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FEYNMAN DIAGRAMS FOR THE YANG-MILLS FIELD

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Feynman and De Witt showed, that the rules must be changed for the calculation of contributions from diagrams with closed loops in the theory of gauge invariant fields. They suggested also a specific recipe for the case of one loop. In this letter we propose a simple method for calculation of the contribution from arbitrary diagrams. The method of Feynman functional integration is used.

It is known, that one can associate the field of the Yang-Mills type with an arbitrary simple group G [1-3]. It is appropriate to describe this field by means of the matricies $B_\mu(x)$ with values in the Lie algebra of this group.

The gauge group consits of the transformations

$$B_\mu \to B_\mu^\Omega = \Omega B_\mu \Omega^{-1} + \epsilon^{-1}\partial_\mu \Omega\Omega^{-1}$$

where $\Omega(x)$ is an arbitrary function with values in the group G.

The Lagrange function

$$\mathcal{L}(x) = -\tfrac{1}{4}\mathrm{Sp}G_{\mu\nu}G^{\mu\nu};\; G_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + \epsilon[B_\mu, B_\nu]$$

is invariant with respect to these transformations. It is clear, that

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

where $\mathcal{L}_0$ is a quadratic form, and $\mathcal{L}_1$ is the sum of trilinear and quartic forms in B. In the quantization of the Feynman type $\mathcal{L}_1$ generates vertices with three and four external lines and $\mathcal{L}_0$ is to define the propagator function. However the form $\mathcal{L}_0$ is singular and the longitudinal part of the propagator can not be found unambiguously. This ambiguity does not influence the physical results in quantum electrodynamics. It seems that Feynman [4] was the first to show that the matter is not so simple in the cases of Yang-Mills and gravitational fields. Namely the contribution of the closed loop diagrams depends essentially on the longitudinal part of the propagator and spoils the transversality and unitarity properties of scattering amplitudes. Feynman himself described the necessary change of rules for calculation the contribution from diagrams with one closed loop. A more detailed derivation of the new rules was given by De Witt [5]. However it seems that nobody gave the generalization of these rules for arbitrary diagrams.

The formal considerations below are to give a simple explanation of the described difficulties and a quite workable recipe to circumvent them.

We know from Feynman [6] that every element of the S-matrix can be written down as the functional integral

$$\langle \mathrm{in} | \mathrm{out} \rangle \sim \int \exp\{\mathrm{i}S[B]\} \prod_x \mathrm{d}B(x)$$

up to an (infinite) normalizing factor. Here $S[B] = \int \mathcal{L}(x)\mathrm{d}x$ is the action functional and one is to integrate over all fields $B(x)$ with the asymptote at $t = x_0 \to \pm\infty$ prescribed by in-and out-states. The diagrams appear naturally in the perturbative calculation of this integral.

In the case of gauge invariant theory it is necessary to transform this integral a little. In fact, we can say, using the natural geometrical language, that the integrand is constant on the "orbits" $B_\mu \to B_\mu^\Omega$ of the gauge group in the manifold of all fields $B_\mu(x)$. It follows that the integral itself is proportional to the volume of this orbit which can be expressed as the integral $\int \prod_x d\Omega(x)$ over all matrices $\Omega(x)$. This integral should be factorized before using the perturbation theory.

There exist several methods for this purpose. The idea of one of them is to integrate over the orbits and some transversal surface. It is appropriate to choose for the latter the "plane" $\partial_\mu B^\mu = 0$. Then the integral reduces to the following

$$\int \exp\{iS[B]\}\ \Delta[B] \prod_x \delta\left(\partial_\mu B^\mu(x)\right) dB(x) \int \prod_x d\Omega(x)$$

where the factor $\prod_x \delta\left(\partial_\mu B^\mu\right)$ symbolizes that we integrate over transverse fields and $\Delta[B]$ is to be chosen such that the condition

$$\Delta[B] \int \prod_x \delta\left(\partial_\mu (B^\mu)^\Omega\right) d\Omega = \text{const}$$

holds. It is the nontriviality of $\Delta[B]$ which distinguishes the theories of Yang-Mills and gravitational fields from quantum electrodynamics.

We must know $\Delta[B]$ only for transverse fields and in this case all contribution to the last integral is given by the neighbourhood of the unit element of the group. After appropriate linearization we come to the condition

$$\Delta[B] \int \prod_x \delta\left(\Box u - \epsilon[B^\mu, \partial_\mu u]\right) du(x) = \text{const}$$

where $\Box$ is the D'Alembert operator and $u(x)$ are functions with values in the Lie algebra of the group G.

Formally $\Delta[B]$ is equal to the determinant of the operator

$$Au = \Box u - \epsilon[B^\mu, \partial_\mu u] \equiv A_0 u - \epsilon V(B) u$$

After extracting the trivial infinite factor $\det A_0$ we obtain the following expression for $\ln\Delta[B]$

$$\ln\Delta[B] = \ln(\det A/\det A_0) = \mathrm{Sp}\ln(1 - \epsilon A_0^{-1} V(B))$$

Developing the right hand side in a power series in ϵ we have the following expressions for the coefficients

$$\int dx_1 \ldots \int dx_n\ \mathrm{Sp}\left(B^{\mu_1}(x_1) \ldots B^{\mu_n}(x_n)\right) \times$$
$$\times\ \partial_{\mu_1} G(x_1 - x_2) \ldots \partial_{\mu_n} G(x_n - x_1)$$

where $G(x)$ is a Green function of the D'Alembert operator. This expression corresponds to the closed loop with the scalar particle propagating along it and interacting with the transverse vector particles according to the law $\sim \epsilon\ \mathrm{Sp}(\varphi[B^\mu \partial_\mu \varphi])$.

There results the diagram technique with the following features:

1. The pure transversal Green function is to be used as a propagator for the vector particles (Landau gauge).
2. It is necessary to take into account the new vertex with two scalar and one vector external line in addition to the ordinary vertices with three and four lines.

Concrete calculations with these changes in the rules give transverse and unitary expressions for the scattering amplitudes.

It must be stressed that the Landau gauge is essential for the new rules. It is connected with the chosen method of extracting the fact or $\int \prod_x d\Omega(x)$.

An other method leads to the expression

$$\int \exp\{iS[B] - \tfrac{1}{2} i \int \mathrm{Sp}(\partial_\mu B^\mu)^2\, dx\}\ \varphi[B] \prod_x dB$$

where the factor $\varphi[B]$ must be found from the condition

$$\varphi[B] \int \exp\{-\tfrac{1}{2} i \int \mathrm{Sp}\left(\partial_\mu (B^\mu)^\Omega\right)^2 dx\} \prod_x d\Omega = \text{const}$$

This integral gives the perturbation series with Feynman propagator, but the calculation of $\varphi[B]$ is more cumbersome than that of $\Delta[B]$.

We conclude with the comment that one can proceed in an analogous way with gravitation theory. The analog for $\Delta[B]$ is the determinant of the Beltrami - Laplace operator in a harmonic coordinate system.

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