu)(\mathrm{d}\sigma)(\mathrm{d}h). \tag{A.9}$$

This is again possible after specifying a suitable normalization of the functional measure as

$$\int (\mathrm{d}h_{\mu\nu}) \exp\left\{-\mathcal{G}(h,h)\right\} = 1, \tag{A.10}$$

where $\mathcal{G}$ is an inner product in the space of symmetric two-tensors, defined as

$$\begin{aligned}\mathcal{G}(h,h) &= \int_x \sqrt{\bar{g}}\,\left(h_{\mu\nu}h^{\mu\nu} + \frac{a}{2}h^2\right)\\ &= \int_x \sqrt{\bar{g}}\,\Bigg[h^{\mathrm{TT}}_{\mu\nu}h^{\mathrm{TT},\mu\nu} + 2\xi_\mu\left(-\bar{\nabla}^2 - \frac{\bar{R}}{d}\right)\xi^\mu \\ &+ \frac{d-1}{d}\sigma\left(-\bar{\nabla}^2\right)\left(-\bar{\nabla}^2 - \frac{\bar{R}}{d-1}\right)\sigma + \left(\frac{1}{d}+\frac{a}{2}\right)h^2\Bigg]\end{aligned} \tag{A.11}$$

in the case of an Einstein type background metric[1]. This yields

$$J=\left(\det_\xi\left(-\bar{\nabla}^2-\frac{R}{d}\right)\right)^{1/2}\left(\det'_\sigma\left(-\bar{\nabla}^2\right)\right)^{1/2}\left(\det_\sigma\left(-\bar{\nabla}^2-\frac{R}{d-1}\right)\right)^{1/2}. \tag{A.12}$$

Note, that the prime has the same meaning and physical interpretation as in the previous case: If σ is constant, it does not contribute to $h_{\mu\nu}$.
For both cases, the decomposition of the general gauge field and the York decomposition of the fluctuation field, appropriate rescalings of the fields ϕ, ξ_μ and σ respectively, help us to cancel the non-trivial Jacobians and to achieve, that all modes have the same mass dimension. For the sake of completeness, we present the rescaled versions of the fields:

$$\hat{\phi}=\sqrt{-\nabla^2}\,\phi \tag{A.13}$$

$$\hat{\xi}_\mu=\sqrt{-\bar{\nabla}^2-\frac{\bar{R}}{d}}\,\xi_\mu \tag{A.14}$$

$$\hat{\sigma}=\sqrt{-\bar{\nabla}^2}\sqrt{-\bar{\nabla}^2-\frac{\bar{R}}{d-1}}\,\sigma. \tag{A.15}$$

The resulting graviton two-point function, after decomposition of the fluctuation field has the following structure:

$$\Gamma^{(2)}_{hh}=\begin{pmatrix}\Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}} & 0 & 0 & 0\\ 0 & \Gamma^{(2)}_{\xi\xi} & 0 & 0\\ 0 & 0 & \Gamma^{(2)}_{h^{\mathrm{Tr}}h^{\mathrm{Tr}}} & \Gamma^{(2)}_{h^{\mathrm{Tr}}\sigma}\\ 0 & 0 & \Gamma^{(2)}_{\sigma h^{\mathrm{Tr}}} & \Gamma^{(2)}_{\sigma\sigma}\end{pmatrix} \tag{A.16}$$

This concludes our discussion of the York decomposition, as a useful tool to simplify calculations in the background field method.

1. A metric is of Einstein type, if $R_{\mu\nu}$ is a constant multiple of $g_{\mu\nu}$, i. e. $R_{\mu\nu}=\frac{1}{d}\mathcal{R}g_{\mu\nu}$.

A.2. Heat-Kernel Techniques

We use heat-kernel techniques to evaluate the r. h. s. of the flow equation (2.20b), where we need to compute the functional trace over functions depending on the Laplacian on a curved background. In general, the method can be understood as a curvature expansion about a flat background.
The general formula to compute such traces is given by

$$\mathrm{Tr}\, f(\Delta) = N \sum\!\!\!\!\!\!\int_{\ell} \rho(\ell) f(\lambda(\ell)), \tag{A.17}$$

with some normalization N, the spectral values $\lambda(\ell)$ and their corresponding multiplicities $\rho(\ell)$.
On flat backgrounds, the computation of (A.17) is simply a standard momentum integral, whereas on curved backgrounds, consider for example a four-sphere $\mathbb{S}^4$ with constant background curvature $r = \frac{\bar{\mathcal{R}}}{k^2} > 0$, the spectrum of the Laplacian is discrete and we need to sum over all spectral values.
For our example of $\mathbb{S}^4$, we have

$$\lambda(\ell) = \frac{\ell(3+\ell)}{12} r \qquad \text{and} \qquad \rho(\ell) = \frac{(2\ell+3)(l+2)!}{6\ell!}. \tag{A.18}$$

The normalization is then given by the inverse of the four-sphere-volume $(V_{\mathbb{S}^4})^{-1} = \frac{k^4 r^2}{384\pi^2}$. This leads us to the formula for our computation of the r.h.s. of the flow equation on a background with constant positive curvature:

$$\mathrm{Tr}\, f(\Delta) = \frac{k^4 r^2}{384\pi^2} \sum_{\ell=0}^{\infty} \frac{(2\ell+3)(\ell+2)!}{6\ell!} f\left(\frac{\ell(3+\ell)}{12} r\right). \tag{A.19}$$

This is called spectral sum. For large curvatures r the convergence of the series is rather fast, whereas in the limit $r \to 0$ one finds exponentially slow convergence.
The master equation for heat kernel computations reads

$$\mathrm{Tr}\, f(\Delta) = \frac{1}{(4\pi)^{\frac{d}{2}}} \left[\mathbf{B}_0(\Delta) Q_2[f(\Delta)] + \mathbf{B}_2(\Delta) Q_1[f(\Delta)]\right] + \mathcal{O}\left(\mathcal{R}^2\right), \tag{A.20}$$

with the heat-kernel coefficients

$$\mathbf{B}_n(\bar{\Delta}) = \int_x \sqrt{g}\ \mathrm{Tr}\, \mathbf{b}_n(\bar{\Delta}) \tag{A.21}$$

and

$$Q_n[f(x)] = \frac{1}{\Gamma(n)} \int \mathrm{d}x\, x^{n-1} f(x). \tag{A.22}$$

For computations on $\mathbb{S}^4$, the values for the heat kernel coefficients $\mathbf{B}_n(\bar{\Delta})$ are presented in the following.

	TT	TV	S
$\mathrm{Tr}\,\mathbf{b}_0$	5	3	1
$\mathrm{Tr}\,\mathbf{b}_2$	$-\frac{5}{6}\mathcal{R}$	$\frac{1}{4}\mathcal{R}$	$\frac{1}{6}\mathcal{R}$

Table A.1.: Heat-kernel coefficients for transverse-traceless tensors (TT), transverse vectors (TV) and scalars (S) for computations on $\mathbb{S}^4$.

The basic idea of the proof of equation (A.17) is based on the Laplace transform

$$f(\Delta) = \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-s\Delta}\,\tilde{f}(s). \tag{A.23}$$

We insert this definition of the Laplace transform into equation (A.17) and find

$$\mathrm{Tr}\,f(\Delta) = \int_0^\infty \mathrm{d}s\,\tilde{f}(s)\,\mathrm{Tr}\,\mathrm{e}^{-s\Delta}\,. \tag{A.24}$$

The trace on the r. h. s. is explicitly the trace of the heat kernel. We expand this term as follows:

$$\mathrm{Tr}\,\mathrm{e}^{-s\Delta} = \frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n=0}^{\infty} s^{\frac{n-d}{2}}\mathbf{B}_n(\Delta). \tag{A.25}$$

This is where the heat-kernel coefficients $\mathbf{B}_n$ become important. We proceed by inserting this expanded version of the heat-kernel trace into equation (A.24) and find:

$$\begin{aligned}
\mathrm{Tr}\,f(\Delta) &= \frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n=0}^{\infty}\mathbf{B}_n(\Delta)\int_0^\infty \mathrm{d}s\,s^{\frac{n-d}{2}}\tilde{f}(s) \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(\frac{d-k}{2}\right)}\mathbf{B}_n(\Delta)\int_0^\infty \mathrm{d}t\,t^{\frac{d-n}{2}-1}f(t) \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}}\sum_{n=0}^{\infty}\mathbf{B}_n(\Delta)Q_{\frac{d-n}{2}}[f(t)].
\end{aligned} \tag{A.26}$$

This completes the derivation of the master equation (A.20) for heat-kernel computations. Note, that we used the definition of the Q-functionals, given in equation (A.22) and the relation $\int_s s^{-x}\tilde{f}(x) = \frac{1}{\Gamma(x)}\int_z z^{x-1}f(z)$.

When investigating matter fields, such as in chapter 5, we often encounter kinetic operators of the form $\tilde{\Delta} = -\nabla^2 \cdot \mathbb{1} + \mathbf{E}$, where $\mathbf{E}$ is a linear map acting on the spacetime and the internal indices of the fields. In this notation, $\mathbb{1}$ has to be understood as the identity

in the respective field space.
If $[\Delta, \mathbf{E}] = 0$[2], we can relate the coefficients of the modified Laplacian $\tilde{\Delta}$ and those of the initially considered operator $-\nabla^2$ via

$$\mathrm{Tr}\, \mathrm{e}^{-s\left(-\nabla^2+\mathbf{E}\right)} = \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{k,l=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} \int_x \sqrt{g}\ \mathrm{Tr}\, \mathbf{b}_k(\Delta) \mathbf{E}^{\ell} s^{k+\ell-2}. \tag{A.27}$$

This results in the following, modified values for the coefficients we are interested in:

$$\begin{aligned} \mathbf{b}_0 &= \mathbb{1} \\ \mathbf{b}_2 &= \frac{\mathcal{R}}{6} \cdot \mathbb{1} - \mathbf{E}. \end{aligned} \tag{A.28}$$

For further study and a more general treatment of the modified Laplacians, including higher order coefficients, [3, 14] are recommended.

2. In the case of $[\Delta, \mathbf{E}] \neq 0$, there would be additional terms including (higher order) commutators of Δ and $\mathbf{E}$ due to the Baker-Campbell-Hausdorff formula.

Appendix B.

Additional calculations

For the sake of completeness, we present some auxiliary calculations and important steps, that were used to obtain the results presented in scope of this work, but were in general too long or unsuitable to be included in the main part.

B.1. Matter calculations

Fermion part

In the part on fermions, we are confronted with operators contracted with gamma matrices, represented as usual in Feynman slash notation. Here, we present the proof of an identity we used in the derivation of the fermion two-point function:

$$\begin{aligned}\not{\nabla}^2 &= \nabla_\mu \nabla_\nu \gamma^\mu \gamma^\nu \\ &= \frac{1}{2} \nabla_\mu \nabla_\nu \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \right) \\ &= g^{\mu\nu} \nabla_\mu \nabla_\nu \\ &= \nabla^2\end{aligned} \tag{B.1}$$

The third line follows from the definition of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$.

Gauge field part

During the computation of the gauge field contribution to the running of G and Λ, we encounter the following term, which can be simplified a lot after a few manipulations:

$$\begin{aligned}
\int g^{\mu\nu} g^{\kappa\lambda} F_{\mu\kappa} F_{\nu\lambda} = \int F_{\mu}{}^{\lambda} F^{\mu}{}_{\lambda} = \int F_{\mu\lambda} F^{\mu\lambda} & \\
&= \int \left(\partial_\mu A_\lambda - \partial_\lambda A_\mu\right) F^{\mu\lambda} + \mathcal{O}\left(A^3\right) \\
&\overset{(\star)}{=} \int 2\partial_\mu A_\lambda F^{\mu\lambda} \\
&= \int 2\partial_\mu A_\lambda \left(\partial^\mu A^\lambda - \partial^\lambda A^\mu\right) \\
&= \int 2\left(\partial_\mu A_\lambda \partial^\mu A^\lambda - \partial_\mu A_\lambda \partial^\lambda A^\mu\right) \\
&\overset{(\dagger)}{=} -\int 2\left(A_\lambda \partial^2 A^\lambda - A_\lambda \partial_\mu \partial^\lambda A^\mu\right) \\
&= \int 2A_\lambda \left[\partial^\mu \partial^\lambda - g^{\mu\lambda}\partial^2\right] A_\mu
\end{aligned} \tag{B.2}$$

For the first non-trivial step ($\star$) we use that $2\partial_\mu A_\lambda = \partial_{(\mu} A_{\lambda)} + \partial_{[\mu} A_{\lambda]}$, where $(\cdots)$ and $[\cdots]$ denote symmetrization and antisymmetrization w. r. t. the indices, respectively. The symmetric part vanishes due to the fact, that $F^{\mu\lambda}$ is antisymmetric under $\mu \rightleftharpoons \lambda$. This allows us to write $2\partial_\mu A_\lambda F^{\mu\lambda} = \partial_{[\mu} A_{\lambda]} F^{\mu\lambda} = \left(\partial_\mu A_\lambda - \partial_\lambda A_\mu\right) F^{\mu\lambda}$. The second non-trivial step ($\dagger$) results from integrating by parts and assuming vanishing boundary terms.

Later on in the gauge field calculation, after specifying the gauge parameter $\xi = 1$, we encounter a commutator of covariant derivatives acting on A_μ. With the definition of the curvature tensor, given in equation (3.10), we find

$$\begin{aligned}
\left[\nabla^\mu, \nabla^\lambda\right] A_\mu &= R_\mu{}^{\rho\mu\lambda} A_\rho \\
&= R^{\rho\lambda} A_\rho
\end{aligned} \tag{B.3}$$

Now, one simply has to rename the dummy indices $\rho \rightleftharpoons \mu$ to find the wanted expression.

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List of Figures

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Declaration of Authorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise.

Heidelberg, 8$^{\text{th}}$ of July 2019

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Mathieu Kaltschmidt