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submitted by

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The Asymptotic Safety approach to Quantum Gravity

This Bachelor thesis has been carried out by

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at the

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at

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under the supervision of

Prof. Dr. Jan M. Pawlowski

The Asymptotic Safety approach to Quantum Gravity

Mathieu Kaltschmidt

Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Zusammenfassung

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Contents

Chapter 1.

Introduction

Throughout this thesis we use units such that $\hbar = c = G \equiv 1$.

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Chapter 2.

Functional Methods in Quantum Field Theory

This chapter introduces the the treatment of Quantum Field Theory using functional methods. The main goal is to derive the flow equation for the effective action functional, the generating functional for the one-particle irreducible (1PI) correlation functions. The flow equation was first derived by Christof Wetterich in 1993 [14]. It is the foundation for our treatment of Quantum Gravity in the Asymptotic Safety approach, discussed in more detail later on.

2.1. Generating Functionals and Correlation Functions

We consider a theory setting of N scalar fields $\varphi_a(x), a \in \{1, \dots, N\}$ in d-dimensional Euclidean space. The corresponding partition sum in presence of sources $J_a(x)$ reads

$$Z[J] = \int \mathcal{D}\varphi \, \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \,. \tag{2.1}$$

The information content of the partition sum results mainly from the classical action functional $\mathcal{S}[\varphi]$, which determines the classical field equations

$$\frac{\delta \mathcal{S}}{\delta \varphi(x)} = 0. \tag{2.2}$$

Notation: The scalar product sums over field components and integrates over all space …

$$J \cdot \varphi = \int_x J_a(x) \, \varphi_a(x) = \int_p \tilde{J}_a(p) \, \tilde{\varphi}_a(p) \tag{2.3}$$

with

$$\int_x = \int_{\mathbb{R}^d} \mathrm{d}^d x \qquad \text{and} \qquad \int_p = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d p}{(2\pi)^d} \tag{2.4}$$

Mean field description:

$$\phi := \langle \varphi \rangle = \frac{1}{Z} \frac{\delta Z}{\delta J}\bigg|_{J=0} = \int \mathcal{D}\varphi \, \varphi \, \mathrm{e}^{-\mathcal{S}[\varphi] + J\cdot\varphi} \tag{2.5}$$

Higher correlations:

$$\langle \varphi_1 \cdots \varphi_n \rangle := \langle \varphi^n \rangle = \frac{1}{Z} \frac{\delta^n Z}{\delta^n J} = \int \mathcal{D}\varphi \; \overbrace{\varphi_1 \cdots \varphi_n}^{:= \varphi^n} \; \mathrm{e}^{-\mathcal{S}[\varphi] + J \cdot \varphi} \tag{2.6}$$

The Schwinger functional W is then defined as

$$Z[J] = \mathrm{e}^{W[J]} \tag{2.7}$$

For the special case of $n = 2$ the correlation function yields the connected 2-point function which is also known as the propagator $G_{ab}(x, y) = G_{\alpha\beta}$ correlating the field φ_a at spacetime point x with the field φ_b at y.

$$\begin{aligned} G_{\alpha\beta} &= \frac{\delta^2 W[J]}{\delta J_\alpha \delta J_\beta} = \frac{\delta}{\delta J_\alpha} \left(\frac{1}{Z} \frac{\delta Z}{\delta J_\beta} \right) \\ &= \frac{1}{Z} \left(\frac{\delta^2 Z}{\delta J_\alpha \delta J_\beta} \right) - \frac{1}{Z^2} \left(\frac{\delta Z}{\delta J_\alpha} \right) \left(\frac{\delta Z}{\delta J_\beta} \right) \\ &= \langle \varphi_\alpha \varphi_\beta \rangle - \phi_\alpha \phi_\beta = \langle \varphi_\alpha \varphi_\beta \rangle_\mathrm{c} \end{aligned} \tag{2.8}$$

The Effective Action:

The effective action can be obtained by performing a Legendre transform of the Schwinger funtional, i. e.:

$$\Gamma[\phi] = \sup_J \left\{ \int_x J(x)\phi(x) - \mathcal{W}[J] \right\} = \int_x J_\mathrm{sub}(x)\phi(x) - \mathcal{W}[J_\mathrm{sub}] \tag{2.9}$$

Quantum equation of motion:

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = J(x) \tag{2.10}$$

Dyson-Schwinger equation:

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = \frac{\delta \mathcal{S}}{\delta \varphi(x)} \left[\varphi = G \cdot \frac{\delta}{\delta \phi} + \phi \right] \tag{2.11}$$

2.2. The Functional Renormalization Group

- Kadanoff Block-Spin model

- maybe visualization of Ising model + phase transitions

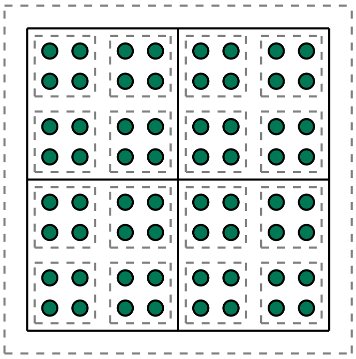

Figure 2.1.: Visualization of the Kadanoff Block-Spin model.[1]

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2.3. Flow Equations for Generating Functionals

We introduce the RG time scale t:

$$\partial_t = \frac{\partial}{\partial \ln(k/\Lambda)} = \frac{k}{\Lambda} \frac{\partial}{\partial(k/\Lambda)} = k\partial_k \tag{2.12}$$

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1. This visualization is inspired by an image provided in the PhD thesis of J.R. Laguna.

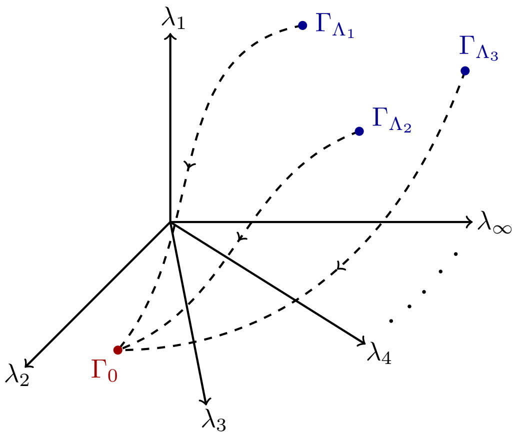
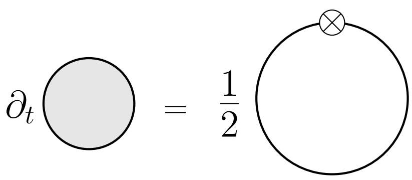


Figure 2.2.: Flow of Γ_k through infinite-dimensional theory space for different regulators.

need for special content, but the length of words should match the language.

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{\Gamma_k^{(2)}[\phi] + R_k}\partial_t R_k\right]$$
$$= \frac{1}{2}\int_p \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(p,-p)\,\partial_t R_k(p^2) \tag{2.13a}$$

This translates directly into the following diagrammic representation:



(2.13b)

where $\otimes = \partial_t R_k$ represents the insertion of the respective regulator.

Regulator properties:

$$R_k(p) \to \begin{cases} k^2 & \text{for } p \to 0 \\ 0 & \text{for } p \to \infty \\ 0 & \text{for } k \to 0 \\ \infty & \text{for } k \to \Lambda \end{cases} \tag{2.14}$$

Chapter 3.

Curved Spacetimes

This section is based on [2, 12].

3.1. An Introduction to Spacetime Geometry

inner product:

$$g(X,Y) = g_{\mu\nu}X^\mu Y^\nu = X^\mu Y_\mu = g^{\mu\nu}X_\mu Y_\nu = X_\mu Y^\mu \tag{3.1}$$

we can use the metric tensor to raise and lower spacetime indices,
Christoffel symbols (connection):

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\mu\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \tag{3.2}$$

Geodesic equation:

$$\int ds = \int d\tau \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau}} \tag{3.3}$$

$$\ddot{x}^\mu + \Gamma^\mu{}_{\sigma\rho}\dot{x}^\sigma\dot{x}^\rho = 0 \tag{3.4}$$

Riemann/Curvature tensor:

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha{}_{\beta\delta} - \partial_\delta\Gamma^\alpha{}_{\beta\gamma} + \Gamma^\epsilon{}_{\beta\delta}\Gamma^\alpha{}_{\epsilon\gamma} - \Gamma^\epsilon{}_{\beta\gamma}\Gamma^\alpha{}_{\epsilon\delta} \tag{3.5}$$

Definition using the commutator of covariant derivatives

$$\left[\nabla_\mu, \nabla_\nu\right] A^\sigma = R^\sigma{}_{\rho\mu\nu}A^\rho \tag{3.6}$$

Contractions of the Curvature tensor:

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = g^{\alpha\beta}R^\beta{}_{\mu\alpha\nu} \tag{3.7}$$

Curvature Scalar:

$$\mathcal{R} = g_{\mu\nu}R^{\mu\nu} = R^\mu{}_\mu \tag{3.8}$$

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3.2. From Geometry to the Einstein Equations

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The Einstein-Hilbert action:

$$\mathcal{S}_{\mathrm{EH}}[g_{\mu\nu}] = \frac{1}{16\pi G} \int_x \sqrt{-\det g_{\mu\nu}} (\mathcal{R} - 2\Lambda) \tag{3.9}$$

Varying this action as usual yields the Einstein equations in absence of matter:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3.10}$$

where we used $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}$.

Diffeomorphism invariance, Lie derivatives:

$$\mathcal{L}_\omega \phi = \omega^\mu \partial^\mu \phi = \omega^\mu \nabla^\mu \phi \tag{3.11}$$

3.3. Gravity with Matter

Energy-Momentum Tensor:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-\det g_{\mu\nu}}} \frac{\delta \mathcal{S}_{\mathrm{matter}}}{\delta g^{\mu\nu}} \tag{3.12}$$

Matter part of the action for a minimally coupled scalar field ϕ:

$$\mathcal{S}_{\mathrm{matter}}[g_{\mu\nu}, \phi] = -\frac{1}{2} \int_x \sqrt{-\det g_{\mu\nu}} \left(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} V(\phi) \right) \tag{3.13}$$

From this, we get the Einstein equations including matter by demanding the variation $\sqrt{-\det g_{\mu\nu}} \frac{\delta \mathcal{S}}{\delta g^{\mu\nu}}$ to vanish. This yields:

$$\frac{1}{8\pi G}\left[\mathcal{R}_{\mu\nu}-\frac{1}{2}(\mathcal{R}-2\Lambda)g_{\mu\nu}\right]=T_{\mu\nu} \tag{3.14}$$

3.4. Perturbative Non-Renormalizability of Gravity

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Functional Renormalization and Quantum Gravity

4.1. RG approach to Quantum Gravity

Flow equation for QG:

$$\partial_t \Gamma_k[\bar{g}, \Phi] = \frac{1}{2} \operatorname{Tr} \, G_{\text{hh}}[\Phi] \, \partial_t R_k - \operatorname{Tr} \, G_{\text{c}\bar{\text{c}}}[\Phi] \, \partial_t R_k \tag{4.1}$$

4.2. Einstein-Hilbert Truncation

We want to solve the Flow equation (4.2) approximately. All terms that are invariant under the imposed symmetry, i. e. invariant under diffeomorphism transformations need to be taken into account.

Easiest truncation takes only the scalar curvature $\mathcal{R}$ and the cosmological constant Λ into account (No higher order terms ...) and was performed by Martin Reuter in 1993 [11].

This truncation reads

$$\Gamma_k = 2\kappa^2 Z_k \int_x \sqrt{\det g} \, [-\mathcal{R} + 2\Lambda_k] + \mathcal{S}_{\text{gf}} + \mathcal{S}_{\text{gh}} \tag{4.2}$$

with

$$\kappa^2 = \frac{1}{32\pi G}, \qquad G_k = G Z_k^{-1} \tag{4.3}$$

Linear gauge fixing F_μ and corresponding ghost term induced by the Faddeev-Popov procedure:

$$\begin{aligned} \mathcal{S}_{\text{gf}} &= \frac{1}{2\alpha} \int_x \sqrt{\det \bar{g}} \, \bar{g}^{\mu\nu} F_\mu F_\nu \\ \mathcal{S}_{\text{gh}} &= \int_x \sqrt{\det \bar{g}} \, \bar{g}^{\mu\mu'} \bar{g}^{\nu\nu'} \bar{c}_{\mu'} \mathcal{M}_{\mu\nu} c_{\nu'} \end{aligned} \tag{4.4}$$

with the Faddeev-Popov operator $\mathcal{M}_{\mu\nu}(\bar{g}, h)$ for the gauge fixing $F_\mu(\bar{g}, h)$.
A linear, de-Donder type gauge fixing with

$$F_\mu = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1+\beta}{4}\bar{\nabla}_\mu h^\nu{}_\nu$$
$$\mathcal{M}_{\mu\nu} = \bar{\nabla}^\rho(g_{\mu\nu}\nabla_\rho + g_{\rho\nu}\nabla_\mu) - \bar{\nabla}_\mu\nabla_\nu, \tag{4.5}$$

is employed, where $\beta = 1$ and $\alpha \to 0$ represents a fixed point of the RG flow. Note, that the limit $\alpha \to 0$ is performed after the gauge fixing process.

anomalous dimension:

$$\eta_g = -\frac{\partial_t Z_k}{Z_k} = -\partial_t \ln Z_k$$

dimensionless renormalized cosmological constant:

$$\lambda_k = \Lambda_k k^{-2}$$

dimensionless renormalized cosmological constant:

$$g_k = G_k k^{d-2} = \frac{G k^{d-2}}{Z_k}$$

corresponding beta function:

$$\beta_g = \partial_t g_k = \left(d - 2 + \eta_g\right) g_k \tag{4.6}$$

maximally symmetric space:

$$\bar{\mathcal{R}}_{\mu\nu} = \frac{1}{d}\,\bar{g}_{\mu\nu}\bar{\mathcal{R}} \tag{4.7}$$

$$\bar{\mathcal{R}}_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)}\,(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})\bar{\mathcal{R}} \tag{4.8}$$

suitable tensor basis:

$$h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \left(\bar{\nabla}_\mu\bar{\nabla}_\nu - \frac{1}{d}\,\bar{g}_{\mu\nu}\bar{\Delta}\right)\sigma + \frac{1}{d}\,\bar{g}_{\mu\nu} h \tag{4.9}$$

As a first approximation, we only take the contribution from the spin-two graviton mode $h^{\mathrm{TT}}_{\mu\nu}$ into account. This is motivated by the fact, that this mode carries the the most degrees

of freedom.
In this setting, we want to solve the Wetterich equation (2.13b) by computing the left hand side and the right hand side separately and extract the β-functions for the Newton coupling g_k and the cosmological constant λ_k by a comparison of all terms of order $\sim \sqrt{\det g}$ and $\sim \sqrt{\det g}\,\mathcal{R}$. Here, only the most important steps of the calculation are presented. For the complete calculation have a look at the Appendix A.
In our spin-two graviton mode approximation, we don't have to deal with the gauge-fixing and ghost parts ocuring in the effective action. The simplified version of equation (4.2) reads

$$\Gamma_{k,h^{\mathrm{TT}}} = 2\kappa^2 Z_k \int_x \sqrt{\det g}\,[-\mathcal{R} + 2\Lambda_k]. \tag{4.10}$$

We start by computing the transverse-traceless graviton two-point function

$$\Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}} = \frac{Z_k}{32\pi}\left(\bar{\Delta} - 2\Lambda_k + \frac{2}{3}\bar{\mathcal{R}}\right). \tag{4.11}$$

Using a regulator of the form

$$R_k = \Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}}\Big|_{\Lambda_k=\bar{\mathcal{R}}=0} \cdot r_k\left(\frac{\bar{\Delta}}{k^2}\right) = \frac{Z_k}{32\pi}\bar{\Delta}\left(\frac{k^2}{\bar{\Delta}} - 1\right)\Theta\left(1 - \frac{\bar{\Delta}}{k^2}\right),$$

with a Litim-type cutoff

$$r_k(x) = \left(\frac{1}{x} - 1\right)\Theta(1-x), \tag{4.12}$$

we are directly able to compute the l.h.s. of the Wetterich equation, i. e. the scale derivative of the effective average action:

$$\partial_t \Gamma_{k,h^{\mathrm{TT}}} = 2\kappa^2 Z_k \int_x \sqrt{\det g}\left\{\eta_g \mathcal{R} + 2\left(k^2(\partial_t \lambda_k) + \Lambda_k(2-\eta_g)\right)\right\} \tag{4.13}$$

One finds the β-function for the Newton coupling without performing the analysis of the Wetterich equation, i. e.

$$\beta_g = \partial_t g_k = \partial_t\left(\frac{G\cdot k^2}{Z_k}\right) = g_k\,(2+\eta_g)\,. \tag{4.14}$$

The computation of the r.h.s. of the flow equation is more complicated because it involves the computation of a trace of a function depending on the Laplacian. We can use heat-kernel techniques to solve such equations. Heat-kernel computations are based on a curvature expansion in powers of the curvature scaler $\mathcal{R}$. For more details, have a look at the appendix (A.2). As a first step, we simplify the trace expression as much as possible.

$$\mathrm{Tr}\left[\frac{1}{\Gamma_k^{(2)}+R_k}\partial_t R_k\right] = \mathrm{Tr}\left[\frac{\partial_t\left(\frac{Z_k}{32\pi}\bar{\Delta}\right)r_k}{\left(\frac{Z_k}{32\pi}\right)\left(\bar{\Delta}+2\Lambda_k+\frac{2}{3}\bar{\mathcal{R}}\right)+\left(\frac{Z_k}{32\pi}\bar{\Delta}\right)r_k}\right]$$
$$= \mathrm{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_g r_k\right)}{\bar{\Delta}(1+r_k)-2\Lambda_k+\frac{2}{3}\bar{\mathcal{R}}}\right] \tag{4.15}$$

We expand this expression around vanishing curvature and get

$$\mathrm{Tr}\left[\frac{1}{\Gamma_k^{(2)}+R_k}\partial_t R_k\right] = \mathrm{Tr}\left[\frac{\bar{\Delta}(\partial_t r_k-\eta_g r_k)}{\bar{\Delta}(1+r_k)-2\Lambda_k}\right] - \frac{2}{3}\bar{\mathcal{R}}\,\mathrm{Tr}\left[\frac{\bar{\Delta}(\partial_t r_k-\eta_g r_k)}{\left(\bar{\Delta}(1+r_k)-2\Lambda_k\right)^2}\right] + \mathcal{O}(\mathcal{R}^2) \tag{4.16}$$

Now we are able to evaluate these two terms separately using heat-kernel techniques. One finds for the first term

$$\mathrm{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_g r_k\right)}{\bar{\Delta}(1+r_k)-2\Lambda_k}\right] = \frac{1}{(4\pi)^2}\int_x \sqrt{\det \bar{g}}\left[5\Phi_2^1(-2\Lambda_k)-\frac{5}{6}\bar{\mathcal{R}}\Phi_1^1(-2\Lambda_k)\right], \tag{4.17}$$

with the threshold functions

$$\Phi_n^p(\omega) = \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z\; z^{n-1}\frac{z(-2zr_k(z)-\eta_g r_k(z))}{(z(1+r_k(z))+\omega)^p}. \tag{4.18}$$

Analogously, the second term in our expansion reads

$$-\frac{2}{3}\bar{\mathcal{R}}\,\mathrm{Tr}\left[\frac{\bar{\Delta}\left(\partial_t r_k-\eta_g r_k\right)}{\left(\bar{\Delta}(1+r_k)-2\Lambda_k\right)^2}\right] = -\frac{10}{3}\frac{\bar{\mathcal{R}}}{(4\pi)^2}\int_x \sqrt{\det \bar{g}}\frac{1-\frac{\eta_g}{6}}{(1-2\lambda_k)^2}. \tag{4.19}$$

For the cosmological constant, comparing the $\int\sqrt{\det g}$ terms yields

$$\beta_\lambda = \partial_t\lambda_k = -4\lambda_k + \frac{\lambda_k}{g_k}\partial_t g_k + \frac{10}{4\pi}g_k\frac{1-\frac{\eta_g}{6}}{1-2\lambda_k}. \tag{4.20}$$

where the anomalous dimension η_g is determined by comparing the $\int\sqrt{\det g}\mathcal{R}$ terms:

$$\eta_g = -\frac{5}{3\pi}\left(\frac{1-\frac{\eta_g}{4}}{1-2\lambda_k}+2\frac{1-\frac{\eta_g}{6}}{(1-2\lambda_k)^2}\right). \tag{4.21}$$

The solution of this system of coupled differential equations is evaluated using Python. We arrive at the following fixed point values for the Newton coupling and the cosmolog-

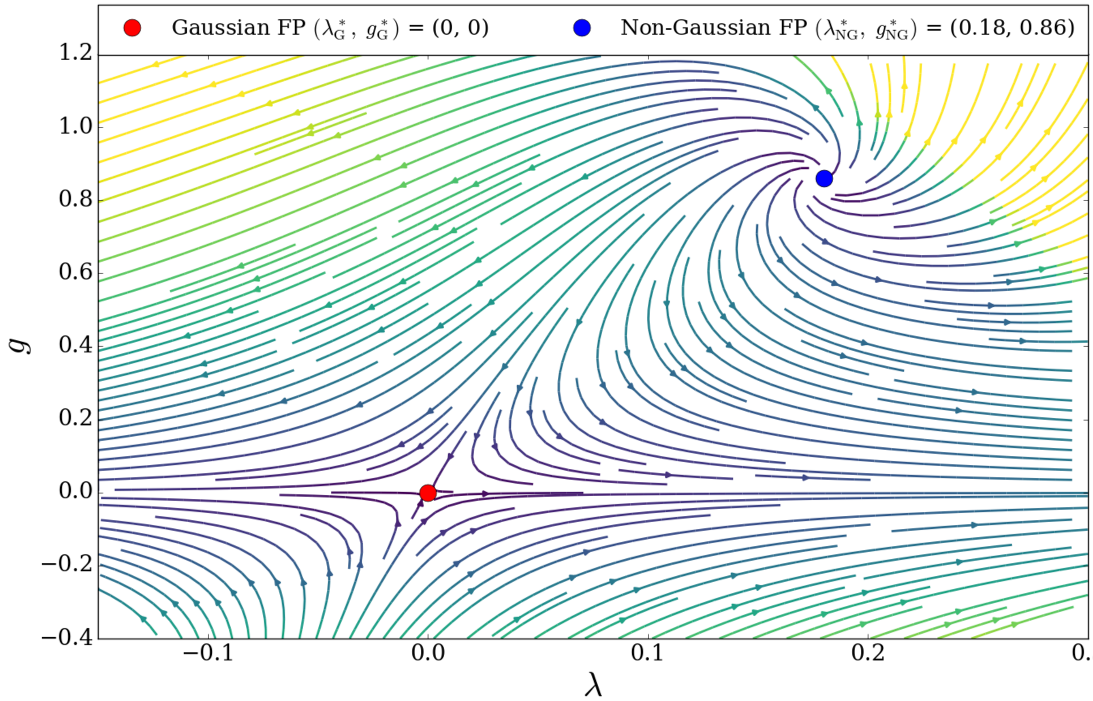

Figure 4.1.: RG flow diagram for the Einstein-Hilbert truncation in TT approximation as computed in this work. The flow points towards the infrared.

ical constant:

$$(g_k^*, \lambda_k^*) = (0.86, 0.18). \tag{4.22}$$

The corresponding critical exponents, i. e. minus the eigenvalues of the stability matrix evaluated at the fixed point, are given by

$$\theta_{1,2} = 2.9 \pm 2.6i \tag{4.23}$$

Chapter 5.

Asymptotic Safety of Gravity-Matter Systems

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5.1. Matter contributions

$$\Gamma_k = \Gamma_{\text{EH}} + \mathcal{S}_{\text{gf}} + \mathcal{S}_{\text{gh}} + \Gamma_{\text{matter}} \tag{5.1}$$

where the different matter contributions come from

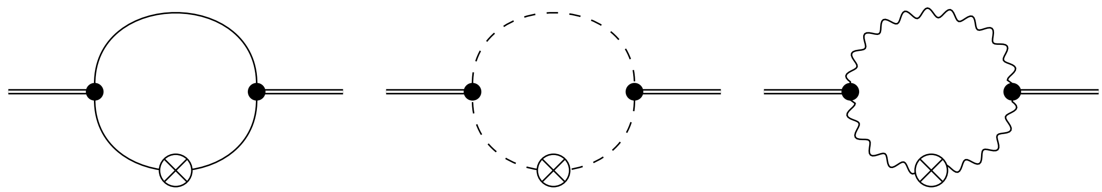
$$\Gamma_{\text{matter}} = \mathcal{S}_{\text{S}} + \mathcal{S}_{\text{D}} + \mathcal{S}_{\text{V}} \tag{5.2}$$

with

$$\mathcal{S}_{\text{S}} = \frac{Z_{\text{S}}}{2} \int_x \sqrt{\det g}\, g^{\mu\nu} \sum_{i=1}^{N_{\text{S}}} \partial_\mu \phi^i \partial_\nu \phi^i \tag{5.3}$$

$$\mathcal{S}_{\text{D}} = i Z_{\text{D}} \int_x \sqrt{\det g} \sum_{i=1}^{N_{\text{D}}} \bar{\psi}^i \nabla\!\!\!/\, \psi^i \tag{5.4}$$

$$\begin{aligned}\mathcal{S}_{\text{V}} = &\frac{Z_{\text{V}}}{4} \int_x \sqrt{\det g} \sum_{i=1}^{N_{\text{V}}} g^{\mu\nu} g^{\kappa\lambda} F^i_{\mu\kappa} F^i_{\nu\lambda} \\ &+ \frac{Z_{\text{V}}}{2\xi} \int_x \sqrt{\det \bar{g}} \sum_{i=1}^{N_{\text{V}}} \left(\bar{g}^{\mu\nu} \bar{D}_\mu A^i_\nu \right)^2 \\ &+ \frac{1}{2} \int_x \sqrt{\det \bar{g}} \sum_{i=1}^{N_{\text{V}}} \bar{c}_i (-\bar{D}^2) c_i\end{aligned} \tag{5.5}$$



(a) Fermions. **(b)** Scalars. **(c)** Gauge Fields.

Figure 5.1.: Different matter contributions to the graviton anomalous dimension η_h.

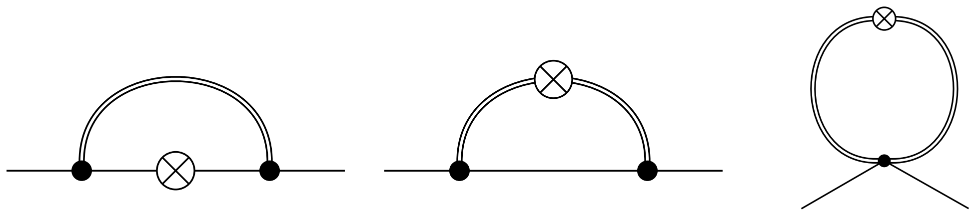


Figure 5.2.: Contributing diagrams to the fermion anomalous dimension η_D. Analogous contributions arise for external scalars and gauge fields to η_S and η_V.

5.2. Fermionic fields

Covariant derivative:

$$\nabla_\mu = \partial_\mu + \frac{1}{8}\left[\gamma^a, \gamma^b\right]\omega_\mu^{ab} \tag{5.6}$$

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Chapter 6.

Summary and Outlook

Appendix A.

Mathematical Appendix

In this part of the appendix we want to present some of the mathematical tools we used during the calculations presented in the scope of this thesis in a more formal manner.

A.1. York Decomposition

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

A.2. Heat-Kernel Techniques

We use heat-kernel techniques to evaluate the r.h.s. of the flow equation (2.13b), where we need to compute traces over functions depending on the Laplacian on a curved background. In general, the method can be understood as a curvature expansion about a flat background.

The formula to compute such traces is given by

$$\mathrm{Tr}\, f(\Delta) = N \sum\!\!\!\!\!\!\int_{\ell} \rho(\ell) f(\lambda(\ell)), \tag{A.1}$$

with some normalization N, the spectral values $\lambda(\ell)$ and their corresponding multiplicities $\rho(\ell)$.

On flat backgrounds, the computation of (A.1) is simply a standard momentum integral, whereas on curved backgrounds, consider for example a four-sphere with constant background curvature $r = \frac{\bar{\mathcal{R}}}{k^2} > 0$, the spectrum of the Laplacian is discrete and we need to sum over all spectral values in (A.1).

For our example of the four-sphere, we have

$$\lambda(\ell) = \frac{\ell(3+\ell)}{12} r \qquad \text{and} \qquad \rho(\ell) = \frac{(2\ell+3)(l+2)!}{6\ell!}. \tag{A.2}$$

and the normalization is given by the inverse of the four-sphere-volume $N = V_{S^4}^{-1} = \frac{k^4 r^2}{384\pi^2}$. This leads us to the formula for our computation of the r.h.s. of the flow equation on a background with constant positive curvature

$$\operatorname{Tr} f(\Delta) = \frac{k^4 r^2}{384\pi^2} \sum_{\ell=0}^{\infty} \frac{(2\ell+3)(\ell+2)!}{6\ell!} f\left(\frac{\ell(3+\ell)}{12} r\right). \tag{A.3}$$

This is called the spectral sum. For large curvatures r the convergence of the series is rather fast, whereas in the limit $r \to 0$ one finds exponentially slow convergence.
The master equation for heat kernel computations reads

$$\operatorname{Tr} f(\Delta) = \frac{1}{(4\pi)^{\frac{d}{2}}} \left[B_0(\Delta) Q_2[f(\Delta)] + B_2(\Delta) Q_1[f(\Delta)]\right] + O\left(\bar{\mathcal{R}}^2\right) \tag{A.4}$$

with the heat-kernel coefficients

$$B_n(\bar{\Delta}) = \int \mathrm{d}^d x \sqrt{\det \bar{g}}\ \operatorname{Tr} b_n(\bar{\Delta}) \tag{A.5}$$

and

$$Q_n[f(x)] = \frac{1}{\Gamma(n)} \int \mathrm{d}x x^{n-1} f(x). \tag{A.6}$$

For our computation on S^4, the traces over the coefficients $b_n(\bar{\Delta})$ are presented in the following.

	TT	TV	S
$\operatorname{Tr} b_0$	5	3	1
$\operatorname{Tr} b_2$	$-\frac{5}{6}\mathcal{R}$	$\frac{1}{4}\mathcal{R}$	$\frac{1}{6}\mathcal{R}$

Table A.1.: Heat-kernel coefficients for transverse-traceless tensors (TT), transverse vectors (TV) and scalars (S) for computations on the four-sphere S^4.

The basic idea of the proof of equation (A.1) is based on the Laplace transform

$$f(\Delta) = \int_0^{\infty} \mathrm{d}s\, \mathrm{e}^{-s\Delta}\, \tilde{f}(s). \tag{A.7}$$

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List of Figures

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Declaration of Authorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise.

Heidelberg, ____________________ ________________________________