Quantum Gravity and the Functional Renormalization Group

The Road towards Asymptotic Safety

MARTIN REUTER AND
FRANK SAUERESSIG

CAMBRIDGE MONOGRAPHS
ON MATHEMATICAL PHYSICS

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During the past two decades the gravitational Asymptotic Safety scenario has undergone a major transition from an exotic possibility to a serious contender as a realistic theory of quantum gravity. It aims at a mathematically consistent quantum description of the gravitational interaction and the geometry of spacetime within the realm of quantum field theory, keeping its predictive power at the highest energies. This volume provides a self-contained pedagogical introduction to Asymptotic Safety and introduces the functional renormalization group techniques used in its investigation, along with the requisite computational techniques. The foundational chapters are followed by an accessible summary of the results obtained thus far. It is the first detailed exposition of Asymptotic Safety, providing a unique introduction to quantum gravity. The text assumes no previous familiarity with the renormalization group and thus serves as an important resource for both practicing researchers and graduate students entering this maturing field.

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Contents

Preface

This book grew out of a series of lectures the authors have given at various universities and summer schools. Its intention is to provide an easily accessible, pedagogical account of the basic conceptual ideas and methods underlying the asymptotic safety approach to quantum gravity. Knowledge of General Relativity and quantum field theory at a master-course level should suffice to follow the exposition. The necessary technical background is developed from the beginning and no previous familiarity with functional renormalization group methods is assumed. As much as possible we try to supplement formal derivations by intuitive arguments. Our hope is that the book provides a valuable resource for graduate students and researchers, enabling them to follow the cutting-edge research in this field and placing it into the broad context.

It is impossible to thank all people here who directly or indirectly contributed to our work on quantum gravity and to this book ultimately. However, M. R. is particularly grateful to Christof Wetterich for the crucial and highly inspiring collaborations during the early days of the functional renormalization group and more than three decades of creative exchange and to Roberto Percacci for sharing the dream about asymptotic safety right from the start and for his decisive work toward making it a reality. It is also a pleasure to thank Alfio Bonanno for an enjoyable collaboration from very early on in an effort to understand the phenomenological consequences of asymptotic safety. Special heartfelt thanks go to Ennio Gozzi and Walter Dittrich for their support and guidance across all of physics, and well beyond.

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List of Abbreviations

ADM	Arnowitt–Deser–Misner
BRST	Becchi–Rouet–Stora–Tyutin
CDT	Causal Dynamical Triangulation
CREH	conformally reduced Einstein–Hilbert
EAA	Effective Average Action
EH	Einstein-Hilbert
FRG	Functional Renormalization Group
FRGE	Functional Renormalization Group Equation
GFP	Gaussian fixed point
GR	General Relativity
IR	infrared
LHS	left-hand side
LPA	local potential approximation
LQG	Loop Quantum Gravity
LSZ	Lehmann–Symanzik–Zimmermann
NGFP	non-Gaussian fixed point
ODE	ordinary differential equation
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QEG	Quantum Einstein Gravity
RG	renormalization group
RHS	right-hand side
UV	ultraviolet

1

A Quantum Field Theory of Gravity

Today we know of four kinds of fundamental interactions which seem to underlie all elementary processes observed in nature. Three of them, the electromagnetic, the weak, and the strong interactions, are combined in the standard model of elementary particle physics, which has received striking experimental confirmation during the past decades. Regarded as a classical field theory, the model employs geometrically natural and mathematically well-understood structures such as connections of the Yang–Mills type, for example. Furthermore, being renormalizable in perturbation theory, we also know how the model can be elevated to the level of a perturbatively defined quantum field theory. Beyond this stage there are ongoing efforts directed toward a non-perturbative definition and evaluation of at least certain sectors of the theory. Here, modern concepts of statistical field theory have proven invaluable. They explain, for instance, how the renormalization properties of the original continuum theory are related to the behavior of appropriate statistical mechanics models on spacetime lattices when they approach a second-order phase transition. These insights opened the door for employing Monte-Carlo simulations as a non-perturbative tool in quantum field theory, and in particular, as a device to test for the "existence" of a theory beyond the realm of perturbation theory.

1.1 Renormalizing the Unrenormalizable

As for our theoretical understanding of the fourth of the fundamental interactions, gravity, the situation is markedly different from the other three forces of nature. With Einstein's general theory of relativity we have a classical field theory at our disposal which is spectacularly successful in explaining gravitational phenomena on scales that span many orders of magnitude. However, when we try to quantize General Relativity (GR) along the same lines as the standard model, we find that this road is blocked since the theory is non-renormalizable within perturbation theory [1, 2]. At higher orders of the loop expansion the calculations must cope with an increasing number of new types of divergences and they all must be absorbed by suitable counter terms. The finite parts of their coefficients are left undetermined by the theory itself and so they must be taken from experiment. While this does not exclude the possibility of computing quantum corrections to predictions of the classical theory, it implies that those corrections involve an increasing number of undetermined parameters as the perturbation order is increased.

As long as one restricts attention to a regime where only a few such new coupling constants play a role, quantized General Relativity (GR) has the status of an *effective quantum field theory* [3–5]. Similar to chiral perturbation theory [6] it makes unambiguous predictions for certain leading quantum corrections. With increasing energy increasingly high loop orders must be included and hence the predictions unavoidably involve a growing number of undetermined parameters. At this stage the theory gradually loses its predictive power, and ultimately it may break down completely. Increasing the order of perturbation theory beyond this point, would then have the paradoxical consequence of diminishing the theory's net predictive power as the hoped-for better precision is more than offset by the new undetermined parameters it introduces.

This loss of predictivity at high-energy or short-distance scales is a strong motivation to search for a *fundamental quantum theory* of gravity, i.e., a theory that is *predictive on all scales* and that admits *potentially large quantum effects.* Ideally this hypothetical theory would contain only a few free parameters whose values are not fixed by the theory itself. Similar to the familiar situation in perturbatively renormalizable models it would express all predictions as well-defined, computable functions of those few measured parameters.

Given that GR is not renormalizable in standard perturbation theory, it has commonly been argued that a satisfactory microscopic theory of the gravitational interaction cannot be set up within the realm of quantum field theory, at least not without adding further symmetries, extra dimensions, or new principles such as holography, for instance. By contrast, the Asymptotic Safety program retains quantum field theory without such additions to the theoretical arena. Instead, it abandons the traditional techniques of perturbation theory, the concepts of perturbative renormalization, and of perturbative renormalizability in particular.

The Asymptotic Safety approach is based on the generalized notion of renormalization shaped by Kadanoff and Wilson [7–9] and the use of a "functional" or "exact" renormalization group (RG) equation. Hence, concepts from modern statistical field theory play a pivotal role. They provide a unified framework for approaching the problems with both continuum and discrete methods.

In the new setting one can conceive of *non-perturbatively renormalizable* quantum field theories, i.e., models free from physically harmful divergences that remain predictive up to the highest energies even though they are non-renormalizable in the perturbative sense.

The idea that there might exist a non-perturbatively renormalizable, or as he called it, "asymptotically safe" quantum theory of gravity was first proposed by S. Weinberg in the late 1970s [10]. At that time no efficient tools to test this scenario were available. However, an epsilon-expansion valid near *two* dimensions indicated that the new path could indeed be viable, at least for an unphysical dimensionality of spacetime. Further encouragement came from certain matter field theories that likewise were not renormalizable within perturbation theory but could be shown to be non-perturbatively renormalizable. In a paper entitled "Renormalizing the Non-renormalizable," Gawedzki and Kupiainen [11] used a $1/N$-expansion to prove the non-perturbative renormalizability of the Gross–Neveu model in three dimensions [12, 13].

The systematic exploration of the Asymptotic Safety scenario in four dimensions began only in the 1990s when powerful functional renormalization group methods became available for the gravitational field [14]. The exposition of these non-perturbative methods and their use in scrutinizing the viability of the Asymptotic Safety route to a fundamental quantum field theory of gravity is the main topic of this book.

1.2 Background Independence

While the various approaches trying to unify the principles of quantum mechanics and General Relativity are based upon rather different physical ideas and are formulated in correspondingly different mathematical frameworks,[1] they all must cope with the problem of Background Independence in one way or another. Whatever the ultimate theory of quantum gravity, a central requirement we impose on it is that it should be Background Independent in the same sense as GR. Loosely speaking, this means the spacetime structure that is actually realized in nature should not be part of the theory's definition but rather arise as a solution to certain dynamical equations [18–20].

In classical General Relativity the spacetime structure is encoded in a Lorentzian metric on a smooth manifold, and this metric, via Einstein's equation, is a dynamical consequence of the matter present in the universe. In quantum gravity we would like to retain the fundamental idea that "matter tells space how to curve, and space tells matter how to move," but describe both "matter" and "space" quantum mechanically. While, today, it is fairly well understood how to set up quantum field theories of matter systems, the open key problem is the quantum mechanical description of "space."

In this book we will mostly explore the possibility of constructing a quantum field theory of gravity in which the spacetime metric carries the dynamical degrees of freedom which we associate to "space." Even though this property is taken

[1] For reviews of the attempts to obtain a quantum theory of gravitation see, for example, [15–17].

from GR, the fundamental dynamics of those metric degrees of freedom is allowed to be different from that in the classical theory.

The quantum theory of gravity we are searching for will be required to respect the following principle of Background Independence: In the formulation of the theory no special metric should play any distinguished role. The actual metric of spacetime should arise as the expectation value of the quantum field (operator) $\widehat{g}_{\mu\nu}$ with respect to some state: $g_{\mu\nu} = \langle \widehat{g}_{\mu\nu} \rangle$.

This requirement is in sharp contradistinction to the traditional setting of quantum field theory of matter systems on Minkowski space whose conceptual foundations heavily rely on the availability of a non-dynamical (rigid) Minkowski spacetime as a background structure.

The principle of Background Independence[2] can be rephrased more precisely as follows. We require that *none of the theory's basic rules and assumptions, and none of its predictions, therefore, may depend on any special metric that has been fixed a priori. All metrics of physical relevance must result from the intrinsic quantum gravitational dynamics.*

A possible objection against this working definition [21] could be as follows: A theory can be made "Background Independent" in the above sense, but nevertheless has a distinguished rigid background if the latter arises as the unique solution to some field equation which is made part of the "basic rules." For instance, rather than introducing a Minkowski background directly one instead imposes the field equation $R^{\mu}{}_{\nu\rho\sigma} = 0$. However, this objection can apply only in a setting where the dynamics, the field equations, can be chosen freely. In asymptotically safe gravity this is impossible since, as we shall see, the dynamical laws are dictated by the fixed-point action. They are thus a prediction rather than an input.

If we try to set up a continuum quantum field theory for the metric itself, even assuming we are given some plausible candidate for a microscopic dynamics, described by, say, a diffeomorphism invariant bare action functional S, then already well before one encounters the notorious problems related to the UV divergences, profound conceptual problems arise. Just to name one, in absence of a rigid background when the metric is dynamical, there is no preferred time direction, for instance, hence no notion of equal time commutators, and clearly the usual rules of quantization cannot be applied straightforwardly.

Many more problems arise when one tries to apply the familiar concepts and calculational methods of quantum field theory to the metric itself without introducing a rigid background structure. Some of them are conceptually deep while others are of a more technical nature.

The problems are particularly severe if one demands that the sought-for theory can also describe potential phases of gravity in which $\langle \widehat{g}_{\mu\nu} \rangle$ is degenerate

[2] Here, and in the following, we write "Background Independence" with capital letters when we refer to this principle rather than simply to the independence of some quantity with respect to the background field.

(non-invertible) or completely vanishing in the most extreme case. Interpreting $\langle \widehat{g}_{\mu\nu} \rangle$ as an order parameter analogous to the magnetization in a magnetic system, a non-degenerate classical metric $\langle \widehat{g}_{\mu\nu} \rangle$ would signal a spontaneous breaking of diffeomorphism invariance that leaves only the stability group of $\langle \widehat{g}_{\mu\nu} \rangle$ unbroken, i.e., the Poincaré group for example, when $\langle \widehat{g}_{\mu\nu} \rangle$ is given by the Minkowski metric. Conversely, $\langle \widehat{g}_{\mu\nu} \rangle \equiv 0$ would then be the hallmark of a phase with completely unbroken diffeomorphism invariance.

The analogy to magnetic systems suggests that this "unbroken phase" is much easier to deal with than those with $\langle \widehat{g}_{\mu\nu} \rangle \neq 0$. However, in practice this is not the case, and again the reason is that the traditional toolbox of quantum field theory as shaped by the requirements of particle or condensed matter physics has very little to offer as soon as $g_{\mu\nu}$ vanishes. The familiar actions for matter fields such as, say, $\int \sqrt{g} g^{\mu\nu} D_\mu \phi D_\nu \phi$ or $\int \sqrt{g} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}$, can no longer be written down since they require an *invertible* $g_{\mu\nu}$, and problems of this kind are clearly only the tip of the iceberg.

A similar difficulty shows up when one tries to conceive an appropriate notion of a "functional renormalization group" in the realm of quantum gravity. In standard field theory on a rigid background spacetime typical regularization schemes (by higher derivative regulators, for example) which are used to make the calculations well defined both in the infrared (IR) and the ultraviolet (UV) make essential use of the metric provided by this background spacetime. As a result, it is not obvious whether and how such schemes can carry over to quantum gravity.

This problem is particularly acute for non-perturbative approaches employing any kind of functional renormalization group equation (FRGE) that would implement a Wilson-like or "exact" renormalization group flow by a repeated coarse graining [22–35].

In conventional Euclidean field theory as it is employed in statistical mechanics, for instance, every such coarse-graining step comes equipped with an associated length scale. In the case of, say, block-spin transformations it measures the size of the spacetime blocks within which the microscopic degrees of freedom were averaged. But when the metric is dynamical and no rigid background is available, this concept becomes highly problematic since it is not clear in terms of which metric one should measure the physical, i.e. proper extension of a given spacetime block.

From a continuum viewpoint, in one way or another all techniques of functional renormalization involve a mode decomposition of the (field) configurations that are summed or integrated over in the partition function or functional integral. In the standard case the modes are often taken to be plane waves, characterized by a momentum vector p_μ. They should be thought of as the eigenfunctions of the Laplacian $\delta^{\mu\nu}\partial_\mu\partial_\nu$. These modes are grouped into two classes then, namely long wavelength (or IR) modes, and short-wavelength (or UV) modes, respectively, depending on whether the Euclidean magnitude of their momentum, $(\delta^{\mu\nu} p_\mu p_\nu)^{1/2}$, is smaller or bigger than a certain value. The short-wavelength modes are then "integrated out" and the resulting effective dynamics of the long-wavelength modes is deduced.

In the absence of an intrinsically given metric comparable to $\delta^{\mu\nu}$ this procedure fails for (at least) the following obvious reasons: There is neither a natural, physically motivated way of choosing the basis of field modes, nor is it clear how to discriminate between IR and UV modes and to fix the order in which the individual modes belonging to some (ad hoc) basis of field modes should be integrated out.

Thus, the potential danger one faces in applying the ideas of coarse graining and RG flows to a continuum formulation of gravity is that in the absence of a naturally provided metric there is a considerable degree of arbitrariness in the flow that might ruin the power the exact RG has otherwise. After all, most of its celebrated successes on both the foundational or conceptual side (understanding the nature of continuum limits, etc.) and the practical side (computing useful effective descriptions of a given fundamental theory) heavily rely on the rule "short wavelengths first, long wavelengths second" when it comes to integrating out degrees of freedom. Trivial as it sounds, nothing the like of it is available in a manifestly Background Independent continuum formulation of gravity.

1.3 All Backgrounds Is No Background

There are two quite different strategies for complying with the requirement of Background Independence:

(i) One can try to define the theory and work out its implications without ever employing a background metric or a similar non-dynamical structure. This is the path taken in Loop Quantum Gravity [36–39] and the discrete approaches to quantum gravity [40–48], for instance, where manifest Background Independence has dramatic consequences for the structure of the theory [49]. As we saw, it seems very hard, if not impossible, to realize it in a continuum field theory, however.[3]

(ii) One takes advantage of an arbitrary classical background metric $\bar{g}_{\mu\nu}$ at the intermediate steps of the quantization, but verifies at the end that no physical prediction depends on which metric was chosen. This *background field method* is at the heart of the continuum-based gravitational average action approach [14], which we shall employ in our investigation of asymptotic safety.

[3] Some of the difficulties are reminiscent of those encountered in the quantization of topological Yang–Mills theories. Even when the classical action can be written down without the need of a metric, the gauge fixing and quantization of the theory usually requires one. Hence, the only way of proving the topological character of some result is to show its independence of the metric chosen.

The two strategies have complementary advantages and disadvantages. Following the path (i), Background Independence is implemented strictly. Hence, it is manifest at all intermediate steps of the constructions and calculations, but one must then cope with the above profound difficulties. Taking the path (ii) instead, Background Independence is not manifest during the intermediate steps and requires an effort to reestablish it at the end. This strategy has the invaluable benefit that basically the entire arsenal of general concepts and technical tools of conventional background-dependent quantum field theory is applicable.

In the simplest variant of the background field method one parameterizes the quantum metric as $\widehat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$ or a non-linear generalization thereof, and then quantizes the fluctuation $\widehat{h}_{\mu\nu}$ in essentially the same way one would quantize a matter field in a classical spacetime with metric $\bar{g}_{\mu\nu}$. In this way, all of the conceptual problems alluded to above, in particular the difficulties related to the construction of regulators, disappear.

Technically the quantization of gravity proceeds then almost as in standard field theory on a rigid classical spacetime, with one essential difference, though: In the latter, one concretely fixes the background $\bar{g}_{\mu\nu}$ typically as $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ or as $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ in the Euclidean case. In "background independent" quantum gravity instead, the metric $\bar{g}_{\mu\nu}$ is never specified concretely. All objects that one has to compute in this setting, generating functionals, say, are functionals of the variable $\bar{g}_{\mu\nu}$.

An example is the effective action $\Gamma[h_{\mu\nu}; \bar{g}_{\mu\nu}]$ that generates the dynamical equations for the expectation value $h_{\mu\nu} \equiv \langle \widehat{h}_{\mu\nu} \rangle$ of $\widehat{h}_{\mu\nu}$ and its higher n-point functions. It depends on both the background metric and the fluctuation expectation value. Similarly all n-point correlation functions of $\widehat{h}_{\mu\nu}$ which one computes from it have a parametric dependence on $\bar{g}_{\mu\nu}$. To stress this fact we sometimes write $h_{\mu\nu}[\bar{g}] \equiv \langle \widehat{h}_{\mu\nu} \rangle_{\bar{g}}$ for the 1-point function, for example.

Thus, in a sense, *the Background Independent quantization of gravity amounts to its quantization on all possible backgrounds simultaneously.*

There are now two metrics in the game which are equally important: the background $\bar{g}_{\mu\nu}$ and the expectation value of the full metric,

$$g_{\mu\nu} \equiv \langle \widehat{g}_{\mu\nu} \rangle = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \qquad h_{\mu\nu} \equiv \langle \widehat{h}_{\mu\nu} \rangle. \tag{1.1}$$

Alternatively we may regard the effective action as a functional of the two metrics rather than $h_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. We define

$$\Gamma[g_{\mu\nu}, \bar{g}_{\mu\nu}] \equiv \Gamma[h_{\mu\nu}; \bar{g}_{\mu\nu}]\Big|_{h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}} . \tag{1.2}$$

Because of the almost symmetric status enjoyed by the two metrics we refer to this setting as the *"bi-metric" approach to the Background Independence problem.*

As for the notion of an "exact renormalization group" in quantum gravity, we will introduce the Effective Average Action (EAA) as a scale-dependent version of the ordinary effective action with a built-in IR cutoff at a variable mass scale

k and derive in particular a functional RG equation for it. As we will explain in more detail below, the construction of the EAA and its RG equation are only possible due to the presence of the classical background spacetime.

Despite the unavoidable bi-metric appearance of the background field method, the expectation value of the microscopic metric, $g_{\mu\nu}$, and the variable background metric $\bar{g}_{\mu\nu}$, enter physical quantities (observables) not independently but instead are constrained by a symmetry requirement. Obviously the full metric $\widehat{g}_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$ is invariant under the *split symmetry transformation* $\boldsymbol{\delta}\widehat{h}_{\mu\nu} = \varepsilon_{\mu\nu}$, $\boldsymbol{\delta}\bar{g}_{\mu\nu} = -\varepsilon_{\mu\nu}$ with an arbitrary symmetric tensor field $\varepsilon_{\mu\nu}$. At the quantum level, this transformation implies Ward identities for the n-point functions and the effective (average) action similar to those implied by gauge or Becchi–Rouet–Stora–Tyutin (BRST) invariance. In either case one must make sure that in the end the quantum theory constructed actually satisfies these Ward identities.

In a way, this is the point where one is paying the price for the many advantages the background field technique brings about. However, it will become clear that while the extra work necessary to implement split symmetry, and thus Background Independence at the quantum level, is a hard technical challenge, it does not involve insoluble problems of principle.

1.4 Asymptotic Safety in a Nutshell

In this section we give a concise preview of what Asymptotic Safety is about. Technical details and refinements are omitted as much as possible. They will be delivered later on in this book.

(1) The problem. The ultimate goal of the Asymptotic Safety program consists in giving a mathematically precise meaning to, and actually compute, functional integrals over "all" spacetime metrics of the form $\int \mathcal{D}\widehat{g}_{\mu\nu} \, \exp\big(iS[\widehat{g}_{\mu\nu}]\big)$, or

$$Z = \int \mathcal{D}\widehat{g}_{\mu\nu} \, e^{-S[\widehat{g}_{\mu\nu}]} \,, \tag{1.3}$$

from which all quantities of physical interest can be deduced then. Here $S[\widehat{g}_{\mu\nu}]$ denotes the classical or, more correctly, the bare action. It is required to be diffeomorphism invariant, but is kept completely arbitrary otherwise. In general it differs from the usual Einstein–Hilbert action. This generality is essential in the Asymptotic Safety program: the viewpoint is that the functional integral would exist only for a certain class of actions S and the task is to identify this class.

(2) The problem, reformulated. Following the approach proposed in [14] one attacks this problem in an indirect way: rather than dealing with the integral per se, one interprets it as the solution of a certain differential equation, a functional renormalization group equation, or "FRGE". The advantage is that, contrary to the functional integral, the FRGE is manifestly well defined. It can be seen as an "evolution equation" in a mathematical sense, defining an infinite dimensional dynamical system in which the RG scale plays the role of time.

Loosely speaking, this reformulation replaces the problem of defining functional integrals by the task of finding evolution histories of the dynamical system that extend to *infinitely late times*. According to the Asymptotic Safety conjecture the dynamical system possesses a fixed point which is approached at late times, yielding well-defined, fully extended evolutions, which in turn tell us how to construct (or "renormalize") the functional integral.

(3) From the functional integral to the FRGE. Let us now be slightly more explicit about the passage from the functional integral to the FRGE.

(3a) Formal character of the integral. Recall that in trying to put the purely formal functional integrals on a solid basis one is confronted with a number of obstacles:

(i) As in every field theory, difficulties arise since one tries to quantize *infinitely many degrees of freedom*. Therefore, at the intermediate steps of the construction one keeps only finitely many of them by introducing cutoffs at very small and very large distances, Λ^{-1} and k^{-1}, respectively. We shall specify their concrete implementation in a moment. The ultraviolet and infrared cutoff scales Λ and k, respectively, have the dimension of a mass, and the original system is recovered for $\Lambda \to \infty$, $k \to 0$.

(ii) We mentioned already that the most severe problem one encounters when trying to quantize gravity is the requirement of *Background Independence*. In the approach to Asymptotic Safety along the lines of [14] we follow the spirit of DeWitt's background field method [50, 51] and introduce a (classical, non-dynamical) background metric $\bar{g}_{\mu\nu}$, which is kept arbitrary. We then decompose the integration variable as $\widehat{g}_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$, or a nonlinear generalization thereof, and replace $\mathcal{D}\widehat{g}_{\mu\nu}$ with an integration over the fluctuation, $\mathcal{D}\widehat{h}_{\mu\nu}$. In this way one arrives at a conceptually easier task, namely the quantization of the matter-like field $\widehat{h}_{\mu\nu}$ in a generic, but classical background $\bar{g}_{\mu\nu}$.

The availability of the background metric is crucial at various stages of the construction of an FRGE. However, the final physical results do not depend on the choice of a specific background.

(iii) As in every gauge theory, the *redundancy of gauge-equivalent field configurations* (diffeomorphic metrics) has to be carefully accounted for. Here we employ the Faddeev–Popov method and add a gauge-fixing term $S_{\text{gf}} \propto \int \sqrt{\bar{g}}\bar{g}^{\mu\nu} F_\mu F_\nu$ to S where $F_\mu \equiv F_\mu(\widehat{h}; \bar{g})$ is chosen such that the condition $F_\mu = 0$ picks a single representative from each gauge orbit. The resulting volume element on orbit space, the Faddeev–Popov determinant, is expressed as a functional integral over Grassmannian ghost fields C^μ and $\bar{C}_\mu$, governed by an action S_{gh}.

In this way (1.3) gets replaced by $\widetilde{Z}[\bar{\Phi}] = \int \mathcal{D}\widehat{\Phi}\, \exp\bigl(-S_{\text{tot}}[\widehat{\Phi}, \bar{\Phi}]\bigr)$. Here the total bare action $S_{\text{tot}} \equiv S + S_{\text{gf}} + S_{\text{gh}}$ depends on the dynamical fields $\widehat{\Phi} \equiv (\widehat{h}_{\mu\nu}, C^{\mu}, \bar{C}_{\mu})$, the background fields $\bar{\Phi} \equiv (\bar{g}_{\mu\nu})$, and possibly also on (both dynamical and background) matter fields, which for simplicity are not included here.

(3b) Standard effective action. Using the gauge fixed and regularized integral we can compute arbitrary ($\bar{\Phi}$-dependent!) expectation values $\langle \mathcal{O}(\widehat{\Phi})\rangle \equiv \widetilde{Z}^{-1} \int \mathcal{D}\widehat{\Phi}\, \mathcal{O}(\widehat{\Phi})\, e^{-S_{\text{tot}}[\widehat{\Phi},\bar{\Phi}]}$; for instance, n-point functions where $\mathcal{O}$ consists of strings $\widehat{\Phi}(x_1)\widehat{\Phi}(x_2)\dots\widehat{\Phi}(x_n)$. For $n=1$ we use the notation $\Phi \equiv \langle\widehat{\Phi}\rangle \equiv (h_{\mu\nu}, \xi^{\mu}, \bar{\xi}_{\mu})$, i.e., the elementary field expectation values are $h_{\mu\nu} \equiv \langle\widehat{h}_{\mu\nu}\rangle$, $\xi^{\mu} \equiv \langle C^{\mu}\rangle$ and $\bar{\xi}_{\mu} \equiv \langle\bar{C}_{\mu}\rangle$. Thus, the full dynamical metric has the expectation value $g_{\mu\nu} \equiv \langle\widehat{g}_{\mu\nu}\rangle = \bar{g}_{\mu\nu} + h_{\mu\nu}$.

The dynamical laws which govern the expectation value $\Phi(x)$ have an elegant description in terms of the *effective action* Γ. It is a functional depending on Φ similar to the classical $S[\Phi]$ to which it reduces in the classical limit. Requiring stationarity, S yields the classical field equation $(\delta S/\delta\Phi)[\Phi_{\text{class}}] = 0$, while Γ gives rise to a quantum mechanical analog satisfied by the expectation values, the *effective field equation* $(\delta\Gamma/\delta\Phi)[\langle\widehat{\Phi}\rangle] = 0$.

If, as in the case at hand, $\Gamma \equiv \Gamma[\Phi, \bar{\Phi}] \equiv \Gamma[h_{\mu\nu}, \xi^{\mu}, \bar{\xi}_{\mu}; \bar{g}_{\mu\nu}]$ also depends on background fields, the solutions of this equation inherit this dependence and thus $h_{\mu\nu} \equiv \langle\widehat{h}_{\mu\nu}\rangle$ functionally depends on $\bar{g}_{\mu\nu}$.

Technically, Γ is obtained from a functional integral with S_{tot} replaced by $S^{J}_{\text{tot}} \equiv S_{\text{tot}} - \int \mathrm{d}x\, J(x)\widehat{\Phi}(x)$. The new term couples the dynamical fields to external, classical sources $J(x)$ and repeated functional differentiation $(\delta/\delta J)^n$ of $\ln \widetilde{Z}[J, \bar{\Phi}]$ yields the n-point functions. In particular, $\Phi = \delta \ln \widetilde{Z}/\delta J$. It is a standard result that $\Gamma[\Phi, \bar{\Phi}]$ equals exactly the Legendre transform of $\ln \widetilde{Z}[J, \bar{\Phi}]$, at fixed background fields $\bar{\Phi}$.

The importance of Γ also resides in the fact that *it is the generating functional of special n-point functions from which all others can be easily reconstructed.* Therefore, finding Γ in a given quantum field theory is often considered equivalent to completely "solving" the theory.

(3c) Notions of gauge invariance. In practical applications of $\Gamma[\Phi, \bar{\Phi}]$ it is advantageous to employ a gauge-breaking condition F_{μ} that fixes a gauge belonging to the distinguished class of the so-called *background gauges*. To see the benefit, recall that the original gauge transformations read $\delta\widehat{g}_{\mu\nu} = \mathcal{L}_v \widehat{g}_{\mu\nu}$ where $\mathcal{L}_v$, denotes the Lie derivative with regards to the vector field v.

When we decompose $\widehat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$ we can distribute the gauge variation of $\widehat{g}_{\mu\nu}$ in different ways over $\bar{g}_{\mu\nu}$ and $\widehat{h}_{\mu\nu}$. In particular, this gives rise to what is known as *quantum gauge transformations*,

$$\boldsymbol{\delta}^{\mathbf{Q}}\widehat{h}_{\mu\nu} = \mathcal{L}_v(\bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu})\,, \quad \boldsymbol{\delta}^{\mathbf{Q}}\bar{g}_{\mu\nu} = 0 \tag{1.4}$$

and *background gauge transformations*,

$$\boldsymbol{\delta}^{\mathbf{B}}\widehat{h}_{\mu\nu} = \mathcal{L}_v \widehat{h}_{\mu\nu}\,, \quad \boldsymbol{\delta}^{\mathbf{B}}\bar{g}_{\mu\nu} = \mathcal{L}_v \bar{g}_{\mu\nu}. \tag{1.5}$$

Note that the "ordinary" or "true" gauge invariance the Faddeev–Popov method has to take care of is the $\boldsymbol{\delta}^{\mathbf{Q}}$-invariance. It must be gauge-fixed by the condition $F_\mu(\widehat{h};\bar{g}) = 0$.

Interestingly enough, there exist F_μ's, a variant of the harmonic coordinate condition, for example, which indeed fix the $\boldsymbol{\delta}^{\mathbf{Q}}$-transformations, but at the same time *transform covariantly under* $\boldsymbol{\delta}^{\mathbf{B}}$*-transformations*: $\boldsymbol{\delta}^{\mathbf{B}} F_\mu = \mathcal{L}_v F_\mu$. They implement the background gauges, and from now on we assume that one of those is employed.

Then, as a consequence, the effective action $\Gamma[\Phi,\bar{\Phi}]$ *is invariant under background-gauge transformations* which include the ghosts: $\boldsymbol{\delta}^{\mathbf{B}}\Gamma[\Phi,\bar{\Phi}] = 0$ where all fields transform as $\boldsymbol{\delta}^{\mathbf{B}}\Phi = \mathcal{L}_v\Phi$, $\boldsymbol{\delta}^{\mathbf{B}}\bar{\Phi} = \mathcal{L}_v\bar{\Phi}$.

We emphasize that this property should not be confused with another notion of "gauge independence," which the above $\Gamma[\Phi,\bar{\Phi}]$ actually does *not* have: It is not independent of which particular F_μ is picked from the class with $\boldsymbol{\delta}^{\mathbf{B}} F_\mu = \mathcal{L}_v F_\mu$. This F_μ-dependence will disappear only at the level of observables.

(3d) Effective Average Action. Heading now toward the concept of a *functional renormalization group for gravity* we recall that the above definition of Γ is based on the functional integral regularized in the IR and UV and hence depends on the corresponding cutoff scales: $\Gamma \equiv \Gamma_{k,\Lambda}[\Phi,\bar{\Phi}]$. It is this object for which we derive a FRGE, more precisely a closed evolution equation governing its dependence on the IR cutoff scale k. This is possible only if the IR regularization is implemented appropriately, as in the so-called *Effective Average Action* (EAA)[22, 23].

The EAA is related to the modified integral,

$$\int \mathcal{D}\widehat{\Phi}\, e^{-S^J_{\text{tot}}}\, e^{-\Delta S_k[\widehat{\Phi},\bar{\Phi}]} \equiv Z_{k,\Lambda}[J,\bar{\Phi}] \tag{1.6}$$

whose second exponential factor in the integrand, containing the *cutoff action* ΔS_k, is designed to achieve the IR regularization. To see how this works, assume the integration variable $\widehat{\Phi} = (\widehat{h}, C, \bar{C})$ is expanded in terms of eigenfunctions φ_p of the covariant tensor Laplacian related to the background metric, $\bar{D}^2 \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$. Writing $-\bar{D}^2\varphi_p = p^2\varphi_p$ we have, symbolically, $\widehat{\Phi}(x) = \sum_p \alpha_p\varphi_p(x)$. The α_p's are generalized Fourier coefficients, and so the functional integration over $\widehat{\Phi}$ amounts to integrating over all α_p:

$$Z_{k,\Lambda}[J,\bar{\Phi}] = \prod_{p^2\in[0,\Lambda^2]} \int_{-\infty}^{\infty} \mathrm{d}\alpha_p\, \exp\left(-{S^J_{\text{tot}}}'[\{\alpha\},\bar{\Phi}]\right). \tag{1.7}$$

Here ${S^J_{\text{tot}}}'$ equals $S^J_{\text{tot}}[\widehat{\Phi},\bar{\Phi}]+\Delta S_k[\widehat{\Phi},\bar{\Phi}]$ with the expansion for $\widehat{\Phi}$ inserted. In (1.7) we implemented the UV regularization by retaining only eigenfunctions (or "modes") corresponding to $-\bar{D}^2$-eigenvalues (or squared "momenta") smaller than Λ^2. The IR contributions, i.e. those corresponding to eigenvalues between $p^2=0$ and about $p^2=k^2$ are cut off smoothly instead, namely by a p^2-dependent suppression factor arising from ΔS_k.

To obtain a structurally simple FRGE, ΔS_k should be chosen quadratic in the dynamical fields. Usually one sets $\Delta S_k=\frac{1}{2}\int \mathrm{d}x\,\widehat{\Phi}\mathcal{R}_k\widehat{\Phi}$ with an operator $\mathcal{R}_k\propto k^2R^{(0)}(-\bar{D}^2/k^2)$ containing a dimensionless function $R^{(0)}$. In the $-\bar{D}^2$-basis we have then $\Delta S_k\propto k^2\sum_p R^{(0)}(p^2/k^2)\alpha_p^2$, which shows that ΔS_k represents a kind of p^2-dependent mass term: A mode with eigenvalue p^2 acquires a (mass)^2 of the order $k^2R^{(0)}(p^2/k^2)$.

We require $R^{(0)}(p^2/k^2)$ to have the qualitative properties of a smeared step function, which, around $p^2/k^2\approx 1$, drops smoothly from $R^{(0)}=1$ for $p^2/k^2\lesssim 1$ to $R^{(0)}=0$ for $p^2/k^2\gtrsim 1$. This achieves precisely the desired IR regularization: In the product over p^2 in (1.7), ΔS_k equips all $\int \mathrm{d}\alpha_p$-integrals pertaining to the *low momentum modes*, i.e., those with $p^2\in[0,k^2]$, with a Gaussian suppression factor $e^{-k^2\alpha_p^2}$ since for such eigenvalues $R^{(0)}(p^2/k^2)\approx 1$. The *high momentum modes*, having $p^2\in[k^2,\Lambda^2]$, yield $R^{(0)}(p^2/k^2)\approx 0$ and thus remain unaffected by ΔS_k.

At least on a flat background, low (high) momentum modes $\varphi_p(x)$ have long (short) wavelengths. Therefore, when one lowers k from $k=\Lambda$ down to $k=0$ one "unsuppresses" modes of increasingly long wavelengths, thus proceeding from the UV to the IR.[4]

Note that at this point the availability of the background metric is crucial for the whole construction: Via the associated covariant Laplacian $\bar{D}^2$ it *defines* which modes are high or low momentum. Hence, tuning k from high to low scales, the background metric also decides in which order the various field modes get integrated out.

The cutoff action for the gravitational field itself has the structure $\Delta S_k\propto \int \mathrm{d}x\sqrt{\bar{g}}\,\widehat{h}_{\mu\nu}R^{(0)}(-\bar{D}^2/k^2)\widehat{h}^{\mu\nu}$ and hence is invariant under general coordinate transformations if both $\widehat{h}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ transform as tensors. This highly desirable property could not have been realized without having the background metric in the game.

This process of encoding the contributions from an increasing number of modes in a scale-dependent or "running" action functional is precisely a *renormalization in the modern sense* due to Wilson [7].

The EAA, $\Gamma_{k,\Lambda}[\Phi,\bar{\Phi}]$, is defined to be the Legendre transform of $\ln Z_{k,\Lambda}[J,\bar{\Phi}]$ given by (1.7), with respect to J, for k, Λ, and $\bar{\Phi}$ fixed (and with $\Delta S_k[\Phi,\bar{\Phi}]$ subtracted from the result of the transformation, which is not essential here).

[4] This is called the "integrating out" of the high momentum modes since in older approaches the low momentum modes were completely discarded, rather than just suppressed.

The EAA has a number of important features not realized in other functional RG approaches:

(i) Since no fluctuation modes are taken into account in the $k = \Lambda \to \infty$ limit, the EAA approaches (essentially) the bare (i.e., unrenormalized) action, $\Gamma_{\Lambda,\Lambda} \sim S_{\text{tot}}$. In the limit $k \to 0$, it yields the standard effective action (with an UV cutoff).

(ii) The EAA satisfies an exact FRGE independent of S and can be computed by integrating this FRGE toward low k, with the initial condition $\Gamma_{\Lambda,\Lambda} = S_{\text{tot}} + \ldots$ at $k = \Lambda$. (The dots stand for a computable correction term which is unimportant here.)

(iii) The functional $\Gamma_{k,\Lambda}[\Phi, \bar{\Phi}]$ is invariant under background gauge transformations $\boldsymbol{\delta}^{\mathbf{B}}$ for all values of the cutoffs. This property is preserved by the FRGE: the RG evolution does not generate $\boldsymbol{\delta}^{\mathbf{B}}$-noninvariant terms.

(iv) The FRGE continues to be well behaved when the UV regularization is removed ($\Lambda \to \infty$). Denoting solutions to the UV cutoff-free FRGE by $\Gamma_k[\Phi, \bar{\Phi}]$, it reads:

$$k\partial_k \Gamma_k[\Phi, \bar{\Phi}] = \frac{1}{2} \operatorname{STr}\left[\left(\Gamma_k^{(2)}[\Phi, \bar{\Phi}] + \mathcal{R}_k[\bar{\Phi}]\right)^{-1} k\partial_k \mathcal{R}_k[\bar{\Phi}]\right]. \tag{1.8}$$

Here STr denotes the functional supertrace, and $\Gamma_k^{(2)}$ stands for the matrix of second functional derivatives of Γ_k with respect to Φ at fixed $\bar{\Phi}$. The interplay of the regulators $\mathcal{R}_k$ in the numerator and denominator ensures that the argument of the supertrace is strongly peaked around $p^2 = k^2$ and vanishes for $p^2 \gg k^2$. As a consequence, the trace is perfectly finite both in the IR and the UV, and this is why sending $\Lambda \to \infty$ was unproblematic.

(v) At least in non-gauge theories, Γ_k is closely related to a generating functional for *field averages* over finite domains of size k^{-1}; hence the name EAA [22, 23]. Thanks to this property, when treated as a *classical* action, Γ_k can provide an effective field theory description of *quantum* physics involving typical momenta near k. This property has been exploited in numerous applications of the EAA to particle and condensed matter physics, but it plays no role in the present context. Rather, it is its interpolating property between S and $\Gamma_{k=0}$ which is instrumental in the Asymptotic Safety program.

(3e) Theory space. The arena in which the RG dynamics takes place is the infinite dimensional *theory space*, $\mathcal{T}$. It consists of all well-behaved action functionals $(\Phi, \bar{\Phi}) \mapsto A[\Phi, \bar{\Phi}]$, which depend on a given set of fields and respect certain subsidiary conditions and symmetry constraints. In metric gravity the "points" of $\mathcal{T}$ are $\boldsymbol{\delta}^{\mathbf{B}}$-invariant functionals $A[g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu]$.

The RHS of the FRGE (1.8) defines a vector field $\boldsymbol{\beta}$ on $\mathcal{T}$. Its natural orientation is such that $\boldsymbol{\beta}$ points from higher to lower momentum scales k, from the

UV to the IR. (This is the direction of increasing "coarse-graining" in which the microscopic dynamics is "averaged" over increasingly large spacetime volumes.) The integral curves of this vector field, $k \mapsto \Gamma_k$, are the *RG trajectories*, and the pair $(\mathcal{T}, \boldsymbol{\beta})$ is called the *RG flow*. It constitutes the dynamical system alluded to earlier.

(3f) Coordinates, a.k.a. coupling constants. One usually assumes that every $A \in \mathcal{T}$ can be expanded as $A[\Phi, \bar{\Phi}] = \sum_{\alpha=1}^{\infty} \bar{u}^\alpha I_\alpha[\Phi, \bar{\Phi}]$ where the set $\{I_\alpha\}$ forms a basis of invariant functionals. Writing the RG trajectory correspondingly, $\Gamma_k[\Phi, \bar{\Phi}] = \sum_{\alpha=1}^{\infty} \bar{u}^\alpha(k) I_\alpha[\Phi, \bar{\Phi}]$, one encounters infinitely many *running coupling constants*, $\bar{u}^\alpha(k)$, whose k-dependence is governed by an infinite coupled system of differential equations: $k\partial_k \bar{u}^\alpha(k) = \bar{\beta}^\alpha(\bar{u}^1, \bar{u}^2, \cdots; k)$. The dimensionful *beta functions* $\bar{\beta}^\alpha$ arise by expanding the RHS of the FRGE: $\frac{1}{2}\,\mathrm{STr}[\cdots] = \sum_{\alpha=1}^{\infty} \bar{\beta}^\alpha I_\alpha[\Phi, \bar{\Phi}]$. The coefficients $\bar{\beta}^\alpha$ are similar to the familiar beta functions of perturbative quantum field theory (where, however, only the finitely many beta functions of the relevant couplings are considered).

Reexpressing the RG equations in terms of dimensionless couplings $u^\alpha \equiv k^{-d_\alpha} \bar{u}^\alpha$ with d_α the canonical mass dimension of $\bar{u}^\alpha$, the resulting *FRGE in component form* is autonomous, i.e,. its β-functions have no explicit k-dependence: $k\partial_k u^\alpha(k) = \beta^\alpha(u^1(k), u^2(k), \cdots)$. The coupling constants $(u^\alpha) \equiv u$ serve as local coordinates on $\mathcal{T}$, and the β^αs are the components of the vector field $\boldsymbol{\beta} \equiv (\beta^\alpha(u))$.

(4) Asymptotic Safety construction of the UV limit. The construction of a quantum field theory involves finding an RG trajectory that is infinitely extended in the sense that it is a curve, entirely within theory space, with well-defined limits $k \to 0$ and $k \to \infty$, respectively. *Asymptotic Safety is a proposal for ensuring the existence of the second limit.* Its crucial prerequisite is a non-trivial RG fixed point on $\mathcal{T}$.

(4a) Fixed points. By definition, at a fixed point of the RG flow the vector field $\boldsymbol{\beta}$ vanishes, $\boldsymbol{\beta} = 0$, so its coordinates $(u_*^\alpha) = u_*$ satisfy the infinitely many conditions $\beta^\alpha(u_*) = 0$. The fixed points *UV critical hypersurface*, $\mathscr{S}_{\mathrm{UV}}$, or synonymously its *unstable manifold*, is defined to consist of all points in $\mathcal{T}$, which are pulled into the fixed point under the inverse RG flow, i.e., for increasing scale k.

Linearizing the flow about u_* one has $k\partial_k u^\alpha(k) = \sum_\gamma B^\alpha{}_\gamma \big(u^\gamma(k) - u_*^\gamma\big)$ with the *stability matrix* $B = (B^\alpha{}_\gamma)$, $B^\alpha{}_\gamma \equiv \partial_\gamma \beta^\alpha(u_*)$. If the eigenvectors of B form a basis, its solution reads $u^\alpha(k) = u_*^\alpha + \sum_I C_I V_I^\alpha \big(\frac{k_0}{k}\big)^{\theta_I}$. Here the C_I's are constants of integration and the V_I's denote the right-eigenvectors of B with eigenvalues $-\theta_I$, i.e., $\sum_\gamma B^\alpha{}_\gamma V_I^\gamma = -\theta_I V_I^\alpha$. In general, B is not symmetric and the *critical exponents* θ_I are complex.

Along eigendirections with Re $\theta_I > 0$ (Re $\theta_I < 0$) deviations from u^α_* grow (shrink) when k is lowered from the UV toward the IR; they are termed *relevant* (*irrelevant*).

A trajectory $u^\alpha(k)$ within $\mathscr{S}_{\rm UV}$, by definition, approaches $u^\alpha(k\to\infty) = u^\alpha_*$ in the UV. For the constants C_I in its linearization this implies that $C_I = 0$ for all I with Re $\theta_I < 0$. Hence, the trajectories in $\mathscr{S}_{\rm UV}$ are labeled by the remaining C_I's related to the critical exponents with Re $\theta_I > 0$. (For simplicity, we assume all Re θ_I non-zero.)

As a consequence, *the dimensionality of the critical hypersurface* $\mathscr{S}_{\rm UV}$ *equals the number of critical exponents with* Re $\theta_I > 0$, i.e., the number of relevant directions.

A fixed point is called *Gaussian* if it corresponds to a free field theory. Its critical exponents agree with the canonical mass dimension of the corresponding operators. A fixed point whose critical exponents differ from the canonical ones is referred to as non-trivial or as a *non-Gaussian fixed point* (NGFP).

(4b) Asymptotically safe RG trajectories. Now we return to the key problem of ensuring the existence of the $k\to\infty$ limit. So let us assume that there is indeed a fixed point of the RG flow on theory space. Then it is sufficient to pick any of the trajectories within its hypersurface $\mathscr{S}_{\rm UV}$ to be sure that the trajectory has a singularity-free UV behavior. In this case it is certain that the trajectory will always hit the fixed point for $k\to\infty$, thus excluding the possibility that it ultimately leaves the space of acceptable actions, $\mathcal{T}$.

There exists a $\dim(\mathscr{S}_{\rm UV})$-parameter family of such "asymptotically safe" trajectories. Assuming the existence of a suitable IR-limit, any of them constitutes a complete family of action functionals, $\big\{\Gamma_k, k\in(0,\infty)\big\}$, which describes the integrating out of *all* modes of the field, and hence may be seen as defining a quantum field theory.

Most probably a UV fixed point is not only sufficient but also necessary for an acceptable theory without divergences. Therefore, in the simplest case when there exists only one, the physically inequivalent asymptotically safe quantum theories one can construct are labeled by the $\dim(\mathscr{S}_{\rm UV})$ parameters characterizing trajectories inside $\mathscr{S}_{\rm UV}$.

Thus, the degree of predictivity of asymptotically safe theories is essentially determined by the number of *relevant* eigendirections. If $\dim(\mathscr{S}_{\rm UV})$ is a finite number, it is sufficient to measure only $\dim(\mathscr{S}_{\rm UV})$ of the couplings $\{u^\alpha(k)\}$ characterizing Γ_k in order to predict the infinitely many others. In particular, at $k=0$ the standard effective action $\Gamma\equiv\Gamma_0$ is obtained which "knows" all possible predictions.

(4c) Background Independence of the second kind. The only input required in order to start the search for asymptotically safe quantum field theories is the theory space $\mathcal{T}$, i.e., the desired field content and the symmetries. It fully determines the structure of the FRGE and, as a consequence, the RG flow and in particular its fixed point properties.

In the EAA approach, the functional flow equation with the UV regulator removed is completely independent of the bare action S in the original, merely formal functional integral from which we started. Hence, the entire RG flow and in particular (if our search is successful) its fixed points enjoy an existence independent of any preconceived bare action, and so the same is true for the asymptotically safe trajectories related to those fixed points.

We refer to this property as *Background Independence of the second kind.* This term expresses the independence of our approach from a preferred *action*, in the same spirit as the familiar Background Independence (of the first kind) refers to the absence of a distinguished *metric*.

(5) Construction of a bare action. By the general properties of the EAA in the presence of an explicit UV regulator, the functional $\Gamma_{\Lambda,\Lambda} \equiv \Gamma_{k=\Lambda,\Lambda}$ for $\Lambda \to \infty$ is closely related to the bare action $S \equiv S_\Lambda$ in the UV regularized functional integral, $\Gamma_{\Lambda,\Lambda} = S_\Lambda + \cdots$ This asymptotic action should not be confused with the limit $\Gamma_{k=\Lambda\to\infty}$ of a solution Γ_k to the *UV regulator-free* flow equation (1.8). Under certain (technical) conditions the two can be related, provided one more piece of input is supplied, namely a UV-regularized measure (and of course a UV-regularization scheme, in the first place). The information contained in an asymptotically safe trajectory Γ_k near the fixed point should then suffice to determine how the Λ-dependent couplings in the bare action $S_{\Lambda\to\infty}$ must be tuned so that, together with the particular measure selected, a mathematically well-behaved functional integral arises in the limit $\Lambda \to \infty$. This last step in the Asymptotic Safety program is sometimes referred to as the *reconstruction problem*, i.e., the task of a posteriori finding a bare, or "classical" theory that reproduces a given effective one.

Thus, contrary to the usual situation in quantum mechanics or quantum field theory the implicitly underlying classical theory is actually *computed* here. It is an output rather than an input, giving special predictive power to the approach. Based on the EAA, the Asymptotic Safety program may be thought of as a systematic research process among quantum theories, rather than the quantization of a classical system known beforehand.

It has become customary to call *Quantum Einstein Gravity*, or QEG, any quantum field theory of metric-based gravity, regardless of its bare action, which is defined by a complete trajectory on the theory space $\mathcal{T} \equiv \mathcal{T}_{\rm QEG}$. So conceptually QEG has a priori no reason to be related to the familiar Einstein–Hilbert action, or any other simple diffeomorphism invariant functional of the metric one might guess naively.

1.5 Continuum or "Atoms of Spacetime"?

We saw that the EAA-based search for a fundamental quantum theory of gravity and the geometry of spacetime makes use of classical or at least classical-looking fields in order to describe expectation values like $\langle \widehat{g}_{\mu\nu}(x)\rangle \equiv g_{\mu\nu}(x)$ and that the entire physical contents of such a theory would be encoded in an effective action functional defined over such fields. This might cause concerns that the whole program is perhaps doomed to failure right from the start because its mathematical setting is still "too classical" and therefore incapable of successfully coping with the presumably very "exotic" physics of strongly non-perturbative gravity at microscopic scales.

Such concerns were nurtured, for instance, by the discovery of quantized area and volume elements in the framework of Loop Quantum Gravity (LQG), which led to the intuitive picture of spacetime being built up from tiny discrete building blocks, the "atoms of spacetime." It is not yet clear whether or not this kind of fundamental discreteness (at the kinematical level) can have any imprint on observable physics. But for the sake of the argument, let us assume it has, and let us explain why this is nevertheless *not* an argument against the continuum-based Asymptotic Safety program.

It is indeed possible to employ the language of continuum functions on classical manifolds in order to describe "quantum spacetimes," which are fundamentally different from those in General Relativity, differentiable manifolds furnished with classical metrics. This probably includes even highly non-classical quantum spacetimes with discrete or lattice-like features.

A proof of principle can already be found in elementary quantum mechanics, namely in the *Wigner–Weyl–Moyal formalism*, which represents all states (density matrices) and operators by ordinary commuting functions on the system's classical phase space manifold. The canonical commutation relations are nevertheless fully accounted for since the operator product gets replaced by a non-commutative star product for such functions.

This is perhaps the simplest instance of a much more general methodology also used in modern non-commutative geometry [52, 53], namely defining and characterizing "non-commutative manifolds" in terms of an algebra of functions defined over them. In the quantum mechanics example this algebra consists of the phase space functions equipped with the star product. It encodes, for example, the well-known partitioning of (2-dimensional) phase space in elementary blocks which realize the fundamental unit for the symplectic volume set by Planck's constant. These are the Planck cells which underlie the familiar counting rule "one state per Planck cell."

It is instructive to see how the continuum language of the classically looking phase space functions manages to express this lattice-like discreteness. In this language, coherent states $|p,q\rangle$ are represented by Gaussian functions[5] on phase space that are peaked at a given point $(p,q) \in \mathbb{R} \times \mathbb{R}$. Because of the one-to-one correspondence between such points and Gaussians we may think of phase space as the set of all Gaussians (with a given width fixed by Planck's constant) which in turn are in one-to-one correspondence with the coherent states $|p,q\rangle$.

Now, note that the set $\left\{|p,q\rangle, p \in \mathbb{R}, q \in \mathbb{R}\right\}$ seen as a Hilbert space basis is *overcomplete*. In order to reduce it to a subset which is complete but not overcomplete, one can restrict the labels p, q of the states $|p,q\rangle$ to the discrete points of the *von Neumann lattice*. It is a regular lattice in the p-q plane whose elementary cell has a volume of precisely one Planck unit [56].

These remarks are certainly not to say that it is really a von Neumann lattice that is at the heart of spacetime on microscopic scales.[6] They make it plausible though that the framework we are going to advocate in this book, namely a continuum-based background field approach, is in principle capable to also cope with potentially quite formidable "deformations" of classical geometry such as the replacement of a continuum by a discrete point set.

In summary, classical functions over a continuum should be seen merely as a *language* to formulate theories, and as such it is largely unrelated to the *physical content* such theories may have.

The discrete and the continuum languages have complementary advantages and disadvantages. The former, in its incarnations of Loop Quantum Gravity or Causal Dynamical Triangulations, for example, is applied most easily in the deep UV, i.e., for a description of the building blocks or "atoms" of spacetime, which may or may not manifest themselves as a physical, observable discreteness in the end. Moving toward the IR in these approaches is comparatively difficult though since assembling many such building blocks amounts to solving an increasingly complex dynamical problem. As a result, proving the emergence of a classical spacetime is one of the hardest problems in these approaches.

Adopting the continuum language the situation is exactly the opposite. While the desired classical regime of the theory is also not for free in this setting, it is much more easily explored by the background field method and a continuum description because its mathematical language is already similar to that of classical General Relativity. In the extreme UV, on the other hand, a continuum-based theory must "work very hard," that is it must be able to correctly handle the large quantum effects which become unavoidable should the microstructure of spacetime be very non-classical.

[5] Like for all state vectors they are the "Weyl symbol" of the associated pure state density matrix, a.k.a. the Wigner function of the pertinent wave function [54, 55].

[6] It is amusing though that the microscopic building blocks were "Planck cells" in a double sense of the word then. For concrete realizations of this idea see also [57, 58].

2

The Functional Renormalization Group

In this chapter we introduce the general idea of the Effective Average Action and we describe in particular the Functional Renormalization Group Equation it gives rise to [22–28, 31–35].[1] To avoid inessential technical complications we choose the simplest setting possible, namely a scalar matter field theory on a non-dynamical, flat Euclidean space. We will mostly focus on those aspects that later on will be important in quantum gravity, referring to the literature for a more exhaustive survey.

2.1 The Concept of the Effective Average Action

We start by considering a single-component real scalar field ϕ, which is defined on the d-dimensional Euclidean spacetime $(\mathbb{R}^d, \delta_{\mu\nu})$. Its dynamics is governed by the bare action $S[\phi]$. Typically the functional S has the structure $S[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \text{interactions} \right\}$, but we will not assume any specific form of S in the following.

In all applications of quantum or statistical field theory the ultimate goal consists in computing expectation values $\langle \mathcal{O} \rangle$ of observables $\mathcal{O}$, given for instance by the (a priori formal) functional integral[2]

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\widehat{\phi}\ \mathcal{O}(\widehat{\phi})\ \exp\big\{ -S[\widehat{\phi}] \big\}. \tag{2.1}$$

The most important example of such expectation values are Green functions, or n-point functions, where $\mathcal{O}$ consists of a string of elementary fields, $\big\langle \widehat{\phi}(x_1) \widehat{\phi}(x_2) \cdots \widehat{\phi}(x_n) \big\rangle$. Generating functionals are a technically convenient way

[1] General references on functional integration include [59–62].

[2] Here and in the following our convention is to use a caret to denote fields which, like $\widehat{\phi}$, serve as integration variables. The same symbol without a caret, like ϕ, will always denote the corresponding expectation value or "classical fields".

of treating all n-point functions in one stroke by representing them as a multiple functional derivative with respect to a "source" function J:

$$\big\langle \widehat{\phi}(x_1)\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_n)\big\rangle = \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}. \tag{2.2}$$

After coupling $\phi(x)$ to a source $J(x)$ we can write down a path integral representation for it, and for $W[J]$, the generating functional of all *connected* Green functions:

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}\widehat{\phi}\ \exp\left\{-S[\widehat{\phi}] + \int \mathrm{d}^d x\, \widehat{\phi}(x) J(x)\right\}. \tag{2.3}$$

A third generating functional is the well-known *standard effective action*, $\Gamma[\phi]$, which is given by the Legendre transform of $W[J]$. It depends on the variable "dual" to the source J, namely the field expectation value $\phi(x) \equiv \big\langle\widehat{\phi}(x)\big\rangle = \delta W[J]/\delta J(x)$.

Here and in the following the notation $\langle\cdots\rangle$ is understood to denote *connected* n-point functions *in the presence of a source*; hence $J(x)$ is not put to zero after the functional differentiation: $\big\langle\widehat{\phi}(x_1)\widehat{\phi}(x_2)\cdots\widehat{\phi}(x_n)\big\rangle = \delta^n W[J]/\delta J(x_1)\cdots\delta J(x_n)$. The functional Γ in turn can be shown to generate all 1-*particle irreducible* Green functions by a similar multiple functional differentiation with respect to $\phi(x)$.

In order to make the functional integral well-defined a UV cutoff is needed. One could replace $\mathbb{R}^d$ by a d-dimensional lattice $\mathbb{Z}^d$, for example. The formal measure $\mathcal{D}\widehat{\phi}$ would then turn into the regularized product $\prod_{x\in\mathbb{Z}^d}\mathrm{d}\widehat{\phi}(x)$. In the following we implicitly assume the presence of some sort of UV regularization, but we leave its detailed form unspecified and use continuum notation for the fields and their Fourier transforms.

The construction of the Effective Average Action [22] starts out from $W_k[J]$, a modified form of the functional $W[J]$ which has a built-in IR cutoff at a variable mass scale, k. This scale is used to classify the Fourier modes of $\widehat{\phi}$ as "short wavelength" or "long wavelength," depending on whether their momentum square $p^2 \equiv p_\mu p^\mu$ is, respectively, larger or smaller than k^2. We would like the modes with $p^2 > k^2$ to contribute without any suppression to the functional integral defining $W_k[J]$, while those with $p^2 < k^2$ should contribute only with a reduced weight or even be suppressed altogether, depending on which variant of the formalism is used. The new functional $W_k[J]$ can be thought of as the conventional generating functional belonging to a modified theory whose bare action $S + \Delta S_k$ includes an additional *cutoff action* $\Delta S_k[\widehat{\phi}]$:

$$Z_k[J] = \exp\big\{W_k[J]\big\} = \int \mathcal{D}\widehat{\phi} \exp\Big\{-S[\widehat{\phi}] - \Delta S_k[\widehat{\phi}] + \int \mathrm{d}^d x\, \widehat{\phi}(x) J(x)\Big\}. \tag{2.4}$$

The factor $\exp\{-\Delta S_k[\widehat{\phi}]\}$ serves the purpose of suppressing the "IR modes" contained in the field $\widehat{\phi}(x)$, i.e., those of its Fourier components which have

$p^2 < k^2$. In momentum space, the new term in the action is taken to be of the form

$$\Delta S_k[\widehat{\phi}] \equiv \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \mathcal{R}_k(p^2)\, |\widehat{\phi}_p|^2, \tag{2.5}$$

with $\mathcal{R}_k(p^2)$ a certain *cutoff function*, and $\widehat{\phi}_p$ the Fourier transform of $\widehat{\phi}(x)$ defined in Appendix A. The precise details of the function $p^2 \mapsto \mathcal{R}_k(p^2)$ are arbitrary to a large extent; what matters is only its limiting behavior for $p^2 \gg k^2$ and $p^2 \ll k^2$, respectively. In the simplest case[3] we require that

$$\mathcal{R}_k(p^2) \approx \begin{cases} k^2 & \text{for } p^2 \ll k^2, \\ 0 & \text{for } p^2 \gg k^2. \end{cases} \tag{2.6}$$

The first condition leads to a suppression of the small momentum modes by a soft mass-like IR cutoff, the second guarantees that the large momentum modes are integrated out in the usual way. Adding ΔS_k to the above bare action leads to

$$\big(S + \Delta S_k\big)[\widehat{\phi}] = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \Big[p^2 + m^2 + \mathcal{R}_k(p^2)\Big]\, |\widehat{\phi}_p|^2 + \text{interactions}. \tag{2.7}$$

Obviously the cutoff function $\mathcal{R}_k(p^2)$ has the interpretation of a momentum dependent mass square which vanishes for $p^2 \gg k^2$, while it assumes the constant value k^2 for $p^2 \ll k^2$. How precisely $\mathcal{R}_k(p^2)$ interpolates between these two regimes is a matter of calculational convenience. In explicit calculations one often uses the exponential cutoff

$$\mathcal{R}_k(p^2) = p^2 \big[\exp(p^2/k^2) - 1\big]^{-1}, \tag{2.8}$$

but many other choices are possible as well [31, 32, 63–65].

One could also suppress the $p^2 \ll k^2$ modes completely. This is achieved by allowing $\mathcal{R}_k(p^2)$ to diverge for $p^2 \ll k^2$ so that $\exp\{-\Delta_k S[\widehat{\phi}]\} \to 0$ for modes with $p^2 \ll k^2$. While this behavior of $\mathcal{R}_k(p^2)$ seems most natural from the viewpoint of a Kadanoff–Wilson type coarse graining, its singular behavior makes the resulting generating functional problematic to deal with technically. For this reason, and since it will still allow us to derive an exact RG equation later on, we prefer to work with a smooth cutoff satisfying (2.6).

At the non-perturbative path integral level, the presence of ΔS_k suppresses the long-wavelength modes by a factor $\exp\{-\frac{1}{2}k^2 \int |\widehat{\phi}|^2\}$. In perturbation theory, instead, according to (2.7), the ΔS_k term leads to the modified propagator $[p^2 + m^2 + \mathcal{R}_k(p^2)]^{-1}$, which equals $[p^2 + m^2 + k^2]^{-1}$ for $p^2 \ll k^2$. Thus, when computing loops with this propagator, k^2 acts indeed as a conventional IR cutoff if $m^2 \ll k^2$. (It plays no role in the opposite limit $m^2 \gg k^2$ in which the physical particle mass acts like an IR cutoff.)

[3] We will discuss a slight generalization of these conditions at the end of this section.

We also note that by replacing p^2 with $-\partial^2$ in the argument of $\mathcal{R}_k(p^2)$ the cutoff action can be written in a way which no longer makes any explicit reference to the Fourier decomposition of $\widehat{\phi}$:

$$\Delta S_k[\widehat{\phi}] = \frac{1}{2}\int \mathrm{d}^d x\, \widehat{\phi}(x)\mathcal{R}_k(-\partial^2)\widehat{\phi}(x). \tag{2.9}$$

The next steps toward the definition of the Effective Average Action are similar to the standard procedure. One defines the (now k-dependent) field expectation value $\phi(x) \equiv \langle\widehat{\phi}(x)\rangle = \delta W_k[J]/\delta J(x)$ and computes the Legendre transform of $W_k[J]$ with respect to $J(x)$, at fixed k. If the functional relationship $\phi = \phi[J]$ can be solved for the source to yield $J = \mathcal{J}_k[\phi]$, it assumes the form

$$\widetilde{\Gamma}_k[\phi] = \int \mathrm{d}^d x\, \mathcal{J}_k[\phi](x)\phi(x) - W_k\big[\mathcal{J}_k[\phi]\big], \tag{2.10}$$

where the notation $\mathcal{J}_k[\phi](x)$ indicates that $\mathcal{J}_k$ is both a functional of the function ϕ and a function of the point x. In the general case, $\widetilde{\Gamma}_k$ is given by the Legendre–Fenchel transform [66]:

$$\widetilde{\Gamma}_k[\phi] = \sup_{J(x)}\left\{\int \mathrm{d}^d x\, J(x)\phi(x) - W_k[J]\right\}. \tag{2.11}$$

The actual *EAA*, denoted by $\Gamma_k[\phi]$, is obtained by subtracting $\Delta S_k[\phi]$ from $\widetilde{\Gamma}_k$:

$$\boxed{\Gamma_k[\phi] \equiv \widetilde{\Gamma}_k[\phi] - \frac{1}{2}\int \mathrm{d}^d x\, \phi(x)\mathcal{R}_k(-\partial^2)\phi(x).} \tag{2.12}$$

The rationale for this specific definition becomes clear when we look at the list of properties enjoyed by the Effective Average Action (EAA):

(1) The EAA gives rise to the *effective field equation*

$$\boxed{\frac{\delta\Gamma_k[\phi]}{\delta\phi(x)} + \mathcal{R}_k(-\partial^2)\phi(x) = J(x).} \tag{2.13}$$

This scale-dependent source–field relationship follows from the definition (2.12) together with $\delta\widetilde{\Gamma}_k[\phi]/\delta\phi(x) = J(x)$. The latter equation is obtained by differentiating (2.10) and noting that the terms that are due to the ϕ-dependence of $\mathcal{J}_k[\phi]$ mutually cancel by virtue of $\phi(y) = \delta W_k[J]/\delta J(y)$:

$$\begin{aligned}\frac{\delta\widetilde{\Gamma}_k}{\delta\phi(x)} &= \mathcal{J}_k[\phi](x) + \int \mathrm{d}^d y\, \phi(y)\frac{\delta\mathcal{J}_k}{\delta\phi(y)} - \int \mathrm{d}^d y\, \frac{\delta W_k}{\delta J(y)}\bigg|_{J=\mathcal{J}_k}\frac{\delta\mathcal{J}_k}{\delta\phi(y)} \\ &= \mathcal{J}_k[\phi](x).\end{aligned} \tag{2.14}$$

(2) The EAA satisfies a *functional renormalization group equation*, namely the Wetterich equation:

$$\boxed{k\frac{\partial}{\partial k}\Gamma_k[\phi]=\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[\phi]+\mathcal{R}_k\right)^{-1}k\frac{\partial}{\partial k}\mathcal{R}_k\right].} \tag{2.15}$$

Here the RHS of (2.15) uses a compact infinite dimensional matrix notation. In the position space representation, the operator $\Gamma_k^{(2)}$ has the matrix elements $\Gamma_k^{(2)}(x,y)\equiv\delta^2\Gamma_k/\delta\phi(x)\delta\phi(y)$, i.e., it is the Hessian of the EAA. Likewise, the cutoff operator has matrix elements $\mathcal{R}_k(x,y)\equiv\mathcal{R}_k(-\partial_x^2)\delta(x-y)$, and the functional trace Tr corresponds to an integral $\int \mathrm{d}^d x$.

At the level of the functional RG equation (2.15) the implicit UV cutoff can be removed trivially now. This is most easily seen in the momentum representation. There the product of $\big(\Gamma_k^{(2)}+\mathcal{R}_k\big)^{-1}$, which equals essentially $\big(p^2+\mathcal{R}_k(p^2)+\cdots\big)^{-1}$, and $k\frac{\partial}{\partial k}\mathcal{R}_k(p^2)$, considered a function of p^2, is significantly different from zero only in a narrow interval centered around $p^2=k^2$. Hence, the trace receives contributions from a thin shell of momenta $p^2\approx k^2$ only and is therefore well convergent both in the UV and IR.

The derivation of the FRGE (2.15) proceeds as follows [22]. Taking the k-derivative of (2.11) with (2.4) and (2.9) inserted one finds

$$k\frac{\partial}{\partial k}\widetilde{\Gamma}_k[\phi]=\frac{1}{2}\int \mathrm{d}^d x\int \mathrm{d}^d y\,\langle\widehat{\phi}(x)\widehat{\phi}(y)\rangle\,k\frac{\partial}{\partial k}\mathcal{R}_k(x,y)\,. \tag{2.16}$$

The angular brackets denote a normalized expectation; for a general observable $\mathcal{O}$ it is given by

$$\langle\mathcal{O}\rangle\equiv e^{-W_k[J]}\int\mathcal{D}\widehat{\phi}\;\mathcal{O}(\widehat{\phi})\;\exp\Big\{-S-\Delta S_k+\int J\widehat{\phi}\Big\}. \tag{2.17}$$

Note that $\langle\mathcal{O}\rangle$ depends on both J and k, but we do not indicate this notationally. Next, it is convenient to introduce the (likewise J and k dependent) connected two-point function $G_{xy}\equiv G(x,y)\equiv\delta^2W_k[J]/\delta J(x)\delta J(y)$ and the ϕ and k dependent Hessian of $\widetilde{\Gamma}_k$: $(\widetilde{\Gamma}_k^{(2)})_{xy}\equiv\delta^2\widetilde{\Gamma}_k[\phi]/\delta\phi(x)\delta\phi(y)$. Since W_k and $\widetilde{\Gamma}_k$ are related by a Legendre transformation one shows in the usual way that

$$\int \mathrm{d}^d z\,\frac{\delta^2 W_k[J]}{\delta J(x)\delta J(z)}\,\frac{\delta^2\widetilde{\Gamma}_k[\phi]}{\delta\phi(z)\delta\phi(y)}=\delta(x-y). \tag{2.18}$$

Hence G and $\widetilde{\Gamma}^{(2)}$ are mutually inverse matrices: $G\widetilde{\Gamma}^{(2)}=1$. Furthermore, taking two J-derivatives of (2.4) one obtains the following decomposition of the two-point function in a connected plus a disconnected part:

$$\big\langle\widehat{\phi}(x)\widehat{\phi}(y)\big\rangle=G(x,y)+\phi(x)\phi(y). \tag{2.19}$$

Substituting this representation for the two-point function into (2.16) we arrive at

$$\partial_k\widetilde{\Gamma}_k[\phi]=\frac{1}{2}\mathrm{Tr}\big[\partial_k\mathcal{R}_kG\big]+\frac{1}{2}\int \mathrm{d}^d x\,\phi(x)\,\partial_k\mathcal{R}_k(-\partial^2)\,\phi(x). \tag{2.20}$$

In terms of the EAA (2.12), this equation boils down to $\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\big[\partial_k \mathcal{R}_k G\big]$. The cancellation of the $\frac{1}{2}\int \phi\, \partial_k \mathcal{R}_k\, \phi$ term is a first motivation for the definition (2.12) where this term is subtracted from the Legendre transform $\widetilde{\Gamma}_k$. The derivation of the FRGE is completed by noting that $G = [\widetilde{\Gamma}^{(2)}]^{-1} = \big(\Gamma_k^{(2)} + \mathcal{R}_k\big)^{-1}$, where the second equality follows by differentiating (2.12): $\Gamma_k^{(2)} = \widetilde{\Gamma}_k^{(2)} - \mathcal{R}_k$.

(3) The EAA satisfies the following integro-differential equation:

$$\boxed{\begin{aligned}\exp\big\{-\Gamma_k[\phi]\big\} = \int \mathcal{D}\widehat{\phi}\, \exp\Big\{-S[\widehat{\phi}] + \int \mathrm{d}^d x\, (\widehat{\phi} - \phi)\frac{\delta \Gamma_k[\phi]}{\delta \phi(x)}\Big\} \\ \times \exp\Big\{-\frac{1}{2}\int \mathrm{d}^d x\, (\widehat{\phi} - \phi)\mathcal{R}_k(-\partial^2)(\widehat{\phi} - \phi)\Big\}.\end{aligned}} \tag{2.21}$$

This equation is easily derived by combining (2.4), (2.10), and (2.12) and by using the effective field equation $\delta \widetilde{\Gamma}_k / \delta \phi = J$.

(4) For $k \to 0$ the EAA approaches the ordinary effective action, $\lim_{k\to 0} \Gamma_k = \Gamma$, and for $k \to \infty$ the bare action, $\Gamma_{k\to\infty} = S$:

$$\boxed{\Gamma \xleftarrow{k\to 0} \Gamma_k \xrightarrow{k\to\infty} S.} \tag{2.22}$$

The $k \to 0$ limit is a direct consequence of (2.6); $\mathcal{R}_k(p^2)$ vanishes for all $p^2 > 0$ when $k \to 0$. The derivation of the $k \to \infty$ limit makes use of the integro-differential equation (2.21). A formal version of the argument is as follows. Since $\mathcal{R}_k(p^2)$ approaches k^2 for $k \to \infty$, the second exponential on the RHS of (2.21) becomes $\exp\{-k^2 \int \mathrm{d}^d x (\widehat{\phi} - \phi)^2\}$ in this limit, which, up to a normalization factor, approaches a delta-functional $\delta[\widehat{\phi} - \phi]$. The $\widehat{\phi}$ integration can be performed trivially then and one ends up with $\lim_{k\to\infty} \Gamma_k[\phi] = S[\phi]$. In a more careful treatment [22] one shows that the saddle-point approximation of the functional integral in (2.21) about the point $\widehat{\phi} = \phi$ becomes exact in the limit $k \to \infty$. As a result, $\lim_{k\to\infty} \Gamma_k$ and S differ at most by the infinite mass limit of a one-loop determinant, which we omit here since it plays no role in typical applications (see [30] for a more detailed discussion).

(5) There exists a second, *equivalent form of the FRGE*:

$$k \frac{\partial}{\partial k}\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[k \frac{D}{Dk} \ln\Big(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\Big)\right]. \tag{2.23}$$

Here D/Dk is a special scale derivative which by definition acts only on the k-dependence of $\mathcal{R}_k$, but not of $\Gamma_k^{(2)}$.

While (2.23) is exact, it is instructive to see how it relates to the *one-loop approximation of the EAA*. The latter is given by the formal expression

$$\Gamma_k[\phi] = S[\phi] + \frac{1}{2}\mathrm{Tr}\ln\Big(S^{(2)}[\phi] + \mathcal{R}_k\Big) + O(2\text{ loops}), \tag{2.24}$$

which one derives most easily from the functional integral (2.4) by going through exactly the same steps as in the standard case [59]. The only difference to be taken into account is the regulator term ΔS_k in the bare action which contributes the term $\big(\Delta S_k\big)^{(2)} = \mathcal{R}_k$ to its Hessian. In this way, the familiar one-loop expression $\ln\det(S^{(2)}) = \mathrm{Tr}\ln(S^{(2)})$ immediately leads to (2.24).

Note that (2.24) makes sense only in the presence of a UV cutoff since the functional trace in (2.24) would not exist otherwise.

Taking a k-derivative on both sides of (2.24) leads to

$$k\frac{\partial}{\partial k}\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[k\frac{D}{Dk}\ln\Big(S^{(2)}[\phi] + \mathcal{R}_k\Big)\right] + O(2\text{ loops}). \tag{2.25}$$

Obviously (2.25) differs only in one respect from the exact equation (2.23): the argument of the logarithm contains $S^{(2)}$ rather than the Hessian of the quantum-corrected action, $\Gamma_k^{(2)}$. In writing down (2.25) we tacitly pulled the k-derivative under the trace. This amounts to a UV regularization since upon taking the scale derivative of the logarithm we get the same suppression factor $\partial\mathcal{R}_k/\partial k$ as in (2.15).

So the conclusion is that the FRGE has almost the structure of a one-loop result, but with two crucial modifications, which render it an exact relation: First, a mode-suppression term ΔS_k is added which serves as a variable IR cut-off, and second, in the one-loop determinant the bare action is replaced with the EAA. The replacement $S^{(2)} \to \Gamma_k^{(2)}$ is an example of what is called an *RG improvement*, a topic to which we will come back to later on.

(6) The FRGE by itself is completely *independent of the bare action S*. The latter comes into play only via its initial condition, $\Gamma_\Lambda = S$. In fact, in the EAA approach, we trade the problem of performing the functional integral for the task of solving the RG equation all the way down from some very high UV scale $k=\Lambda$, $\Lambda\to\infty$, where the initial condition $\Gamma_\Lambda = S$ is imposed, down to $k=0$, where the EAA equals the ordinary effective action Γ, the object which we actually would like to know; see Figure 2.1 for an illustration.

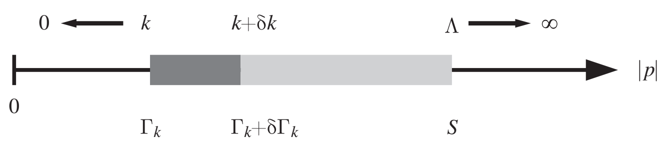


Figure 2.1. The figure shows the $|p|$-axis corresponding to the momenta of the field modes $\propto \exp(ip_\mu x^\mu)$. Solving the FRGE for Γ_k subject to initial conditions imposed at $k=\Lambda$ amounts to "integrating out" the momentum eigenmodes with $|p|\in[k,\Lambda]$. Changing the IR cutoff infinitesimally from k to $k+\delta k$, the EAA changes by the amount $\delta\Gamma_k = \frac{1}{2}\mathrm{Tr}\big[\big(\Gamma_k^{(2)} + \mathcal{R}_k\big)^{-1}\,\partial_k\mathcal{R}_k\big]\delta k$, according to the FRGE.

(7) Since the FRGE depends on the cutoff operator $\mathcal{R}_k$ chosen, so do its solutions in general. The change of $\Gamma_k[\phi]$ caused by a small "deformation" $\mathcal{R}_k \to \mathcal{R}_k + \delta\mathcal{R}_k$ is governed by the exact functional equation

$$\boxed{\delta\Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\delta\mathcal{R}_k\right].} \tag{2.26}$$

The derivation of this relation parallels the derivation of the FRGE.

(8) The EAA is closely related to a generating functional for correlators of *field averages*, hence its name. To see this, consider an arbitrary field configuration $\widehat{\phi}$ and associate the following new field to it:

$$\boxed{\phi(x) = \int \mathrm{d}^d y\, F_k(x-y)\,\widehat{\phi}(y).} \tag{2.27}$$

Here F_k denotes a spherically symmetric and normalized weight function, i.e., $F_k \geq 0$ and $\int \mathrm{d}^d x F_k(x) = 1$, which is nonzero and approximately constant at small distances $|x-y| \ll k^{-1}$, and vanishes for $|x-y| \gg k^{-1}$. Near $|x-y| \approx k^{-1}$ it interpolates between the two regimes smoothly. As a result, the value of ϕ at the point x is a (weighted) average of the original field $\widehat{\phi}$ over a ball in the Euclidean spacetime, which is centered at x and has a radius of order k^{-1}. The average value field ϕ is said to be obtained from the original one, $\widehat{\phi}$, by a process of *coarse graining*. While ϕ and $\widehat{\phi}$ show the same features at large distance scales, ϕ is lacking the finer short-distance details of $\widehat{\phi}$ that got "averaged away".

If we express $\widehat{\phi}(x)$ and $\phi(x)$ in terms of their Fourier transforms $\widehat{\phi}_p$ and ϕ_p, respectively, the convolution in (2.27) turns into a simple multiplication in momentum space:

$$\boxed{\phi_p = f_k(p^2)\,\widehat{\phi}_p.} \tag{2.28}$$

Back in position space it amounts to applying the operator $f_k(-\partial^2)$:

$$\boxed{\phi(x) = f_k(-\partial^2)\,\widehat{\phi}(x).} \tag{2.29}$$

Here f_k is related to the kernel $F_k(x-y)$ by

$$F_k(x-y) = f_k(-\partial^2)\,\delta(x-y) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} f_k(p^2)\; e^{ip(x-y)}. \tag{2.30}$$

For example, if we choose a Gaussian weight function in position space, $F_k(x-y) = (2\pi)^{-\frac{d}{2}} k^d \exp\!\big(-\frac{1}{2}k^2(x-y)^2\big)$, its counterpart in momentum space is Gaussian as well:

$$f_k(p^2) = \exp\!\big(-\tfrac{p^2}{2k^2}\big). \tag{2.31}$$

Using (2.31) in (2.28) makes it obvious that the low-momentum Fourier components of ϕ and $\widehat{\phi}$ coincide essentially, while the average field has vanishing high-momentum components: $\phi_p = \widehat{\phi}_p$ if $p^2 \ll k^2$, and $\phi_p = 0$ if $p^2 \gg k^2$, as it should be.

Next, we turn to ensembles of field configurations and define a partition function which is restricted to fields giving rise to a prescribed average field:

$$e^{-\Gamma_k^{\mathrm{AA}}[\phi]} \equiv \int \mathcal{D}\widehat{\phi}\, \delta[\phi - f_k(-\partial^2)\widehat{\phi}]\, e^{-S[\widehat{\phi}]}. \tag{2.32}$$

Clearly, the (proper) average action Γ_k^{AA} is the continuum analog of a Kadanoff–Wilson block spin action, with $\widehat{\phi}$ and ϕ playing the role of spin configurations on the fine and blocked lattice, respectively [7, 9]. Note that the original partition function $Z = \int \mathcal{D}\widehat{\phi}\ e^{-S[\widehat{\phi}]}$ can be recovered from any of the functionals Γ_k^{AA}, $k \in [0, \infty)$, by integrating over the average field, $Z = \int \mathcal{D}\phi\ e^{-\Gamma_k^{\mathrm{AA}[\phi]}}$.

In the continuum it is highly non-trivial to make sense of the delta-functional in (2.32), which, formally, consists of an infinite product of Dirac delta-functions, one for each point of spacetime. In [67, 68] the authors proposed approximating those functions by a Gaussian of finite width, whereby the width is allowed to depend on the momentum carried by the fields. As a result, (2.32) becomes $e^{-\Gamma_k^{\mathrm{AA}}} = \int \mathcal{D}\widehat{\phi}\ e^{-S-S_{\text{constraint}}}$, with an additional contribution to S, the "constraint action" $S_{\text{constraint}}$. It has the general structure

$$S_{\text{constraint}} = \frac{1}{2}\int \mathrm{d}^d x \left(\phi - f_k(-\partial^2)\widehat{\phi}\right) \mathcal{K} \left(\phi - f_k(-\partial^2)\widehat{\phi}\right). \tag{2.33}$$

Herein $\mathcal{K}$ is an operator which controls how precisely the width of the approximating Gaussian depends on the momentum.

We interpret the formal equation (2.32) as follows now:

$$e^{-\Gamma_k^{\mathrm{AA}}[\phi]} = e^{-\frac{1}{2}\int \mathrm{d}^d x\, \phi \mathcal{K} \phi} \int \mathcal{D}\widehat{\phi}\ e^{-S[\widehat{\phi}] - \frac{1}{2}\int \mathrm{d}^d x\, \widehat{\phi} \mathcal{R}_k \widehat{\phi} + \int \mathrm{d}^d x\, J \widehat{\phi}}. \tag{2.34}$$

Here we denoted $J \equiv \mathcal{K} f_k(-\partial^2)\phi$, and tentatively set $\mathcal{R}_k \equiv f_k(-\partial^2)^2 \mathcal{K}$. In fact, if we assume that $\mathcal{K}$ is such that the $\mathcal{R}_k$ defined in this way satisfies the requirements for a cutoff operator, the functional integral in (2.34) coincides exactly with the one representing W_k in (2.4) above, whence $\Gamma_k^{\mathrm{AA}}[\phi] = \frac{1}{2}\int \mathrm{d}^d x\, \phi \mathcal{K} \phi - W_k[J]\big|_{J=\mathcal{K} f_k \phi}$. Thus, we observe that (up to an explicitly known term $\propto \phi \mathcal{K} \phi$) the action $\Gamma_k^{\mathrm{AA}}[\phi]$ is given by the generating functional related to the EAA, $W_k[J]$, evaluated at a ϕ-dependent source. So in essence, Γ_k^{AA} and Γ_k are related by a Legendre transformation.

In [68] the choice $\mathcal{K} = (-\partial^2)\left[1 - f_k(-\partial^2)^2\right]^{-1}$ has been advocated. With the weight function (2.31) it entails $\mathcal{R}_k = (-\partial^2)\left[\exp(-\partial^2/k^2) - 1\right]^{-1}$, which equals precisely the cutoff operator (2.8) written in position space.

(9) The continuum functional Γ_k^{AA} is known as the (proper) *Average Action* [67, 68].[4] It has been generalized to gauge fields [25, 26] and was employed in a number of practical calculations [70–75].

[4] Its non-derivative part at $k = 0$ coincides with the so-called *constraint effective potential* [69].

Historically, in the search for a non-perturbative renormalization group in the continuum, the *Average Action* preceded the *EAA*, Γ_k, reviewed above and introduced in [22].

In this book we focus almost entirely on the EAA, which is more suitable for our purposes. Nevertheless, keeping in mind its relation to the "old" average action helps understanding in what sense precisely Γ_k is the result of averaging over increasingly larger volumes in spacetime when k is lowered.

2.2 Theory Space and Its Truncation

The first step toward making the initial value problem of the FRGE precise consists in fixing the space of functionals over which the equation is supposed to be defined, and from which eligible initial points Γ_Λ are drawn, for example. It has become customary to refer to this space comprising, in a sense, all action functionals that are possible in principle, as *theory space* $\mathcal{T}$. This space contains bare actions, the standard effective actions, and Effective Average Actions on a completely equal footing.

2.2.1 The Space of Actions

To illustrate the concept of a theory space, let us return to the above scalar example and define $\mathcal{T}$ by the following requirements:

(i) All functionals $A[\,\cdot\,] \in \mathcal{T}$ are $\mathbb{Z}_2$-symmetric, i.e., $A[\phi] = A[-\phi]$ for any ϕ, and invariant under $\mathsf{O}(d)$ transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, the Euclidean analog of Lorentz transformations, i.e., $A[\phi] = A[\phi']$ for all fields ϕ and ϕ' related by the scalar transformation law $\phi'(x) = \phi(\Lambda^{-1}x)$.

(ii) The space $\mathcal{T}$ contains all functionals that can occur in a derivative expansion, i.e., arbitrary field monomials, integrated over spacetime, consisting of any number of fields and derivatives of any order which act on them in all ways possible.

Thus, schematically, elements of theory space would look as follows:

$$\begin{aligned} A[\phi] \;=\; \int \mathrm{d}^d x \Big\{ & \bar{u}_{0,0} + \bar{u}_{0,2}\,\phi^2(x) + \bar{u}_{0,4}\,\phi^4(x) + \bar{u}_{0,6}\,\phi^6(x) + \cdots \\ & + \bar{u}_{2,0}\left(\partial_\mu\phi\right)^2 + \bar{u}_{2,2}\,\phi^2\left(\partial_\mu\phi\right)^2 + \bar{u}_{2,4}\,\phi^4\left(\partial_\mu\phi\right)^2 + \cdots \\ & + \bar{u}_{4,0}\,\partial_\mu\phi\,\partial^\mu\phi\,\partial_\nu\phi\,\partial^\nu\phi + \cdots \\ & \vdots \\ & + \cdots + \bar{u}_{m,n,p}\left(\partial_\mu\phi\,\partial^\mu\phi\right)^m \phi^n \left(\partial_\nu\,\partial^\nu\phi\right)^p + \cdots \\ & \vdots \;\ldots \Big\}. \end{aligned} \tag{2.35}$$

We do not attempt a systematic enumeration of all possible terms here. However, from (2.35) it should be clear that the generic $A \in \mathcal{T}$ has an expansion of the form

$$A[\phi] = \sum_\alpha \bar{u}^\alpha \, I_\alpha[\phi]. \tag{2.36}$$

The infinitely many "basis functionals" $I_\alpha[\,\cdot\,]$ are monomials built from powers of the field and its derivatives, all evaluated at the same point x, and integrated over spacetime. The coefficients $\bar{u}^\alpha$ should be thought of as the "components" of the functional $A[\,\cdot\,]$ with respect to the basis $\{I_\alpha\}$. The set of coefficients $\{\bar{u}^\alpha\}$, often referred to as *generalized coupling constants* can also be seen as local coordinates[5] on the "manifold" $\mathcal{T}$. Indeed, there is an obvious one-to-one correspondence between functionals, regarded usually as maps $\phi \mapsto A[\phi]$ and sets $\{\bar{u}^\alpha\}$:

$$\mathcal{T} \ni A \longleftrightarrow \{\bar{u}^\alpha\}. \tag{2.37}$$

2.2.2 The FRGE in Component Form

Let us return to the FRGE, (2.15). Its solutions, called the *renormalization group trajectories*, are one-parameter families of actions Γ_k of the type (2.35). For every fixed value of k they can be expanded as in (2.36):

$$\Gamma_k[\phi] = \sum_\alpha \bar{u}^\alpha(k) \, I_\alpha[\phi]. \tag{2.38}$$

The scale dependence of Γ_k is now carried by the *running coupling constants*, $\bar{u}^\alpha(k)$. Note that the basis $\{I_\alpha\}$ is independent of k. Geometrically speaking, the RG trajectories are parametrized curves on $\mathcal{T}$.

(1) By virtue of the correspondence (2.37) the abstract form of the FRGE in (2.15) is equivalent to a coupled system of differential equations for the functions $k \mapsto \bar{u}^\alpha(k)$. To derive it, we insert the expansion (2.38) into (2.15) and obtain[6] the following equation, valid at any field argument ϕ:

$$\sum_\alpha k\partial_k \, \bar{u}^\alpha(k) \, I_\alpha[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\sum_\alpha \bar{u}^\alpha(k) \, I_\alpha^{(2)}[\phi] + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]. \tag{2.39}$$

While the RHS of this equation is an extremely complicated functional of ϕ, at least in principle we can apply the standard algorithms for derivative expansions

[5] The notation $\bar{u}^\alpha$ with an overbar is chosen for later convenience. It indicates that the generic $\bar{u}^\alpha$ has a non-zero canonical dimension. Later on we will slightly modify the definition of $\mathcal{T}$ and employ a dimensionless variant of the couplings as coordinates. For the time being this difference is inessential.

[6] Here and in the following the superscript (2) denotes the Hessian of the corresponding functional. We also abbreviate $\partial_k \equiv \partial/\partial k$.

to it. Since, by its very definition, $\mathcal{T}$ contains all functionals that could possibly be produced by those algorithms, it follows that the ϕ-dependence of $\mathrm{Tr}[\cdots]$ can be expanded in the basis $\{I_\alpha\}$. Denoting the corresponding coefficients $\bar{b}^\alpha$, (2.39) then reads:

$$\sum_\alpha k\partial_k\, \bar{u}^\alpha(k)\, I_\alpha[\phi] = \sum_\alpha \bar{b}^\alpha\big(\bar{u}(k);k\big) I_\alpha[\phi]. \tag{2.40}$$

The functions $\bar{b}^\alpha$ have both an implicit and explicit k-dependence. The former is due to their dependence on the set of all running couplings, $\bar{u}(k) \equiv \{\bar{u}^\alpha(k)\}$, and the latter stems from the operator $\mathcal{R}_k$ under the trace. Since the I_α's are linearly independent we may equate their coefficients on both sides of (2.40), yielding

$$\boxed{k\partial_k\, \bar{u}^\alpha(k) = \bar{b}^\alpha\big(\bar{u}(k);k\big) \quad \forall\alpha.} \tag{2.41}$$

This is the coupled system of infinitely many ordinary differential equations which determines the RG trajectories. We will refer to it as the *FRGE in component form* and to $\bar{b}^\alpha$ as the *beta function* of the running coupling constant $\bar{u}^\alpha(k)$.[7]

(2) The reader might wonder whether the enormous complexity of the system (2.41), consisting of infinitely many differential equations, is really unavoidable, and if so, what do we get in return if we really try to come to terms with the system? After all, in perturbative field theory RG equations are used, too, but they are way simpler and in particular there are just a few of them [76].

The answer to the first question is indeed affirmative: We really do need the infinite set of coupled equations in general; with any attempt of reducing it to some subset we lose information. Intuitively it is quite plausible that the equations for $\{\bar{u}^\alpha(k)\}$ cannot be anything simple since, within the framework of the EAA, the running couplings parameterize an entire functional, namely Γ_k. Now, even if we impose initial conditions at $k=\Lambda$ and choose Γ_Λ as simple as, say,

$$\Gamma_\Lambda[\phi] = \int \mathrm{d}^4x\, \Big\{\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m_\Lambda^2\phi^2 + \frac{\lambda_\Lambda}{4!}\phi^4\Big\}, \tag{2.42}$$

the FRGE tells us that, at any lower scale $k<\Lambda$, the EAA will look as complicated as the functional in (2.35). Integrating out the field modes with $|p|\in[k,\Lambda]$ generates a plethora of complicated terms (with higher derivatives and higher powers of ϕ) that were not present initially. This is actually as it must be since, by construction, the running functional Γ_k approaches the ordinary effective action Γ in the limit $k\to 0$, and from standard quantum field theory we know that a typical effective action is anything but simple. Usually Γ will contain all sorts of "exotic" terms which never appear in the bare action of a perturbatively renormalizable

[7] We reserve the standard notation β^α for the beta function of the dimensionless coupling to be introduced later.

theory; a typical example in the scalar model at hand is the Coleman–Weinberg potential, $\phi^4 \ln(\phi)$ [77].

When applied to one of the familiar, perturbatively renormalizable theories, the EAA trajectory $k \mapsto \Gamma_k$ may be regarded as a smooth interpolation between the bare action at $k = \Lambda \to \infty$, and the effective one at $k = 0$. Hence, *the RG evolution of the EAA describes how the complexity of* Γ *emerges out of the simplicity of* S.

(3) The effective action Γ is a concise way of summarizing all predictions of a field theory model. Thus, loosely speaking, computing Γ amounts to completely "solving" this model. This answers the second question above: If we manage to find the solution to the FRGE subject to the initial condition $\Gamma_\Lambda = S$ we have derived all possible predictions of the model, i.e., we have "solved" this theory. It is clear therefore that the FRGE should have a level of complexity comparable to that of the functional integral we started from.

This property of the FRGE, providing a complete "solution of the theory", is in sharp contradistinction to the various types of RG equations used in perturbation theory [78]. They are finite in number and control only the coupling constants related to "renormalizable interactions." Therefore, the amount of information encapsulated in the solutions to their RG equations is insufficient to fully determine Γ.

2.2.3 Truncations of Theory Space

Up to this point, our construction did not involve any approximation. However, when it comes to actually solving the FRGE or the system (2.41) we clearly must resort to some sort of approximation. An obvious possibility is perturbation theory in one or several couplings which are assumed small; in this way one could recover the RG equations of perturbative renormalization theory, for instance [79].

Here we will adopt a different strategy, the *method of truncated theory spaces*, which has the essential advantage that it can yield *non-perturbative approximate solutions*. They can go beyond the realm of perturbation theory by summing up contributions from all orders of the small coupling expansion or the loop expansion, for instance. Such solutions might even depend on the small parameters in which perturbation theory is trying to expand in a non-analytical way so that at any finite order perturbation theory is "blind" to the corresponding contributions.

The basic idea is to project the exact RG evolution on $\mathcal{T}$ as described by (2.41) onto a simpler, often finite dimensional subspace $\mathcal{T}_{\text{trunc}} \subset \mathcal{T}$. Ideally, the *truncated theory space*, $\mathcal{T}_{\text{trunc}}$, should be chosen such that, on the one side, it is general enough for the projected evolution to retain the essential physical features of its exact precursor, and on the other side, is simple enough to make the necessary calculations technically feasible.

In the simplest case, the projection from $\mathcal{T}$ onto a, say, N-dimensional subspace $\mathcal{T}_{\text{trunc}}$ proceeds as follows. To (partly) specify a truncation, we make an ansatz for the functionals in the subspace:

$$\boxed{\Gamma_k[\phi] = \sum_{i=1}^{N} \bar{u}^i(k)\, I_i[\phi].} \tag{2.43}$$

Here $\{I_i[\,\cdot\,], i=1,\cdots,N\}$ is a certain subset of the full basis $\{I_\alpha[\,\cdot\,]\}$, which is selected such that it spans the subspace chosen. Inserting (2.43) into the exact FRGE we are confronted with the following relation:

$$\sum_{j=1}^{N} k\partial_k\, \bar{u}^j(k)\, I_j[\phi] \stackrel{?}{=} \frac{1}{2}\mathrm{Tr}\left[\Big(\sum_{j=1}^{N} \bar{u}^j(k)\, I_j^{(2)}[\phi] + \mathcal{R}_k\Big)^{-1} k\partial_k \mathcal{R}_k\right]. \tag{2.44}$$

As it stands, this equation will have no solution for the N functions $\bar{u}^j(k)$ in general. The reason is that even though only the basis elements I_j spanning $\mathcal{T}_{\text{trunc}}$ appear under the functional trace $\mathrm{Tr}\big[\cdots\big]$, its derivative expansion will generate terms outside $\mathcal{T}_{\text{trunc}}$ as well:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\big[\cdots\big] \;=\; & \sum_{i=1}^{N} \bar{b}^i\Big(\big\{\bar{u}^j\big\}_{j=1,\cdots,N}; k\Big)\, I_i[\phi] \\ & + \sum_{\alpha,\, I_\alpha \notin \mathcal{T}_{\text{trunc}}} \bar{b}^\alpha\Big(\big\{\bar{u}^j\big\}_{j=1,\cdots,N}; k\Big)\, I_\alpha[\phi].\end{aligned} \tag{2.45}$$

Typically such terms are generated by expanding the propagator in terms of background quantities. Upon inserting (2.45) into (2.44), those terms on the RHS of the resulting equation which involve basis elements $I_\alpha \notin \mathcal{T}_{\text{trunc}}$ have no counterparts on its LHS.

At this point an additional assumption is needed in order to render the system of equations consistent. One assumes that it is legitimate to replace the prefactors of all invariants outside $\mathcal{T}_{\text{trunc}}$ by zero:

$$\boxed{\bar{b}^\alpha \mapsto 0 \quad \text{for all } \alpha \text{ such that } I_\alpha \notin \mathcal{T}_{\text{trunc}}.} \tag{2.46}$$

With (2.43) and (2.46) the FRGE boils down to a system of N coupled equations:

$$k\partial_k\, \bar{u}^j(k) = \bar{b}^j\Big(\bar{u}^1(k), \cdots, \bar{u}^N(k); k\Big) \quad \forall j = 1, \cdots, N. \tag{2.47}$$

At least when N is a (small) finite number one can investigate this system with standard methods and try to justify the assumption underlying the truncation.

Several comments are in order here.

(i) **The projection.** The rule (2.46) describes what we referred to as the "projection" of the RG dynamics from $\mathcal{T}$ onto $\mathcal{T}_{\text{trunc}}$; together with the specification of $\mathcal{T}_{\text{trunc}}$ in (2.43) it defines a certain truncation of theory space.

As for specifying a truncation uniquely there is a subtlety which is overlooked sometimes and at first sight might seem surprising perhaps.

Namely, the beta functions for the couplings retained, $\bar{u}^j$, depend not only on the choice of a basis on $\mathcal{T}_{\text{trunc}}$, *but also on how the other basis elements* $I_\alpha \notin \mathcal{T}_{\text{trunc}}$ *are chosen* even though their coupling constants are set to zero by the truncation ansatz.

To see the reason note that theory space in general does not carry a natural inner product that would define, in particular, a notion of orthogonality and orthogonal projections. While the restricted setting of the derivative expansion adopted here does equip $\mathcal{T}$ with the structure of a vector space, it still does not supply a natural inner product.

As a consequence, given an arbitrary point $P \in \mathcal{T}$, which does not happen to lie on $\mathcal{T}_{\text{trunc}}$, it is the matter of an explicit *definition* to specify its image under the projection, $P_{\text{proj}} \in \mathcal{T}_{\text{trunc}}$. The choice of a *projection map* has the same conceptual status as the choice of a subspace, $\mathcal{T}_{\text{trunc}}$, but is logically independent from it.

A projection is conveniently specified in terms of an *adapted basis* on $\mathcal{T}$: a subset of its basis vectors spans $\mathcal{T}_{\text{trunc}}$, and we define the projection to be parallel to all those basis vectors that are not tangent to $\mathcal{T}_{\text{trunc}}$. With other words, the coordinates of $P_{\text{proj}} \in \mathcal{T}_{\text{trunc}}$ are obtained by setting to zero all coordinates of its preimage $P \in \mathcal{T}$ except those related to the subbasis spanning $\mathcal{T}_{\text{trunc}}$. This is exactly what is done in (2.46).

In Figure 2.2 this is illustrated for the simple example of the one-dimensional truncation of a 2D theory space. In Figure 2.2a, the subspace $\mathcal{T}_{\text{trunc}}$ is fixed by picking a single basis vector on the plane; it defines the "u^1-direction" and identifies $\mathcal{T}_{\text{trunc}}$ with the "u^1-axis". No notion of a projection is available yet. In Figure 2.2b a second basis vector is selected which gives rise to the "u^2-axis". The projection $\mathcal{T} \to \mathcal{T}_{\text{trunc}}$, $P \mapsto P_{\text{proj}}$, is defined to be

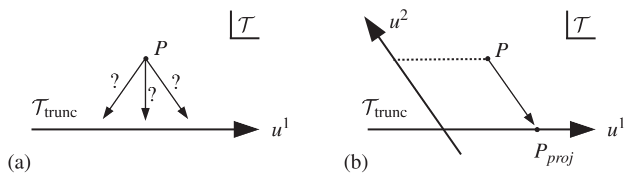


Figure 2.2. A 1-dimensional truncation of the 2-dimensional plane. (a) The definition of a first basis vector specifies the truncation subspace. (b) The definition of a second basis vector defines a projection onto this subspace.

parallel to this axis. It is clear then that the location of P_{proj} on $\mathcal{T}_{\text{trunc}}$, i.e., its u^1-coordinate, depends on our choice of the u^2-axis.

(ii) **Projected solution vs. solution of the projected equation.** Let $k \mapsto \Gamma_k$ be an exact solution of the FRGE on the untruncated theory space $\mathcal{T}$. Furthermore, let γ be the set of all points visited by this trajectory. We can visualize it as an oriented curve on $\mathcal{T}$, as shown in Figure 2.3.

Now we choose a truncation by specifying a subspace $\mathcal{T}_{\text{trunc}}$ plus a projection. By applying the latter to the exact RG trajectory γ we obtain its projection in $\mathcal{T}_{\text{trunc}}$, which we denote by γ_{proj}.

The truncation chosen gives rise to an approximate RG equation of the type (2.47). Let us solve this approximate equation subject to the condition that it passes through a point $P_0 \in \mathcal{T}_{\text{trunc}}$, which is also visited by the exact γ. We denote this resulting (exact!) solution to the approximate RG equation by γ_{trunc}.

We emphasize that, typically, γ, γ_{proj}, and γ_{trunc}, are three distinct curves. While γ can explore the full theory space in principle, both γ_{proj} and γ_{trunc} "live" on $\mathcal{T}_{\text{trunc}}$ by definition. In general, γ_{proj} and γ_{trunc} are different from one another, and this is of course what makes the truncation, at best, an approximation.

The trajectories γ and γ_{proj} are both exact, the only difference being that γ_{proj} represents only partial information about the running coupling constants, namely only about those which coordinatize $\mathcal{T}_{\text{trunc}}$. The third

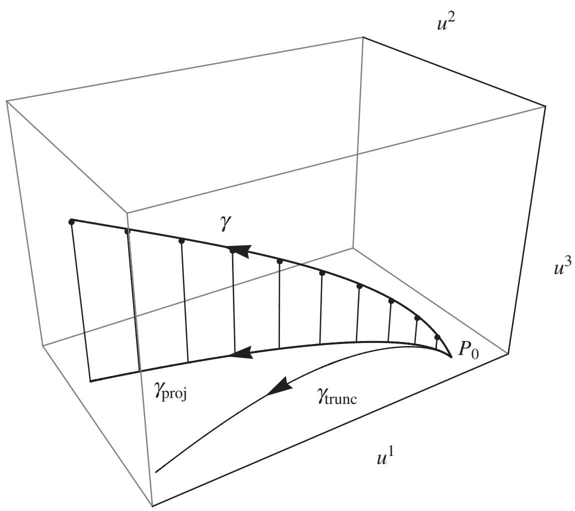


Figure 2.3. A 2-dimensional truncation (u^1–u^2-plane) of a 3-dimensional space. The exact trajectory γ, its projection γ_{proj}, and γ_{trunc} pass through a common point P_0 on $\mathcal{T}_{\text{trunc}}$. While γ gradually departs from $\mathcal{T}_{\text{trunc}}$, γ_{proj} and γ_{trunc} stay close to each other if the truncation is reliable.

curve, γ_{trunc}, is an approximation (to γ_{proj}) since it is the solution of an incomplete system of differential equations which is ignorant of the exact form of the RG dynamics; see Figure 2.3 for an illustration.

Thus, a *reliable truncation* is characterized by an approximate equality of γ_{proj} and γ_{trunc}, at least within a certain interval of scales k. It is in general a difficult but clearly important task to judge the reliability of a given truncation. We will address this issue repeatedly later on.

2.2.4 The Local Potential Approximation

The local potential approximation (LPA) is a prototypical example of a truncation of the scalar theory space discussed above. The "points" $A[\,\cdot\,] \in \mathcal{T}$ have the general structure indicated in (2.35). The LPA deals with an infinite-dimensional subspace $\mathcal{T}_{\text{trunc}}$. It retains from (2.35) all the non-derivative terms of the first line, and the first term in the second, i.e., $\bar{u}_{2,0}\left(\partial_\mu \phi\right)^2$. Thus, the generic $A[\,\cdot\,] \in \mathcal{T}_{\text{trunc}}$ has the form

$$A[\phi] = \int \mathrm{d}^d x \left\{\bar{u}_{2,0}\, \partial_\mu \phi(x)\, \partial^\mu \phi(x) + U(\phi(x))\right\}. \tag{2.48}$$

Here all possible zero-derivative terms are collected in the potential function $U(\,\cdot\,)$.

The projection onto $\mathcal{T}_{\text{trunc}}$ is defined in the spirit of the derivative expansion: The projected functional $A_{\text{proj}} \in \mathcal{T}_{\text{trunc}}$ related to a general $A \in \mathcal{T}$ is obtained by inserting for ϕ a plane wave, expanding $A[\phi]$ in powers of its amplitude and momentum, and setting to zero all terms except those of zeroth order in the momentum (and any order in the amplitude), and those of second order in both the momentum and the amplitude.

Thus, the general form of the EAA on the subspace reads

$$\boxed{\Gamma_k[\phi] = \int \mathrm{d}^d x \left\{\frac{1}{2} Z_k\, \partial_\mu \phi(x)\, \partial^\mu \phi(x) + U_k(\phi(x))\right\}.} \tag{2.49}$$

Here U_k denotes the scale-dependent *effective average potential*, and we also introduced the customary notation $Z_k \equiv 2\bar{u}_{2,0}(k)$. Actually the ansatz (2.49) with a running prefactor of the kinetic term, Z_k, amounts already to an extended version of the LPA. The LPA proper, on which we will focus in the following, makes the additional assumption that the k-dependence of Z_k is actually weak, and it sets $Z_k \equiv 1$ in the ansatz (2.49). As a result, the only k-dependence that has to be tracked in the (projected) FRGE is that of the running potential function $U_k(\,\cdot\,)$.

To derive a differential equation for U_k, we insert (2.49) with $Z_k \equiv 1$ into the exact form of the FRGE, (2.15), and obtain in a first step:

$$\int \mathrm{d}^d x\, k\partial_k\, U_k(\phi(x)) = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]\Bigg|_{\text{proj}}. \tag{2.50}$$

Here we indicated symbolically that the RHS of (2.50), regarded as a functional of ϕ, must be projected onto $\mathcal{T}_{\text{trunc}}$ ultimately. The Hessian of the truncation ansatz, to be inserted on the RHS of (2.50), is given by the second functional derivatives of (2.49):

$$\frac{\delta^2 \Gamma_k[\phi]}{\delta\phi(x)\delta\phi(y)} = \Big[-\partial^2 + U_k''(\phi(x))\Big]\delta(x-y). \tag{2.51}$$

A prime will always denote the derivative with respect to the argument, so here $U_k'' \equiv \partial^2 U_k/\partial\phi^2$. In the following we interpret $\Gamma_k^{(2)}$ as the ϕ-dependent differential operator acting on the δ-distribution in (2.51):

$$\Gamma_k^{(2)}[\phi] = -\partial^2 + U_k''(\phi(x)). \tag{2.52}$$

To set up the RG equation for $U_k(\,\cdot\,)$, we first introduce a cutoff function according to the prescription (2.7). Subsequently we perform the projection $\mathcal{T} \to \mathcal{T}_{\text{trunc}}$, $A \mapsto A_{\text{proj}}$ of the special functional that is defined by the RHS of (2.50) with (2.52):

$$A[\phi] \equiv \frac{1}{2}\text{Tr}\left[\Big(-\partial^2 + \mathcal{R}_k(-\partial^2) + U_k''(\phi(x))\Big)^{-1} k\partial_k\,\mathcal{R}_k(-\partial^2)\right]. \tag{2.53}$$

This calculation proceeds as follows.

The functionals in $\mathcal{T}_{\text{trunc}}$ are space integrals over $Z_k\big(\partial_\mu\phi\big)^2$ plus all possible non-derivative terms built from $\phi(x)$. Since the standard LPA makes the additional approximation $Z_k \equiv 1$ we actually do not need to know the component of the projection $A_{\text{proj}}[\,\cdot\,]$ along the $\big(\partial_\mu\phi\big)^2$-direction of $\mathcal{T}_{\text{trunc}}$, so it suffices to find the non-derivative terms of $A_{\text{proj}}[\,\cdot\,]$. They are precisely those which, upon inserting a plane wave into $A[\,\cdot\,]$, are of zeroth order in the plane wave's momentum. Hence, it is convenient to insert a constant field configuration $\phi(x) = \text{const} \equiv C$ since it sets to zero the uninteresting terms automatically. As a consequence, the operator under the trace (2.53) loses its explicit x-dependence and is easily diagonalized in the momentum eigen-basis of the Laplacian on $\mathbb{R}^d$. Thus, for any real value C of the constant field:

$$A[C] = \frac{1}{2}\int \mathrm{d}^d x \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{k\partial_k\,\mathcal{R}_k(p^2)}{p^2 + \mathcal{R}_k(p^2) + U_k''(C)}. \tag{2.54}$$

Furthermore, it follows that $A_{\text{proj}}[C] = A[C]$, since the original functional $A \in \mathcal{T}$ and its projection $A_{\text{proj}} \in \mathcal{T}_{\text{trunc}}$ agree on constant functions.

Let us emphasize at this point that usually A_{proj} and A are assumed to have the same domain, i.e., $A_{\text{proj}}[\phi]$ and $A[\phi]$ are defined over the same space of ϕ's. So setting $\phi = \text{const}$ for projection purposes is not meant to imply that A_{proj} is defined for constant fields only. Rather, the correct interpretation is that $A_{\text{proj}}[\phi(\,\cdot\,)]$ for any, in general non-constant $\phi \equiv \phi(x)$, has the same structure as $A_{\text{proj}}[C]$, but with C replaced by $\phi(x)$:

$$A_{\text{proj}}[\phi(\,\cdot\,)] = A[C]\Big|_{C\to\phi(x)} + a_k \int \mathrm{d}^d x\,\big(\partial_\mu\phi\big)^2. \tag{2.55}$$

For completeness we also added the $\left(\partial_\mu \phi\right)^2$-component of the projection, so that (2.55) covers all terms contained in the ansatz (2.49). Since the scale dependence of Z_k will not be considered in the sequel, we will not compute the coefficient a_k here and refer to Appendix A of [80] for the details of its derivation.

In fact, the assumption $Z_k \equiv 1$, implying $\partial_k Z_k = 0$, amounts to assuming that a_k is small and we may replace $a_k \to 0$. This becomes obvious when we insert (2.55) into the FRGE (2.50): Comparing the terms under the space integral, we see that the term $a_k \left(\partial_\mu \phi\right)^2$ on the RHS has no counterpart on the LHS to match with since $\partial_k Z_k = 0$ was used there; thus consistency is achieved if a_k is negligibly small.[8]

Furthermore, equating the non-derivative terms under the x-integral yields the desired relation for the potential:

$$k\partial_k U_k(\phi) = \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{k\partial_k \mathcal{R}_k(p^2)}{p^2 + \mathcal{R}_k(p^2) + U_k''(\phi)}. \tag{2.56}$$

At this point it has become immaterial whether ϕ is a constant number or a function of x: (2.56) is a partial differential equation involving a first-order scale and a second-order ϕ-derivative, which makes no reference to spacetime points.

Still with a general cutoff function $\mathcal{R}_k$, we can simplify (2.56) by performing the angular part of the momentum integration. The relevant identity, valid for any function F with suitable falloff behavior, reads

$$\frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, F(p_\mu p^\mu) = v_d \int_0^\infty \mathrm{d}w\, w^{\frac{d}{2}-1}\, F(w) \tag{2.57}$$

with the dimension-dependent constant

$$v_d \equiv \left[2\,(4\pi)^{\frac{d}{2}}\,\Gamma\!\left(\tfrac{d}{2}\right)\right]^{-1}. \tag{2.58}$$

Special values of v_d include $v_2 = \frac{1}{8\pi}$, $v_3 = \frac{1}{8\pi^2}$, $v_4 = \frac{1}{32\pi^2}$.

This yields the following projected flow equation:

$$\boxed{k\partial_k U_k(\phi) = v_d \int_0^\infty \mathrm{d}w\, w^{\frac{d}{2}-1}\, \frac{k\partial_k \mathcal{R}_k(w)}{w + \mathcal{R}_k(w) + \partial_\phi^2 U_k(\phi)}.} \tag{2.59}$$

The partial differential equation (2.59) is the main result of the LPA. It has a number of rather typical properties which we shall re-encounter frequently later on in quantum gravity, albeit in a far more complicated setting. Next, we discuss some of them.

(1) $\boldsymbol{\mathcal{R}_k}$ dependence. The RG equation (2.59) depends manifestly on the cutoff function chosen, and the same is true for most properties of its solutions. Properties of, or quantities derived from, a solution which are equal for all functions $\mathcal{R}_k$

[8] For a detailed discussion of the circumstances under which this is the case we must refer to the literature [31].

with the asymptotic behavior (2.6) are called *universal*. For instance, observables must be universal quantities in this sense.

(2) Explicit form. The w-integration in (2.59) can be performed analytically for a number of special choices of $\mathcal{R}_k$, including, for example, the "optimized" cutoff function [63],

$$\mathcal{R}_k(p^2) = (k^2 - p^2)\,\Theta(k^2 - p^2), \tag{2.60}$$

which gives rise to the partial differential equation

$$\boxed{k\partial_k\, U_k(\phi) = \frac{4v_d}{d}\,\frac{k^{d+2}}{k^2 + \partial_\phi^2\, U_k(\phi)}.} \tag{2.61}$$

While this equation is of a non-standard form, a number of techniques have been developed to analyze and solve it. This includes, in particular, numerical [31] or pseudospectral methods [81, 82].

(3) Approaching convexity. Since Γ_k equals the usual effective action Γ in the limit $k \to 0$, the non-derivative part of the former, $U_k(\phi)$, should approach the non-derivative part of the latter, the familiar effective potential [59]: $V_{\text{eff}}(\phi) = \lim_{k\to 0} U_k(\phi)$.

Furthermore, by its very definition Γ is given by a Legendre transform, and the same is true for V_{eff}, by restriction to constant fields. Standard properties of the Legendre transformation [66] thus imply that Γ is a convex functional, and V_{eff} a convex function, $V''_{\text{eff}}(\phi) \geq 0$.

This should be contrasted with Γ_k and U_k at non-zero scales $k > 0$ where the ΔS_k-term in (2.12) prevents them from being pure Legendre transforms. As a consequence, they have no reason to be convex as long as $k > 0$, but they must become so in the limit $k \to 0$. Indeed, in this limit $\Delta S_k[\phi]$ vanishes.

While these properties apply to the exact EAA and its likewise exact non-derivative part, they can easily get ruined by inadequate approximations. Convexity poses a notorious problem in perturbation theory (loop expansion); see, for instance, [59]. Therefore, it is even more remarkable that the LPA flow equation (2.61) is powerful enough to correctly describe the *approach to convexity*.

This is demonstrated in Figure 2.4, which shows the effective average potential $U_k(\phi)$ at various scales. It is obtained by numerically integrating (2.61) "downward", i.e., toward decreasing values of k. The initial potential imposed at $k = \Lambda$ was chosen to be "W-shaped": $U_\Lambda(\phi) = -c_1\phi^2 + c_2\phi^4$ with $c_{1,2} > 0$. When k is lowered at first the two minima of the "W" move inward, then come to a halt at a certain value $\pm\phi_{\text{min}}$, and the non-convex part of the curve flattens more and more, until it finally approaches a function $U_0(\phi)$ which is indeed convex, and *constant* for ϕ between $-\phi_{\text{min}}$ and $+\phi_{\text{min}}$.

This is, in fact, precisely what one expects for the exact effective potential in the phase where the $\mathbb{Z}_2$ symmetry under $\phi \to -\phi$ is spontaneously broken; see [69] and [31] for a detailed discussion.

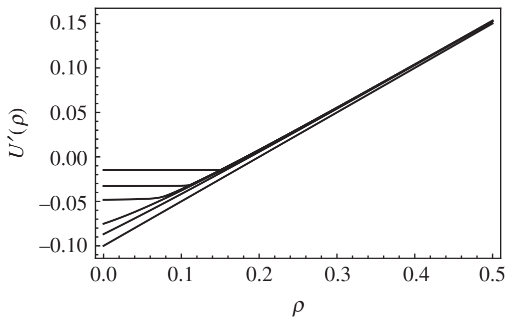


Figure 2.4. Plot of the first derivative effective average potential, $U'(\rho)$ as a function of $\rho = \frac{1}{2}\phi^2$ in $d = 3$. Moving from the bottom to the top, the curves are obtained for decreasing RG times $\ln(k/\Lambda) = 0, -0.5, -1, -1.5, -1.7, -2, -2.1$. Starting from $U_\Lambda = -0.05\phi^2 + 0.0625\phi^4$, the potential turns convex for decreasing RG scale k. (Obtained by solving the LPA equation via pseudospectral methods [82].)

(4) Dimensionless variables. The example of (2.61) illustrates that the differential equation for $U_k(\phi)$ has an *explicit dependence on the scale, k*. It can be removed though, by re-expressing it in terms of a dimensionless potential and a dimensionless field variable, which are obtained from the corresponding dimensionful quantities by multiplying them with appropriate powers of the cutoff scale.

Assigning the canonical mass dimensions, in d spacetime dimensions,

$$\begin{aligned} &[x^\mu] = -1, \qquad [\partial_\mu] = +1, \qquad [k] = +1, \\ &[S] = [\Gamma] = [\Gamma_k] = 0 \implies [U_k] = d,\ [\phi] = \tfrac{d-2}{2}, \end{aligned} \tag{2.62}$$

the dimensionless field and its potential are given by, respectively,

$$\begin{aligned} \varphi &\equiv k^{-\frac{d-2}{2}}\phi \\ u_k(\varphi) &\equiv k^{-d}\, U_k\big(k^{\frac{d-2}{2}}\varphi\big) \equiv k^{-d}\, U_k(\phi). \end{aligned} \tag{2.63}$$

Furthermore, we introduce a dimensionless version of $\mathcal{R}_k$, namely the *shape function* $R^{(0)}$:

$$\mathcal{R}_k(p^2) = k^2 R^{(0)}\big(\tfrac{p^2}{k^2}\big). \tag{2.64}$$

By virtue of (2.6), $R^{(0)}$ is an essentially arbitrary dimensionless function, depending on a dimensionless argument $z \equiv p^2/k^2$, and required to comply with the asymptotics:

$$R^{(0)}(z) \longrightarrow 0 \quad \text{for } z \to \infty, \quad \text{and} \quad R^{(0)}(z) \longrightarrow 1 \quad \text{for } z \to 0, \tag{2.65}$$

whereby the transition between the two regimes occurs near $z \approx 1$.

We also employ the dimensionless *renormalization group "time"*:

$$t \equiv \ln(k/k_0), \tag{2.66}$$

which takes over the role of k as the evolution parameter. (The fixed mass parameter k_0 just defines the units in which k is measured.)

In terms of the dimensionless variables, the general LPA equation is

$$\begin{aligned}\partial_t u_k(\varphi) + d u_k(\varphi) - \frac{1}{2}(d-2)\,\varphi\, u_k'(\varphi)& \\ = 2v_d \int_0^\infty & \mathrm{d}z\, z^{\frac{d}{2}-1}\, \frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{z + R^{(0)}(z) + u_k''(\varphi)}.\end{aligned} \tag{2.67}$$

The example (2.61) which employs the cutoff (2.60), or equivalently the shape function

$$R^{(0)}(z) = (1-z)\,\Theta(1-z), \tag{2.68}$$

assumes the following form now:

$$\boxed{\partial_t u_k(\varphi) + d u_k(\varphi) - \frac{1}{2}(d-2)\,\varphi\, u_k'(\varphi) = \frac{4v_d}{d}\,\frac{1}{1+u_k''(\varphi)}.} \tag{2.69}$$

We see that (2.67) and (2.69) are indeed *autonomous* equations, i.e., they have no explicit dependence on k, or t, any longer. (While the running potential is actually a function of t now, it is customary to keep using the notation u_k.)

(5) Fixed points. As for solutions $u_k(\varphi)$ to the dimensionless LPA equation, a particularly interesting class are the *fixed point solutions* which satisfy (2.67) with $\partial_t u_k(\varphi) = 0$. These special potentials will be denoted $u_*(\varphi)$. Picking the shape function (2.68) they satisfy the following ordinary differential equation, if they exist:

$$\boxed{d u_*(\varphi) - \frac{1}{2}(d-2)\,\varphi\, u_*'(\varphi) = \frac{4v_d}{d}\,\frac{1}{1+u_*''(\varphi)}.} \tag{2.70}$$

There is one obvious solution to this equation which exists in any number of spacetime dimensions, namely the trivial, or *Gaussian fixed point* (GFP) with the φ-independent potential $u_*(\varphi) = \frac{4v_d}{d^2} = \text{const}$. This fixed point is termed "Gaussian" since the only important field monomial in the corresponding action is the quadratic kinetic term $\frac{1}{2}\int \left(\partial_\mu \phi\right)^2$, and this makes the functional integral over ϕ a Gaussian one.

Concerning non-trivial, or "non-Gaussian," fixed points with potentials that actually depend on φ the situation strongly depends on the dimensionality of spacetime: There are infinitely many such fixed points in two dimensions, there is one (the Wilson–Fisher fixed point) in three dimensions, but none is known to exist in four dimensions.

Note that the dimensionful potential $U_k(\phi) \equiv k^d u_k\big(k^{-\frac{d-2}{2}}\phi\big)$ related to a fixed point u_* is by no means scale independent. It "runs" in a particularly simple way:

$$U_k(\phi) = k^d\, u_*\big(k^{-\frac{d-2}{2}}\,\phi\big). \tag{2.71}$$

We will re-encounter this kind of behavior in the case of quantum gravity later on.

Fixed points in scalar systems have been investigated by many authors, using a variety of methods. This is an important topic in its own right and which is beyond the scope of this book though. The interested reader is referred to the pedagogical introduction in [66] and the comprehensive list of references therein.

(6) Power series expansion. In (2.59) or (2.61) the LPA equation for $U_k(\phi)$ has the appearance of a partial differential equation in two variables, k and ϕ. We can obtain an alternative representation in "component form" by expanding the potential in a power series:

$$U_k(\phi) = \sum_{n=0}^{\infty} \bar{\lambda}_{2n}(k)\, \phi^{2n}. \tag{2.72}$$

In this manner the partial differential equation becomes a system of ordinary ones for the expansion coefficients $\bar{\lambda}_{2n} \equiv \bar{u}_{0,2n}$:

$$k\partial_k\, \bar{\lambda}_{2n}(k) = \bar{b}_{2n}\Big(\bar{\lambda}_0(k), \bar{\lambda}_2(k), \cdots; k\Big). \tag{2.73}$$

Since

$$[\bar{\lambda}_{2n}] = [\bar{b}_{2n}] = d - (d-2)n, \tag{2.74}$$

the corresponding dimensionless coupling constants are

$$\lambda_{2n}(k) = k^{(d-2)n-d}\, \bar{\lambda}_{2n}(k). \tag{2.75}$$

Their RG equations read

$$\partial_t \lambda_{2n}(k) = \beta_{2n}\Big(\lambda_0(k), \lambda_2(k), \lambda_4(k), \cdots\Big) \tag{2.76}$$

with the dimensionless beta functions

$$\beta_{2n}\big(\lambda_0, \lambda_2, \cdots\big) \equiv \big[(d-2)n - d\big]\lambda_{2n} + k^{(d-2)n-d}\, \bar{b}_{2n}, \tag{2.77}$$

where the $\bar{\lambda}$'s in the argument of $\bar{b}_{2n}$ are now to be expressed in terms of their dimensionless pendants.

The functions (2.77) have a structure which is typical of all β-functions we will encounter in this book. They consist of two parts: First, a classical, or canonical term linear in the respective coupling which owes its existence to using the variable scale k as the unit in which all dimensionful quantities are expressed, and second, a dedimensionalized $\bar{b}$-function; the latter describes the true renormalization effects that are caused by the statistical or quantum fluctuations.

The explicit form of the $\bar{b}'_{2n}$s is found by inserting (2.72) into (2.59), expanding in ϕ^2, and comparing equal powers of ϕ^2. The structure of the first few examples looks as follows:

$$\begin{aligned}
\bar{b}_0 &= \mathcal{J}_1, && (2.78)\\
\bar{b}_2 &= -12\,\bar{\lambda}_4\,\mathcal{J}_2, && (2.79)\\
\bar{b}_4 &= -30\,\bar{\lambda}_6\,\mathcal{J}_2 + 144\,(\bar{\lambda}_4)^2\mathcal{J}_3, && (2.80)\\
\bar{b}_6 &= -56\,\bar{\lambda}_8\,\mathcal{J}_2 + 720\,\bar{\lambda}_4\bar{\lambda}_6\,\mathcal{J}_3 - 1728\,(\bar{\lambda}_4)^3\mathcal{J}_4, && (2.81)\\
\bar{b}_8 &= -90\,\bar{\lambda}_{10}\,\mathcal{J}_2 + 1344\,\bar{\lambda}_4\bar{\lambda}_8\,\mathcal{J}_3 + 900\,(\bar{\lambda}_6)^2\mathcal{J}_3 \\
&\quad -12960\,\bar{\lambda}_6(\bar{\lambda}_4)^2\mathcal{J}_4 + 20736\,(\bar{\lambda}_4)^4\mathcal{J}_5, && (2.82)\\
\bar{b}_{10} &= \cdots\,. && (2.83)
\end{aligned}$$

Here the $\mathcal{J}'$s are a family of convergent momentum integrals. They depend on $\bar{\lambda}_2 = k^2\lambda_2$, but on no further coupling constants:

$$\begin{aligned}
\mathcal{J}_n &\equiv v_d \int_0^\infty \mathrm{d}w\, w^{\frac{d}{2}-1}\,\frac{k\partial_k\,\mathcal{R}_k(w)}{\big[w+\mathcal{R}_k(w)+2\bar{\lambda}_2\big]^n} \\
&= 2\,v_d\,k^{d+2-2n}\int_0^\infty \mathrm{d}z\,z^{\frac{d}{2}-1}\,\frac{R^{(0)}(z)-zR^{(0)\prime}(z)}{\big[z+R^{(0)}(z)+2\lambda_2\big]^n}. \qquad (2.84)
\end{aligned}$$

Generically these integrals depend on the cutoff function in a non-trivial way. Exceptions occur, however, at $\lambda_2 = 0$ if $n = (d+2)/2$. The integrand turns out to be a total derivative then, and it is sufficient to exploit that $R^{(0)}(0) = 1$ and $R^{(0)}(\infty) = 0$ in order to evaluate the integral:

$$\begin{aligned}
\mathcal{J}_{\frac{d+2}{2}}\Big|_{\lambda_2=0} &= 2\,v_d\int_0^\infty \mathrm{d}z\,z^{\frac{d}{2}-1}\,\frac{R^{(0)}(z)-zR^{(0)\prime}(z)}{\big[z+R^{(0)}(z)\big]^{\frac{d+2}{2}}} \\
&= 2\,v_d\int_0^\infty \mathrm{d}z\,\frac{d}{dz}\left(\frac{2}{d}\,\frac{z^{\frac{d}{2}}}{\big[z+R^{(0)}(z)\big]^{\frac{d}{2}}}\right) \\
&= \frac{4v_d}{d}\left[\left(\frac{z}{z+R^{(0)}(z)}\right)^{\frac{d}{2}}\right]_0^\infty \\
&= \frac{4v_d}{d}. \qquad (2.85)
\end{aligned}$$

The coefficients $\mathcal{J}_{\frac{d+2}{2}}$ at $\lambda_2 = 0$ are examples of *universal* quantities which are insensitive to the details of the cutoff function. They are dimensionless, pure numbers.

It should be stressed, however, that universality is rather the exception than the rule; typical coefficients in $\bar{b}$-, or β-functions, are indeed sensitive to the full profile of the function $R^{(0)}(z)$. This dependence on unphysical features of the regulator makes it quite clear that, in the EAA approach, the majority of the beta functions and running couplings cannot have an immediate physical meaning per se. In general, the cutoff scheme dependence will drop out only at a late stage in the calculation of observables. Up to this point a concrete shape function must be employed, but any such function with the correct asymptotics will do the job.

With (2.68), for example, the above coefficients $\mathcal{J}_n$ for arbitrary n and d, and any value of λ_2, are explicitly given by

$$\mathcal{J}_n = 4\, v_d\, d^{-1}\, k^{d+2-2n} \Big(\frac{1}{1+2\lambda_2}\Big)^n. \tag{2.86}$$

Because of the general structure of the $\bar{b}$-functions in (2.78)–(2.83), we observe that the hierarchy of equations $k\partial_k\, \bar{\lambda}_{2n} = \bar{b}_{2n}$ does not close at any finite order. The scale derivative of a given coupling constant, apart from its own value, depends on couplings which are both lower and higher up in the hierarchy. In (2.82), for instance, it is seen that $k\partial_k\, \bar{\lambda}_8$ depends on $\big(\bar{\lambda}_2, \bar{\lambda}_4, \bar{\lambda}_6\big)$, on $\bar{\lambda}_8$ itself, but also on $\bar{\lambda}_{10}$. Thus, we cannot hope to easily find an exact solution to these equations.

(7) Polynomial truncations. Under certain conditions it can be legitimate to use a truncation which is simpler than the full-fledged LPA and to replace the power series (2.72) by a polynomial, $U_k(\phi) = \sum_{n=0}^{N} \bar{\lambda}_{2n}(k)\, \phi^{2n}$, $N < \infty$.

Note that trying to set $\bar{\lambda}_{2n}(k) = 0$ for $n > N$ at all scales assumes not only that the couplings thus discarded are negligibly small; it also imposes constraints *on the couplings retained*, namely that $\bar{b}_{2n} \approx 0$ for $n > N$, and this also constrains $\bar{\lambda}'_{2n}$s at $n \le N$. This latter requirement must be met for consistency since otherwise a coupling that is small at one scale could grow large at another.

If $\bar{\lambda}_{2n}$ and $\bar{b}_{2n}$ can be replaced by zero for $n > N$, the hierarchy indeed collapses to its first $N+1$ equations.

As an example, let us consider the case $N = 2$, i.e., a quadratic polynomial:

$$U_k(\phi) = \bar{\lambda}_0(k) + \bar{\lambda}_2(k)\phi^2 + \bar{\lambda}_4(k)\phi^4. \tag{2.87}$$

Now, according to (2.81), there are three terms contributing to $\bar{b}_6$. The first two of them, containing $\bar{\lambda}_8$ and $\bar{\lambda}_6$, do indeed vanish upon setting the higher couplings to zero, but not so the third term $\propto (\bar{\lambda}_4)^3 \mathcal{J}_4$. Demanding that it can be neglected obviously implies conditions on $\bar{\lambda}_4$ and $\bar{\lambda}_2$. Once a solution to the RG equations of the polynomial truncation has been found one must check *a posteriori* whether or not this solution does indeed meet these conditions.

With the ansatz (2.87), what remains are the three equations:

$$k\partial_k\, \bar{\lambda}_0 = \mathcal{J}_1, \tag{2.88}$$

$$k\partial_k\, \bar{\lambda}_2 = -12\, \bar{\lambda}_4\, \mathcal{J}_2, \tag{2.89}$$

$$k\partial_k\, \bar{\lambda}_4 = 144\, (\bar{\lambda}_4)^2\, \mathcal{J}_3. \tag{2.90}$$

Noting that the $\mathcal{J}'$s depend on $\bar{\lambda}_2$ we see that the equations for $\bar{\lambda}_2$ and $\bar{\lambda}_4$ are indeed coupled, while the equation for $\bar{\lambda}_0$ stands alone and can be integrated straightforwardly once $\bar{\lambda}_2(k)$ is known.

Let us stick to $d=4$ now; $\bar{\lambda}_0$ and $\bar{\lambda}_2$ have canonical mass dimension 4 and 2 then, and $\bar{\lambda}_4 \equiv \lambda_4$ is dimensionless, hence $\beta_4 \equiv \bar{b}_4$. Let us furthermore specialize for the regime where the running mass, i.e., $(2\bar{\lambda}_2)^{\frac{1}{2}}$, is much smaller than the cutoff:

$$\bar{\lambda}_2 \ll k^2 \quad \Longleftrightarrow \quad \lambda_2 \ll 1. \tag{2.91}$$

Then letting $\bar{\lambda}_2 \to 0$ in $\mathcal{J}_3$, the equation for the quartic coupling λ_4 gets separated from that of the mass, and the coefficient in its β-function happens to be one of the universal quantities (2.85), namely $\mathcal{J}_3(0) = v_4 = \frac{1}{32\pi^2}$. As a consequence, the resulting RG equation,

$$k\partial_k\, \lambda_4 = \beta_4(\lambda_4) = \frac{18}{(2\pi)^2}\lambda_4^2, \tag{2.92}$$

is seen to have the same form and to contain the same numerical constant as the familiar result for the RG equation in *one-loop perturbation theory.* (Usually the latter is derived using dimensional regularization (minimal subtraction) and k is to be identified with the mass scale μ, see [76].)

Again, the coincidence of some $\bar{b}$-function descending from the FRGE with a perturbative β-function is a very rare exception. In general, this happens only in the case of *universal* coefficients. In fact, it takes as little as going beyond the regime of small masses, allowing for arbitrary λ_2, to obtain a $R^{(0)}$-dependent answer which has no direct analog in perturbation theory. And, what is even more important, the superficial similarity of the three differential equations (2.88)–(2.90) with the perturbative RG equation of ϕ^4-theory arises only because we employed a rather severe approximation of the full FRGE, namely the LPA and on top of it the restriction to a quartic potential.

Adopting the special shape function (2.68), equation (2.86) provides us with the mass dependence of β_4:

$$\beta_4(\lambda_2, \lambda_4) = \frac{18}{(2\pi)^2}\frac{1}{(1+2\lambda_2)^3}\lambda_4^2. \tag{2.93}$$

In comparison with the formula for $\lambda_2 = 0$ the new feature of this β-function is the factor

$$\frac{1}{1+2\lambda_2} = \frac{k^2}{k^2+2\bar{\lambda}_2} = \begin{cases} 1 & \text{if } k \gg |2\bar{\lambda}_2|^{\frac{1}{2}} \\ 0 & \text{if } k \ll |2\bar{\lambda}_2|^{\frac{1}{2}}. \end{cases} \tag{2.94}$$

This factor makes our result (2.93) powerful enough to quantitatively describe the phenomenon known as the *decoupling of heavy modes.*[9]

Equation (2.93) displays a *threshold* at the (mass)^2-scale set by $2\bar{\lambda}_2(k)$. For k values well above the threshold, β_4 is the same function as for a massless field, but

[9] See [83] for its description within perturbation theory and [84] for the example of QED.

it approaches zero when the cutoff scale falls below the threshold. As a result, when we integrate the RG equations with (2.93) for decreasing k, the quartic coupling stops running; it "decouples" when k approaches the threshold from above: $\partial_k \lambda_4(k) \to 0$. In the following we will see how we can take advantage of this decoupling phenomenon as a powerful computational tool.

A more detailed discussion of the solutions to the scalar RG equations would lead us too far afield, and thus we refer to the literature here [31].

2.2.5 Adjusted Cutoffs

The first step in going beyond the local potential approximation consists in giving up the restriction $Z_k \equiv 1$, thus allowing for a running wave function normalization constant Z_k in (2.49). Using truncations of this type one should employ a slightly different normalization of the cutoff function, such that $\mathcal{R}_k(p^2) = Z_k k^2$ if $p^2 \ll k^2$. As a result, $\mathcal{R}_k$ combines in the FRGE with $\Gamma_k^{(2)}$ to the inverse propagator $\Gamma_k^{(2)} + \mathcal{R}_k = Z_k(p^2 + k^2) + \cdots$, which is valid for low momentum modes. This is precisely as it should be if we want the IR cutoff to give a squared mass of the size k^2, rather than k^2/Z_k, to the low momentum modes.

This adjustment is particularly important in more complicated systems with more than one field. Typically their kinetic terms will contain different Z_k-factors, and so we must make sure that all fields are cut off at precisely the same scale, namely k^2. This is achieved by a cutoff of the type

$$\mathcal{R}_k(p^2) = \mathcal{Z}_k \, k^2 \, R^{(0)}(\tfrac{p^2}{k^2}). \tag{2.95}$$

As before, $R^{(0)}$ is a shape function satisfying the normalized boundary conditions $R^{(0)}(0) = 1$ and $R^{(0)}(\infty) = 0$. Furthermore, $\mathcal{Z}_k$ is a matrix in field space which is adjusted to the normalization of the various fields in such a way that they are all cut off at $p^2 = k^2$. If the field modes of type "i" have an inverse propagator $Z_k^{(i)} p^2 + \cdots$ and the modes do not mix, $\mathcal{Z}_k$ is a diagonal matrix with $\mathcal{Z}_k = Z_k^{(i)}$ in the sector of the "i" modes.

2.3 Decoupling

In this section we return to the decoupling mechanism which we encountered earlier already and demonstrate how it can be exploited in order to deduce the presence of certain terms in the EAA without actually going through the explicit EAA-based calculation [70, 85].

For the sake of simplicity we employ again the example of a single real scalar field on a flat 4-dimensional space whose Effective Average Action is a solution to the flow equation

$$k\partial_k \, \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \left[\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k \right)^{-1} k\partial_k \, \mathcal{R}_k \right]. \tag{2.96}$$

For the purposes of the present discussion it is sufficient to work at the level of the LPA ansatz for Γ_k. After a slight simplification of the notation it reads:

$$\Gamma_k[\phi] = \int \mathrm{d}^4x \left\{ \frac{1}{2} Z(k) \partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2} \bar{m}^2(k) \phi^2 + \frac{1}{12} \lambda(k) \phi^4 + \cdots \right\}. \tag{2.97}$$

To start with, we neglect the running of the kinetic term and approximate $Z(k) \equiv 1$. For functionals of this type, and in a momentum basis where $-\partial^2 \equiv p^2$, the denominator appearing under the trace of (2.96) reads

$$\Gamma_k^{(2)} + \mathcal{R}_k = p^2 + \bar{m}^2(k) + k^2 + \lambda(k)\phi^2 + \cdots . \tag{2.98}$$

To make the discussion as transparent as possible we inserted a mass-type cutoff $\mathcal{R}_k = k^2$ here even though the corresponding profile function is not strictly admissible in the sense of the requirements (2.65).[10]

In a diagrammatic loop calculation of Γ_k the inverse of (2.98) appears as the effective propagator in all loops. It contains the IR cutoff at the scale k. For the specific choice (2.98) this adds the k^2-term to the running mass parameter $\bar{m}^2(k)$. Therefore, the p_μ-modes (plane waves) are integrated out fully only in the domain $p^2 \gtrsim \bar{m}^2 + k^2 + \lambda\phi^2 + \cdots$. In the opposite case, all loop contributions are suppressed because of the non-zero *effective mass square* given by $\bar{m}^2 + k^2 + \lambda\phi^2 + \cdots$. It receives contributions from both the "artificial" cutoff k^2 introduced by hand and the "physical" cutoff terms $\bar{m}^2(k) + \lambda(k)\phi^2 + \cdots$. As a consequence, Γ_k can display a significant dependence on k in the regime $k^2 \gtrsim \bar{m}^2(k) + \lambda(k)\phi^2 + \cdots$ only. Otherwise, k^2 is negligible relative to $\bar{m}^2 + \lambda\phi^2 + \cdots$ in all propagators; it is then the physical cutoff scale $\bar{m}^2 + \lambda\phi^2 + \cdots$ rather than k^2 which delimits the range of p^2-modes, which are integrated out.

Typically, for k very large, k^2 is larger than the physical cutoffs so that Γ_k "runs" very fast. Lowering k it might happen that the running of $\bar{m}(k)$ comes to a halt at some $k = k_{\mathrm{dec}}$. For $k < k_{\mathrm{dec}}$ the "artificial" cutoff k becomes smaller than $\bar{m}(k) = \text{const}$. At this point the physical mass starts playing the role of the actual cutoff; its effect overrides that of k so that Γ_k becomes approximately independent of k for $k < k_{\mathrm{dec}}$. As a result, $\Gamma_k \approx \Gamma_{k_{\mathrm{dec}}}$ for all k below the *threshold scale* given by k_{dec}. In particular, the ordinary effective action $\Gamma = \Gamma_0$ does not differ from $\Gamma_{k_{\mathrm{dec}}}$ significantly. This is the prototype of a decoupling or threshold phenomenon.

The situation is even more interesting when $\bar{m}^2$ is negligible and k^2 competes with $\lambda\phi^2$ for the role of the actual cutoff. (Here we assume that ϕ is x-independent.) The running of Γ_k, evaluated at a fixed ϕ, stops once $k \lesssim k_{\mathrm{dec}}(\phi)$. The field-dependent decoupling scale is determined by the implicit equation

[10] It is not difficult to convince oneself that a general cutoff function $\mathcal{R}_k(p^2)$ would lead to identical conclusions and that they are properties of the exact RG flow valid beyond the LPA truncation.

$k_{\text{dec}}^2 = \lambda\big(k_{\text{dec}}\big)\,\phi^2$ and decoupling occurs for sufficiently large values of ϕ. The RG evolution below k_{dec} is negligible then and

$$\Gamma[\phi] = \Gamma_k[\phi]\Big|_{k=k_{\text{dec}}(\phi)}. \tag{2.99}$$

This relationship represents a valuable practical tool. Under favorable circumstances it allows us to go beyond the truncation (2.97) without having to derive and solve a more complicated flow equation. Indeed, because of the additional ϕ-dependence which comes into play via $k_{\text{dec}}(\phi)$, (2.99) can predict certain terms that must be present in Γ, but were not included into the truncation ansatz.

A simple example illustrates this point. For k large, the quartic truncation for the potential gives rise to a logarithmic running of the ϕ^4-coupling: $\lambda(k) \propto \ln(k)$. As a result, (2.99) suggests that Γ should contain a term $\propto \ln\big(k_{\text{dec}}(\phi)\big)\,\phi^4$. Since, in leading order of the implicit equation for the decoupling scale, $k_{\text{dec}} \propto \phi$, this leads us to the *prediction of a $\phi^4\ln(\phi)$-term* in the conventional effective action. This prediction, including the prefactor of the term, is known to be correct actually: the Coleman–Weinberg potential of massless ϕ^4-theory does indeed contain a $\phi^4\ln(\phi)$-term [76, 77]. Note that this term admits no power series expansion in ϕ, so it lies outside the space of functionals spanned by the original ansatz (2.97).

This example nicely illustrates the power of the decoupling arguments. They can be applied even when ϕ is taken x-dependent as it is necessary for computing n-point functions by differentiating $\Gamma_k[\phi]$. The running inverse propagator, say, is given by $\Gamma_k^{(2)}(x-y) = \delta^2\Gamma_k/\delta\phi(x)\delta\phi(y)$. Here a new competitor for the role of the actual cutoff scale enters the game: the momentum q conjugate to the distance $x-y$. When q serves as the acting IR cutoff, the running of $\widetilde{\Gamma}_k^{(2)}(q)$, the Fourier transform of $\Gamma_k^{(2)}(x-y)$, stops once k^2 is smaller than $k_{\text{dec}}^2 = q^2$. Hence $\widetilde{\Gamma}_k^{(2)}(q) \approx \widetilde{\Gamma}_k^{(2)}(q)\Big|_{k=\sqrt{q^2}}$ for $k^2 \lesssim q^2$, provided no other physical scales intervene. As a result, if one has computed the running wave function renormalization Z_k in the truncation (2.97) one is able to predict an inverse propagator of the type $Z\big(\sqrt{q^2}\big)\,q^2$ in the standard effective action. Note that $\phi\,Z\big(\sqrt{-\partial^2}\big)\,\partial^2\phi$ corresponds to a *non-local* term in Γ, even though the truncation ansatz was perfectly local. (See [70] for an example.)

The methodology of expressing k in terms of dynamical quantities, such as physical masses, field values, momenta, etc., is referred to as a *renormalization group improvement*. Based upon the above ideas on decoupling, it can be a powerful tool, a kind of "shortcut" from the UV to the IR, provided one is able to identify the actual physical cutoff mechanism without laboriously solving for the full RG flow. Typically this requires a detailed case-by-case study of the various physical scales displayed by the system at hand. The identification of the physical cutoff is greatly facilitated by symmetry arguments and, in particular, near a scale-invariant fixed point, by dimensional analysis. But even in the generic case it is sometimes possible to get a handle on the essential physics by a careful analysis of the eligible cutoff candidates in $\Gamma_k^{(2)}$. We will come back to this issue when we try to find the quantum gravity corrections to black holes and cosmological spacetimes.

2.4 Background Fields

Later on when we address the difficulties specific to quantum gravity and not shared by matter field theories on a classical spacetime, the concept of background fields [51] will play a pivotal role. While the background technique can be applied to such matter theories as well, in typical calculations of particle or condensed matter physics this is avoided usually. Occasionally it has been used in Yang–Mills theory where, still not compulsory, it offers the convenient option of a gauge-invariant generating functional [86, 87]. In this section we illustrate the approach by means of a scalar "toy system" where it appears to be a triviality almost. Nevertheless, it is helpful to see the background method at work in a setting which is much simpler than quantum gravity [59, 88, 89].

(1) The EAA in the background formalism, $\Gamma_k^{\rm B}$, is based on a variant of the generating functional in (2.4), designated as $W_k^{\rm B}[J;\bar{\phi}]$. The essential modification consists in writing the variable of integration, $\widehat{\phi}$, as the sum of a prescribed, non-dynamical or "background" field $\bar{\phi}$, and a dynamical or "fluctuations" field, $\widehat{\varphi}$:

$$\widehat{\phi}(x) = \bar{\phi}(x) + \widehat{\varphi}(x). \tag{2.100}$$

We emphasize that $\widehat{\varphi}$ is by no means assumed to be numerically small relative to $\bar{\phi}$, and that no expansion in $\widehat{\varphi}$ is implied by (2.100). Indeed, in the background approach $\widehat{\varphi}$ serves as the new integration variable. By virtue of the translation invariance of the measure $\mathcal{D}\widehat{\phi}$ we have $\mathcal{D}\widehat{\phi} = \mathcal{D}\widehat{\varphi}$, and so we define the new, now $\bar{\phi}(\,\cdot\,)$-dependent, generating functional by

$$\begin{aligned} Z_k^{\rm B}[J;\bar{\phi}] \equiv e^{W_k^{\rm B}[J;\bar{\phi}]} = \int \mathcal{D}\widehat{\varphi}\ \exp\Big\{ & -S[\bar{\phi}+\widehat{\varphi}] - \Delta S_k^{\rm B}[\widehat{\varphi};\bar{\phi}] \\ & + \int \mathrm{d}^d x\, \widehat{\varphi}(x) J(x) \Big\}. \end{aligned} \tag{2.101}$$

The new cutoff action possesses the structure

$$\Delta S_k^{\rm B}[\widehat{\varphi};\bar{\phi}] = \frac{1}{2}\int \mathrm{d}^d x\, \widehat{\varphi}(x)\, \mathcal{R}_k^{\rm B}[\bar{\phi}]\, \widehat{\varphi}(x), \tag{2.102}$$

whereby $\mathcal{R}_k^{\rm B}[\bar{\phi}]$ is an operator acting on $\widehat{\varphi}$, which has the same general properties as the ones we employed earlier. The only novelty is that it may be chosen $\bar{\phi}$-dependent now.

As the notation indicates, the generating functionals in the background setting, $Z_k^{\mathrm{B}}[J;\bar{\phi}]$ and $W_k^{\mathrm{B}}[J;\bar{\phi}]$, respectively, will in general depend on our choice for the background field $\bar{\phi}(x)$. In fact, (2.101) differs from its "non-background" analog in two ways that go beyond simply shifting a dummy variable:

(i) The source J couples to the fluctuation $\widehat{\varphi}$ only, i.e., not to the full field $\bar{\phi}+\widehat{\varphi}$. As a consequence, multiple differentiation of W_k^{B} with respect to J gives rise to connected correlation functions $\langle\widehat{\varphi}(x_1)\widehat{\varphi}(x_2)\cdots\widehat{\varphi}(x_n)\rangle$ that apply in the presence of an external field, namely $\bar{\phi}$. The one-point function $\langle\widehat{\varphi}(x)\rangle$, for example, reads in full detail

$$\frac{\delta W_k^{\mathrm{B}}}{\delta J(x)}[J;\bar{\phi}]=\varphi_k[J;\bar{\phi}](x). \tag{2.103}$$

Often we omit the plethora of arguments and write $\langle\widehat{\varphi}(x)\rangle=\varphi(x)$ simply. Furthermore, $\bar{\phi}$ is a classical, i.e., non-fluctuating field, implying $\langle\bar{\phi}\rangle=\bar{\phi}$, and thus the expectation value of the complete field is recovered as

$$\phi(x)\equiv\langle\widehat{\phi}(x)\rangle=\bar{\phi}(x)+\langle\widehat{\varphi}(x)\rangle\equiv\bar{\phi}(x)+\varphi(x). \tag{2.104}$$

The actual motivation for the seemingly ad hoc $\widehat{\varphi}\,J$-coupling in the definition (2.101) would become clear only in more complicated theories containing Yang–Mills fields, for instance. For a gauge field, the decomposition analogous to (2.100) reads $\widehat{A}_\mu(x)=\bar{A}_\mu(x)+\widehat{a}_\mu(x)$ whereby both the full field, $\widehat{A}_\mu$, and the background field, $\bar{A}_\mu$, gauge-transform as a connection, i.e., *inhomogeneously*. It then follows that $\widehat{a}_\mu$ must transform homogeneously, i.e., in a tensorial way. Coupling $\widehat{a}_\mu$ rather than $\widehat{A}_\mu$ to the source is therefore a first crucial step in ultimately arriving at a gauge-covariant formalism.

(ii) The new cutoff action is designed to produce a mass term for the low momentum modes of $\widehat{\varphi}$ rather than $\widehat{\phi}$. In (2.102), ΔS_k^{B} is bilinear in the fluctuation field and depends on $\bar{\phi}$ at most via $\mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]$. In fact, in scalar field theory on a classical spacetime there is no cogent reason that would enforce a $\bar{\phi}$-dependent cutoff operator; a choice of the form $\mathcal{R}_k^{\mathrm{B}}=k^2R^{(0)}(-\partial^2/k^2)$ is also the most natural option here. However, admitting also $\bar{\phi}$-dependent operators brings our scalar play ground closer to quantum gravity where deep general principles will make it impossible to construct cutoff actions that would not depend on the background field.

The EAA of the background approach, $\Gamma_k^{\mathrm{B}}[\varphi;\bar{\phi}]$, is defined in analogy with $\Gamma_k[\phi]$ as the Legendre transform of $W_k^{\mathrm{B}}[J;\bar{\phi}]$ with respect to the source with the cutoff action subtracted. In this way we replace J by the expectation value of the fluctuation, $\varphi=\langle\widehat{\varphi}\rangle$, as the independent variable. In the non-singular case when the field-source relationship (2.103), i.e., $\varphi_k[J;\bar{\phi}](x)=\varphi(x)$ with $\varphi(x)$ given, can be solved for the source as $J(x)=\mathcal{J}_k^{\mathrm{B}}[\varphi;\bar{\phi}](x)$, we have explicitly

$$\boxed{\begin{aligned}\Gamma_k^{\mathrm{B}}[\varphi;\bar{\phi}] \equiv \int \mathrm{d}^d x\, \varphi(x)\, \mathcal{J}_k^{\mathrm{B}}[\varphi;\bar{\phi}](x) - W_k^{\mathrm{B}}\big[\mathcal{J}_k^{\mathrm{B}}[\varphi;\bar{\phi}];\bar{\phi}\big] \\ - \frac{1}{2}\int \mathrm{d}^d x\, \varphi(x)\, \mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]\, \varphi(x).\end{aligned}} \tag{2.105}$$

Obviously the EAA has inherited a parametric $\bar{\phi}$-dependence from W_k^{B}. (Note that the background field is just a spectator in the Legendre transformation.)

(2) The properties of Γ_k^{B} are easily established following the steps outlined in Section 2.1. For example, by differentiating (2.105) with respect to φ we obtain the corresponding effective field equation:

$$\frac{\delta \Gamma_k^{\mathrm{B}}}{\delta \varphi(x)}[\varphi;\bar{\phi}] + \mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]\varphi(x) = J(x). \tag{2.106}$$

This equation expresses the relationship between sources and fields in a form which is conjugate to (2.103). Furthermore, the EAA in the background setting is easily seen to satisfy an FRGE of the form

$$\partial_k \Gamma_k^{\mathrm{B}}[\varphi;\bar{\phi}] = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{\mathrm{B}(2)}[\varphi;\bar{\phi}] + \mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]\right)^{-1} \partial_k \mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]\right]. \tag{2.107}$$

Here $\Gamma_k^{\mathrm{B}(2)}$ stands for the Hessian matrix that comprises all second derivatives with respect to φ at fixed $\bar{\phi}$:

$$\Gamma_k^{\mathrm{B}(2)}[\varphi;\bar{\phi}](x,y) \equiv \frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}\Gamma_k^{\mathrm{B}}[\varphi;\bar{\phi}]. \tag{2.108}$$

(3) Sometimes it is helpful to think of the EAA as depending on the two independent fields $\phi \equiv \bar{\phi} + \varphi$ and $\bar{\phi}$, rather than φ together with $\bar{\phi}$, and to employ the redefined action functional[11]

$$\boxed{\Gamma_k^{\mathrm{B}}[\phi,\bar{\phi}] \equiv \Gamma_k^{\mathrm{B}}[\varphi;\bar{\phi}]\Big|_{\varphi=\phi-\bar{\phi}}.} \tag{2.109}$$

Consider, for instance, the special case when $\Gamma_k^{\mathrm{B}}[\varphi;\bar{\phi}]$ happens to depend on the fields φ and $\bar{\phi}$ only via their sum, the full, undecomposed field $\phi \equiv \langle\widehat{\phi}\rangle \equiv \bar{\phi} + \langle\widehat{\varphi}\rangle \equiv \bar{\phi} + \varphi$. The new functional $\Gamma_k^{\mathrm{B}}[\phi,\bar{\phi}]$ is independent of its second argument then:

$$\Gamma_k^{\mathrm{B}}[\varphi;\bar{\phi}] = F[\bar{\phi}+\varphi] \Longleftrightarrow \Gamma_k^{\mathrm{B}}[\phi,\bar{\phi}] = F[\phi]. \tag{2.110}$$

[11] The two functionals in (2.109) are distinguished by the “comma vs. semicolon” notation that has been customary for the two variants of the background EAA since very early on [23, 24].

Therefore, a non-trivial dependence of $\Gamma_k^{\mathrm{B}}[\phi,\bar{\phi}]$ on $\bar{\phi}$, at fixed ϕ indicates an *extra background field dependence* over and above the one that matches the dependence on φ so that it can be joined with it, resulting in a complete field $\bar{\phi}+\varphi\equiv\phi$.

A given field ϕ can be decomposed into background and fluctuation fields in infinitely many different ways. Different decompositions are related by the *background-quantum field split symmetry transformations*, involving an arbitrary function α:

$$\delta_\alpha\varphi(x)=\alpha(x)\ ,\quad \delta_\alpha\bar{\phi}(x)=-\alpha(x). \tag{2.111}$$

Obviously the complete field ϕ is invariant under such split transformations, $\delta_\alpha\phi=\delta_\alpha\varphi+\delta_\alpha\bar{\phi}=0$, and the same holds true for any functional of ϕ alone. In fact, the response of the EAA to a split symmetry transformations, $\delta_\alpha\Gamma_k^{\mathrm{B}}[\phi,\bar{\phi}]\equiv\Gamma_k^{\mathrm{B}}[\phi,\bar{\phi}-\alpha]-\Gamma_k^{\mathrm{B}}[\phi,\bar{\phi}]$, is a measure for its extra background-field dependence. If we require for instance that $\delta_\alpha\Gamma_k^{\mathrm{B}}=0\ \forall\ \alpha$, it follows that the EAA has no extra $\bar{\phi}$ dependence.

More generally, one can derive a *Ward identity for the split symmetry* by applying (2.111) under the functional integral and working out the consequences for the EAA. In the scalar toy model the result has a similar structure as the flow equation [90]:

$$\boxed{\frac{\delta}{\delta\bar{\phi}(y)}\Gamma_k^{\mathrm{B}}[\phi,\bar{\phi}]=\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{\mathrm{B}(2)}[\phi,\bar{\phi}]+\mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]\right)^{-1}\frac{\delta\mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]}{\delta\bar{\phi}(y)}\right].} \tag{2.112}$$

This kind of split symmetry Ward identity was first derived in [29] for Yang–Mills theory. Its applications include testing the reliability of truncations [90]. As they are derived from the same functional integral, consistency between Ward identity and FRGE is guaranteed for exact RG trajectories, but within truncations it may be violated to some extent.

(4) Despite the differences noted in **(i)**, **(ii)**, the background approach and the one without background fields are strictly equivalent in the scalar case. This equivalence is made manifest by the following relationship between their respective generating functionals:

$$W_k^{\mathrm{B}}[J;\bar{\phi}]=-\int\mathrm{d}^dx\Big(J\bar{\phi}+\frac{1}{2}\bar{\phi}\,\mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]\,\bar{\phi}\Big)+W_k'\big[J+\mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]\,\bar{\phi}\big]. \tag{2.113}$$

Herein, $W_k'[J]$, which appears with a shifted argument in (2.113), represents the usual generating functional of the non-background formalism, albeit with the possibly $\bar{\phi}$-dependent cutoff operator $\mathcal{R}_k^{\mathrm{B}}$ built into it:

$$\begin{aligned} e^{W_k'[J]}\equiv\int\mathcal{D}\widehat{\phi}\ \exp\bigg(&-S[\widehat{\phi}]-\frac{1}{2}\int\mathrm{d}^dx\ \widehat{\phi}\,\mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]\,\widehat{\phi}\\ &+\int\mathrm{d}^dx\,J(x)\widehat{\phi}(x)\bigg)\,. \end{aligned} \tag{2.114}$$

Exactly as outlined in Section 2.1, we associate an EAA, in this context denoted $\Gamma'_k[\phi]$, to the above $W'_k[J]$.

The primes at W'_k, Γ'_k, and at the quantities derived from them are meant to remind us of the "weak" $\bar{\phi}$-dependence those quantities will acquire if we decide to employ a cutoff $\mathcal{R}^{\mathrm{B}}_k$, which depends on the background field. If instead we select a $\bar{\phi}$-independent one (which is an option in the scalar toy model but not in quantum gravity) all primes may be omitted and the corresponding quantities will be exactly those of the non-background approach.

Differentiating both sides of (2.113) with respect to $J(x)$ we are led to

$$\frac{\delta W^{\mathrm{B}}_k}{\delta J(x)}[J;\bar{\phi}] + \bar{\phi}(x) = \frac{\delta W'_k}{\delta J(x)}\big[J + \mathcal{R}^{\mathrm{B}}_k[\bar{\phi}]\,\bar{\phi}\big]. \tag{2.115}$$

The interpretation of this identity becomes clear when we recall (2.103) and the analogous equation for W'_k,

$$\frac{\delta W'_k}{\delta J(x)}[J] = \phi_k[J](x). \tag{2.116}$$

Namely, for J and $\bar{\phi}$ fixed, (2.115) says that the expectation values of $\widehat{\varphi}$ and $\widehat{\phi}$, respectively, are related by

$$\boxed{\varphi_k[J;\bar{\phi}](x) + \bar{\phi}(x) = \phi'_k\big[J + \mathcal{R}^{\mathrm{B}}_k[\bar{\phi}]\,\bar{\phi}\big](x).} \tag{2.117}$$

Obviously this formula is the "effective" analog of the decomposition $\widehat{\phi} = \bar{\phi} + \widehat{\varphi}$. It applies at the level of expectation values, however, rather than to the bare quantities.

We can obtain a conjugate form of the result (2.117) if we fix the expectation values of the fields and ask which source produces them:

$$\mathcal{J}^{\mathrm{B}}_k[\varphi;\bar{\phi}](x) + \mathcal{R}^{\mathrm{B}}_k[\bar{\phi}]\,\bar{\phi}(x) = \mathcal{J}'_k[\bar{\phi} + \varphi](x). \tag{2.118}$$

Here $\mathcal{J}^{\mathrm{B}}_k[\,\cdot\,](\,\cdot\,)$ is the same functional as above, namely the solution to (2.103), but we read (2.118) as an equation for J in which $\varphi_k[J;\bar{\phi}] \equiv \varphi$ is fixed. Analogously the solution to (2.116), with $\phi_k[J] \equiv \phi$ fixed, is denoted $\mathcal{J}'_k[\phi]$. On the RHS of (2.118) the latter functional is evaluated at $\bar{\phi} + \varphi = \phi$.

Equipped with the tool of (2.118) it is easy to express the EAA of the background field approach, Γ^{B}_k, in terms of Γ'_k, the ordinary EAA with a $\bar{\phi}$-dependent cutoff possibly. After the dust has settled we find

$$\boxed{\Gamma^{\mathrm{B}}_k[\varphi;\bar{\phi}] = \Gamma'_k[\varphi + \bar{\phi}].} \tag{2.119}$$

This result looks even simpler if we rewrite it in terms of the alternate version of the EAA, $\Gamma^{\mathrm{B}}_k[\phi,\bar{\phi}] \equiv \Gamma^{\mathrm{B}}_k[\phi - \bar{\phi};\bar{\phi}]$, which regards ϕ and $\bar{\phi}$ as the independent arguments:

$$\boxed{\Gamma^{\mathrm{B}}_k[\phi,\bar{\phi}] = \Gamma'_k[\phi].} \tag{2.120}$$

(5) The interpretation of (2.119) is particularly clear in the limit $k \to 0$ with $\mathcal{R}_k^{\mathrm{B}} \to 0$. Then $\Gamma_0' = \Gamma_0 = \Gamma$ coincides with the standard effective action of the non-background formalism. According to (2.119) it may be written as $\Gamma[\varphi + \bar{\phi}] = \Gamma_0^{\mathrm{B}}[\varphi; \bar{\phi}]$, or after setting $\varphi \equiv 0$:

$$\Gamma[\bar{\phi}] = \Gamma_0^{\mathrm{B}}[0; \bar{\phi}]. \tag{2.121}$$

This simple statement shows that the standard one particle irreducible Green functions given by multiple derivatives of Γ can also be obtained by differentiating Γ_0^{B} *with respect to the background field*, at $\varphi = 0$:

$$\frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} = \frac{\delta^n}{\delta\bar{\phi}(x_1)\cdots\delta\bar{\phi}(x_n)} \Gamma_0^{\mathrm{B}}[\varphi; \bar{\phi}]\bigg|_{\bar{\phi}=\phi, \varphi=0}. \tag{2.122}$$

As a consequence, the entire physical contents of the theory is encapsulated in the background EAA *for a vanishing expectation value of the dynamical field*, i.e., in the restricted map $\bar{\phi} \mapsto \Gamma_0^{\mathrm{B}}[0; \bar{\phi}]$.

This, then, is one of the main advantages of the background field method: it permits a diagrammatic computation of the relevant effective action by summing *vacuum* graphs only, i.e., graphs without external lines of the quantum fields.

(6) Returning to (2.119) in the functional RG context we see that at scales $k > 0$ there is a minor difference between $\Gamma_k^{\mathrm{B}}[0; \bar{\phi}]$ and the non-background EAA, the reason being that Γ_k' employs the possibly $\bar{\phi}$-dependent cutoff operator $\mathcal{R}_k^{\mathrm{B}}[\bar{\phi}]$. This is its only $\bar{\phi}$-dependence though, and since this $\bar{\phi}$-dependence merges into the intrinsic arbitrariness of the cutoff it follows that Γ_k and Γ_k', and hence $\Gamma_k^{\mathrm{B}}\big|_{\varphi=0}$, can differ at most in unphysical features which will cancel out somewhere on the way to observable quantities.

As an aside we mention that in gauge theories there exists another source of such unphysical differences between the analogs of Γ_k' and Γ_k if Γ_k^{B} employs a *background field-dependent gauge-fixing condition.* As Γ_k' inherits this gauge fixing, it acquires a corresponding dependence on the background gauge field which, in this case, merges into the general arbitrariness of the gauge-fixing condition.

As a consequence, the difference cannot have any influence on quantities that are insensitive to the gauge-fixing condition, observables in particular.

A generic n-point function, on the other hand, is gauge-fixing dependent and in this way acquires in particular a certain dependence on the background gauge field. But, for example, in constructing scattering matrix elements from it via the LSZ method, which in particular includes going "on shell", this dependence disappears so that Γ, Γ', and Γ_0^{B} become indeed fully equivalent at the observable level [87, 91].

3

The Asymptotic Safety Mechanism

In this chapter we give an elementary introduction to the general ideas Asymptotic Safety rests upon without embarking on the particular intricacies of gravity. Our presentation emphasizes the geometry of theory space and its natural structures while the discussion of the primarily technical aspects of quantum field theory is kept to a bare minimum.

3.1 Geometry and RG Dynamics on Theory Space

We consider a general set of fields $\widehat{\Phi} = \{\widehat{\Phi}^1, \widehat{\Phi}^2, \cdots\}$ with arbitrary spin and Grassmann parity, along with the corresponding background fields $\bar{\Phi} = \{\bar{\Phi}^1, \bar{\Phi}^2, \cdots\}$ and fluctuations $\widehat{\varphi} \equiv \widehat{\Phi} - \bar{\Phi}$. Their expectation values are collectively denoted $\Phi = \langle \widehat{\Phi} \rangle$ and $\varphi = \langle \widehat{\varphi} \rangle = \Phi - \bar{\Phi}$, respectively. We assume that the effective dynamics of this system of fields is governed by an Effective Average Action (EAA) of the form $\Gamma_k[\varphi; \bar{\Phi}] \equiv \Gamma_k[\Phi, \bar{\Phi}]$.

In the previous chapter we described in detail how to construct the EAA in the special case of scalar fields. Since the discussion in the present chapter will mostly deal with the "kinematic" aspects of the RG evolution on theory space, it is at this point sufficient to know that we can indeed generalize the EAA to arbitrary systems of (matter, gauge, gravitational, etc.) fields while retaining most of the general properties relevant to the scalar case. The specific difficulties posed by gauge and gravitational fields will not yet play an important role, and we postpone their discussion to the next chapter. Suffice it to say that even for general field systems, which include quantized gravity, an appropriately defined action functional Γ_k will ultimately turn out to satisfy a closed functional RG equation again, and the structural appearance of this FRGE is analogous to that of the scalar flow equation from the previous chapter:

$$k\partial_k \, \Gamma_k[\Phi, \bar{\Phi}] = \frac{1}{2} \mathrm{STr} \left[\left(\Gamma_k^{(2)}[\Phi, \bar{\Phi}] + \mathcal{R}_k[\bar{\Phi}] \right)^{-1} k\partial_k \, \mathcal{R}_k[\bar{\Phi}] \right] . \tag{3.1}$$

Here $\Gamma_k^{(2)}$ denotes the Hessian matrix with respect to all dynamical fields at fixed background fields, $\delta^2\Gamma_k/\delta\Phi^i(x)\delta\Phi^j(y)$, and the *supertrace* STr includes a summation over the field components and takes care of extra minus signs occurring for Grassmann fields.

We assume that, besides the nature of the field multiplets, all additional requirements to be satisfied by the EAA (symmetries, etc.) have been specified in some way. This concerns, in particular, the function spaces from which Φ and $\bar{\Phi}$ are drawn, henceforth denoted $\mathcal{F}$ and $\bar{\mathcal{F}}$, respectively. These data define the (dimensionful) theory space $\mathcal{T}$. Its "points" $A \in \mathcal{T}$ are action functionals $A : \mathcal{F} \times \bar{\mathcal{F}} \to \mathbb{R}$, $(\Phi, \bar{\Phi}) \mapsto A[\Phi, \bar{\Phi}]$ with certain prescribed (symmetry, regularity, etc.) properties.

3.1.1 The Beta Functional

When written in the pointwise style of (3.1) the FRGE expresses an equality between two real numbers which must hold true for all pairs of fields $(\Phi, \bar{\Phi}) \in \mathcal{F} \times \bar{\mathcal{F}}$. We may equivalently regard it as an equality of functionals and write it compactly as

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]. \tag{3.2}$$

As we shall see, the most geometric way to think of the FRGE is as follows. For every fixed value of k we consider the map

$$\begin{aligned} \overline{\mathcal{B}}_k : \mathcal{T} \times \mathcal{F} \times \bar{\mathcal{F}} &\longrightarrow \mathbb{R}, \\ (A, \Phi, \bar{\Phi}) &\longmapsto \overline{\mathcal{B}}_k\{A\}[\Phi, \bar{\Phi}], \end{aligned} \tag{3.3}$$

which returns a real number for every given action and every pair of field arguments, and is defined explicitly by

$$\overline{\mathcal{B}}_k\{A\}[\Phi, \bar{\Phi}] \equiv \frac{1}{2}\mathrm{STr}\left[\left(A^{(2)}[\Phi, \bar{\Phi}] + \mathcal{R}_k[\bar{\Phi}]\right)^{-1} k\partial_k \mathcal{R}_k[\bar{\Phi}]\right]. \tag{3.4}$$

The number $\overline{\mathcal{B}}_k\{A\}[\Phi, \bar{\Phi}]$ should be seen as a certain action functional, denoted $\overline{\mathcal{B}}_k\{A\}$, which is being evaluated at the point $(\Phi, \bar{\Phi})$.

Thus $\overline{\mathcal{B}}_k : A \mapsto \overline{\mathcal{B}}_k\{A\}$ constitutes a map of actions onto actions. For lack of a better word we refer to $\overline{\mathcal{B}}_k$ as the *beta functional.*

The usefulness of the definitions (3.3) and (3.4) is obvious, of course. It casts the FRGE into the form

$$k\partial_k \Gamma_k[\Phi, \bar{\Phi}] = \overline{\mathcal{B}}_k\{A\}[\Phi, \bar{\Phi}] \quad \forall\ (\Phi, \bar{\Phi}), \tag{3.5}$$

which is equivalent to the functional relationship

$$\boxed{k\partial_k \Gamma_k = \overline{\mathcal{B}}_k\{\Gamma_k\}}. \tag{3.6}$$

In particular, for structural considerations the equation (3.6), and the interpretation that comes with it, is the most convenient way to express the FRGE.

3.1.2 From the Beta Functional to the Beta Functions

Let us assume that $\mathcal{T}$ has the structure of a vector space, and let $\{I_\alpha\}$ denote a complete set of (k-independent!) basis functionals $I_\alpha : (\Phi, \bar{\Phi}) \mapsto I_\alpha[\Phi, \bar{\Phi}]$. Every $A \in \mathcal{T}$ has an expansion $A = \sum_\alpha \bar{u}^\alpha I_\alpha$ with real components $\bar{u}^\alpha$ then.

In the case of the EAA, the components, or *generalized coupling constants*, as they are called here, carry the entire scale dependence:

$$\Gamma_k = \sum_\alpha \bar{u}^\alpha(k)\, I_\alpha. \tag{3.7}$$

To derive RG equations for the k-dependence of the running coupling constants we insert (3.7) into the functional equation (3.6):

$$\sum_\alpha k\partial_k\, \bar{u}^\alpha(k)\, I_\alpha = \overline{\mathcal{B}}_k\Big\{\sum_\alpha \bar{u}^\alpha(k)\, I_\alpha\Big\}. \tag{3.8}$$

As the map $\overline{\mathcal{B}}_k\{\,\cdot\,\}$ returns an action functional, and since the I_αs form a basis of theory space, we know that the RHS of (3.8) possesses a unique expansion in this basis:

$$\overline{\mathcal{B}}_k\Big\{\sum_\alpha \bar{u}^\alpha(k)\, I_\alpha\Big\} = \sum_\alpha \bar{b}^\alpha\big(\bar{u}(k); k\big)\, I_\alpha. \tag{3.9}$$

Here $\bar{u}(k) \equiv \{\bar{u}^\alpha(k)\}$ denotes the list of all running coupling constants. Upon using (3.9) in (3.8) and equating coefficients of equal basis elements we are led to the *exact RG equation in component form*:

$$\boxed{k\partial_k\, \bar{u}^\alpha(k) = \bar{b}^\alpha\big(\bar{u}(k); k\big) \quad \forall\ \alpha.} \tag{3.10}$$

Note that besides an implicit k-dependence via $\bar{u}(k)$, the functions $\bar{b}^\alpha$ explicitly depend on k since the map $\overline{\mathcal{B}}_k$ does.

We note that in the present context one might regard $\Gamma_k[\,\cdot\,]$ and $k\partial_k\,\Gamma_k[\,\cdot\,]$ simply as generating functionals for two infinite families of functions, namely their components $\{\bar{u}^\alpha\}$ and $\{\bar{b}^\alpha\}$, respectively. This viewpoint deemphasizes, or at least shifts, the role of the fields; they are needed merely in order to describe the mathematical nature of the basis elements I_α and the space they span.

3.1.3 The Cutoff Scale as a Unit

A generic running coupling constant $\bar{u}^\alpha$ will have a nonzero canonical mass dimension $[\bar{u}^\alpha] \equiv d_\alpha$. Since $\Gamma_k[\Phi, \bar{\Phi}]$ is dimensionless so are all terms $\bar{u}^\alpha\, I_\alpha[\Phi, \bar{\Phi}]$

in its component representation and thus $\big[I_\alpha[\Phi,\bar{\Phi}]\big] = -[\bar{u}^\alpha] = -d_\alpha$. Furthermore, (3.10) shows that $\bar{b}^\alpha$ must have the same dimension as the coupling it belongs to: $[\bar{b}^\alpha] = d_\alpha$.

Let us now switch to a formalism in which all dimensionful quantities, couplings, fields, beta functions, etc., are expressed in units of the cutoff scale, k, or appropriate powers thereof.

(1) First, we introduce dimensionless coupling constants:

$$u^\alpha \equiv k^{-d_\alpha}\, \bar{u}^\alpha. \tag{3.11}$$

Usually u^α is considered a function of the RG time $t \equiv \ln(\frac{k}{k_0})$, which is also dimensionless; it involves an arbitrary, fixed reference scale k_0. Thus, inserting $\bar{u}^\alpha(k) = k^{d_\alpha} u^\alpha(t)$ into (3.10), we arrive at the *exact RG equation in dimensionless component form*:

$$\boxed{\partial_t u^\alpha(t) = \beta^\alpha\big(u(t)\big).} \tag{3.12}$$

It involves the *dimensionless β-functions* given by

$$\boxed{\beta^\alpha(u) = -d_\alpha u^\alpha + b^\alpha(u).} \tag{3.13}$$

The b^αs without the overbar are defined as

$$b^\alpha(u) = k^{-d_\alpha}\, \bar{b}^\alpha\big(\{k^{d_\alpha} u^\alpha\}; k\big). \tag{3.14}$$

Here all explicit factors of k on the RHS must cancel, the reason being that this term is dimensionless, k is the only dimensionful quantity available and so the term is of zeroth order in k.

The new β-functions depend only on $u \equiv \{u^\alpha\}$ and when $u \equiv u(t)$ is inserted on the RHS of (3.12) they of course acquire an implicit scale dependence. This shows that, in contrast to the original dimensionful couplings, the dimensionless ones satisfy a system of *autonomous* differential equations.

(2) In general, the various fields in the sets $\Phi = \{\Phi^i\}$, $\varphi = \{\varphi^i\}$, and $\bar{\Phi} = \{\bar{\Phi}^i\}$ come with different canonical dimensions $\delta_i \equiv [\Phi^i] = [\varphi^i] = [\bar{\Phi}^i]$. We define dimensionless fields in the obvious way:

$$\widetilde{\Phi}^i \equiv k^{-\delta_i}\, \Phi^i\,, \quad \widetilde{\varphi}^i \equiv k^{-\delta_i}\, \varphi^i\,, \quad \widetilde{\bar{\Phi}}^i \equiv k^{-\delta_i}\, \bar{\Phi}^i\,. \tag{3.15}$$

For the sets $\widetilde{\Phi} \equiv \{\widetilde{\Phi}^i\}$, etc., we use the convenient shorthand notations $\widetilde{\Phi} = k^{-[\Phi]}\Phi$, $\widetilde{\varphi} = k^{-[\varphi]}\varphi$, and $\widetilde{\bar{\Phi}} = k^{-[\bar{\Phi}]}\bar{\Phi}$, where $[\Phi] = \{[\Phi^i]\}$ and, by definition of the linear split, $[\Phi] = [\varphi] = [\bar{\Phi}]$.

We already know that the basis functionals $I_\alpha[\Phi,\bar{\Phi}]$ consist of monomials built from the dimensionful fields in such a way that their dimensions sum up to the overall value $\big[I_\alpha[\Phi,\bar\Phi]\big]=-d_\alpha$. Since the fields are the only[1] dimensionful quantities available out of which the I_αs can be constructed, dimensional analysis implies that

$$I_\alpha\big[c^{[\Phi]}\Phi, c^{[\bar\Phi]}\bar\Phi\big]=c^{-d_\alpha}I_\alpha[\Phi,\bar\Phi] \tag{3.16}$$

for any constant $c>0$. Evaluating this relation at $c=k^{-1}$ we find

$$\boxed{I_\alpha[\widetilde\Phi,\widetilde{\bar\Phi}]\equiv I_\alpha\big[k^{-[\Phi]}\Phi, k^{-[\bar\Phi]}\bar\Phi\big]=k^{d_\alpha}I_\alpha[\Phi,\bar\Phi].} \tag{3.17}$$

The quantity (3.17), in any of the three versions, is dimensionless. Thus we learn that when we insert dimensionless field arguments into the basis functionals in place of the original Φ and $\bar\Phi$, the I_αs lose their nonzero mass dimension: $\big[I_\alpha[\widetilde\Phi,\widetilde{\bar\Phi}]\big]=0$.

For later use we also remark that by acting with $c\,\partial_c$ on (3.16) and setting $c=1$ thereafter we obtain the useful identity

$$X I_\alpha[\widetilde\Phi,\widetilde{\bar\Phi}]=-d_\alpha\, I_\alpha[\widetilde\Phi,\widetilde{\bar\Phi}] \tag{3.18}$$

with the operator

$$X=\sum_i \delta_i X_i\,,\qquad X_i\equiv\int \mathrm{d}^d x\,\left\{\widetilde\Phi^i(x)\frac{\delta}{\delta\widetilde\Phi^i(x)}+\widetilde{\bar\Phi}^i(x)\frac{\delta}{\delta\widetilde{\bar\Phi}^i(x)}\right\}. \tag{3.19}$$

When applied to a given field monomial the operator X_i counts the number of "i"-type fields contained in it. Note that while equation (3.18) is written entirely in terms of dimensionless quantities, the eigenvalue $(-d_\alpha)$ on its RHS equals exactly the would-be canonical dimension of I_α, i.e., the one it assumes when dimensionful fields are plugged into it.

(3) If parametrized by the dimensionless couplings, the EAA reads

$$\Gamma_k[\Phi,\bar\Phi]=\sum_\alpha u^\alpha(k)\,k^{d_\alpha}\,I_\alpha[\Phi,\bar\Phi] \tag{3.20}$$

or, after making use of (3.17),

$$\begin{aligned}\Gamma_k[\Phi,\bar\Phi]&=\sum_\alpha u^\alpha(k)\,I_\alpha\big[k^{-[\Phi]}\Phi,k^{-[\bar\Phi]}\bar\Phi\big]\\&=\sum_\alpha u^\alpha(k)\,I_\alpha[\widetilde\Phi,\widetilde{\bar\Phi}].\end{aligned} \tag{3.21}$$

[1] Note that, by definition, I_α does not contain any coupling constants. Furthermore, according to the assignment of dimensions we use in this section all *coordinates* x^μ *are dimensionless.* This comes at the price of considering the metric, even when non-dynamical, as a field. Carrying a nonzero dimension, the transition to dimensionless fields, (3.15), entails a nontrivial rescaling, $\widetilde{\bar g}_{\mu\nu}=k^2\bar g_{\mu\nu}$.

Let us employ the set of dimensionless running couplings $\{u^\alpha(k)\}$ appearing in (3.20) in order to define a new action functional $\mathcal{A}_k : (\widetilde{\Phi}, \widetilde{\bar{\Phi}}) \mapsto \mathcal{A}_k[\widetilde{\Phi}, \widetilde{\bar{\Phi}}]$ according to

$$\boxed{\mathcal{A}_k[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] \equiv \sum_\alpha u^\alpha(k)\, I_\alpha[\widetilde{\Phi}, \widetilde{\bar{\Phi}}].} \tag{3.22}$$

Here $\widetilde{\Phi}$ and $\widetilde{\bar{\Phi}}$ should be thought of as dimensionless fields that can be chosen freely. They are the independent arguments of $\mathcal{A}_k$ and thus play the same role for $\mathcal{A}_k$ as the dimensionful ones for Γ_k.

If some pair of dimensionless arguments $(\widetilde{\Phi}, \widetilde{\bar{\Phi}})$ is fixed first, and then a dimensionful pair $(\Phi, \bar{\Phi})$ is associated to it by virtue of (3.15), equation (3.21) amounts to

$$\boxed{\mathcal{A}_k[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] = \Gamma_k[\Phi, \bar{\Phi}].} \tag{3.23}$$

Despite the numerical equality expressed by this relation, $\mathcal{A}_k$ *and* Γ_k *are different functionals.* The independent arguments of the former (latter) are the dimensionless (dimensionful) fields.

If corresponding sets of dimensionless and dimensionful fields are related by the transformation (3.15), then $\mathcal{A}_k$ attributes the k-dependence of this transformation to $(\Phi, \bar{\Phi})$, while from the perspective of Γ_k it is the dimensionless pair $(\widetilde{\Phi}, \widetilde{\bar{\Phi}})$ that carries the scale dependence. As a consequence, when we take the k-derivatives of $\mathcal{A}_k$ and Γ_k, keeping their respective field arguments fixed, we obtain different results:

$$k\partial_k\, \mathcal{A}_k[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] \;=\; \sum_\alpha k\partial_k\, u^\alpha(k)\, I_\alpha[\widetilde{\Phi}, \widetilde{\bar{\Phi}}], \tag{3.24}$$

$$k\partial_k\, \Gamma_k[\Phi, \bar{\Phi}] \;=\; \sum_\alpha \Big\{ k\partial_k\, u^\alpha(k) + d_\alpha u^\alpha(k) \Big\}\, k^{d_\alpha}\, I_\alpha[\Phi, \bar{\Phi}]. \tag{3.25}$$

The extra term proportional to $d_\alpha u^\alpha(k)$ in (3.25) stems from the derivative of the factor k^{d_α} in (3.20).

(4) If we insert the explicit β-functions (3.13) and (3.14) into (3.25) we are led back to

$$k\partial_k\, \Gamma_k[\Phi, \bar{\Phi}] = \sum_\alpha \bar{b}^\alpha\big(\bar{u}(k); k\big)\, I_\alpha[\Phi, \bar{\Phi}]. \tag{3.26}$$

This equation appeared above where it defined the expansion coefficients $\bar{b}_\alpha$. The analogous equation at the dimensionless level follows from (3.24) by inserting (3.12):

$$k\partial_k\, \mathcal{A}_k[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] = \sum_\alpha \beta^\alpha\big(u(k)\big)\, I_\alpha[\widetilde{\Phi}, \widetilde{\bar{\Phi}}]. \tag{3.27}$$

Obviously the scale-derivatives of Γ_k and $\mathcal{A}_k$ may be interpreted as generating functionals for the functions $\{\bar{b}^\alpha\}$ and $\{\beta^\alpha\}$, respectively.

Finally we set up a truly functional RG equation also in the dimensionless setting; it evolves actions *per se* rather than their basis-dependent components. It is defined on the theory space proper, namely a space of functionals

$$\mathcal{A} : \mathcal{F} \times \bar{\mathcal{F}} \longrightarrow \mathbb{R} , \quad (\widetilde{\Phi}, \widetilde{\bar{\Phi}}) \longmapsto \mathcal{A}[\widetilde{\Phi}, \widetilde{\bar{\Phi}}], \tag{3.28}$$

which, abstractly, is not different from the "dimensionful" theory space considered earlier, just the assignment of dimensions has changed.

Henceforth we refer to this reinterpreted space of actions depending on dimensionless fields as "the" *theory space*. Keeping in mind that the dimensionless theory space is isomorphic to the dimensionful one introduced at the beginning of this section, we denote it by $\mathcal{T}$ as well.

Inserting (3.13) with (3.14) into (3.27) we obtain

$$k\partial_k \mathcal{A}_k = -\sum_\alpha u^\alpha(k)\, d_\alpha\, I_\alpha + \sum_\alpha \bar{b}^\alpha\big(\bar{u}(k); k\big)\, k^{-d_\alpha}\, I_\alpha . \tag{3.29}$$

Taking advantage of the identity (3.18) the first term on the RHS of this equation is seen to be equal to $X \sum_\alpha u^\alpha(k) I_\alpha = X\mathcal{A}_k$, while the second term equals $\overline{\mathcal{B}}_k$ up to various rescalings. The latter are written most conveniently with the help of the simple map

$$\sigma_k : (\Phi, \bar{\Phi}) \mapsto \big(k^{-[\Phi]}\Phi, k^{-[\bar{\Phi}]}\bar{\Phi}\big), \tag{3.30}$$

which turns dimensionful into dimensionless fields.

In this manner we obtain the final result for the *functional RG equation governing the EAA of the dimensionless fields*, namely

$$\boxed{\partial_t \mathcal{A}_k \equiv \boldsymbol{\beta}\{\mathcal{A}_k\}} \tag{3.31}$$

with the *beta functional of the dimensionless approach*:

$$\boxed{\boldsymbol{\beta}\{\mathcal{A}\} = X\mathcal{A} + \mathcal{B}\{\mathcal{A}\}.} \tag{3.32}$$

Here, X is the counting operator (3.19) which depends on the field dimensions δ_i, and $\mathcal{B}$ denotes the dimensionless counterpart of $\overline{\mathcal{B}}_k$,

$$\mathcal{B}\{\mathcal{A}\}[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] \equiv \overline{\mathcal{B}}_k\{\mathcal{A} \circ \sigma_k\}\big[\sigma_k^{-1}(\widetilde{\Phi}, \widetilde{\bar{\Phi}})\big]. \tag{3.33}$$

The new map $\mathcal{B}$ has no explicit scale dependence.[2] As a result, the beta functional (3.32) is also k-independent, so that (3.31) is indeed an autonomous, basis-independent evolution equation. After fixing a basis it would lead back to the component equations (3.12), (3.13), of course.

[2] Cf. the reasoning in **(1)** above. Note also that we returned to the RG time as a manifestly dimensionless evolution parameter ($t = \ln k + \text{const}$, $k\partial_k = \partial_t$). For simplicity we keep using the notation $\mathcal{A}_k$ though.

3.1.4 The Geometric Interpretation

The evolution of the Effective Average Action $\mathcal{A}_k$ resulting from a change of the RG scale has an appealing geometric interpretation as a dynamical system on an infinite dimensional space.

(A) Finite dimensional flows. To make this interpretation explicit, let us first consider an arbitrary smooth, n-dimensional manifold, $\mathcal{M}_n$. We furnish it with a system of local coordinates, $\xi \equiv \left(\xi^i\right), i=1,\cdots,n$, giving rise to a corresponding basis $\partial_i \equiv \frac{\partial}{\partial \xi^i}$ in the tangent spaces $T_\xi \mathcal{M}_n$, and a dual basis $\mathrm{d}\xi^i$ in $T^*_\xi \mathcal{M}_n$. Furthermore, we assume that in addition to the manifold structure we are given a vector field V on $\mathcal{M}_n$, locally represented as $V(\xi)=V^i(\xi)\partial_i$.

The pair $(\mathcal{M}_n, V)$, which consists of a manifold and a vector field, is referred to as a *flow*. This setup plays a central role in the general theory of dynamical systems, e.g., [92–96].

A well-known special case is Hamilton's version of classical mechanics, where the (now even dimensional) manifold $\mathcal{M}_n$ plays the role of the system's *phase space*. Hamiltonian dynamical systems are special in the sense that the admissible vector fields V are "symplectic gradients" $V^i = J^{ij}\partial_j H$. Here the Hamilton function H is an arbitrary scalar on $\mathcal{M}_n$ and the Poisson tensor J^{ij} is an independent ingredient that defines the mechanical system along with H and the manifold structure. In local Darboux coordinates $\xi^i=(q,p)$ and for $n=2$ its components are $J^{12}=-J^{21}=1,\ J^{11}=J^{22}=0$. The vector field $V\equiv V_H$ associated with H then reads

$$V_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \tag{3.34}$$

and similarly for more than one degree of freedom.

The name "flow" for the pair $(\mathcal{M}_n, V)$ has then been coined since those data are precisely what is needed in order to set up the "flow equation"

$$\boxed{\dot{\xi}(\lambda) \equiv \partial_\lambda \xi(\lambda) = V(\xi(\lambda)).} \tag{3.35}$$

This equation determines the integral curves $\lambda \mapsto \xi(\lambda)$ the vector field V gives rise to. By definition, their "velocity" $\dot{\xi}(\lambda) \in T_{\xi(\lambda)}\mathcal{M}_n$ is tangent to V at all points visited by the curve.

Denoting the solution to (3.35) subject to the initial condition $\xi(0)=\xi_0$ by $\lambda \mapsto \xi(\lambda;\xi_0)$, we may think of $\xi(\lambda;\xi_0)$ as the location of a fictitious particle at "time" λ moving on $\mathcal{M}_n$ and released from ξ_0 at the time $\lambda=0$. Conversely we can fix the final time, setting $\lambda=1$, say, and ask how the position of the particle at this time depends on its initial position at $\lambda=0$. Allowing ξ_0 to be any point of the manifold, answering this question amounts to constructing the map $\mathcal{M}_n \to \mathcal{M}_n$, $\xi_0 \mapsto \xi(1;\xi_0)$ by solving the differential equation subject to all possible initial conditions. Under appropriate conditions this map is a diffeomorphism. It may be interpreted (actively) as describing the motion of a swarm comprising many particles released from different initial points, or (passively) as a general coordinate transformation on $\mathcal{M}_n$.

In Hamiltonian mechanics the flow equation (3.35) refers to the physical time $\lambda \equiv t$ and uses a symplectic vector field of the form (3.34). At the level of its components it is nothing else than Hamilton's equations of motion, $\dot{q} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial q$. In this particular case the above diffeomorphism is *volume preserving* (by Liouville's theorem). A generic flow $(\mathcal{M}_n, V)$ will *not* have this property.

(B) The RG flow. Returning now to the Effective Average Action of the dimensionless setting, $\mathcal{A}_k \equiv \mathcal{A}(t)$, we see that the RG equation which it satisfies,

$$\partial_t \mathcal{A}(t) = \boldsymbol{\beta}\{\mathcal{A}(t)\}\,, \tag{3.36}$$

has the same structure as the flow equation (3.35), at least formally. The role of the evolution parameter λ is played by the RG time, the "manifold" on which the dynamics take place is the theory space $\mathcal{T}$ of "all" actions $\mathcal{A} : \mathcal{F} \times \bar{\mathcal{F}} \to \mathbb{R}$, and $\boldsymbol{\beta}\{\mathcal{A}\} \equiv X\mathcal{A} + \mathcal{B}\{\mathcal{A}\}$ with $\mathcal{B}$ defined in (3.33) takes the place of the vector field V.

Invoking this formal correspondence we would interpret the map $\boldsymbol{\beta}$ when evaluated at some point $\mathcal{A} \in \mathcal{T}$ as a vector in the "tangent space to $\mathcal{T}$ at $\mathcal{A}$," and instead of $V(\xi) \in T_\xi \mathcal{M}_n$ we would write $\boldsymbol{\beta}\{\mathcal{A}\} \in T_{\mathcal{A}}\mathcal{T}$. At the level of this correspondence the customary term *renormalization group flow* then indeed refers to a flow in the above sense:

$$\boxed{\text{RG flow of the EAA} : (\mathcal{T}, \boldsymbol{\beta}).} \tag{3.37}$$

It should be clear that the "RG flow of the EAA" goes significantly beyond the mathematically well studied finite dimensional case. Theory space is infinite dimensional and it is notoriously difficult to make the notion of a "space of all actions" mathematically precise. Therefore, we cannot expect that the results established for finite-dimensional systems $(\mathcal{M}_n, V)$ carry over straightforwardly to the RG case, at least not completely.

The drastic nature of the step from $(\mathcal{M}_n, V)$ to $(\mathcal{T}, \boldsymbol{\beta})$ is perhaps appreciated best by recalling that $\partial_t \mathcal{A}(t) = \boldsymbol{\beta}\{\mathcal{A}(t)\}$ expresses the equality of two functionals and is hence equivalent to $\partial_t \mathcal{A}(t)[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] = \boldsymbol{\beta}\{\mathcal{A}(t)\}[\widetilde{\Phi}, \widetilde{\bar{\Phi}}]$ for all field configurations $(\widetilde{\Phi}, \widetilde{\bar{\Phi}}) \in \mathcal{F} \times \bar{\mathcal{F}}$. Introducing the "index notation" $\partial_t \mathcal{A}^{(\widetilde{\Phi}, \widetilde{\bar{\Phi}})}(t) \equiv \partial_t \mathcal{A}(t)[\widetilde{\Phi}, \widetilde{\bar{\Phi}}]$, and similarly for $\boldsymbol{\beta}$, the equation assumes the somewhat arcane-looking form

$$\partial_t \mathcal{A}^{(\widetilde{\Phi}, \widetilde{\bar{\Phi}})}(t) = \beta^{(\widetilde{\Phi}, \widetilde{\bar{\Phi}})}\{\mathcal{A}(t)\} \quad \forall\ (\widetilde{\Phi}, \widetilde{\bar{\Phi}}) \in \mathcal{F} \times \bar{\mathcal{F}}. \tag{3.38}$$

We would like to interpret this equation as the component form of (3.36). The analogy to the finite-dimensional equation

$$\partial_\lambda \xi^i(\lambda) = V^i(\xi(\lambda)) \quad \forall\ i \in \{1, 2, \cdots, n\}, \tag{3.39}$$

then suggests that there is a correspondence between the vector index i on the one side, assuming finitely many discrete values only, and full-fledged field configurations on the other side:

$$i \longleftrightarrow (\widetilde{\Phi}, \widetilde{\bar{\Phi}}). \tag{3.40}$$

Thus, loosely speaking, the tangent space $T_{\mathcal{A}}\mathcal{T}$ would have as many dimensions as there are fields in $\mathcal{F} \times \bar{\mathcal{F}}$.

It is certainly not easy to give a mathematically precise meaning to such statements and to find an appropriate infinite-dimensional generalization of the classical geometrical concepts in full generality. It also seems clear that this will not be possible without imposing additional technical conditions on the various ingredients of the setup. Those conditions should be strong enough to allow for proving general mathematical theorems, but at the same time not so strong as to rule out some or, possibly, all applications of physical interest.

Here and in the following we make first of all the technical assumption that $\mathcal{T}$ has at least the structure of a *vector space*. This setting will be sufficient for all our general discussions and explicit examples. We invoked this requirement already earlier on when we assumed that every $\mathcal{A} \in \mathcal{T}$ can be expanded in a set of basis functionals, $\mathcal{A} = \sum_\alpha u^\alpha I_\alpha$. Moreover, we already discussed a nontrivial example where we saw that this is indeed possible, namely the framework of the derivative expansion.

Under this assumption we can identify all tangent spaces $T_{\mathcal{A}}\mathcal{T}$ with $\mathcal{T}$ itself and reinterpret any point $\mathcal{A} = \sum_\alpha u^\alpha I_\alpha \in \mathcal{T}$ as a vector, which we view as the "position vector" of the point $\mathcal{A}$, having components $u \equiv (u^\alpha)$, with respect to the basis chosen ("I-basis").

Let us recast the expansion $\mathcal{A}[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] = \sum_\alpha u^\alpha I_\alpha[\widetilde{\Phi}, \widetilde{\bar{\Phi}}]$ in the "vector index" language used in (3.38). Setting $I_\alpha^{(\widetilde{\Phi}, \widetilde{\bar{\Phi}})} \equiv I_\alpha[\widetilde{\Phi}, \widetilde{\bar{\Phi}}]$ this leads us to interpret

$$\mathcal{A}^{(\widetilde{\Phi}, \widetilde{\bar{\Phi}})} = \sum_\alpha u^\alpha \, I_\alpha^{(\widetilde{\Phi}, \widetilde{\bar{\Phi}})} \tag{3.41}$$

as the components of the position vector in the "field basis." Seen from a formal perspective, $\big(I_\alpha^{(\widetilde{\Phi}, \widetilde{\bar{\Phi}})}\big)$ is an infinite-dimensional matrix whose rows and columns are labeled by "indices" $(\widetilde{\Phi}, \widetilde{\bar{\Phi}})$ and α, respectively, and which describes a basis change from the I-basis to the field basis. The completeness of the functionals I_α assumed, this matrix is invertible and so we can freely switch back and forth between the two bases, trading the $(\widetilde{\Phi}, \widetilde{\bar{\Phi}})$-index for α and vice versa.

In particular, we can now regard $i \leftrightarrow (\widetilde{\Phi}, \widetilde{\bar{\Phi}})$ as a correspondence between the vector indices i and α, and we may apply the basis change (3.41) to both sides of the "arcane" equation (3.38). In this way we end up with the expected result,

$$\partial_t u^\alpha(t) = \beta^\alpha(u(t)) \quad \forall \ \alpha, \tag{3.42}$$

where $\partial_t u^\alpha$ and $\beta^\alpha(u(t))$ are the components in the I-basis of $\partial_t \mathcal{A}$ and $\boldsymbol{\beta}\{\mathcal{A}(t)\}$, respectively, and $u^\alpha(t)$ are the coordinates of the point $\mathcal{A}(t)$, or the components of its position vector.

The components of $\boldsymbol{\beta}$ in (3.38) and (3.42), respectively, are related by a matrix multiplication with the transformation matrix $\big(I_\alpha^{(\widetilde{\Phi},\widetilde{\bar{\Phi}})}\big)$:

$$\beta^{(\widetilde{\Phi},\widetilde{\bar{\Phi}})}\{\mathcal{A}\} \equiv \boldsymbol{\beta}\{\mathcal{A}\}[\widetilde{\Phi},\widetilde{\bar{\Phi}}] = \sum_\alpha \beta^\alpha(u) I_\alpha[\widetilde{\Phi},\widetilde{\bar{\Phi}}] \equiv \sum_\alpha \beta^\alpha I_\alpha^{(\widetilde{\Phi},\widetilde{\bar{\Phi}})}. \tag{3.43}$$

The first and the last equalities in (3.43) merely amount to a trivial change of notation, while the second one represents the expansion of a certain functional in a basis, namely $\boldsymbol{\beta}\{\mathcal{A}\}[\,\cdot\,]$, yielding $\sum_\alpha \beta^\alpha(u) I_\alpha[\,\cdot\,]$. The point $\mathcal{A}$, having coordinates u, is held fixed thereby.

Of course (3.42) is not new, but precisely the FRGE in component form (3.12) from which we started. However, we went through this rederivation since it is often quite illuminating to think of the flow equation in terms of these differential geometric concepts and the formal calculus they entail.

For example, at first sight the set of all beta functions $\{u \mapsto \beta^\alpha(u)\}$ appears to be a mathematical object of a quite different nature than the supertrace functional $\mathrm{STr}[\cdots]$ on the RHS of the FRGE. However, in the geometric interpretation the latter is essentially the same as $\beta^{(\widetilde{\Phi},\widetilde{\bar{\Phi}})}\{\mathcal{A}_k\}$, which, as (3.43) shows, is related to $\beta^\alpha(u(k))$ simply by a change of basis, though admittedly a colossal one.

(C) General coordinate transformations on $\mathcal{T}$. The functional RG equation $\partial_t \mathcal{A}(t) = \boldsymbol{\beta}\{\mathcal{A}(t)\}$ enjoys a manifestly coordinate independent status. While its component form $\partial_t u^\alpha(t) = \beta^\alpha(u(t))$ depends of course on the system of coordinates chosen, it is *form invariant* under t-independent general coordinate transformations on theory space, $u \mapsto u'(u)$. Since under this transformation, $\partial_t u'^\alpha = \left(\frac{\partial u'^\alpha}{\partial u^\gamma}\right) \partial_t u^\gamma$, the transformed RG equation is seen to be of the same form as the old one, $\partial_t u'^\alpha(t) = \beta'^\alpha(u'(t))$, provided the beta functions transform according to

$$\boxed{\beta'^\alpha(u') = \frac{\partial u'^\alpha}{\partial u^\gamma}\,\beta^\gamma(u),} \tag{3.44}$$

which is exactly the transformation law satisfied by the components of a vector field.

Note that theory space carries no natural metric or connection, and therefore there is no intrinsically given covariant derivative with which we could act on $\boldsymbol{\beta}$. The partial derivatives of the components β^α,

$$B^\alpha{}_\gamma(u) \equiv \frac{\partial}{\partial u^\gamma} \beta^\alpha(u)\,, \tag{3.45}$$

transform at a generic point u of $\mathcal{T}$ like

$$B'^{\alpha}{}_{\gamma}(u') = \frac{\partial u'^{\alpha}}{\partial u^{\kappa}} B^{\kappa}{}_{\delta}(u) \frac{\partial u^{\delta}}{\partial u'^{\gamma}} + \beta^{\kappa}(u) \frac{\partial u^{\delta}}{\partial u'^{\gamma}} \frac{\partial^2 u'^{\alpha}}{\partial u^{\delta} \partial u^{\kappa}} \,. \tag{3.46}$$

This transformation law is non-tensorial unless the second term on the RHS of (3.46) vanishes for some reason. There is an important situation where this is indeed the case, namely when $\boldsymbol{\beta}$ happens to vanish at the particular point considered, $\beta^{\alpha}(u) = 0$. Then the matrix $B = (B^{\alpha}{}_{\gamma})$ evaluated at this special point does indeed transform as a $(1,1)$-tensor under a change of coordinates on $\mathcal{T}$. This simple fact will become relevant when we discuss the critical exponents of fixed points in the next section.

(D) RG-time independence of $\boldsymbol{\beta}$. Throughout our interpretation of the functional RG equation as an infinite-dimensional extension of $\partial_\lambda \xi^i(\lambda) = V^i(\xi(\lambda))$ we tacitly exploited already that the RHS of this equation has only an *implicit* dependence on λ, namely via $\xi(\lambda)$. This property does not come for free. The original FRGE in (3.1), i.e., the one written in terms of dimensionful quantities does *not* have this property. It is rather of the type $\partial_\lambda \xi^i(\lambda) = V^i(\xi(\lambda); \lambda)$ where the vector field $V(\,\cdot\,; \lambda)$ has an *explicit* λ-dependence as well. In the FRGE it is due to the explicitly k-dependent $\overline{\mathcal{B}}_k$ of (3.4) with the components $\bar{b}^{\alpha}(\,\cdot\,; k)$; see (3.9). But following the steps leading from the dimensionful differential equations (3.10) to the dimensionless ones in (3.12) and (3.13), it becomes clear that the explicit scale dependence disappears, and $\boldsymbol{\beta}$ becomes RG time independent, if we *employ k as the universal unit of all dimensionful quantities.* This is the reason why we usually prefer the $\mathcal{A}_k$-variant of the EAA over Γ_k.

(E) Background independence of the first and second kind. Looking at the explicit definition of $\boldsymbol{\beta}$ given in (3.32) with (3.4) and (3.19), it is obvious that on a given theory space (i.e., for a fixed set of fields, symmetries, etc.) it is only $\mathcal{R}_k$ that remains to be specified before we can evaluate $\boldsymbol{\beta}$ explicitly. The cutoff operator is indeed arbitrary to a large extent. While physical observables must be independent of it, the RG flow, i.e., $\boldsymbol{\beta}$ and the functions $\beta^{\alpha}(u^1, u^2, \cdots)$, will usually depend on it.

Having decided for some $\mathcal{R}_k$ we can "turn the crank" and compute the functional traces on the RHS of the FRGE. The vector field $\boldsymbol{\beta}$ is found then by evaluating the components $\boldsymbol{\beta}\{\mathcal{A}\}[\widetilde{\Phi}, \bar{\widetilde{\Phi}}]$ of (3.32) for all points $\mathcal{A}$ of the theory space and all points $(\widetilde{\Phi}, \bar{\widetilde{\Phi}})$ of field space. This shows that nothing more than *the arguments plugged into* $\boldsymbol{\beta}\{\,\cdot\,\}[\,\cdot\,]$ appear in the trace computations: the functional $\mathcal{A}$ from the first "slot" and the fields $(\widetilde{\Phi}, \bar{\widetilde{\Phi}})$ from the second.

This simple observation has an important consequence: Neither does $\boldsymbol{\beta}$ depend on any special field configuration, i.e., a distinguished point of field space, nor does it depend on any special action functional, i.e., a distinguished point in theory space.

We refer to these properties of $\boldsymbol{\beta}$ as *Background Independence of the first and the second kind,* respectively.

These names are motivated by the traditional use of the term "Background Independence" in quantum gravity where it expresses that no special spacetime and no metric are singled out. This is an example of "Background Independence of the first kind." In fact, when it comes to applying the EAA formalism to gravity, the fields $(\widetilde{\Phi}, \widetilde{\bar{\Phi}})$ will include both a dynamical and a background metric.

3.2 Fixed Points

Fixed points are particularly important and distinguished "inhabitants" of theory space. They play a prominent role in virtually any RG application, and quantum gravity is no exception in this respect. In this section we shall discuss some of their general properties as a preparation for our exposition of Asymptotic Safety later on.

3.2.1 The Zeros of $\boldsymbol{\beta}$

By definition, a *fixed point* in the dimensionless theory space, $\mathcal{A}_* \in \mathcal{T}$, is a point at which the vector field $\boldsymbol{\beta}$ has a zero: $\boldsymbol{\beta}\{\mathcal{A}_*\} = 0$. At a fixed point the "velocity" of any RG trajectory will vanish: $\partial_t \mathcal{A}(t) = \boldsymbol{\beta}\{\mathcal{A}_*\} = 0$.

Depending on whether one prefers the field or component language the condition for a fixed point reads, more explicitly, either

$$\boldsymbol{\beta}\{\mathcal{A}_*\}[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] = 0 \quad \forall\ (\widetilde{\Phi}, \widetilde{\bar{\Phi}}) \tag{3.47}$$

or

$$\boxed{\beta^\alpha(u_*) = 0 \quad \forall\ \alpha\,.} \tag{3.48}$$

Within the LPA approximation we saw already a concrete example of (3.47), namely the differential equation (2.70) for the potential $u_*(\varphi)$. Instead, in (3.48), where $u_* \equiv \{u_*^\alpha\}$ denotes the components of $\mathcal{A}_*$, i.e.,

$$\mathcal{A}_*[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] = \sum_\alpha u_*^\alpha\, I_\alpha[\widetilde{\Phi}, \widetilde{\bar{\Phi}}]\,, \tag{3.49}$$

the fixed-point condition has the appearance of infinitely many coupled equations for infinitely many unknowns: $\beta^\alpha(u_*^1, u_*^2, u_*^3, \cdots) = 0$, $\alpha = 1, 2, 3, \cdots$. It is a highly nontrivial question whether this system of equations does actually admit solutions, and if so, what their characteristic properties are.

Notice that it is the dimensionless couplings that are required to become scale independent: $u^\alpha(k) \equiv k^{-d_\alpha}\, \bar{u}^\alpha(k) = \text{const} \equiv u_*^\alpha$. The dimensionful ones continue to be k-dependent even at a fixed point, but their RG evolution is very simple there. It amounts to a set of power laws

$$\boxed{\bar{u}^\alpha(k) = u_*^\alpha\, k^{d_\alpha}\,,} \tag{3.50}$$

in which all couplings scale according to their canonical dimensions d_α.

Every fixed point (FP) gives rise to a special RG trajectory, the *fixed point trajectory*. At the dimensionful level it is described by the running action

$$k \mapsto \Gamma_k^{\mathrm{FP}}[\Phi, \bar{\Phi}] = \sum_\alpha u_*^\alpha\, k^{d_\alpha}\, I_\alpha[\Phi, \bar{\Phi}]\,. \tag{3.51}$$

This is indeed a nontrivial curve on the (dimensionful) theory space. In the dimensionless setting, the analogous action reads

$$k \mapsto \mathcal{A}_k^{\mathrm{FP}}[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] = \sum_\alpha u_*^\alpha\, I_\alpha[\widetilde{\Phi}, \widetilde{\bar{\Phi}}] = \mathcal{A}_*[\widetilde{\Phi}, \widetilde{\bar{\Phi}}]\,. \tag{3.52}$$

Logically, $\mathcal{A}_k^{\mathrm{FP}}$ is also defined for all $k \in [0, \infty)$, but it happens to be time independent and so the curve degenerates to a point on the dimensionless theory space, $\mathcal{T}$. In the dynamical system interpretation $\mathcal{A}_k^{\mathrm{FP}}$ corresponds to a particle that sits at one and the same point at all times. This is possible since at the point $\mathcal{A}_*$ the "force" $\boldsymbol{\beta}$ acting on the particle vanishes.

With the fixed-point trajectories Γ_k^{FP} and $\mathcal{A}_k^{\mathrm{FP}}$ we have found an example of an exact solution to the dimensionful and dimensionless forms of the FRGE, respectively.

3.2.2 The Linearized Flow

Next, we ask about the behavior of RG trajectories that stay close to the fixed point for a while without necessarily sitting on top of it at all times. To answer the question we Taylor expand the vector field $\boldsymbol{\beta}$ to first order in deviations from the fixed point u_*. The exact FRGE $k\partial_k\, u^\alpha = \beta^\alpha(u)$ then leads to the following linear differential equation:

$$k\partial_k\, u^\alpha(k) = \sum_\gamma B^\alpha{}_\gamma \big(u^\gamma(k) - u_*^\gamma\big). \tag{3.53}$$

Here we encounter the *stability matrix* $B \equiv (B^\alpha{}_\gamma)$ of the fixed point under consideration:

$$B^\alpha{}_\gamma = \frac{\partial}{\partial u^\gamma}\beta^\alpha\bigg|_{u=u_*}\,. \tag{3.54}$$

Generically, this matrix has no reason to be symmetric. As a consequence, we may not expect that it possesses a complete system of eigenvectors in general, and even if it does, the left and right eigenvectors may be different. Furthermore, its eigenvalues are not necessarily real [96].

Rather than discussing all cases that could occur we assume here and in the following that B does indeed admit a complete system of right eigenvectors $V_J \equiv (V_J^\alpha)$, $J = 1, 2, 3, \cdots$:

$$\boxed{\sum_\gamma B^\alpha{}_\gamma V_J^\gamma = -\theta_J V_J^\alpha\,, \quad \forall\ J.} \tag{3.55}$$

In all RG investigations of quantum gravity, up to now, this assumption indeed turned out to be satisfied. Nevertheless, the stability matrices that occur are typically not symmetric, and as a consequence, the eigenvalues $(-\theta_J)$ often have a nonzero imaginary part, and the eigenvectors are not orthogonal. The complex numbers θ_J are referred to as stability exponents, *critical exponents*, or scaling exponents.

Equipped with a complete system $\{V_J\}$, it is easy to write down the general solution to the linearized RG equation:

$$\boxed{u^\alpha(k) = u_*^\alpha + \sum_J C_J V_J^\alpha \left(\frac{k_0}{k}\right)^{\theta_J}.} \tag{3.56}$$

Here, the C_Js denote constants of integration which can be expressed in terms of the initial conditions, and k_0 is a fixed scale which we identify with the one present in $t \equiv \ln\left(\frac{k}{k_0}\right)$. Note that if some critical exponent θ_J is not real, the solution (3.56) contains an oscillatory component:

$$\left(\frac{k_0}{k}\right)^{\theta_J} = e^{-\theta_J t} = \Big[\cos(\operatorname{Im}\theta_J t) - i\sin(\operatorname{Im}\theta_J t)\Big]\left(\frac{k_0}{k}\right)^{\operatorname{Re}\theta_J}. \tag{3.57}$$

In a physically relevant solution the C_Js are such that all imaginary parts cancel in the sum (3.56).

When we move forward along the trajectory (3.56), i.e., when we decrease k, equation (3.57) implies that the component of $u(k) - u_*$ in the direction of a given vector V_J will increase if $\operatorname{Re}(\theta_J) > 0$, decrease if $\operatorname{Re}(\theta_J) < 0$, and is purely oscillatory if $\operatorname{Re}(\theta_J) = 0$. The basis vector (or "scaling operator," as it is sometimes called in this context) is then termed *relevant*, *irrelevant*, or *marginal*, respectively.

So "relevant" is synonymous to "growing under downward evolution" from the UV to the IR. Conversely, under upward evolution (increasing k) all contributions to the sum in (3.56) belonging to relevant directions get damped, while those along irrelevant directions blow up if the pertinent coefficients C_J are nonzero.

Since, by assumption, the vectors $\{V_J\}$ form a complete system, they provide us with a (not necessarily orthogonal) basis of the tangent space to $\mathcal{T}$ at the fixed point, $T_{\mathcal{A}_*}\mathcal{T}$. The stability matrix supplies a natural decomposition of $T_{\mathcal{A}_*}\mathcal{T}$ in a *relevant, irrelevant, and marginal subspace*, which are spanned by the V_Js with $\operatorname{Re}(\theta_J) > 0$, $\operatorname{Re}(\theta_J) < 0$, and $\operatorname{Re}(\theta_J) = 0$, respectively.

3.2.3 Universality of the Critical Exponents

An important property of the critical exponents θ_J is that they depend neither on the system of coordinates on $\mathcal{T}$, $\{u^\alpha\}$, nor on the cutoff scheme, i.e., on the operator $\mathcal{R}_k$: They are *universal quantities* [97]. As an immediate consequence, when we decompose the tangent space $T_{\mathcal{A}_*}\mathcal{T}$ in relevant, marginal, and irrelevant subspaces the respective dimensionalities of those subspaces are likewise universal.

(1) Coordinate independence. The coordinate independence of the θ_Js is easy to see. In (3.46) we wrote down how the partial derivatives $B^\alpha{}_\gamma(u) \equiv \frac{\partial}{\partial u^\gamma}\beta^\alpha(u)$ respond to a general coordinate transformation on theory space, $u \mapsto u'(u)$. There we mentioned that their transformation law becomes tensorial at points where $\boldsymbol{\beta}$ vanishes, i.e., at fixed points. At a fixed point, $B^\alpha{}_\gamma(u_*) \equiv B^\alpha{}_\gamma$ in (3.45) coincides exactly with the above stability matrix. From (3.46) we infer that in going from the old coordinates u^α to the new ones, u'^α, the corresponding stability matrices B and B' are related according to

$$B'^\alpha{}_\gamma(u'_*) = \frac{\partial u'^\alpha}{\partial u^\kappa} B^\kappa{}_\delta(u_*) \frac{\partial u^\delta}{\partial u'^\gamma}. \tag{3.58}$$

Here u_* and u'_* are the old and new fixed-point coordinates, respectively, $\beta'^\alpha(u'_*) = 0 = \beta^\alpha(u_*)$. Thus B and B' are connected by a similarity transformation, $B' = SBS^{-1}$, $S \equiv \big(\frac{\partial}{\partial u^\gamma} u'^\alpha(u_*)\big)$, which implies that their spectra coincide. Therefore, we can be sure to find the same critical exponents, no matter which coordinate system we use.

(2) Scheme independence. The scheme independence of the critical exponents, i.e., their independence of the cutoff operator $\mathcal{R}_k$, is a consequence of their coordinate independence. The reason is that a deformation $\mathcal{R}_k \to \mathcal{R}_k + \delta\mathcal{R}_k$ modifies the linear flow at u_* by not more than a coordinate transformation.

This can be seen most directly from the functional relation (2.26) whose general structure carries over from scalars to general field systems.

Proceeding in complete analogy with the derivation of the FRGE in component form, we may expand $\Gamma_k = \sum_\alpha \bar{u}^\alpha(k)\, I_\alpha$ and rewrite (2.26) in terms of the dimensionless cousins of the $\bar{u}^\alpha$s. The resulting equations tell us how the individual couplings respond to the change $\mathcal{R}_k \to \mathcal{R}_k + \delta\mathcal{R}_k$. They assume the form

$$\delta u^\alpha(k) = f^\alpha\big(u(k)\big), \tag{3.59}$$

where the f^αs depend linearly on $\delta\mathcal{R}_k$. These equations do not couple different scales and describe an "instantaneous" change of the RG trajectories, $u^\alpha(k) \to u^\alpha(k) + \delta u^\alpha(k)$. Hence, for $k \to \infty$ and $u(k) \to u_*$ in the case of interest, we see that modifying the cutoff can be compensated by a (local) coordinate transformation $u^\alpha \to u^\alpha + f^\alpha$, which is known to leave the spectrum of the stability matrix unaltered.

3.2.4 The Ultraviolet Critical Hypersurface

By definition, the *unstable manifold*, or synonymously the *UV critical hypersurface* $\mathscr{S}_{\text{UV}}$ of a given fixed point consists of all points $A \in \mathcal{T}$ that are "pulled" into the fixed point under the inverse flow, i.e., when k is increased toward infinity.

Suitable regularity conditions assumed, this set of points indeed constitutes a submanifold of theory space. It should be visualized as a curved surface embedded in $\mathcal{T}$, which, while tangent to the unstable, or relevant subspace of $T_{\mathcal{A}_*}\mathcal{T}$ precisely at the fixed point $\mathcal{A}_*$, describes the true behavior of the flow lines even far away from the fixed point where the linearized differential equation loses its validity.

Clearly, if a trajectory $k \mapsto u(k)$ constituting a solution to the exact RG equation is such that $u(k_1) \in \mathscr{S}_{\text{UV}}$ at one scale k_1, then $u(k) \in \mathscr{S}_{\text{UV}}$ for all other scales as well.

Let us determine the dimensionality of $\mathscr{S}_{\text{UV}}$. Close to the fixed point, the general solution to the linear RG equation given in (3.56) provides a valid description of all trajectories, whether they run inside $\mathscr{S}_{\text{UV}}$ or not. Now, letting $k \to \infty$ in (3.56), those terms in the sum that correspond to the "relevant" eigenvectors with $\text{Re}(\theta_J) > 0$ vanish in this limit independently of the values of the associated coefficients C_J. The terms associated with the "irrelevant" directions, having $\text{Re}(\theta_J) < 0$, will diverge, however, unless their coefficients C_J are zero. Finally, in marginal directions linearized trajectories deviate in the limit $k \to \infty$ from u_* by a finite, but nonzero amount should their C_Js be nonzero.

Hence, it follows that trajectories running inside $\mathscr{S}_{\text{UV}}$, close to the fixed point, are approximated by (3.56) with $C_J = 0$ for all J which refer to an irrelevant or marginal eigenvector V_J:

$$\boxed{u^\alpha(k) = u^\alpha_* + \sum_{\text{Re}(\theta_J)>0} C_J V^\alpha_J \left(\frac{k_0}{k}\right)^{\theta_J}} \quad \text{on } \mathscr{S}_{\text{UV}}. \tag{3.60}$$

Here the sum is restricted to include the "relevant" directions only. All trajectories (3.60) indeed have the property that $\lim_{k\to\infty} u^\alpha(k) = u^\alpha_*$ for any choice of the remaining, in general, nonzero coefficients C_J.

Conversely, near $\mathcal{A}_*$, every trajectory inside $\mathscr{S}_{\text{UV}}$ has the form (3.60) for some choice of these coefficients. As a consequence, $\dim(\mathscr{S}_{\text{UV}})$ equals the number of independent C_Js that can still be chosen freely, or, stated differently, the dimensionality of the "relevant" subspace of $T_{\mathcal{A}_*}\mathcal{T}$. Therefore, we obtain the important result that the dimension of $\mathscr{S}_{\text{UV}}$ equals the number of eigenvalues of B which possess a positive real part:

$$\boxed{\Delta_{\text{UV}} \equiv \dim(\mathscr{S}_{\text{UV}}) = \#\big\{\theta_J \big| \,\text{Re}(\theta_J) > 0\big\}.} \tag{3.61}$$

Henceforth we denote the number of relevant directions by Δ_{UV}.

At this point it is not obvious that the notion of a "number of relevant eigenvectors" makes any sense, given that B is an infinite-dimensional matrix. Later on we shall see however that for a typical fixed point in the physical systems we are familiar with, Δ_{UV} is indeed a finite, and usually *small*, number.

3.2.5 Gaussian vs. non-Gaussian Fixed Points

To begin with a simple example leading to a trivial, or Gaussian fixed point (GFP), consider a set of β-functions which behaves like

$$\beta^\alpha(u) = -d_\alpha u^\alpha + O(u^2) \tag{3.62}$$

for small values of the couplings. According to (3.13) this amounts to $b^\alpha(u) = O(u^2)$ and $d_\alpha \equiv [\bar{u}^\alpha]$. In this case, $u_*^\alpha = 0$ for all α with $d_\alpha \neq 0$ is an obvious solution to $\beta^\alpha(u_*) = 0$. The exact stability matrix at this fixed point is very simple, namely $B^\alpha{}_\gamma = -d_\alpha \delta^\alpha_\gamma$. It happens to be diagonal and its negative eigenvalues, the critical exponents, equal exactly the canonical mass dimensions of the couplings $\bar{u}^\alpha$:

$$\boxed{\theta_\alpha = d_\alpha \quad \text{at a GFP.}} \tag{3.63}$$

This equality is the motivation for defining the θs with the explicit minus sign in (3.55).

In this book we adopt the definition proposed in [98] and refer to *any* fixed point whose critical exponents are all equal to the canonical mass dimensions as a *Gaussian fixed point.* Conversely, at a *non-Gaussian fixed point* (NGFP), by definition, some or all critical exponents differ from their canonical value.

While the defining property (3.63) is a coordinate-independent statement, the beta functions near a GFP are usually more complicated than (3.62) when a generic system of coordinates $\{u^\alpha\}$ is used. A linear change of variables (diagonalization of $B^\alpha{}_\gamma$) is then necessary to find the critical exponents.

3.2.6 Linearization about the Scalar GFP

In our investigation of scalar field theory within the local potential approximation we encountered an example of a Gaussian fixed point. In the LPA, the fixed-point condition for the dimensionless potential $u_*(\varphi)$ is given by the (ordinary) differential equation (2.70), and we remarked already that it admits the constant solution $u_*(\varphi) = 4v_d/d^2 = \text{const.}$

(1) Let us compute the critical exponents and the scaling operators for this fixed point. The calculation can be done either in an algebraic way by expanding the potential as in (2.72) and solving for the coefficients λ_{2n} individually, or "in one stroke" by using the (partial) differential equation for the complete potential function.

Here we follow the second strategy and make the following linearization ansatz for the running dimensionless potential:

$$u_k(\varphi) = u_* + Y(\varphi)\left(\frac{k_0}{k}\right)^{\theta} \equiv u_* + Y(\varphi)e^{-\theta t}. \tag{3.64}$$

We insert (3.64) into the flow equation (2.67) and expand to linear order in Y, obtaining

$$\boxed{\left[-\sigma_d\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} + \left(\frac{d-2}{2}\right)\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi} - d\right]Y(\varphi) = (-\theta)Y(\varphi)} \tag{3.65}$$

with the dimension-dependent positive constant

$$\sigma_d \equiv 2v_d\int_0^\infty \mathrm{d}z\, z^{\frac{d}{2}-1}\,\frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{\left[z + R^{(0)}(z)\right]^2}. \tag{3.66}$$

The differential equation (3.65) represents an infinite-dimensional eigenvalue equation $BY = (-\theta)Y$ for the eigenvectors Y and eigenvalues $(-\theta)$. The role of the stability matrix is played by the second-order differential operator on the LHS of (3.65).

Assuming $d > 2$ from now on the analysis of (3.65) simplifies by redefining the function Y in the following way:

$$Y(\varphi) \equiv e^{\frac{x^2}{4}}\psi(x) \quad \text{with} \quad x \equiv \sqrt{\tfrac{d-2}{2\sigma_d}}\,\varphi. \tag{3.67}$$

In terms of ψ, the above eigenvalue condition boils down to an equation that looks like the time-independent Schrödinger equation of a harmonic oscillator:

$$\boxed{-\psi''(x) + \frac{1}{2}x^2\psi(x) = E\psi(x)} \tag{3.68}$$

with the identification

$$E \equiv \frac{1}{2} + 2\frac{d-\theta}{d-2}. \tag{3.69}$$

(2) At this point a very general question arises which we shall re-encounter in gravity later on. Namely, what precisely is the space of functions the deviations from the fixed point are supposed to live in? Or concretely, what are the subsidiary or boundary conditions to be imposed on eligible solutions to (3.68)?

If we interpret the eigenvalue problem (3.68) as a quantum mechanical Schrödinger equation, we know that ψ must be square-integrable. However, in the present case ψ is essentially a potential, and there is no obvious physical necessity for this potential to be square-integrable.

(3) Nevertheless, let us begin our discussion by finding all solutions to (3.68) with $\int_{-\infty}^{\infty}\mathrm{d}x|\psi|^2 < \infty$. From quantum mechanics we know that such solutions exist only for a discrete set of energy eigenvalues, $E = E_n \equiv n + \frac{1}{2}$, $n = 0, 1, 2, \cdots$, and that the corresponding eigenfunctions are Hermite polynomials times an exponential, $\psi_n(x) \propto H_n(\frac{x}{\sqrt{2}})e^{-\frac{x^2}{4}}$. This implies the following solutions to the original equation (3.65):

$$\boxed{\begin{aligned} Y_n(\varphi) &\propto H_n\left(\tfrac{1}{2}\sqrt{\tfrac{d-2}{\sigma_d}}\varphi\right), \\ \theta_n &= d - \left(\tfrac{d}{2} - 1\right) n \end{aligned}} \tag{3.70}$$

We observe that the scaling operators are given by polynomials of degree n. They are even (odd) functions of φ if n is even (odd). If we impose symmetry under $\varphi \to -\varphi$, only the even eigenfunctions $Y_{2n}(\varphi)$ are permitted, and so the general solution to the linearized flow equation is the following linear superposition analogous to (3.56):

$$u_k(\varphi) = u_* + \sum_{n=0}^{\infty} C_{2n} Y_{2n}(\varphi) \left(\frac{k_0}{k}\right)^{\theta_{2n}}. \tag{3.71}$$

The pertinent scaling exponents are given by $\theta_{2n} = d - (d-2)n$. A comparison with (2.74) reveals that the θ_{2n}s are precisely the canonical mass dimensions of the coupling constants $\bar{\lambda}_{2n}$ multiplying the basis functionals $I_{2n}[\phi] \equiv \int \mathrm{d}^d x \phi^{2n}(x)$. Thus, having shown that all critical exponents equal the classical dimensions, $\theta_{2n} = [\bar{\lambda}_{2n}]$, we have established that the fixed point $u_*(\varphi) = \text{const}$ is indeed a *Gaussian* one.

(4) We also observe that the vast majority of eigenvectors are associated with *irrelevant* directions since $\theta_{2n} = d - (d-2)n$ is negative for most values of n. Relevant directions ($\theta_{2n} > 0$) can occur only as long as n is smaller than $d/(d-2)$. In $d = 4$ dimensions, e.g., there exist two relevant, one marginal, and infinitely many irrelevant directions. This is summarized in Table 3.1.

In the third column of the table we show the general structure of the eigenfunctions, suppressing numerical coefficients.

Table 3.1 *Relevant, marginal, and irrelevant scaling fields (eigen-directions) at the scalar GFP in $d = 4$ dimensions according to the LPA*

n	$\theta_{2n} = 4 - 2n$	$Y_{2n}(\varphi)$	Type
0	4	1	Relevant
1	2	$\varphi^2 + 1$	Relevant
2	0	$\varphi^4 + \varphi^2 + 1$	Marginal
3	–2	$\varphi^6 + \varphi^4 + \varphi^2 + 1$	Irrelevant
$\vdots$	$\vdots$	$\vdots$	”

Thus, for the special example of the GFP, and within the local potential approximation, we see explicitly that *"relevant" eigendirections are indeed an exception, while irrelevant ones are the rule.*

(5) Actually it is easy to generalize this result beyond the LPA. If we invoke a double expansion of Γ_k in both the number of fields and the number of derivatives, additionally including all terms $\bar{u}_m \int \mathrm{d}^4x\, \phi(\partial_\mu \partial^\mu)^m \phi$, for example, into the action ($m = 2, 3, 4, \cdots$), then the new higher derivative terms are all irrelevant. In this example the running couplings have the dimension $[\bar{u}_m] = 2 - 2m$, which can only become smaller by increasing m.

This is also the general situation: Given a monomial containing n powers of ϕ and $2m$ derivatives acting in an arbitrary way on the fields, it is sufficient to know that $[\partial_\mu] = +1$ in order to conclude that adding derivatives unavoidably decreases the dimension of the coupling and, as a consequence, the corresponding eigenvalue of the stability matrix. Hence the "most relevant" monomials of this kind are those with only two derivatives.

(6) So far we have focused on *square integrable* solutions to the Schrödinger-like eigenvalue equation (3.68). For completeness we mention that if we relax this condition a continuous infinity of further solutions will arise. For every $\theta \in \mathbb{R}$ there exists an eigenfunction

$$\psi_\theta(x) = e^{-\frac{x^2}{4}}\, {}_1F_1\left(-\tfrac{d-\theta}{d-2}, \tfrac{1}{2}, \tfrac{x^2}{2}\right), \tag{3.72}$$

where ${}_1F_1$ denotes the confluent hypergeometric (or Kummer's) function. While the solution (3.72) is even under $x \to -x$, there also exists a similar odd one. For all $\theta \neq \theta_n$ the functions $\psi_\theta(x)$ and the corresponding $Y_\theta(\varphi)$ grow very rapidly for $x \to \pm\infty$ or $\varphi \to \pm\infty$:

$$\psi_\theta(x) \sim e^{\frac{x^2}{2}}, \quad Y_\theta(\varphi) \sim \exp\left(+\left(\tfrac{d-2}{4\sigma_d}\right)\varphi^2\right). \tag{3.73}$$

This asymptotic behavior makes it obvious that, for $\theta \neq \theta_n$, all Y_θs are *non-polynomial* functions of the field. As a result, perturbation theory is unable to detect these so called "Halpern-Huang directions." Hence their discovery, by non-perturbative methods, came as quite a surprise [99]. Their status and physical role are not yet fully understood [100, 101].

3.3 Asymptotic Safety: The Basic Idea

After the detailed preparations in the previous sections we are now in the position to outline the key idea of Asymptotic Safety in a concise way. The goal of this section is to describe the basic picture in a language as geometric as possible. The discussion of the technical issues related to the investigation of this scenario is relegated to subsequent chapters of this book.

At first let us recall the main questions which the Asymptotic Safety program as a whole wants to address:

(i) Given the fact that functional integrals such as $\int \mathcal{D}\widehat{\Phi}\, e^{-S[\hat{\Phi},\bar{\Phi}]}$ are mathematically ill defined and make sense only in the presence of a UV cutoff, we would like to find out for which measure and bare actions S the limit of an infinite UV cutoff can be taken consistently. This limit must give rise to a quantum field theory which respects all basic physical principles which we consider indispensable (Hilbert space positivity, Background Independence, etc.).

Perturbation theory can give only a partial answer here; it identifies the *perturbatively renormalizable* actions. However, many potentially interesting cases (including quantum gravity) lie beyond the scope of perturbation theory. Hence an appropriate notion of *non-perturbative renormalizability* is called for.

(ii) For the class of bare actions that can have an acceptable limit we would like to know whether it is unique, and if not, to what extent there are ambiguities, and most importantly how the bare parameters in S must depend on the cutoff when the UV limit is taken.

(iii) Once it is known how to take the limit(s), we are confronted with a well-defined, but presumably very complicated, functional integral which may contain non-perturbative information. Therefore, we need a corresponding non-perturbative calculational method in order to evaluate it and derive predictions for observable quantities.

As we shall see, the framework of the Effective Average Action is well suited for exploring all three of the above problems in a self-contained manner. In the EAA framework, they amount to the following three steps in the analysis of the RG flow, which we assume as given for the time being:

(i) Determine all fixed points of the RG flow.
(ii) Compute the linearized flow at a given fixed point and determine its subspace of "relevant" directions.
(iii) Compute the nonlinear extension of the latter, i.e., the fixed point's entire UV critical hypersurface, $\mathscr{S}_{\text{UV}}$.

Let us now describe the various parts of this program in more detail.

3.3.1 The Input: $\mathcal{T}$ and β

Our search for non-perturbatively renormalizable theories begins by specifying a theory space of well-behaved action functionals, $\mathcal{T}$. In this step we fix the nature of the fields on which the actions are supposed to depend, the symmetry, covariance, and regularity properties of the actions and their field arguments, as well as any additional subsidiary conditions we might want the functionals to satisfy. As a further, purely technical input we also pick a cutoff operator $\mathcal{R}_k$.

Given these "input data" we have all that is needed in order to write down the associated FRGE and the associated vector field $\boldsymbol{\beta}$ which determines the RG flow on $\mathcal{T}$.

3.3.2 The Quest for Complete Trajectories

As discussed, the FRGE is not plagued by any UV divergences. In the previous chapter we derived the flow equation from a functional integral which was equipped with a UV cutoff at some scale Λ. While it is unclear whether or not the integral *per se* admits a meaningful limit for $\Lambda \to \infty$, we saw that this limit can be taken straightforwardly at the level of the flow equation.

So, if the FRGE is perfectly finite, where have all the notorious questions related to the existence of a UV limit gone? The answer is simple: Now they manifest themselves as questions about the nature of the *solutions* to the FRGE, i.e., the RG trajectories.

The construction of a quantum field theory with the UV cutoff removed amounts to integrating out *all* field modes. At the level of Γ_k this means lowering the IR cutoff from "$k = \infty$" to $k = 0$. In fact, we saw already that $\Gamma_{k\to\infty}$ is closely related to the bare action S, while, at the endpoint of the RG trajectory, $\Gamma_{k=0}$ equals the ordinary effective action.

Hence, if we require that there is a fundamental quantum field where all field modes can be integrated out, it is necessary that there exists a well defined action functional Γ_k for each value of the scale parameter $k \in [0, \infty)$.

Geometrically speaking, and using the dimensionless language now, these actions constitute a *complete RG trajectory*, i.e., a trajectory that never leaves the space $\mathcal{T}$ of "acceptable" actions, neither at finite scales, nor in the limits $k \to \infty$ or $k \to 0$, respectively.

Thus, the EAA-based search for the (non-)perturbatively renormalizable theories, i.e., those admitting a meaningful limit $k \to \infty$, essentially consists in finding all complete RG trajectories which exist on the theory space chosen: $\mathcal{A}_\bullet : [0, \infty) \to \mathcal{T}, k \mapsto \mathcal{A}_k$.

This reformulation turns the original renormalization problem into a question about the *long-time behavior of trajectories* in the infinite-dimensional dynamical system related to $(\mathcal{T}, \boldsymbol{\beta})$.

As such, it is on a par with many other important, but notoriously difficult, problems in physics and mathematics. They range from establishing ergodicity or the various forms of chaoticity to problems in biology, medicine, and economics, to mention just a few of the fields [94].

3.3.3 The "Asymptotic Safety Scenario" of the UV Limit

When trying to construct complete RG trajectories the most severe difficulty one must overcome is to make sure that the trajectories do not leave theory space in the limit $k \to \infty$, i.e., develop an unacceptable behavior at high momentum scales.

This dangerous possibility is the new incarnation of the old UV renormalization problem.

Given the previous discussion, it suggests itself to use a fixed point of the RG flow in order to control the $k \to \infty$ limit of the complete RG trajectories. In fact, Asymptotic Freedom (well-known from QCD, for example) is a realization of this idea restricted to the case where the underlying fixed point is the Gaussian one. For the Einstein–Hilbert action, Asymptotic Freedom fails, however, due to the negative mass dimension of Newton's coupling. Phrased differently, General Relativity is not contained in the unstable manifold of the corresponding GFP.

In his seminal paper [10], Steven Weinberg formulated a conjecture about the existence of the infinite cutoff limit in quantum gravity, which became known as the *Asymptotic Safety Scenario.*

This conjecture has been the main inspiration for the whole Asymptotic Safety program in its modern EAA-based form, and also gave the name to this entire direction of research.

Weinberg's conjecture consists of two parts. In a somewhat simplified form (refinements will be discussed later), and in the same language as used above, they read as follows:

(W1) The theory space contains a non-trivial fixed point, and this fixed point has a low-dimensional UV critical hypersurface.

(W2) Every RG trajectory that does *not* hit this fixed point in the UV limit ultimately leaves theory space, developing unacceptable singular behavior at high scales.

If both (W1) and (W2) are true, a given RG trajectory has an acceptable UV limit, and hence signals the existence of a "non-perturbatively renormalizable" field theory, if and only if its endpoint in the ultraviolet is given by the *non-trivial fixed point of the RG flow.* The theory is then safe from the threat of divergences in physical quantities at asymptotically high momentum scales since we know for sure that when $k \to \infty$ the trajectory always stays (increasingly) close to an inner point of theory space, namely the fixed point. This behavior is exactly what gave rise to the name "Asymptotic Safety."

Let the functional $\mathcal{A}_* \in \mathcal{T}$, having components u_*^α, be a nontrivial fixed-point action, i.e., $\beta^\alpha(u_*) = 0 \ \forall \alpha$, and furthermore let the trajectory $k \mapsto u^\alpha(k)$ be an "asymptotically safe" one, i.e., $\lim_{k\to\infty} u^\alpha(k) = u_*^\alpha$. It is clear then that this trajectory has a very small "velocity" $\partial_t u^\alpha = \beta^\alpha$ when it approaches the fixed point since, directly at $\mathcal{A}_*$, the β-functions vanish completely. As a result, the trajectory "uses up" an infinite amount of RG time while staying within an infinitesimally small neighborhood of $\mathcal{A}_*$.

During this eternal approach of the fixed point's exact location, the dimensionless Effective Average Action $\mathcal{A}_k$ almost does not move, while at the dimensionful level the components of Γ_k evolve according to the power laws $\bar{u}^\alpha(k) = u_*^\alpha\, k^{d_\alpha}$, as we saw in Section 3.2.1.

3.3.4 Predictive Power

It is essential to understand in what sense and to which degree theories based on asymptotically safe RG trajectories can have predictive power.

(1) Such theories are intended to constitute *fundamental quantum field theories* which remain consistent and predictive even at the highest momentum scales. This distinguishes them from *effective field theories*, which require an increasing number of coupling constants at high momenta that are in no way constrained theoretically and therefore must be taken from experiments.

As we explain next, Asymptotic Safety can "tame" this plethora of infinitely many undetermined constants and reduce them to just a few parameters that must be extracted from experiments. The situation is then comparable to an ordinary perturbatively renormalizable theory, a prime example being QED, which has only two theoretically undetermined parameters, namely the electron's mass and charge.

(2) According to (W1), theory space contains a fixed point, $\mathcal{A}_*$. As described in Section 3.2.2, we explore the RG flow in its vicinity by linearizing it about $\mathcal{A}_*$. This leads to a decomposition of the tangent space $T_{\mathcal{A}_*}\mathcal{T}$ into the subspaces spanned by the relevant, marginal, and irrelevant eigendirections of the stability matrix. These subspaces correspond to critical exponents θ_J with positive, vanishing, and negative real parts, respectively.

Furthermore, (W1) and (W2) imply that every complete trajectory runs into the fixed point when k is increased toward infinity, or in other words, when we proceed along the trajectory in the direction of the *inverse* RG flow, i.e., in the direction from the IR to the UV.

Recalling the discussion in Section 3.2.4 we see that this is tantamount to saying that *a trajectory is complete if, and only if, it lies in the UV critical hypersurface of an RG fixed point.*

Indeed, in Section 3.2.4 we defined the set of all points $\mathcal{A} \in \mathcal{T}$ that are dragged into a fixed point $\mathcal{A}_*$ under the inverse flow to constitute the UV critical surface $\mathscr{S}_{\mathrm{UV}} \equiv \mathscr{S}_{\mathrm{UV}}(\mathcal{A}_*)$.

(3) If $u^\alpha(k_1)$ is an arbitrary point on a complete trajectory $k \mapsto u^\alpha(k)$, this point is situated within the UV critical surface when $k = k_1$. By the very definition of $\mathscr{S}_{\mathrm{UV}}$ this also applies to all other points visited by $u^\alpha(k)$ at higher and lower scales k.

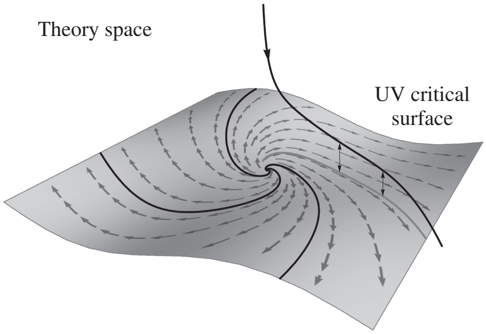


Figure 3.1. A two-dimensional UV critical hypersurface embedded in a three-dimensional theory space

In the limit $k \to \infty$, when the linearization about $\mathcal{A}_*$ is a good approximation, the trajectory $u^\alpha(k)$ is therefore given by the sum (3.60). It involves only terms belonging to the *relevant* subspace.

The situation is illustrated in Figure 3.1, which shows examples of trajectories that hit $\mathcal{A}_*$ for $k \to \infty$, as well as others that run outside $\mathscr{S}_{\mathrm{UV}}$. The latter fail to reach the fixed point and get driven away from it under the inverse flow.[3] According to (W2), these trajectories will leave theory space ultimately when $k \to \infty$ and thus they do not correspond to quantum field theories with a continuum limit.

(4) Let us switch for a moment from the inverse flow (increasing k) to the forward flow (decreasing k). So now we follow the trajectories of Figure 3.1 in the direction of the arrows. From this perspective, trajectories outside $\mathscr{S}_{\mathrm{UV}}$ are attracted by $\mathcal{A}_*$ toward $\mathscr{S}_{\mathrm{UV}}$ along the irrelevant directions. Conversely, the trajectories inside $\mathscr{S}_{\mathrm{UV}}$, those corresponding to field theories admitting a continuum limit, are pushed away from the fixed point along one of its relevant directions.

From this geometric picture and the assumed validity of (W1) and (W2) we can derive an estimate of how many complete trajectories and therefore inequivalent fundamental quantum field theories can exist.

According to the discussion leading to (3.61) the dimensionality of the UV-critical hypersurface, $\dim(\mathscr{S}_{\mathrm{UV}}) \equiv \Delta_{\mathrm{UV}}$, is given by the number of relevant directions, i.e., the number of θ_Js with $\mathrm{Re}(\theta_J) > 0$, which, according to (W1), is a small finite number.

[3] Recall again that the (forward) flow amounts to *decreasing* the mass scale k, while the inverse flow means increasing it. Note also that in this book (and most parts of the literature) the arrows in all figures point in the *forward* direction, i.e., toward *smaller* values of k.

As a consequence, *there exists a* $(\Delta_{\text{UV}} - 1)$*-parameter family of RG trajectories approaching* $\mathcal{A}_*$ *in the UV*, since we must specify $(\Delta_{\text{UV}} - 1)$ constants of integration in order to pick a specific trajectory on $\mathscr{S}_{\text{UV}}$.

In the linear approximation, these constants of integration are the C_J's in (3.56), up to an irrelevant common factor that can be absorbed into the reference scale k_0. This leaves us with the $(\Delta_{\text{UV}} - 1)$ free parameters which label asymptotically safe trajectories and their inequivalent quantum theories.

(5) In practice one could find those trajectories as follows. Within $\mathscr{S}_{\text{UV}}$, we draw a small Δ_{UV}-dimensional ball of radius ε around $\mathcal{A}_*$; its surface is a $(\Delta_{\text{UV}} - 1)$-sphere. Then we pick a point on this sphere, use it as an initial condition for the RG equation, and solve the flow equation in the forward direction toward smaller values of k, proceeding from the UV to the IR. Finally taking the limit $\epsilon \to 0$ we obtain a trajectory which is indeed complete, the existence of the limit being guaranteed by the (conjectured) validity of (W1) and (W2). We can repeat the procedure with all other points of the $(\Delta_{\text{UV}} - 1)$-sphere, which shows that its points are in a one-to-one correspondence with the asymptotically safe trajectories the fixed point gives rise to.

So, again, the conclusion is that the family of UV complete, or "fundamental" quantum theories that can be obtained by the Asymptotic Safety construction is labeled by $(\Delta_{\text{UV}} - 1)$ parameters. It is therefore necessary to perform a corresponding number of measurements in order to find out which one of those theories is actually realized in nature.

(6) In this scenario there is a simple way of *visualizing the totality of all predictions* that are implied by a given fixed point: they are encoded in the way $\mathscr{S}_{\text{UV}}$ is imbedded into theory space. Near the fixed point, $\mathscr{S}_{\text{UV}}$ is linearly approximated by the "relevant" subspace of $T_{\mathcal{A}_*}\mathcal{T}$. But away from $\mathcal{A}_*$ when the nonlinearities in the beta functions become significant, this linear subspace of the "tangent plane" gets deformed and becomes a complicated curved surface in general.

Locally the embedding of $\mathscr{S}_{\text{UV}}$ into $\mathcal{T}$ is parametrized by functions of the form

$$\left(w^1, w^2, \cdots, w^{\Delta_{\text{UV}}}\right) \mapsto u^\alpha \left(w^1, w^2, \cdots, w^{\Delta_{\text{UV}}}\right). \tag{3.74}$$

We may use the parameters $w = \left(w^i\right), i = 1, \ldots, \Delta_{\text{UV}}$, as local coordinates on $\mathscr{S}_{\text{UV}}$, and then describe the complete trajectories on $\mathscr{S}_{\text{UV}}$ concisely as $k \mapsto w^i(k)$.

Thus, since on all scales the physical system under consideration must be represented by a point on the, by assumption, low-dimensional hypersurface $\mathscr{S}_{\text{UV}}$, knowledge of the few parameters $w^1, \cdots, w^{\Delta_{\text{UV}}}$ is sufficient to determine all the infinitely many couplings u^α by using the embedding map (3.74). This makes it clear that, geometrically speaking, the shape of $\mathscr{S}_{\text{UV}}$ inside $\mathcal{T}$ encapsulates all possible predictions due to $\mathcal{A}_*$.

Note that the predictions that can be made on the basis of *a single* given asymptotically safe theory, based on a certain trajectory $k \mapsto w(k)$, is only a subset of all possible predictions compatible with Asymptotic Safety with respect to $\mathcal{A}_*$ since this specific theory evaluates the embedding functions only along a line, $k \mapsto u^\alpha(w(k))$. On the other hand, the shape of $\mathscr{S}_{\text{UV}}$ is really a geometrical way of summarizing the predictions of *all possible* asymptotically safe theories that are constructable at $\mathcal{A}_*$.

(7) Of course, there is also the possibility that nature does not adhere to the principle of Asymptotic Safety, which could actually be tested by setting up experiments searching for violations of the embedding maps (3.74). Thus the Asymptotic Safety hypothesis is falsifiable, at least in principle.

The wealth of information contained in the embedding maps (3.74) is indeed the main difference between an effective field theory and an asymptotically safe fundamental one [102, 103]. In an effective field theory the RG trajectories explore increasingly higher dimensional subspaces of theory space at higher scales, while in Asymptotic Safety the trajectories will always stay on a low-dimensional submanifold. This is precisely what "tames" the flood of phenomenological parameters that tend to render effective theories inefficient at high scales.

3.3.5 Generalizations and Refinements

Above we have drawn a slightly simplified picture of the Asymptotic Safety approach, trying to be as geometrical and intuitive as possible. In concrete realizations of this framework a number of generalizations, refinements, and additional technical assumptions might be required. In this subsection we discuss some of them.

(1) Essential vs. inessential couplings. It might happen that some of the couplings u^α do not affect the explicit expressions for observables. Therefore, the observables may have a well-defined UV-limit even if these special couplings diverge for $k \to \infty$ rather than assume fixed-point values. Following Weinberg [10] we refer to such couplings that can be tolerated to diverge in the UV limit without ruining the theory's predictivity and mathematical consistency as *inessential couplings*. In standard field theory a well-known example of inessential couplings are the wave function renormalizations associated with different fields. They do not enter observables like reaction rates or cross sections and can be changed by field redefinitions. Conversely, all other couplings, which are required to approach finite fixed-point values, are termed *essential couplings*.

As a consequence, the coupling constants appearing in the general picture of Asymptotic Safety developed above should include the essential ones only.

(2) Several fixed points. In general there can be more than one fixed point on a given theory space of (essential) couplings which is suitable for constructing an asymptotically safe UV limit. In this case the theory space supports multiple “universality classes” of qualitatively different quantum field theories based on different UV fixed points.

(3) Infinite dimensional $\mathscr{S}_{\mathbf{UV}}$. In concrete examples one also might encounter fixed points with an infinite-dimensional UV critical hypersurface. Even in this case Asymptotic Safety may result in predictive theories if the number of predictions grows faster than linearly with Δ_{UV} when $\Delta_{\mathrm{UV}} \to \infty$. (We refer to [98] for a detailed discussion and an example of this possibility.)

Furthermore, there may be additional physics principles beyond renormalizability (stability criteria, unitarity requirements, etc.) which put additional restrictions on the admissible RG trajectories within $\mathcal{S}_{\mathrm{UV}}$ and allow to fix a priori free parameters on theoretical grounds.

3.4 A First Example: Gravity in $2 + \varepsilon$ Dimensions

Returning now to quantum field theories of gravity it is certainly far from obvious that one can define a nonperturbative infinite cutoff limit à la Asymptotic Safety. In particular there is no general reason why there should exist an RG fixed point with the right properties.

Nonetheless, already at the time of Weinberg’s seminal paper [10], a simple toy model was available which displayed such a fixed point, namely gravity in $d = 2 + \varepsilon$ dimensions, with a small $\varepsilon > 0$.

In exactly two dimensions, Newton’s constant is dimensionless and thus a theory based on the bare action $\propto \frac{1}{G} \int \mathrm{d}^d x \sqrt{g} R(g)$ with $d = 2$ is formally power counting renormalizable in perturbation theory. However, in $d = 2$, this action happens to be a topological invariant, essentially the Euler number, and so in order to maintain a nontrivial kinetic term for the metric fluctuations the best we can do is to compute the beta functions in $d \neq 2$ dimensions and study their behavior near $d = 2$ thereafter.

In the “pre-FRGE” era, dimensional regularization had been used for this purpose, which gives rise to two types of poles in $1/\varepsilon$. Besides the usual ones due to UV divergences there are also “kinematical poles” which originate from the vanishing of the kinetic term for $\varepsilon \to 0$. Absorbing both types of divergences in renormalized quantities, the bare Newton constant is turned into a floating coupling constant $G(\mu)$, where μ is the familiar normalization mass scale of the dimensional regularization scheme. Since μ is not an observable, only the floating of Newton’s constant relative to another observable has intrinsic meaning; in practice, the cosmological constant or certain matter couplings have been used for this purpose.

A number of perturbative calculations had been performed in $2+\varepsilon$ dimensions. They differed in their precise computational setup, but they all led to the following Callan–Symanzik equation at one-loop order of perturbation theory:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g(\mu) = \big(\varepsilon - bg\big)\, g \equiv \beta^{\mathrm{pert}}(g). \tag{3.75}$$

Here $g(\mu) \equiv \mu^{\varepsilon} G(\mu)$ is the dimensionless Newton constant, and the coefficient b in the perturbative beta function is dimensionless as well.

While all calculations agree that b is *positive* for pure gravity, one can distinguish three different physical settings based on the value of b they give rise to. Within each of these settings b is universal though.

For quantum gravity coupled to n_s real, minimally coupled scalar matter fields the results are

$$\begin{aligned} b_1 &= \tfrac{2}{3}(1-n_s) && [10,\ 104,\ 105] \\ b_2 &= \tfrac{2}{3}(19-n_s) && [10, 106\text{–}109] \\ b_3 &= \tfrac{2}{3}(25-n_s) && [110\text{–}116] \end{aligned} \tag{3.76}$$

Obviously the metric fluctuations and the scalar fields always drive $g(\mu)$ in *opposite* directions, but the graviton contribution is different in the three cases: it can offset 1, 19, or 25 scalars, respectively. (Later on the very same numbers reappeared also in FRGE-based calculations.)

The explanation for the differences among the groups are two critical choices which underlie any of the calculations:

First, for spacetimes with a boundary, a scale-dependent $G(\mu)$ can be deduced from both the coefficient of the "bulk" gravitational action $\int \mathrm{d}^{2+\varepsilon}x\sqrt{g}R$ and the corresponding Gibbons-Hawking surface term. In general, the resulting beta functions are found to disagree. In fact, b_1 above was obtained from the surface term, while b_2 and b_3 are obtained from the bulk action instead.

Second, the calculations may employ different field parametrizations for the gravitational fluctuations: the derivation of b_1 and b_2 employs the linear background split $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, while b_3 is obtained with the exponential parametrization $g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^{\rho}{}_{\nu}$. In either case, $\bar{g}_{\mu\nu}$ is a classical background metric and $h = (h_{\mu\nu})$ is the elementary quantum field.

The deeper meaning of these classes of beta functions is still not fully understood; see [117] for a discussion.

The perturbative RG equation (3.75) displays two fixed points, a trivial "Gaussian" one at $g_* = 0$ and, remarkably enough, also a non-trivial one of the kind we are looking for in relation to Asymptotic Safety. It is located at

$$g_* = b^{-1}\varepsilon \tag{3.77}$$

and thus merges with the trivial one in the limit $\varepsilon \to 0$. The non-vanishing fixed-point value of Newton's constant is indicative of an interacting, or "non-Gaussian" theory at small but nonzero values of ε. Since g_* can be made arbitrary small, one may hope that perturbation theory is actually reliable in this case so that the perturbative fixed point (3.77) "lifts" to a full-fledged NGFP on the infinite-dimensional theory space.

On the basis of this result (3.77), Weinberg argued that gravity in $2+\varepsilon$ dimensions could indeed be asymptotically safe [10].

Since at the time he proposed Asymptotic Safety the modern functional RG methods were not available yet, extensions to the non-perturbative level and/or to four spacetime dimensions were not possible. Thus the perturbatively accessible fixed point in $d = 2 + \varepsilon$ dimensions remained for a long time the only evidence in support of the conjecture that pure quantum gravity is non-perturbatively renormalizable. This only changed with the advent of the Effective Average Action for gravity, which is introduced in the next chapter.

4

A Functional Renormalization Group for Gravity

The purpose of this chapter is to motivate and explain a particular proposal for a generally covariant and Background Independent notion of "coarse graining" or "averaging" that is applicable in a gravitational context, to implement it concretely in a suitably constructed scale dependent action functional, and finally to establish an exact renormalization group equation for the functional thus defined [14].

This setting, employing the *gravitational Effective Average Action*, has been used in almost all modern investigations of Asymptotic Safety.[1] Nevertheless, one should emphasize that there exists no such thing as *the* functional renormalization group for gravity. In principle, other approaches and formulations are conceivable as well.

Even if one focuses, as we do here, on continuum-based approaches and tries to follow the example of the scalar EAA from Chapter 2 as closely as possible it is not immediately obvious how to proceed. When one leaves the safe terrain of matter field theories on classical spacetimes, a considerable number of general concepts and methods become inapplicable. Typically there exists no "canonical" solution to these problems that would indicate the way to go. Thus the gravitational EAA as we introduce it here arose in a long process of trial and error as a compromise between practical applicability and mathematical elegance or "naturalness."

It should also be stressed that finding a generally covariant and Background Independent generalization of the exact renormalization group is of great importance in its own right and should not only be seen in connection to Asymptotic Safety. Its potential applications go well beyond the search for asymptotically safe theories. For instance, the entire framework of the gravitational EAA also offers useful tools for the effective field theory approach to gravity. The EAA

[1] Notable exceptions are the non-standard perturbative approach in [118–120] and an exact treatment of the restricted functional integral over all metrics with two Killing vector fields [121–123]. (See also [98, 124].)

defines an effective field theory at a certain scale, and the corresponding flow equation can be used to "evolve" it to another scale. In fact, the exploration of effective infrared theories of gravity provided a second strong motivation for the original work [14] independently of Asymptotic Safety.

As we will see explicitly later in this chapter, one of the advantages of the gravitational EAA framework is that it completely decouples the notorious UV problems of quantum gravity from the "foundational" questions related to the search for a suitable notion of an exact renormalization group in gravity, within a fully Background Independent and generally covariant setting. Intuitively speaking the reason for this decoupling is the difference between the functional integral and the FRGE-based quantization methods which we observed already: Working with the functional integral directly, the UV divergences are "defects" of the basic mathematical object itself. The indirect way via the FRGE bypasses these difficulties: The FRGE is perfectly finite, and the UV divergences only resurface at the level of its *solutions*.

For this reason, the problems related to UV divergences will not play an important role in the present chapter. Instead, the focus is on the *conceptual differences* between simple non-gauge matter theories on rigid spacetimes, such as the scalar theory from Chapter 2, and gravity. These differences are mostly due to the requirements of Background Independence and general covariance.

4.1 What is "Metric" Quantum Gravity?

In this chapter we are going to explain all relevant aspects by means of *metric quantum gravity* as the so far best studied example. Thus the first question to be clarified is what exactly should be "metric" in the quantum theory we are aiming at.

A successful quantum theory of gravity should be able to reproduce classical General Relativity in a certain limit. We know that classical gravity can be described in a large variety of physically equivalent but mathematically different ways, employing different field variables for instances (metric, tetrads, spin-connection, etc.). Hence, at the quantum level, there is potentially a similar multitude of different theories, based on different variables. These theories may or may not be equivalent beyond the classical regime. In this sense, the weakest requirement on a theory that deserves being called "metric" would be roughly as follows: It is a quantum theory containing a certain operator $\widehat{\boldsymbol{g}}_{\mu\nu}(x)$, symmetric in μ and ν, which in some states $|\Psi\rangle$ possesses an expectation value $\langle\Psi|\widehat{\boldsymbol{g}}_{\mu\nu}(x)|\Psi\rangle \equiv g_{\mu\nu}(x)$ with the properties of a classical Riemannian metric, i.e., it is a symmetric $(0,2)$ tensor field defining a non-degenerate positive-definite bilinear form.[2]

[2] Unless stated otherwise we assume an Euclidean signature, but the same remarks apply to the Lorentzian case *mutatis mutandis*.

Here it is left open whether $\widehat{\boldsymbol{g}}_{\mu\nu}$ is an elementary field variable, or a composite operator built of elementary fields. The most familiar example of the latter case is the representation $\widehat{\boldsymbol{g}}_{\mu\nu} = \widehat{\boldsymbol{e}}^a{}_\mu \widehat{\boldsymbol{e}}^b{}_\nu \eta_{ab}$ with the tetrad field $\widehat{e}^a{}_\mu$ now considered the elementary one.[3] At this stage we make our definition of "metric quantum gravity" more precise and require that $\widehat{\boldsymbol{g}}_{\mu\nu}$ is *not* a composite operator.

At the level of Lagrangian functional integrals, we would then integrate over commuting functions $\widehat{g}_{\mu\nu}$ which take the place of the metric operator $\widehat{\boldsymbol{g}}_{\mu\nu}$.[4] As a consequence we would now say that "metric quantum gravity," by definition, is a theory defined via an integral of the form $\int \mathcal{D}\widehat{g}_{\mu\nu} \exp(-S)$ in which the commuting analog of the metric operator, $\widehat{g}_{\mu\nu}$, serves as the variable of integration.

Henceforth we shall use the term *metric quantum gravity*, or synonymously *Quantum Einstein Gravity*, in this general and still somewhat vague sense. It is vague not only because the ultimate "theory" will depend on how the UV limit of the functional integral (if any) is taken, but also for the following more mundane reason.

Recall that above we insisted that there exist *some* states $|\Psi\rangle$ in which $g_{\mu\nu}(x) \equiv \langle\Psi|\widehat{\boldsymbol{g}}_{\mu\nu}(x)|\Psi\rangle$ has the properties of a non-degenerate metric. In the functional integral language this expectation value reads

$$g_{\mu\nu}(x) = \int \mathcal{D}\widehat{g}_{\alpha\beta}\, \widehat{g}_{\mu\nu}(x)\, e^{-S[\widehat{g}]} \tag{4.1}$$

with suitable boundary conditions imposed and sources included in S. In general, one would expect that besides the "classically appearing" states $|\Psi\rangle$ there exist also other physical states, say $|\widetilde{\Psi}\rangle$, in which the expectation value of $\widehat{\boldsymbol{g}}_{\mu\nu}$ has no interpretation as a classical metric. For example, $g_{\mu\nu} = \langle\widetilde{\Psi}|\widehat{\boldsymbol{g}}_{\mu\nu}|\widetilde{\Psi}\rangle$ may not be smooth, or the matrix $g_{\mu\nu}$ may degenerate, $\det(g_{\mu\nu}) = 0$, so that neither a volume element $\sqrt{g}$ nor an inverse metric $g^{\mu\nu}$ exists. One could even imagine a state $|\Psi_0\rangle$ in which the expectation value of the metric operator vanishes identically all over spacetime:

$$\langle\Psi_0|\widehat{\boldsymbol{g}}_{\mu\nu}(x)|\Psi_0\rangle = 0. \tag{4.2}$$

We would then expect the gravitational system to be in a *state of unbroken diffeomorphism invariance.*

These terms allude to an analogy with spontaneous symmetry breaking in magnetic systems. The correspondence is between the magnetization density $\widehat{\boldsymbol{M}}$ and the metric $\widehat{\boldsymbol{g}}_{\mu\nu}$, respectively. If they develop an expectation value this can lead to a breakdown of the original vacuum symmetry, namely invariance under spatial SO(3) rotations in the first, and general coordinate transformations in the second case.

[3] We are cavalier here concerning the definition of operator products at the same point.

[4] According to the conventions used throughout this book both integration variables and operators are indicated by a hat. The difference will be clear from the context and no confusion should arise.

From this perspective, gravity with $\langle\Psi|\widehat{\boldsymbol{g}}_{\mu\nu}|\Psi\rangle \equiv \eta_{\mu\nu}$ being equal to the Minkowski metric amounts to a broken phase of the underlying theory. The "vacuum condensate" $\eta_{\mu\nu}$ plays a role analogous to the spontaneous magnetization of a ferromagnet. It leaves a small subset of coordinate transformations unbroken, namely those that form the Poincaré group.

The possibility of an unbroken phase of gravity might appear somewhat "exotic" at first sight, but there are also good reasons to believe that it is actually not, and that the unbroken phase even admits to the simpler description [125]. After all, in typical examples of spontaneous symmetry breaking in particle and condensed matter physics the much harder problem is usually to analyze the broken phase. In the case of gravity we are perhaps biased as we have no experimental or observational access to the unbroken phase.

The reason for mentioning the possibility of an unbroken phase at this point of our discussion is because in the above we agreed that it is plausible to assume that the integral (4.1) for the expectation value $g_{\mu\nu}(x)$ is a bona fide classical metric in some states (i.e., for some boundary conditions, sources in S, etc.), but not in others. Therefore, when we leave the state unspecified, the expectation value $g_{\mu\nu}(x)$ is at best a symmetric *tensor field*, but not a *metric*, since it does not necessarily satisfy the additional requirements of constituting a non-degenerate positive-definite quadratic form.

Thus accepting that the left-hand side of (4.1) is "just" a tensor field without additional properties, it then is natural to ask the question: What precisely are the properties of the functions $\widehat{g}_{\mu\nu}$ on the right-hand side of (4.1) over which we perform the functional integral? Are they metrics, and therefore non-degenerate by definition, or generic tensor fields that can degenerate in all possible ways, or perhaps something in between with only some "mild" form of degeneracy allowed?

Technically speaking it would seem easiest to integrate over generic tensors since they form a vector space. If instead we restrict the integral to true metrics then the domain of integration is a complicated curved submanifold thereof. In the first case "$\mathcal{D}\widehat{g}_{\mu\nu}$" could be a simple translational invariant measure, but in the second case we would have to cover the manifold of all metrics with a nontrivial atlas of coordinate charts and express the functional integral as the sum of their respective contributions [126]. On each chart there are local coordinates $\widehat{p}_1(\,\cdot\,)$, $\widehat{p}_2(\,\cdot\,)$, $\cdots$, $\widehat{p}_{d(d+1)/2}(\,\cdot\,)$ then, themselves functions over spacetime, which parametrize the genuine metrics in a suitable way: $\widehat{g}_{\mu\nu} \equiv \widehat{g}_{\mu\nu}(\widehat{p}_1, \widehat{p}_2, \cdots)$. In this case "$\mathcal{D}\widehat{g}_{\mu\nu}$" would boil down to $\prod_i \mathcal{D}\widehat{p}_i$ times the Jacobian factor implied by the parametrization.

At this point there is no way of answering the question about the "correct" or "best" way of choosing the integration variables. For the time being we therefore subsume all of the above options under the heading of metric quantum gravity. When compared to the direct construction of the functional integral, the flow-equation based approach has the advantage that the choice of integration variables can be deferred until a later stage of development. At this point computational results are already available which may guide us toward the physically interesting choices.

When we try to develop a renormalization group for gravity along the lines of the scalar EAA in Chapter 2, the main role of the functional integral consists of determining the theory space to be considered, i.e., the space $\mathcal{T}$ of all "well-behaved" actions $A[\,\cdot\,]$ on which the RG flow is to take place. In particular, the type of field arguments on which the actions depend and the symmetries get fixed in this manner.

In the case at hand, the natural first guess about the theory space is that its points $A \equiv A[g_{\mu\nu}]$ should be functionals of the expectation value $g_{\mu\nu} \equiv \langle \widehat{g}_{\mu\nu} \rangle$ that are invariant under general coordinate transformations if $g_{\mu\nu}$ is transformed tensorially under such transformations.[5] However, this guess turns out to be too naive ultimately, for reasons which we shall discuss in detail in the next section.

To summarize, let us be explicit about the desired properties of the yet-to-be found theory of metric quantum gravity, or Quantum Einstein Gravity (QEG). These properties will be the *input* for our RG-based search strategy:

(i) The gravitational degrees of freedom are carried by a symmetric tensor field $\widehat{g}_{\mu\nu}$ which acts as an elementary field variable.
(ii) The theory is a gauge theory, covariant under gauge transformations implied by the spacetime diffeomorphisms (general coordinate transformations).
(iii) The theory is Background Independent.

These three properties are inspired by the *formal structure* of classical General Relativity. In fact, they are all that we take over from GR; in particular, the microscopic *dynamics* of $\widehat{g}_{\mu\nu}$ is still completely open. The name Quantum Einstein Gravity refers only to this formal structure but not, for instance, to the use of the Einstein–Hilbert action. It plays no conceptually distinguished role in the Asymptotic Safety program. This is in marked contrast to other quantum gravity programs such as canonical quantum gravity [39].

4.2 What is "Coarse Graining" in Gravity?

In the previous chapter we emphasized that for non-gravitational theories, given a theory space, the form of the FRGE and, as a result, the vector field $\boldsymbol{\beta}$, are basically completely fixed, so that the RG flow and in particular its fixed point structure can be studied without any further input. In the case of gravity, however, it is much harder to make this program work in a concrete way.

In this section we are going to discuss the new problems which arise specifically in quantum gravity when one tries to develop a renormalization group framework

[5] This tentative definition is insensitive to the allowed degeneration properties of $g_{\mu\nu}$, which could be imposed in addition.

by following the route sketched in Chapter 2 that led to the scalar Effective Average Action.

4.2.1 Background Independence

The theory of quantum gravity we are aiming at should be formulated in a Background Independent way. It should *explain* rather than presuppose the existence and the properties of the spacetime we live in. Hence, no special spacetime manifold, in particular no special causal, let alone Riemannian structure, should play a distinguished role at the fundamental level of the theory.

Moreover, we might require the theory to also cover phases in which the metric can degenerate or even have a vanishing expectation value.

Since almost our entire repertoire of quantum field theory methods relies on the availability of an externally given spacetime manifold, usually Minkowski space, this is a severe conceptual problem. In fact, it is *the* central challenge for basically all approaches to quantum gravity, in one guise or another [16, 36–38, 127].

To give an extreme example of the dramatic consequences it may entail in connection with an EAA, let us consider again the unbroken phase of a (hypothetical) metric theory based on a theory space consisting of action functionals $A[g_{\mu\nu}]$ whose domain includes fields $g_{\mu\nu}$ which vanish identically: $g_{\mu\nu}=0$. When we try to write down examples of such actions we soon realize how extremely difficult this is as we have neither a volume element $\sqrt{g}$ nor an inverse matrix $g^{\mu\nu}$ at our disposal.

To get an impression of the problem it suffices to leave the gravitational dynamics aside for a moment and simply try to construct matter Lagrangians, which remain meaningful when $g_{\mu\nu}=0$. This requirement rules out basically all actions we are familiar with from theories like the Standard Model: Scalar potential terms like $\int \mathrm{d}^d x \sqrt{g}\, V(\phi)$ vanish identically, the scalar kinetic term $\int \mathrm{d}^d x \sqrt{g}\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is undefined when $g_{\mu\nu}\to 0$ since $g^{\mu\nu}$ does not exist then, and for the same reason the Yang–Mills action $\int \mathrm{d}^d x \sqrt{g}\, g^{\mu\nu} g^{\alpha\beta} F^a_{\mu\alpha} F^a_{\nu\beta}$ no longer makes sense. Likewise, the Dirac action $\int \mathrm{d}^d x\, (\det e)\, \bar{\psi}\gamma^a e^\mu_a D_\mu \psi$ is ruled out as the inverse tetrad e^μ_a, the "square root" of $g^{\mu\nu}$, is undefined now.

Even if we (tentatively) insist that $g_{\mu\nu}$ must be non-degenerate everywhere, other problems still remain, most importantly those related to the "averaging" of field configurations, an issue we discuss in detail below.

At this point a remark is in order concerning the mathematical structures that constitute a "background", and to what extent the approach to quantum gravity we are going to develop will eventually be independent of those structures.

At the most basic level, "spacetime" for us is a set of points, called the "events," with no further structure. Then, at the next higher, slightly more refined level of description we *assume* that this set is a topological space, and that it carries a differentiable structure. These assumptions will not be proven or justified; their motivation resides solely in their importance for classical physics.

The theory we try to construct will only be able to deal with, and make predictions for, those levels in the description of spacetime that go beyond its (assumed) property of being a differentiable manifold. In particular, it can “decide” about the Riemannian structure this manifold also may or may not carry in addition. The theory renders the Riemannian structure dynamical. In contrast, the topological and the differentiable structures are given a priori and thus cannot be “explained.”

Therefore, strictly speaking a “background”, or more precisely a background spacetime, consists of a topological, differentiable and a Riemannian structure (metric). The form of “Background Independence” we are actually able to achieve refers to independence with respect to a (preferred) metric only. All of the following discussions of asymptotically safe gravity are based on an *externally given manifold with unchangeable topological and differential structure.*

4.2.2 Gauge Invariance

The theory we are searching for is supposed to share a number of properties with GR: The elementary field carrying the gravitational degrees of freedom is a tensor field $g_{\mu\nu}$, it is a gauge theory covariant under spacetime diffeomorphisms, and it is Background Independent. In this section we discuss specifically the implications of gauge invariance.

(1) General coordinate transformations. In Section 4.1 we already raised the question about how the gauge theory aspects affect the correct choice of the theory space of QEG, $\mathcal{T} \equiv \mathcal{T}_{\text{QEG}}$. The gauge transformations in question are (infinitesimal) coordinate transformations generated by arbitrary vector fields v^μ which act on $g_{\mu\nu}$ by the Lie derivative $\mathcal{L}_v$:

$$\boxed{\delta g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu} \equiv v^\rho \partial_\rho g_{\mu\nu} + \partial_\mu v^\rho g_{\rho\nu} + \partial_\nu v^\rho g_{\rho\mu}.} \tag{4.3}$$

Again, referring back to General Relativity, we observe that the underlying Einstein–Hilbert action $S_{\text{EH}}[g_{\mu\nu}]$ is gauge invariant in the sense that $\delta S_{\text{EH}} = 0$ under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, and we are thus motivated to define $\mathcal{T}_{\text{QEG}}$ as the theory space of all invariant functionals:

$$\left\{ A[g_{\mu\nu}] \middle| \delta A = 0 \text{ under } \delta g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu} \right\} \equiv \mathcal{T}_{\text{inv}}. \tag{4.4}$$

Natural and simple as it seems, this choice would cause considerable problems once we try to set up a gravitational RG framework along the lines of the scalar EAA.

(2) Zero modes and FRGE. To see the origin of the difficulties, recall that in the scalar theory the k-dependence of the EAA is governed by the Wetterich equation $\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\Big[\big(\Gamma_k^{(2)} + \mathcal{R}_k\big)^{-1}\partial_t \mathcal{R}_k\Big]$. If we try to apply an analogous equation to a running action $\Gamma_k \in \mathcal{T}_{\rm inv}$ on our tentative theory space (4.4) we find immediately that there is a clash between the required diffeomorphism invariance of $A[g_{\mu\nu}]$ and the general structure of this FRGE. This clash occurs even if we restrict $g_{\mu\nu}$ to be non-degenerate.

The reason is as follows. Naively replacing $\Gamma_k[\phi]$ by some $\Gamma_k[g_{\mu\nu}]$, the Hessian operator $\Gamma_k^{(2)}$ under the trace in the FRGE is concretely given by the second derivatives *of a diffeomorphism invariant functional*, $\frac{\delta^2}{\delta g_{\mu\nu}(x)\delta g_{\alpha\beta}(y)}\Gamma_k[g]$. This implies that $\Gamma_k^{(2)}$ will possess zero modes if it is evaluated "on shell." There exists an infinite-dimensional subspace of the space traced over on which $\Gamma_k^{(2)}$ is zero effectively and thus contributes an unacceptable divergent piece $\mathrm{Tr}\big[\mathcal{R}_k^{-1}\partial_t \mathcal{R}_k\big] = \mathrm{Tr}\big[\partial_t \ln \mathcal{R}_k\big]$ to the right-hand side of the flow equation.

Those zero modes are a direct consequence of diffeomorphism invariance: Consider an arbitrary invariant functional $A[g]$. It satisfies $\delta A \equiv A[g+\delta g] - A[g] = 0$, which, after Taylor expanding to first order in v^μ, implies

$$\int \mathrm{d}^d x \frac{\delta A[g]}{\delta g_{\mu\nu}(x)} \mathcal{L}_v g_{\mu\nu}(x) = 0. \tag{4.5}$$

Taking another functional derivative of (4.5) yields

$$\int \mathrm{d}^d x \frac{\delta^2 A[g]}{\delta g_{\alpha\beta}(y)\delta g_{\mu\nu}(x)} \mathcal{L}_v g_{\mu\nu}(x) = -\int \mathrm{d}^d x \frac{\delta A[g]}{\delta g_{\mu\nu}(x)} \frac{\delta}{\delta g_{\alpha\beta}(y)} \mathcal{L}_v g_{\mu\nu}(x). \tag{4.6}$$

This equation holds true for all tensor fields $g_{\mu\nu}$. Let us now specialize them for the stationary points of A, i.e., those satisfying the "equation of motion" $\frac{\delta A[g]}{\delta g_{\mu\nu}} = 0$. For them,

$$\int \mathrm{d}^d x \frac{\delta^2 A[g]}{\delta g_{\alpha\beta}(y)\delta g_{\mu\nu}(x)} \mathcal{L}_v g_{\mu\nu}(x) = 0. \tag{4.7}$$

This equation expresses the fact that "on shell," i.e., when the corresponding equations of motion are fulfilled, the Hessian of an invariant functional A has eigenvectors with eigenvalue zero, namely the "gauge modes" $\mathcal{L}_v g_{\mu\nu}$ stemming from arbitrary vector fields v^μ. This is what we wanted to show.

(3) Gauge fixing. The above reasoning is the gravitational analog of a well-known argument in quantum electrodynamics and Yang–Mills theory. It suggests that some sort of gauge-fixed flow equation is called for that does not suffer from the problems due to gauge zero modes.

We take this as motivation to reconsider the derivation of a flow equation, but this time starting out from a *gauge fixed functional integral.*

In principle, it is straightforward to use the Faddeev–Popov trick in order to "divide out" the redundant integration over gauge equivalent fields. The method parallels that in quantum electrodynamics or Yang–Mills theory, and like there, the extended bare action that emerges in this way is invariant under appropriate BRST transformations, which in turn give rise to certain Ward identities.

As we will see, *the BRST Ward identities constitute the symmetry constraint that codetermines the elements of the actual theory space* $\mathcal{T}_{\text{QEG}}$.

(4) Manifest invariance of $\boldsymbol{\Gamma}$. Opting for a gauge-fixed functional integral as the starting point for the derivation of a flow equation, while solving one problem, leads to a new one, namely the issue of *manifest* gauge invariance of the actions $A \in \mathcal{T}_{\text{QEG}}$.

Considering generic functionals $A[T_1, T_2, T_3, \cdots]$, which may depend on any set of tensor fields $T_1, T_2, \cdots$ we say that A is *manifestly gauge invariant* if it does not change under the transformations where all the tensor arguments simultaneously are changed by an amount $\delta T_i = \mathcal{L}_v T_i$, $i = 1, 2, \cdots$.[6]

For example, if $A \equiv A[g_{\mu\nu}]$ as before, then the functionals in $\mathcal{T}_{\text{inv}}$ of (4.4) are manifestly gauge invariant in this sense.

Let us leave the functional RG aside for a moment, and let us also sidestep the difficulties related to Background Independence by postulating some ad hoc background metric (for flat Euclidean space, say) relative to which we quantize the fluctuations of $\widehat{g}_{\mu\nu}$. The problem that arises then is that if we straightforwardly apply the Faddeev–Popov trick to $\int \mathcal{D}\widehat{g}_{\mu\nu}\, e^{-S}$ and follow the lines of typical perturbative calculations in particle physics [128], we end up with an 1PI-generating functional, an effective action $\Gamma[g_{\mu\nu}]$, which is *not manifestly gauge invariant.*

The reason is exactly the same as in the standard and perhaps better-known functional integral quantization of Yang–Mills theories. If one gauge fixes the functional integral with an ordinary (covariant) gauge-fixing condition like $\partial^\mu A^a_\mu = 0$,[7] couples the non-abelian gauge field A^a_μ to a source, and constructs the ordinary effective action, the resulting functional $\Gamma[A^a_\mu]$ is *not* invariant under the gauge transformations of A^a_μ, $A^a_\mu \mapsto A^a_\mu + D^{ab}_\mu(A)\,\omega^b$. The straightforwardly defined $\Gamma[A^a_\mu]$ communicates the fixed gauge from the bare to the effective level [128].

Only at the level of the observable quantities deduced from $\Gamma[A^a_\mu]$ (S-matrix elements, for instance) gauge invariance is recovered. So from the physical point of view there is nothing wrong about the non-invariance of $\Gamma[A^a_\mu]$. All gauge-variant features of $\Gamma[A^a_\mu]$ are forgotten along the way to the observables. In perturbative particle physics calculations one is therefore, more often than not, willing to live with this non-invariance as only a minor deficiency of $\Gamma[A^a_\mu]$.

[6] The explicit form of the Lie derivative $\mathcal{L}_v$ depends on the type of the tensor, of course. Fields T_i with odd Grassmann parity are included here as well.

[7] In this section indices $a, b, \cdots$ refer to the adjoint representation of the Yang–Mills gauge group. Gauge group generators in some given representation are denoted T^a. Our conventions follow [88]. In particular the generators T^a, satisfying $[T^a, T^b] = i f^{abc} T^c$, are taken hermitian.

In a non-perturbative functional RG setting the requirements are somewhat different though. Assume we manage in some way to introduce a scale-dependent version of this effective action, $\Gamma_k[A_\mu^a]$, then it will also not display manifest gauge invariance either. When it comes to computing RG flows this is a severe disadvantage, however, since one can build many more gauge non-invariant action monomials from A_μ^a than invariant ones. Therefore, the structure of the RG flow is much simpler in an invariant than in a non-invariant framework. In the former case, truncations that achieve a given level of accuracy will require much fewer terms than in the latter.

These observations apply to gravity and Yang–Mills theory alike. The BRST quantization of $\widehat{g}_{\mu\nu}$ relative to a fixed background metric leads to a $\Gamma[g_{\mu\nu}]$, which is not manifestly gauge invariant. Introducing an ad hoc IR cutoff in this situation can make things even worse.

In the later chapters of this book we will see repeatedly that the limiting factor of practical RG calculations in quantum gravity is always the theory's enormous algebraic complexity. For this reason one should take advantage of all available devices that can simplify the structure of the RG flow and its truncations.

Motivated by these remarks, we decide to "repair" the non-invariance of the naive $\Gamma[g_{\mu\nu}]$. In the following we describe a scale-dependent action Γ_k which enjoys manifest gauge invariance.

We emphasize already at this point that the manifest gauge invariance is not a substitute for the BRST Ward identities mentioned in **(3)**. *The actions belonging to $\mathcal{T}_{\text{QEG}}$ will be constrained by manifest gauge invariance and the Ward identities independently.*

4.2.3 Toward a Notion of Coarse Graining

A particularly hard problem in gravity consists in establishing a suitable generalized concept of "coarse graining" which respects Background Independence and diffeomorphism invariance. Deeply rooted conceptually, the problem pervades both the continuum and discrete approaches.[8]

A functional RG equation which implements the, yet to be found, new generalized concept of coarse graining should, on the one hand, be as simple as possible, but on the other hand should also involve a type of scale-dependent action which is as close to observable physics as possible; "simple," i.e., computationally feasible approximations, should suffice in order to extract the physical essence of the RG flow.

Before embarking on gravity we consider Yang–Mills theory on a flat Euclidean spacetime in order to disentangle the problems arising from gauge (diffeomorphism) invariance, from those due to Background Independence.

[8] See [129–131] for a corresponding discussion within Loop Quantum Gravity.

In Yang–Mills theory a "coarse graining" based on a naive Fourier decomposition of $A^a_\mu(x)$ is not gauge covariant and hence cannot be physically meaningful. In fact, if one were to gauge transform a slowly varying $A^a_\mu(x)$ using a parameter function $\omega^a(x)$ with a fast x-variation, a gauge field with a fast x-variation would arise which, however, still describes the same physics.

In a non-gauge theory coarse graining is usually performed by expanding the field in terms of eigenfunctions of the (positive) operator $-\partial^2$ and declaring its eigenmodes to be of "long" or "short" wavelength depending on whether the corresponding eigenvalue p^2 is smaller or larger than a given k^2.

In a Yang–Mills theory there is an obvious clash between this way of distinguishing "coarse" and "fine" and the basic requirement of gauge covariance, since the ordinary Laplacian ∂^2 is not covariant under gauge transformations.

It seems therefore natural that in a gauge theory ∂^2 should be replaced by the *covariant Laplacian* $D^2 = \delta^{\mu\nu} D_\mu D_\nu$. It involves the covariant derivative $D_\mu \equiv \partial_\mu - iA_\mu$ with the Lie algebra-valued field $A_\mu \equiv A^a_\mu T^a$ in a, for the time being, arbitrary representation of the gauge group generators T^a.

While from a differential geometric point of view replacing ∂^2 by D^2 is only too natural, it ruins to some extent the possibility of intuitively grasping the difference between a "fine" and a "coarse" field configuration. Our intuition for such matters stems from the classical example of Fourier analysis and synthesis that is naturally connected to the operator ∂^2 with its plane-wave eigenfunctions. But this "free field"-like situation is atypical as the relevant operator, ∂^2, happens to be *field independent*. Its non-abelian pendant $D^2 \equiv (\partial_\mu - iA_\mu)^2$ has an explicit dependence on the dynamical field A_μ instead.

Moreover, when we go on along this route and consider the eigenvalue problem of the field-dependent operator,

$$-D^2(A)\,\psi_n = E_n\,\psi_n \tag{4.8}$$

we are confronted with eigenfunctions $\psi_n \equiv \psi_n(A)$ and eigenvalues $E_n \equiv E_n(A)$ which have inherited the parametric dependence on the gauge field. This field dependence causes a number of severe problems, as we discuss next.

(1) Mode functions. Gauge covariance comes at the price of eigenfunctions that are no longer simple plane waves, but rather reflect the details of the underlying gauge-field configuration, including its gauge dependence. Typically the ψ_ns will not possess any (translation, rotation, etc.) symmetries, unlike the plane waves that are momentum eigenstates. Moreover, the relation between the eigenvalue E_n and the position space dependence displayed by the associated ψ_n is usually far from obvious. In the case of $-\partial^2$ the eigenvalue $E \equiv p^2$ of a given eigenfunction can easily be read off "with the bare eye" from the single length scale featured by the corresponding $\psi(x) \propto \exp(ip \cdot x)$, namely its wavelength $2\pi/\sqrt{p^2}$. For a generic $\psi_n(A)$ this is usually impossible.

(2) Ordered sets of modes. Let us assume now that a meaningful notion of nonabelian coarse graining can be based on the identification

$$\text{“fine vs. coarse”} \longleftrightarrow \text{“large vs. small eigenvalues of } -D^2(A)\text{”}. \tag{4.9}$$

For A_μ fixed, we order the (real, positive) eigenvalues[9] of $-D^2(A)$ in an ascending sequence and determine the concomitant ordering of the eigenfunctions:

$$0 \leq E_1(A) \leq E_2(A) \leq \cdots \leq E_n(A) \leq \cdots \Rightarrow \psi_1(A), \psi_2(A), \cdots, \psi_n(A), \cdots. \tag{4.10}$$

As n increases, the associated sequence of eigenfunctions proceeds from ψs considered “coarse” to increasingly “fine” ones.

This sequence of eigenfunctions can be thought of as an ordered set[10] of “templates,” which we can match against the field configuration to be coarse grained.

This concept of coarse vs. fine is a gauge-invariant notion. In fact, if ψ_n is an eigenfunction with eigenvalue E_n for the gauge field A_μ, then $\psi'_n = U\psi_n$ is an eigenfunction with the same eigenvalue for the gauge-transformed field $A'_\mu = UA_\mu U^{-1} - i(\partial_\mu U)U^{-1}$ with $U \equiv \exp(-i\omega^a(x)T^a)$. As a consequence, the order of the ψs in (4.10) depends only on the gauge equivalence class A_μ belongs to.

(3) Field-dependent bases. Under certain technical conditions the eigenfunctions obtained with a given gauge field, $\{\psi_n(A)\}$, form a complete set. Hence, each A_μ gives rise to its own basis.

We may use any of the bases $\{\psi_n(A)\}$ in order to expand arbitrary matter fields $\Psi(x)$ that are coupled to A_μ, provided they transform as $\Psi \mapsto \Psi' = U\Psi$, i.e., homogeneously and in the T^a-representation. Then,

$$\Psi(x) = \sum_n c_n(A)\, \psi_n(x; A). \tag{4.11}$$

By assumption, Ψ is independent of A_μ. Therefore, its expansion coefficients $c_n(A)$ must depend on the gauge field in order to compensate for the A_μ-dependence of the basis functions.

Given the expansion (4.11), a natural notion of *coarse-graining matter fields* emerges: It simply consists in truncating the summation in (4.11) at some large $n = n_{\text{max}}$, thus omitting extremely “fine” basis functions from the expansion.

Conversely, we can introduce the IR cutoff necessary for the formulation of an FRGE by omitting from the sum (4.11) the “coarse” terms below a certain $n = n_{\text{min}}$.

[9] For the sake of this argument we assume a discrete spectrum.

[10] In the case of degenerate eigenvalues it is only partially ordered. In general, degeneracies are unlikely to occur however since a generic field configuration $A^a_\mu(x)$ will break all symmetries.

Note that we cannot apply this strategy to the Yang–Mills field itself. Being a connection, it transforms inhomogeneously under gauge transformations, and so unlike the matter fields Ψ, the gauge field admits no covariant expansion of the form (4.11).

The actually difficult problem in Yang–Mills theory is finding a suitable notion of coarse graining or averaging for the connection. Before returning to it let us take a closer look at the matter fields.

(4) Invariant cutoff for matter fields. Based on this definition of "fine vs. course" it is now straightforward to introduce an EAA for quantum matter fields Ψ interacting with an *external*, i.e., a classical gauge field that is not quantized. The relevant partition function

$$Z^{\mathrm{M}}[A] = \int \mathcal{D}\widehat{\Psi}\ e^{-S^{\mathrm{M}}[\widehat{\Psi}, A]} \tag{4.12}$$

is invariant under gauge transformations of A_μ if $S^{\mathrm{M}}[\widehat{\Psi}, A]$ is invariant under combined $\widehat{\Psi}$- and A-transformations (in absence of anomalies).

Following Chapter 2, it is not difficult to equip the integral over $\widehat{\Psi}$ with a regulator term which suppresses the IR modes, *while preserving its gauge invariance*:

$$Z_k^{\mathrm{M}}[A] = \int \mathcal{D}\widehat{\Psi}\, e^{-S^{\mathrm{M}}[\widehat{\Psi}, A]}\, e^{-\Delta S_k^{\mathrm{M}}[\widehat{\Psi}, A]}. \tag{4.13}$$

Every cutoff action of the type

$$\Delta S_k^{\mathrm{M}}[\widehat{\Psi}, A] = \frac{1}{2} k^2 \int \mathrm{d}^d x \left(\widehat{\Psi}, R^{(0)}\big(-\tfrac{D^2(A)}{k^2}\big)\, \widehat{\Psi} \right), \tag{4.14}$$

where the covariant derivatives are in the representation of $\widehat{\Psi}$, and $(\,\cdot\,,\,\cdot\,)$ denotes an invariant inner product, will retain gauge invariance. Furthermore, $R^{(0)}(\,\cdot\,)$ is a shape function obeying (2.65). As a result, the factor $e^{-\Delta S_k}$ suppresses precisely the $\widehat{\Psi}$-modes with eigenvalues $E_n \lesssim k^2$.

So we see that for the matter fields it is indeed possible to implement a covariant notion of coarse graining at the (formal) functional integral level. Moreover, it is straightforward to start out from Z_k^{M}, define the corresponding EAA, and derive its FRGE. The steps parallel those for the scalar theory, and lead to a (parametrically A-dependent) flow equation with the same structure as in the scalar case.

For later comparison we emphasize that the RG flow obtained by varying the parameter k in (4.13), or the equivalent EAA, admits a clearcut interpretation:

(i) The *absolute* scale of the cutoff parameter k enjoys an invariant meaning: it is a particular point in the gauge-invariant spectrum of a gauge-covariant operator.

(ii) Concerning the significance of the *relative* order of any two scales k_1 and $k_2 < k_1$, we can be sure that when we lower the cutoff and proceed from the UV to the IR the mode with eigenvalue $E = k_1^2$ is integrated out first, the other with k_2^2 second. Decreasing the parameter k from infinity down to zero, the individual modes of Ψ are integrated out[11] exactly according to the fine vs. coarse classification in (4.10), from right to left.

While at first sight they seem relatively trivial and self-evident, these two properties are quite "precious" actually. They do not readily generalize to Yang–Mills and gravitational fields, as we shall see in a moment.

Again for later comparison, let us also note that the second property above is seen most explicitly by rewriting the functional integral over the functions Ψ as an integration over all their expansion coefficients $\{c_n\}$ appearing in (4.11). Formally this amounts to a unitary transformation, with a unit Jacobian, for any basis $\{\psi_n(A)\}$, i.e., any gauge field A_μ. Hence, somewhat symbolically,

$$Z_k^{\mathrm{M}}[A] = \int \prod_n \mathrm{d}c_n \, e^{-S^{\mathrm{M}}[\{c_n\},A]} \, e^{-\frac{1}{2}k^2 \sum_n R^{(0)}\left(\frac{E_n}{k^2}\right) c_n^2}. \tag{4.15}$$

The second exponential in this equation makes it manifest that the mode ψ_n gets unsuppressed exactly when k^2 passes the eigenvalue E_n.

(5) The core of the problem. The proposed notion of coarse graining applies only to fields transforming homogeneously under gauge transformations, $\Psi' = U\Psi$, thus excluding the most interesting case of the gauge field itself.

A first idea that comes to mind is that, perhaps, one should coarse grain a given A_μ-configuration indirectly via the field strength $F_{\mu\nu} \equiv F_{\mu\nu}^a T^a$. This has the advantage that $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$ transforms homogeneously with U in the adjoint representation of the gauge group. Being given some A_μ field configuration, we would then compute its field strength $F_{\mu\nu} \equiv F_{\mu\nu}(A)$, coarse grain it in some way to obtain a certain $F_{\mu\nu}^{\text{coarse}}$, and then search for a new connection A_μ^{coarse} which reproduces it, $F_{\mu\nu}^{\text{coarse}} = F_{\mu\nu}(A_\mu^{\text{coarse}})$.

Unfortunately, at least for individual A_μ configurations, this simple idea is not viable. At this point we mention only one of several reasons: In non-abelian gauge theories, contrary to abelian ones, not all gauge potentials A_μ that produce a given $F_{\mu\nu}$ are gauge equivalent, i.e., can be related by a gauge transformation. Because of this so-called *Wu–Yang ambiguity* [132] the computed $F_{\mu\nu}^{\text{coarse}}$ fails to determine A_μ^{coarse} uniquely modulo gauge transformations.

A second approach one might try is to give up the ambition of coarse graining individual configurations as we did for $\widehat{\Psi}$, but rather jump ahead and modify the functional integral over $\widehat{A}_\mu^a$ right away by some sort of nonstandard mode suppression term $\Delta S_k^{\mathrm{YM}}[\widehat{A}]$. The hope would be that a clever choice of $\Delta S_k^{\mathrm{YM}}[\widehat{A}]$

[11] Recall that getting "integrated out" has the concrete meaning of "becoming unsuppressed" when $R^{(0)}$ crosses over from $R^{(0)} \approx 1$ to $R^{(0)} \approx 0$.

might give rise to an interesting and useful RG flow, even though the separation of modes (suppressed vs. unsuppressed) might not have much to do with their "fineness" but perhaps some other feature.

Since the cutoff action should be gauge invariant, the main building blocks are the field-strength tensor and covariant derivatives acting upon it. A simple example is

$$\Delta S_k^{\mathrm{YM}}[\widehat{A}] = \frac{1}{4} k^2 \int \mathrm{d}^d x\, F^a_{\mu\nu}(\widehat{A}) R^{(0)}\big(-\tfrac{D^2(\widehat{A})}{k^2}\big)^{ab} F^{b\mu\nu}(\widehat{A}) \tag{4.16}$$

with $R^{(0)}$ a shape function with the standard characteristics. The cutoff action (4.16) is perfectly gauge invariant. Adding it to the bare action $S[A]$ we are led to the functional integral (before gauge fixing)

$$Z_k^{\mathrm{YM}} = \int \mathcal{D}\widehat{A}\, e^{-S[\widehat{A}]}\, e^{-\Delta S_k^{\mathrm{YM}}[\widehat{A}]}. \tag{4.17}$$

At first sight this scale-dependent integral seems to be similar to (4.13) for matter fields with $F_{\mu\nu}(\widehat{A})$ in place of $\widehat{\Psi}$. However, as soon as we try to base a full-fledged functional renormalization on it, severe difficulties emerge.

The first problem concerns the interpretation of (4.17). What is it really that gets suppressed by ΔS_k^{YM}, and what is the significance of some value of k being smaller than another one?

For a fixed configuration $\widehat{A}_\mu$ contributing to the integral we might solve the tensorial eigenvalue equation for $-D^2(\widehat{A})$ in the adjoint representation of the gauge group, the one appropriate for $F_{\mu\nu}$, and then imitate (4.11) and expand it; symbolically,

$$F^a_{\mu\nu}(\widehat{A}) = \sum_n c_n(\widehat{A})\, \psi_n(\widehat{A})^a_{\mu\nu}. \tag{4.18}$$

This time *the same* field $\widehat{A}_\mu$ enters both the operator $D^2(\widehat{A})$ whose eigenbasis $\{\psi_n(\widehat{A})\}$ is used for the expansion and the function to be expanded. This completely obscures the interpretation of such an eigenfunction expansion, if any, and suggests again that the indirect coarse graining of an $\widehat{A}_\mu$ via its $F_{\mu\nu}(\widehat{A})$ is a dead end.

More importantly, when we try to functionally integrate over the gauge field, we clearly cannot trade the integration variable $\widehat{A}_\mu(x)$ for the collection of expansion coefficients $\{c_n\}$, as we did for $\widehat{\Psi}$ in the derivation of (4.15). However, as we discussed in the previous section, only after bringing the partition function to the form (4.15) can we get a clear understanding of the order in which field configurations are integrated out along RG trajectories, cf. property (ii) in **(4)** above. Concerning Z_k^{YM} it is meaningless to say that "at scale k^2 the eigenmodes with eigenvalue $E = k^2$ are integrated out", since *at each point of the domain of integration the pertinent operator $D^2(\widehat{A})$ is a different one.*

Even if we decide to ignore these interpretational issues and go ahead, trying to derive an FRGE on the basis of Z_k^{YM}, we soon realize that the present approach leads to a disaster from the practical point of view.

The reason is that $\Delta S_k^{\rm YM}[\widehat{A}]$, contrary to its scalar precursor, is no longer *bilinear in the integration variable* $\widehat{A}_\mu$. Higher order terms in $\widehat{A}_\mu$ arise both because of the $\widehat{A}$-dependence in the argument of $R^{(0)}$ and the $\partial\widehat{A}+\widehat{A}^2$ structure of the nonabelian field strength. As a consequence $\partial_t \Delta S_k[\widehat{A}]$, too fails to be bilinear, and so it is impossible to express its expectation value $\langle \partial_t \Delta S_k[\widehat{A}]\rangle$, the analog of (2.16), solely in terms of the two-point function and its inverse, $\widetilde{\Gamma}_k^{(2)} \equiv \Gamma_k^{(2)} + \mathcal{R}_k$. Recall that this possibility was instrumental in the derivation of the scalar FRGE, cf. the steps leading from (2.16) to (2.20). In fact, the bilinearity of ΔS_k was the deeper reason why the FRGE turned out to contain only second-order functional derivatives and no higher ones.

Employing instead our candidate (4.16) or any similar gauge invariant cutoff action, we can reexpress $\langle \partial_t \Delta S_k[\widehat{A}]\rangle$ in terms of Γ_k itself at best at the price of utilizing not only $\Gamma_k^{(2)}$ but also the plethora of all higher n-point functions $\Gamma_k^{(n)}$. This would amount to an extremely complicated FRGE in which arbitrarily high derivatives $\frac{\delta^n \Gamma_k}{\delta A \cdots \delta A}$ appear on the right-hand side.

Obviously there is a rather generic clash between a simple structure of the FRGE, building on the propagators derived from Γ_k only, and gauge invariance, which enforces the presence of the quantum field in places where it does not appear in non-gauge theories.

Since the prospects for making sense of an infinite-order functional differential equation seem to be rather dim, progress will depend on overcoming this *"infinite-order problem"* in one way or another.

(6) Gravity: problems and options. So far we have considered separately the typical problems that arise from the requirements of Background Independence and gauge invariance. If we now return to our main goal, finding a generalization of the Wilsonian renormalization group that continues to make sense in the realm of quantum gravity, we must deal with both types of difficulties in combination.

The overall problem presents itself somewhat differently in typical discrete and continuum approaches.

(6a) Discrete approaches to quantum gravity based on any sort of "atoms of spacetime" exhibit a certain similarity to the many body problems of condensed matter physics. Concepts of the exact RG naturally lend themselves to a description of the (real or purely mathematical) process in which large clusters of atoms build up iteratively by the aggregation of further such atoms. There is a clearcut notion of an RG step and hence of the direction of the RG flow in this case. Determining the degree of "infraredness" of a given cluster, i.e., its position on the RG trajectory, is thus a simple matter of *counting*.

The same is true if one considers an infinite "solid" or "crystal" made of such atoms and follows the familiar idea of block spin transformations. One would combine a few of the atoms into blocks, determine the properties of the entire blocks as implied by the given properties of the atoms, and then declare those blocks with their "renormalized" properties to be the new "atoms" for the next RG step. Again, if the rules for the "blocking" step are given, a clear notion of an exact RG flow emerges. The direction of the RG trajectories defines a *relative* notion of coarse vs. fine blocks, and also here, finding their location on the trajectory only requires elementary counting, of the number of blocking steps in this case.

As these two examples illustrate, within appropriate discrete formulations of quantum gravity it may be rather obvious, at least in principle, how to formulate a natural notion of a coarse-graining step and the resulting renormalization group *without invoking ad hoc background structures.*

It goes without saying that it is by no means straightforward to implement this kind of RG transformations at a technically explicit level in spin-foam models or similar theories, but first results have been obtained [133, 134].

(6b) Let us now return to the continuum-based brand of the exact renormalization group, the functional RG, and more concretely, let us try to modify formal functional integrals like $\int \mathcal{D}\widehat{g}_{\mu\nu}\, e^{-S[\widehat{g}]}$ by some, yet-to-be found Background Independent and generally covariant "mode" suppression factor with similar properties as $e^{-\Delta S_k}$ in the scalar case.

Unquestionably, it seems natural to replace the standard Laplacian $\partial^2 \equiv \delta^{\mu\nu}\partial_\mu\partial_\nu$ with a covariant one, something like, say, $\mathsf{D}^2 \equiv \mathsf{g}^{\mu\nu}\mathsf{D}_\mu\mathsf{D}_\nu$. This immediately raises the questions: what is the tensor $\mathsf{g}^{\mu\nu}$ and how should one define the connection entering the covariant derivative D_μ? Taking Background Independence literally, the only building block available for this construction is the tensor $\widehat{g}_{\mu\nu}$ itself.

The simplest option is to identify $\mathsf{g}^{\mu\nu}$ with the inverse of the matrix $\widehat{g}_{\mu\nu}$ and $\mathsf{D}_\mu \equiv D_\mu(\widehat{g})$ with the associated Levi–Civita connection. This would make D^2 the Laplace–Beltrami operator related to the dynamical field, the integration variable $\widehat{g}_{\mu\nu}$:

$$D^2(\widehat{g}) \equiv \widehat{g}^{\mu\nu} D_\mu(\widehat{g}) D_\nu(\widehat{g}). \tag{4.19}$$

This is not a viable road though if our ambition is to functionally integrate over all symmetric tensor fields $\widehat{g}_{\mu\nu}$, and not only metrics that are non-degenerate by definition. Above we argued that it is desirable to be open-minded about including arbitarily degenerate, i.e., non-invertiable $\widehat{g}_{\mu\nu}$s into the integration, but for those (4.19) is undefined.

Thus, let us be modest at this point and assume that it is actually meaningful to integrate over genuine metrics only, so that (4.19) does make sense. Then we can proceed as in Yang–Mills theory, and consider the eigenvalue problem of $D^2(\widehat{g})$, acting on tensor fields of any type:

$$-D^2(\widehat{g})\,\psi_n = E_n\,\psi_n. \tag{4.20}$$

Leaving technical issues aside, the eigenfunctions $\{\psi_n(\widehat{g})\}$ form a metric-dependent basis, and we may expand arbitrary matter fields as

$$\Psi(x) = \sum_n c_n(\widehat{g})\,\psi_n(x;\widehat{g}). \tag{4.21}$$

With this expansion we have all tools that are needed to introduce an EAA for matter fields interacting with an external gravitational field, via

$$Z^{\mathrm{M}}[\widehat{g}] = \int \mathcal{D}\widehat{\Psi}\; e^{-S^{\mathrm{M}}[\widehat{\Psi},\widehat{g}]}. \tag{4.22}$$

The steps are analogous to those in Yang–Mills theory leading from (4.12) to the representation (4.15) of the modified partition function. The latter representation was instrumental in establishing the absolute and relative meaning of the cutoff scale k, and the discussion is perfectly analogous for quantum matter fields interacting with external gravitational rather than Yang–Mills fields.

While this leads to the desired diffeomorphism-invariant notion of coarse graining matter fields, the other key problem, Background Independence, is not addressed yet since from the point of view of Ψ, the field $\widehat{g}_{\mu\nu}$ is a given, fixed classical metric.

(6c) Thus we are back to the core of the problem, the coarse graining of $\widehat{g}_{\mu\nu}$ itself. At first sight the situation seems more favorable than in Yang–Mills theory, since in metric gravity the elementary field variable transforms under the pertinent gauge transformations as a tensor rather than a connection. Hence, as far as gauge invariance is concerned, nothing would forbid us to write down straightforward mode-suppression terms like

$$\Delta S_k \propto k^2 \int \mathrm{d}^d x \sqrt{\widehat{g}}\, \widehat{g}^{\mu\nu} R^{(0)}\big(-\tfrac{D^2(\widehat{g})}{k^2}\big)\, \widehat{g}_{\mu\nu} \tag{4.23}$$

since D^2 can act on $\widehat{g}_{\mu\nu}$ directly. This is an illusion, though, since the covariant derivative employed is metric-compatible with $\widehat{g}_{\mu\nu}$, and thus $R^{(0)}$ gets evaluated at zero argument for *any* metric. Using (2.65), $R^{(0)}(-\frac{D^2(\widehat{g})}{k^2})\,\widehat{g}_{\mu\nu} = \widehat{g}_{\mu\nu}$. Hence, (4.23) is actually nothing more than a contribution to the cosmological constant.

(6d) Another option is to suppress $\widehat{g}_{\mu\nu}$-configurations in the functional integral according to their curvature; for instance, using terms involving the Riemann tensor, like

$$\Delta S_k \propto k^2 \int \mathrm{d}^d x \sqrt{\widehat{g}}\, R(\widehat{g})^{\mu\nu\rho\sigma} R^{(0)}\big(-\tfrac{D^2(\widehat{g})}{k^2}\big)\, R(\widehat{g})_{\mu\nu\rho\sigma}. \tag{4.24}$$

Indeed, it would be quite interesting, and could teach us a lot about quantum gravity to investigate the k-dependence of the modified partition function $Z_k = \int \mathcal{D}\widehat{g}_{\mu\nu} \exp\big(-S[\widehat{g}] - \Delta S_k[\widehat{g}]\big)$ with cutoff actions like (4.24).

However, in order to explore this sort of RG flow the typical strategies of the scalar EAA approach are of little help: ΔS_k depends on the dynamical fields, the components of $\widehat{g}_{\mu\nu}$, in a complicated non-polynomial manner, and so an FRGE derived from Z_k that is modeled along the scalar example would not obey a sufficiently simple, practically useful FRGE with only finitely many functional derivatives. It would suffer from the same *infinite-order problem* we encountered already in Yang–Mills theory.

4.2.4 The Background Field Method

In the preceding sections we outlined the difficulties and obstacles specific to quantum gravity. They arise from the basic requirements of Background Independence and general covariance, and also because no distinguished, natural notion of coarse graining suggests itself in this setting.

We saw that there is no unique ideal way to define *the* renormalization group for quantum gravity that would apply to all, or at least several of the different approaches one would like to compare on the basis of their RG properties.

At a concrete technical level the options for defining what “coarse graining” is to be are quite different in discrete- and continuum-based approaches. But even when we focus only on continuum approaches and search for a physically meaningful scale-dependent functional “Γ_k” roughly similar to the scalar EAA, there is still a variety of routes one may follow.

Experience has shown that the most useful definition of “Γ_k” hardly can be found by abstract reasoning but rather as a subtle *compromise between desirable formal properties of the functional on one side and its practical computability on the other*. As such, it can only emerge in a process of trial and error. In the rest of this chapter we describe the result of this process, the gravitational Effective Average Action proposed in [14].

(1) Three problems, one solution. The construction of the gravitational EAA begins with a critical decision with far-reaching consequences, namely the decision to approach the Background Independence problem using the *background field method*.

Surprisingly enough, what at first sounds like a contradiction in terms, namely to use background fields in order to implement Background Independence, does not only solve the first of the three main problems discussed above, but the other two, coarse graining and gauge invariance, as well: The background field method supplies an almost canonical, natural notion of coarse graining, and, by a standard technique (background-type gauge fixing), it allows us to define the associated scale-dependent action as a manifestly diffeomorphism invariant functional.

In the simpler case of Yang–Mills theory when Background Independence is not an issue the background field method likewise leads to both a gauge-covariant notion of coarse graining, and to an EAA with manifest gauge invariance. We will not go into the details of the Yang–Mills case here but refer the reader to the original articles on the subject: The EAA for abelian and non-abelian gauge fields (coupled to matter) was introduced in a series of papers [23–26]. In the first two of them, [25, 26], the proper "*average action*" for gauge theories was constructed, while the related "*Effective Average Action*" for Yang–Mills theory was introduced in [23, 24]. Early applications include [27–29], and for reviews see [33–35].[12]

(2) All backgrounds = no background! Pioneered by Schwinger and later on in particular by DeWitt [50, 51], the basic idea behind the background field approach is to decompose, in one way or another, the dynamical field $\widehat{g}_{\mu\nu}(x)$ as the "sum" of a given classical background metric $\bar{g}_{\mu\nu}(x)$ "plus" a dynamical quantum field $\widehat{h}_{\mu\nu}(x)$ that is then promoted to an operator, in canonical quantization, or is integrated over in the corresponding functional integral. This amounts to quantizing the fluctuation degrees of freedom, $\widehat{h}_{\mu\nu}$, similar to a matter field, *on a given classical spacetime manifold* which carries a bona fide metric $\bar{g}_{\mu\nu}$.

As a result, we free ourselves of all the conceptual difficulties specifically related to the Background Independent quantization of gravity and may resort to well-developed methods and calculational tools of standard, i.e., background-dependent, quantum field theory.

The problems that originate in the quantization of spacetime itself have not disappeared, however; they resurface in the (logically) second step of the procedure: To achieve Background Independence we must now *repeat the quantization of $\widehat{h}_{\mu\nu}$ for all background metrics $\bar{g}_{\mu\nu}$*.[13] Only the totality of *all* quantum theories together, describing the dynamics of $\widehat{h}_{\mu\nu}$ on *arbitrary, but classical* background spacetimes, contains sufficient information to implicitly account for a single "quantum spacetime."

So, loosely speaking, the background-field method allows us to reinterpret the quantum theory of spacetime itself as a set of infinitely many "ordinary" quantum field theories of a matter-like field on given classical spacetimes. These matter-like theories are distinguished by a label which identifies the background spacetime manifold in question. Besides its topological and differentiable structure, which we consider fixed here, the label specifies the metric it is equipped with, $\bar{g}_{\mu\nu}$.

At least in principle, we can now use the established methods of background-dependent quantum field theory to "solve" the theory of the $\widehat{h}_{\mu\nu}$ field *on one*

[12] Note that the (older, proper) "average action" [67] and the "Effective Average Action" [22] are two differently defined scale-dependent functionals. While the latter is the type of running action considered in this book exclusively, the former is closer in spirit to an iterated Kadanoff–Wilson block spin transformation; in the case of scalar fields it may be thought of as a scale-dependent generalization of the "constraint effective potential" [69].

[13] A mathematically more precise meaning of "all background metrics" will emerge later in the discussion.

fixed background, and then repeat this conceptually clear step *for all possible backgrounds.*

The outcome of the corresponding calculations are $\bar{g}$-dependent expectation values $\langle\mathcal{O}(\widehat{h})\rangle_{\bar{g}}$, for example, and in particular n-point functions like $\langle\widehat{h}(x_1)\cdots\widehat{h}(x_n)\rangle_{\bar{g}}$. The latter can be encoded in a generating functional $\Gamma[h;\bar{g}]$, which, besides $h\equiv\langle\widehat{h}\rangle$, depends on a second argument, $\bar{g}$. As for the domain of this *background-effective action*, we tentatively assume that the space of backgrounds for which it can be defined, $\bar{\mathcal{F}}\equiv\{\bar{g}_{\mu\nu}\}$, is roughly as large as the space from which the dynamical metric is drawn.

Because of the comparable importance enjoyed by $\bar{g}_{\mu\nu}$ and $\langle\widehat{g}_{\mu\nu}\rangle\equiv g_{\mu\nu}$ this effective action is sometimes said to be of the *bi-metric* type.

(3) The natural notion of coarse graining. Having interpreted the quantum theory of spacetime itself as the totality of all $\widehat{h}_{\mu\nu}$-theories on a fixed background, there is an obvious natural notion of coarse graining now. Within the theory with fixed "label" $\bar{g}_{\mu\nu}$, it is based, in the sense explained above, on the *ordered set of eigenmodes of* $D^2(\bar{g})\equiv\bar{D}^2\equiv\bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ where $\bar{D}_\mu$ involves the Christoffel symbols of $\bar{g}_{\mu\nu}$:

$$-D^2(\bar{g})\,h^{(n)}_{\mu\nu}=E_n\,h^{(n)}_{\mu\nu}. \tag{4.25}$$

Now $h_{\mu\nu}$ appears as a matter field similar to Ψ above, which at this point is completely unrelated to $\bar{g}_{\mu\nu}$. Hence, we are no longer in the confusing and almost circular situation described above where the very same field that should be coarse grained occurs also in the covariant Laplacian defining the modes.

In each of the fixed-$\bar{g}$ theories the background metric implies a clearcut concept of "long-wavelength" and "short-wavelength" modes of $\widehat{h}_{\mu\nu}$, or corresponding low-momentum and high-momentum modes.

Furthermore, as we will see, the discrimination between the two types of modes under the functional integral can be achieved by means of a cutoff action ΔS_k which is bilinear in $\widehat{h}_{\mu\nu}$, and thus also solves the "infinite order problem" we were facing before.

(4) Parametrizing $\widehat{g}_{\mu\nu}$. The background field approach expresses the functional integral over $\widehat{g}_{\mu\nu}$ as an integral over a new field, $\widehat{h}_{\mu\nu}$, which is then considered the elementary quantum mechanical variable. Technically, this step depends on whether or not the field configurations $\widehat{g}_{\mu\nu}$ contributing to the original integral are allowed to degenerate.

(4a) If the integral $\int\mathcal{D}\widehat{g}_{\mu\nu}\cdots$ extends over *all symmetric tensor fields*, including arbitrarily degenerate matrices and even $\widehat{g}_{\mu\nu}=0$, the domain of the $\widehat{g}$-integration is easily parametrized in terms of fluctuations $\widehat{h}_{\mu\nu}$ using a linear split:

$$\widehat{g}_{\mu\nu}=\bar{g}_{\mu\nu}+\widehat{h}_{\mu\nu}. \tag{4.26}$$

The functional integral $\int \mathcal{D}\widehat{g}_{\mu\nu}\cdots \equiv \int \mathcal{D}\widehat{h}_{\mu\nu}\cdots$ involves an unconstrained integration over all symmetric tensors $\widehat{h}_{\mu\nu}$ then, with a translation-invariant measure like in standard matter field theories.

(4b) If $\int \mathcal{D}\widehat{g}_{\mu\nu}\cdots$ extends only over the subset of *all metrics* with a given signature, the requirement of non-degeneracy implies certain nonlinear constraints among the components of $\widehat{g}_{\mu\nu}$, in particular $\det(\widehat{g}_{\mu\nu}) \neq 0$. In this case, the domain of integration, $\mathcal{F} \equiv \{\text{nondeg., sym. tensor fields } \widehat{g}_{\mu\nu}\}$, is no longer a linear space but a curved infinite-dimensional "manifold". The task then consists in finding a suitable parametrization $\widehat{g} = \widehat{g}(\widehat{h}; \bar{g})$ of this manifold in terms of unconstrained fields $\widehat{h}_{\mu\nu}$. A typical example for such a parametrization is provided by the exponential split (4.33) constructed below.

Regarding the differential geometry of the infinite dimensional manifold $\mathcal{F}$, the functions $\widehat{g}$ are merely coordinates on some local patch of $\mathcal{F}$, i.e., no tensorial quantities. Instead, we would prefer $\widehat{h}$ to behave as a vector, since only then a functional integral over $\widehat{h}$ will be of the familiar type we know from matter fields [135].

At least in the finite-dimensional case, there is a well-known general construction, the *exponential map*, which supplies precisely the sought-for parametrization [136]. It interprets $\widehat{h}$ as a tangent vector to the field space $\mathcal{F}$ at some base point $\bar{g}$, that is, $\widehat{h} \in T_{\bar{g}}\mathcal{F}$, and then parametrizes arbitrary points $\widehat{g} \in \mathcal{F}$ in a neighborhood of $\bar{g}$ by setting up a one-to-one correspondence between those points on $\mathcal{F}$, and vectors $\widehat{h}$ in a fixed tangent space, namely $T_{\bar{g}}\mathcal{F}$. Of course, the base point $\bar{g}$ is identified with the background metric in our case.

To write down this correspondence concretely let us switch to a condensed index notation and identify $\phi^i \equiv \widehat{g}_{\mu\nu}(x)$, $\bar{\phi}^i \equiv \bar{g}_{\mu\nu}(x)$, $h^i \equiv \widehat{h}_{\mu\nu}(x)$. Indices $i, j, k, \cdots$ will combine the spacetime indices $\mu, \nu, \cdots$ with the continuous x coordinates, and the "summation" over repeated Latin indices is understood to include an integration over x.

Let us assume now that the manifold in question,[14] in our case $\mathcal{F}$, is equipped with an arbitrary linear connection, Γ^k_{ij}, so that we may talk about geodesics on $\mathcal{F}$, i.e., parametrized curves $\tau \mapsto \phi(\tau)$ satisfying

$$\ddot{\phi}^i(\tau) + \Gamma^i_{jk}(\phi(\tau))\,\dot{\phi}^j(\tau)\dot{\phi}^k(\tau) = 0. \tag{4.27}$$

Here dots denote derivatives with respect to the parameter τ.

After having fixed a base point $\bar{\phi} \in \mathcal{F}$, we now associate arbitrary points $\phi \in \mathcal{F}$ in a neighborhood of $\bar{\phi}$ to vectors $h \in T_{\bar{\phi}}\mathcal{F}$ in a neighborhood of the origin $0 \in T_{\bar{\phi}}\mathcal{F}$

[14] In GR the exponential map makes its appearance in setting up systems of Riemann normal coordinates. In this case the manifold is spacetime itself instead of $\mathcal{F}$, and Γ^k_{ij} is the Levi–Civita connection of its metric [137]. For a similar application on symplectic manifolds (phase spaces) see [138].

by demanding that ϕ and $\bar{\phi}$ are connected by the segment of a geodesic, $\{\phi(\tau), \tau \in [0,1]\}$, i.e.,

$$\phi^i(\tau = 0) = \bar{\phi}^i, \quad \phi^i(\tau = 1) = \phi^i \tag{4.28}$$

with the "initial velocity" given by the vector

$$\dot{\phi}^i(\tau = 0) = h^i. \tag{4.29}$$

The relationship $h \leftrightarrow \phi$ provided by the exponential map is indeed a diffeomorphism of some neighborhood of the origin in $T_{\bar{\phi}}\mathcal{F}$ with a neighborhood of $\bar{\phi}$ on $\mathcal{F}$ [136]. It can be worked out explicitly by Taylor expanding the geodesic at the initial point,

$$\phi^i(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\partial_\tau^n \phi^i(\tau) \Big|_{\tau=0} \right) \tau^n \tag{4.30}$$

and recursively eliminating all higher derivatives by means of the geodesic equation. This expresses the right-hand side of (4.30) in terms of $\phi(0)$ and $h = \dot{\phi}(0)$ only. In this manner we obtain at $\tau = 1$ the desired formula for the endpoint ϕ, namely

$$\boxed{\phi^i = \bar{\phi}^i + H^i(h; \bar{\phi})} \tag{4.31}$$

with the series representation

$$\boxed{\begin{aligned} H^i(h; \bar{\phi}) = h^i &- \frac{1}{2!} \bar{\Gamma}^i_{jk} h^j h^k \\ &- \frac{1}{3!} \left(\bar{\Gamma}^i_{jk,l} - \bar{\Gamma}^m_{lk} \bar{\Gamma}^i_{mj} - \bar{\Gamma}^m_{lj} \bar{\Gamma}^i_{km} \right) h^j h^k h^l + O(h^4). \end{aligned}} \tag{4.32}$$

Here the overbar indicates that the connection coefficients and their derivatives are evaluated at $\bar{\phi}$.

(4c) The next step consists of using the *nonlinear background quantum field split* given by (4.31) in order to replace the $\widehat{\phi}$-integral with an integration over $\widehat{h}$, taking the corresponding Jacobian factor properly into account.

Note that while the connection Γ^i_{jk} has not been specified yet, there are various natural options.

In background-dependent field theory, there exist well-developed techniques for the quantization of *nonlinear σ-models*, pioneered by Honerkamp [139], Friedan [140], and others [141, 142], based on the normal coordinate expansion (4.32). In this case the situation simplifies since the ϕ^is are coordinates on the model's (finite-dimensional) target space only, rather than the huge space of all fields, $\mathcal{F}$. Nevertheless, many of the methods developed for the perturbative quantization of nonlinear σ-models carry over to the infinite-dimensional context of the space of all metrics, $\mathcal{F}$.[15]

[15] The relation to the σ-models becomes particularly clear if one imposes Killing vector constraints on the metrics in $\mathcal{F}$; see [121–123] for an in-depth analysis of the σ-model describing the quantum theory of metrics with two Killing symmetries.

In σ-models with a Riemannian target space the natural choice of Γ^k_{ij} is clearly the Levi–Civita connection of its target space metric.

On the field space underlying General Relativity, $\mathcal{F}$, a number of possibilities are conceivable, for instance the Levi–Civita connection pertaining to the family of ultralocal DeWitt metrics on $\mathcal{F}$ [126, 143], or the Vilkovisky–DeWitt connection [144], which is adapted to the action of the gauge group on the points of $\mathcal{F}$.

Furthermore, there even exists a choice of the connection for which the general exponential map boils down to, literally, the exponential of the matrix $h^\mu{}_\nu = \bar{g}^{\mu\alpha} h_{\alpha\nu}$ [145–147]:

$$g = \bar{g}\, e^{\bar{g}^{-1}h} \iff g_{\mu\nu} = \bar{g}_{\mu\rho}\big(e^h\big)^\rho{}_\nu. \tag{4.33}$$

This *exponential parametrization* establishes a one-to-one correspondence between symmetric tensors $h_{\mu\nu}$ and metrics $g_{\mu\nu}$ for Euclidean signature, but it fails to do so in the Lorentzian case [146].

(5) The split symmetry. The base point $\bar{\phi}$ is chosen freely on the manifold, and so it is natural to ask how the vector h representing a given $\phi \in \mathcal{F}$ responds to a change of $\bar{\phi}$. From (4.31) it follows that the pairs $(\bar{\phi}, h)$ and $(\bar{\phi} + \boldsymbol{\delta}\bar{\phi}, h + \boldsymbol{\delta} h)$, where $h \in T_{\bar{\phi}}\mathcal{F}$ and $(h + \boldsymbol{\delta} h) \in T_{\bar{\phi}+\boldsymbol{\delta}\bar{\phi}}\mathcal{F}$, respectively, describe the same point ϕ on the manifold, if

$$\boxed{\begin{aligned} &\boldsymbol{\delta}\bar{\phi}^i = -\varepsilon^i, \\ &\boldsymbol{\delta} h^i = -F^i_k(h;\bar{\phi})\,\varepsilon^k, \end{aligned}} \tag{4.34}$$

where ε^i is an infinitesimal vector and F^i_k is determined by the requirement $\boldsymbol{\delta}\phi^i = 0$,[16] i.e.,

$$\delta^i_k + \frac{\partial H^i}{\partial \bar{\phi}^k} + F^l_k \frac{\partial H^i}{\partial h^l} = 0. \tag{4.35}$$

The equations (4.34) constitute the general, i.e., nonlinear form of what is known as a *background quantum field split symmetry transformation*, or simply "split transformation" for short [140].

Given the series representation of H^i in (4.32) we can work out the solution to (4.35) as a similar Taylor series in h:

$$F^i_k(h;\bar{\phi}) = -\delta^i_k - \Gamma^i_{kl}(\bar{\phi}) h^l + O(h^2). \tag{4.36}$$

Furthermore, it is not difficult to infer from (4.35) that the functions F^i_k satisfy the relation

$$\boxed{\frac{\partial F^i_k}{\partial \bar{\phi}^j} - \frac{\partial F^i_j}{\partial \bar{\phi}^k} + \frac{\partial F^i_k}{\partial h^l} F^l_j - \frac{\partial F^i_j}{\partial h^l} F^l_k = 0,} \tag{4.37}$$

which can be thought of as a kind of zero-curvature condition. In fact, the relation (4.37) implies that the split symmetry transformations (4.34) form an

[16] We use the notation of finite-dimensional manifolds here. Otherwise, all derivatives should be replaced by functional derivatives.

abelian group. Thus, their order does not matter when performing consecutive transformations.

Friedan [140] showed that, in more geometrical terms, $F^i_k(h;\bar{\phi})$ has the interpretation of a *nonlinear flat connection* on the tangent bundle. It defines a parallel transport of vectors h from $T_{\bar{\phi}}\mathcal{F}$ to another tangent space, at, say, $\bar{\phi}'$ such that the original h together with $\bar{\phi}$, and the transported vector together with $\bar{\phi}'$, describe one and the same point ϕ on the manifold.

Assume we are given a set of functions $T_{i_1\cdots i_n}(h;\bar{\phi})$ with an arbitrary dependence on h and $\bar{\phi}$, and we ask whether they are the components of a unique, globally well-defined tensor field on $\mathcal{F}$, expressed via the exponential map based at $\bar{\phi}$. The condition for this to be the case is that those functions must be covariantly constant with respect to the nonlinear connection $F^i_k(h;\bar{\phi})$. Explicitly [140, 142, 141]:

$$\boxed{\frac{\partial}{\partial\bar{\phi}^k}T_{i_1\cdots i_n} + F^l_k\frac{\partial}{\partial h^l}T_{i_1\cdots i_n} + \sum_{r=1}^{n} T_{i_1\cdots i_{r-1}j i_{r+1}\cdots i_n}\frac{\partial}{\partial h^{i_r}}F^j_k = 0.} \tag{4.38}$$

Analogous *globality conditions* can be set up for more general tensors and spinors.[17] If T carries no indices, (4.38) simply states that $T(h;\bar{\phi}) = S(\bar{\phi} + H(h;\bar{\phi}))$ for some globally defined scalar S.

It is clear that if the manifold $\mathcal{F}$ in question happens to support a trivial connection with $\Gamma^i_{jk} \equiv 0$, the expansions (4.32) and (4.36) lead us back to the linear split, $\phi^i = \bar{\phi}^i + H^i$, where $H^i = h^i$, with its obvious split transformation $\boldsymbol{\delta}\bar{\phi}^i = -\varepsilon^i$, $\boldsymbol{\delta}h^i = \varepsilon^i$. We mentioned already that this split is appropriate to coordinatize the unconstrained space of all tensor fields, but not the manifold of metrics:

$$\boxed{\begin{aligned} \widehat{g}_{\mu\nu} &= \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}, \\ \boldsymbol{\delta}\bar{g}_{\mu\nu} &= -\varepsilon_{\mu\nu}, \quad \boldsymbol{\delta}\widehat{h}_{\mu\nu} = \varepsilon_{\mu\nu}. \end{aligned}} \tag{4.39}$$

In the sequel we will mostly discuss this case where unconstrained symmetric tensors rather than genuine metrics are integrated over. The motivation is first of all simplicity and the fact that it has been employed in the vast majority of Asymptotic Safety studies.

It is clear that scrutinizing more advanced theory spaces by taking into account higher orders of the normal coordinate expansion will be a central future research direction. First results along these lines, employing the exponential parametrization, were reported in [145]. For a systematic examination of different ways to parametrize the graviton field and the related functional measures we refer to [148, 149].

Nevertheless, we emphasize that the linear split is important not only as a first approximation to a nonlinear expansion but *also in its own right*. It is well

[17] For the corresponding discussion of metaplectic spinor fields see [138].

conceivable that the integral over unconstrained $\widehat{h}_{\mu\nu}$s possesses an Asymptotically Safe continuum limit different from the one over metrics, i.e., that it probes another "universality class" of fundamental theories. (See [117] for a discussion of this point.)

4.3 Introducing an EAA for Gravity

In this section we describe in detail the Effective Average Action for gravity, focusing mostly on those aspects that are different from the scalar case outlined in Chapter 2. The presentation follows [14] to which we refer for further details.

4.3.1 The Gauge-Fixed Functional Integral

We start out from a functional integral over tensors $\widehat{g}_{\mu\nu}$ of the form

$$\int \mathcal{D}\widehat{g}_{\mu\nu}\, \exp\left(-S[\widehat{g}_{\mu\nu}] + \text{source terms}\right). \tag{4.40}$$

The fields $\widehat{g}_{\mu\nu}$ live on a given spacetime manifold with a fixed topological class and smooth differential structure. The bare action $S[\widehat{g}_{\mu\nu}]$ is assumed to be invariant under general coordinate transformations which act on the metric according to

$$\delta\widehat{g}_{\mu\nu} = \mathcal{L}_v \widehat{g}_{\mu\nu} \equiv v^\rho \partial_\rho \widehat{g}_{\mu\nu} + \partial_\mu v^\rho \widehat{g}_{\rho\nu} + \partial_\nu v^\rho \widehat{g}_{\rho\mu}. \tag{4.41}$$

Here $\mathcal{L}_v$ denotes the Lie derivative with respect to the generating vector field v^μ.

Furthermore, to render the ensuing mathematical manipulations well defined we assume that the functional integral has been UV-regularized in some way. We shall not need to specify how this is done explicitly at this point. It is sufficient to keep the corresponding cutoff implicit.

Invoking the linear background split for explicitness, we decompose the variable of integration according to $\widehat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is a fixed background metric. We interpret the measure $\mathcal{D}\widehat{g}_{\mu\nu}$ as $\mathcal{D}\widehat{h}_{\mu\nu}$, which leaves us with $\int \mathcal{D}\widehat{h}_{\mu\nu}\, \exp(-S[\bar{g} + \widehat{h}] + \cdots)$ then.[18]

The next step consists in "cutting out" gauge-equivalent field configurations from this functional integral. The gauge transformations which we have to gauge-fix under the integral over $\widehat{h}_{\mu\nu}$ are the so-called *"true," or "quantum" gauge transformations* $\boldsymbol{\delta}^{\mathrm{Q}}$. They follow from (4.41) under the assumption that $\bar{g}_{\mu\nu}$ does not change:

$$\boxed{\boldsymbol{\delta}^{\mathrm{Q}}\widehat{h}_{\mu\nu} = \mathcal{L}_v \widehat{g}_{\mu\nu} = \mathcal{L}_v\big(\bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}\big), \quad \boldsymbol{\delta}^{\mathrm{Q}}\bar{g}_{\mu\nu} = 0.} \tag{4.42}$$

[18] The construction outlined in this section is easily adapted to other decompositions of $\widehat{g}_{\mu\nu}$ into background and fluctuation fields. In particular, there are no additional conceptual difficulties when constructing the corresponding EAAs.

Picking any gauge-fixing function $F_\mu(\widehat{h};\bar{g})$, the familiar Faddeev–Popov trick can be applied straightforwardly to this type of gauge invariance [91, 143, 150–152]. It leads us to the modified integral

$$\boxed{\int \mathcal{D}\widehat{h}_{\mu\nu}\mathcal{D}C^\mu\mathcal{D}\bar{C}_\mu \, \exp\Big(-S[\bar{g}+\widehat{h}] - S_{\mathrm{gf}}[\widehat{h};\bar{g}] - S_{\mathrm{gh}}[\widehat{h},C,\bar{C};\bar{g}]\Big).} \tag{4.43}$$

Let us explain its various ingredients.

First of all, the classical action $S[\widehat{g}] \equiv S[\bar{g}+\widehat{h}]$ was augmented by a gauge-fixing term, S_{gf}, which has the general structure

$$S_{\mathrm{gf}}[\widehat{h};\bar{g}] = \frac{1}{2\alpha}\int \mathrm{d}^d x \, \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} F_\mu(\widehat{h};\bar{g}) F_\nu(\widehat{h};\bar{g}). \tag{4.44}$$

The gauge-fixing function $F_\mu(\widehat{h};\bar{g})$ is still quite arbitrary here. In particular, we allow it to depend on the background metric $\bar{g}$.

Furthermore, S_{gh} is the action for the associated Faddeev–Popov ghost and antighost fields, which we denote C^μ and $\bar{C}_\mu$, respectively:

$$S_{\mathrm{gh}}[\widehat{h},C,\bar{C};\bar{g}] = -\kappa^{-1}\int \mathrm{d}^d x \, \sqrt{\bar{g}}\, \bar{C}_\mu \, \bar{g}^{\mu\nu} \, \frac{\partial F_\nu}{\partial \widehat{h}_{\alpha\beta}} \, \mathcal{L}_C\big(\bar{g}_{\alpha\beta} + \widehat{h}_{\alpha\beta}\big). \tag{4.45}$$

This action $S_{\mathrm{gh}} \propto \int \mathrm{d}^d x \, \bar{C}\mathcal{M}C$ displays the ghost kinetic operator $\mathcal{M}$ and is obtained along the same lines as in the familiar case of Yang–Mills theory: one first applies a gauge transformation $\boldsymbol{\delta}^{\mathrm{Q}}$ to F_μ, then contracts the result with $\bar{C}_\mu$, and finally replaces the parameters v^μ by the anticommuting field C^μ [153].

Functional integration over the ghost C^μ and the likewise anticommuting antighost $\bar{C}_\mu$ exponentiates the Faddeev–Popov determinant, then

$$\det\big[\mathcal{M}\big] = \det\left[\frac{\delta F_\mu}{\delta v^\nu}\right] = \int \mathcal{D}C^\mu\mathcal{D}\bar{C}_\mu \, e^{-\int \bar{C}\mathcal{M}C}, \tag{4.46}$$

whereby the integration rules for Grassmann variables are applied. The determinant of the infinite-dimensional "matrix" $\mathcal{M}$ may be seen as the volume element on the "gauge slice" of fields $\widehat{h}$ satisfying the gauge condition $F_\mu(\widehat{h};\bar{g}) = 0$.

For the time being, the gauge-fixing parameter α and the ghost normalization κ are arbitrary constants. Since κ has the dimension of a mass we will often set in the following

$$\kappa \equiv (32\pi\bar{G})^{-\frac{1}{2}}, \tag{4.47}$$

where $\bar{G}$ is a constant reference value of Newton's constant.

4.3.2 Classical BRST Invariance

The gauge-fixed action under the integral (4.43) is invariant under the following BRST transformation:

$$\begin{aligned} \boldsymbol{\delta}_\varepsilon^{\text{BRST}} \widehat{h}_{\mu\nu} &= \varepsilon\kappa^{-2}\mathcal{L}_C \widehat{g}_{\mu\nu} = \varepsilon\kappa^{-2}\mathcal{L}_C\left(\bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}\right), \\ \boldsymbol{\delta}_\varepsilon^{\text{BRST}} \bar{g}_{\mu\nu} &= 0, \\ \boldsymbol{\delta}_\varepsilon^{\text{BRST}} C^\mu &= \varepsilon\kappa^{-2} C^\nu \partial_\nu C^\mu, \\ \boldsymbol{\delta}_\varepsilon^{\text{BRST}} \bar{C}_\mu &= \varepsilon(\alpha\kappa)^{-1} F_\mu. \end{aligned} \tag{4.48}$$

Here ε is an anticommuting, x-independent parameter. It is not difficult to check that the total action is indeed invariant:

$$\boldsymbol{\delta}_\varepsilon^{\text{BRST}}\Big(S[\bar{g}+\widehat{h}] + S_{\text{gf}}[\widehat{h};\bar{g}] + S_{\text{gh}}[\widehat{h}, C, \bar{C}; \bar{g}]\Big) = 0. \tag{4.49}$$

Furthermore, one can verify that $\boldsymbol{\delta}^{\text{BRST}}$ is nilpotent, even "off-shell," when acting on $\widehat{g}_{\mu\nu}$, $\bar{g}_{\mu\nu}$, $\widehat{h}_{\mu\nu}$, and C^μ, while $\left(\boldsymbol{\delta}^{\text{BRST}}\right)^2 \bar{C}_\mu$ vanishes only when the classical equation of motion for the antighost field is used.

4.3.3 Background-Type Gauge-Fixing Conditions

As we will see more explicitly in a moment, using a generic gauge-fixing condition $F_\mu(\widehat{h};\bar{g}) = 0$, the (ordinary) effective action resulting from the above gauge-fixed functional integral fails to be a diffeomorphism-invariant functional of its arguments [87, 91, 143, 150–152].

But the situation is different for the special class of the so-called *gauge-fixing conditions of the background type.* While – as any gauge-fixing condition does – they break the invariance under the true gauge transformations (4.42), their defining property is to be invariant under the so-called *background gauge transformations* defined by

$$\boxed{\boldsymbol{\delta}^{\text{B}} \widehat{h}_{\mu\nu} = \mathcal{L}_v \widehat{h}_{\mu\nu}, \quad \boldsymbol{\delta}^{\text{B}} \bar{g}_{\mu\nu} = \mathcal{L}_v \bar{g}_{\mu\nu}.} \tag{4.50}$$

The key idea is that the full metric $\widehat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$ transforms in the required fashion (4.41) both under the background gauge transformations (4.50) and the true, quantum gauge transformations (4.42): $\boldsymbol{\delta}^{\text{B}}\widehat{g}_{\mu\nu} = \boldsymbol{\delta}^{\text{Q}}\widehat{g}_{\mu\nu} = \mathcal{L}_v \widehat{g}_{\mu\nu}$. The transformations $\boldsymbol{\delta}^{\text{Q}}$ and $\boldsymbol{\delta}^{\text{B}}$ distribute the given variation $\delta\widehat{g}_{\mu\nu} = \mathcal{L}_v\widehat{g}_{\mu\nu}$ of $\widehat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$ differently over $\bar{g}_{\mu\nu}$ and $\widehat{h}_{\mu\nu}$. The quantum gauge transformations $\boldsymbol{\delta}^{\text{Q}}$ keep $\bar{g}_{\mu\nu}$ unchanged and ascribe the entire variation of $\widehat{g}_{\mu\nu}$ to $\widehat{h}_{\mu\nu}$, while the background transformations $\boldsymbol{\delta}^{\text{B}}$ of (4.50) change both $\bar{g}_{\mu\nu}$ and $\widehat{h}_{\mu\nu}$, but only by *their own* Lie derivative.

These rules imply $\boldsymbol{\delta}^{\text{B}} F_\mu = \mathcal{L}_v F_\mu$, and as a consequence, both the classical action and the gauge-fixing action are invariant under $\boldsymbol{\delta}^{\text{B}}$:

$$\boldsymbol{\delta}^{\text{B}} S[\bar{g}+\widehat{h}] = 0, \quad \boldsymbol{\delta}^{\text{B}} S_{\text{gf}}[\widehat{h};\bar{g}] = 0. \tag{4.51}$$

Of course, one of the consequences entailed by the differences between $\boldsymbol{\delta}^{\mathrm{Q}}$ and $\boldsymbol{\delta}^{\mathrm{B}}$ is that S_{gf}, despite its $\boldsymbol{\delta}^{\mathrm{B}}$-invariance, is not invariant under $\boldsymbol{\delta}^{\mathrm{Q}}$-transformations: after all, its very raison d'être is to break the quantum gauge invariance!

It can be verified that for every background-type choice of $F_\mu(\widehat{h};\bar{g})$, the ghost action $S_{\mathrm{gh}}[\widehat{h},C,\bar{C};\bar{g}]$ is also invariant under the extended background gauge transformations given by (4.50) together with the rules

$$\boxed{\boldsymbol{\delta}^{\mathrm{B}} C^\mu = \mathcal{L}_v C^\mu, \quad \boldsymbol{\delta}^{\mathrm{B}} \bar{C}_\mu = \mathcal{L}_v \bar{C}_\mu,} \tag{4.52}$$

for the ghost and antighost fields, respectively. So, taking everything together we may conclude that the total action including the gauge-fixing and ghost terms is invariant under background gauge transformations:

$$\boldsymbol{\delta}^{\mathrm{B}}\Big(S[\bar{g}+\widehat{h}] + S_{\mathrm{gf}}[\widehat{h};\bar{g}] + S_{\mathrm{gh}}[\widehat{h},C,\bar{C};\bar{g}]\Big) = 0. \tag{4.53}$$

This crucial property, among others, will allow us to set up a fully $\boldsymbol{\delta}^{\mathrm{B}}$-covariant Effective Average Action.

4.3.4 Generalized Harmonic Gauges

Clearly there exist many possible gauge-fixing actions $S_{\mathrm{gf}}[\widehat{h};\bar{g}]$ of the form (4.44) which break (4.42) and at the same time are invariant under (4.50). It is convenient to choose gauge functions of the form $F_\mu \propto \mathcal{F}_\mu{}^{\alpha\beta}[\bar{g}]\,\widehat{h}_{\alpha\beta}$. They are *linear* in the metric fluctuation, and employ some $\boldsymbol{\delta}^{\mathrm{B}}$-covariant operator $\mathcal{F}_\mu{}^{\alpha\beta}$ acting on it.

An example of this sort which has often been employed in practical calculations is the one-parameter family of *generalized harmonic gauge conditions*:

$$F_\mu(\widehat{h};\bar{g}) \equiv \sqrt{2}\,\kappa\,\mathcal{F}_\mu{}^{\alpha\beta}[\bar{g}]\,\widehat{h}_{\alpha\beta} \quad \text{with} \quad \mathcal{F}_\mu{}^{\alpha\beta} = \delta^\beta_\mu \bar{g}^{\alpha\gamma}\bar{D}_\gamma - \varpi \bar{g}^{\alpha\beta}\bar{D}_\mu. \tag{4.54}$$

Here ϖ is a free parameter, and $\mathcal{F}_\mu{}^{\alpha\beta}$ is taken to be a first-order differential operator built from the covariant derivative $\bar{D}_\mu$, which involves the Christoffel symbols $\bar{\Gamma}^\rho_{\mu\nu}$ of the background metric.

For the value $\varpi = \frac{1}{2}$, (4.54) indeed reduces to the background version of the harmonic coordinate condition [91, 143, 150–152]: on a flat background with $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$, for example, the condition $F_\mu = 0$ assumes the familiar form $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h_\mu{}^\mu$.

The ghost action for the gauge condition (4.54) reads

$$S_{\mathrm{gh}}[\widehat{h},C,\bar{C};\bar{g}] = -\sqrt{2}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,\bar{C}_\mu\,\mathcal{M}[\widehat{g},\bar{g}]^\mu{}_\nu\,C^\nu. \tag{4.55}$$

The Faddeev–Popov operator involves the metric-compatible covariant derivatives constructed from both $\widehat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$:

$$\mathcal{M}[\widehat{g},\bar{g}]^\mu{}_\nu = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda\big(\widehat{g}_{\rho\nu}D_\sigma + \widehat{g}_{\sigma\nu}D_\rho\big) - 2\varpi\,\bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda\widehat{g}_{\sigma\nu}D_\rho. \tag{4.56}$$

As always, the covariant derivatives related to $\widehat{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are denoted D_μ and $\bar{D}_\mu$, respectively.

4.3.5 The Mode Suppression Term

Next, we equip the integral (4.43) with an infrared regulator, concretely with a higher-derivative mode suppression that damps the contribution of the low-momentum modes to the functional integral, while leaving the high-momentum modes untouched. Thanks to the presence of the background metric, a natural notion of "low-momentum" vs. "high-momentum" is available now.

In analogy with the flat-space and non-gauge case, where the momentum square equals the eigenvalue of the negative Laplacian $-\partial^2$, we now define the "momentum square" to be the negative eigenvalue of the covariant tensor Laplacian $\bar{D}^2 \equiv \bar{g}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu$ built from the background metric. Low- and high-momentum modes, by definition, are those eigenfunctions of $-\bar{D}^2$ whose eigenvalues are smaller or larger, respectively, than a given scale k^2.

To implement the IR regulator we multiply the integrand of (4.43) by the extra factor

$$e^{-\Delta S_k[\widehat{h}, C, \bar{C}; \bar{g}]}. \tag{4.57}$$

The cutoff action ΔS_k suppresses the low-momentum modes of the metric fluctuations $\widehat{h}_{\mu\nu}$ and the Faddeev–Popov ghosts.

To avoid the fatal "infinite-order problem" we take another critical decision at this point, namely we insist that ΔS_k must be *bilinear in the dynamical fields*, i.e., in the integration variables $\widehat{h}, C, \bar{C}$:

$$\Delta S_k = \frac{\kappa^2}{2} \int \mathrm{d}^d x \sqrt{\bar{g}}\, \widehat{h}_{\mu\nu}\, \mathcal{R}_k^{\mathrm{grav}}[\bar{g}]^{\mu\nu\rho\sigma}\, \widehat{h}_{\rho\sigma} + \sqrt{2} \int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{C}_\mu\, \mathcal{R}_k^{\mathrm{gh}}[\bar{g}]\, C^\mu. \tag{4.58}$$

As we saw already, this decision sacrifices geometric naturalness to some extent in favor of practical computability.

The $\bar{g}_{\mu\nu}$-dependent cutoff operators $\mathcal{R}_k^{\mathrm{grav}}$ and $\mathcal{R}_k^{\mathrm{gh}}$ must be designed in such a way that they achieve the desired discrimination between high-momentum and low-momentum modes: Eigenmodes of $-\bar{D}^2$ with eigenvalues $p^2 \gg k^2$ are left untouched, whereas modes with small eigenvalues $p^2 \ll k^2$ get suppressed. Both operators, $\mathcal{R}_k^{\mathrm{grav}}$ and $\mathcal{R}_k^{\mathrm{gh}}$, have the same general structure $\mathcal{R}_k[\bar{g}] = \mathcal{Z}_k k^2 R^{(0)}(-\bar{D}^2/k^2)$, where the dimensionless shape function $R^{(0)}$ interpolates smoothly between $R^{(0)}(0) = 1$ and $R^{(0)}(\infty) = 0$; see (2.65).

The factors $\mathcal{Z}_k$ are different for the graviton and the ghost cutoff. They are determined by the condition that all fluctuation spectra must be cut off at precisely the same value k^2. This is the case, for example, if $\mathcal{R}_k$ combines with $\Gamma_k^{(2)}$ to an inverse propagator like $\Gamma_k^{(2)} + \mathcal{R}_k = \mathcal{Z}_k(p^2 + k^2 R^{(0)}(p^2/k^2)) + \cdots$. For low-momentum modes it gives rise to a $(\text{mass})^2$ of order k^2, rather than the wrong value $(\mathcal{Z}_k)^{-1} k^2$. Following our earlier discussion of adjusted cutoffs in Section 2.5 this is precisely as it should be.

Applying this construction principle in the ghost sector, $\mathcal{Z}_k \equiv Z_k^{\text{gh}}$ is a pure number, whereas for the metric fluctuation $\mathcal{Z}_k \equiv \mathcal{Z}_k^{\text{grav}}$ is a certain tensor constructed from the background metric. We will be more explicit about the adjustment of the $\mathcal{Z}_k$s when we discuss concrete examples later on.

A first general property of ΔS_k which is of importance for the practical applicability of the resulting RG equations is that the modes of $\widehat{h}_{\mu\nu}$ and the ghosts are organized according to their eigenvalues with respect to the *background* Laplace operator $\bar{D}^2 = \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ rather than $D^2 = g^{\mu\nu}D_\mu D_\nu$, which pertains to the full quantum metric. Using $\bar{D}^2$, the functional ΔS_k is indeed bilinear in the quantum fields and thus will give rise to a flow equation, which contains only *second* functional derivatives of Γ_k but no higher ones.

A second general property of ΔS_k which significantly simplifies the resulting EAA formalism is its invariance under the background gauge transformations (4.50) with (4.58):

$$\boxed{\boldsymbol{\delta}^{\text{B}}\Delta S_k[\widehat{h}, C, \bar{C}; \bar{g}] = 0.} \tag{4.59}$$

This property further motivates the particular form (4.58) we have given to ΔS_k.

On the other hand, for scales $k \neq 0$ the cutoff action is *not* BRST invariant:

$$\boldsymbol{\delta}_\varepsilon^{\text{BRST}}\Delta S_k[\widehat{h}, C, \bar{C}; \bar{g}] \neq 0. \tag{4.60}$$

BRST invariance can only be recovered in the limit $k \to 0$.

4.3.6 Sources and Expectation Values

Now we promote the integral (4.43) with the extra factor (4.57) included to a generating functional and define:

$$\boxed{\begin{aligned}&\exp\big(W_k[t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu; \beta^{\mu\nu}, \tau_\mu; \bar{g}_{\mu\nu}]\big)\\ &\quad = \int \mathcal{D}\widehat{h}_{\mu\nu}\mathcal{D}C^\mu\mathcal{D}\bar{C}_\mu \,\exp\Big(-S[\bar{g} + \widehat{h}] - S_{\text{gf}}[\widehat{h}; \bar{g}]\\ &\qquad - S_{\text{gh}}[\widehat{h}, C, \bar{C}; \bar{g}] - \Delta S_k[\widehat{h}, C, \bar{C}; \bar{g}] - S_{\text{source}}\Big).\end{aligned}} \tag{4.61}$$

Here the last part of the action,

$$\begin{aligned} S_{\text{source}} = -\int \mathrm{d}^d x\, \sqrt{\bar{g}}\,\Big\{ & t^{\mu\nu}\widehat{h}_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^\mu \bar{C}_\mu \\ & + \beta^{\mu\nu}\mathcal{L}_C\left(\bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}\right) + \tau_\mu C^\nu \partial_\nu C^\mu \Big\} \end{aligned} \tag{4.62}$$

contains sources for the three dynamical fields $\widehat{h}_{\mu\nu}$, C^μ, and $\bar{C}_\mu$, which are denoted by $t^{\mu\nu}$, $\bar{\sigma}_\mu$, and σ^μ, respectively. The sources for the ghosts are anticommuting objects themselves. The two additional source functions $\beta^{\mu\nu}$ and τ_μ couple to the BRST variations of $\widehat{h}_{\mu\nu}$ and C^μ, respectively. We do not need them at this point, but later on they will play a role in the formulation of the Ward identities that represent the BRST invariance at the quantum level.

This leads us to a somewhat lengthy list of arguments for the generating functional:

$$W_k[t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu;\beta^{\mu\nu},\tau_\mu;\bar{g}_{\mu\nu}]\equiv W_k[J;J^{\text{BRST}};\bar{g}_{\mu\nu}]. \tag{4.63}$$

Functional derivatives of W_k with respect to $J\equiv(t^{\mu\nu},\sigma^\mu,\bar{\sigma}_\mu)$ and $J^{\text{BRST}}\equiv(\beta^{\mu\nu},\tau_\mu)$ generate the expectation values of the dynamical fields and BRST variations, respectively. Besides on these sources, W_k also functionally depends on the background metric.

The expectation values of $\widehat{\varphi}\equiv(\widehat{h}_{\mu\nu},C^\mu,\bar{C}_\mu)$ are denoted by $\varphi\equiv(h_{\mu\nu},\xi^\mu,\bar{\xi}_\mu)$ and we have explicitly

$$h_{\mu\nu}=\big\langle\widehat{h}_{\mu\nu}\big\rangle=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta t^{\mu\nu}},\ \ \xi^\mu=\langle C^\mu\rangle=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta\bar{\sigma}_\mu},\ \ \bar{\xi}_\mu=\big\langle\bar{C}_\mu\big\rangle=\frac{1}{\sqrt{\bar{g}}}\frac{\delta W_k}{\delta\sigma^\mu}, \tag{4.64}$$

or, in a more compact notation,

$$\varphi^i(x)=\big\langle\widehat{\varphi}^i(x)\big\rangle=\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta}{\delta J_i(x)}W_k[J;J^{\text{BRST}};\bar{g}]. \tag{4.65}$$

The explicit factors of $\sqrt{\bar{g}}$ in (4.64) and (4.65) guarantee that both the sources and the fields transform under general coordinate transformations as proper tensors rather than tensor densities.

Let us now switch from the sources to the field expectation values as the independent variables. Correspondingly we introduce the Legendre transform $\widetilde{\Gamma}_k$ of W_k with respect to the dynamical sources J. All other variables W_k depends on, J^{BRST}, $\bar{g}_{\mu\nu}$, and k, are just spectators in this transformation and are kept fixed.

In the general case $\widetilde{\Gamma}_k$ is given by the Legendre–Fenchel supremum formula (2.11). In the regular case when the relations given by (4.65), $\varphi\equiv\varphi[J;J^{\text{BRST}};\bar{g}]$, can be solved for J, it assumes the form

$$\widetilde{\Gamma}_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}]=\int\mathrm{d}^dx\,\sqrt{\bar{g}}\,\big\{t^{\mu\nu}h_{\mu\nu}+\bar{\sigma}_\mu\xi^\mu+\sigma^\mu\bar{\xi}_\mu\big\}-W_k[t,\sigma,\bar{\sigma};\beta,\tau;\bar{g}] \tag{4.66}$$

with the inverted φ-J-relationship $J\equiv(t,\sigma,\bar{\sigma})=J_k[\varphi;J^{\text{BRST}};\bar{g}]$ substituted on the right-hand side.

As usual, the Legendre transform gives rise to source–field relations,

$$\frac{\delta\widetilde{\Gamma}_k}{\delta h_{\mu\nu}}=\sqrt{\bar{g}}\,t^{\mu\nu},\quad\frac{\delta\widetilde{\Gamma}_k}{\delta\bar{\xi}_\mu}=-\sqrt{\bar{g}}\,\sigma^\mu,\quad\frac{\delta\widetilde{\Gamma}_k}{\delta\xi^\mu}=-\sqrt{\bar{g}}\,\bar{\sigma}_\mu, \tag{4.67}$$

which are "dual" to the original ones in (4.64).

4.3.7 Formal Definition of the Gravitational EAA

Having explained all the ingredients which go into the functional integral (4.61) for the generating functional W_k we finally introduce the actual Effective Average Action for metric gravity. It is obtained from the Legendre transform $\widetilde{\Gamma}_k$ of W_k by subtracting the cutoff action ΔS_k with the classical fields inserted:

$$\boxed{\Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}] = \widetilde{\Gamma}_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}] - \Delta S_k[h,\xi,\bar{\xi};\bar{g}].} \tag{4.68}$$

The relations (4.61), (4.66), and (4.68) constitute what one might call the *formal* definition of the EAA since it is based on a functional integral which only makes mathematical sense in the presence of a UV regulator, and this regulator is implicit in all of the above equations.

At this point it is an open question, whether, or how, the UV regulator can be removed consistently. Before we can address this issue we first derive a number of general properties of the gravitational EAA, or more precisely, of the *EAA in presence of the UV regulator*, unless otherwise stated.

A remark concerning the notations we use for the EAA might be in place here. It is natural to introduce the expectation value of the full metric,

$$g_{\mu\nu}(x) = \langle \widehat{g}_{\mu\nu}(x) \rangle \equiv \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \tag{4.69}$$

as the classical analogue of the quantum metric $\widehat{g}_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \widehat{h}_{\mu\nu}$, and to consider Γ_k as a functional of $g_{\mu\nu}$ rather than $h_{\mu\nu}$:

$$\boxed{\Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau] \equiv \Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}]\Big|_{h=g-\bar{g}}.} \tag{4.70}$$

Apart from the ghosts and BRST sources, the action $\Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}]$, by definition, depends on the fluctuations' expectation value $h_{\mu\nu}$ and the background metric as the independent field variables. This should be contrasted with the second variant of the EAA, $\Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau]$, which, by definition, depends on two fully fledged metrics, namely the expectation value of the quantum metric, $\langle \widehat{g}_{\mu\nu}(x) \rangle \equiv g_{\mu\nu}(x)$ and the background metric $\bar{g}_{\mu\nu}(x)$.

The "semicolon variant" of the EAA, $\Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}]$, emphasizes the point of view that the fluctuations of the metric, $h_{\mu\nu}$, may be regarded as *matter-like excitations on a classical spacetime with metric* $\bar{g}_{\mu\nu}$.

Instead, the "comma variant" $\Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau] \equiv \Gamma_k[\bar{g}+h,\bar{g},\xi,\bar{\xi};\beta,\tau]$, makes it explicit that the EAA suffers from an *extra* $\bar{g}$*-dependence* over and above the one that combines with h to build up a full metric $g = \bar{g} + h$. In fact, it is precisely the second argument of $\Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau]$ that amounts to the extra $\bar{g}$-dependence.

This seemingly trivial difference of the viewpoint will become important when we discuss the restoration of background independence later on. It is no exaggeration to say that, in the EAA framework, the most profound and deeply rooted difference between the standard matter quantum field theory on a classical spacetime and quantum gravity is the *inevitability of an extra Background Dependence.*

4.4 Properties of the Gravitational EAA

Let us now collect the main features of the EAA introduced in the previous section. We begin by streamlining our notation a bit. Note that $\bar{g}_{\mu\nu}$ is the only background field in the case at hand. In principle, one could also introduce background fields for the ghosts, but this is unnecessary here.

For brevity we shall often use the following collective notation for the various fields, their expectation values and sources:

$$\begin{aligned} \widehat{\varphi}^i &\equiv \big(\widehat{h}_{\mu\nu}, C^\mu, \bar{C}_\mu\big) && \text{integration variable,} \\ \bar{\Phi}^i &\equiv \big(\bar{g}_{\mu\nu}, 0, 0\big) && \text{background field,} \\ \varphi^i &\equiv \big(h_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu\big) && \text{expectation value (fluctuation),} \\ \Phi^i &\equiv \big(g_{\mu\nu}, \xi^\mu, \bar{\xi}_\mu\big) && \text{expectation value (full field),} \\ J_i &\equiv \big(t^{\mu\nu}, \bar{\sigma}_\mu, \sigma^\mu\big) && \text{sources for } \widehat{\varphi}, \\ J^{\text{BRST}} &\equiv \big(\beta^{\mu\nu}, \tau_\mu\big) && \text{sources for } \big(\boldsymbol{\delta}^{\text{BRST}} h_{\mu\nu}, \boldsymbol{\delta}^{\text{BRST}} C^\mu\big)\,. \end{aligned} \tag{4.71}$$

Thus the EAA reads $\Gamma_k\big[\varphi; J^{\text{BRST}}; \bar{\Phi}\big]$ or $\Gamma_k\big[\Phi, \bar{\Phi}; J^{\text{BRST}}\big]$, respectively.

(1) Background gauge invariance. The functionals $\widetilde{\Gamma}_k$ and Γ_k are invariant under infinitesimal general coordinate transformations where all its arguments transform as tensors of the corresponding rank:

$$\boxed{\Gamma_k\big[\varphi + \mathcal{L}_v\varphi; J^{\text{BRST}} + \mathcal{L}_v J^{\text{BRST}}; \bar{\Phi} + \mathcal{L}_v\bar{\Phi}\big] = \Gamma_k\big[\varphi; J^{\text{BRST}}; \bar{\Phi}\big].} \tag{4.72}$$

Note that (4.72) expresses the invariance of Γ_k under background gauge transformations $\boldsymbol{\delta}^{\text{B}}$ lifted to the level of the expectation values. Hence, contrary to the "quantum gauge transformation" (4.42), also the background metric transforms nontrivially here: $\delta\bar{g}_{\mu\nu} = \mathcal{L}_v\bar{g}_{\mu\nu}$.

Equation (4.72) is a consequence of the invariance enjoyed by the generating functional $W_k[J; J^{\text{BRST}}; \bar{\Phi}]$:

$$W_k[J + \mathcal{L}_v J; J^{\text{BRST}} + \mathcal{L}_v J^{\text{BRST}}; \bar{\Phi} + \mathcal{L}_v\bar{\Phi}] = W_k[J; J^{\text{BRST}}; \bar{\Phi}]. \tag{4.73}$$

This property follows from the functional integral (4.61) if one performs a compensating transformation (4.50), (4.52) on the integration variables $\widehat{h}_{\mu\nu}$, C^μ, $\bar{C}_\mu$, and uses the $\boldsymbol{\delta}^{\text{B}}$-invariance of $S[\bar{g} + \widehat{h}]$, S_{gf}, S_{gh}, and ΔS_k, respectively. At this point we also assume that the functional measure in (4.61) is diffeomorphism invariant.

The background gauge invariance of Γ_k, expressed in (4.72), is highly welcome from the practical point of view. It implies that if the initial functional contains only $\boldsymbol{\delta}^{\mathrm{B}}$-invariant terms, its RG evolution will not generate non-invariant terms. Very often this reduces the number of terms to be retained in a reliable truncation ansatz quite considerably.

(2) The infrared limit. Since the $\mathcal{R}_k$s vanish for $k=0$, the limit $k\to 0$ of $\Gamma_k[g_{\mu\nu},\bar{g}_{\mu\nu},\xi^\mu,\bar{\xi}_\mu]$ leads us to the standard effective action. It still depends on two metrics, though. The "ordinary" effective action $\Gamma[g_{\mu\nu}]$, having only one metric argument is obtained from this functional by setting $\bar{g}_{\mu\nu}=g_{\mu\nu}$, or equivalently $h_{\mu\nu}=0$ [87, 91, 143, 150–152]:

$$\begin{aligned}\Gamma[g_{\mu\nu}] &\equiv \lim_{k\to 0}\Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau]\Big|_{g=\bar{g},\,\xi=0=\bar{\xi},\,\beta=0=\tau}\\ &\equiv \lim_{k\to 0}\Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}]\Big|_{h=0,\,\xi=0=\bar{\xi},\,\beta=0=\tau}.\end{aligned}\tag{4.74}$$

This action brings about the "magic property" of the background field formalism: a priori the 1PI n-point functions of the metric are obtained by an n-fold functional differentiation of $\Gamma_0[h,0,0;0,0;\bar{g}]$ with respect to $h_{\mu\nu}$. Hereby $\bar{g}_{\mu\nu}$ is kept fixed; the background metric acts simply as an externally prescribed function which specifies the form of the gauge-fixing condition. Hence, the functional Γ_0 and the resulting *off-shell* Green functions do depend on $\bar{g}_{\mu\nu}$, while the *on-shell* Green functions, related to observable scattering amplitudes, do not depend on $\bar{g}_{\mu\nu}$. In this respect, $\bar{g}_{\mu\nu}$ plays a role that is conceptually similar to the gauge parameter α in the standard approach.

Remarkably enough, the very same on-shell Green functions can be obtained by differentiating the functional $\Gamma[g_{\mu\nu}]$ of (4.74) with respect to $g_{\mu\nu}$, or equivalently $\Gamma_0[h=0,0,0;0,0;\bar{g}=g]$, *with respect to its $\bar{g}$ argument.* (In this context, "on-shell" means that the metric satisfies the effective field equation $\delta\Gamma_0[g]/\delta g_{\mu\nu}=0$.)

(3) The functional RG equation. The derivation of an exact evolution equation for Γ_k proceeds as follows. Taking a derivative of the functional integral (4.61) with respect to the renormalization group time $t\equiv\ln k$ one obtains, in matrix notation:

$$-\partial_t W_k = \frac{1}{2}\,\mathrm{Tr}\left[\left\langle\widehat{h}\otimes\widehat{h}\right\rangle\left(\partial_t\widehat{\mathcal{R}}_k\right)_{hh}\right] - \mathrm{Tr}\left[\left\langle\bar{C}\otimes C\right\rangle\left(\partial_t\widehat{\mathcal{R}}_k\right)_{\bar{\xi}\xi}\right].\tag{4.75}$$

Here $\widehat{\mathcal{R}}_k$ is a matrix in field space whose non-zero entries are taken to be:

$$\begin{aligned}\left(\widehat{\mathcal{R}}_k\right)_{hh}^{\mu\nu\rho\sigma} &= \kappa^2\left(\mathcal{R}_k^{\mathrm{grav}}[\bar{g}]\right)^{\mu\nu\rho\sigma},\\ \left(\widehat{\mathcal{R}}_k\right)_{\bar{\xi}\xi} &= \sqrt{2}\,\mathcal{R}_k^{\mathrm{gh}}[\bar{g}].\end{aligned}\tag{4.76}$$

The right-hand side of (4.75) can be re-expressed directly in terms of Γ_k by noting that the connected two-point function

$$\begin{aligned} G^{ij}(x,y) &\equiv \big\langle \widehat{\varphi}^i(x)\widehat{\varphi}^j(y)\big\rangle - \varphi^i(x)\varphi^j(y) \\ &= \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}}\frac{\delta^2 W_k}{\delta J_i(x)\delta J_j(y)} \end{aligned} \tag{4.77}$$

and the Hessian of the Legendre transform $\widetilde{\Gamma}_k$

$$\big(\widetilde{\Gamma}_k^{(2)}\big)_{ij}(x,y) \equiv \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}}\frac{\delta^2\widetilde{\Gamma}_k}{\delta\varphi^i(x)\delta\varphi^j(y)} \tag{4.78}$$

are inverse matrices in the sense that

$$\int \mathrm{d}^d y\,\sqrt{\bar{g}(y)}\,G^{ij}(x,y)\big(\widetilde{\Gamma}_k^{(2)}\big)_{jl}(y,z) = \delta^i_l\,\frac{\delta(x-z)}{\sqrt{\bar{g}(z)}}. \tag{4.79}$$

Thus we arrive at the following exact functional RG equation for the gravitational Effective Average Action:

$$\boxed{\begin{aligned} &\partial_t\Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}] \\ &\quad= \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \widehat{\mathcal{R}}_k\right)^{-1}_{hh}\left(\partial_t\widehat{\mathcal{R}}_k\right)_{hh}\right] \\ &\quad\quad - \frac{1}{2}\,\mathrm{Tr}\left[\left\{\left(\Gamma_k^{(2)} + \widehat{\mathcal{R}}_k\right)^{-1}_{\bar{\xi}\xi} - \left(\Gamma_k^{(2)} + \widehat{\mathcal{R}}_k\right)^{-1}_{\xi\bar{\xi}}\right\}\left(\partial_t\widehat{\mathcal{R}}_k\right)_{\bar{\xi}\xi}\right]. \end{aligned}} \tag{4.80}$$

If one wants to express the right-hand side of this equation in terms of position-space matrix elements then the integration implied by the functional traces has to be interpreted as $\int \mathrm{d}^d x\sqrt{\bar{g}(x)}$, and correspondingly the matrix elements of the Hessian operator $\Gamma_k^{(2)}$ are given by

$$\big(\Gamma_k^{(2)}\big)_{ij}(x,y) \equiv \big\langle x,i\big|\Gamma_k^{(2)}\big|y,j\big\rangle = \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}}\frac{\delta^2\Gamma_k}{\delta\varphi^i(x)\delta\varphi^j(y)}. \tag{4.81}$$

The analogous matrix elements in the ghost sector are defined in terms of *left* derivatives; for example,

$$\Big(\big(\Gamma_k^{(2)}\big)_{\bar{\xi}\xi}\Big)_\mu{}^\nu(x,y) = \frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta}{\delta\xi^\mu(x)}\frac{1}{\sqrt{\bar{g}(y)}}\frac{\delta}{\delta\bar{\xi}_\nu(y)}\Gamma_k. \tag{4.82}$$

For any cutoff operator $\widehat{\mathcal{R}}_k$ with the correct qualitative properties the traces on the right-hand side of (4.80) are well convergent, both in the IR and the UV. By virtue of the factor $\partial_t\widehat{\mathcal{R}}_k$, the dominant contributions to the traces, now interpreted as a sum over eigenvalues, come from a narrow band of generalized momenta centered around k. Large momenta are exponentially suppressed.

Therefore, at this point, it is easily possible to *remove the (implicit) UV regulator* that we introduced above in order to render the derivation of the FRGE from the functional integral meaningful.

Henceforth we regard (4.80) as a *UV cutoff-free flow equation* which has lost all memory about the cutoff. The equation describes the change of Γ_k when its scale is lowered from k to $k - \mathrm{d}k$ and it is, in this sense, a "local" statement in eigenvalue space.

(4) The exact integro-differential equation. From the definition of the EAA in terms of the formal functional integral one easily derives the following, likewise formal, exact integro-differential equation satisfied by the gravitational average action:

$$\boxed{\begin{aligned}&\exp\bigl(-\Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}]\bigr)\\ &\quad=\int\mathcal{D}\widehat{h}\mathcal{D}C\mathcal{D}\bar{C}\,\exp\Biggl[-\widetilde{S}[\widehat{h},C,\bar{C};\beta,\tau;\bar{g}]\\ &\qquad+\int\mathrm{d}^dx\left\{\bigl(\widehat{h}_{\mu\nu}-h_{\mu\nu}\bigr)\frac{\delta\Gamma_k}{\delta h_{\mu\nu}}+\bigl(C^\mu-\xi^\mu\bigr)\frac{\delta\Gamma_k}{\delta\xi^\mu}+\bigl(\bar{C}_\mu-\bar{\xi}_\mu\bigr)\frac{\delta\Gamma_k}{\delta\bar{\xi}_\mu}\right\}\Biggr]\\ &\qquad\times\exp\bigl(-\Delta S_k[\widehat{h}-h,C-\xi,\bar{C}-\bar{\xi};\bar{g}]\bigr).\end{aligned}}\tag{4.83}$$

The action $\widetilde{S}$ which appears under the above integral is given by:

$$\widetilde{S}\equiv S+S_{\mathrm{gf}}+S_{\mathrm{gh}}-\int\mathrm{d}^dx\,\sqrt{\bar{g}}\left\{\beta^{\mu\nu}\mathcal{L}_C\bigl(\bar{g}_{\mu\nu}+\widehat{h}_{\mu\nu}\bigr)+\tau_\mu C^\nu\partial_\nu C^\mu\right\}.\tag{4.84}$$

Contrary to the FRGE, this functional integro-differential equation for Γ_k requires the presence of a UV regularization in order to be well defined.

(5) The modified BRST Ward identities. We mentioned already that the classical action plus the gauge-fixing and ghost terms are invariant under the BRST transformations (4.48). Therefore, the BRST variation of the total bare action $S_{\mathrm{tot}}\equiv S+S_{\mathrm{gf}}+S_{\mathrm{gh}}+\Delta S_k+S_{\mathrm{sources}}$ receives contributions only from the cutoff and the source terms.

If we apply a BRST transformation to the functional integral defining W_k, and assume that its measure is BRST invariant, we obtain

$$\langle\boldsymbol{\delta}_\varepsilon^{\mathrm{BRST}}S_{\mathrm{sources}}+\boldsymbol{\delta}_\varepsilon^{\mathrm{BRST}}\Delta S_k\rangle=0,\tag{4.85}$$

where the expectation value is to be interpreted as

$$\langle\mathcal{O}\rangle\equiv e^{-W_k}\int\mathcal{D}\widehat{h}\mathcal{D}C\mathcal{D}\bar{C}\,\mathcal{O}\,e^{-S_{\mathrm{tot}}}.\tag{4.86}$$

Our goal is to convert (4.85) to a statement about the average action Γ_k.

Because the BRST transformation (4.48) is off-shell nilpotent when acting on $\widehat{h}_{\mu\nu}$ and on C^μ (but not on $\bar{C}_\mu$) one has:

$$\begin{aligned}\boldsymbol{\delta}_\varepsilon^{\text{BRST}} S_{\text{sources}} = -\varepsilon\kappa^{-2}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,\Big\{ & t^{\mu\nu}\mathcal{L}_C\big(\bar{g}_{\mu\nu}+\widehat{h}_{\mu\nu}\big) \\ & -\bar{\sigma}_\mu C^\nu\partial_\nu C^\mu - \kappa\alpha^{-1}\sigma^\mu F_\mu(\bar{g},\widehat{h})\Big\}.\end{aligned} \tag{4.87}$$

If we take the expectation value of (4.87) and then express W_k in terms of Γ_k we are led to

$$\langle\boldsymbol{\delta}_\varepsilon^{\text{BRST}} S_{\text{sources}}\rangle = \frac{\varepsilon}{\kappa^2}\int \mathrm{d}^d x\frac{1}{\sqrt{\bar{g}(x)}}\left\{\frac{\delta\Gamma'_k}{\delta h_{\mu\nu}}\frac{\delta\Gamma'_k}{\delta\beta^{\mu\nu}}+\frac{\delta\Gamma'_k}{\delta\xi^\mu}\frac{\delta\Gamma'_k}{\delta\tau_\mu}\right\}+\frac{\varepsilon}{\kappa^2}\widetilde{Y}_k, \tag{4.88}$$

with

$$\widetilde{Y}_k \equiv \int \mathrm{d}^d x\left\{\frac{1}{\sqrt{\bar{g}}}\left(\frac{\delta\Delta S_k}{\delta h_{\mu\nu}}\frac{\delta\Gamma'_k}{\delta\beta^{\mu\nu}}+\frac{\delta\Delta S_k}{\delta\xi^\mu}\frac{\delta\Gamma'_k}{\delta\tau_\mu}\right)-\sqrt{2}\frac{\kappa}{\alpha}\sqrt{\bar{g}}F_\mu(\bar{g},h)\mathcal{R}_k^{\text{gh}}\xi^\mu\right\}. \tag{4.89}$$

Here we introduced the convenient abbreviation:

$$\Gamma'_k \equiv \Gamma_k - S_{\text{gf}}[h;\bar{g}]. \tag{4.90}$$

We also exploited the antighost's quantum equation of motion: $\big\langle\delta S_{\text{tot}}/\delta\bar{C}_\mu\big\rangle = 0$, which can be cast in the form

$$\left[\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta}{\delta\bar{\xi}_\mu(x)}-\sqrt{2}\,\bar{g}^{\mu\nu}\mathcal{F}_\nu{}^{\rho\sigma}\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta}{\delta\beta^{\rho\sigma}(x)}\right]\Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}]=0. \tag{4.91}$$

Furthermore, the BRST variation of the cutoff action gives rise to:

$$\langle\boldsymbol{\delta}_\varepsilon^{\text{BRST}}\Delta S_k\rangle = -\frac{\varepsilon}{\kappa^2}\left(Y_k+\widetilde{Y}_k\right), \tag{4.92}$$

with the following traces on its right-hand side:

$$\begin{aligned} Y_k \equiv \kappa^2\,\mathrm{Tr}&\left[(\mathcal{R}_k^{\text{grav}})^{\mu\nu\rho\sigma}\left(\Gamma_k^{(2)}+\widehat{\mathcal{R}}_k\right)^{-1}_{h_{\rho\sigma}\varphi}\frac{\delta^2\Gamma_k}{\sqrt{\bar{g}}\delta\varphi\sqrt{\bar{g}}\delta\beta^{\mu\nu}}\right] \\ &-\sqrt{2}\,\mathrm{Tr}\left[\mathcal{R}_k^{\text{gh}}\left(\Gamma_k^{(2)}+\widehat{\mathcal{R}}_k\right)^{-1}_{\bar{\xi}_\mu\varphi}\frac{\delta^2\Gamma_k}{\sqrt{\bar{g}}\delta\varphi\sqrt{\bar{g}}\delta\tau_\mu}\right] \\ &-2\alpha^{-1}\kappa^2\,\mathrm{Tr}\left[\mathcal{R}_k^{\text{gh}}\mathcal{F}_\mu{}^{\rho\sigma}\left(\Gamma_k^{(2)}+\widehat{\mathcal{R}}_k\right)^{-1}_{h_{\rho\sigma}\xi^\mu}\right].\end{aligned} \tag{4.93}$$

Herein the field components $\varphi \equiv \big\{h,\xi,\bar{\xi}\big\}$ are summed over.

From (4.88) and (4.92) we obtain the *modified BRST Ward identities* in their final form:

$$\boxed{\int \mathrm{d}^d x\frac{1}{\sqrt{\bar{g}(x)}}\left\{\frac{\delta\Gamma'_k}{\delta h_{\mu\nu}}\frac{\delta\Gamma'_k}{\delta\beta^{\mu\nu}}+\frac{\delta\Gamma'_k}{\delta\xi^\mu}\frac{\delta\Gamma'_k}{\delta\tau_\mu}\right\}=Y_k.} \tag{4.94}$$

Equation (4.94) has to be compared to the ordinary gravitational Ward identities [153] which are similar to (4.94) but have a vanishing right-hand side.

The contribution Y_k is due to the cutoff and therefore it vanishes for $k \to 0$ since $\mathcal{R}_k \propto k^2 \to 0$ in this limit. Hence the standard effective action $\Gamma = \Gamma_0$ indeed obeys the familiar BRST identities,

$$\int \mathrm{d}^d x \frac{1}{\sqrt{\bar{g}(x)}} \left\{ \frac{\delta \Gamma'}{\delta h_{\mu\nu}} \frac{\delta \Gamma'}{\delta \beta^{\mu\nu}} + \frac{\delta \Gamma'}{\delta \xi^\mu} \frac{\delta \Gamma'}{\delta \tau_\mu} \right\} = 0, \tag{4.95}$$

where $\Gamma' \equiv \Gamma - S_{\mathrm{gf}}$. Thus standard BRST invariance is restored for $k \to 0$.

(6) The Ward identity for split symmetry. The amount of "extra" $\bar{g}$-dependence the EAA is suffering from is governed by the following exact functional equation:

$$\boxed{\begin{aligned} &\frac{\delta}{\delta \bar{g}_{\mu\nu}(x)} \Gamma_k[g, \bar{g}, \xi, \bar{\xi}; \beta, \tau] \\ &\quad = \frac{1}{2} \mathrm{STr} \left[\left(\Gamma_k^{(2)}[g, \bar{g}, \cdots] + \mathcal{R}_k[\bar{g}] \right)^{-1} \frac{\delta \widetilde{S}^{(2)}[g, \bar{g}, \cdots]}{\delta \bar{g}_{\mu\nu}(x)} \right]. \end{aligned}} \tag{4.96}$$

It should be noted that this equation is written in terms of the functional $\Gamma_k[g, \bar{g}, \cdots]$ for which $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are the independent variables. Therefore, the derivative (4.96) really measures only that dependence on $\bar{g}_{\mu\nu}$, which does *not* combine with $h_{\mu\nu}$ and thus is "extra."

The relation (4.96) can be seen as the Ward identity associated with a linear split symmetry transformation.[19] The $\bar{g}$-derivative given by (4.96) would vanish identically if there was no breaking of split symmetry. This requires the Hessian $\widetilde{S}^{(2)}[g, \bar{g}, \cdots]$ to be independent of $\bar{g}$, which is clearly impossible to achieve because of the unavoidable split symmetry breaking by the coarse-graining operator $\Delta S_k^{(2)} = \mathcal{R}_k[\bar{g}]$ and the gauge-fixing sector.

(7) Ultraviolet behavior. In Section 4.1 we saw already in the scalar case that the exact integral equation satisfied by Γ_k is a convenient starting point for deducing its behavior at large values of k.

For a heuristic derivation one observes that when $k \to \infty$ the last exponential in (4.83) becomes proportional to a δ-functional, which equates the quantum fields $(\widehat{h}, C, \bar{C})$ to their classical counterparts:

$$e^{-\Delta S_k} \underset{k \to \infty}{\sim} \delta[\widehat{h} - h]\, \delta[C - \xi]\, \delta[\bar{C} - \bar{\xi}]. \tag{4.97}$$

[19] This type of Ward identity was first introduced in the context of Yang–Mills theory [29]. See Appendix A of [29] for a detailed derivation and discussion. It was first applied to gravity in [90].

As a consequence, the Effective Average Action for $k \to \infty$ reads:

$$\boxed{\begin{aligned}\Gamma_k[h,\xi,\bar{\xi};\beta,\tau;\bar{g}] = S[\bar{g}+h] + S_{\text{gf}}[h;\bar{g}] + S_{\text{gh}}[h,\xi,\bar{\xi};\bar{g}] \\ - \int \text{d}^d x\, \sqrt{\bar{g}}\, \big\{\beta^{\mu\nu}\mathcal{L}_\xi\big(\bar{g}_{\mu\nu}+h_{\mu\nu}\big) + \tau_\mu \xi^\nu \partial_\nu \xi^\mu\big\} + \cdots \,.\end{aligned}} \tag{4.98}$$

At the level of the restricted functional,

$$\bar{\Gamma}_k[g_{\mu\nu}] \equiv \Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau]\Big|_{g=\bar{g},\, \xi=0=\bar{\xi},\, \beta=0=\tau} \tag{4.99}$$

the UV behavior (4.98) boils down to:

$$\bar{\Gamma}_k[g_{\mu\nu}] = S[g_{\mu\nu}] + \cdots \,. \tag{4.100}$$

A more precise treatment employs a saddle-point approximation of the functional integral appearing in (4.83) and also determines the subleading terms which are indicated by the dots in the above equations. We will come back to this point in Chapter 8.

(8) Extended theory space and bi-metric character of the EAA. With the restricted effective action (4.99) we succeeded in constructing a diffeomorphism-invariant generating functional for the n-point functions of the metric which depends only on a single metric. Thanks to (4.72), the actions $\Gamma[g_{\mu\nu}]$ and $\bar{\Gamma}_k[g_{\mu\nu}]$ are indeed invariant under general coordinate transformations, $\delta g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu}$.

However, the simplified functional $\bar{\Gamma}_k[g_{\mu\nu}]$ is afflicted with the drawback that it does not satisfy any closed functional RG equation we could use to compute it without obtaining the full-fledged Γ_k first.

As a consequence, the actual RG evolution must be performed at the level of the much more complicated functional $\Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau]$. Only *after* having solved the FRGE we may equate the two metrics, i.e., let $\bar{g} = g$ in the list of arguments of Γ_k, along with setting the ghosts and BRST sources to zero.

As a result, the *true theory space of metric gravity*,

$$\mathcal{T} \equiv \big\{A[g,\bar{g},\xi,\bar{\xi};\beta,\tau]\big\}\,, \tag{4.101}$$

consists of functionals which depend not only on one, but rather two metrics, and in addition on Faddeev–Popov ghosts and BRST sources. Thus, unavoidably, the running action functionals are of the bi-metric type. Furthermore, the actions $A \in \mathcal{T}$ are required to comply with a set of constraints, including the invariance under background gauge transformations, (4.72), the BRST Ward identities (4.94), and the Ward identities for the split symmetry (4.96).

These constraints are "instantaneous" conditions on the form of the EAA; all quantities are evaluated at the same RG time $t = \ln(k/k_0)$. Importantly enough, they are consistent with the RG time evolution given by the flow equation since they are derived from the same underlying functional integral.

Coming back to the flow equation it is suggestive to abbreviate the FRGE (4.80) as

$$\boxed{\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \widehat{\mathcal{R}}_k\right)^{-1} \partial_t \widehat{\mathcal{R}}_k\right],} \tag{4.102}$$

where the supertrace "STr" takes care of the minus sign in (4.80) due to the odd Grassmann parity of the ghosts, and also implies a summation over the various fields, $\{h, \xi, \bar{\xi}\}$.

This makes it obvious that we did reach one of our goals, namely to preserve the simple formal structure displayed by the scalar flow equation even in the realm of Background Independent quantum gravity.

The *structural simplicity of the flow equation* comes at a considerable price, however, namely a highly *complicated space of functionals* on which it must be defined, and solved ultimately.

(9) Running field equations and self-consistent backgrounds. The EAA gives rise to a scale-dependent source–field relationship which includes an explicit cutoff term linear in the fluctuation field. From (4.67) with (4.68) we obtain, in compact notation,[20]

$$\boxed{\frac{1}{\sqrt{\bar{g}(x)}}\frac{\delta \Gamma_k[\varphi;\bar{\Phi}]}{\delta \varphi^i(x)} + \mathcal{R}_k[\bar{\Phi}]^i{}_j\, \varphi^j(x) = J^i(x).} \tag{4.103}$$

These coupled field equations generalize the classical Euler–Lagrange equations and determine the expectation values of the metric fluctuations and the ghosts in dependence on the external sources, the background, and the RG scale. The solutions $\varphi^i(x) \equiv \langle \widehat{\varphi}^i(x)\rangle = \varphi_k^i[J;\bar{\Phi}](x)$ describe how the quantum fluctuations respond to changes of the environment they are placed in. The latter is modeled by the background field $\bar{\Phi}$ and the sources J.

The magnitude of the quantum fluctuations decides to what extent the dynamically determined, actual expectation value $\langle\widehat{\Phi}\rangle$ deviates from the background which is prescribed externally:

$$\langle \widehat{\Phi}^i \rangle = \bar{\Phi}^i + \varphi_k^i[J;\bar{\Phi}]. \tag{4.104}$$

We are particularly interested in the distinguished backgrounds in which the quantum fluctuations, at $J=0$, are "too weak" to move $\langle\widehat{\Phi}\rangle$ away from $\bar{\Phi}$. By definition, such *self-consistent backgrounds* $\bar{\Phi} \equiv \bar{\Phi}^{\mathrm{sc}}$ have the property that $\varphi_k[J=0;\bar{\Phi}=\bar{\Phi}^{\mathrm{sc}}]=0$, whence $\langle\widehat{\Phi}\rangle = \bar{\Phi}^{\mathrm{sc}}$. In particular, in a self-consistent background, $\langle \widehat{h}_{\mu\nu}\rangle = 0$, and so $\langle \widehat{g}_{\mu\nu}\rangle = \bar{g}_{\mu\nu}^{\mathrm{sc}}$ [154].

20 For simplicity we suppress the BRST sources here.

A background is self-consistent if, with this background inserted, the field equation (4.103) for $J \equiv 0$ admits the solution $\varphi(x) \equiv 0$. It satisfies the *tadpole equation*, expressing a vanishing one-point function:

$$\boxed{\frac{\delta}{\delta\varphi^i(x)}\Gamma_k[\varphi;\bar{\Phi}]\bigg|_{\varphi=0;\bar{\Phi}=\bar{\Phi}_k^{\mathrm{sc}}} = 0.} \tag{4.105}$$

It is clear that in general solutions to (4.105) will have an explicit scale dependence, hence the notation $\bar{\Phi}_k^{\mathrm{sc}}$.

Let us be slightly more specific and focus on solutions involving vanishing ghost fields, $\xi^\mu = 0 = \bar{\xi}_\mu$.[21] Then self-consistent background metrics g_k^{sc} satisfy

$$\boxed{\frac{\delta}{\delta h_{\mu\nu}(x)}\Gamma_k[h,0,0;\bar{g}]\bigg|_{h=0;\bar{g}=\bar{g}_k^{\mathrm{sc}}} = 0.} \tag{4.106}$$

Note again that this is not a differential equation for $h_{\mu\nu}$, but a constraint on $\bar{g}_{\mu\nu}$. The tadpole equation (4.106) often serves as a powerful tool for extracting physical information from the EAA.

This completes our summary of the main properties enjoyed by the gravitational average action. In later chapters we will come back to them repeatedly.

[21] Note that $\xi^\mu = 0 = \bar{\xi}_\mu$ is always a possible solution of (4.103) since the ghost-charge neutrality of Γ_k forces ghost and antighost fields to occur in pairs.

5

Truncations of Single-Metric Type

In quantum gravity, as in basically all applications of the functional renormalization group, finding non-perturbative solutions to the pertinent flow equations is an extremely difficult task. It may come as a surprise that, *solving* the RG equations in gravity is usually the easier part of the problem: The true challenge consists in *deriving* them in an explicit form in the first place.

In fact, both for deep conceptual reasons related to the implementation of general covariance and Background Independence, for instance, and substantial technical difficulties, it is particularly hard to actually compute explicit RG flows from the rather abstract, albeit exact, functional differential equation for the gravitational Effective Average Action. The tensor character of the fields renders the analytical evaluation of the functional traces in the FRGE significantly more demanding than in the scalar theory encountered in Chapter 2 or similar matter field theories in flat space.

In practice, both on a rigid spacetime and in quantum gravity, the most important strategy for finding approximate, while still non-perturbative, solutions to the flow equation is the *truncation of theory space*. Hereby one projects the exact flow on the infinite-dimensional theory space onto a "smaller," technically tractable subspace, often parametrized by just a few generalized couplings, and then, possibly after some iterations of "trial and error," tries to prove that the truncated flow thus obtained approximates the exact one sufficiently well.

As a starting point, a still rather general class of truncations for the gravitational EAA has been proposed in [14], and for more than a decade all examples worked out explicitly were based on truncations belonging to this class. They are referred to as *single-metric truncations*, for a reason that will become clear in a moment.

In this chapter we first explain the general structure of the single-metric truncations, in Section 5.1. The subsequent sections then discuss a number of representative examples and the results achieved with them. The discussion focuses on the question of whether and to what extent those truncated RG flows support the Asymptotic Safety hypothesis in four-dimensional (and more generally, d dimensional) quantum gravity.

5.1 General Classes of Truncations

5.1.1 Freezing the Ghost Sector

As a first step toward simplifying the field dependence of the gravitational average action one can try to neglect the evolution of the ghost action. This amounts to making an ansatz of the following form:

$$\boxed{\begin{aligned}\Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau] = \Gamma_k^{\text{grav}}[g,\bar{g}] + S_{\text{gf}}[g-\bar{g};\bar{g}] + S_{\text{gh}}[g-\bar{g},\xi,\bar{\xi};\bar{g}]\\ - \int \mathrm{d}^d x\,\sqrt{\bar{g}}\big\{\beta^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu} + \tau_\mu \xi^\nu \partial_\nu \xi^\mu\big\}.\end{aligned}} \tag{5.1}$$

In (5.1) we pulled out the classical gauge-fixing and ghost actions S_{gf} and S_{gh} from Γ_k, and also assumed that the coupling to the BRST variations, for all k, has the same form as in the bare action. The remaining functional Γ_k^{grav} is allowed to depend on both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$.

Without loss of generality it can be decomposed as

$$\Gamma_k^{\text{grav}}[g,\bar{g}] \equiv \bar{\Gamma}_k[g] + \widehat{\Gamma}_k[g,\bar{g}], \tag{5.2}$$

where $\bar{\Gamma}_k[g] \equiv \Gamma_k^{\text{grav}}[g,g]$ is the restriction of Γ_k^{grav} to the diagonal, and $\widehat{\Gamma}_k \equiv \Gamma_k^{\text{grav}} - \bar{\Gamma}_k$ denotes the resulting remainder. Note that $\bar{\Gamma}_k$ is the same functional we encountered in (4.99),

$$\bar{\Gamma}_k[g] = \Gamma_k[g,g,0,0;0,0], \tag{5.3}$$

and it approaches the one in (4.74) for $k \to 0$. Furthermore, by definition, $\widehat{\Gamma}_k$ vanishes whenever its two arguments are equal:

$$\widehat{\Gamma}_k[g,g] = 0. \tag{5.4}$$

Note also that $\widehat{\Gamma}_k$ contains in particular quantum corrections to the gauge-fixing term, which also vanishes for $\bar{g} = g$.

At this point $\Gamma_k^{\text{grav}}[g,\bar{g}]$ is still a rather general "bi-metric" functional, i.e., it depends on two independent metrics, $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. However, it is constrained by the requirement that it must be invariant under the background gauge transformations (4.50), i.e.,

$$\boxed{\boldsymbol{\delta}^{\mathrm{B}}\Gamma_k^{\text{grav}}[g,\bar{g}] = 0.} \tag{5.5}$$

As a consequence, $\bar{\Gamma}_k[g]$ and $\widehat{\Gamma}_k[g,\bar{g}]$ are also $\boldsymbol{\delta}^{\mathrm{B}}$-invariant.

5.1.2 A Class of Bi-Metric Truncations

Let us now list a number of further properties of the special class of *bi-metric truncations* based on (5.1) satisfying (5.5):

(i) All actions of the form (5.1) are invariant under the background gauge transformations (4.50) with (4.52):

$$\boldsymbol{\delta}^{\mathrm{B}}\Gamma_k[g,\bar{g},\xi,\bar{\xi};\beta,\tau]=0. \tag{5.6}$$

(ii) The ansatz (5.1) complies with the large-k behavior (4.98) if, to lowest loop order,

$$\bar{\Gamma}_k[g]=S[g]+\cdots\,,\quad \widehat{\Gamma}_k[g,\bar{g}]=0+\cdots\quad \text{for } k\to\infty. \tag{5.7}$$

(iii) The ansatz (5.1) satisfies the antighost quantum equation of motion, (4.91), exactly.

(iv) Substituting the ansatz (5.1) into (4.94) one finds that the BRST Ward identity implies a non-trivial constraint on the $\widehat{\Gamma}_k$-part of Γ_k^{grav}:

$$\int \mathrm{d}^d x\, \frac{\delta\widehat{\Gamma}_k[g,\bar{g}]}{\delta g_{\mu\nu}(x)}\,\mathcal{L}_\xi g_{\mu\nu}=-Y_k. \tag{5.8}$$

Obviously the functional differential operator applied to $\widehat{\Gamma}_k[g,\bar{g}]$ on the left-hand side of (5.8) causes a gauge transformation on the first argument of $\widehat{\Gamma}_k[g,\bar{g}]$ only. Despite the $\boldsymbol{\delta}^{\mathrm{B}}$-invariance of $\widehat{\Gamma}_k$, the result can be non-zero because of the "extra $\bar{g}$-dependence" $\widehat{\Gamma}_k[g,\bar{g}]$ may possess.

We observe that $\widehat{\Gamma}_k=0$ is a good approximation provided we may neglect Y_k on the right-hand side of (5.8). The traces which define Y_k amount to loop integrals, and if we think in terms of a loop expansion, Y_k amounts to a higher loop effect and may be neglected in a first approximation.

At the non-perturbative level one can still try to set $\widehat{\Gamma}_k=0$ and investigate the consequences in concrete examples. In Yang–Mills theory the analogous truncation has been tested under more controlled conditions, and it led to rather encouraging results [22, 23]. In the following sections we perform various explicit calculations of the gravitational EAA in this approximation.

(v) The Ward identity for the split symmetry, (4.96), when projected down onto Γ_k^{grav}, yields likewise a rather complicated condition on the $\widehat{\Gamma}_k$-part; it has the general structure $(\frac{\delta}{\delta\bar{g}_{\mu\nu}})\widehat{\Gamma}_k[g,\bar{g}]=\frac{1}{2}\,\mathrm{STr}[\cdots]$. Like (5.8), it is satisfied by $\widehat{\Gamma}_k=0$ if the corrections from the supertrace can be neglected.

(vi) Inserting the truncation ansatz (5.1) into the exact FRGE (4.80) and discarding all types of terms absent from the ansatz we obtain the following *reduced functional RG equation*:

$$\boxed{\begin{aligned}\partial_t \Gamma_k[g,\bar{g}] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}] + \mathcal{R}_k^{\mathrm{grav}}[\bar{g}]\right)^{-1} \partial_t \mathcal{R}_k^{\mathrm{grav}}[\bar{g}]\right] \\ - \mathrm{Tr}\left[\left(-\mathcal{M}[g,\bar{g}] + \mathcal{R}_k^{\mathrm{gh}}[\bar{g}]\right)^{-1} \partial_t \mathcal{R}_k^{\mathrm{gh}}[\bar{g}]\right].\end{aligned}} \tag{5.9}$$

The first trace on the right-hand side of this equation is due to the metric fluctuations, $h_{\mu\nu}$, and the second one stems from the Faddeev–Popov ghosts. The equation is written in terms of the bi-metric action

$$\begin{aligned}\Gamma_k[g,\bar{g}] &\equiv \Gamma_k[g,\bar{g},0,0;0,0] \\ &= \Gamma_k^{\mathrm{grav}}[g,\bar{g}] + S_{\mathrm{gf}}[g-\bar{g};\bar{g}] \\ &= \bar{\Gamma}_k[g] + S_{\mathrm{gf}}[g-\bar{g};\bar{g}] + \widehat{\Gamma}_k[g,\bar{g}].\end{aligned} \tag{5.10}$$

Furthermore, $\Gamma_k^{(2)}$ denotes the Hessian of $\Gamma_k[g,\bar{g}]$ with respect to $g_{\mu\nu}$ at fixed $\bar{g}_{\mu\nu}$. Symbolically, $\Gamma_k^{(2)}[g,\bar{g}] \propto \left(\frac{\delta}{\delta g}\right)^2 \Gamma_k[g,\bar{g}]$. Choosing the generalized harmonic coordinate condition, for example, the classical kinetic operator of the ghosts, $\mathcal{M}$, is given by (4.56).

(vii) In Chapter 4 we discussed *self-consistent backgrounds* and saw in particular that those with vanishing ghosts are determined by the simplified tadpole equation (4.106). For the family of actions (5.1), (5.2) it assumes the form [154]

$$\boxed{\frac{\delta}{\delta g_{\mu\nu}(x)}\left\{\bar{\Gamma}_k[g] + \widehat{\Gamma}_k[g,\bar{g}]\right\}\Big|_{g=\bar{g};\bar{g}=\bar{g}_k^{\mathrm{sc}}} = 0.} \tag{5.11}$$

Note that the gauge-fixing term $S_{\mathrm{gf}} \propto \int (\mathcal{F}h)^2$ drops out from this equation since, being bilinear in $h \equiv g - \bar{g}$, its derivative vanishes at $h = 0$.

5.1.3 The Class of Single-Metric Truncations

Solving the reduced functional RG equation (5.9) is still not an easy task. Therefore, it seems advisable to restrict the truncation ansatz even further for a first analysis of this equation. A well-motivated subclass of the functionals (5.1) are the so-called *single-metric truncations* in which one either sets $\widehat{\Gamma}_k[g,\bar{g}]$ to zero exactly, or at most allows it to be of the same form as S_{gf}, though with a different prefactor:

$$\boxed{\widehat{\Gamma}_k[g,\bar{g}] \equiv 0 \qquad \text{or} \qquad \widehat{\Gamma}_k \propto S_{\mathrm{gf}}.} \tag{5.12}$$

Adopting (5.12), the only scale dependence in the EAA enters via $\bar{\Gamma}_k[g]$, and thus the truncated theory space boils down to diffeomorphism-invariant functionals of a *single* metric: $\mathcal{T}_{\text{trunc}} = \{\bar{A}[g] | \boldsymbol{\delta}^{\text{B}} \bar{A}[g] = 0\}$.

The definition of the truncation is completed by specifying how a generic functional $A[g, \bar{g}]$ is projected onto the subspace $\mathcal{T}_{\text{trunc}}$. We define this projection to act on bi-metric functionals by equating the two metrics,

$$A[g, \bar{g}] \mapsto A[g, \bar{g}]\Big|_{\bar{g}=g} \equiv \bar{A}[g] \tag{5.13}$$

or equivalently in the "semicolon notation," by setting the fluctuation field to zero:

$$A[h; \bar{g}] \mapsto A[h; \bar{g}]\Big|_{h=0,\, \bar{g}=g} \equiv \bar{A}[g]. \tag{5.14}$$

Note that since $\bar{A} \in \mathcal{T}_{\text{trunc}}$ depends on one field only, it does not matter whether we denote this argument by $g_{\mu\nu}$ or $\bar{g}_{\mu\nu}$. Therefore, replacing $\bar{g}$ by g in (5.14) is only a trivial renaming of the independent variable.

When we apply this projection to both sides of the reduced FRGE (5.9) we obtain a flow equation for the single-metric functional. The only place in this equation where the intrinsic bi-metric nature of the gravitational EAA still plays a role is the gauge-fixing term. Under the first trace in (5.9) we need the projected Hessian of $\Gamma_k[g, \bar{g}] = \bar{\Gamma}_k + S_{\text{gf}}$, i.e., $\Gamma_k^{(2)}[g, \bar{g}] = \bar{\Gamma}_k^{(2)}[g] + S_{\text{gf}}^{(2)}[g - \bar{g}; \bar{g}]$, evaluated at $h \equiv g - \bar{g} = 0$. Projecting the gauge-fixing term we encounter at first $S_{\text{gf}}^{(2)}[0; \bar{g}]$ which must be read as $\left(\frac{\delta}{\delta h}\right)^2 S_{\text{gf}}[h; \bar{g}]\big|_{h=0}$: the differentiation with respect to $h_{\mu\nu}$ is performed prior to setting $h_{\mu\nu} = 0$, of course. As a second, trivial step one may rename $g \to \bar{g}$, so that then, symbolically, $S_{\text{gf}}^{(2)}[0; g] \propto \left(\frac{\delta}{\delta h}\right)^2 S_{\text{gf}}[h; \bar{g}]\big|_{h=0,\, \bar{g}=g}$.

In this manner (5.9) projects onto the following *functional RG equation for the single-metric EAA*:

$$\boxed{\begin{aligned} \partial_t \bar{\Gamma}_k[g] = \frac{1}{2} \operatorname{Tr} \left[\left(\kappa^{-2} \bar{\Gamma}_k^{(2)}[g] + \kappa^{-2} S_{\text{gf}}^{(2)}[0; g] + \mathcal{R}_k^{\text{grav}}[g] \right)^{-1} \partial_t \mathcal{R}_k^{\text{grav}}[g] \right] \\ - \operatorname{Tr} \left[\left(-\mathcal{M}[g, g] + \mathcal{R}_k^{\text{gh}}[g] \right)^{-1} \partial_t \mathcal{R}_k^{\text{gh}}[g] \right]. \end{aligned}} \tag{5.15}$$

This equation inherits the background gauge invariance from its exact precursor. As a consequence, if a solution is invariant under general coordinate transformations at one scale, $\boldsymbol{\delta}^{\text{B}} \bar{\Gamma}_k[g] = 0$, it is so at any other scale as well.

As it stands, (5.15) applies to the case when $\widehat{\Gamma}_k$ is put to zero exactly. If instead $\widehat{\Gamma}_k$ is allowed to be proportional to S_{gf}, the prefactor of $S_{\text{gf}}^{(2)}$ in (5.15) changes correspondingly.

The ultimate justification of the truncation leading to (5.15) will consist in systematically enlarging the subspace $\mathcal{T}_{\text{trunc}}$ and demonstrating that the results stabilize at a certain point so that a not too large subspace $\mathcal{T}_{\text{trunc}}$ is already sufficient to achieve the desired level of accuracy.

There is a heuristic argument which indicates that the single-metric truncations are a sensible first guess to study: According to (ii) above, $\widehat{\Gamma}_k$ does indeed vanish for $k \to \infty$ and so it is plausible to try setting $\widehat{\Gamma}_k = 0$ also at lower scales. We know that this is in conflict with the BRST Ward identity (5.8), which tells us that $\widehat{\Gamma}_k = 0$ is only a good approximation provided we may neglect Y_k. The functional traces which define Y_k amount to loop integrals, so if we think in terms of a systematic loop expansion Y_k is indeed a higher-order effect and may be neglected in a first approximation.

At the non-perturbative level one can still employ the ansatz $\widehat{\Gamma}_k = 0$, try to justify it a posteriori by non-perturbative means, and investigate its consequences in concrete examples. In the next section we perform an explicit calculation in this approximation.

5.2 The Einstein–Hilbert Truncation

In this section we illustrate the use of the reduced FRGE (5.9) using the perhaps most natural example of a single-metric truncation. It is inspired by the classical Einstein–Hilbert action, in d dimensions:

$$S = \frac{1}{16\pi G} \int \mathrm{d}^d x \sqrt{g} \left\{ -R(g) + 2\Lambda \right\}. \tag{5.16}$$

We are going to use a truncation that replaces in (5.16) the classical Newton constant G and cosmological constant Λ by k-dependent functions G_k and $\bar{\lambda}_k$, respectively. We then determine the scale dependence of these running coupling constants by solving the flow equation (5.9) with an ansatz of the single-metric type:

$$\Gamma_k[g, \bar{g}] \equiv \bar{\Gamma}_k[g] + \widehat{\Gamma}_k + S_{\text{gf}}, \tag{5.17}$$

with $\widehat{\Gamma}_k \propto S_{\text{gf}}$. Explicitly, the Einstein–Hilbert truncation amounts to the choice

$$\boxed{\begin{aligned} \bar{\Gamma}_k[g] &= \frac{1}{16\pi G_k} \int \mathrm{d}^d x \sqrt{g} \left\{ -R(g) + 2\bar{\lambda}_k \right\}, \\ \widehat{\Gamma}_k[g, \bar{g}] &= \alpha \left(\tfrac{\bar{G}}{G_k} - \tfrac{1}{\alpha} \right) S_{\text{gf}}[g - \bar{g}; \bar{g}]. \end{aligned}} \tag{5.18}$$

The first equation of (5.18) is the expected k-dependent generalization of the classical Einstein–Hilbert action, while the second one exploits the possibility to change the gauge sector by a non-zero $\widehat{\Gamma}_k$ proportional to the classical S_{gf}. This choice for $\widehat{\Gamma}_k$ is motivated by the algebraic simplifications it will entail later on.

In (5.18), α is the gauge-fixing parameter that occurs in the prefactor of the gauge-fixing term, $S_{\text{gf}} \equiv \frac{1}{2\alpha} \int \mathrm{d}^d x \sqrt{\bar{g}}\, \bar{g}^{\mu\nu} F_\mu F_\nu$, and $\bar{G}$ denotes the k-independent reference value of the (dimensionful) Newton constant which we had introduced into the gauge-fixing functions of (4.54) via the constant mass parameter

$$\kappa \equiv (32\pi \bar{G})^{-\frac{1}{2}}. \tag{5.19}$$

With the above ansatz the calculation of the RG flow of G_k and $\bar{\lambda}_k$ will be simplest if one adopts a special member of the family of gauges $F_\mu[h;\bar{g}] = \sqrt{2}\kappa\, \mathcal{F}_\mu{}^{\alpha\beta}[\bar{g}]h_{\alpha\beta}$ with $\mathcal{F}_\mu{}^{\alpha\beta}[\bar{g}]h_{\alpha\beta} \equiv \bar{D}^\nu h_{\mu\nu} - \varpi\, \bar{D}_\mu\, \bar{g}^{\alpha\beta}h_{\alpha\beta}$. Letting $\varpi = \frac{1}{2}$, which amounts to the standard harmonic gauge, and choosing in addition the gauge-fixing parameter $\alpha = 1$, simplifies the algebra considerably.

It remains to clarify why we are allowed to freely add any convenient $\widehat{\Gamma}_k \propto S_{\text{gf}}$ to the truncation ansatz in the first place. After all, the chosen $\widehat{\Gamma}_k$ replaces the constant gauge-fixing parameter α in the prefactor of the original S_{gf} by the scale-dependent ratio $G_k/\bar{G}$. However, $S_{\text{gf}}[h;\bar{g}]$ is *bilinear* in the fluctuation field $h \equiv g - \bar{g}$, hence "bi-metric," and therefore, by the single-metric hypothesis, the scale dependence of its coefficient is assumed to have negligible, or acceptably small, impact on $\bar{\Gamma}_k$, i.e., on G_k and $\bar{\lambda}_k$. Hence, even if α is now k-dependent with a k-dependence inherited from G_k, which is probably not the correct one, it is consistent within the single-metric setting to neglect this effect. (Only in a much more ambitious bi-metric calculation would one include a running gauge-fixing action $\frac{1}{2\alpha_k}\int \mathrm{d}^d x\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}F_\mu F_\nu$ in the ansatz and then *compute* the actual running of α_k using the FRGE; see [155].)

It is convenient to refer the running Newton constant to its fixed reference value $\bar{G}$ by means of the dimensionless function $Z_{\mathrm{N}k} \equiv \bar{G}/G_k$:

$$G_k \equiv \frac{\bar{G}}{Z_{\mathrm{N}k}}. \tag{5.20}$$

The complete truncation ansatz then assumes the following form:

$$\boxed{\begin{aligned}\Gamma_k[g,\bar{g}] = 2\kappa^2 Z_{\mathrm{N}k} &\int \mathrm{d}^d x\,\sqrt{g}\,\big\{-R(g) + 2\bar{\lambda}_k\big\}\\ &+ \kappa^2 Z_{\mathrm{N}k}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}\big(\mathcal{F}_\mu{}^{\alpha\beta}g_{\alpha\beta}\big)\big(\mathcal{F}_\nu{}^{\rho\sigma}g_{\rho\sigma}\big).\end{aligned}} \tag{5.21}$$

In writing the gauge-fixing term we exploited that $\mathcal{F}_\mu{}^{\alpha\beta}g_{\alpha\beta} = \mathcal{F}_\mu{}^{\alpha\beta}h_{\alpha\beta}$ since $\bar{D}_\mu\bar{g}_{\alpha\beta} = 0$.

5.2.1 Derivation of the β-Functions for g and λ

In order to determine the functions $Z_{\mathrm{N}k}$ and $\bar{\lambda}_k$ we must project the evolution equation on the space spanned by the operators $\int\sqrt{g}$ and $\int\sqrt{g}R$. After having inserted the ansatz into the reduced evolution equation (5.9) and performed the

$g_{\mu\nu}$-derivatives we may set $\bar{g}_{\mu\nu} = g_{\mu\nu}$ so that the gauge-fixing term in (5.21) vanishes. As a result, the left-hand side of the evolution equation reads:

$$\partial_t \Gamma_k[g,g] = 2\kappa^2 \int \mathrm{d}^d x \sqrt{g} \left[-R(g)\partial_t Z_{\mathrm{N}k} + 2\partial_t \left(Z_{\mathrm{N}k} \bar{\lambda}_k \right) \right]. \tag{5.22}$$

On the right-hand side of (5.9) we must perform a derivative expansion and retain only the terms proportional to the two invariants $\int \sqrt{g}$ and $\int \sqrt{g}R$, respectively. Equating the result to (5.22) we can read off the sought-for system of ordinary differential equations for $Z_{\mathrm{N}k}$ and $\bar{\lambda}_k$.

(1) Under the trace appearing in the reduced FRGE we need the second functional derivative of $\Gamma_k[g,\bar{g}] \equiv \bar{\Gamma}_k + \widehat{\Gamma}_k + S_{\mathrm{gf}}$ at fixed $\bar{g}_{\mu\nu}$. We expand

$$\Gamma_k[\bar{g}+h,\bar{g}] = \Gamma_k[\bar{g},\bar{g}] + O(h) + \Gamma_k^{\mathrm{quad}}[h;\bar{g}] + O(h^3), \tag{5.23}$$

$$\Gamma_k^{\mathrm{quad}}[h;\bar{g}] = Z_{\mathrm{N}k}\kappa^2 \int \mathrm{d}^d x \sqrt{\bar{g}}\, h_{\mu\nu} \left[-K^{\mu\nu}{}_{\rho\sigma} \bar{D}^2 + U^{\mu\nu}{}_{\rho\sigma} \right] h^{\rho\sigma}. \tag{5.24}$$

Here and in the following, indices are always raised and lowered with $\bar{g}_{\mu\nu}$. Furthermore, the tensors K and U are given by

$$K^{\mu\nu}{}_{\rho\sigma} = \frac{1}{4}\left[\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma} \right] \tag{5.25}$$

and

$$\begin{aligned} U^{\mu\nu}{}_{\rho\sigma} = &\frac{1}{4}\left[\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma} \right] \left(\bar{R} - 2\bar{\lambda}_k \right) + \frac{1}{2}\left[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu} \right] \\ &- \frac{1}{4}\left[\delta^\mu_\rho \bar{R}^\nu{}_\sigma + \delta^\mu_\sigma \bar{R}^\nu{}_\rho + \delta^\nu_\rho \bar{R}^\mu{}_\sigma + \delta^\nu_\sigma \bar{R}^\mu{}_\rho \right] - \frac{1}{2}\left[\bar{R}^\nu{}_\rho{}^\mu{}_\sigma + \bar{R}^\nu{}_\sigma{}^\mu{}_\rho \right]. \end{aligned} \tag{5.26}$$

In (5.26) all geometrical quantities with an overbar are constructed from the background metric.[1]

(2) In order to partially diagonalize the quadratic form (5.24) we decompose $h_{\mu\nu}$ as the sum of a traceless tensor $\mathring{h}_{\mu\nu}$ and a trace part involving $\phi \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$:

$$h_{\mu\nu} = \mathring{h}_{\mu\nu} + d^{-1}\bar{g}_{\mu\nu}\phi, \quad \bar{g}^{\mu\nu}\mathring{h}_{\mu\nu} = 0. \tag{5.27}$$

As a consequence, (5.24) becomes

$$\begin{aligned} \Gamma_k^{\mathrm{quad}}[h;\bar{g}] = Z_{\mathrm{N}k}\kappa^2 \int \mathrm{d}^d x \sqrt{\bar{g}} \Bigg\{ &\frac{1}{2}\mathring{h}_{\mu\nu} \Big[-\bar{D}^2 - 2\bar{\lambda}_k + \bar{R} \Big] \mathring{h}^{\mu\nu} \\ &- \left(\frac{d-2}{4d} \right) \phi \left[-\bar{D}^2 - 2\bar{\lambda}_k + \frac{d-4}{d}\bar{R} \right] \phi \\ &- \bar{R}_{\mu\nu}\mathring{h}^{\nu\rho}\mathring{h}^\mu{}_\rho + \bar{R}_{\alpha\beta\nu\mu}\mathring{h}^{\beta\nu}\mathring{h}^{\alpha\mu} + \frac{d-4}{d}\phi\bar{R}_{\mu\nu}\mathring{h}^{\mu\nu} \Bigg\}. \end{aligned} \tag{5.28}$$

[1] Our conventions for curvature tensors are summarized in Appendix A.

(3) The differential equations for $Z_{\mathrm{N}k}$ and $\bar{\lambda}_k$ are obtained by comparing the coefficients of $\int \sqrt{g}$ and $\int \sqrt{g}R$ on both sides of the evolution equation at $\bar{g}_{\mu\nu} = g_{\mu\nu}$. For this purpose we may insert an arbitrary family of metrics $\bar{g}_{\mu\nu}$ which is general enough to identify the terms $\int \sqrt{\bar{g}}$ and $\int \sqrt{\bar{g}}\bar{R}$ and to distinguish them from higher-order terms in the derivative expansion, such as $\int \sqrt{\bar{g}}\bar{R}^2$ or $\int \sqrt{\bar{g}}\bar{R}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu\bar{R}$, for instance.

We exploit this freedom by assuming that $\bar{g}_{\mu\nu}$ corresponds to a *maximally symmetric space* so that its Riemann and Ricci tensor, respectively, have the structure [156]

$$\begin{aligned}\bar{R}_{\mu\nu\rho\sigma} &= \frac{1}{d(d-1)}\left[\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\right]\bar{R},\\ \bar{R}_{\mu\nu} &= \frac{1}{d}\bar{g}_{\mu\nu}\bar{R}.\end{aligned} \tag{5.29}$$

From now on the curvature scalar $\bar{R}$ parametrizes the family of metrics inserted, and it should be regarded as an externally prescribed number rather than a functional of the metric.

For a maximally symmetric background the quadratic action reads

$$\begin{aligned}\Gamma_k^{\mathrm{quad}}[h;\bar{g}] = \frac{1}{2}Z_{\mathrm{N}k}\kappa^2 \int \mathrm{d}^d x \Big\{ &\mathring{h}_{\mu\nu}\left[-\bar{D}^2 - 2\bar{\lambda}_k + C_T\bar{R}\right]\mathring{h}^{\mu\nu}\\ &- \left(\frac{d-2}{2d}\right)\phi\left[-\bar{D}^2 - 2\bar{\lambda}_k + C_S\bar{R}\right]\phi\Big\},\end{aligned} \tag{5.30}$$

with the dimension-dependent constants

$$C_T \equiv \frac{d(d-3)+4}{d(d-1)}, \quad C_S \equiv \frac{d-4}{d}. \tag{5.31}$$

(4) Before continuing we need to specify the precise form of the cutoff operators $\mathcal{R}_k^{\mathrm{grav}}$ and $\mathcal{R}_k^{\mathrm{gh}}$ which describe the transition from the high-momentum to the low-momentum regime. Either of them has the structure

$$\mathcal{R}_k[\bar{g}] = \mathcal{Z}_k k^2 R^{(0)}\left(-\tfrac{\bar{D}^2}{k^2}\right) \tag{5.32}$$

whereby the matrix $\mathcal{Z}_k$ should be adjusted in such a way that for every low-momentum mode the cutoff combines with the kinetic term of this mode to $(-\bar{D}^2 + k^2)$ times a constant.

Looking at (5.30) we see that the respective kinetic terms for $\mathring{h}_{\mu\nu}$ and ϕ differ by a factor of $-(d-2)/2d$. This suggests the following choice:

$$(\mathcal{Z}_k^{\mathrm{grav}})^{\mu\nu\rho\sigma} = \left[(\mathbf{1} - P_\phi)^{\mu\nu\rho\sigma} - \frac{d-2}{2}P_\phi^{\mu\nu\rho\sigma}\right]Z_{\mathrm{N}k}. \tag{5.33}$$

Here $(\mathbf{1})_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\delta_\mu^\rho\delta_\nu^\sigma + \delta_\nu^\rho\delta_\mu^\sigma\right)$ is the unit on the space of symmetric 2-tensors and

$$(P_\phi)_{\mu\nu}{}^{\rho\sigma} = \frac{\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}}{d} \tag{5.34}$$

is the projector on the trace part of the metric. For the traceless tensor, (5.33) yields $\mathcal{Z}_k^{\text{grav}} = Z_{\text{N}k}\mathbf{1}$, while for ϕ the different normalization is taken into account.

With this cutoff we obtain the following operators in the $\mathring{h}$ and the ϕ-sector, respectively:

$$\begin{aligned}\left(\kappa^{-2}\Gamma_k^{(2)}[g,g]+\mathcal{R}_k^{\text{grav}}\right)_{\mathring{h}\mathring{h}} &= Z_{\text{N}k}\left[-D^2+k^2R^{(0)}\left(-\tfrac{D^2}{k^2}\right)-2\bar{\lambda}_k+C_T R\right]\\ \left(\kappa^{-2}\Gamma_k^{(2)}[g,g]+\mathcal{R}_k^{\text{grav}}\right)_{\phi\phi} &= -\frac{d-2}{2d}\\ &\quad\times Z_{\text{N}k}\left[-D^2+k^2R^{(0)}\left(-\tfrac{D^2}{k^2}\right)-2\bar{\lambda}_k+C_S R\right].\end{aligned} \tag{5.35}$$

At this point we may set $\bar{g}=g$, thus omitting the bars from the metric and the curvature in the following.

(5) The last ingredient that is required before we can assemble the entire RG equation is the Faddeev–Popov operator. From (4.56) one obtains, at $\bar{g}=g$,

$$\mathcal{M}[g,g]^\mu{}_\nu = \delta^\mu_\nu D^2 + R^\mu{}_\nu = -\delta^\mu_\nu\left[-D^2+C_V R\right], \tag{5.36}$$

with the constant

$$C_V \equiv -\frac{1}{d}. \tag{5.37}$$

The second equality in (5.36) follows upon using (5.29) for a maximally symmetric background. Since we neglect all renormalization effects in the ghost action we set $\mathcal{Z}_k^{\text{gh}} \equiv 1$ in $\mathcal{R}_k^{\text{gh}}$ and obtain

$$-\mathcal{M}+\mathcal{R}_k^{\text{gh}} = -D^2+k^2R^{(0)}\left(-\tfrac{D^2}{k^2}\right)+C_V R. \tag{5.38}$$

(6) Equipped with the cutoff and Hessian operator (5.36) and (5.38) we can now return to the FRGE.

Let us write $\mathcal{S}_k(R)$ for the right-hand side of the reduced RG equation (5.9) evaluated at $\bar{g}=g$. Inserting (5.36) and (5.38) there we arrive at

$$\begin{aligned}\mathcal{S}_k(R) = {}&\text{Tr}_T\left[\mathcal{N}(\mathcal{A}+C_T R)^{-1}\right]+\text{Tr}_S\left[\mathcal{N}(\mathcal{A}+C_S R)^{-1}\right]\\ &-2\,\text{Tr}_V\left[\mathcal{N}_0(\mathcal{A}_0+C_V R)^{-1}\right],\end{aligned} \tag{5.39}$$

with the two operators

$$\begin{aligned}\mathcal{A} &\equiv -D^2+k^2R^{(0)}\left(-\tfrac{D^2}{k^2}\right)-2\bar{\lambda}_k,\\ \mathcal{N} &\equiv (2Z_{\text{N}k})^{-1}\,\partial_t\left[Z_{\text{N}k}k^2R^{(0)}\left(-\tfrac{D^2}{k^2}\right)\right]\\ &= \left[1-\frac{1}{2}\eta_{\text{N}}(k)\right]k^2R^{(0)}\left(-\tfrac{D^2}{k^2}\right)+D^2R^{(0)\prime}\left(-\tfrac{D^2}{k^2}\right).\end{aligned} \tag{5.40}$$

Here a prime denotes the derivative with respect to the argument, and

$$\boxed{\eta_{\mathrm{N}} \equiv -\partial_t \ln Z_{\mathrm{N}k}} \tag{5.41}$$

is the anomalous dimension related to the running Newton constant. Furthermore, the operators $\mathcal{N}_0$ and $\mathcal{A}_0$ are defined similarly to (5.40) but with $\bar{\lambda}_k = 0$ and $Z_{\mathrm{N}k} = 1$, i.e., $\eta_{\mathrm{N}}(k) = 0$.

The right-hand side of the flow equation, (5.39), involves traces of functions of the covariant Laplacian $D^2 \equiv g^{\mu\nu} D_\mu D_\nu$ acting on traceless symmetric tensors ("T"), scalars ("S"), and vectors ("V"). Because we need only the zeroth and the first order in the curvature scalar we can expand in R:

$$\begin{aligned}\mathcal{S}_k(R) = {} & \mathrm{Tr}_T \left[\mathcal{N}\mathcal{A}^{-1}\right] + \mathrm{Tr}_S \left[\mathcal{N}\mathcal{A}^{-1}\right] - 2\,\mathrm{Tr}_V \left[\mathcal{N}_0\mathcal{A}_0^{-1}\right] \\ & - R\left(C_T\,\mathrm{Tr}_T \left[\mathcal{N}\mathcal{A}^{-2}\right] + C_S\,\mathrm{Tr}_S \left[\mathcal{N}\mathcal{A}^{-2}\right] - 2C_V\,\mathrm{Tr}_V \left[\mathcal{N}_0\mathcal{A}_0^{-2}\right]\right) \\ & + O(R^2).\end{aligned} \tag{5.42}$$

(7) We evaluate the traces in (5.42) by taking advantage of the familiar heat kernel expansion[2]

$$\mathrm{Tr}\left[e^{-isD^2}\right] = \left(\frac{i}{4\pi s}\right)^{d/2} \mathrm{tr}(I) \int \mathrm{d}^d x\,\sqrt{g}\,\left\{1 - \frac{1}{6} isR\right\} + O(R^2). \tag{5.43}$$

Here I denotes the unit matrix of the space of fields on which D^2 acts. Hence, $\mathrm{tr}(I)$ is the number of independent field components. In the case at hand we encounter in particular

$$\begin{aligned}\mathrm{tr}_S(I) &= 1, \\ \mathrm{tr}_V(I) &= d, \\ \mathrm{tr}_T(I) &= \frac{1}{2}(d-1)(d+2).\end{aligned} \tag{5.44}$$

As for generic functions of the covariant Laplacian, let us consider an arbitrary function W with Fourier transform $\widetilde{W}$, i.e., $W(z) = \int_{-\infty}^{\infty} \mathrm{d}s \widetilde{W}(s) e^{isz}$. Then the expansion of the trace

$$\mathrm{Tr}[W(-D^2)] = \int_{-\infty}^{\infty} \mathrm{d}s\,\widetilde{W}(s)\,\mathrm{Tr}\left[e^{-isD^2}\right] \tag{5.45}$$

is given by [14]:

$$\boxed{\begin{aligned}\mathrm{Tr}[W(-D^2)] = (4\pi)^{-d/2}\,\mathrm{tr}(I) \Big\{ & Q_{d/2}[W] \int \mathrm{d}^d x\,\sqrt{g} \\ & + \frac{1}{6} Q_{d/2-1}[W] \int \mathrm{d}^d x\,\sqrt{g} R \Big\} + O(R^2).\end{aligned}} \tag{5.46}$$

[2] See the Appendix for a brief review of the heat kernel and Laplace transform techniques needed here.

Here we introduced the "Q-functionals"

$$Q_n[W] \equiv \int_{-\infty}^{\infty} \mathrm{d}s\, (-is)^{-n} \widetilde{W}(s). \tag{5.47}$$

It is not difficult to reexpress (5.47) in terms of the original function $W(z)$ itself. We obtain [14]:

$$\boxed{\begin{aligned} &Q_0[W] = W(0), \\ &Q_n[W] = \frac{1}{\Gamma(n)} \int_0^{\infty} \mathrm{d}z\, z^{n-1} W(z) \qquad \text{for } n > 0, \\ &Q_n[W] = \Big(-\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{|n|} W(z)\Big|_{z=0} \qquad \text{for } n < 0. \end{aligned}} \tag{5.48}$$

(8) The next step toward the beta functions consists of using (5.46) in order to evaluate the traces in (5.42) and to combine the resulting expression $\mathcal{S}_k(R)$ with the left-hand side of the evolution equation, (5.22).

Comparing the coefficients of $\int \sqrt{g}$ we read off the following equation:

$$\begin{aligned} \partial_t \left(Z_{\mathrm{N}k} \bar{\lambda}_k\right) = (4\kappa^2)^{-1} (4\pi)^{-d/2} \Big\{ &\mathrm{tr}_T(I) Q_{d/2}[\mathcal{N}/\mathcal{A}] \\ &+ \mathrm{tr}_S(I) Q_{d/2}[\mathcal{N}/\mathcal{A}] - 2\, \mathrm{tr}_V(I) Q_{d/2}[\mathcal{N}_0/\mathcal{A}_0] \Big\}. \end{aligned} \tag{5.49}$$

Likewise, equating the prefactors of $\int \sqrt{g} R$ gives rise to:

$$\begin{aligned} \partial_t Z_{\mathrm{N}k} = &-(12\kappa^2)^{-1} (4\pi)^{-d/2} \\ &\times \Big[\mathrm{tr}_T(I) \big\{ Q_{d/2-1}[\mathcal{N}/\mathcal{A}] - 6 C_T Q_{d/2}[\mathcal{N}/\mathcal{A}^2] \big\} \\ &+ \mathrm{tr}_S(I) \big\{ Q_{d/2-1}[\mathcal{N}/\mathcal{A}] - 6 C_S Q_{d/2}[\mathcal{N}/\mathcal{A}^2] \big\} \\ &- 2\, \mathrm{tr}_V(I) \big\{ Q_{d/2-1}[\mathcal{N}_0/\mathcal{A}_0] - 6 C_S Q_{d/2}[\mathcal{N}_0/\mathcal{A}_0^2] \big\} \Big]. \end{aligned} \tag{5.50}$$

In (5.49) and (5.50), $\mathcal{N}$ and $\mathcal{A}$ are considered functions of the real variable z now which replaces $-D^2$ in (5.40).

To proceed it is convenient to introduce the following *normalized threshold functions* [14]:

$$\boxed{\begin{aligned} \Phi_n^p(w) &\equiv \frac{1}{\Gamma(n)} \int_0^{\infty} \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z) - z R^{(0)\prime}(z)}{\left[z + R^{(0)}(z) + w\right]^p}, \\ \widetilde{\Phi}_n^p(w) &\equiv \frac{1}{\Gamma(n)} \int_0^{\infty} \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z)}{\left[z + R^{(0)}(z) + w\right]^p}. \end{aligned}} \tag{5.51}$$

Here $p = 1, 2, 3, \cdots$, and $n \geq 0$. A careful evaluation of the limit $n \searrow 0$ yields:

$$\Phi_0^p(w) = \widetilde{\Phi}_0^p(w) = (1 + w)^{-p}. \tag{5.52}$$

After having decided about a concrete cutoff shape function $R^{(0)}(z)$ it is straightforward to perform the z-integrations and thus obtain the threshold functions related to this shape function; see Appendix E for examples.

When expressed in terms of these threshold functions, (5.49) assumes the form

$$\begin{aligned}\partial_t\left(Z_{\mathrm{N}k}\bar{\lambda}_k\right) = (16\kappa^2)^{-1}(4\pi)^{-d/2}k^d\Big[&2d(d+1)\Phi^1_{d/2}\left(-\tfrac{2\bar{\lambda}_k}{k^2}\right)\\ &-8d\Phi^1_{d/2}(0)-d(d+1)\eta_{\mathrm{N}}\widetilde{\Phi}^1_{d/2}\left(-\tfrac{2\bar{\lambda}_k}{k^2}\right)\Big].\end{aligned} \tag{5.53}$$

Similarly the second equation (5.50) becomes

$$\begin{aligned}\partial_t Z_{\mathrm{N}k} = &-(24\kappa^2)^{-1}(4\pi)^{-d/2}k^{d-2}\\ &\times\Big[d(d+1)\Big\{\Phi^1_{d/2-1}\left(-\tfrac{2\bar{\lambda}_k}{k^2}\right)-\frac{1}{2}\eta_{\mathrm{N}}\widetilde{\Phi}^1_{d/2-1}\left(-\tfrac{2\bar{\lambda}_k}{k^2}\right)\Big\}\\ &-6d(d-1)\Big\{\Phi^2_{d/2}\left(-\tfrac{2\bar{\lambda}_k}{k^2}\right)-\frac{1}{2}\eta_{\mathrm{N}}\widetilde{\Phi}^2_{d/2}\left(-\tfrac{2\bar{\lambda}_k}{k^2}\right)\Big\}\\ &-4d\Phi^1_{d/2-1}(0)-24\Phi^2_{d/2}(0)\Big].\end{aligned} \tag{5.54}$$

At this point we could integrate the two equations (5.53), (5.54) and obtain the running dimensionful Newton and cosmological constant, respectively.

5.2.2 Structure of the β-Functions for g and λ

Let us switch to *dimensionless coupling constants* now. We introduce the dimensionless running Newton constant

$$g_k \equiv k^{d-2}G_k \equiv k^{d-2}Z_{\mathrm{N}k}^{-1}\bar{G} \tag{5.55}$$

and the dimensionless cosmological constant:

$$\lambda_k \equiv k^{-2}\bar{\lambda}_k. \tag{5.56}$$

Recall that $G_k \equiv Z_{\mathrm{N}k}^{-1}\bar{G}$ is the *dimensionful* running Newton constant at scale k, and that $[G_k] = 2-d$ and $[\bar{\lambda}_k] = 2$ in d spacetime dimensions.

Applying ∂_t to (5.55) we are led to the following *RG equation for the dimensionless Newton constant*:

$$\boxed{\partial_t g_k = \left[d-2+\eta_{\mathrm{N}}(g_k,\lambda_k)\right]g_k.} \tag{5.57}$$

From (5.54) we obtain an implicit equation which allows us to determine the anomalous dimension η_{N} as a function of g_k and λ_k:

$$\eta_{\mathrm{N}}(k) = g_k\,B_1(\lambda_k)+\eta_{\mathrm{N}}(k)\,g_k\,B_2(\lambda_k). \tag{5.58}$$

The coefficient functions B_1 and B_2 read explicitly:

$$\boxed{\begin{aligned} B_1(\lambda_k) \equiv{}& \frac{1}{3}(4\pi)^{1-d/2}\times \\ &\times\Big[d(d+1)\Phi^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\Phi^2_{d/2}(-2\lambda_k) \\ &- 4d\Phi^1_{d/2-1}(0) - 24\Phi^2_{d/2}(0)\Big], \\ B_2(\lambda_k) \equiv{}& -\frac{1}{6}(4\pi)^{1-d/2}\times \\ &\times\Big[d(d+1)\widetilde{\Phi}^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\widetilde{\Phi}^2_{d/2}(-2\lambda_k)\Big].\end{aligned}} \tag{5.59}$$

We can easily solve the relation (5.58) for η_{N} in terms of g_k and λ_k:

$$\boxed{\eta_{\mathrm{N}}(g_k,\lambda_k) = \frac{g_k B_1(\lambda_k)}{1 - g_k B_2(\lambda_k)}.} \tag{5.60}$$

Note that the anomalous dimension has no explicit k-dependence. It depends on the scale only via g_k and λ_k.

As for the cosmological constant, the scale derivative of λ_k is related to (5.53) according to

$$\partial_t \lambda_k = -(2-\eta_{\mathrm{N}})\lambda_k + 32\pi g_k \kappa^2 k^{-d} \partial_t(Z_{\mathrm{N}k}\bar{\lambda}_k) \tag{5.61}$$

so that we obtain the following *RG equation for the cosmological constant*:

$$\boxed{\begin{aligned} \partial_t\lambda_k = \beta_\lambda(g_k,\lambda_k) ={}& -(2-\eta_{\mathrm{N}})\lambda_k + \frac{1}{2}g_k(4\pi)^{1-d/2} \\ &\times\Big[2d(d+1)\Phi^1_{d/2}(-2\lambda_k) - 8d\Phi^1_{d/2}(0) \\ &- d(d+1)\eta_{\mathrm{N}}\widetilde{\Phi}^1_{d/2}(-2\lambda_k)\Big].\end{aligned}} \tag{5.62}$$

The pair of equations (5.57) and (5.62) constitutes a system of coupled differential equations for the dimensionless Newton and cosmological constant.

This system of RG equations has the general structure

$$\boxed{\begin{aligned} &\partial_t g_k = \beta_g(g_k,\lambda_k) \equiv \big[d-2+\eta_{\mathrm{N}}(g_k,\lambda_k)\big]g_k, \\ &\partial_t\lambda_k = \beta_\lambda(g_k,\lambda_k).\end{aligned}} \tag{5.63}$$

On its right-hand side it displays the two-component vector field $\boldsymbol{\beta} = (\beta_\lambda, \beta_g)$ comprised of the dimensionless β functions β_g and β_λ, respectively. As expected on general grounds, the system has indeed become autonomous after switching to dimensionless couplings.

The g_k and λ_k dependence of the anomalous dimension $\eta_{\mathrm{N}}(g_k,\lambda_k)$ has the structure of (5.60) whereby the two functions B_1 and B_2 are explicitly given by (5.59). Furthermore, β_λ is defined by the right-hand side of (5.62).

As we can see, the non-trivial content of this system of RG equations is encoded in the three functions $B_1(\lambda)$, $B_2(\lambda)$, and $\beta_\lambda(g,\lambda)$, respectively, which we expressed in terms of the standardized threshold functions Φ^p_n and $\widetilde{\Phi}^p_n$ defined in (5.51).

(1) Threshold functions and scheme dependence. The functions Φ^p_n and $\widetilde{\Phi}^p_n$ are manifestly cutoff-scheme dependent via the shape function $R^{(0)}$. The latter is required to interpolate smoothly between $R^{(0)}(0)=1$ and $R^{(0)}(\infty)=0$ but is quite arbitrary otherwise. In addition, typical regulators $R^{(0)}$ do not vanish too quickly for small values of its argument, i.e., $z+R^{(0)}(z)\geq 1$, $\forall z\geq 0$.

The definition of the threshold functions (5.51) immediately implies that $\Phi^p_n(w)$, $\widetilde{\Phi}^p_n(w)\to 0$ for $w\to\infty$.[3] Furthermore, one notes by examining the denominator of the integrands that the functions $\Phi^p_n(w)$ and $\widetilde{\Phi}^p_n(w)$ are well defined only for arguments $w>-1$. For $w<-1$ the denominator vanishes for some z in the region of integration, yielding a non-integrable singularity. Therefore, $\Phi^p_n(w)$ and $\widetilde{\Phi}^p_n(w)$ are finite and well defined for $w\in(-1,\infty)$ only.

In β_g and β_λ the threshold functions are being evaluated at $w=-2\lambda$. As a consequence, *the beta functions are well defined for* $\lambda<\frac{1}{2}$ *only.*

(2) Sharp cutoff. A particularly convenient choice of $R^{(0)}(z)$ for which the z-integration can be done analytically is the *sharp cutoff* (sc) defined in (E.4). It leads to a perfectly explicit final form of the RG equations for spacetimes of any dimensionality d.

Upon substituting the threshold functions (E.5), the corresponding functions B_1, B_2, and β_λ read as follows:

$$\boxed{\begin{aligned} B_1(\lambda)^{\text{sc}} &= \frac{1}{3}(4\pi)^{1-d/2}\Big\{-\frac{d(d+1)}{\Gamma(d/2-1)}\ln(1-2\lambda)+d(d-3)\varphi_{d/2-1}\\ &\qquad -\frac{6d(d-1)}{\Gamma(d/2)}\frac{1}{1-2\lambda}-\frac{24}{\Gamma(d/2)}\Big\},\\ B_2(\lambda)^{\text{sc}} &= -\frac{1}{6}(4\pi)^{1-d/2}\frac{d(d+1)}{\Gamma(d/2)},\\ \beta_\lambda(\lambda,g)^{\text{sc}} &= -(2-\eta_{\text{N}}^{\text{sc}})\lambda+\frac{1}{2}(4\pi)^{1-d/2}g\Big[-\frac{2d(d+1)}{\Gamma(d/2)}\ln(1-2\lambda)\\ &\qquad +2d(d-3)\varphi_{d/2}-\frac{d(d+1)}{\Gamma(d/2+1)}\eta_{\text{N}}^{\text{sc}}\Big]. \end{aligned}} \tag{5.64}$$

[3] If $z+R^{(0)}(z)\geq\varepsilon$ we may use $\frac{1}{[z+R^{(0)}(z)+w]^p}\leq\frac{1}{[\varepsilon+w]^p}$ under the z-integrals of (5.51). Since the right-hand side of this estimate is independent of z, we may pull it out of the integral, which still converges then thanks to the fall-off properties of $R^{(0)}$. As a consequence, this prefactor causes the threshold functions (with $p>0$) to vanish in the limit $w\to\infty$.

As always, $\eta_{\mathrm{N}}(g,\lambda)$ is of the form (5.60), where the functions $B_{1,2} \equiv B_{1,2}^{\mathrm{sc}}$ are now given by the above expressions. Furthermore, the φ_ns are numerical constants whose values are given in (E.12).

Note that the sharp cutoff is special in the sense that $B_2(\lambda)^{\mathrm{sc}}$ turns out to be a constant, i.e., independent of λ.

In the following sections we analyze the RG flow implied by the Einstein–Hilbert truncation in detail. We close this section with a number of further comments.

(3) Resummed perturbative contributions. The above RG equations while approximate are nevertheless of a manifestly non-perturbative nature. Taylor expanding for example (5.60) with respect to the dimensionless Newton constant, we obtain

$$\eta_{\mathrm{N}}(g,\lambda) = B_1(\lambda) \sum_{n=0}^{\infty} g^{n+1} \big(B_2(\lambda)\big)^n. \tag{5.65}$$

Obviously the Einstein–Hilbert truncation sums up certain contributions to η_{N} and $\beta_g \equiv [d-2+\eta_{\mathrm{N}}]g$, which are of arbitrarily high order in the coupling constant g.

(4) The "$\mathcal{Z}=\zeta$"-rule. Let us return to the idea of an adjusted cutoff, which was briefly mentioned in Section 5.2.5.

There we described a rule for fixing the matrix $\mathcal{Z}_k$ that appears in the ansatz for the cutoff operator, $\mathcal{R}_k = \mathcal{Z}_k k^2 R^{(0)}(-D^2/k^2)$. We applied this rule in the above calculation where it led to the form (5.33) of the submatrix $\mathcal{Z}_k^{\mathrm{grav}}$ for the graviton sector. However, there is a rather subtle issue related to the choice (5.33), as we discuss next.

Let us try to apply the rule formulated in Section 5.2.5 to the cutoff for $h_{\mu\nu}$. We leave the truncation arbitrary for a moment and concentrate on the contribution of a fixed mode χ contained in $h_{\mu\nu}$. We assume that χ is an eigenfunction of $\Gamma_k^{(2)}$ with eigenvalue $\zeta_k p^2$, where p^2 is a positive eigenvalue of some covariant kinetic operator, typically of the form $-\bar{D}^2 + \bar{R}$*-terms*. For theories with $\Gamma_k^{(2)} > 0$, the wave-function normalization ζ_k is positive. In this case our general rule is to set $\mathcal{Z}_k = \zeta_k$ because this guarantees that for the low-momentum modes the effective inverse propagators $\Gamma_k^{(2)} + \mathcal{R}_k$ become $\zeta_k(p^2+k^2)$, as it should be.

In the Einstein–Hilbert truncation, the condition $\Gamma_k^{(2)} > 0$ is met for the traceless part of the metric fluctuation, $\mathring{h}_{\mu\nu}$. From (5.30) we can read off the corresponding ζ_k, and it is found to be positive.

The trace part ϕ, on the other hand, is seen to possess a *negative* normalization constant ζ_k in all dimensions $d>2$. So the important question is *how do we choose $\mathcal{Z}_k$ if ζ_k is negative*?

If we continue to use $\mathcal{Z}_k = \zeta_k$, the FRGE is perfectly well defined because the inverse propagator $-|\zeta_k|(p^2+k^2)$ never vanishes, and the functional traces on its right-hand side do not suffer from any IR problems. For instance, if we write down their perturbative expansion we see that all propagators are correctly cut off in the IR, and that loop momenta smaller than k are properly suppressed. Moreover, due to the projective structure of the trace arguments, the overall minus sign drops out so that the conformal mode contributes like a scalar field with a standard kinetic term.

On the other hand, if we set $\mathcal{Z}_k = -\zeta_k$, we are led to $-|\zeta_k|(p^2 - k^2)$, which introduces a spurios singularity at $p^2 = k^2$, and so the cutoff operator fails to make the theory IR finite in this case.

At first sight the choice $\mathcal{Z}_k = |\zeta_k|$ might have appeared the more natural one because only if $\mathcal{Z}_k > 0$ the factor $\exp(-\Delta S_k) \sim \exp\big(-\int \mathcal{R}_k \phi^2\big)$ is a damped exponential which suppresses the low momentum modes in the usual way.

Nevertheless, there are good reasons to adopt the rule of setting $\mathcal{Z}_k = \zeta_k$ *for either sign of* ζ_k.

As for the conformal factor of the metric, our above choice (5.33) complies with this rule. At least in the Einstein–Hilbert truncation the evolution equations are well defined when $\mathcal{Z}_k = \zeta_k$ (but not with $\mathcal{Z}_k = |\zeta_k|$) even though it appears even more difficult than usual to give a meaning to the functional integral itself then. In the case $\mathcal{Z}_k = \zeta_k < 0$ the factor $\exp\big(+\int |\mathcal{R}_k| \phi^2\big)$ is a growing exponential and it might seem that this enhances rather than suppresses the low momentum modes. However, as suggested by the perturbative argument above, this conclusion is too naive probably. (If one invokes, for example, Hawking's prescription of rotating the contour of integration over ϕ so that it is parallel to the imaginary axis, both the kinetic term and the cutoff lead to damped exponentials.)

The "$\mathcal{Z} = \zeta$"-rule may find further justification when one deals with quantum mechanics-type functional integrals over oscillating exponentials e^{iS} rather than e^{-S}. For the FRGE this change is by far less dramatic than for the integral. Apart from the obvious substitutions $\Gamma_k \to -i\Gamma_k$, $\mathcal{R}_k \to -i\mathcal{R}_k$, the evolution equation remains structurally unaltered. For $\mathcal{Z}_k = \zeta_k$ it has all desired features, and $\zeta_k < 0$ does not seem to pose a special problem.

5.3 Properties of the Einstein–Hilbert Flow

In order to gain a first understanding of the RG trajectories from the Einstein–Hilbert truncation, arising as the solutions of the system (5.63), we begin by studying the two possible "subtruncations" which it admits.

At first, in Section 5.3.1 we will omit the cosmological constant term in the ansatz for Γ_k, and solve the remaining RG equation for Newton's constant alone. Conversely, in Section 5.3.2, we freeze the running of G_k, setting $G_k \equiv \text{const}$ in the $\int \sqrt{g} R$-term, and study the resulting simplified scale dependence of the cosmological constant.

In Section 5.3.3 and 5.3.4 we then explore the fixed points of the full coupled system for g_k and λ_k, and analyze its entire phase portrait.

Finally, after also introducing some close "relatives" of the Einstein–Hilbert truncation, and discussing the available methods to judge the reliability of truncations, we summarize the status of the Asymptotic Safety conjecture in light of those findings in Section 5.3.7.

From now on we focus on four spacetime dimensions ($d=4$). The generalization to $d\neq 4$ would be straightforward; we will comment on the corresponding results later on.

5.3.1 The Subtruncation of Vanishing λ

Let us consider the one-dimensional truncation with $g(k)$ as the only running coupling constant. Omitting the term $\bar{\lambda}(k)\int \mathrm{d}^d x\sqrt{g}$ from the ansatz for Γ_k gives rise to a single differential equation for Newton's constant then. It is obtained from (5.63) by letting $\lambda\equiv 0$ in the first equation, and discarding the second one.[4] Thus we are left with

$$\boxed{k\partial_k g(k)=\big[2+\eta_{\mathrm{N}}(g(k))\big]g(k)\equiv\beta_g(g(k)),}\tag{5.66}$$

with η_{N} and β_g given by

$$\boxed{\eta_{\mathrm{N}}(g)=\frac{B_1\,g}{1-B_2\,g}\Longleftrightarrow\beta_g(g)=2g\frac{1-\omega'\,g}{1-B_2\,g}.}\tag{5.67}$$

At this level, B_1 and B_2 denote simple, albeit cutoff-dependent constants:

$$\begin{aligned}B_1&\equiv B_1(0)=-\frac{1}{3\pi}\big[24\Phi^2_2(0)-\Phi^1_1(0)\big],\\ B_2&\equiv B_2(0)=\frac{1}{6\pi}\big[18\widetilde{\Phi}^2_2(0)-5\widetilde{\Phi}^1_1(0)\big].\end{aligned}\tag{5.68}$$

For convenience we also abbreviate

$$\omega\equiv-\frac{1}{2}B_1,\qquad \omega'\equiv\omega+B_2.\tag{5.69}$$

Picking, for example, the exponential cutoff (E.1) we have explicitly

$$\begin{aligned}\Phi^1_1(0)^{\exp}&=\frac{\pi^2}{6},\qquad &\Phi^2_2(0)^{\exp}&=1,\\ \widetilde{\Phi}^1_1(0)^{\exp}&=1,\qquad &\widetilde{\Phi}^2_2(0)^{\exp}&=\frac{1}{2},\end{aligned}\tag{5.70}$$

which entails the following two *positive* constants:

$$\omega=\frac{4}{\pi}\Big(1-\frac{\pi^2}{144}\Big),\qquad B_2=\frac{2}{3\pi}.\tag{5.71}$$

[4] We shall often write $g(k)\equiv g_k$, $\lambda(k)\equiv\lambda_k$, etc., for RG trajectories, $k\mapsto\big(g(k),\lambda(k)\big)$, and (g,λ) for the coordinates on the dimensionless theory space.

While the precise value of ω is cutoff dependent, it turns out that

$$\boxed{\omega \equiv -\frac{1}{2} B_1(0) > 0} \tag{5.72}$$

for any admissible choice of the shape function $R^{(0)}$. The relevance of a *positive* value of ω will become clear in a moment.

(1) We observe that $\beta_g(g)$ has two zeros, namely at $g=0$ and $g=\frac{1}{\omega'}$, respectively. As a result, the evolution equation (5.66) displays two fixed points g_*, $\beta(g_*)=0$, namely an infrared-attractive *Gaussian fixed point* at $g_*^{\text{GFP}}=0$ and a ultraviolet-attractive *non-Gaussian fixed point* at

$$g_*^{\text{NGFP}} = \frac{1}{\omega'}. \tag{5.73}$$

The non-Gaussian fixed point (NGFP) separates a weak coupling regime ($g < g_*^{\text{NGFP}}$) from a strong coupling regime where $g > g_*^{\text{NGFP}}$. Since the β-function is positive for $g \in [0, g_*^{\text{NGFP}}]$ and negative otherwise the solutions $g(k)$, the "renormalization group trajectories," fall into the following three classes:

(i) Trajectories with $g(k) < 0$ for all k. They are attracted toward g_*^{GFP} for $k \to 0$.
(ii) Trajectories with $g(k) > g_*^{\text{NGFP}}$ for all k. They are attracted toward g_*^{NGFP} for $k \to \infty$.
(iii) Trajectories with $g(k) \in [0, g_*^{\text{NGFP}}]$ for all k. They are attracted toward $g_*^{\text{GFP}}=0$ for $k \to 0$ and towards g_*^{NGFP} for $k \to \infty$.

Only the trajectories in (iii) are relevant to us. We will not allow for a negative Newton constant, and we also discard the solutions of (ii) as they are entirely within the strong coupling region and do not connect to a classical large distance regime.

(2) The differential equation (5.66) with (5.67) is easily integrated analytically, yielding

$$\frac{g}{(1-\omega' g)^{\frac{\omega}{\omega'}}} = \frac{g(k_0)}{\left[1-\omega' g(k_0)\right]^{\frac{\omega}{\omega'}}} \left(\frac{k}{k_0}\right)^2. \tag{5.74}$$

This expression cannot be solved for $g=g(k)$ in closed form. However, it is obvious that this solution indeed interpolates between the IR behavior $g(k) \propto k^2$ for $k^2 \to 0$ and $g(k) \to \frac{1}{\omega'}$ for $k \to \infty$.

In order to obtain an approximate analytic expression for the running Newton constant we observe that the ratio ω'/ω is actually very close to unity. With the exponential shape function one has $\omega \approx 1.2$, $B_2 \approx 0.21$, $\omega' \approx 1.4$, $g_*^{\text{NGFP}} \approx 0.71$ so that $\omega'/\omega \approx 1.18$ is indeed not far from unity.

Replacing $\omega'/\omega \to 1$ in (5.74) yields a rather accurate approximation with the same general features as the exact solution. With this approximation we can easily solve (5.74):

$$g(k) = \frac{g(k_0)k^2}{\omega g(k_0)k^2 + [1 - \omega g(k_0)]k_0^2}. \tag{5.75}$$

This function is actually an *exact* solution to the renormalization group equation containing the approximate anomalous dimension

$$\boxed{\eta_{\mathrm{N}}(g) = -2\omega g + O(g^2),} \tag{5.76}$$

which is nothing but the first term in the perturbation series of (5.65):

$$\eta_{\mathrm{N}}(g) = -2\omega g \left[1 + \sum_{n=1}^{\infty} (B_2\, g)^n\right]. \tag{5.77}$$

It is reassuring that for the trajectory (5.75) the product $B_2\, g(k)$ remains negligibly small for all values of k. It assumes its largest value at the UV fixed point where $B_2\, g_*^{\mathrm{NGFP}} \approx 0.15$. Thus the linear approximation $\eta_{\mathrm{N}} \propto g$ and the resulting equation (5.75) provide us with a reliable approximation to the exact solution of (5.66) and (5.67).

Within this approximation, the fixed point coordinate (5.73) gets replaced with $g_*^{\mathrm{NGFP}} = \frac{1}{\omega}$, and (5.75) indeed approaches this value asymptotically:

$$\lim_{k\to\infty} g(k) = g_*^{\mathrm{NGFP}}. \tag{5.78}$$

(3) We mentioned already that the constant ω is always positive. This implies a negative anomalous dimension for the physically relevant case of a positive Newton constant:

$$\boxed{\eta_{\mathrm{N}} < 0 \quad \text{if} \quad g > 0} \quad \text{(antiscreening).} \tag{5.79}$$

Recalling the definition of the anomalous dimension,

$$\eta_{\mathrm{N}} \equiv -k\partial_k \, \ln Z_{\mathrm{N}k} \equiv -k\partial_k \, \ln \tfrac{\bar{G}}{G_k} = \frac{k\partial_k G_k}{G_k}, \tag{5.80}$$

we therefore conclude that *the dimensionful Newton constant is a decreasing function of the scale k.*

This behavior predicted by the Einstein–Hilbert truncation, first obtained in [14], is known as *gravitational antiscreening*: Newton's constant G_k decreases as k increases, it is small in the UV, and grows larger as we evolve it toward the infrared.

The sign of this effect is the same as for the non-abelian gauge coupling in Yang-Mills theory and it is opposite to the one in QED.

Interpreting high momentum scales k as short distances, we see that gravity is antiscreening in the sense that at large distances Newton's constant is larger than at small distances. This fits well with the intuitive picture that the gravitational charge, the mass, is not screened by quantum fluctuations but rather receives an additional positive contribution from the virtual particles and graviton excitations surrounding it. This behavior is opposite to the one found in QED where virtual charges screen the electromagnetic field.

(4) In terms of the dimensionful Newton constant $G(k) \equiv g(k)/k^2$ the solution (5.75) reads

$$G(k) = \frac{G(k_0)}{1 + \omega G(k_0)[k^2 - k_0^2]}. \tag{5.81}$$

Henceforth, we set $k_0 = 0$ for the reference scale and identify $G_0 \equiv G(0)$ with the observed value of Newton's constant in the extreme IR. This leads to the following very simple, but instructive formula for the running dimensionful Newton constant [157]:

$$\boxed{G(k) = \frac{G_0}{1 + \omega\, G_0\, k^2}.} \tag{5.82}$$

This equation describes a smooth, monotonic interpolation between a *classical regime* in the infrared where, approximately, $G(k) = \text{const} = G_0$, and a *fixed point regime* in the ultraviolet where instead $g(k) = \text{const} = g_*^{\text{NGFP}}$. The crossover between those regimes takes place when $k \approx m_{\text{Pl}}$, i.e., near the *Planck scale* defined by $m_{\text{Pl}} \equiv G_0^{-1/2}$.

For $k \ll m_{\text{Pl}}$ we may expand (5.82),

$$\boxed{\begin{aligned} G(k) &= G_0 - \omega G_0^2 k^2 + O(G_0^3 k^4) \\ &\equiv G_0 \left[1 - \omega \left(\frac{k}{m_{\text{Pl}}} \right)^2 + O\left(\frac{k^4}{m_{\text{Pl}}^4} \right) \right]. \end{aligned}} \tag{5.83}$$

Obviously the leading quantum corrections at low scales are suppressed by a factor of $\left(\frac{k}{m_{\text{Pl}}}\right)^2$.

In the other extreme, for $k \gg m_{\text{Pl}}$, the fixed point behavior prevails, i.e., $G(k)$ "forgets" its IR value G_0 and evolves according to the power law:

$$\boxed{G(k) = \frac{1}{\omega} \frac{1}{k^2} = \frac{g_*^{\text{NGFP}}}{k^2}.} \tag{5.84}$$

As it should be, the corresponding dimensionless Newton constant $g(k) \equiv k^2 G(k)$ approaches the constant value g_*^{NGFP} there.

(5) Even at the exact, i.e., untruncated, level and in all truncations more general than the present one, the existence of a NGFP, for purely dimensional reasons, always leads to the power law $G(k) = g_*^{\text{NGFP}} k^{2-d}$ near the fixed point, i.e., $G(k) \propto k^{-2}$ in $d = 4$. According to the definition (5.80), the k^{-2}-behavior of the dimensionful Newton constant is tantamount to the fixed point value of the anomalous dimension

$$\boxed{\eta_{\mathrm{N}}^{\mathrm{NGFP}} = -2.} \tag{5.85}$$

Therefore, by continuity, we always have $\eta_{\mathrm{N}} < 0$, hence antiscreening, on trajectories near to a NGFP. However, there is no general reason that would guarantee η_{N} to be negative for all scales along a given trajectory. This happened to be the case for the trajectories we computed above, but as we will see later on, more general (bi-metric) truncations can change the picture for $k \to 0$.

5.3.2 The Subtruncation of Constant G

Turning to the second specialization of the Einstein–Hilbert system of RG equations, we now consider a segment of an RG trajectory, which is near the classical regime so that we may ignore the k-dependence of the dimensionful Newton constant on all scales of interest. We retain the $\int \sqrt{g}R$-term in the truncation ansatz, however, furnishing it with a constant prefactor in which $G(k) \equiv \mathrm{const} \equiv G_0$. At the dimensionless level this translates into $g(k) = G_0 k^2$. This simple function is then inserted into the RG equation (5.62) for the cosmological constant.

Exploiting that $\eta_{\mathrm{N}} = 0$ when G is constant we obtain the following differential equation for the dimensionful cosmological constant, $\bar{\lambda}_k$:

$$k\partial_k \, \bar{\lambda}(k) = \frac{1}{2\pi} k^4 G_0 \left[10\Phi_2^1 \left(-\tfrac{2\bar{\lambda}(k)}{k^2} \right) - 8\Phi_2^1(0) \right]. \tag{5.86}$$

We observe that the value of $\bar{\lambda}(k)$ backreacts in a potentially complicated way on its own RG running via the threshold function Φ_2^1. This possibility is typical of the FRGE-based approach; it is not found in a standard perturbative treatment.

Equation (5.86) simplifies further when $\bar{\lambda} \ll k^2$ so that it is legitimate to replace $\bar{\lambda}(k)/k^2 \to 0$ in the argument of the threshold function. The equation is easy to integrate then. Imposing initial conditions at an arbitrary scale k_1, the solution reads

$$\boxed{\bar{\lambda}(k) = \frac{1}{4\pi} \Phi_2^1(0) G_0 \left[k^4 - k_1^4 \right] + \bar{\lambda}(k_1).} \tag{5.87}$$

This equation describes how the cosmological constant changes upon integrating out the metric fluctuations, or “virtual gravitons,” which have momenta between k and k_1.

(1) With (5.87) we have recovered the well-known, quartically cutoff-dependent cosmological constant $\bar{\lambda}(k) \propto k^4$ which one encounters in basically any quantum field theory model when performing perturbative loop calculations. Nevertheless, more often than not it is simply ignored as one is interested in physics on a non-dynamical flat spacetime only, or one compensates it by a finetuned counter term (which amounts to the same), or one tries to find regularization schemes that put it to zero formally (as in a version of dimensional regularization).

The perhaps best known calculation leading to the quartic cutoff dependence is based on the following semiclassical argument, presumably due to Pauli [158]:

A non-zero cosmological constant $\bar{\lambda} \equiv (8\pi G_0)\rho_{\text{vac}}$ represents a certain vacuum energy density ρ_{vac}. The vacuum energy in turn receives contributions from all (matter, gauge, graviton, etc.) field modes, each of them supplying its ground state energy "$\frac{1}{2}\hbar\omega(\boldsymbol{p})$." The total vacuum energy is then identified with a sum (integral) over all modes, i.e., all 3-momenta $\boldsymbol{p}$. For a massless free field with $\omega(\boldsymbol{p}) = |\boldsymbol{p}|$, for example, we obtain

$$\rho_{\text{vac}} = \frac{\hbar}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, |\boldsymbol{p}| + \text{const.} \tag{5.88}$$

This integral is highly UV divergent and requires regularization. Restricting $|\boldsymbol{p}|$ to a finite interval one finds a quartic dependence of ρ_{vac} on both the UV and the IR cutoff. This is indeed what we also found in (5.87).

Even if we restrict the integration to field modes between $|\boldsymbol{p}| = 0$ and, say, the scale of atomic physics only, the contribution to $\bar{\lambda}$ which one obtains is already larger than its measured value by many orders of magnitude. As a result, the corresponding counter term, the constant in (5.88), must be finetuned at an enormous level of precision, which is usually considered "unnatural."

With this finetuning we encounter one face of the notorious *cosmological constant problem* [159, 158].

We will return to this issue repeatedly. Even if Asymptotic Safety *per se* does not solve the problem fully, it puts it into a new perspective that will become clear eventually.

At this point we continue with a number of remarks related to (5.86) and (5.87).

(2) Taken at face value, (5.87) entails the same finetuning issue as the mode-summation argument: Leaving the possibility of an asymptotically safe UV limit aside for a moment, we adjust the constant of integration in (5.87) at the Planck scale, $k_1 = m_{\text{Pl}} \equiv G_0^{-1/2}$ and assume that at this scale the cosmological constant assumes a "natural" value of the order of the Planck scale, i.e., $\bar{\lambda}(m_{\text{Pl}}) = C m_{\text{Pl}}^2$ with a dimensionless constant $C = O(1)$. From (5.87) we then obtain the following value at $k = 0$:

$$\bar{\lambda}(0) = \left(C - \frac{1}{4\pi} \Phi_2^1(0) \right) m_{\text{Pl}}^2. \tag{5.89}$$

For the sake of argument we further assume that $\bar{\lambda}(0)$ and $G_0 \equiv 1/m_{\text{Pl}}^2$ can be identified with the cosmological constant and Newton constant measured in real nature. Then $\bar{\lambda}(0)$ is known to be about 120 orders of magnitude smaller than m_{Pl}^2. Hence the constant C must be fixed in such a way that $\big(C - \frac{1}{4\pi}\Phi_2^1(0)\big) \approx 10^{-120}$, whereby both C and $(1/4\pi)\Phi_2^1(0)$ are of order unity. This implies that the initial value $\bar{\lambda}(k_1)$ must be finetuned with a precision of about 120 digits, and this, again, amounts to the "cosmological constant problem."

(3) However, there are various reasons to believe that within the FRGE and Asymptotic Safety context the above reasoning is way too simplistic and perhaps even wrong. First of all, (5.87) is the result of drastic truncations and approximations. It might be that this formula for $\bar{\lambda}(k)$ is incorrect even at the qualitative level. In fact, later on we shall see that the Einstein–Hilbert truncation is likely to become questionable at *too low* momentum scales, i.e., *in the infrared.*

(4) Furthermore, even if (5.87) was qualitatively correct, $\bar{\lambda}(k)$ may not be straightforwardly inserted into the standard Einstein field equation in order to relate it to, say, the observed expansion rate of the universe.

It is obvious that $\bar{\lambda}(k)$ cannot have the status of an observable, neither in the approximation (5.87), *nor at the exact level*, since the "RG trajectory" $\bar{\lambda}(k)$ is manifestly cutoff-scheme dependent.

In (5.87), the prefactor $\Phi_2^1(0)$ depends on the shape function $R^{(0)}$, and the exact RG equations that determine $\bar{\lambda}(k)$ together with all other couplings contain further, likewise non-universal threshold functions. It is therefore structurally impossible to relate $\bar{\lambda}(k)$ to an observable such as the Hubble rate by means of the standard Einstein equation since, by itself, it "knows" nothing about our choice of $R^{(0)}$, and hence cannot compensate for the unphysical $R^{(0)}$-dependence of $\bar{\lambda}(k)$. This indicates that a qualitatively reliable and correct way to proceed from running couplings (here $\bar{\lambda}$ only) to observables (the Hubble rate) is actually more involved and requires further running couplings.

(5) This example illustrates the general situation: Generically, the individual running coupling constants have no direct physical relevance, they are no observables, in particular they are cutoff scheme and gauge fixing dependent. Quantities with the status of an observable depend on very special combinations of several such couplings in which the scheme and gauge dependencies mutually cancel (at least in an exact treatment).

A typical example of such "universal" quantities are the critical exponents of fixed points, the topic to which we turn next.

5.3.3 Gaussian and non-Gaussian Fixed Points

Now we return to the full system of RG equations (5.63) that determines both $g(k)$ and $\lambda(k)$. We begin by searching for fixed points on the two-dimensional g-λ-theory space. Their coordinates (g_*, λ_*) would be common zeros of the two beta functions:

$$\begin{aligned} &\beta_g(g_*, \lambda_*) \equiv \left[d - 2 + \eta_{\mathrm{N}}(g_*, \lambda_*)\right] g_* = 0, \\ &\beta_\lambda(g_*, \lambda_*) = 0. \end{aligned} \tag{5.90}$$

The first one of these conditions is satisfied if either $g_* = 0$ or $\big[d - 2 + \eta_{\rm N}(g_*, \lambda_*)\big] = 0$. Both possibilities are indeed realized in the Einstein–Hilbert truncation. They lead to a Gaussian fixed point (GFP) and a non-Gaussian fixed point (NGFP), which are, respectively, located at,

$$\begin{aligned} \left(g_*^{\rm GFP}, \lambda_*^{\rm GFP}\right) &= 0, \\ \left(g_*^{\rm NGFP}, \lambda_*^{\rm NGFP}\right) &\neq 0. \end{aligned} \tag{5.91}$$

Concerning the GFP, it is obvious from the beta function on the right-hand side of (5.62) that $g_* = 0$ together with $\lambda_* = 0$ is indeed also a zero of β_λ.

(1) Coordinates of the NGFP. When searching for non-trivial fixed points it is convenient to start from the condition

$$\boxed{\eta_{\rm N}\left(g_*^{\rm NGFP}, \lambda_*^{\rm NGFP}\right) = 2 - d.} \tag{5.92}$$

Using (5.60) this relation can be solved for g_* as a function of λ_*:

$$g_*^{\rm NGFP}\left(\lambda_*^{\rm NGFP}\right) = \frac{d-2}{(d-2)B_2\left(\lambda_*^{\rm NGFP}\right) - B_1\left(\lambda_*^{\rm NGFP}\right)}. \tag{5.93}$$

We exploit (5.93) in order to eliminate the g-dependence of β_λ at the non-trivial fixed point. So it remains to solve

$$\beta_\lambda\Big(g_*^{\rm NGFP}\left(\lambda_*^{\rm NGFP}\right), \lambda_*^{\rm NGFP}\Big) = 0 \tag{5.94}$$

for $\lambda_*^{\rm NGFP}$. Due to its complicated structure this equation can only be investigated numerically.

In $d = 4$, and with the sharp cutoff, the numerical analysis establishes the existence of precisely one non-Gaussian fixed point. It is located at

$$\boxed{\lambda_*^{\rm NGFP} = 0.330, \qquad g_*^{\rm NGFP} = 0.403 \qquad \text{(sc)}.} \tag{5.95}$$

An interesting characteristic property of fixed points is the value which the anomalous dimension assumes there. In the case at hand, the GFP has $\eta_{\rm N} = 0$, while the NGFP displays a non-zero, in fact large, integer anomalous dimension, $\eta_{\rm N} = 2 - d$. In particular, in four dimensions: $\eta_{\rm N}\left(g_*^{\rm NGFP}, \lambda_*^{\rm NGFP}\right) = -2$.

(2) Linearization. In order to determine the stability properties of the fixed points and to compute their critical exponents we linearize the flow of the couplings $u^i \equiv (u^1, u^2) \equiv (\lambda, g)$, $i = 1, 2$, about $u_*^i = (\lambda_*, g_*)$:

$$\partial_t u^i = \sum_{j=1,2} B^i{}_j (u^j - u_*^j), \quad B \equiv [B^i{}_j] = \begin{bmatrix} \frac{\partial\beta_\lambda}{\partial\lambda} & \frac{\partial\beta_\lambda}{\partial g} \\ \frac{\partial\beta_g}{\partial\lambda} & \frac{\partial\beta_g}{\partial g} \end{bmatrix}. \tag{5.96}$$

All derivatives defining the matrix elements $B^i{}_j$ are taken at $u = u_*$.

The partial derivatives of the β functions are easily found by making use of the following recursion formulas satisfied by the threshold functions:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}w}\Phi^p_n(w)=-p\Phi^{p+1}_n(w),\qquad \frac{\mathrm{d}}{\mathrm{d}w}\widetilde{\Phi}^p_n(w)=-p\widetilde{\Phi}^{p+1}_n(w).}\tag{5.97}$$

These relations are easily proven by interchanging the derivative with respect to w with the z-integration in the integral representations (5.51).

For any shape function $R^{(0)}$, and at a generic point (λ,g) we thus obtain:

$$\begin{aligned}
\frac{\partial\beta_\lambda}{\partial\lambda}=&-(2-\eta_\mathrm{N})+\Big(\lambda-\frac{g}{2}(4\pi)^{1-d/2}d(d+1)\widetilde{\Phi}^1_{d/2}(-2\lambda)\Big)\frac{\partial\eta_\mathrm{N}}{\partial\lambda}\\
&+\frac{g}{2}(4\pi)^{1-d/2}\Big(4d(d+1)\Phi^2_{d/2}(-2\lambda)-2d(d+1)\eta_\mathrm{N}\tilde{\Phi}^2_{d/2}(-2\lambda)\Big)\,,\\
\frac{\partial\beta_\lambda}{\partial g}=&\Big(\lambda-\frac{g}{2}(4\pi)^{1-d/2}d(d+1)\widetilde{\Phi}^1_{d/2}(-2\lambda)\Big)\frac{\partial\eta_\mathrm{N}}{\partial g}+\frac{1}{2}(4\pi)^{1-d/2}\times\\
&\times\Big(2d(d+1)\Phi^1_{d/2}(-2\lambda)-8d\Phi^1_{d/2}(0)-d(d+1)\eta_\mathrm{N}\widetilde{\Phi}^1_{d/2}(-2\lambda)\Big)\,,\\
\frac{\partial\beta_g}{\partial\lambda}=&\frac{g^2}{1-gB_2(\lambda)}\Big(B_1'(\lambda)+\eta_\mathrm{N}B_2'(\lambda)\Big),\\
\frac{\partial\beta_g}{\partial g}=&\,d-2+\left(2+\frac{gB_2(\lambda)}{1-gB_2(\lambda)}\right)\eta_\mathrm{N}.
\end{aligned}\tag{5.98}$$

Furthermore, the partial derivatives of η_N are given by

$$\begin{aligned}
\frac{\partial\eta_\mathrm{N}}{\partial g}&=\left(\frac{1}{g}+\frac{B_2(\lambda)}{1-gB_2(\lambda)}\right)\eta_\mathrm{N},\\
\frac{\partial\eta_\mathrm{N}}{\partial\lambda}&=\frac{g}{1-gB_2(\lambda)}\Big(B_1'(\lambda)+\eta_\mathrm{N}B_2'(\lambda)\Big).
\end{aligned}\tag{5.99}$$

Here $B'_{1,2}(\lambda)$ denotes the derivatives of $B_{1,2}(\lambda)$ with respect to λ:

$$\begin{aligned}
B_1'(\lambda)&=\frac{1}{3}(4\pi)^{1-d/2}\Big(2d(d+1)\Phi^2_{d/2-1}(-2\lambda)-24d(d-1)\Phi^3_{d/2}(-2\lambda)\Big)\,,\\
B_2'(\lambda)&=-\frac{1}{6}(4\pi)^{1-d/2}\Big(2d(d+1)\widetilde{\Phi}^2_{d/2-1}(-2\lambda)-24d(d-1)\widetilde{\Phi}^3_{d/2}(-2\lambda)\Big)\,.
\end{aligned}\tag{5.100}$$

Since these equations make no explicit use of the fixed point values λ_* and g_* they can be used to investigate the trivial and the non-trivial fixed point alike. We will discuss them in turn.

As we mentioned in Chapter 3 the eigenvalues and right eigenvectors of B, evaluated at the corresponding fixed point, determine its critical exponents and scaling fields, respectively. Since, generically, B is not symmetric, its eigenvalues are not real, and the eigenvectors are not orthogonal in general.

We define the stability coefficients θ^I, $I=1,2$, as the negative eigenvalues of B satisfying the equation $BV^I=-\theta^IV^I$, where V^I are the right eigenvectors of B.

(3) Critical exponents (GFP). Substituting $\lambda_* = 0$, $g_* = 0$ into (5.96) the stability matrix simplifies to

$$B_{\text{GFP}} = \begin{bmatrix} -2 & (4\pi)^{1-d/2} d(d-3)\Phi^1_{d/2}(0) \\ 0 & d-2 \end{bmatrix}. \tag{5.101}$$

Diagonalizing (5.101) leads to the following *two real* stability coefficients with their corresponding right eigenvectors:

$$\begin{aligned} \theta_1 &= 2 \quad \text{with} \quad V^1 = (1,0)^T, \\ \theta_2 &= 2-d \quad \text{with} \quad V^2 = \Big((4\pi)^{1-d/2}(d-3)\Phi^1_{d/2}(0), 1\Big)^T. \end{aligned} \tag{5.102}$$

Here, finally, we see that as we have anticipated the fixed point at $g_* = \lambda_* = 0$ is indeed a "Gaussian" one according to our general definition in Chapter 3: The two critical exponents $\theta_1 = 2$ and $\theta_2 = 2-d$ equal precisely the canonical ones, i.e., the mass dimensions of the running couplings in question, $\bar{\lambda}(k)$ and $G(k)$, respectively.

Note also that while V^1 points exactly in the λ-direction, the second "scaling field" V^2 is slightly rotated relative to the g-axis. Interestingly enough, this rotation is absent in precisely three spacetime dimensions.

In $d=4$, the positive (negative) sign of $\theta_1 = +2$ ($\theta_2 = -2$) indicates that the GFP is repulsive (attractive)[5] in the direction of V^1 (V^2).

This behavior is seen explicitly if we use the eigenvalues and -vectors in order to write down the linearized solution:

$$\begin{aligned} \lambda(k) &\approx \alpha_1 \frac{M^2}{k^2} + \alpha_2 (4\pi)^{1-d/2}(d-3)\Phi^1_{d/2}(0)\frac{k^{d-2}}{M^{d-2}}, \\ g(k) &\approx \alpha_2 \frac{k^{d-2}}{M^{d-2}}. \end{aligned} \tag{5.103}$$

Here α_1 and α_2 are constants of integration allowing one to adjust the solution to given initial conditions. They are made dimensionless by separating off an appropriate power of the mass scale M, which is thus not independent of α_1 and α_2.

Switching to dimensionful coupling constants equations (5.103) read:

$$\begin{aligned} G(k) &\approx G_0, \\ \bar{\lambda}(k) &\approx \bar{\lambda}_0 + (4\pi)^{1-d/2}(d-3)\Phi^1_{d/2}(0) G_0 k^d. \end{aligned} \tag{5.104}$$

Here we chose $M = m_{\text{Pl}}$ by setting $\alpha_2 = 1$, and we identified $\alpha_1 = \bar{\lambda}_0 / m_{\text{Pl}}^2$. We see that both $G(k)$ and $\bar{\lambda}(k)$ run toward constant and non-zero values G_0 and $\bar{\lambda}_0$ in

[5] Recall that in the nomenclature used here the terms repulsive and attractive are synonymous to "IR repulsive" and "IR attractive," respectively, indicating whether a certain eigen-perturbation increases or decreases *when k is lowered.*

the limit $k \to 0$, unless we set $\alpha_1 = 0$, which means $\bar{\lambda}_0 = 0$. There is no special reason for α_1 to be zero, however. So we see that $\bar{\lambda}_0$ depends on the free parameter α_1 and therefore on the trajectory considered. Hence, the Gaussian fixed point does not determine the value of the cosmological constant in the infrared, $\bar{\lambda}_0$.

(4) Critical exponents (NGFP). The computation of the critical exponents for the NGFP is slightly more tedious. By using equations (5.64) and (5.60) for the sharp cutoff we can compute the entries of the stability matrix analytically. But since the fixed-point coordinates are the result of a numerical calculation, the diagonalization of B must also be performed numerically.

For the NGFP one finds a complex conjugate pair of critical exponents

$$\theta^1 \equiv \theta' + i\theta'' \quad \text{and} \quad \theta^2 \equiv \theta' - i\theta'', \tag{5.105}$$

where, with the sharp cutoff [160],

$$\theta' = 1.941, \qquad \theta'' = 3.147 \qquad \text{(sc)}. \tag{5.106}$$

As the real part of θ^1 and θ^2 is positive, the NGFP is seen to have two relevant eigendirections, meaning that it is repulsive (or equivalently UV attractive) in both directions. As a result, the two-parameter Einstein–Hilbert truncation leads to a two-dimensional UV critical hypersurface: $\Delta_{\text{UV}} = \dim \mathscr{S}_{\text{UV}} = 2$.

The general solution to the linearized flow equations reads, with the RG time, $t \equiv \ln(k/k_0)$

$$\begin{aligned}\big(\lambda(k), g(k)\big)^T = (\lambda_*, g_*)^T + 2\Big\{&[\operatorname{Re} C \cos(\theta'' t) + \operatorname{Im} C \sin(\theta'' t)] \operatorname{Re} V \\ &+ [\operatorname{Re} C \sin(\theta'' t) - \operatorname{Im} C \cos(\theta'' t)] \operatorname{Im} V\Big\} e^{-\theta' t}.\end{aligned} \tag{5.107}$$

Here $C \equiv C_1 = (C_2)^*$ is an arbitrary complex number labeling the trajectory under consideration, and $V \equiv V^1 = (V^2)^*$ denotes the right-eigenvector of B with eigenvalue $-\theta_1 = -\theta_2^*$.

Due to the positivity of θ', all trajectories are seen to hit the fixed point as t is sent to infinity. The non-vanishing imaginary part θ'' has no impact on the stability. However, it influences the shape of the trajectories: they spiral around, and ultimately into, the fixed point for $k \to \infty$.

Solving the full, non-linear flow equations [160] shows that the asymptotic scaling region, where the linearization (5.107) is valid, extends from $k = \infty$ down to about $k \approx m_{\text{Pl}}$ with the Planck mass defined as $m_{\text{Pl}} \equiv G_0^{-1/2}$. Here the Planck scale m_{Pl} seems to play a role similar to the dynamically generated mass scale Λ_{QCD} of QCD for example: it marks the lower boundary of the asymptotic scaling region.

5.3.4 The Phase Portrait

The information supplied by the linearizations about the two fixed points is almost sufficient in order to draw the qualitative structure of the vector field $\boldsymbol{\beta} = (\beta_\lambda, \beta_g)$ and its integral curves, the RG trajectories.

Alternatively one can solve the coupled differential equations numerically, which leads to the phase portrait shown in Figure 5.1. It was first obtained in [160] using the sharp cutoff, which has the advantage that all threshold functions can be evaluated analytically.

While in [160] the system of RG equations has been investigated in detail for any dimensionality of spacetime, d, here we focus on $d = 4$ dimensions first.

(1) According to (5.64) and (5.60) the sharp cutoff (with $s = 1$, see Appendix E) leads to the following explicit and comparatively simple-looking RG equations:

$$\boxed{\begin{aligned} \partial_t \lambda_k &= -(2-\eta_{\mathrm{N}})\lambda_k - \frac{g_k}{\pi}\left[5\ln(1-2\lambda_k) - 2\zeta(3) + \frac{5}{2}\eta_{\mathrm{N}}\right], \\ \partial_t g_k &= (2+\eta_{\mathrm{N}}) g_k \quad \text{where} \\ \eta_{\mathrm{N}} &= -\frac{2g_k}{6\pi + 5g_k}\left[\frac{18}{1-2\lambda_k} + 5\ln(1-2\lambda_k) - \zeta(2) + 6\right]. \end{aligned}} \tag{5.108}$$

Note that the beta functions in (5.108) have poles of first order as well as logarithmic singularities at $\lambda = \frac{1}{2}$. These divergences are the concrete incarnation,

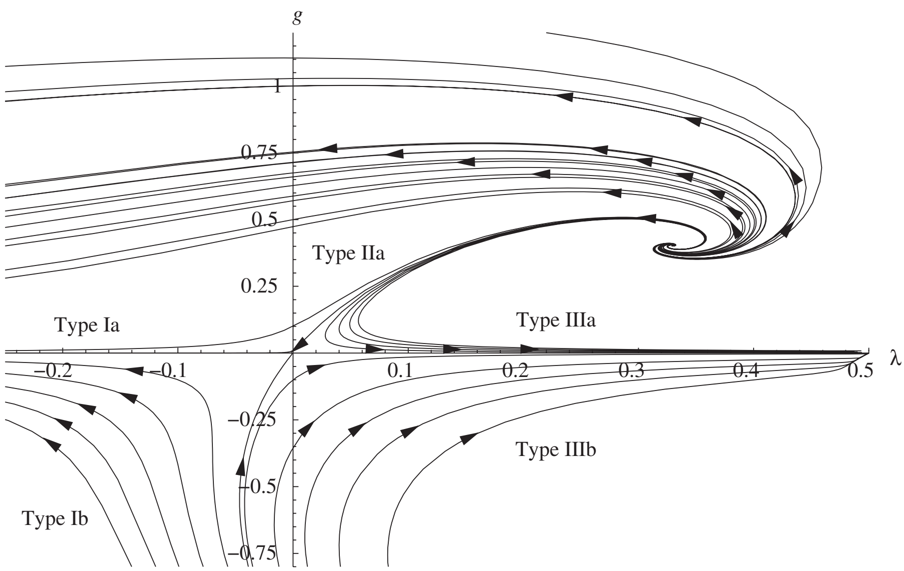


Figure 5.1. The phase portrait of the RG flow in the g-λ-plane. The arrows point in the direction of decreasing k. (Taken from [160].)

for the sharp cutoff, of the divergences at $w=-1$ that plague $\Phi_n^p(w)$ and $\widetilde{\Phi}_n^p(w)$ for any $R^{(0)}$. As a result, trajectories hitting the $\lambda=\frac{1}{2}$-line terminate there at a non-zero value of k.

(2) Figure 5.1 displays a number of typical RG trajectories obtained by numerically integrating (5.108). The RG flow is indeed seen to be dominated by two fixed points: the GFP at $g_*=\lambda_*=0$, and the NGFP with $g_*>0$ and $\lambda_*>0$.

There exist *three types of trajectories emanating from the NGFP*: trajectories of Type Ia and Type IIIa which run toward negative and positive cosmological constants, respectively, and a single trajectory of Type IIa (also known as the *separatrix*), which hits the GFP for $k\to 0$.

The separatrix describes a "crossover" from the NGFP to the GFP, and it separates the qualitatively different trajectories of Type Ia and Type IIIa, respectively.[6]

The UV behavior of any of these trajectories is governed by the NGFP: for $k\to\infty$, they approach and asymptotically hit this point.

(3) Since it is the dimensionless coupling constants $g(k)$ and $\lambda(k)$ that approach constant, non-zero values at the NGFP the corresponding dimensionful quantities keep running for $k\to\infty$ according to the power laws

$$G(k)=g_*^{\mathrm{NGFP}}k^{-2}, \qquad \bar{\lambda}(k)=\lambda_*^{\mathrm{NGFP}}k^2. \tag{5.109}$$

Hence, *for $k\to\infty$, the dimensionful Newton constant vanishes*, while the cosmological constant diverges. This confirms the behavior of Newton's constant we already saw in the first subtruncation.

(4) The onset of the asymptotic scaling law (5.109) is shown explicitly in Figure 5.2, which displays the k-dependence of both the dimensionless and the dimensionful Newton and cosmological constant along the separatrix. The dimensionful quantities, including k itself, are expressed in Planckian units, the Planck mass being defined in terms of the value of G at $k=0$, that is, $m_{\mathrm{Pl}}=G(k=0)^{-1/2}$.

In Figure 5.2 we can see that it is indeed near $k\approx m_{\mathrm{Pl}}$ where the trajectory "crosses over" from the vicinity of the GFP, where $G,\bar{\lambda}$ are approximately independent of k, to the asymptotic scaling regime governed by the NGFP where instead $g,\lambda=\mathrm{const}$.

(5) There is one feature of the RG flow that cannot be seen in Figures 5.1 and 5.2, namely that, according to the Einstein–Hilbert truncation, the trajectories of Type IIIa terminate at a non-zero value of k at the moment they reach the $\lambda=\frac{1}{2}$-line. The beta functions develop singularities there so that the differential

[6] For a complete classification of all trajectories, including also "exotic" ones disconnected from the classical regime, see [160].

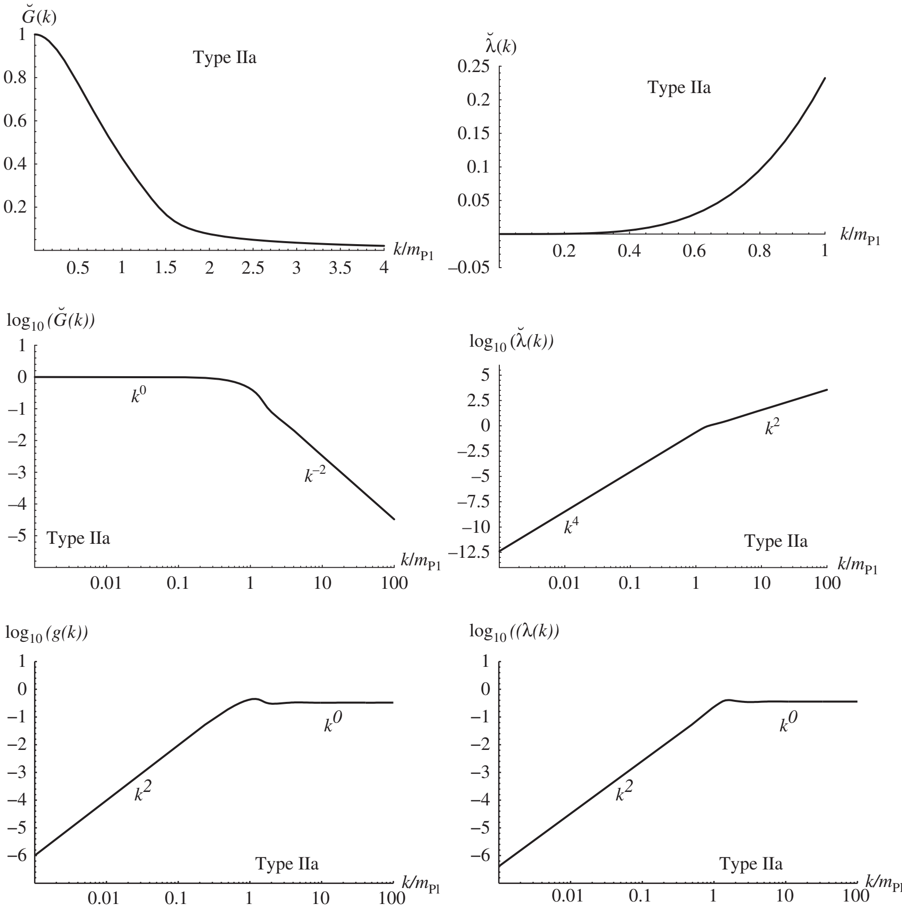


Figure 5.2. Scaling laws for the crossover trajectory IIa, the separatrix. The diagrams show the k-dependence of both the dimensionless couplings g and λ and the dimensionful ones expressed in Planck units, $\breve{G} = G\, m_{\rm Pl}^2$, $\breve{\lambda} = \bar{\lambda}/m_{\rm Pl}^2$, $m_{\rm Pl} \equiv G_0^{-1/2}$. For $k/m_{\rm Pl} \ll 0.1$ and $k/m_{\rm Pl} \gg 10$ the scale dependence is seen to be governed by the scaling laws of the GFP and the NGFP, respectively. The transition between these power laws occurs at $k \approx m_{\rm Pl}$. (Taken from [160].)

equations cannot be integrated beyond this point. Therefore, the $\lambda = \frac{1}{2}$-line can be seen as a natural boundary of the g-λ theory space.

(6) These general properties of the flow are confirmed by the diagrams displaying the anomalous dimension $\eta_{\rm N}(k) \equiv \eta_{\rm N}\big(\lambda(k), g(k)\big)$ evaluated along the sample trajectories of Type Ia, Type IIa, and Type IIIa shown in Figure 5.3.

In the region governed by the non-trivial fixed point it is seen that $\eta_{\rm N} \approx -2$, while $\eta_{\rm N} \approx 0$ in the infrared region. Recalling that $\eta_{\rm N}^* = -2$ and $\eta_{\rm N}^* = 0$ at the NGFP and the GFP in $d = 4$, respectively, this is exactly what we expect.

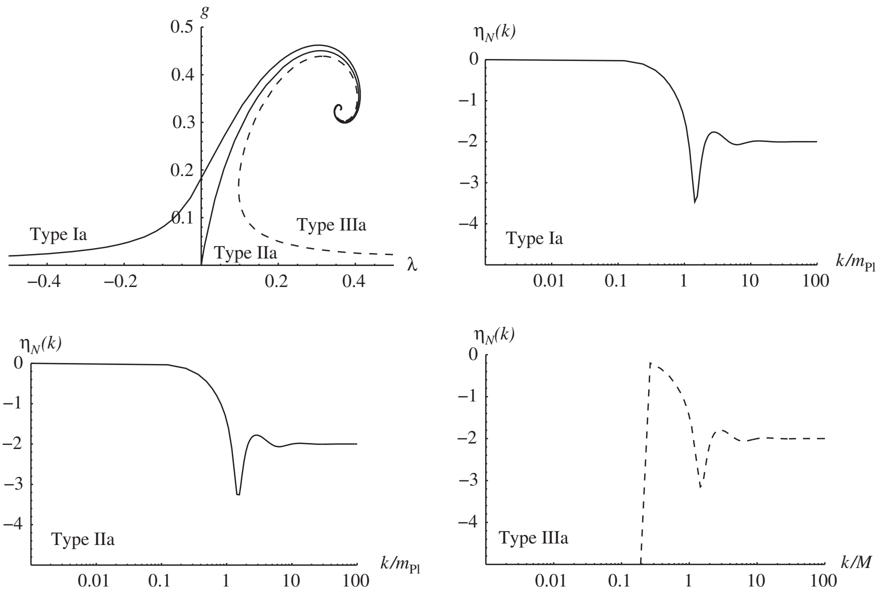


Figure 5.3. Anomalous dimension η_{N} evaluated along the Type Ia, IIa, and IIIa trajectories shown in the first diagram. (Taken from [160].)

The termination of the Type IIIa trajectory is accompanied by a steep decrease of η_{N}, caused by the divergence of $B_1(\lambda)^{\mathrm{sc}}$ at constant $B_2(\lambda)^{\mathrm{sc}}$ for λ approaching the boundary line $\lambda = \frac{1}{2}$. (This can easily be checked from (5.64).)

The "IR divergence" of η_{N} for the trajectories of Type IIIa suggests that the Einstein–Hilbert truncation may not be sufficient to describe the RG flow close to the boundary line $\lambda = \frac{1}{2}$.

5.3.5 Close Relatives of the EH Truncation

Before turning to a detailed assessment of the reliability of the Einstein–Hilbert (EH) truncation and the implications for Asymptotic Safety let us first introduce a number of immediate generalizations it gave rise to. To put the EH truncation into the broader context, we will analyze it further together with those generalizations.

The natural next step beyond the Einstein–Hilbert truncation consists in generalizing the functional $\bar{\Gamma}_k[g]$, while keeping the gauge-fixing and ghost sector classical, as in (5.1). During the RG evolution the flow generates all possible diffeomorphism-invariant terms in $\bar{\Gamma}_k[g]$ which one can construct from $g_{\mu\nu}$. Typical invariants contain strings of curvature tensors, and covariant derivatives acting upon them, with any number of tensors and derivatives; see Appendix B for examples and [161] for a classification of such terms.

Let us now list a number of ansätze for $\bar{\Gamma}_k[g]$ that have been investigated in the literature, always accompanied by single-metric type gauge-fixing and ghost terms.

Considering those ansätze the reader should always keep in mind that *the canonical mass dimension of a field monomial is not a reliable measure for its importance*, as this would be the case in perturbation theory.

We are aiming at describing a non-perturbative fixed point with potentially large anomalous dimensions. Thus every new truncation ansatz usually amounts to a "step in the dark" on the largely uncharted theory space.

(1) The R^2-truncation. The first truncation of this class that was worked out completely [162, 163] is the R^2 truncation defined by (5.10) with the same $\widehat{\Gamma}_k$ as before, and the (curvature)^2 action:

$$\boxed{\bar{\Gamma}_k[g] = \int \mathrm{d}^d x \, \sqrt{g} \left\{ \frac{1}{16\pi G_k} \left[-R + 2\bar{\lambda}_k \right] + \bar{\beta}_k R^2 \right\}.} \tag{5.110}$$

In this case the truncated theory space is three-dimensional. Its natural dimensionless coordinates are (g, λ, β), where $\beta_k \equiv k^{4-d} \bar{\beta}_k$, while g and λ are the usual dimensionless Newton and cosmological constant. Even though (5.110) contains only one additional invariant, the derivation of the corresponding RG equations is far more complicated than in the Einstein–Hilbert case. As the beta functions for the three couplings would fill many pages we cannot reproduce them here but summarize the results obtained with them below.

(2) The $R^2 + C^2$-truncation. A natural extension of the R^2 truncation consists of retaining *all* gravitational four-derivative terms in the truncation subspace:

$$\boxed{\begin{aligned} \bar{\Gamma}_k[g] = \int \mathrm{d}^4 x \, \sqrt{g} \bigg\{ & \frac{1}{16\pi G_k} \left[-R + 2\bar{\lambda}_k \right] - \frac{\omega_k}{3\sigma_k} R^2 \\ & + \frac{1}{2\sigma_k} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{\theta_k}{\sigma_k} E \bigg\}. \end{aligned}} \tag{5.111}$$

Here $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ denotes the square of the (conformal) Weyl tensor, and $E = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{2}{3} R^2$ is the integrand of the (topological) Gauss–Bonnet term in four dimensions. Using this ansatz and the FRGE, the perturbative one-loop beta functions for higher-derivative gravity have been reanalyzed in [119, 164–167], while the first non-perturbative results have been obtained in [168–170].

The key ingredient in the projection of the flow onto the space of actions (5.111) was the generalization of the background metric $\bar{g}$ from the maximally symmetric spheres S^d, that had been employed in the R^2-truncation, to a *generic*

Einstein space. Inserting an S^d metric, the RG equations collapse into a single flow equation for $\beta_k = -\frac{\omega_k}{3\sigma_k} + \frac{\theta_k}{6\sigma_k}$. While working with a generic Einstein metric allows one to find the beta functions for two independent combinations of the four-derivative couplings:

$$\beta_k = -\frac{\omega_k}{3\sigma_k} + \frac{\theta_k}{6\sigma_k}, \qquad \gamma_k = \frac{1}{2\sigma_k} + \frac{\theta_k}{\sigma_k}. \tag{5.112}$$

Ideally, the new class of backgrounds should be general enough to disentangle the coefficients multiplying R^2, C^2, and E and, at the same time, simple enough to avoid the appearance of non-minimal higher-derivative differential operators inside the functional trace, which would be difficult to deal with. While the S^d backgrounds are insufficient in the former respect, a generic compact Einstein background (without Killing or conformal Killing vectors and without boundary for simplicity), satisfying $\bar{R}_{\mu\nu} = \frac{\bar{R}}{4}\bar{g}_{\mu\nu}$, meets both criteria and allows us to determine the non-perturbative beta functions of the linear combinations (5.112).

And surprisingly enough, projecting the flow equation resulting from the ansatz (5.111) with a generic Einstein background, the differential operators appearing on its right-hand side are found to organize themselves into *second order* differential operators of the Lichnerowicz form,

$$\begin{aligned} \Delta_{2L}\phi_{\mu\nu} &\equiv -\bar{D}^2\phi_{\mu\nu} - 2\bar{R}_{\mu\ \nu}^{\ \alpha\ \beta}\phi_{\alpha\beta}, \\ \Delta_{1L}\phi_\mu &\equiv -\bar{D}^2\phi_\mu - \bar{R}_{\mu\nu}\phi^\nu, \\ \Delta_{0L}\phi &\equiv -\bar{D}^2\phi, \end{aligned} \tag{5.113}$$

which, moreover, commute with all the other curvature terms inside the trace. This feature makes the traces amenable to standard heat kernel techniques for minimal second-order differential operators.

(3) Polynomial $\boldsymbol{f(R)}$ truncations. A different direction to go from the EH ansatz is to allow for arbitrary powers of the curvature scalar, but instead discard all other types of invariants in order to keep the calculational problem manageable. The truncation ansatz for $\bar{\Gamma}_k[g]$ contains an arbitrary function of R then:

$$\boxed{\bar{\Gamma}_k[g] = \int \mathrm{d}^d x\,\sqrt{g}\,f_k(R).} \tag{5.114}$$

Later on we will allow $f_k(\,\cdot\,)$ to be a running function of a rather general type. Here we specialize to polynomials, which may have any degree, N, though:

$$\boxed{f_k(R) = \sum_{n=0}^{N} u_n(k)\,k^d\left(\frac{R}{k^2}\right)^n.} \tag{5.115}$$

In this case the truncated theory space is $(N+1)$-dimensional and can be coordinatized by the already dimensionless coefficients $(u_0, u_1, \cdots, u_N)$. Clearly, if $N=1$ or $N=2$ we are back to the EH and R^2-truncations, respectively.

(4) General gauge-fixing parameter $\boldsymbol{\alpha}$. There are also results concerning the gauge-fixing term. Even if one uses the ansatz (5.10) for $\Gamma_k[g, \bar{g}]$ in which the gauge-fixing term has the classical (or more appropriately, bare) *structure* one should in principle still treat its *prefactor* as a running coupling constant: $\alpha \equiv \alpha_k$.

While the beta function of α is not easily computed from the FRGE, there is a simple argument which allows us to bypass this calculation.

In non-perturbative Yang-Mills theory, and in perturbative quantum gravity, $\alpha = \alpha_k = 0$ is known to be a fixed point of the α evolution. The following reasoning suggests that the same is true within the non-perturbative FRGE approach to gravity. In the formal functional integral the limit $\alpha \to 0$ corresponds to a sharp implementation of the gauge-fixing condition, i.e., $\exp(-S_{\text{gf}})$ becomes proportional to $\delta[F_\mu]$. The domain of the $\int \mathcal{D}\widehat{h}_{\mu\nu}$ integration consists of those $\widehat{h}_{\mu\nu}$s which satisfy the gauge-fixing condition exactly, $F_\mu = 0$. Adding the IR cutoff at k amounts to suppressing some of the $\widehat{h}_{\mu\nu}$ modes while retaining the others. But since all of them satisfy $F_\mu = 0$, a variation of k cannot change the domain of the $\widehat{h}_{\mu\nu}$ integration. The delta functional $\delta[F_\mu]$ continues to be present for any value of k if it was there originally. As a consequence, α vanishes for all k, i.e., $\alpha = 0$ is a fixed point of the α evolution [171]. For gravity, this fixed point has been confirmed in [155].

As a consequence, it is possible to mimic the dynamical treatment of a running α by setting the gauge-fixing parameter to the constant value $\alpha = 0$.

The calculation for $\alpha = 0$ is more involved than at $\alpha = 1$, but for the Einstein–Hilbert truncation the α-dependence of β_g and β_λ, for arbitrary constant α, has been found in [172, 173]. The (curvature)^2-truncations could be analyzed only in the simpler $\alpha = 1$ gauge, but the results from the Einstein–Hilbert truncation suggest that the fixed-point quantities of interest do not change much between $\alpha = 0$ and $\alpha = 1$ [163, 173].

(5) Including scale-dependent couplings in the ghost sector. Generalizations of the classical ghost action including a scale-dependent wave-function renormalization or a scale-dependent ghost-curvature coupling have been considered in [174, 175] and [176], respectively. Both extensions confirm the phase diagram obtained from the single-metric Einstein–Hilbert truncation, justifying the approximation of a classical ghost sector a posteriori.

5.3.6 Testing the Reliability of Truncations

Since the method of the truncated theory spaces does not come with a natural small expansion parameter, and since in general the relative importance of the various terms in the ansatz for Γ_k is not simply related to their canonical dimension, a crucial part of the method consists of testing the qualitative reliability, and the quantitative precision that can be achieved with a given truncation.

Generally speaking we can distinguish two types of tests, namely those that apply to individual truncations, and those that address several truncations jointly.

(1) The first type of tests examines the reliability or "robustness" of the predictions obtained within a given truncation under changes of the various *unphysical elements in the computational setup*, the cutoff operator being the prime example. The idea behind this method is quite simple.

Let us assume we know from general principles that a certain quantity $\mathcal{Q}$ that can be computed from the RG flow in some way is *universal*, i.e., cutoff scheme independent. Hence, if we change the cutoff operator from $\mathcal{R}_k$ to $\mathcal{R}_k + \delta\mathcal{R}_k$, the theory's exact answer for this quantity will not change. Now, typically, approximations and truncations destroy this property of scheme independence, i.e., approximate computations of $\mathcal{Q}$ are $\mathcal{R}_k$-dependent, and in fact, they are even more so the poorer the quality of the underlying approximation.

This observation provides us with a necessary condition for the reliability of a truncation (or any sort of approximation), which can be checked straightforwardly: If a quantity that is known to be universal at the exact level changes under a deformation of $\mathcal{R}_k$ by a certain amount, then typical errors of the pertinent calculation are at least of the order of this amount, or larger even.

An example of quantities known to be cutoff scheme-independent when determined exactly are the critical exponents of a fixed point. In Section 3.2 we demonstrated already that the θ_α's are invariant under general coordinate transformations on theory space, and as a consequence, invariant under changes of $\mathcal{R}_k$.

At a less general level, for Einstein–Hilbert-like truncations, the dimensionless combination $g_* \lambda_*^{(d/2-1)}$ is also approximately scheme independent and often has been studied in its dependence on $\mathcal{R}_k$.

(2) In practice one can modify $\mathcal{R}_k$ both by changing its matrix structure in field space and the shape function $R^{(0)}$ it involves. As for the latter, typical choices of $R^{(0)}$ that have been used in quantum gravity include:

(i) The *exponential cutoff* [14]:

$$R^{(0)}(z)^{\exp} = \frac{z}{e^z - 1} \equiv R^{(0)}(z;1)^{\exp}. \tag{5.116}$$

(ii) The one-parameter family of *generalized exponential cutoffs* [177, 14]:

$$R^{(0)}(z;s)^{\exp} = \frac{sz}{e^{sz} - 1} \qquad (s > 0). \tag{5.117}$$

(iii) The one-parameter family of *cutoffs with compact support* [163, 173], with shape parameter $b \in [0, \frac{3}{2})$:

$$R^{(0)}(z;b)^{\text{cp}} = \begin{cases} 1 & \text{if } z \leq b, \\ \exp\left[\left(z - \frac{3}{2}\right)^{-1} \exp\left[\left(b - z\right)^{-1}\right]\right] & \text{if } b < z < \frac{3}{2}, \\ 0 & \text{if } z \geq \frac{3}{2}. \end{cases} \tag{5.118}$$

(iv) The *sharp cutoff* [160, 178, 179]:

$$R^{(0)}(z)^{\text{sc}} = \widehat{R}\,\Theta(1-z), \qquad \widehat{R} \to \infty \tag{5.119}$$

and its one-parameter family of generalizations $R^{(0)}(z;s)^{\text{sc}}$, described in Appendix E.

(v) The *optimized cutoff* [180]:

$$R^{(0)}(z)^{\text{opt}} = (1-z)\Theta(1-z). \tag{5.120}$$

For checking the degree of scheme independence exhibited by a certain quantity one may perform both “small” changes of $R^{(0)}$ by varying the free parameters in one of the families, as well as “large” changes by switching from one type to another. Among the families of cutoffs the generalized sharp cutoff $R^{(0)}(z;s)^{\text{sc}}$ is particularly convenient since it has both a tunable free parameter and allows for an analytic evaluation of the threshold functions Φ and $\widetilde{\Phi}$ [160].

(3) Another test that can be applied to truncations individually suggests itself when the truncated Γ_k contains a certain coupling constant, say $u(k)$, which is scale dependent at the exact level, but whose scale dependence is neglected as part of the approximations that define the truncation. If we set $u(k) = \text{const} \equiv u^{(0)}$ with an arbitrary constant $u^{(0)}$, the test consists in checking how the quantities of interest vary with the value of $u^{(0)}$ chosen in the calculation. Then, again, the “error bar” to be attached to those quantities should be at least of the order of their variation with $u^{(0)}$, unless further information is available.

The prototype of a coupling of this kind is the running gauge-fixing constant $\alpha(k)$. Treating it as k-independent amounts to an approximation on top of a truncation. Letting $\alpha(k) = \text{const} \equiv \alpha$ it is instructive to see how the answers provided by the truncation depend on the constant α. In this way one gets a first impression of the precision that can be expected.

(4) The tests discussed up to now refer to individual truncations. While they are comparatively easy to apply, it is clear that further justification of a given truncation, or a hierarchy of nested truncations, must come from a stepwise extension of the truncated theory space. Including further invariants into Γ_k one should observe a certain degree of stabilization or “convergence” of the physical predictions.

Calculations of this sort, while straightforward conceptually, are *extremely* hard in practice. Usually every enlargement of the truncated gravitational theory space entails new computational challenges which are often quite formidable and require developing completely new strategies and calculational techniques. The problems are mostly due to the functional traces in the FRGE which must be projected analytically on the chosen $\mathcal{T}_{\text{trunc}}$ and whose evaluation is notoriously difficult.

In the next subsection we apply this type of reliability test to the hierarchy of nested polynomial $f(R)$-truncations (5.115) whose members are labeled by the degree of the polynomial, N.

5.3.7 Evidence for Asymptotic Safety

Next, we list and compare the results obtained with the Einstein–Hilbert truncation and its various relatives, detailing in particular how they score in the reliability tests, and collecting the pieces of evidence which they contribute in support of the Asymptotic Safety hypothesis.

(1) Evidence from the EH truncation. The Einstein–Hilbert truncation predicts the existence of a NGFP with exactly the properties needed for the Asymptotic Safety construction. Of course, the immediate question is whether this NGFP is the projection of a true fixed point in the exact theory or whether it is merely the artifact of an insufficient approximation.

We start by summarizing the properties of the NGFP in the Einstein–Hilbert truncation itself. Except for point (v) below, the results refer to $d = 4$.

All findings listed are independent pieces of evidence supporting the conjecture that *QEG is indeed asymptotically safe in four dimensions.*

- **(i) Universal existence (EH):** The non-Gaussian fixed point exists for all cutoff schemes and shape functions implemented to date. It seems impossible to find an admissible cutoff which destroys the fixed point in $d = 4$. This result is highly non-trivial since in higher dimensions ($d \gtrsim 5$) the existence of the NGFP depends on the cutoff chosen [160].
- **(ii) Positive Newton constant (EH):** While the position of the fixed point is scheme dependent, all cutoffs yield *positive* values of g_* and λ_*. A negative g_* might have been problematic for stability reasons, but there is no mechanism in the flow equation which would exclude it on general grounds.
- **(iii) Stability (EH):** For any cutoff employed, the NGFP is found to be UV attractive in both directions of the λ-g-plane. Linearizing the flow equation we always obtain a pair of complex conjugate critical exponents, $\theta_1 = \theta_2^*$, and they have a positive real part $\theta' > 0$ for all cutoffs.
- **(iv) Scheme and gauge dependence (EH):** Analyzing the cutoff-scheme dependence of θ', θ'', and $g_*\lambda_*$ as a measure for the reliability of the

truncation, the critical exponents were found to be reasonably constant within about a factor of 2 within the one-parameter family of exponential cutoffs (5.117). For $\alpha = 1$ and $\alpha = 0$, for instance, they assume values in the ranges $1.4 \lesssim \theta' \lesssim 1.8$, $2.3 \lesssim \theta'' \lesssim 4$ and $1.7 \lesssim \theta' \lesssim 2.1$, $2.5 \lesssim \theta'' \lesssim 5$, respectively. The universality properties of the product $g_* \lambda_*$ are even more impressive. Despite the rather strong scheme dependence of g_* and λ_* separately, their product has almost no visible s-dependence for not too small values of s. Its value is

$$g_* \lambda_* \approx \begin{cases} 0.12 & \text{for } \alpha = 1, \\ 0.14 & \text{for } \alpha = 0. \end{cases} \tag{5.121}$$

The difference between the "correct" (i.e., fixed point) value of the gauge parameter, $\alpha = 0$, and the technically more convenient, $\alpha = 1$, are at the level of about 10 to 20 percent. Modifying the coarse-graining operator appearing in the argument of $\mathcal{R}_k$ gives rise to variations of a similar magnitude [181].

(v) Lower and higher dimensions (EH): The beta functions implied by the FRGE are continuous functions of the spacetime dimensionality and it is instructive to analyze them for $d \neq 4$.

In [14] it has been shown that for $d = 2 + \varepsilon$, $|\varepsilon| \ll 1$, the FRGE reproduces Weinberg's [10] fixed point for Newton's constant, $g_* = \frac{3}{38}\varepsilon$, and in addition supplies a corresponding fixed-point value for the cosmological constant, $\lambda_* = -\frac{3}{38}\Phi_1^1(0)\varepsilon$.

The g_* value from the FRGE, $g_* = b^{-1}\varepsilon$ with $b = \frac{2}{3} \cdot 19$, agrees with the perturbative result for pure gravity, b_2 in (3.76).

The latter had been found using the same linear background split which also underlies the FRGE described here. Furthermore, it turned out that the analogous FRGE with an *exponential background split* also reproduces the corresponding perturbative coefficient, b_3 in (3.76), namely $b = \frac{2}{3} \cdot 25$ [145].

After the FRGE-based beta function for the *Gibbons-Hawking surface term* had been derived [182] it also turned out [183, 117] that the non-perturbative flow equation even reproduces the perturbative answer, b_1 in (3.76), that pertains to the Newton constant in the surface term (and a linear background split).

In each of the three calculations the coefficient b is perfectly independent of $R^{(0)}$, i.e., a universal quantity, in this sense.

In the sequel we return to the linear split and the "bulk" Newton constant.

For arbitrary d, and a generic cutoff, the RG flow is found to be quantitatively similar to the four-dimensional one for all d smaller than a certain critical dimension d_{crit}, above which the existence or non-existence of the NGFP becomes cutoff dependent. The critical dimension is scheme dependent, but for any admissible cutoff it lies well above $d = 4$. As d approaches d_{crit} from below, the scheme dependence of the universal quantities increases drastically, indicating that the EH-truncation becomes insufficient near d_{crit}.

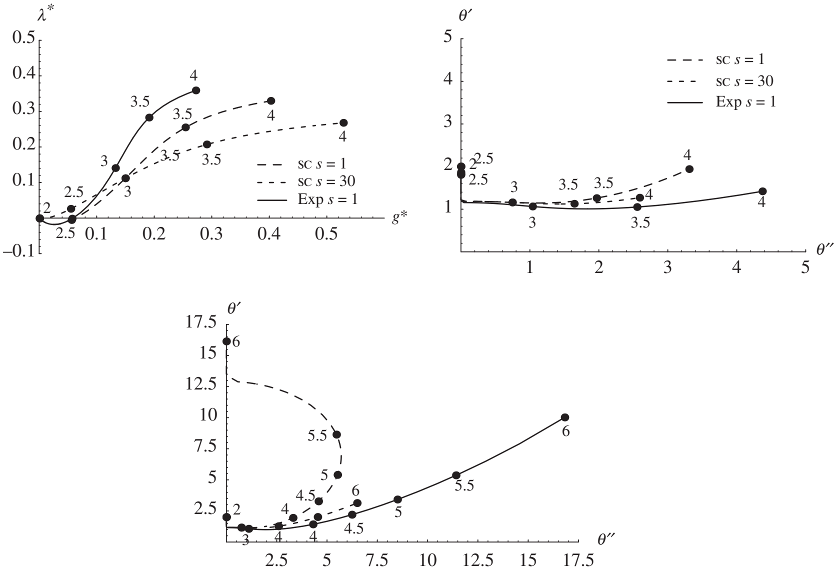


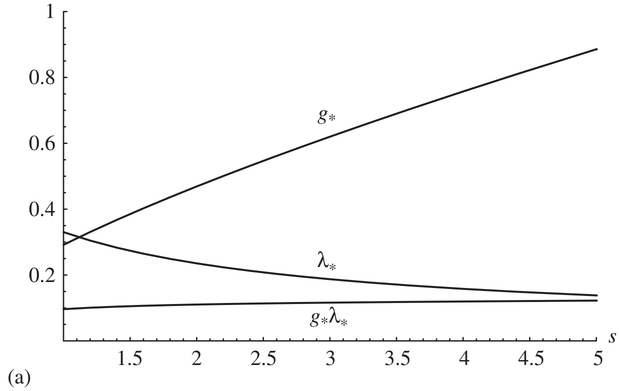
Figure 5.4. Comparison of λ_*, g_*, θ', and θ'' for different cutoff functions in dependence on the dimension d. Two versions of the sharp cutoff (sc) and the exponential cutoff with $s = 1$ (Exp) have been employed. The upper diagrams show that for $2 + \varepsilon \leq d \leq 4$ the cutoff-scheme dependence of the results is rather small. The lower diagram shows that increasing d beyond about 5 leads to a significant difference in the results for θ', θ'' obtained with different cutoff functions. (Taken from [160].)

In Figure 5.4 we show the d-dependence of g_*, λ_*, θ', and θ'' for two versions of the sharp cutoff (with $s = 1$ and $s = 30$, respectively) and for the exponential cutoff with $s = 1$. For $2 + \varepsilon \leq d \leq 4$ the scheme dependence of the critical exponents is seen to be rather weak; it becomes appreciable only near $d \approx 6$ [160].

Figure 5.4 suggests that the Einstein–Hilbert truncation in $d = 4$ performs almost as well as near $d = 2$. (Its validity can be slightly extended toward larger dimensionalities by optimizing the shape function [180, 184].)

(2) Evidence from the R^2 truncation. The ultimate justification of a given truncation consists in proving that if one adds further terms to the ansatz, its predictions do not change much. The first step toward testing the robustness of the Einstein–Hilbert truncation against the inclusion of other invariants has been taken in [162, 163] where the beta functions for the three generalized couplings g, λ and β entering into the R^2-truncation of (5.110) have been derived and analyzed. Concerning Asymptotic Safety, the main results are as follows.

(i) Position of the NGFP (R^2): Also on the basis of the generalized truncation (5.110) the NGFP is found to exist for all admissible cutoffs. Figure 5.5



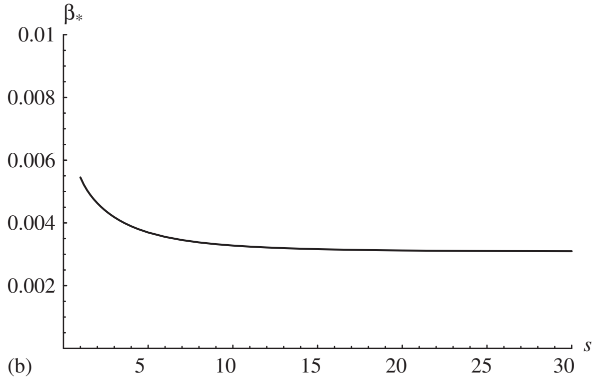


Figure 5.5. (a) g_*, λ_*, and $g_*\lambda_*$ as functions of s for $1 \leq s \leq 5$, and (b) β_* as a function of s for $1 \leq s \leq 30$, using the family of exponential shape functions. (Taken from [162].)

shows its coordinates $(\lambda_*, g_*, \beta_*)$ for the family of exponential shape functions (5.117). For every shape parameter s, the values of λ_* and g_* are almost the same as in the Einstein–Hilbert truncation. In particular, the product $g_*\lambda_*$ is constant with a very high accuracy. For $s = 1$, for instance, one obtains $(\lambda_*, g_*) = (0.359, 0.272)$ from the Einstein–Hilbert truncation and $(\lambda_*, g_*, \beta_*) = (0.330, 0.292, 0.005)$ from the generalized truncation.

It is remarkable that β_* is always significantly smaller than λ_* and g_*, thus "disturbing" the earlier analysis on the basis of the EH-truncation only very little. Within the limited precision of our calculation this means that in the three-dimensional parameter space the fixed point practically lies on the λ-g-plane with $\beta = 0$, i.e., on the parameter space of the Einstein–Hilbert subtruncation.

(ii) Eigenvalues and vectors ($\boldsymbol{R^2}$): The NGFP of the R^2-truncation proves to be UV attractive in any of the three directions of the (λ, g, β)-space for all cutoffs used. The linearized flow in its vicinity is always governed by a pair of complex conjugate critical exponents $\theta_1 = \theta' + i\theta'' = \theta_2^*$ with $\theta' > 0$ and a single real, positive critical exponent $\theta_3 > 0$. For the exponential shape function with $s = 1$, for instance, we find $\theta' = 2.15$, $\theta'' = 3.79$, $\theta_3 = 28.8$. The first two eigenperturbations again yield spiral type trajectories, while the third one amounts to a straight line.[7]

For any cutoff, the numerical results display a number of robust features that are quite remarkable. They all indicate that, *close to the NGFP, the RG flow is rather well approximated by the pure Einstein–Hilbert subtruncation*:

(a) The eigenvectors associated with the spiraling directions span a plane which virtually coincides with the g-λ-subspace at $\beta = 0$, i.e., with the parameter space of the Einstein–Hilbert truncation. As a consequence, the corresponding normal modes are essentially the same trajectories as the "old" normal modes already found without the R^2-term. The corresponding θ'- and θ'' values also coincide within the scheme dependence.

(b) The additional eigenvalue θ_3 introduced by the R^2 term is significantly larger than θ'. When a trajectory approaches the fixed point from below ($t \to \infty$), the "old" normal modes are proportional to $\exp(-\theta' t)$, but the new one is proportional to $\exp(-\theta_3 t)$, so that it decays much quicker. For every trajectory running into the fixed point we therefore find that once t is sufficiently large the trajectory practically lies entirely in the $\beta = 0$-plane. Due to the large value of θ_3, the new scaling field is highly "relevant." However, when we start at the fixed point ($t = \infty$) and lower t it is only at the low energy scale $k \approx m_{\rm Pl}$ ($t \approx 0$) that $\exp(-\theta_3 t)$ reaches unity, and only then, i.e., far away from the fixed point, the new scaling field starts growing rapidly.

Thus it seems that *close to the fixed point the RG flow is essentially two-dimensional*, and that *the two-dimensional flow is well approximated by the RG equations of the EH truncation*.

In Figure 5.6 we show a typical trajectory which has all three normal modes excited with equal strength. All the way down from $k = \infty$ to about $k = m_{\rm Pl}$ it is indeed confined to a very thin box surrounding the $\beta = 0$-plane.

(iii) Scheme dependence (R^2): The scheme dependence of the critical exponents and of the product $g_* \lambda_*$ turns out to be of the same order of magnitude as in the case of the Einstein–Hilbert truncation. Figure 5.7 shows the cutoff dependence of the critical exponents, using the family of shape functions (5.117). For the cutoffs employed, θ' and θ'' assume values in the ranges $2.1 \lesssim \theta' \lesssim 3.4$ and $3.1 \lesssim \theta'' \lesssim 4.3$, respectively. While the scheme dependence of θ'' is weaker than in the case of the Einstein–Hilbert truncation one

[7] The value obtained for θ_3 obtained within the R^2-truncation is rather large. It stabilizes quickly to $\theta_3 \approx 1.5$ once higher powers of R are included; see Table 5.3.

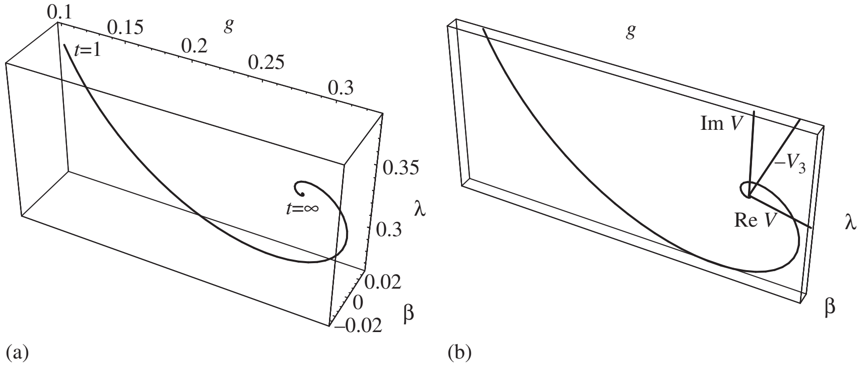


Figure 5.6. Trajectory of the linearized flow equation obtained from the R^2-truncation for $1 \leq t = \ln(\frac{k}{k_0}) < \infty$. In (b) we depict the eigendirections and the "box" to which the trajectory is confined. (Taken from [162].)

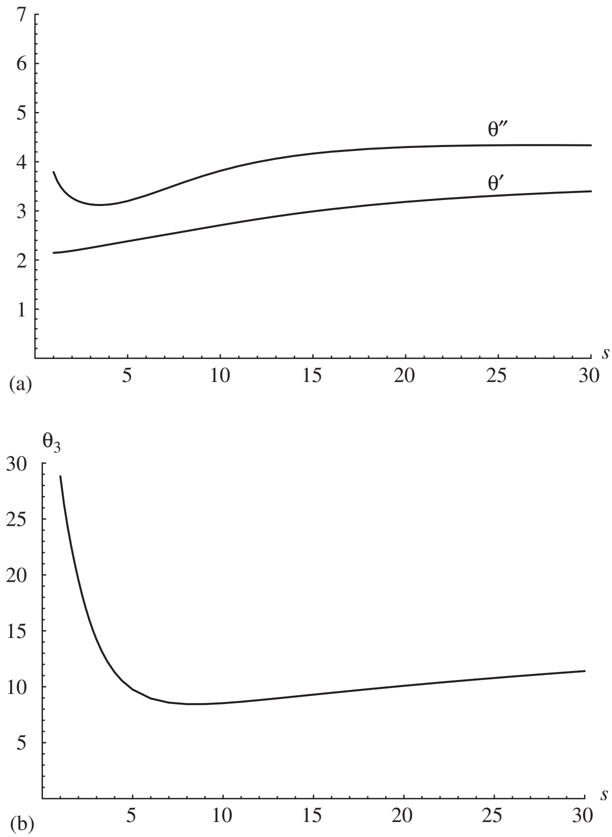


Figure 5.7. (a) $\theta' = \mathrm{Re}\,\theta_1$ and $\theta'' = \mathrm{Im}\,\theta_1$, and (b) θ_3 as functions of s, using the family of exponential shape functions. (Taken from [162].)

finds that it is slightly larger for θ'. The exponent θ_3 suffers from relatively strong variations as the cutoff is changed, $8.4 \lesssim \theta_3 \lesssim 28.8$, but it is always significantly larger than θ'.

The product $g_* \lambda_*$ again exhibits an extremely weak scheme dependence. Figure 5.5(a) displays $g_* \lambda_*$ as a function of s. It is impressive to see how the cutoff dependences of g_* and λ_* cancel almost perfectly. Figure 5.5(a) suggests the universal value $g_* \lambda_* \approx 0.14$. Comparing this value to those obtained from the Einstein–Hilbert truncation we find that it differs slightly from the one based on the same gauge $\alpha = 1$. The deviation is of the same size as the difference between the $\alpha = 0$- and the $\alpha = 1$-results of the Einstein–Hilbert truncation.

(iv) Dimensionality of $\mathscr{S}_{\mathrm{UV}}$ ($\boldsymbol{R^2}$): Since the real parts of θ_1, θ_2, and θ_3 are positive throughout, all three scaling fields are relevant, which suggests that $\Delta_{\mathrm{UV}} \equiv \dim \mathscr{S}_{\mathrm{UV}}$ is *at least* equal to 3.

According to the canonical dimensional analysis, the (curvature)^n invariants in four spacetime dimensions are classically marginal for $n = 2$ and irrelevant for $n > 2$. So, if the classical scaling dimensions were a reliable guideline near the NGFP even, we would expect no further relevant directions to arise when we generalize the R^2-truncation by adding terms of canonical dimension 6 or higher to it. However, there could be large non-classical contributions to the scaling dimensions so that there might be relevant operators perhaps even beyond $n = 2$. Within the R^2-truncation it is therefore not yet possible to determine their number, Δ_{UV}, at least not in $d = 4$.

But nevertheless, as Weinberg pointed out already [10], it is hardly conceivable that the quantum effects are powerful enough to change the signs of *infinitely many*, arbitrarily large (negative) classical scaling dimensions. Therefore, we expect Δ_{UV} to be a *finite* number at least.

If correct, this expectation will manifest itself as follows. Let us consider a sequence of increasingly refined truncations with a growing dimensionality of the truncated theory space, $d_{\mathrm{trunc}} \equiv \dim \mathcal{T}_{\mathrm{trunc}}$. Then, as long as the truncation is still comparatively simple, we expect to find only relevant directions, hence $\Delta_{\mathrm{UV}} = d_{\mathrm{trunc}}$, but from a certain point onward Δ_{UV} should stabilize and no longer grow with increasing d_{trunc}. At this point, once $\Delta_{\mathrm{UV}} < d_{\mathrm{trunc}}$, the truncated flow is precise enough to describe a "low" dimensional hypersurface embedded in a "high" dimensional theory space. This allows us to make $d_{\mathrm{trunc}} - \Delta_{\mathrm{UV}} > 0$ predictions after having measured only Δ_{UV} couplings.

The first confirmation of this picture, the demonstration that enlarging the truncation ultimately adds only irrelevant scaling fields, came from the R^2-calculation in $d = 2 + \varepsilon$ dimensions [163]. There the dimensional count is shifted downward by two units. In this case we indeed find that already the

Table 5.1 *Fixed point coordinates and critical exponents of the R^2 truncation in $2+\varepsilon$ dimensions. Only the respective leading order in ε of these quantities is shown. The negative value $\theta_3 < 0$ implies that $\mathscr{S}_{\mathrm{UV}}$ is an (only!) two-dimensional surface in the three-dimensional theory space. (From [163].)*

s	λ_*	g_*	β_*	θ_1	θ_2	θ_3
1	-0.131ε	0.087ε	-0.083	2	0.963ε	-1.968
5	-0.055ε	0.092ε	-0.312	2	0.955ε	-1.955
10	-0.035ε	0.095ε	-0.592	2	0.955ε	-1.956

third scaling field is *irrelevant*, for any cutoff employed: $\theta_3 < 0$. In Table 5.1 we present the corresponding numerical results for selected values of the shape parameter s and in the leading order of ε.

For all cutoffs used we obtain three *real* critical exponents; the first two are positive and the third is negative. This suggests that in $d = 2+\varepsilon$ the dimensionality of $\mathscr{S}_{\mathrm{UV}}$ is likely to be as small as $\Delta_{\mathrm{UV}} = 2$. Hence, measuring only two free parameters, the renormalized Newton constant G_0 and the renormalized cosmological constant $\bar{\lambda}_0$, for instance, we are able to to predict the third coupling.

(3) Evidence from polynomial $f(R)$ truncations. Next, we summarize the main results obtained with the $(N+1)$-parameter truncations of (5.114) with (5.115).

(i) Position of the NGFP ($f(R), d=4$): The generalized RG equations, too, give rise to a NGFP with $g_* > 0, \lambda_* > 0$, whose N-dependent position is shown in Table 5.2. The product $g_*\lambda_* = u_0/(32\pi(u_1)^2)$ displayed in the last column is remarkably constant. It is in excellent agreement with the Einstein–Hilbert truncation (5.121), and the R^2-truncation, Figure 5.5(a). Only the value obtained in the case $N=2$ shows a mild deviation from the R^2-computations in [163], which can, most probably, be attributed to the use of a different gauge-fixing procedure, cutoff shape function, and ansatz for $\widehat{\Gamma}_k$.

(ii) Dimensionality of $\mathscr{S}_{\mathrm{UV}}$ ($f(R), d=4$): The critical exponents resulting from the stability analysis of the NGFP in the polynomial $f(R)$-truncation are summarized in Table 5.3. We see that only *three* of the eigendirections associated to the NGFP are relevant, i.e., UV attractive. Including higher-derivative terms R^n, $n \geq 3$ in the truncation creates irrelevant directions only. Thus, consistent with the R^2-truncation, we find $\Delta_{\mathrm{UV}} = 3$, i.e., the UV-critical surface associated with the fixed point is a three-dimensional submanifold in the $(N+1)$-dimensional theory space. Its dimensionality Δ_{UV} is stable with respect to increasing N [185, 186].

Table 5.2 *Location of the NGFP in the polynomial $f(R)$-truncation (5.115) of degree N. The table illustrates the stabilization of the fixed point coordinates as N increases. (From [185].)*

N	u_0^*	u_1^*	u_2^*	u_3^*	u_4^*	u_5^*	u_6^*	$g^*\lambda^*$
1	0.00523	−0.0202						0.127
2	0.00333	−0.0125	0.00149					0.211
3	0.00518	−0.0196	0.00070	−0.0104				0.134
4	0.00505	−0.0206	0.00026	−0.0120	−0.0101			0.118
5	0.00506	−0.0206	0.00023	−0.0105	−0.0096	−0.00455		0.119
6	0.00504	−0.0208	0.00012	−0.0110	−0.0109	−0.00473	0.00238	0.116

Table 5.3 *Stabilization of the critical exponents of the NGFP for increasing dimension $N+1$ of the truncation subspace. The first two critical exponents form a complex pair: $\theta' \pm i\theta''$. (From [185].)*

N	θ'	θ''	θ_2	θ_3	θ_4	θ_5	θ_6
1	2.38	−2.17					
2	1.26	−2.44	27.0				
3	2.67	−2.26	2.07	−4.42			
4	2.83	−2.42	1.54	−4.28	−5.09		
5	2.57	−2.67	1.73	−4.40	$-3.97+4.57i$	$-3.97-4.57i$	
6	2.39	−2.38	1.51	−4.16	$-4.67+6.08i$	$-4.67-6.08i$	−8.67

A specific RG trajectory tracing out this surface is determined by fixing the three relevant couplings. These describe in which direction tangent to $\mathscr{S}_{\rm UV}$ the trajectory moves away from the fixed point. All remaining couplings, the irrelevant ones, are predictions from Asymptotic Safety.

Subsequently, these results have been extended to $N=8$ [187] and more recently to $N=35$ [188], providing even stronger evidence for the robustness of the RG flow under the inclusion of further invariants constructed from the Ricci scalar.

(4) Evidence from the R^2+C^2-truncation in $d=4$. The results obtained on the basis of the ansatz (5.111) can be summarized as follows.

(i) Existence of the NGFP (R^2+C^2): The R^2+C^2-truncation predicts a NGFP with positive Newton and cosmological constant at

$$g_* = 1.960,\ \ \lambda_* = 0.218,\ \ \beta_* = 0.008,\ \ \gamma_* = -0.005. \tag{5.122}$$

The finite values for β_* and γ_* also imply a non-zero value of σ_*, via (5.112).

This should be contrasted to the one-loop result $\sigma_* = 0$ obtained from the perturbative quantization of fourth-order gravity. Obviously the non-perturbative corrections captured by the FRGE modify the fixed point properties in such a way that *Asymptotic Freedom is replaced with Asymptotic Safety*.

(ii) Stability properties ($\boldsymbol{R^2 + C^2}$): Linearizing the RG flow at the fixed point (5.122) along the lines of Section 5.2.2 the critical exponents are found as

$$\theta_0 = 2.51, \quad \theta_1 = 1.69, \quad \theta_2 = 8.40, \quad \theta_3 = -2.11. \tag{5.123}$$

We observe that *the inclusion of the C^2-term leads to entirely real stability coefficients*. This is in marked contrast to the complex stability coefficients and the corresponding spiraling approach to the NGFP characteristic for $f(R)$-type truncations.

Moreover, (5.123) shows that the "new" direction added by extending the R^2-truncation is UV-repulsive, i.e., irrelevant. Thus, in a four-dimensional truncation space, $\mathscr{S}_{\rm UV}$ remains three-dimensional: $\Delta_{\rm UV} = 3$, $d_{\rm trunc} = 4$. The condition for a trajectory being inside the UV-critical surface then imposes one constraint among the coupling constants. It can be used to express, say, γ in terms of the other coupling constants contained in the ansatz. In the linear regime we can do this even in closed form:

$$\boxed{\gamma = \gamma(g, \lambda, \beta) = -0.116 + 0.030\, \lambda\, g^{-1} + 0.049\, g^{-1} + 11.06\, \beta.} \tag{5.124}$$

This relationship parametrizes the hyperplane tangent to $\mathscr{S}_{\rm UV}$ at the NGFP. The non-linear structure of $\mathscr{S}_{\rm UV}$ away from the fixed point can be determined numerically.

Comparing the results (5.122) and (5.123) to their counterparts in the Einstein–Hilbert and R^2-truncation, the C^2-term is seen to lead to a moderate shift of the NGFP properties. In particular, the product $g_* \lambda_*$ turns out enhanced by about a factor of three, $g_* \lambda_* = 0.427$, and all stability coefficients of the fixed point are rendered real. Thus, the C^2-term affects the fixed point structure more drastically than the inclusion of the R^2-term, or working with different cutoff schemes or gauge-fixing functions within the Einstein–Hilbert truncation.

Summary: The above results strongly suggest that the non-Gaussian fixed point occurring in the Einstein–Hilbert truncation *is not a truncation artifact* but rather the projection of a genuine fixed point of the full RG flow on the exact theory space.

The fixed point and all its qualitative properties are stable against variations of the cutoff and the inclusion of further invariants in the truncation. It is particularly remarkable that within the scheme dependence the additional R^2-term has essentially no impact on the fixed point. Moreover, truncations involving higher-order polynomials in R or the tensor structure C^2 fully confirm the qualitative picture suggested by the simpler Einstein–Hilbert truncation and provide a strong indication that the corresponding quantum field theories are characterized only by a *finite number of free parameters*.

We interpret the above results and their mutual consistency as quite non-trivial indications supporting the conjecture that four-dimensional QEG possesses a fixed point with precisely the properties needed for Asymptotic Safety.

6

Bi-Metric Truncations

Obviously, there are many directions in which the truncations described in the previous chapter can be extended. We may include more complicated invariants into $\bar{\Gamma}_k[g]$, for instance, or generalize the ghost and gauge-fixing terms, or allow for a nontrivial *extra* background field dependence of the EAA, making it a genuine bi-metric action. It is the latter kind of generalization we are going to discuss in this chapter.

In Sections 5.1.1 and 5.1.3 we already introduced a broad class of bi-metric truncations. The pertinent ansatz for the EAA (5.1) consisted of a $\boldsymbol{\delta}^{\mathbf{B}}$-invariant functional $\Gamma_k^{\text{grav}}[g, \bar{g}]$, plus the classical gauge-fixing and ghost terms with, by assumption, scale-independent prefactors. So far we have analyzed only a special subsector of those truncations, namely the so-called single-metric truncations described in Section 5.1.3. While they are less general and therefore probably less reliable than a generic EAA of the form (5.1), they have the great advantage of a considerably simplified RG equation. In the single-metric sector, the FRGE boils down to the functional differential equation (5.15) for a running action with only one metric argument, $\bar{\Gamma}_k[g]$.

In this section we go beyond the single-metric approximation and consider generic truncation ansätze of the type (5.1). We are thus forced to cope with the considerably more complicated FRGE in (5.9) which RG evolves a generic functional $\Gamma_k^{\text{grav}}[g, \bar{g}]$ of two independent metrics.

6.1 Level Expansion of Γ_k^{grav}

Often it is natural to regard Γ_k^{grav} as depending on the pair $h, \bar{g}$ rather than $g, \bar{g}$ as the set of independent field variables:

$$\Gamma_k^{\text{grav}}[h; \bar{g}] \equiv \Gamma_k^{\text{grav}}[g, \bar{g}]\Big|_{g=\bar{g}+h}. \tag{6.1}$$

Furthermore, if $\Gamma_k^{\text{grav}}[h; \bar{g}]$ admits a functional Taylor series expansion in $h_{\mu\nu}$ about $h_{\mu\nu} = 0$, we can classify the terms contributing to $\Gamma_k^{\text{grav}}[h; \bar{g}]$ according to their *level*, i.e., their degree of homogeneity in the fluctuation field h. The EAA then admits a *level expansion* of the form

$$\Gamma_k^{\text{grav}}[h;\bar{g}] = \sum_{p=0}^{\infty} \check{\Gamma}_k^p[h;\bar{g}], \tag{6.2}$$

where the level-p contribution, $\check{\Gamma}_k^p$, by definition, is homogeneous with respect to h, i.e., $\check{\Gamma}_k^p[ch;\bar{g}] = c^p \check{\Gamma}_k^p[h;\bar{g}]$ for any constant $c > 0$. In a symbolic notion, $\check{\Gamma}_k^p[h;\bar{g}] \sim \check{\gamma}_k^p[\bar{g}] \cdot (h_{\mu\nu})^p$ where the dot represents a summation and integration over the discrete and continuous indices carried by the p factors of h, and $\check{\gamma}_k^p[\bar{g}]$ is a tensor-valued functional which depends only on the background metric. At the level $p = 1$, for example, we have explicitly:

$$\check{\Gamma}_k^1[h;\bar{g}] = \int \mathrm{d}^d x \, \sqrt{\bar{g}} \, h_{\mu\nu}(x) \, \check{\gamma}_k^1[\bar{g}]^{\mu\nu}(x). \tag{6.3}$$

By virtue of the level expansion, we may consider the bi-metric action $\Gamma_k^{\text{grav}}[g,\bar{g}]$ equivalent to an infinite family of single-metric functionals, $\{\check{\gamma}_k^p[\bar{g}]\}_{p=0,1,2,\cdots}$, carrying a non-trivial tensor structure.

When we insert the level expansion into the FRGE and project on a fixed level p, we see that $\partial_t \check{\Gamma}_k^p$ gets equated to an expression exclusively involving $\{\check{\Gamma}_k^q \,|\, q = 2, \cdots, p+2\}$. Hence, the noncanonical parts of all level-p beta functions depend on the level-q couplings, with $q = 2, \cdots, p+2$, only. Generically the FRGE amounts to an infinite hierarchy of equations $\partial_t \check{\Gamma}_k^p = \cdots$ for $p = 0, 1, 2, 3, \cdots$ which does not terminate at any finite level. Only if it was possible to realize split-symmetry exactly this tower of equations collapses to a single equation that governs all levels.

6.2 Level One and the Tadpole Equation

In the expansion (6.2) the level $p = 1$ is of special interest since it supplies the tadpole equation with the help of which self-consistent backgrounds can be computed. According to (5.11) with $\Gamma_k^{\text{grav}} \equiv \bar{\Gamma}_k + \widehat{\Gamma}_k$ it reads for the present class of actions, and in the sector with vanishing ghosts:

$$\frac{\delta}{\delta h_{\mu\nu}(x)} \Gamma_k^{\text{grav}}[h;\bar{g}] \Big|_{h=0;\bar{g}=\bar{g}_k^{\text{sc}}} = 0. \tag{6.4}$$

Inserting the expansion (6.2) into (6.4) we see that the tadpole equation is tantamount to

$$\boxed{\check{\gamma}_k^1[\bar{g}]^{\mu\nu} \Big|_{\bar{g}=\bar{g}_k^{\text{sc}}} = 0.} \tag{6.5}$$

This condition can be regarded as a generalization of Einstein's equation in classical relativity. Indeed, if we identify $\Gamma_k^{\text{grav}}[h;\bar{g}]$ with the Einstein–Hilbert term $\int \mathrm{d}^d x \sqrt{g} R(g)|_{g=\bar{g}+h}$, the left-hand side of (6.5) is nothing but the standard Einstein tensor $\bar{G}^{\mu\nu}$.

Nevertheless, a word of caution might be appropriate here. It is well known that the usual $G^{\mu\nu}$ has vanishing covariant divergence, $D_\mu G^{\mu\nu} = 0$, and that the same is true also for all generalizations of $G^{\mu\nu}$ that can be obtained as the functional derivative of an action $S[g]$, which, first, depends only on one metric and, second, is invariant under diffeomorphisms:

$$D_\mu \left[\frac{1}{\sqrt{g}} \frac{\delta S[g]}{\delta g_{\mu\nu}(x)} \right] = 0. \tag{6.6}$$

The standard argument leading to this conclusion[1] fails however, if we replace $S[g]$ by a generic $\Gamma_k^{\mathrm{grav}}[g, \bar{g}] \equiv \bar{\Gamma}_k[g] + \widehat{\Gamma}_k[g, \bar{g}]$. Returning to the "comma notation" where g and $\bar{g}$ are the independent variables, (6.2) and (6.3) tell us that

$$\check{\gamma}_k^1[g]^{\mu\nu}(x) \propto \frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta g_{\mu\nu}(x)} \Gamma_k^{\mathrm{grav}}[g, \bar{g}] \Big|_{\bar{g}=g}. \tag{6.7}$$

The generalized "Einstein tensor" $\check{\gamma}_k^1$ is indeed a functional derivative but of an action which depends on an additional field. Stated differently, it suffers from an extra background field dependence, and this extra dependence entails that in general

$$D_\mu \check{\gamma}_k^1[g]^{\mu\nu} \neq 0. \tag{6.8}$$

Only the more restrictive single-metric class of truncations which has $\widehat{\Gamma}_k \propto S_{\mathrm{gf}}$ or, more generally, $\Gamma_k^{\mathrm{grav}}[g, \bar{g}] = \bar{\Gamma}_k[g] + O((g - \bar{g})^2)$ yields a conserved tensor, $D_\mu \check{\gamma}_k^1[g]^{\mu\nu} = 0$.

The non-conservation (6.8) is easy to understand: it expresses that while the total 4-momentum is conserved, the dynamical metric can exchange momentum with all other fields present, in particular with matter fields, if included, and with the "extra" component of $\bar{g}_{\mu\nu}$.

When matter is coupled to gravity its energy-momentum tensor $T^{\mu\nu}$ replaces the zero on the right-hand side of (6.5). The condition for the consistency of the classical field equation, $D_\mu(g) T^{\mu\nu} = 0$, is then replaced by a more complicated condition involving $\bar{g}_{\mu\nu}$. (A detailed discussion of this point can be found in [189].)

At the observable level the "extra" part of the background metric will carry no physical energy and momentum, however. At $k = 0$, in the gauge-invariant sector, the restored split symmetry must erase all traces of the background metric we introduced as an intermediate technical device.

[1] Under an infinitesimal transformation $\delta g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu}$ or, equivalently, $\delta g_{\mu\nu} = D_\mu v_\nu + D_\nu v_\mu$ the functional $S[g]$ changes by an amount $\delta S = \int \mathrm{d}^d x \, \delta g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} = 2 \int \mathrm{d}^d x \sqrt{g} \, (D_\mu v_\nu) \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} = -2 \int \mathrm{d}^d x \sqrt{g} \, v_\nu D_\mu \left(\frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \right)$. Since v_ν is arbitrary, invariance ($\delta S = 0$) implies (6.6).

6.3 A Bi-Metric Einstein–Hilbert Ansatz

The analysis of truncations with a non-trivial extra $\bar{g}$-dependence only started relatively recently, the reason being that the derivation of projected flow equations in the presence of two independent metrics $g, \bar{g}$ tends to be a rather formidable task. In order to project and concretely evaluate the functional traces in the FRGE one has to carefully disentangle g and $\bar{g}$-dependencies, but no ready-made efficient techniques are available for such calculations.[2]

The bi-metric truncations were introduced and tested within conformally reduced gravity in [90]. Subsequently they were applied to matter-induced gravity in [189], and finally, in [190, 191], to fully dynamical Quantum Einstein Gravity. Special features of the RG flow in full gravity were analyzed further in [154, 192], namely the (dis-)appearance of antiscreening in QEG and the existence of a certain c-function like quantity, respectively. Building on the level expansion (or synonymously vertex expansion), flows of the Effective Average Action on a flat, Euclidean background have been studied in [193–197] while the generalization to curved backgrounds has been discussed in [155].

In the sequel we mostly focus on the investigations in [190, 191] which are based on the following special truncation ansatz,[3] referred to as a *bi-metric Einstein–Hilbert truncation*:

$$\boxed{\begin{aligned}\Gamma_k^{\text{grav}}[g,\bar{g}] =& -\frac{1}{16\pi G_k^{\text{Dyn}}}\int \mathrm{d}^d x\,\sqrt{g}\,\Big(R(g) - 2\Lambda_k^{\text{Dyn}}\Big)\\ &-\frac{1}{16\pi G_k^{\text{B}}}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,\Big(R(\bar{g}) - 2\Lambda_k^{\text{B}}\Big).\end{aligned}} \tag{6.9}$$

The action (6.9) consists of two structurally identical parts of the Einstein–Hilbert type which depend on different metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. The ansatz involves a total of 4 running coupling constants: the pair G_k^{Dyn}, Λ_k^{Dyn} in the "dynamical," i.e., $g_{\mu\nu}$-dependent sector and G_k^{B}, Λ_k^{B} in the pure background part.

While the two invariants of the *single-metric* Einstein–Hilbert truncation, $\int\sqrt{g}R(g)$ and $\int\sqrt{g}$, are the only terms with at most two derivatives of the metric, one can easily write down many more $\boldsymbol{\delta}^{\mathbf{B}}$-invariant zero- and second-derivative terms depending on g and $\bar{g}$ than those appearing in (6.9). Examples include $\int\sqrt{g}\,R(\bar{g})$, $\int\sqrt{\bar{g}}\,R(g)$, $\int\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}R_{\mu\nu}(g)$, etc., or more complicated mixtures of the two metrics. In fact, there exists already an infinite variety of non-derivative terms of the form $\int\sqrt{g}\,Y(g_{\mu\nu},\bar{g}_{\mu\nu})$, where Y is an arbitrary function.[4]

[2] An exception are bi-metric computations on a flat background. In this case one may resort to momentum space techniques developed within the framework of standard quantum field theory.

[3] In addition to the ansatz (6.9), [190] also considered the mixed bi-metric invariant $\int \mathrm{d}^d x\sqrt{\bar{g}}\,\big(\sqrt{\bar{g}}/\sqrt{g}\big)^n$, $n\in\mathbb{N}$, which will be omitted here.

[4] In the case of matter-induced gravity, the infinite-dimensional RG flow of the full function $Y\equiv Y_k$ has been analyzed in [189]. The generalization to the gravitational setting has been carried out in [198].

Like in the single-metric case, we expect large anomalous dimensions to occur. Hence, the importance of a given term in the ansatz cannot be estimated beforehand by canonical power counting. As before, an important part of the analysis will consist in the validation of the truncation a posteriori.

6.4 The Plethora of Couplings

In order to generate a list of all invariants that can occur in a bi-metric EAA, it is advantageous to employ the $h, \bar{g}$-language and the level-expanded form of $\Gamma_k^{\text{grav}}[h; \bar{g}]$, represented by the coefficients $\{\check{\gamma}_k^p[\bar{g}]\}_{p=0,1,2,\cdots}$.

As they depend on one metric only we can then derivative-expand the $\check{\gamma}$s with respect to $\bar{g}$ in the usual way.[5] This provides us with a double expansion of $\Gamma_k^{\text{grav}}[h; \bar{g}]$ in h and $\bar{g}$ from which, at least in principle, a complete set of bi-metric functionals can be read off. More precisely, this set may be considered complete, i.e., a basis of theory space, provided the functionals to be expanded, say the traces in the FRGE, are evaluated in conformity with this algorithm (level expansion with subsequent derivative expansion in $\bar{g}$).

(1) In the "semicolon language," we can set up truncations for $\Gamma_k^{\text{grav}}[h; \bar{g}]$ by retaining some of the invariants in the list generated, while discarding others. As an, admittedly not quite typical, example let us recast the double Einstein–Hilbert action (6.9) in a form that makes its level structure explicit. Inserting $g = \bar{g} + h$ into (6.9) and expanding in h the first few terms read

$$\begin{aligned}
\Gamma_k^{\text{grav}}[h; \bar{g}] = & -\frac{1}{16\pi G_k^{(0)}} \int \mathrm{d}^d x \sqrt{\bar{g}} \left(R(\bar{g}) - 2\Lambda_k^{(0)} \right) \\
& + \frac{1}{16\pi G_k^{(1)}} \int \mathrm{d}^d x \sqrt{\bar{g}} \left[\bar{G}^{\mu\nu} + \Lambda_k^{(1)} \bar{g}^{\mu\nu} \right] h_{\mu\nu} \\
& + \frac{1}{16\pi G_k^{(2)}} \int \mathrm{d}^d x \sqrt{\bar{g}}\, h_{\mu\nu} \left[-K^{\mu\nu\rho\sigma} \left(\bar{D}^2 + 2\Lambda_k^{(2)} \right) + U^{\mu\nu\rho\sigma} \right] h_{\rho\sigma} \\
& + O(h^3).
\end{aligned} \tag{6.10}$$

Here $\bar{G}^{\mu\nu}$ is the Einstein tensor for the background metric and $K^{\mu\nu\rho\sigma}$ and $U^{\mu\nu\rho\sigma}$ are given in (5.25) and (5.26) except that $\bar{\lambda}_k$ is set to zero in $U^{\mu\nu\rho\sigma}$.

In writing down (6.10) we designated the couplings appearing in a term of level p by the superscript ${}^{(p)}$. Since (6.10) descends from (6.9), the coefficients $G_k^{(p)}$ and $\Lambda_k^{(p)}$ are, of course, not all independent. They can be expressed in terms of

[5] For a partial classification of such monomials see [199].

the four couplings in (6.9):

$$G_k^{(0)} = \frac{G_k^{\text{Dyn}} G_k^{\text{B}}}{G_k^{\text{Dyn}} + G_k^{\text{B}}}, \qquad G_k^{(1)} = G_k^{(2)} = \cdots = G_k^{\text{Dyn}},$$
$$\Lambda_k^{(0)} = \frac{\Lambda_k^{\text{Dyn}}\, G_k^{\text{B}} + \Lambda_k^{\text{B}}\, G_k^{\text{Dyn}}}{G_k^{\text{Dyn}} + G_k^{\text{B}}}, \qquad \Lambda_k^{(1)} = \Lambda_k^{(2)} = \cdots = \Lambda_k^{\text{Dyn}}. \tag{6.11}$$

(2) If we wanted to go *beyond the double EH truncation* we could relax these relations and consider, say, $G_k^{(1)}$ and $G_k^{(2)}$, as two independent functions of k. Note, however, that even if we treat all $G_k^{(p)}$, $\Lambda_k^{(p)}$, $p = 0, 1, 2, \cdots, \infty$, as independent we are still far from exhausting all possible invariants.

For example, there exists a further level-one term that also involves two derivatives of the background metric. Including it into (6.10) the level-one part of the EAA would generalize to

$$\frac{1}{16\pi G_k^{(1)}} \int \mathrm{d}^d x \sqrt{\bar{g}} \left[\bar{G}^{\mu\nu} - \tfrac{1}{2} E_k\, \bar{g}^{\mu\nu} \bar{R} + \Lambda_k^{(1)} \bar{g}^{\mu\nu} \right] h_{\mu\nu}, \tag{6.12}$$

with an independent coupling constant E_k. In this case the tensor $\check{\boldsymbol{\gamma}}_k^1[\bar{g}]^{\mu\nu}$ is, up to a constant, given by $\bar{G}^{\mu\nu} - (1/2) E_k \bar{g}^{\mu\nu} \bar{R} + \Lambda_k^{(1)} \bar{g}^{\mu\nu}$. While it has indeed vanishing covariant divergence $\bar{D}_\mu \check{\boldsymbol{\gamma}}_k^1[\bar{g}]^{\mu\nu} = 0$ if $E_k = 0$, the new term is seen to spoil this property, $\bar{D}_\mu \check{\boldsymbol{\gamma}}_k^1[\bar{g}]^\mu{}_\nu \propto E_k \partial_\nu \bar{R}$.

When we invoke the single-metric approximation,

$$\Gamma_k^{\text{grav}}[g, \bar{g}] = \bar{\Gamma}_k[g] \quad \text{(single-metric approx.)}, \tag{6.13}$$

the E_k-term does not occur. And consistent with that, the $\boldsymbol{\delta}^{\mathbf{B}}$-invariance of $\bar{\Gamma}_k[g]$ implies by the usual argument that $\check{\boldsymbol{\gamma}}_k^1[\bar{g}]^{\mu\nu} \propto \frac{\delta}{\delta g_{\mu\nu}} \bar{\Gamma}_k[g]|_{g=\bar{g}}$ is covariantly conserved then: $\bar{D}_\mu \check{\boldsymbol{\gamma}}_k^1[\bar{g}]^{\mu\nu} = 0$.

(3) Specializing (6.10) for the *single-metric case* of (6.13), the Newton and cosmological constants of *all* levels, including $p = 0$, become equal:

$$G_k^{(0)} = G_k^{(1)} = G_k^{(2)} = \cdots \quad \text{and} \quad \Lambda_k^{(0)} = \Lambda_k^{(1)} = \Lambda_k^{(2)} = \cdots\,. \tag{6.14}$$

In the language of equation (6.9) this special case amounts to

$$\frac{1}{G_k^{\text{B}}} = 0 \quad \text{and} \quad \frac{\Lambda_k^{\text{B}}}{G_k^{\text{B}}} = 0. \tag{6.15}$$

Then $G_k^{\text{Dyn}} = G_k^{(p)}$, $\Lambda_k^{\text{Dyn}} = \Lambda_k^{(p)}$ for all $p = 0, 1, 2, \cdots, \infty$.

(4) The defining properties of theory space include a number of constraints, in particular the Ward identity for split symmetry given by (4.96). In the truncation at hand it reduces to an equation of the form $\frac{\delta}{\delta \bar{g}_{\mu\nu}(x)} \Gamma_k^{\text{grav}}[g,\bar{g}] = (1/2)\,\text{STr}[\cdots]$ with a complicated right-hand side that involves Γ_k^{grav} itself. Projecting the Ward identity on the space of action functionals (6.10) results in an infinite set of relations among all $G_k^{(p)}$ and $\Lambda_k^{(p)}$, respectively.

At the lowest order, i.e., when we neglect loop contributions, $\text{STr}[\cdots] \to 0$, those relations are nothing but (6.14), or equivalently (6.15). They characterize the limiting case of vanishing extra $\bar{g}$-dependence.

6.5 Results from the Bi-Metric EH Truncation

In the sequel our presentation will mostly follow [191]. We consider the four-parameter ansatz (6.9), or equivalently its level-expanded version, subject to the relations (6.11). This amounts to setting $E_k = 0$,[6] in particular.

After multiplying the G_ks and Λ_ks by the usual factors of k^{d-2} and k^{-2}, respectively, the RG trajectories are parametrized by the four dimensionless couplings $(g_k^{\text{Dyn}}, \lambda_k^{\text{Dyn}}, g_k^{\text{B}}, \lambda_k^{\text{B}})$. Their RG equations read:

$$\begin{aligned}
\partial_t g_k^{\text{Dyn}} &= \beta_g^{\text{Dyn}}(g_k^{\text{Dyn}}, \lambda_k^{\text{Dyn}}) \equiv \big[d-2+\eta^{\text{Dyn}}(g_k^{\text{Dyn}}, \lambda_k^{\text{Dyn}})\big]\, g_k^{\text{Dyn}},\\
\partial_t \lambda_k^{\text{Dyn}} &= \beta_\lambda^{\text{Dyn}}(g_k^{\text{Dyn}}, \lambda_k^{\text{Dyn}}),\\
\partial_t g_k^{\text{B}} &= \beta_g^{\text{B}}(g_k^{\text{Dyn}}, \lambda_k^{\text{Dyn}}, g_k^{\text{B}}) \equiv \big[d-2+\eta^{\text{B}}(g_k^{\text{Dyn}}, \lambda_k^{\text{Dyn}}, g_k^{\text{B}})\big]\, g_k^{\text{B}},\\
\partial_t \lambda_k^{\text{B}} &= \beta_\lambda^{\text{B}}(g_k^{\text{Dyn}}, \lambda_k^{\text{Dyn}}, g_k^{\text{B}}, \lambda_k^{\text{B}}).
\end{aligned} \tag{6.16}$$

Generalizing (5.80), the anomalous dimensions η^{Dyn} and η^{B} are given by the logarithmic scale derivatives of G_k^{Dyn} and G_k^{B}, respectively. Notably the two equations for the "Dyn" sector close among themselves and do not contain the "B" couplings, while conversely the "Dyn" couplings do appear in the beta functions of the "B" sector. This triangular structure can be understood by noticing that terms containing the background couplings are, by definition, independent of the fluctuation fields so that they can not appear in the right-hand side of the flow equation.

The equivalent system of equations in level language can be written in terms of the four independent couplings $(g_k^{(0)}, \lambda_k^{(0)}, g_k^{(1)}, \lambda_k^{(1)})$ since, within the present truncation, all couplings at higher levels are equal to those at level one: $g_k^{(p)} = g_k^{(1)}$ and $\lambda_k^{(p)} = \lambda_k^{(1)}$ for all $p \geq 1$.

The β-functions for the bi-metric Einstein–Hilbert ansatz have been derived in [190, 191] for two different but, as it turned out, essentially equivalent projections. They differ with respect to the gauge-fixing conditions and parameters, as well as the field parametrization in use. In [190] the operator $\mathcal{R}_k$ was adapted to a transverse-traceless (TT) decomposed field basis for $h_{\mu\nu}$ (cf. Appendix F), while in [191] no TT decomposition was necessary.

[6] An analysis of the RG behavior of E_k within matter-induced gravity can be found in [189].

The explicit formulae for the β-functions are too lengthy to be reproduced here so that we must limit ourselves to a qualitative discussion of the RG flow they give rise to. Again, we specialize to $d=4$ dimensions.

6.5.1 The 2D Flow on the "Dyn" Subspace

On the four-dimensional parameter space $\mathcal{T} \equiv \{(g^{\text{Dyn}}, \lambda^{\text{Dyn}}, g^{\text{B}}, \lambda^{\text{B}})\}$ the RG flow decomposes hierarchically according to $(g^{\text{Dyn}}, \lambda^{\text{Dyn}}) \to g^{\text{B}} \to \lambda^{\text{B}}$. The "Dyn" equations closing among themselves allows us to first compute the flow on the dynamical subspace $\mathcal{T}_{\text{Dyn}} \equiv \{(g^{\text{Dyn}}, \lambda^{\text{Dyn}})\}$ without reference to the background couplings yet. Hence, the solution to the first two equations in (6.16) yields the exact projection of the four-dimensional flow.

The two-dimensional projection onto the g^{Dyn}-λ^{Dyn}-plane is shown in Figure 6.1. It is strikingly similar to the phase portrait of the corresponding single-metric truncation in Figure 5.1: The g^{Dyn}-λ^{Dyn}-plane displays both a Gaussian fixed point and a non-Gaussian one, denoted by **G$^{\text{Dyn}}$-FP** and **NG$^{\text{Dyn}}$-FP**, respectively. Moreover, we can identify the same classes of trajectories as in the single-metric discussion, namely those of Type Ia, IIa, and IIIa, depending on whether the cosmological constant is heading for $-\infty$, 0, or $+\infty$ in the IR. Typical trajectories display a generalized crossover transition which connects a fixed point in the UV to a classical regime in the IR. The latter is located on the trajectory's lower, almost horizontal, branch where $g \ll 1$.

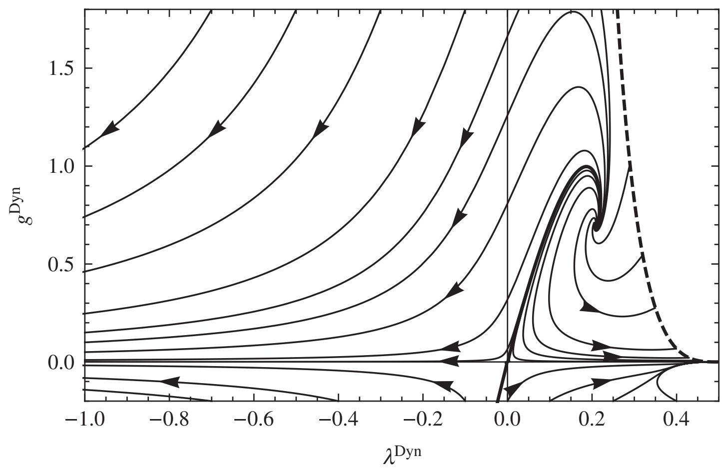


Figure 6.1. Phase portrait on the g^{Dyn}-λ^{Dyn}-plane obtained by projecting the four-dimensional bi-metric flow. This projection is qualitatively identical with the corresponding single-metric Einstein–Hilbert flow, displaying in particular the projection of a 4-dimensional non-Gaussian fixed point. (Adapted from [192].)

The accessible part of theory space has a boundary, a line on which the beta functions diverge. In Figure 6.1, and similar diagrams that will follow, this boundary is indicated by a dashed line.

The properties of $\mathbf{NG^{Dyn}}$**-FP** are qualitatively identical and numerically similar to those of the NGFP found earlier in the single-metric truncations. The critical exponents of $\mathbf{NG^{Dyn}}$**-FP** form a complex conjugate pair with a non-zero imaginary part leading to spiral shaped trajectories. The fixed point is UV attractive in both directions. The precise numerical values of the critical exponents still differ appreciably for the two different projections used in conjunction with the ansatz (6.9). According to [190] and [191] they are $4.47 \pm 4.23i$ and $3.6 \pm 4.3i$, respectively.

6.5.2 Screening vs. Antiscreening

There is an important difference between the single metric and the g^{Dyn}-λ^{Dyn} flow which cannot be seen in the phase portraits: While according to the single-metric EH truncation the anomalous dimension $\eta_{\mathrm{N}}(g,\lambda)$ is always negative for $g>0$, this is not the case for its bi-metric counterpart.

The anomalous dimension belonging to the dynamical Newton constant $\eta^{\mathrm{Dyn}}(g^{\mathrm{Dyn}},\lambda^{\mathrm{Dyn}}) \equiv \partial_t \ln G_k^{\mathrm{Dyn}}$ is found to change its sign at a certain critical value of the dimensionless cosmological constant, $\lambda_{\mathrm{crit}}^{\mathrm{Dyn}}$. On the half-space with $g^{\mathrm{Dyn}} > 0$, the anomalous dimension is *positive* in the domain where $\lambda^{\mathrm{Dyn}} < \lambda_{\mathrm{crit}}^{\mathrm{Dyn}}$, and negative when $\lambda^{\mathrm{Dyn}} > \lambda_{\mathrm{crit}}^{\mathrm{Dyn}}$; see Figure 6.2.

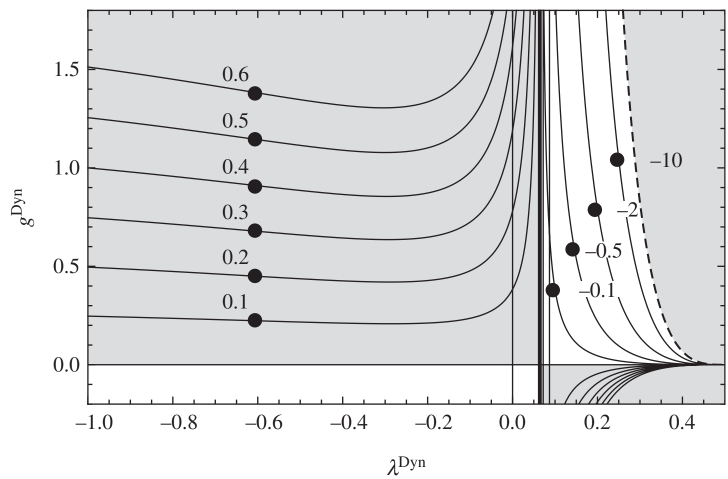


Figure 6.2. The contour plot shows the lines of constant anomalous dimension η^{Dyn} implied by the bi-metric calculation [191, 192]. The shaded regions correspond to $\eta^{\mathrm{Dyn}} > 0$. The function η^{Dyn} changes its sign on a straight line $\lambda^{\mathrm{Dyn}} = \lambda_{\mathrm{crit}}^{\mathrm{Dyn}}$ with $\lambda_{\mathrm{crit}}^{\mathrm{Dyn}} > 0$. Near the Gaussian Fixed Point at the origin, η^{Dyn} is seen to be positive. (Adapted from [192].)

Since $\lambda^{\text{Dyn}}_{\text{crit}} > 0$, the former domain contains a vicinity of the Gaussian Fixed Point, whereas the NGFP is located within the latter. After all, for purely dimensional reasons, η^{Dyn} must assume the value -2 directly at the NGFP; see (6.16) with $d = 4$.

The dynamical Newton constant, G^{Dyn}_k, increases (decreases) with decreasing RG scale k, hence indicating a regime of gravitational antiscreening (screening), provided $\eta^{\text{Dyn}} < 0$ ($\eta^{\text{Dyn}} > 0$).

In contrast to all earlier single-metric based calculations which unanimously find a negative anomalous dimension *everywhere*, the double Einstein–Hilbert truncation predicts that the gravitational antiscreening is lost at sufficiently small values of the dimensionless cosmological constant. In the semiclassical regime near the Gaussian Fixed Point $g^{\text{Dyn}}_* = 0 = \lambda^{\text{Dyn}}_*$, for example, G^{Dyn}_k shows a screening behavior: When we start out at $k = 0$ and increase k, we first pass a regime (extending to about the Planck scale) in which G^{Dyn}_k *grows* slightly and then reaches a maximum before it ultimately approaches zero according to the fixed point scaling behavior $G^{\text{Dyn}}_k = g^{\text{Dyn}}_* k^{-2}$, $k \to \infty$.

On the other hand, according to the single-metric Einstein–Hilbert truncation, G_k is still almost constant at the scales where G^{Dyn}_k grows.

6.5.3 Flow on the 4D Parameter Space

Let us now return to the system of all four differential equations. Every given two-dimensional trajectory $k \mapsto (g^{\text{Dyn}}_k, \lambda^{\text{Dyn}}_k)$, together with initial conditions for the background couplings g^{B} and λ^{B}, fixes a unique trajectory on the $(g^{\text{Dyn}}, \lambda^{\text{Dyn}}, g^{\text{B}}, \lambda^{\text{B}})$-parameter space. Analyzing the corresponding four-dimensional flow it turns out that the fixed point $\mathbf{NG^{Dyn}}$**-FP** discussed above “lifts” to two fixed points in four dimensions, $\mathbf{G^B} \oplus \mathbf{NG^{Dyn}}$**-FP** and $\mathbf{NG^B} \oplus \mathbf{NG^{Dyn}}$**-FP**. As the notation indicates, they are, respectively, Gaussian and non-Gaussian with respect to the background couplings.

The doubly non-Gaussian Fixed Point $\mathbf{NG^B} \oplus \mathbf{NG^{Dyn}}$**-FP** can be seen as the bi-metric refinement of the familiar single-metric fixed point in the Einstein–Hilbert truncation. It is the natural candidate for the construction of an asymptotically safe UV limit. Within the present bi-metric truncation, it has four relevant (i.e., UV-attractive) directions on the $(g^{\text{Dyn}}, \lambda^{\text{Dyn}}, g^{\text{B}}, \lambda^{\text{B}})$-parameter space.

6.5.4 Imposing Split Symmetry

Every point of the $(g^{\text{Dyn}}, \lambda^{\text{Dyn}}, g^{\text{B}}, \lambda^{\text{B}})$-parameter space represents a specific action of the form (5.1). It complies with some, but not yet all, defining properties the functionals forming the true theory space must possess. While it is $\boldsymbol{\delta}^{\mathbf{B}}$-invariant and satisfies the BRST Ward identities to the lowest order, it does not satisfy the Ward identity for split symmetry in general; see Section 5.1.3.

Both types of Ward identities are known to be consistent with the RG flow in an exact calculation, i.e., if they are fulfilled at one point of an RG trajectory, they hold at any other point of the trajectory as well. On the other hand, in a truncated calculation, consistency is expected to be attainable at best in an approximate sense.

Investigations of this issue are particularly demanding technically since the functional traces that occur in the Ward identities are even more complicated than those of the FRGE. Therefore, the first investigation of the role played by the split symmetry Ward identities [191] was based upon a leading-order approximation analogous to the one invoked for the BRST identities; it consists of neglecting the respective $\mathrm{STr}[\cdots]$ terms which are due to higher loop contributions.[7] At the end of Section 6.4 we mentioned already that the split symmetry Ward identities assume the simple form (6.14) or, equivalently, (6.15) then.

Those relations were imposed on the initial conditions of the RG trajectories at some low infrared scale k_{IR} and it was then checked numerically how well they are satisfied at other scales.

Concerning the existence of asymptotically safe RG trajectories the main result is the following:

Since the $\mathbf{NG^B} \oplus \mathbf{NG^{Dyn}}$**-FP** is UV-attractive in all four directions, its UV-critical hypersurface $\mathscr{S}_{\mathrm{UV}}$ is 4-dimensional as long as one ignores the constraints coming from the Ward identities. Hence, within the truncation considered, there exists a four-parameter family of trajectories which emanate from the fixed point in the UV and stay on $\mathscr{S}_{\mathrm{UV}}$ while heading for the IR. Generically, they will not satisfy the constraints there. However, a detailed analysis has shown [191] that *there exists a two-parameter subfamily of trajectories which are both asymptotically safe and satisfy the constraints due to the Ward identities.*

Thus, effectively, the dimensionality of the true theory space and of $\mathscr{S}_{\mathrm{UV}}$ gets lowered from 4 to 2 by imposing split symmetry or, more generally speaking, Background Independence.

The significance of these results resides in the fact that they demonstrate, at least within a truncation, that *the two key requirements of Asymptotic Safety and Background Independence can be attained simultaneously*; they are not mutually exclusive.

6.5.5 Discussion

A detailed comparison of the single- and the bi-metric Einstein–Hilbert truncation reveals that, quite unexpectedly, the former is a rather precise approximation to the latter *in the vicinity of the non-Gaussian fixed point.*

[7] For first steps toward solving the FRGE and modified split symmetry Ward-identities simultaneously see [200, 201].

In the far IR the split-symmetry restoring trajectories, by construction, give rise to another regime in which the two truncations agree well. However, in between there are even qualitative differences, for instance with respect to the sign of the dynamical anomalous dimension η^{Dyn}.

Furthermore, substantial quantitative differences between the critical exponents in both settings are found, despite the "miraculous" precision of the single-metric truncation near the NGFP. They clearly show the limitations of the single-metric approximation.

The general lesson that emerges from a comprehensive reliability analysis [154, 191, 192] is that from a certain degree of precision onward it no longer makes sense to keep including further invariants in the truncation ansatz that are built from the dynamical metric alone. Rather, to the same extent we allow the Effective Average Action to depend on $g_{\mu\nu}$ in a more complicated way, also its dependence on the background metric $\bar{g}_{\mu\nu}$ must be generalized.

While the general picture conveyed by the single-metric truncations seems correct concerning the crucial question of an asymptotically safe UV limit, they are insufficient for quantitatively precise calculations, the determination of critical exponents being a prominent example.

Finally we refer the reader to [202] for the analysis of a different bi-metric truncation that takes the influence of the anomalous dimensions of the $h_{\mu\nu}$ and ghost-fields into account. Within this truncation, a NGFP suitable for Asymptotic Safety is also found.

7

Conformally Reduced Gravity

Conformal reduction in the sense we use the term here describes an approximation scheme in which only the conformal modes of the metric are quantized, i.e., integrated over in the functional integral; all other types of metric fluctuations are neglected, leading to considerable technical simplifications. This approximation may be applied over and above any given truncation of theory space, the prime example being the *conformally reduced Einstein–Hilbert truncation* (CREH).

Reduced gravity theories are an instructive theoretical laboratory, which, thanks to its mathematical simplicity, has led to a number of valuable general insights that carry over to the full-fledged theory where all fluctuations are quantized. The CREH model illuminates in particular, in a setting as transparent as possible, how the special status of the gravitational field leads to RG flows that are substantially different from those of typical matter systems.

The conformally reduced FRGE framework was introduced in [80] and [203, 204], respectively, where the CREH truncation and an LPA generalization thereof were analyzed. The reduced setting has often served as a testbed for new developments. Bi-metric truncations [90], for example, or the "reconstruction" of a regularized functional integral from the NGFP [205, 206], were first explored in this setting. The investigation of Asymptotic Safety on infinite-dimensional theory spaces [203] also started with conformally reduced gravity. Many of the new techniques required by the $f(R)$ truncations, for example, were developed in this context.

Going beyond the CREH model one might also explore the effective dynamics of the conformal factor within full-fledged QEG, i.e., with all fluctuation modes quantized. At this level, the implications of Asymptotic Safety have been analyzed in [207, 208].

7.1 From Gravity to ϕ^4 Theory[1]

To prepare the stage for the CREH truncation we start out from the Einstein–Hilbert action

$$S_{\mathrm{EH}}[g_{\mu\nu}] = -\frac{1}{16\pi G}\int \mathrm{d}^d x\,\sqrt{g}\,\big(R(g) - 2\Lambda\big) \tag{7.1}$$

and assume that its argument, $g_{\mu\nu}$, equals a certain conformal factor times a fixed, non-dynamical reference metric $\mathring{g}_{\mu\nu}$. We would like to parameterize this conformal factor in terms of a scalar function $\phi(x)$ in such a way that the kinetic term for ϕ has the standard form $\propto(\partial_\mu\phi)^2$. This is indeed possible, for any dimensionality of spacetime, provided one expresses the conformal factor appropriately in terms of the independent field variable, ϕ. In fact, introducing ϕ according to [209],

$$g_{\mu\nu} = \phi^{2\nu(d)}\,\mathring{g}_{\mu\nu}, \tag{7.2}$$

with the dimension-dependent exponent

$$\nu(d) = \frac{2}{d-2}, \tag{7.3}$$

yields the following result for S_{EH} evaluated on metrics of the form (7.2):

$$\begin{aligned} S_{\mathrm{EH}}[\phi] = -\frac{1}{8\pi\,\xi(d)\,G}\int \mathrm{d}^d x\,\sqrt{\mathring{g}}\,\Big(&\tfrac{1}{2}\,\mathring{g}^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}\,\xi(d)\,\mathring{R}\,\phi^2 \\ &- \xi(d)\,\Lambda\,\phi^{2d/(d-2)}\Big). \end{aligned} \tag{7.4}$$

Here $\mathring{R}$ denotes the curvature scalar of the reference metric $\mathring{g}_{\mu\nu}$, and

$$\xi(d) \equiv \frac{d-2}{4(d-1)}. \tag{7.5}$$

Employing an exponent ν different from (7.3) the kinetic term in $S_{\mathrm{EH}}[\phi]$ would fail to be bilinear in the field ϕ.

In *four* dimensions, for example, we have $\nu(4)=1$ and $\xi(4)=\frac{1}{6}$ so that

$$\boxed{g_{\mu\nu} = \phi^2\,\mathring{g}_{\mu\nu}} \tag{7.6}$$

converts the Einstein–Hilbert action to a kind of "ϕ^4-theory":

$$\boxed{S_{\mathrm{EH}}[\phi] = -\frac{3}{4\pi G}\int \mathrm{d}^d x\,\sqrt{\mathring{g}}\,\Big(\tfrac{1}{2}\,\mathring{g}^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{12}\mathring{R}\,\phi^2 - \tfrac{1}{6}\Lambda\,\phi^4\Big)\,.} \tag{7.7}$$

[1] Our presentation in Sections 7.1–7.4 mostly follows [80] to which the reader is referred for further details.

We refer to the action (7.4) and its special case (7.7) as the conformally reduced Einstein–Hilbert or "CREH" action.

For $d > 2$, the case we will always assume in the following, the kinetic term in (7.4) is *negative definite* due to the "wrong sign" of its prefactor. As a result, the action is unbounded below: with a $\phi(x)$ which varies sufficiently rapidly, $S_{\mathrm{EH}}[\phi]$ becomes arbitrarily negative. This is the notorious *conformal factor instability.*

Quantizing the conformal factor in the CREH approximation[2] with the bare action $S_{\mathrm{EH}}[\phi]$ thus seems similar to quantizing a scalar theory with an action of the type

$$S[\phi] = c \int \mathrm{d}^4x \left\{ -\tfrac{1}{2} (\partial\phi)^2 + U(\phi) \right\}, \tag{7.8}$$

where c is a positive constant. For the sake of the argument let us assume that $\mathring{g}_{\mu\nu} = \delta_{\mu\nu}$ is the flat metric on $\mathbb{R}^4$. Then S_{EH} of (7.7) is indeed of the form (7.8) with the potential $U(\phi) = \frac{1}{6} \Lambda\, \phi^4$ and $c = \frac{3}{4\pi G} > 0$. If the cosmological constant is positive, as in the case which will be relevant later on, the potential term in (7.8) is *positive* definite.

Now we would like to understand the quantum theory based upon the functional integral

$$I \equiv \int \mathcal{D}\widehat{\phi}\; e^{i\widetilde{S}[\widehat{\phi}]}, \tag{7.9}$$

where $\widetilde{S}$ is the Wick rotated version of S, with the Lorentzian $(\partial\widehat{\phi})^2 \equiv \eta^{\mu\nu}\, \partial_\mu\widehat{\phi}\, \partial_\nu\widehat{\phi}$. One would expect that in this theory the wrong sign of the kinetic term drives the condensation of spatially inhomogeneous (x-dependent) modes, i.e., the formation of a "kinetic condensate" [222]. The amplitude of the inhomogeneous modes cannot grow unboundedly though since this would cost potential energy.

Next, consider the closely related theory with the "inverted" action $S_{\mathrm{inv}}[\phi] \equiv -S[\phi]$. Thus,

$$S_{\mathrm{inv}}[\phi] = c \int \mathrm{d}^4x \left\{ +\tfrac{1}{2} (\partial\phi)^2 + V(\phi) \right\} \tag{7.10}$$

involves the negative potential $V(\phi) \equiv -U(\phi) \leq 0$. In pulling out a global minus sign from S the instability has been shifted from the kinetic to the potential term. According to the picture conveyed by S_{inv}, the kinetic energy now assumes its minimum for x-independent field configurations $\phi = \mathrm{const}$, while the new potential $V(\phi) = -\frac{1}{6}\Lambda\, \phi^4$ is unbounded below when $\Lambda > 0$ so that it causes an instability.

Even though S and S_{inv} appear to be plagued by instabilities of a rather different nature, they should describe the same physics (possibly up to a time

[2] For the quantization of a more general bare action $S[\phi]$ including the non-local terms induced by the conformal anomaly of matter fields [210] we refer to the work of Antoniadis, Mazur, and Mottola [211–217]. Implications for the cosmological constant problem are discussed in [218–221].

reflection). Indeed, the path integrals involving S and S_{inv}, respectively, are related by complex conjugation:

$$I_{\text{inv}} \equiv \int \mathcal{D}\widehat{\phi}\; e^{-i\widetilde{S}[\widehat{\phi}]} = I^*. \tag{7.11}$$

One refers to the formulations in terms of S and S_{inv} as the *original picture* and the *inverted picture*, respectively. Due to the projective nature of the Wetterich equation (2.15) one expects that the overall sign drops out so that both pictures lead to identical RG flows.

So we see that for *pure* gravity in the CREH approximation the "wrong" sign of the kinetic term can be traded for an upside down potential. The FRGE formalism we are going to set up will effectively correspond to the inverted picture, employing an action with a positive kinetic, but negative potential term when $\Lambda > 0$.

The scalar ϕ^4-theory with a negative coupling constant was already discussed by Symanzik [223] in the early 1970s. He showed that the coupling strength decreases at short distances, thus providing the first example of an *asymptotically free* quantum field theory [224].

7.2 The Untruncated Conformally Reduced Theory

Let us now turn to a general reduced quantum theory of the conformal factor and the associated FRGE, which is still exact in the sense that no truncation beyond the reduction is performed.

Thus, we are led to investigate the "toy integral" $\int \mathcal{D}\widehat{\phi}\, e^{-S[\widehat{\phi}]}$ instead of the full $\int \mathcal{D}\widehat{g}_{\mu\nu}\, e^{-S[\widehat{g}_{\mu\nu}]}$, where the integration is only over metrics conformal to the reference metric[3]:

$$\widehat{g}_{\mu\nu} = \widehat{\phi}^{2\nu(d)}\, \mathring{g}_{\mu\nu}. \tag{7.12}$$

Here the microscopic, or quantum field, i.e., the field that would be promoted to an operator in canonical quantization, is denoted by $\widehat{\phi}$. It differs from the actual conformal factor, which is a certain power of it, $\widehat{\phi}^{2\nu(d)}$. As a result, the functional measure for $\widehat{\phi}$ as it descends from the full integral differs from the usual translation-invariant measure $\mathcal{D}\widehat{\phi}$ by a Jacobian that is neglected in the simplest form of the CREH model.[4] Yet, the quantum conformal factor is a *composite* field now,[5] and so its expectation value is not directly related to that of the elementary field [229].

[3] To make contact with the notation used in the literature [80, 203] one should make the following replacements in the above and following equations: $\mathring{g}_{\mu\nu} \to \widehat{g}_{\mu\nu}$, $\widehat{\phi} \to \chi$, $\widehat{\varphi} \to f$, $\varphi \to \bar{f}$, $\bar{\phi} \to \chi_{\text{B}}$, $a \to \varphi$.

[4] For a discussion of the measure see in particular [135, 225, 226] and, in relation to CDT, [47, 227, 228].

[5] If $d = 4$ and $\mathring{g}_{\mu\nu} = \delta_{\mu\nu}$ we may interpret the composite operator representing the metric as the product of two conformally reduced tetrad fields $\widehat{e}^a_\mu = \widehat{\phi}\,\delta^a_\mu$, since in this case $\widehat{g}_{\mu\nu} = \widehat{\phi}^2\,\delta_{\mu\nu} = \widehat{e}^a_\mu \widehat{e}^b_\nu\,\delta_{ab}$.

To make the analogy to full gravity as explicit as possible we compute the $\widehat{\phi}$-integral in the indirect way by means of using a functional RG equation employing the background field technique. More precisely, the field variable integrated over is split linearly,

$$\widehat{\phi} = \bar{\phi} + \widehat{\varphi}, \tag{7.13}$$

with a classical background $\bar{\phi}$ and a dynamical fluctuation field $\widehat{\varphi}$.

The starting point for the derivation of the FRGE is the following generating functional with an infrared cutoff:

$$\begin{aligned}\exp\bigl(W_k[J;\bar{\phi}]\bigr) = \int \mathcal{D}\widehat{\varphi}\, \exp\Bigl(&-S[\bar{\phi}+\widehat{\varphi}] - \Delta S_k[\widehat{\varphi};\bar{\phi}] \\ &+ \int \mathrm{d}^d x\, \sqrt{\mathring{g}}\, J(x)\widehat{\varphi}(x)\Bigr).\end{aligned} \tag{7.14}$$

To mimic the situation in full gravity, the mode suppression term ΔS_k is chosen bilinear in the fluctuation rather than in the full field, and the cutoff operator $\mathcal{R}_k \equiv \mathcal{R}_k[\bar{\phi}]$ is allowed to depend on the background field:

$$\Delta S_k[\widehat{\varphi};\bar{\phi}] = \frac{1}{2} \int \mathrm{d}^d x\, \sqrt{\mathring{g}}\, \widehat{\varphi}(x)\, \mathcal{R}_k[\bar{\phi}]\, \widehat{\varphi}(x). \tag{7.15}$$

By the same series of steps as in Chapter 2, the generating functional (7.14) leads to the corresponding Effective Average Action $\Gamma_k[\varphi;\bar{\phi}]$ and its FRGE. It is similar to that of a matter field theory on a non-dynamical spacetime, albeit a possibly curved one with arbitrary metric $\mathring{g}_{\mu\nu}$:

$$k\partial_k \Gamma_k[\varphi;\bar{\phi}] = \frac{1}{2}\, \mathrm{Tr}\left[\Bigl(\Gamma_k^{(2)}[\varphi;\bar{\phi}] + \mathcal{R}_k[\bar{\phi}]\Bigr)^{-1} k\partial_k \mathcal{R}_k[\bar{\phi}]\right] \tag{7.16}$$

Here, in bra-ket notation, $\mathrm{Tr}(\cdots) \equiv \int \mathrm{d}^d x \sqrt{\mathring{g}} \langle x|(\cdots)|x\rangle$, and the Hessian operator is given by $\langle x|\Gamma_k^{(2)}[\varphi;\bar{\phi}]|y\rangle = \frac{1}{\sqrt{\mathring{g}(x)}\sqrt{\mathring{g}(y)}} \frac{\delta^2 \Gamma_k[\varphi;\bar{\phi}]}{\delta\varphi(x)\delta\varphi(y)}$.

For the field expectation values with respect to the above functional integral we use the notation $\varphi \equiv \langle\widehat{\varphi}\rangle$ as well as $\phi \equiv \langle\widehat{\phi}\rangle = \bar{\phi} + \langle\widehat{\varphi}\rangle \equiv \bar{\phi} + \varphi$. Compared to the full gravity theory the analogous field variables are $g_{\mu\nu} \leftrightarrow \phi$, $\bar{g}_{\mu\nu} \leftrightarrow \bar{\phi}$, $h_{\mu\nu} \leftrightarrow \varphi$. However, there is no counterpart of $\mathring{g}_{\mu\nu}$ in the full-fledged quantum gravity theory.

Also note that *the linear split of* $\widehat{\phi}$ *amounts to a non-linear parametrization of the full dynamical metric in terms of* $\widehat{\varphi}$ if we write the conformal factor as in (7.12) with (7.13).

Besides the reference metric $\mathring{g}_{\mu\nu}$, we must distinguish the following conformally reduced metrics now: the background metric,

$$\bar{g}_{\mu\nu} = \bar{\phi}^{2\nu(d)}\, \mathring{g}_{\mu\nu}, \tag{7.17}$$

the quantum metric, a non-linear function of $\widehat{\phi}$,

$$\widehat{g}_{\mu\nu} = \widehat{\phi}^{2\nu(d)}\, \mathring{g}_{\mu\nu} = (\bar{\phi} + \widehat{\varphi})^{2\nu(d)}\, \mathring{g}_{\mu\nu}, \tag{7.18}$$

the expectation value of the quantum metric,

$$g_{\mu\nu} \equiv \langle \widehat{g}_{\mu\nu} \rangle = \langle \widehat{\phi}^{2\nu(d)} \rangle \, \mathring{g}_{\mu\nu} = \langle (\bar{\phi} + \widehat{\varphi})^{2\nu(d)} \rangle \, \mathring{g}_{\mu\nu}, \tag{7.19}$$

and a novel metric with conformal factor $\phi^{2\nu(d)}$:

$$\breve{g}_{\mu\nu} \equiv \phi^{2\nu(d)} \, \mathring{g}_{\mu\nu} \equiv (\bar{\phi} + \varphi)^{2\nu(d)} \, \mathring{g}_{\mu\nu}. \tag{7.20}$$

In general, $g_{\mu\nu}$ and $\breve{g}_{\mu\nu}$ differ, but they are approximately equal if the quantum fluctuations are negligible.

In $d = 4$ where $\nu = 1$, for instance, we have

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \left[2 \, \bar{\phi} \, \langle \widehat{\varphi} \rangle + \langle \widehat{\varphi}^2 \rangle \right] \mathring{g}_{\mu\nu}, \\ \breve{g}_{\mu\nu} &= \bar{g}_{\mu\nu} + \left[2 \, \bar{\phi} \, \langle \widehat{\varphi} \rangle + \langle \widehat{\varphi} \rangle^2 \right] \mathring{g}_{\mu\nu}. \end{aligned} \tag{7.21}$$

Hence, $g_{\mu\nu} - \breve{g}_{\mu\nu} = \left[\langle \widehat{\varphi}^2 \rangle - \langle \widehat{\varphi} \rangle^2 \right] \mathring{g}_{\mu\nu}$ is proportional to the variance of φ so that $g_{\mu\nu}$ and $\breve{g}_{\mu\nu}$ are not too different if the fluctuations of φ are small. In order to make this statement precise one would have to give a mathematical meaning to the expectation value of the operator product $\widehat{\varphi}^2$ with both operators at the same point, something we will not attempt here; see [229]. But notice that $\breve{g}_{\mu\nu}$ reduces to $\bar{g}_{\mu\nu}$ if $\varphi = 0$. For $g_{\mu\nu}$ this is not necessarily the case since $g_{\mu\nu} = \bar{g}_{\mu\nu} + \langle \widehat{\varphi}^2 \rangle \, \mathring{g}_{\mu\nu}$.

7.3 Coarse Graining Operators: Gravity vs. Matter

The FRGE (7.16) is almost the same as the one governing the RG flow of a scalar matter field on a classical spacetime with metric $\delta_{\mu\nu}$, (2.107). The only minor difference is the replacement of $\delta_{\mu\nu}$ by a possibly more general but likewise non-dynamical metric $\mathring{g}_{\mu\nu}$. So, what can this simple system teach us about quantum gravity?

This brings us to an important issue at the heart of the gravitational Effective Average Action and its interpretation: the interplay of Background Independence and the deeper reason for the existence of a non-Gaussian fixed point.

(1) Recall that the mass scale k parameterizing $\Gamma_k[g, \bar{g}, \cdots]$ equals a special eigenvalue of a certain $\bar{g}_{\mu\nu}$-dependent cutoff operator, the covariant Laplacian typically. Namely, k^2 is the eigenvalue of the "last" eigenmode that is integrated out without suppression.

We insist that the same property holds true in the conformally reduced setting as well. This is a non-trivial decision since, besides $\bar{g}_{\mu\nu} \equiv \bar{\phi}^2 \mathring{g}_{\mu\nu}$, we could also use $\mathring{g}_{\mu\nu}$ itself in order to construct the operator which discriminates the modes. But this option would not have an interpretation at the exact level though.

To find the correct $\mathcal{R}_k$ in (7.15), we think of the integration variable $\widehat{\varphi}$ in (7.14) as expanded in terms of the eigenfunctions of the Laplace–Beltrami operator pertaining to $\bar{g}_{\mu\nu}$:

$$\bar{\Box} \equiv \frac{1}{\sqrt{\bar{g}}} \, \partial_\mu \, \sqrt{\bar{g}} \, \bar{g}^{\mu\nu} \, \partial_\nu. \tag{7.22}$$

As always, $\mathcal{R}_k$ must suppress modes with eigenvalues smaller than k^2 by giving them a mass of the order k, while those with larger eigenvalues are unsuppressed. In the simplest case when the $\widehat{\varphi}$-modes have a kinetic operator proportional to $\bar{\Box}$ itself, the rule is that the correct $\mathcal{R}_k$, when added to $\Gamma_k^{(2)}$, causes the replacement

$$\boxed{(-\bar{\Box}) \longrightarrow (-\bar{\Box}) + k^2 R^{(0)}\big(-\tfrac{\bar{\Box}}{k^2}\big)} \quad \text{(gravity)}, \tag{7.23}$$

with $R^{(0)}(z)$ an arbitrary shape function satisfying (2.65). Hence, the effective inverse propagators of the long- and short-wavelength modes are $-\bar{\Box} + k^2$ and $-\bar{\Box}$, respectively, and the long/short-transition is at the $-\bar{\Box}$-eigenvalue k^2, as it should be.

In this manner the mass scale k acquires a "quasiphysical" status. More precisely, its inverse k^{-1} has the interpretation of a *proper length with respect to the metric* $\bar{g}_{\mu\nu}$. At least for simple geometries, this length is directly related to the "diameter," ℓ, of the spacetime volumes averaged over by the coarse-graining.

(2) The coarse-graining scale $\ell = \ell(k)$ corresponding to a given value of k can be estimated by investigating the properties of the specific $-\bar{\Box}$-eigenfunction that possesses the eigenvalue k^2, the so-called *cutoff mode* [230, 231]: one determines its typical scale of variation with respect to x (a period, say) and converts this coordinate length to a physical, i.e., proper, length by using $\bar{g}_{\mu\nu}$. The result, $\ell(k)$, is an approximate measure for the extension of the spacetime volumes up to which the dynamics has been coarse-grained. If $\bar{g}_{\mu\nu}$ is close to a flat metric, $\ell(k)$ equals approximately π/k. (See [230, 231] for a detailed discussion.)

In this sense the background metric $\bar{g}_{\mu\nu}$, or rather its conformal factor $\bar{\phi}$, determines the physical, i.e., proper scale of k. Furthermore, for self-consistent backgrounds we have $\langle \widehat{g}_{\mu\nu} \rangle = \bar{g}_{\mu\nu}$. So we can even go one step further and say that k is proper, at the level of expectation values, with respect to the dynamical quantum metric.

(3) While the above choice of $\mathcal{R}_k$, based on the rule (7.23), is the correct one in quantum gravity, the standard quantization that treats ϕ as an ordinary scalar matter field uses a different cutoff, namely one based on the Laplace–Beltrami operator pertaining to the reference metric $\mathring{g}_{\mu\nu}$, $\mathring{\Box} \equiv \mathring{g}^{-1/2}\, \partial_\mu\, \mathring{g}^{1/2}\, \mathring{g}^{\mu\nu}\, \partial_\nu$. For a matter field, $\mathcal{R}_k$ is usually designed to implement the replacement:

$$\boxed{(-\mathring{\Box}) \longrightarrow (-\mathring{\Box}) + k^2 R^{(0)}\big(-\tfrac{\mathring{\Box}}{k^2}\big).} \quad \text{(matter)} \tag{7.24}$$

As a consequence the proper scale of k is now determined by the metric $\mathring{g}_{\mu\nu}$, i.e., a metric that has no physical meaning within Background Independent quantum gravity. It "knows" nothing about the true "on-shell" metrics of spacetime, like the self-consistent background metrics, for example, which are determined by the dynamics such that, in $d = 4$, $2\bar{\phi}\langle\widehat{\varphi}\rangle + \langle\widehat{\varphi}^2\rangle = 0$.

The scheme (7.24) is the correct choice if one interprets ϕ as a standard matter field on a non-dynamical spacetime with metric $\mathring{g}_{\mu\nu}$, on flat space ($\mathring{g}_{\mu\nu} = \delta_{\mu\nu}$), for instance. The average action formalism based on (7.24) then reproduces in particular all the familiar results of perturbation theory, the $\ln(k)$-running of the quartic coupling in ϕ^4-theory, for example.

(4) Contrary to $\bar{g}_{\mu\nu}$, the metric $\mathring{g}_{\mu\nu}$ is an absolutely rigid object. The reference metric never gets changed and thus constitutes an "absolute element" in the RG formalism *which destroys its Background Independence.* This is the reason it should not be used for gravity. Only the cutoff that employs $\bar{g}_{\mu\nu}$ respects Background Independence.

As we will see, the RG flow based on the $\bar{\Box}$-scheme (7.23) is extremely different from the one for standard scalars. The reason is that, via the $\bar{\phi}$ dependence of $\bar{\Box}$, *the gravitational field itself sets the scale of* k. The difference between (7.23) and (7.24) becomes manifest when we recall that the Laplacians of $\mathring{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu} = \bar{\phi}^{2\nu(d)}\, \mathring{g}_{\mu\nu}$ are related by

$$\bar{\Box} = \bar{\phi}^{-2\nu(d)}\, \mathring{\Box} + O(\partial\bar{\phi}). \tag{7.25}$$

The factor $\bar{\phi}^{-2\nu(d)}$ leads to considerable modifications of the RG flow within the conformally reduced Einstein–Hilbert truncation, the topic we discuss next.

7.4 The Reduced Einstein–Hilbert Truncation

So far we have applied the conformal reduction to the exact, untruncated theory. In order to be able to perform explicit calculations, we now combine it with a truncation of theory space. As a first example of the single-metric type, we turn to the conformally reduced Einstein–Hilbert truncation (CREH). For simplicity we specialize for $d = 4$ in the sequel.

(1) As an ansatz for $\Gamma_k[\varphi;\bar{\phi}]$ we choose $S_{\mathrm{EH}}[\bar{\phi}+\varphi]$ from (7.7) with a k-dependent Newton constant G_k and cosmological constant Λ_k:

$$\begin{aligned}\Gamma_k[\varphi;\bar{\phi}] = -\frac{3}{4\pi G_k}\int \mathrm{d}^4x\,\sqrt{\mathring{g}}\,\Big\{&-\tfrac{1}{2}\,(\bar{\phi}+\varphi)\,\mathring{\Box}\,(\bar{\phi}+\varphi)\\ &+\tfrac{1}{12}\mathring{R}\,(\bar{\phi}+\varphi)^2 - \tfrac{1}{6}\Lambda_k\,(\bar{\phi}+\varphi)^4\Big\}.\end{aligned} \tag{7.26}$$

From (7.26) we obtain the Hessian operator

$$\Gamma_k^{(2)}[\varphi;\bar{\phi}] = -\frac{3}{4\pi G_k}\Big\{-\mathring{\Box}_x + \tfrac{1}{6}\mathring{R}(x) - 2\Lambda_k\big(\bar{\phi}(x)+\varphi(x)\big)^2\Big\}, \tag{7.27}$$

to which we come back in a moment.

(2) To find the beta functions of G_k and Λ_k we insert (7.26) into (7.16) and perform a derivative and field expansion of the functional trace, retaining only terms proportional to the monomials $\phi\mathring{\Box}\phi$, $\mathring{R}\phi^2$, and ϕ^4, respectively, where $\phi \equiv \bar{\phi} + \varphi$. It is sufficient to perform the projection at a constant background field $\bar{\phi}(x) \equiv \bar{\phi}$.

Since G_k appears both in the kinetic and the $\mathring{R}\phi^2$-term of the potential in the ansatz (7.26), it is possible to compute the anomalous dimensions $\eta_{\rm N} \equiv k\partial_k \ln G_k$ in two different ways, namely either from the kinetic or the potential term. The respective anomalous dimensions, $\eta_{\rm N}^{\rm kin}$ and $\eta_{\rm N}^{\rm pot}$, have no reason to be exactly equal given the approximations we made, but they should at least turn out comparable if the truncation is reliable.

(3) In quantum gravity the cutoff must be imposed on the spectrum of $\bar{\Box}$, not that of $\mathring{\Box}$. Since $\bar{\phi} = \text{const}$ in the case at hand, the two operators are related by

$$\mathring{\Box} = \bar{\phi}^2\,\bar{\Box} \tag{7.28}$$

so that we may re-express $\Gamma_k^{(2)}$ as

$$\Gamma_k^{(2)}[\varphi;\bar{\phi}] = -\frac{3}{4\pi G_k}\Big\{-\bar{\phi}^2\,\bar{\Box} + \tfrac{1}{6}\mathring{R} - 2\Lambda_k(\bar{\phi}+\varphi)^2\Big\}. \tag{7.29}$$

Now we fix $\mathcal{R}_k$ in a way that effects the replacement (7.23):

$$\begin{aligned}\Gamma_k^{(2)}[\varphi;\bar{\phi}] &+ \mathcal{R}_k[\bar{\phi}]\\ &= -\frac{3}{4\pi G_k}\Big\{\bar{\phi}^2\big[-\bar{\Box} + k^2R^{(0)}\big(-\tfrac{\bar{\Box}}{k^2}\big)\big] + \tfrac{1}{6}\mathring{R} - 2\Lambda_k(\bar{\phi}+\varphi)^2\Big\}.\end{aligned} \tag{7.30}$$

As a consequence, the cutoff operator acquires an *explicit dependence on the background field*:

$$\boxed{\mathcal{R}_k[\bar{\phi}] = -\frac{3}{4\pi G_k}\,\bar{\phi}^2\,k^2R^{(0)}\big(-\tfrac{\bar{\Box}}{k^2}\big) = -\frac{3}{4\pi G_k}\,\bar{\phi}^2\,k^2R^{(0)}\big(-\tfrac{\mathring{\Box}}{\bar{\phi}^2k^2}\big).} \tag{7.31}$$

By contrast, installing the cutoff in the spectrum of $\mathring{\Box}$ instead and applying the rule (7.24), we find the cutoff operator for standard matter fields,

$$\boxed{\mathcal{R}_k^{\rm standard} = -\frac{3}{4\pi G_k}\,k^2R^{(0)}\big(-\tfrac{\mathring{\Box}}{k^2}\big).} \tag{7.32}$$

This operator is field independent.

Even though the field present in the operator (7.31) is not the dynamical one, ϕ, but the background, $\bar{\phi}$, the two RG equations based on $\mathcal{R}_k[\bar{\phi}]$ and $\mathcal{R}_k^{\rm standard}$, are quite different. In fact, in projecting out the monomials $\phi\mathring{\Box}\phi$, $\mathring{R}\phi^2$, and ϕ^4 from the FRGE we set $\phi = \bar{\phi}$ after computing the Hessian so that ϕ and $\bar{\phi}$ are indistinguishable from that point on.

Hence, in the quantum gravity setting, there are additional ϕ-dependent terms which affect the beta functions. They originate from the cutoff operator $\mathcal{R}_k$ and are absent in the standard matter field theory.

(4) Note also the overall minus sign displayed by $\Gamma_k^{(2)}$ in (7.27) and, as a consequence, $\mathcal{R}_k$ in (7.31), which is a reflection of the conformal factor instability. However, upon inserting them into the flow equation, the minus signs cancels from $\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\partial_k \mathcal{R}_k$.

This is exactly the point where in the FRG setting the transition from the "original" to the "inverted" picture takes place: the negative kinetic term is traded for an upside-down potential.

(5) The detailed derivation of the beta functions for $g_k \equiv k^{d-2} G_k$ and $\lambda_k \equiv k^{-2}\Lambda_k$, with an arbitrary shape function and for any spacetime dimensionality, can be found in [80]. Here we only quote the results for the optimized shape function (5.120) and $d = 4$.

The coupled RG equations have the same structure as in the full Einstein–Hilbert truncation: $k\partial_k\, g_k = \left[2 + \eta_{\mathrm{N}}(g_k, \lambda_k)\right] g_k$, $k\partial_k\, \lambda_k = \beta_\lambda(g_k, \lambda_k)$. For the anomalous dimension coming from the kinetic term we obtain

$$\eta_{\mathrm{N}}^{(\mathrm{kin})}(g, \lambda) = -\frac{2}{3\pi} \frac{g\lambda^2}{(1 - 2\lambda)^4}. \tag{7.33}$$

The one derived from the potential has the familiar structure

$$\eta_{\mathrm{N}}^{(\mathrm{pot})}(g, \lambda) = \frac{g\, B_1(\lambda)}{1 - g\, B_2(\lambda)}, \tag{7.34}$$

with the following B-functions:

$$\begin{aligned} B_1(\lambda) &= \frac{1}{3\pi}\left(\frac{1}{4} - \lambda\right)\frac{1}{(1 - 2\lambda)^2}, \\ B_2(\lambda) &= -\frac{1}{12\pi}\left(\frac{1}{3} - \lambda\right)\frac{1}{(1 - 2\lambda)^2}. \end{aligned} \tag{7.35}$$

For β_λ one finds

$$\beta_\lambda(g, \lambda) = -\left(2 - \eta_{\mathrm{N}}\right)\lambda + \frac{g}{4\pi}\left(1 - \frac{1}{6}\eta_{\mathrm{N}}\right)\frac{1}{1 - 2\lambda}, \tag{7.36}$$

where either $\eta_{\mathrm{N}}^{(\mathrm{kin})}$ or $\eta_{\mathrm{N}}^{(\mathrm{pot})}$ is to be inserted for η_{N}.

(6) The solutions to the above RG equations deviate considerably from those in a standard scalar matter theory, which employ the operator $\mathcal{R}_k^{\mathrm{standard}}$. Here is a particularly striking example.

Let us consider an RG trajectory in a regime where the anomalous dimension is small so that we may approximate $\eta_{\mathrm{N}} = 0$. Then $G_k = \mathrm{const} \equiv G_0$, and the equation for λ_k involves the correspondingly simplified β-function (7.36) with $g_k \equiv G_0\, k^2$. In terms of the dimensionful cosmological constant $\Lambda_k \equiv k^2 \lambda_k$ this RG equation reads:

$$k\partial_k\, \Lambda_k = \frac{G_0}{4\pi} \frac{k^6}{k^2 - 2\Lambda_k}. \tag{7.37}$$

In particular, when $\Lambda_k \ll k^2$ it simplifies to

$$k\partial_k\, \Lambda_k = \frac{1}{4\pi}\, G_0\, k^4. \tag{7.38}$$

Obviously *the RG equations of the CREH truncation imply a quartic running of the cosmological constant* as long as Λ is small and G is approximately constant.

This behavior is exactly the same as the one found on the basis of the full-fledged Einstein–Hilbert truncation, (5.87).

From the quantum gravity perspective it is unsurprising, being the well known result of summing up the zero-point energies of all field modes.

On the other hand, from the matter field perspective, the quartic running *is* a surprise. In the CREH ansatz for Γ_k the cosmological constant Λ_k plays the role of a ϕ^4-coupling constant which, quite unusually, behaves as $\Lambda_k \propto k^4$ here. This comparatively strong scale dependence has to be contrasted with the much weaker, merely logarithmic k-dependence one finds with $\mathcal{R}_k^{\mathrm{standard}}$ in an ordinary scalar matter theory on a 4-dimensional flat spacetime.

The origin of this difference is clear: *The conformal factor determines the proper scale of the cutoff, while a scalar matter field does not.*

In this manner the distinguished status of the conformal factor comes into play. We emphasized already that if the coarse-graining scale is to be given a proper as well as Background Independent meaning, k must be a cutoff in the spectrum of the *background field dependent* operator $\bar{\Box}$. This enforces the substitution rule (7.23) which differs from the one for standard matter. The rule (7.23) in turn is crucial for the occurrence of a NGFP. This establishes the deep connection between Background Independence and Asymptotic Safety we alluded to above.

(7) Analyzing the RG flow numerically, the main results are as follows.

On the g-λ-plane, the RG equations of the CREH truncation give rise to a phase portrait which, at the qualitative level, is *identical* to the one in Figure 5.1 that was obtained in the full Einstein–Hilbert truncation. This striking similarity between the reduced and the full flow occurs with both variants of the anomalous dimension, $\eta_{\mathrm{N}}^{(\mathrm{kin})}$ and $\eta_{\mathrm{N}}^{(\mathrm{pot})}$, respectively.

It is particularly astonishing that the CREH flow, coming from a simple scalar-looking theory, displays a non-Gaussian fixed point. The product of its coordinate values g_* and λ_* in the "kin" and "pot" schemes are $(g_*\lambda_*)^{(\mathrm{kin})} \approx 1.296$, and $(g_*\lambda_*)^{(\mathrm{pot})} \approx 1.106$, respectively.

The unexpectedly small discrepancy between them is of the same order of magnitude as the uncertainty in the full EH truncation. This suggests that the CREH truncation could have a certain degree of *internal* consistency. On the other hand these values differ substantially from the result in the full EH truncation, which is $(g_*\lambda_*)^{(\text{full EH})} \approx 0.137$ for the same cutoff. But this is no reason for concern, since we are comparing theories with a different number of fields here.

The critical exponents of the NGFP show a stronger variation. They form complex conjugate pairs, $\theta^{(\text{kin})} = 4.0 \pm 6.2\, i$ and $\theta^{(\text{pot})} = 1.47 \pm 9.30\, i$, which should be compared to $\theta^{(\text{full EH})} = 1.48 \pm 3.04\, i$. Both CREH calculations agree on a positive real part. Hence, in conformity with the full EH truncation, the NGFP is UV attractive in both directions.

These findings are consistent with the general expectation that the CREH model could represent a theory sharing certain properties with full quantum gravity. But it should not be regarded as an *approximation* thereof, though.

For further details on the CREH truncation the reader is referred to the literature [80]. Within the finite-dimensional (and single-metric) framework, conformally reduced gravity has been analyzed further in [232] using a more advanced truncation which involves four-derivative terms.

7.5 An Infinite-Dimensional Truncation

Among other developments, conformally reduced gravity also allowed for the first investigation of Asymptotic Safety on a theory space, which, while truncated, is still *infinite dimensional* [203].

The corresponding truncation is similar to the local potential approximation (LPA) which we encountered in Section 2.2.4 for standard scalar matter fields. The ansatz reads:

$$\begin{aligned}\Gamma_k[\varphi;\bar{\phi}] = &+\frac{3}{8\pi G_k}\int \mathrm{d}^4x\,\sqrt{\mathring{g}}\,\left(\bar{\phi}+\varphi\right)\mathring{\Box}\left(\bar{\phi}+\varphi\right)\\ &+\int \mathrm{d}^4x\,\sqrt{\mathring{g}}\,U_k\left(\bar{\phi}+\varphi\right).\end{aligned} \tag{7.39}$$

Here $U_k(\,\cdot\,)$ is an arbitrary, k-dependent potential function for $\bar{\phi}+\varphi \equiv \phi$, generalizing the simple polynomial $U_k^{\text{CREH}}(\phi) = -\frac{3}{48\pi G_k}\big(\mathring{R}\phi^2 - 2\Lambda_k\phi^4\big)$ used above.

With the ansatz (7.39) the projected FRGE consists of a *partial* differential equation coupled to an ordinary one. Together they determine the scale dependence of $U_k(\,\cdot\,)$ and G_k, respectively. So the corresponding truncated theory space is indeed infinite dimensional. We coordinatize it by pairs $\big(g, Y(\,\cdot\,)\big)$ where g denotes the familiar dimensionless Newton constant and $Y(\,\cdot\,)$ is a dimensionless potential depending on a likewise dimensionless field, $k\phi \equiv a$:

$$U_k(\phi) \equiv -\frac{3}{4\pi g_k}\,Y_k(a), \quad a \equiv k\phi. \tag{7.40}$$

Using the same operator $\mathcal{R}_k$ as for the CREH truncation, the flow is governed by the partial differential equation

$$\boxed{\begin{aligned} k\partial_k Y_k(a) = {} & (2+\eta_\mathrm{N})\, Y_k(a) - a\, Y_k'(a) \\ & - \frac{g_k}{2\pi} \frac{\left(1 - \frac{\eta_\mathrm{N}}{2}\right) a^2 \rho(a) + \frac{1}{2}\eta_\mathrm{N}\, \widetilde{\rho}(a)}{a^2 + Y_k''(a)} . \end{aligned}} \tag{7.41}$$

It must be solved together with $k\partial_k g_k = \left[2 + \eta_\mathrm{N}\big(g_k, [Y_k]\big)\right] g_k$.

The $Y(\,\cdot\,)$-dependent anomalous dimension, read off from the kinetic term, is explicitly given by

$$\boxed{\eta_\mathrm{N}\big(g_k, [Y_k]\big) = -\frac{g_k}{24\pi} \frac{\left[a_1^3\, Y_k'''(a_1)\right]^2}{\left[a_1^2 + Y_k''(a_1)\right]^4} .} \tag{7.42}$$

Here a_1 is a normalization point that ultimately will be sent to infinity, $a_1 \to \infty$.

The partial differential equation (7.41) contains the two functions

$$\rho(a) \equiv \mathrm{Tr}\Big[\Theta(a^2 + \mathring{\Box})\Big], \quad \widetilde{\rho}(a) \equiv \mathrm{Tr}\Big[(-\mathring{\Box})\Theta(a^2 + \mathring{\Box})\Big], \tag{7.43}$$

which are sensitive to the spectrum of $\mathring{\Box}$, the Laplace–Beltrami operator pertaining to the reference metric $\mathring{g}_{\mu\nu}$. This brings us to the first general lesson we can learn from this truncation.

(1) Topology dependence. The truncation at hand is the first one we encounter that allows us to see a phenomenon which we expect on general grounds, namely that *the RG flow depends on the topology of the spacetime manifold.*

Considering manifolds of topology S^4 and $\mathbb{R}^4$ as examples, the spectral functions ρ, $\widetilde{\rho}$ have been calculated [203] with $\mathring{g}_{\mu\nu}$, given by the metric of a round unit sphere[6] in the first, and by $\delta_{\mu\nu}$ in the second case.

(i) The spectrum of the Laplacian on the unit sphere S^4 is well known. The eigenvalues $\mathcal{E}_n$ of $(-\mathring{\Box})$ and their degeneracies D_n, $n = 0, 1, 2, \cdots$, for $(-\mathring{\Box})$ acting on scalars are given by [233–235]

$$\mathcal{E}_n = n(n+3), \quad D_n = \tfrac{1}{6}\,(n+1)(n+2)(2n+3). \tag{7.44}$$

To evaluate ρ and $\widetilde{\rho}$ we note that

$$\rho(a) = \sum_{n=0}^{\infty} D_n\, \Theta(a^2 - \mathcal{E}_n), \quad \widetilde{\rho}(a) = \sum_{n=0}^{\infty} \mathcal{E}_n\, D_n\, \Theta(a^2 - \mathcal{E}_n) \tag{7.45}$$

[6] The radius of this sphere is inessential. A change could be absorbed by a redefinition of the field variable.

are functions of the form

$$\rho(a) = J_4\big(N(a)\big), \quad \widetilde{\rho}(a) = \widetilde{J}_4\big(N(a)\big), \tag{7.46}$$

where $N(a)$ is the largest positive integer n such that $\mathcal{E}_n = n(n+3) < a^2$, and

$$J_4(N) \equiv \sum_{n=0}^{N} D_n, \quad \widetilde{J}_4(N) \equiv \sum_{n=0}^{N} \mathcal{E}_n\, D_n. \tag{7.47}$$

The finite sums can be worked out explicitly:

$$\begin{aligned} J_4(N) &= \tfrac{1}{12}\, N^4 + \tfrac{2}{3}\, N^3 + \tfrac{23}{12}\, N^2 + \tfrac{7}{3}\, N + 1, \\ \widetilde{J}_4(N) &= \tfrac{1}{18}\, N^6 + \tfrac{2}{3}\, N^5 + \tfrac{55}{18}\, N^4 + \tfrac{20}{3}\, N^3 + \tfrac{62}{9}\, N^2 + \tfrac{8}{3}\, N. \end{aligned} \tag{7.48}$$

Since the largest contributing eigenvalue has the quantum number

$$N(a) \approx \begin{cases} a & \text{for } a \gg 1, \\ 0 & \text{for } a \ll 1, \end{cases} \tag{7.49}$$

we observe that for very large and small dimensionless fields, respectively,

$$\rho(a) \approx \begin{cases} \frac{1}{12}\, a^4 & \text{for } a \gg 1 \\ 1 & \text{for } a \ll 1 \end{cases}, \quad \widetilde{\rho}(a) \approx \begin{cases} \frac{1}{18}\, a^6 & \text{for } a \gg 1 \\ 0 & \text{for } a \ll 1 \end{cases}. \tag{7.50}$$

On S^4 the spectrum of $\mathring{\Box}$ is completely discrete. As a consequence, the exact formulas for $\rho(a)$ and $\widetilde{\rho}(a)$ display discontinuities at the a-values at which $N(a)$ jumps. To analyze the RG equation smooth functions $\rho(a)$ and $\widetilde{\rho}(a)$ clearly would be easier to deal with. For this reason it was proposed to adopt a certain *smoothing procedure* that replaces the original functions ρ and $\widetilde{\rho}$ by smooth interpolating functions [203].

The way of performing this interpolation, or smoothing, is in no way unique. Choosing a specific smoothing procedure has the same conceptual status as choosing a particular cutoff function $R^{(0)}$: It amounts to specifying how precisely the transition from the high- to the low-momentum regime takes place, i.e., how the modes are suppressed when their eigenvalue is below the threshold value given by k. Observable quantities derived from the RG flow must be independent of both $R^{(0)}$ and the smoothing procedure.

Denoting the smoothened functions by ρ and $\widetilde{\rho}$ again, the numerical analysis has shown that for most purposes the following approximation suggested by (7.50) is perfectly sufficient:

$$\rho(a) = 1 + \tfrac{1}{12}\, a^4, \quad \widetilde{\rho}(a) = \tfrac{1}{18}\, a^6. \tag{7.51}$$

(ii) Turning to the case of $\mathbb{R}^4$, with $\mathring{g}_{\mu\nu} = \delta_{\mu\nu}$, the spectrum of $\mathring{\Box}$ is continuous, and the functions ρ and $\widetilde{\rho}$ are easily evaluated in the plane wave eigenbasis. One finds that, for all values of a, $\rho(a) \propto a^4$ and $\widetilde{\rho}(a) \propto a^6$. Here we omitted a constant factor proportional to the infinite volume of $\mathbb{R}^4$. Choosing a normalization convention consistent with the one for S^4, the partial differential equation on $\mathbb{R}^4$ becomes

$$\boxed{\begin{aligned} k\partial_k Y_k(a) = (2+\eta_{\mathrm{N}})\, Y_k(a) - a\, Y_k'(a) \\ - \frac{g_k}{24\pi}\left(1 - \tfrac{1}{6}\,\eta_{\mathrm{N}}\right) \frac{a^6}{a^2 + Y_k''(a)}. \end{aligned}} \tag{7.52}$$

This equation coincides with the corresponding S^4 result if in the latter the $a \gg 1$-approximations from (7.50) are used for *all* values of a. Indeed, for $a \gg 1$ the spectral sums of S^4 are dominated by many densely spaced eigenvalues forming a quasicontinuum.

The difference between $\mathbb{R}^4$ and S^4 is most pronounced for small values of a. There the finite volume of S^4 and the resulting discreteness of the spectrum strongly modify the spectral density ρ. On flat space it is proportional to a^4, while on the sphere it approaches $\rho(0) = 1$ for $a \ll 1$.

For $a \ll 1$ the functional trace in the FRGE of S^4 is dominated by a single eigenvalue, namely the zero mode of $\mathring{\Box}$. For the $\mathbb{R}^4$ topology, on the other hand, the $\rho \sim a^4$-behavior extends down to $a = 0$.

(2) Fixed functions. The RG flow on the infinite-dimensional theory space possesses both a Gaussian and a non-Gaussian fixed point $\big(g_*, Y_*(\,\cdot\,)\big)$. The "fixed function" $Y_*(\,\cdot\,)$ is determined by the *ordinary* differential equation one obtains by setting $\partial_k Y_k \equiv 0$.

The GFP is characterized by $g_* = 0$ together with the fixed point potential $Y_*(a) = c\, a^2$ with $c = 0$ for $\mathbb{R}^4$ and $c = 1$ for S^4.

For the $\mathbb{R}^4$ topology, the result for the NGFP turns out to be exactly the same as we found earlier in the CREH approximation, namely $g_* = g_*^{\mathrm{CREH}} \approx 4.650$ and $Y_*(a) = y_* - \frac{1}{6}\,\lambda_*\, a^4$, where $\lambda_* = \lambda_*^{\mathrm{CREH}} \approx 0.279$. The constant y_* is not determined by the flow equation.

Except for this constant, the infinite-dimensional RG equations have not taken advantage of the possibility to generalize the fixed point potential beyond the functional form of the CREH approximation. This robustness under extensions of the truncation is quite remarkable.

For the S^4 topology the situation is different, and the properties of the NGFP show significant deviations from the earlier CREH analysis. In this case the fixed point potential can only be determined numerically. In Figure 7.1 we display the result obtained with the smooth spectral functions (7.51).

(3) Infinite-dimensional stability analysis. In order to determine the fixed point's critical exponents and scaling fields, we insert

$$\begin{aligned} g_k &= g_* + \varepsilon\, y_g\, e^{-\theta t}, \\ Y_k(a) &= Y_*(a) + \varepsilon\, \Upsilon(a)\, e^{-\theta t} \end{aligned} \tag{7.53}$$

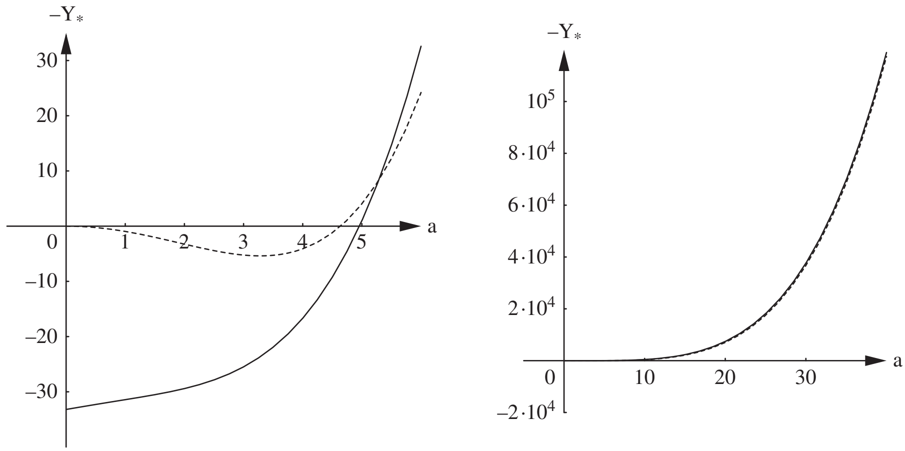


Figure 7.1. The negative fixed point potential $-Y_*(a)$ for S^4. The dashed line is the CREH potential with the same value for λ_*. The figure on the left-hand side magnifies the small-a region of the one on the right-hand side; the latter shows that Y_* coincides with the CREH potential asymptotically. (Taken from [203].)

into the RG equations and expand them to first order in ε. The resulting condition on the allowed values of θ and the pairs $\big(y_g, \Upsilon(\,\cdot\,)\big)$, which generalize the eigenvectors of the stability matrix, involves a linear ordinary differential equation. (See Section 2.2.4 for a similar discussion.)

Focusing on the NGFP and the $\mathbb{R}^4$ topology from now on, the main results are as follows.

There exist only two scaling fields, $\big(y_g \neq 0, \Upsilon(\,\cdot\,)\big)$, with a non-vanishing g component. Their only other non-vanishing component points in the a^4-direction of the function space $\{Y(\,\cdot\,)\}$, which we can identify with the λ-direction of the CREH truncation. Those two scaling fields are relevant, have complex critical exponents with $\theta' = 4$, $\theta'' \approx 6.18$, and coincide *exactly* with those of the CREH approximation.

All other scaling fields, $\big(0, \Upsilon(\,\cdot\,)\big)$, have vanishing g-component and correspond to perturbations of the potential alone. There are both relevant and irrelevant fields of this type. Generically they correspond to non-polynomial oscillatory functions like

$$Y_k(a) - Y_*(a) \propto a^{n'}\, e^{-\theta' t}\, \cos(n'' \ln a - \theta'' t + \delta)\,, \tag{7.54}$$

which arise as the real and imaginary parts of complex powers, $\Upsilon(a) \propto a^n$, where $n \equiv n' + in''$ and, as usual, $\theta \equiv \theta' + i\theta''$. The eigenvalue condition provides the relationship $\theta \equiv \theta(n)$, and moreover it restricts the exponent to be in a certain domain of the complex plane, $n \in \mathcal{D} \subset \mathbb{C}$.[7]

[7] A graphical representation of $\mathcal{D}$ can be found in [203].

It is unclear a priori whether one should admit all $n \in \mathcal{D}$, or only the subset of, say, all real, or all integer exponents. *Depending on which scaling fields (eigenfunctions) are considered admissible, the number of relevant and irrelevant directions changes.* This affects in particular the dimensionality of the UV-critical manifold, $\mathscr{S}_{\mathrm{UV}}$.

If we were to admit all $n \in \mathcal{D}$, we would find infinitely many relevant scaling fields, i.e., $\mathscr{S}_{\mathrm{UV}}$ is infinite dimensional. On the other hand, $\dim(\mathscr{S}_{\mathrm{UV}})$ is finite if n is restricted to the integers in $\mathcal{D}$.

(4) Precise specification of theory space. These findings bring us to another general lesson which is nicely illustrated by the model system at hand.

Namely, at a certain point of the Asymptotic Safety program we must decide about the *mathematically precise specification of theory space, $\mathcal{T}$*. Only then can we distinguish "acceptable" and "unacceptable" action functionals, depending on whether or not they are elements of $\mathcal{T}$.

Finding the optimum theory space is an involved problem: $\mathcal{T}$ should be small enough to eliminate all actions with physically harmful singular behavior, but at the same time large enough to contain as many asymptotically safe and complete RG trajectories as possible. Finding the optimum space will require a certain understanding of the observables in the theory and their dependence on the running couplings (which we are lacking for the time being).

At this point the mathematical setting of the linear differential equation for the scaling fields will also become fully concrete so that one can meaningfully talk about the number of relevant eigendirections. Loosely speaking, the scaling fields live in the tangent space to $\mathcal{T}$, a notion that must be made precise in this step.

Let us consider the following general situation exemplified by the present truncation. Assume that we start out from some preliminary "trial" theory space $\mathcal{T}_0$ and, in the ideal case, are able to compute all RG trajectories on it. We then discover that there exists a subspace $\mathcal{T}' \subset \mathcal{T}_0$ which is left invariant by the flow, i.e., if a trajectory begins on $\mathcal{T}'$ it will stay on it for ever. If $\mathcal{T}'$ furthermore contains the complete and asymptotically safe trajectories we are after, it is then possible to declare $\mathcal{T}'$ the actual theory space.

Narrowing down the set of eligible actions from $\mathcal{T}_0$ to $\mathcal{T}'$ can, but not necessarily must, lead to enhanced predictive power of the theory constructed. The decisive question is whether there is a *conclusive physical reason for excluding the actions in $\mathcal{T}_0 \setminus \mathcal{T}'$*.

In our example, the space of all potentials, $\{Y(\,\cdot\,)\}$, plays the role of $\mathcal{T}_0$. It contains functions that are indeed never generated by the flow if they are absent initially. They include potentials that grow faster than $Y(a) \propto a^4$ for $a \to \infty$, or examples like $Y(a) \propto a^{-p}$, $p = 1, 2, 3, \cdots$, which are singular at $a = 0$.

In [203] various possible restrictions of the function spaces $\{Y(\,\cdot\,)\}$ and $\{\Upsilon(\,\cdot\,)\}$ at the linearized level were considered as examples of $\mathcal{T}'$. They lead to the differing results for $\dim(\mathscr{S}_{\rm UV})$ which we mentioned in **(3)** above. We will not go into the details here. Suffice it to say that no compelling physical arguments preferring one definition of theory space over the other are known for the time being.

(5) Phases of unbroken diffeomorphism invariance. If one numerically solves the non-linear RG equations one can distinguish two essentially different classes of potentials $U_k(\phi)$ along the RG trajectories, namely potentials having their minimum at $\phi \neq 0$ or $\phi = 0$, respectively.

Recalling that $g_{\mu\nu} \equiv \langle \widehat{g}_{\mu\nu} \rangle = \phi^2 \, \delta_{\mu\nu}$, it is tempting to interpret ϕ as an "order parameter" whose vanishing (non-vanishing) expectation value indicates a phase of gravity with unbroken (broken) diffeomorphism invariance. If one lowers k, "phase transitions" from one regime to the other are observed. They can be of both first- and second-order type [203].

7.6 Bi-Metric Truncations and the Split Ward Identity

The truncated actions for conformally reduced gravity which we considered in the previous sections all depend on φ and $\bar{\phi}$ via the combination $\varphi + \bar{\phi} \equiv \phi$ only. In the $(g, \bar{g})$-, or here, $(\phi, \bar{\phi})$-language, the functional $\Gamma_k[\phi, \bar{\phi}] \equiv \Gamma_k[\varphi; \bar{\phi}]\big|_{\varphi = \phi - \bar{\phi}}$ is completely independent of $\bar{\phi}$. It thus amounts to the conformally reduced variant of a single-metric truncation.

As a first orientation beyond the single-metric level the following "bi-field" ansatz was considered in [90]:[8]

$$\Gamma_k[\phi, \bar{\phi}] = + \frac{3}{8\pi} \int \mathrm{d}^4 x \, \sqrt{\mathring{g}} \, \left\{ \frac{1}{G_k^{\rm Dyn}} \, \phi \, \mathring{\Box} \, \phi + \frac{1}{G_k^{\rm B}} \, \bar{\phi} \, \mathring{\Box} \, \bar{\phi} \right\} + \int \mathrm{d}^4 x \, \sqrt{\mathring{g}} \, U_k(\phi, \bar{\phi}). \tag{7.55}$$

This action leaves room for an "extra" background field dependence by an independent kinetic term for $\bar{\phi}$ and its appearance in the potential. It covers (the conformally reduced version of) the bi-metric Einstein–Hilbert truncation (6.9) in which case

$$U_k(\phi, \bar{\phi}) = \frac{\Lambda_k^{\rm Dyn}}{8\pi G_k^{\rm Dyn}} \phi^4 + \frac{\Lambda_k^{\rm B}}{8\pi G_k^{\rm B}} \bar{\phi}^4. \tag{7.56}$$

(We specialize for the $\mathbb{R}^4$ topology in this section.)

The FRGE projected on the ansatz (7.55) yields two independent anomalous dimensions $\eta_{\rm N}^{\rm Dyn}$, $\eta_{\rm N}^{\rm B}$ related to $g_k^{\rm Dyn} \equiv k^2 G_k^{\rm Dyn}$ and $g_k^{\rm B} \equiv k^2 G_k^{\rm B}$, respectively, as well as a partial differential equation for the potential $Y_k(a, b) \equiv -\frac{4\pi}{3} \, g_k \, U_k(\phi, \bar{\phi})$.

[8] In [75, 236, 237] an earlier, similar calculation with two independent conformal factors in a perturbatively renormalizable gravity model can be found.

Here $a \equiv k\phi$ and $b \equiv k\bar{\phi}$ denote the dimensionless field variables. The partial differential equation reads explicitly:

$$\big(k\partial_k + a\partial_a + b\partial_b - \eta_{\mathrm{N}}^{\mathrm{Dyn}} - 2\big)Y_k(a,b) = -\frac{g_k^{\mathrm{Dyn}}}{24\pi}\Big(1 - \frac{\eta_{\mathrm{N}}^{\mathrm{Dyn}}}{6}\Big)\frac{b^6}{b^2 + \partial_a^2 Y_k(a,b)}. \tag{7.57}$$

The solutions of the coupled equations have been studied both within a finite-dimensional "subtruncation" of U_k and on the infinite-dimensional $\big(g^{\mathrm{Dyn}}, g^{\mathrm{B}}, Y(\,\cdot\,,\,\cdot\,)\big)$-theory space. Thereby the essential qualitative results of the earlier analysis could be confirmed. A detailed discussion can be found in [90].

More interestingly, within the same setting also a first analysis of the *Ward identity for split symmetry* (4.96), has been possible [90]. In conformally reduced gravity it assumes the form:

$$\boxed{\frac{\delta}{\delta\bar{\phi}(x)}\Gamma_k[\phi,\bar{\phi}] = \frac{1}{2}\,\mathrm{Tr}\left[\Big(\Gamma_k^{(2)}[\phi,\bar{\phi}] + \mathcal{R}_k[\bar{\phi}]\Big)^{-1}\frac{\delta}{\delta\bar{\phi}(x)}\mathcal{R}_k[\bar{\phi}]\right].} \tag{7.58}$$

For the truncation (7.55), at the dimensionless level, this Ward identity boils down to:

$$b\frac{\partial}{\partial b}\,Y_k(a,b) = -\frac{g_k^{\mathrm{Dyn}}}{24\pi}\frac{b^6}{b^2 + \partial_a^2 Y_k(a,b)}. \tag{7.59}$$

While its structure is similar to the RG equation (7.57), the Ward identity is seen to be an "instantaneous" relation, i.e., it involves no scale derivative.

If it was not for the truncation, the Ward identity would hold true at any point of an RG trajectory if it holds at one of its points. Hence, the mutual consistency of the truncated Ward identity and the flow equation, both considered at the same level of accuracy, can shed light on the reliability of the truncation.

As the RG trajectories are approximate common solutions of both the flow equation and the Ward identity, the same should be true for any combination thereof. It is therefore tempting to combine (7.57) and (7.59) in such a way that the complicated term involving $\partial_a^2 Y_k(a,b)$ in the denominator disappears [90]:

$$\left[k\partial_k - \Big(2 + \eta_{\mathrm{N}}^{\mathrm{Dyn}}\Big) + a\partial_a + \frac{1}{6}\eta_{\mathrm{N}}^{\mathrm{Dyn}}\,b\partial_b\right]Y_k(a,b) = 0. \tag{7.60}$$

It is particularly instructive to consider (7.60) at the fixed points, $\partial_k Y_k = 0$. At the GFP we have $g_*^{\mathrm{Dyn}} = 0$ and $\eta_{\mathrm{N}}^{\mathrm{Dyn}} = 0$. Hence, (7.59) tells us that $b\partial_b\,Y_*(a,b) = 0$ and (7.60) yields $[a\partial_a - 2]\,Y_*(a,b) = 0$. Taken together these relations imply that

$$Y_*^{\mathrm{GFP}}(a,b) = c\,a^2, \tag{7.61}$$

where c is a constant. Note that this fixed point potential depends only on the dynamical field a, not on the background.

At a NGFP, instead, one has $\eta_{\mathrm{N}}^{\mathrm{Dyn}}=-2$, and (7.60) boils down to

$$\left[a\partial_a - \frac{1}{3}b\partial_b\right] Y_*^{\mathrm{NGFP}}(a,b)=0. \tag{7.62}$$

This equation when taken at face value would imply that Y_*^{NGFP} is a function of a single variable, $b^3 a$. However, there exists no *exact* fixed point solution to the flow equation (7.57) of the form $Y_*^{\mathrm{NGFP}}(a,b)=f(b^3 a)$.

These examples illustrate a general difficulty: As the flow equation and the Ward identity are the result of approximations, we should be prepared to find spurious contradictions or inconsistencies if we commit the mistake and consider them true *exactly*. Clearly it is somewhat subtle to meaningfully talk about "approximate consistency," and so one must resort to a detailed case-by-case analysis.

The qualitative conclusion we can draw from (7.62) is that, at a NGFP, the fixed point potential Y_*^{NGFP} is likely to involve both the dynamical field a and the background b, with a clear trend for the background field to be predominant. In fact, if we change the normalization of $\mathcal{R}_k$ and replace $1/G_k^{\mathrm{Dyn}}$ in its prefactor by a constant, the $\eta_{\mathrm{N}}^{\mathrm{Dyn}}/6$ term disappears from the flow equation and from (7.60). As a consequence, (7.62) gets replaced by $a\partial_a\, Y_*(a,b)=0$, implying that Y_*^{NGFP} is a function of b alone!

As we emphasized above, we are dealing with approximate solutions to approximate equations so that this result cannot be taken at face value. Yet, it highlights the importance of the background field for the structure of the fixed point action.[9]

Further quantitative tests of the compatibility between the two equations compared the values of the fixed point coordinates as predicted by (7.57) and (7.59), respectively, within a finite-dimensional subtruncation. In this context, indications of a non-trivial "conspiracy" between the two equations have indeed been observed.

For further details concerning the CREH model we must refer to the literature [90]. Related work with a somewhat different scope includes a comprehensive conceptual discussion and explicit analysis of the RG flow in Weyl-invariant theories [239–241] and an investigation of the background-scale Ward identity [201].

[9] Further investigations along these lines can be found in [238].

8

The Reconstruction Problem

This chapter is devoted to the so-called reconstruction problem, i.e., the problem of finding a UV regularized functional integral, which, first, has a mathematically well-defined continuum limit and, second, reproduces a given RG trajectory, a complete solution to the UV-cutoff-free FRGE.

8.1 In Search of a Bare Theory

In order to put the reconstruction problem into a proper perspective, let us recall that we started out from an ill-defined functional integral that is plagued by short-distance singularities. We equipped this integral with a UV regulator, then associated a FRGE to the regularized integral, and finally observed that *in the flow equation* the regulator is actually unnecessary and can be removed almost trivially.

(1) After this last step, our efforts toward a quantum theory of gravity completely decoupled from the initial, and rather formal, functional integral. While we did employ the integral as inspiration for a promising theory space and for the structure of the FRGE, our actual calculations were all based on the *UV-cutoff-free flow equation.* We accepted this equation as the starting point of all mathematically rigorous developments, because it replaces the problem of giving a meaning to the functional integral by a new one which, we believe, is easier to handle, namely finding fully extended RG trajectories $\{\Gamma_k\}_{k\in[0,\infty)}$.

The consequence is that, even if we have a complete RG trajectory at our disposal, we are unable to specify a regularized functional integral without additional input. The information that is missing consists of a prescription for regularizing the measure along with the bare action, $\mathcal{D}\widehat{g} \to \mathcal{D}_\Lambda\widehat{g}, S \to S_\Lambda$, and a rule for sending $\Lambda \to \infty$ in

$$\int \mathcal{D}_\Lambda \widehat{g}\; e^{-S_\Lambda[\widehat{g}]}, \tag{8.1}$$

which guarantees a meaningful limit. The integral (8.1) depends on a number of Λ-dependent "bare" parameters, and the rule would include information about how one must finetune the Λ-dependence in order to achieve this limit.

(2) At this point the decision to base our search for consistent theories within the Asymptotic Safety program on a variant of the *effective action*, namely the EAA, rather than a *running bare action* becomes significant.

The latter option would seem natural in the Wilson–Kadanoff renormalization group describing a discrete sequence of successive coarse-graining steps consisting of a block-spin transformation, for instance [7–9]. For continuum-based gravity and gauge theories we rejected this route as it entails profound difficulties related to gauge invariance.

Note that a true Wilsonian action, S_Λ^{W}, has the status of a *bare* action, i.e., it is meant to be used under a regularized path integral. It depends on the UV cutoff, and the dependence on Λ is governed by an RG equation which differs from the one for the EAA both conceptually and structurally.

There is indeed a considerable difference between the renormalization groups behind the EAA and the Wilsonian approach:

In a sense, S_Λ^{W} for different values of Λ is a set of actions *for the same system*: the Green functions have to be computed from S_Λ^{W} by a further functional integration over the low momentum modes, and this integration renders them independent of Λ. By contrast, the EAA can be thought of as the standard effective action *for a family of different systems*, namely those with the bare actions $S_\Lambda + \Delta S_k$ labeled by $k \in [0, \Lambda)$. The corresponding n-point functions, computed simply as functional derivatives of Γ_k without any further integration, are scale dependent and constitute an effective field theory description valid near the scale k [31].

Also these general differences between the EAA and a genuine Wilson action make it quite clear that our approach cannot by itself imply a regularized path integral.

(3) At first sight one might think that the functional integral is dispensable and that a complete RG trajectory Γ_k suffices to extract all properties of interest from the corresponding quantum field theory; after all, its n-point functions are all provided by the derivatives of Γ_k at $k = 0$.

Nevertheless, there are a number of reasons why the presentation of the asymptotically safe theory as a bona fide functional integral is desirable.

(a) Working with the EAA alone we have no information about the microscopic (or "classical") system whose standard quantization would give rise to this particular effective action. The functional integral representation of an asymptotically safe theory will allow the reconstruction of the *microscopic degrees of freedom* that we implicitly integrated out in solving the FRGE as well as their fundamental interactions.

Based on the action resulting from the path integral we can reconstruct the Hamiltonian description by a kind of generalized Legendre transformation. From this phase space formulation we can in turn read off a classical dynamical system whose quantization (also by other methods, e.g., canonically [242]) leads to the given effective action. We expect that this classical system is rather complicated so that it may not be guessed easily.

(b) Many physically interesting observables involve *composite operators* which do not appear in Γ_k, either because they are not even elements of the exact theory space, or they are absent as the consequence of a truncation. Typical examples include geometrically motivated length or volume operators [229]. Their expectation values and correlation functions cannot be inferred from the usual EAA. However, a functional integral representation would allow us to extend the scope of the EAA and its renormalization group beyond the correlators of the *elementary* quantum fields. (See [229, 243] for a detailed discussion.)

(c) Ultimately we would like to understand *the relation of QEG with other approaches* to quantum gravity, such as canonical quantization [242], Loop Quantum Gravity [36–39], or Monte Carlo simulations of statistical systems [244], for example. In these approaches the bare action plays a central role.

In the Monte Carlo simulations of the Regge or dynamical triangulations formulation, for instance, the starting point is a regularized path integral involving some discretized version of S_Λ. In order to take the continuum limit one must finetune the bare parameters in S_Λ in a suitable way. If one is interested in the asymptotic scaling, for instance, and wants to compare the analytic QEG predictions to the way the continuum is approached in the simulations, one must convert the Γ_k-trajectory to a S_Λ-trajectory first.

The map from Γ_k to S_Λ depends explicitly on how the path integral is discretized; so *each alternative formulation of QEG associates its own regularized measure and bare action S_Λ to one and the same Γ_k-trajectory.*

(4) It should be clear from these remarks that the reconstruction step of the Asymptotic Safety program must begin with a decision about, first, the *nature of the microscopic variables* integrated or summed over, and second, a *UV regularization* of the corresponding functional integral or partition function.

The sought-for map relating effective and bare couplings will strongly depend on these two choices.

In the sequel of this chapter we illustrate these ideas using the specific example in which the microscopic variables are chosen to be continuum fields $\widehat{g}_{\mu\nu}(x)$ and the UV regularization is realized by a sharp mode cutoff.

We show how to find a corresponding regularized functional integral which reproduces a given asymptotically safe RG trajectory $\{\Gamma_k,\, 0 \leq k < \infty\}$, a solution of the FRGE without UV regulator. In particular, we determine how the bare couplings parameterizing S_Λ must behave for $\Lambda \to \infty$ in order to yield a continuum limit in accordance with the fixed point of the EAA.

Our presentation follows the work in [205, 206].

8.2 The EAA in the Presence of a UV Cutoff

As a preparation we briefly revisit the derivation of the EAA and its flow equation from the functional integral, paying special attention to the regularization in the UV now. In order to make the integral mathematically well defined we install a UV regulator at a cutoff scale Λ and then define an Effective Average Action on the basis of the regularized integral. To indicate the presence of the UV cutoff we denote it $\Gamma_{k,\Lambda}$.

Many regularization schemes are conceivable here. For concreteness we use a "finite-mode regularization," which will also be ideally suited for implementing Background Independence in QEG.

To avoid inessential complications we first consider a simple scalar field on flat space as we did in Chapter 2.

(1) So, let $\widehat{\phi}(x)$ be a real scalar matter field on a flat d-dimensional Euclidean spacetime. In order to discretize momentum space we compactify it to a d-torus. The eigenfunctions of the Laplacian $\hat{p}^2 \equiv -\delta^{\mu\nu}\partial_\mu\partial_\nu$ are plane waves $u_p(x) \propto \exp(ip\cdot x)$ with discrete momenta p_μ and eigenvalues p^2. For a given cutoff scale Λ there are only finitely many eigenfunctions with $|p| \equiv \sqrt{p^2} \leq \Lambda$. We regularize the path integral in the UV by restricting the integration to those modes. Therefore, the field $\widehat{\phi}$ and its source J have expansions:

$$\widehat{\phi}(x) = \sum_{|p|\in[0,\Lambda]} \alpha_p\, u_p(x) \quad \text{and} \quad J(x) = \sum_{|p|\in[0,\Lambda]} J_p\, u_p(x). \tag{8.2}$$

We define a UV-regularized generating functional:

$$\exp\big(W_{k,\Lambda}[J]\big) \equiv \int \mathcal{D}_\Lambda\widehat{\phi}\, \exp\left(-S_\Lambda[\widehat{\phi}] - \Delta S_k[\widehat{\phi}] + \int \mathrm{d}^d x\, J(x)\widehat{\phi}(x)\right). \tag{8.3}$$

The notation in (8.3) is symbolic. Its right-hand side involves only finitely many integrations and thus is not a genuine functional integral. The measure $\mathcal{D}_\Lambda\widehat{\phi}$ stands for an integration over the Fourier coefficients α_p with p^2 below Λ^2:

$$\int \mathcal{D}_\Lambda\widehat{\phi} = \prod_{|p|\in[0,\Lambda]} \int_{-\infty}^{\infty} \mathrm{d}\alpha_p\, M^{-[\alpha_p]}. \tag{8.4}$$

The arbitrary mass parameter M was introduced in order to give the canonical dimension zero to (8.4). Furthermore, as always in the EAA construction, the IR modes with $|p| < k$ are suppressed in $W_{k,\Lambda}$ by means of using the cutoff action $\Delta S_k[\widehat{\phi}] = \frac{1}{2}\int \mathrm{d}^d x\, \widehat{\phi}(x)\mathcal{R}_k(\hat{p}^2)\widehat{\phi}(x)$.

In (8.3) the bare action S_Λ is allowed to depend on the UV cutoff. Ultimately we would like to fix this Λ-dependence in such a way that, for every finite k and J, the path integral has a well-defined limit for $\Lambda \to \infty$.

We introduce the UV regular EAA as

$$\Gamma_{k,\Lambda}[\phi] \equiv \widetilde{\Gamma}_{k,\Lambda}[\phi] - \frac{1}{2}\int \mathrm{d}^d x\, \phi(x) \mathcal{R}_k(\hat{p}^2)\phi(x), \tag{8.5}$$

where $\widetilde{\Gamma}_{k,\Lambda}[\phi]$ is the Legendre transform of $W_{k,\Lambda}[J]$ with respect to the finitely many variables of J and $\phi = \{\phi_p\}_{|p|\in[0,\Lambda]}$ is the expectation value $\phi \equiv \langle \widehat{\phi} \rangle$ obtained by differentiating $W_{k,\Lambda}[J]$ with respect to the source J, cf. Section 2.1.

(2) Repeating the steps described in Section 2.1 one finds that $\Gamma_{k,\Lambda}$ satisfies the following exact FRGE:

$$k\partial_k \Gamma_{k,\Lambda}[\phi] = \frac{1}{2}\,\mathrm{Tr}_\Lambda \left[\left(\Gamma^{(2)}_{k,\Lambda}[\phi] + \mathcal{R}_k \right)^{-1} k\partial_k \mathcal{R}_k \right]. \tag{8.6}$$

Here, Tr_Λ denotes the trace restricted to the subspace spanned by the eigenfunctions of $\hat{p}^2$ with eigenvalues smaller than Λ^2:

$$\mathrm{Tr}_\Lambda[\cdots] = \mathrm{Tr}\left[\Theta(\Lambda^2 - \hat{p}^2)[\cdots]\right]. \tag{8.7}$$

The UV-regularized EAA satisfies the integro-differential equation

$$\begin{aligned} \exp\left(-\Gamma_{k,\Lambda}[\phi]\right) = \int \mathcal{D}_\Lambda f \exp\Big(&- S_\Lambda[\phi + f] + \int \mathrm{d}^d x\, f(x) \frac{\delta \Gamma_{k,\Lambda}[\phi]}{\delta \phi(x)} \\ &- \frac{1}{2}\int \mathrm{d}^d x\, f(x) \mathcal{R}_k(\hat{p}^2) f(x) \Big), \end{aligned} \tag{8.8}$$

with the measure (8.4) and the fluctuation field $f(x) \equiv \widehat{\phi}(x) - \phi(x)$. Equation (8.8) will be instrumental in the next section.

(3) The question arises about whether the UV cutoff can be removed from the FRGE. Indeed, for this to be possible, it is sufficient to assume that the IR-cutoff operator is chosen such that $k\partial_k \mathcal{R}_k(p^2)$ decreases rapidly for large p^2 so that the trace in the flow equation (8.6) does not receive contributions from eigenvalues $p^2 \gg k^2$ and hence continues to exist in the limit $\Lambda \to \infty$. As a result, the "Λ-free" FRGE without UV cutoff, valid for all $k \geq 0$, takes the familiar form:

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma^{(2)}_k[\phi] + \mathcal{R}_k(\hat{p}^2) \right)^{-1} k\partial_k \mathcal{R}_k(\hat{p}^2) \right]. \tag{8.9}$$

We denote the solutions of (8.9) as $\{\Gamma_k,\, 0 \leq k < \infty\}$ and those of the FRGE (8.6) with UV cutoff as $\{\Gamma_{k,\Lambda},\, 0 \leq k < \Lambda\}$.

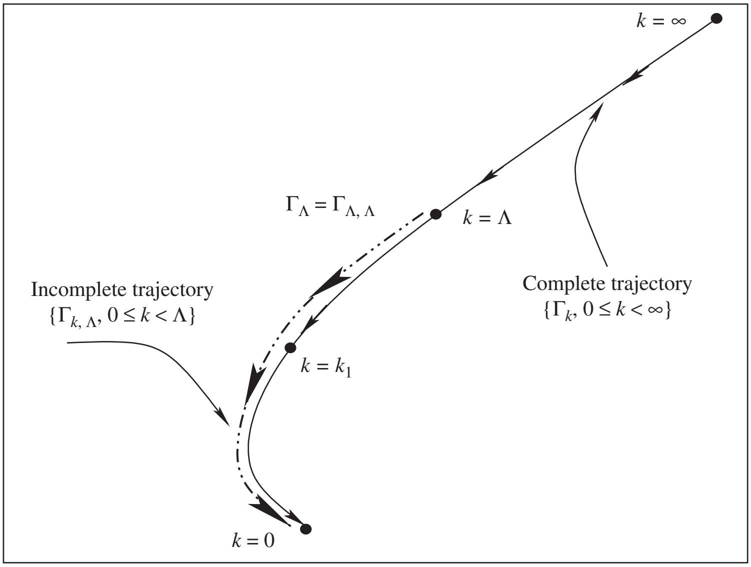


Figure 8.1. With the optimized cutoff, every complete solution to the Λ-free FRGE gives rise to a solution of the FRGE with UV cutoff, valid up to any value of Λ.

(4) Remarkably, the flow equations for Γ_k and $\Gamma_{k,\Lambda}$ are essentially identical as long as $k \ll \Lambda$ [205]. Generically, when k approaches Λ from below, there exist some small corrections due to the UV cutoff which affect $\Gamma_{k,\Lambda}$ and cause it to differ from Γ_k. However, it is always possible to choose a special IR cutoff $\mathcal{R}_k(p^2)$ such that those corrections vanish, an example being the optimized cutoff $\mathcal{R}_k(p^2) = (k^2 - p^2)\Theta(k^2 - p^2)$.

As a result, the functional $\Gamma_{k,\Lambda}$ satisfies the same FRGE as Γ_k but is defined in the interval $k \le \Lambda$ only. For identical initial conditions, the relationship between the respective solutions of the two flow equations is thus quite simple:

$$\boxed{\Gamma_{k,\Lambda} = \Gamma_k \quad \text{when} \quad 0 \le k \le \Lambda.} \tag{8.10}$$

For a fixed, but arbitrary, finite scale Λ, $\{\Gamma_{k,\Lambda},\ 0 \le k < \Lambda\}$ is the restriction of the complete solution $\{\Gamma_k,\ 0 \le k < \infty\}$ to the interval $k < \Lambda$. Thus, sending $\Lambda \to \infty$ in (8.10) is a trivial step. The situation is illustrated in Figure 8.1.

8.3 A One-Loop Reconstruction Formula

The problem we want to address now is how we can determine the Λ-dependence of the bare action S_Λ, given some complete solution of the Λ-free flow equation, $\{\Gamma_k,\ k \in [0, \infty)\}$. According to (8.10), this complete solution implies a solution with a UV cutoff: $\{\Gamma_{k,\Lambda},\ k \in [0,\Lambda]\}$. Setting $k=\Lambda$ we have in particular $\Gamma_{\Lambda,\Lambda}=\Gamma_\Lambda$ or, more explicitly,

$$\Gamma_{k=\Lambda,\Lambda} = \Gamma_{k=\Lambda}. \tag{8.11}$$

Thus, knowing Γ_k for all k means that we know $\Gamma_{\Lambda,\Lambda}$ for all Λ.

Next, we explain how the bare action S_Λ can be found once Γ_k is given. Setting $k=\Lambda$ we know $\Gamma_{\Lambda,\Lambda}=\Gamma_\Lambda$ and since Γ_k is a solution for all values of k, we may assume that $\Gamma_{\Lambda,\Lambda}$ is known for all values of Λ.

(1) Using (8.8) we can obtain the desired relation between Γ_k and S_Λ. For this purpose we evaluate the functional integral on the right-hand side of (8.8) by a saddle point expansion. Let the fluctuation field be $f(x) \equiv f_0(x) + h(x)$ where f_0 is the stationary point of the total action:

$$S_{\text{tot}}[f;\phi] \equiv S_\Lambda[\phi+f] - \int \mathrm{d}^d x\, f(x) \frac{\delta \Gamma_{k,\Lambda}[\phi]}{\delta\phi(x)} + \frac{1}{2}\int \mathrm{d}^d x\, f(x) \mathcal{R}_k(\hat{p}^2) f(x). \tag{8.12}$$

Now, expanding S_{tot} to second order in h and performing the Gaussian integral over h we obtain the following relationship between the bare and the Effective Average Action:

$$\begin{aligned}\Gamma_{k,\Lambda}[\phi] = S_\Lambda[\phi+f_0] &- \int \mathrm{d}^d x\, f_0 \frac{\delta\Gamma_{k,\Lambda}[\phi]}{\delta\phi} + \frac{1}{2}\int \mathrm{d}^d x\, f_0 \mathcal{R}_k f_0 \\ &+ \frac{1}{2}\,\mathrm{Tr}_\Lambda \ln\left[\left(\frac{\delta^2 S_\Lambda[\phi+f_0]}{\delta\phi^2} + \mathcal{R}_k\right) M^{-2}\right] + \cdots \quad .\end{aligned} \tag{8.13}$$

Noting that also the stationary point f_0 has an expansion in powers of the loop-counting parameter $\hbar$, (8.13) yields, in a slightly symbolic notation:

$$\begin{aligned}&\Gamma_{k,\Lambda}[\phi] - S_\Lambda[\phi] \\ &\quad = -\int f_0 \frac{\delta}{\delta\phi}\Big(\Gamma_{k,\Lambda} - S_\Lambda\Big)[\phi] + \frac{1}{2}\int f_0 \left(S_\Lambda^{(2)}[\phi] + \mathcal{R}_k\right) f_0 \\ &\qquad + O(f_0^3) \\ &\qquad + \frac{\hbar}{2}\,\mathrm{Tr}_\Lambda \ln\left\{\left[S_\Lambda^{(2)}[\phi] + S_\Lambda^{(3)}[\phi] f_0 + S_\Lambda^{(4)}[\phi] f_0 f_0 + \cdots + \mathcal{R}_k\right] M^{-2}\right\} \\ &\qquad + O(\hbar^2) \quad .\end{aligned} \tag{8.14}$$

Together with the expansion of the stationarity condition $(\delta S_{\text{tot}}/\delta f)[f_0]=0$ the above equation is solved self-consistently if $f_0 = 0 + O(\hbar)$ and $\Gamma_{k,\Lambda}[\phi] - S_\Lambda[\phi] = O(\hbar)$. This leads to the following one-loop formula for the difference between the average and the bare action:

$$\Gamma_{k,\Lambda}[\phi] - S_\Lambda[\phi] = \frac{\hbar}{2}\,\mathrm{Tr}_\Lambda \ln\left\{\left[S_\Lambda^{(2)}[\phi] + \mathcal{R}_k\right] M^{-2}\right\}. \tag{8.15}$$

Setting $k = \Lambda$ we arrive at the final result:

$$\boxed{\Gamma_{\Lambda,\Lambda}[\phi] - S_\Lambda[\phi] = \frac{\hbar}{2}\,\mathrm{Tr}_\Lambda \ln\left\{\left[S_\Lambda^{(2)}[\phi] + \mathcal{R}_\Lambda\right] M^{-2}\right\}.} \tag{8.16}$$

This is a differential equation which determines S_Λ. It tells us how the bare action S_Λ must depend on Λ in order to give rise to the prescribed $\Gamma_{\Lambda,\Lambda}$. It will be our main tool for finding the path integral that belongs to a known solution of the FRGE.

(2) A comment concerning the mass parameter M might be in order here. Even though M was introduced in (8.4) only in order to make the measure dimensionless, it has a non-trivial impact on the solution of (8.16) for the bare action S_Λ. Indeed, different choices of M can lead to quite different actions, but all of them are physically equivalent. In a sense, changing M amounts to shifting contributions from the measure into the bare action and vice versa. This illustrates the general fact that neither $\int \mathcal{D}_\Lambda \widehat{\phi}$ nor e^{-S_Λ} have a physical meaning separately, only their combination has. (See [205] for a detailed discussion.)

(3) In QEG, the analogous construction begins with the UV-regularized functional integral:

$$\int \mathcal{D}_\Lambda \widehat{h}\, \mathcal{D}_\Lambda C\, \mathcal{D}_\Lambda \bar{C}\, \exp\left(-\widetilde{S}_\Lambda[\widehat{h}, C, \bar{C}; \bar{g}] - \Delta S_k[\widehat{h}, C, \bar{C}; \bar{g}]\right). \tag{8.17}$$

Here the total bare action $\widetilde{S}_\Lambda \equiv S_\Lambda + S_{\mathrm{gf},\Lambda} + S_{\mathrm{gh},\Lambda}$ depends on Λ and includes the gauge-fixing term $S_{\mathrm{gf},\Lambda}$ and the ghost action $S_{\mathrm{gh},\Lambda}$. The UV cutoff is implemented by restricting the expansion of $(\widehat{h}, C, \bar{C})$ in terms of eigenfunctions of the background-covariant Laplacian $-\bar{D}^2 \equiv -\bar{g}^{\mu\nu}\bar{D}_\mu \bar{D}_\nu$ to those with eigenvalues κ smaller than a given Λ^2. Hence, the measure reads in analogy with (8.4),

$$\int \mathcal{D}_\Lambda \widehat{h} = \prod_{\kappa\in[0,\Lambda^2]} \prod_m \int_{-\infty}^{\infty} \mathrm{d}h_{\kappa m}\, M^{-[h_{\kappa m}]}, \tag{8.18}$$

and likewise for the ghosts. Here, m is a degeneracy index, and $h_{\kappa m}$ are the expansion coefficients of $\widehat{h}_{\mu\nu}(x)$.

Note that at this point the background metric $\bar{g}_{\mu\nu}(x)$ is instrumental not only for the IR cutoff and the gauge fixing but also for implementing the UV cutoff in a $\boldsymbol{\delta}^{\mathbf{B}}$-covariant fashion.

The remaining steps are analogous to the scalar case. One considers the UV-regular EAA $\Gamma_{k,\Lambda}[h, \xi, \bar{\xi}; \bar{g}]$ pertaining to the integral (8.17) and shows that it satisfies a FRGE of the form (8.6), albeit with a modified supertrace that carries an explicit $\bar{g}$-dependence:

$$\mathrm{STr}_\Lambda[\cdots] \equiv \mathrm{STr}\left[\Theta(\Lambda^2 + \bar{D}^2)[\cdots]\right]. \tag{8.19}$$

Denoting, as usual, solutions to the Λ-free FRGE as $\Gamma_k[h, \xi, \bar{\xi}; \bar{g}]$, we establish that at $k = \Lambda \to \infty$, and for an appropriate $\mathcal{R}_k$,

$$\Gamma_{\Lambda,\Lambda}[h, \xi, \bar{\xi}; \bar{g}] = \Gamma_\Lambda[h, \xi, \bar{\xi}; \bar{g}], \tag{8.20}$$

and that the latter functional is related to the total bare action by

$$\boxed{\Gamma_{\Lambda,\Lambda}[h, \xi, \bar{\xi}; \bar{g}] = \widetilde{S}_\Lambda[h, \xi, \bar{\xi}; \bar{g}] + \frac{1}{2}\,\mathrm{STr}_\Lambda \ln\left[\left(\widetilde{S}^{(2)}_\Lambda + \mathcal{R}_\Lambda\right)[h, \xi, \bar{\xi}; \bar{g}]\,\mathcal{N}^{-1}\right].} \tag{8.21}$$

Here $\mathcal{N}$ is a block diagonal normalization matrix, equal to M^d and M^2 in the graviton and the ghost sector, respectively.

8.4 The Twofold Einstein–Hilbert Truncation

Solving the functional differential equation (8.21) for $\widetilde{S}_\Lambda[h, \xi, \bar{\xi}; \bar{g}]$ is clearly difficult. In practice one has to restrict the spaces of actions from which Γ_k and $\widetilde{S}_\Lambda$ are drawn by truncating them to a tractable number of invariants. The simplest possibility, which we discuss now, is to employ the Einstein–Hilbert truncation for both the effective *and the bare* action.

As in Section 5.2 we make the ansatz

$$\begin{aligned}\Gamma_k[g, \bar{g}, \xi, \bar{\xi}] = &- \frac{1}{16\pi G_k}\int \mathrm{d}^d x \sqrt{g}\,\big(R(g) - 2\bar{\lambda}_k\big) + S_{\mathrm{gh}}[g - \bar{g}, \xi, \bar{\xi}; \bar{g}] \\ &+ \frac{1}{32\pi G_k}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}\big(\mathcal{F}^{\alpha\beta}_\mu g_{\alpha\beta}\big)\big(\mathcal{F}^{\rho\sigma}_\nu g_{\rho\sigma}\big).\end{aligned} \tag{8.22}$$

The last term of (8.22) is the gauge-fixing action, with $\alpha = 1$, for the harmonic coordinate condition, involving $\mathcal{F}^{\alpha\beta}_\mu \equiv \delta^\beta_\mu \bar{g}^{\alpha\gamma}\bar{D}_\gamma - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{D}_\mu$. Furthermore, we make an analogous ansatz for the bare action:

$$\begin{aligned}\widetilde{S}_\Lambda[g, \bar{g}, \xi, \bar{\xi}] = &- \frac{1}{16\pi \check{G}_\Lambda}\int \mathrm{d}^d x \sqrt{g}\,\big(R(g) - 2\check{\bar{\lambda}}_\Lambda\big) + S_{\mathrm{gh}}[g - \bar{g}, \xi, \bar{\xi}; \bar{g}] \\ &+ \frac{1}{32\pi \check{G}_\Lambda}\int \mathrm{d}^d x\,\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}\big(\mathcal{F}^{\alpha\beta}_\mu g_{\alpha\beta}\big)\big(\mathcal{F}^{\rho\sigma}_\nu g_{\rho\sigma}\big).\end{aligned} \tag{8.23}$$

The EAA (8.22) contains the familiar running parameters G_k and $\bar{\lambda}_k$. The corresponding *bare* Newton and cosmological constant are denoted by $\check{G}_\Lambda$ and $\check{\bar{\lambda}}_\Lambda$, respectively.

(1) Computing the Hessians and setting $\xi = \bar{\xi} = 0$ and $\bar{g} = g$ afterwards, the supertrace in (8.21) has a derivative expansion of the form

$$\begin{aligned}&\frac{1}{2}\,\mathrm{STr}_\Lambda \ln\left[\left(\widetilde{S}^{(2)}_\Lambda + \mathcal{R}_\Lambda\right)[0, 0, 0; \bar{g}]\,\mathcal{N}^{-1}\right] \\ &\quad = B_0 \Lambda^d \int \mathrm{d}^d x\,\sqrt{g} + B_1 \Lambda^{d-2}\int \mathrm{d}^d x\,\sqrt{g}R(g) + \cdots.\end{aligned} \tag{8.24}$$

The dimensionless coefficients B_0 and B_1 can be evaluated with the same heat kernel methods used in Chapter 5. In $d = 4$ one finds [205]:

$$
\begin{aligned}
B_0 &= \frac{1}{32\pi^2}\Big[5\ln(1-2\check{\lambda}_\Lambda) - 5\ln(\check{g}_\Lambda) + Q_\Lambda\Big],\\
B_1 &= \frac{1}{3}B_0 + \Delta B_1,\\
\Delta B_1 &\equiv \frac{1}{16\pi^2}\,\frac{2-\check{\lambda}_\Lambda}{1-2\check{\lambda}_\Lambda},\\
Q_\Lambda &\equiv 12\ln\Big(\frac{\Lambda}{M}\Big) + b_0,\\
b_0 &\equiv -5\ln(32\pi) - \ln 2 \quad .
\end{aligned}
\tag{8.25}
$$

Using (8.24) in (8.21) and equating the coefficients of the independent invariants we obtain two equations relating the effective to the bare parameters:

$$
\frac{1}{G_\Lambda} - \frac{1}{\check{G}_\Lambda} = -16\pi B_1 \Lambda^{d-2}, \qquad \frac{\bar{\lambda}_\Lambda}{G_\Lambda} - \frac{\check{\bar{\lambda}}_\Lambda}{\check{G}_\Lambda} = 8\pi B_0 \Lambda^d. \tag{8.26}
$$

In terms of the dimensionless quantities defined by $g_\Lambda \equiv \Lambda^{d-2}G_\Lambda$, $\check{g}_\Lambda \equiv \Lambda^{d-2}\check{G}_\Lambda$, and analogously for the two cosmological constants, we get

$$
\boxed{\frac{1}{g_\Lambda} - \frac{1}{\check{g}_\Lambda} = -16\pi B_1, \qquad \frac{\lambda_\Lambda}{g_\Lambda} - \frac{\check{\lambda}}{\check{g}_\Lambda} = 8\pi B_0.} \tag{8.27}
$$

This algebraic system of equations contains the information we are interested in. It allows us to determine $\check{g}_\Lambda$ and $\check{\lambda}_\Lambda$ for given g_Λ and λ_Λ.

(2) While it is impossible to solve the system (8.27) analytically, the numerical analysis performed in [205] established a well-defined map $(g,\lambda) \mapsto (\check{g},\check{\lambda})$ for all $g > 0$ and $\lambda < \frac{1}{2}$. The analysis covered a wide range of values for the constant $Q = 12\ln c + b_0$ which labels different choices for the measure.[1]

(3) By this map, the fixed point behavior $\lim_{k\to\infty}(g_k,\lambda_k) = (g_*,\lambda_*)$ on the effective side is mapped onto an analogous fixed point behavior at the bare level. The image of the GFP is always at $\check{g}_* = \check{\lambda}_* = 0$, while the coordinates of the "bare" NGFP, $\check{g}_*$ and $\check{\lambda}_*$, depend on the value of Q.

[1] Initially the map $(g,\lambda) \mapsto (\check{g},\check{\lambda})$ is explicitly Λ-dependent because of the parameter $Q_\Lambda \equiv 12\ln(\frac{\Lambda}{M}) + b_0$. This Λ-dependence can be removed by including appropriate factors of the UV cutoff into the measure. In the present discussion we set $M = \frac{\Lambda}{c}$ with an arbitrary $c > 0$, hence the quantity $Q \equiv Q_{cM} = 12\ln c + b_0$ becomes a Λ-independent constant. As a result, the map $(g,\lambda) \mapsto (\check{g},\check{\lambda})$ has no explicit dependence on any (UV or IR) cutoff.

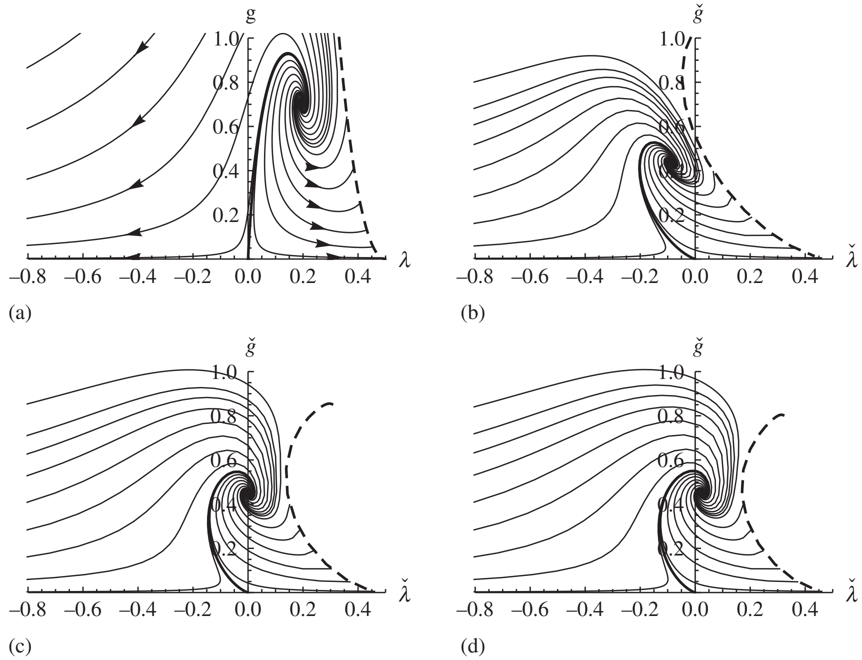


Figure 8.2. The diagram (a) shows the phase portrait of the effective RG flow on the (g, λ)-plane. The other diagrams are its image on the $(\breve{g}, \check{\lambda})$-plane of bare parameters for three different values of the parameter Q characterizing the measure, namely (b) $Q = +1$, (c) $Q = -0.1167$ where $\check{\lambda}_* = 0$, and (d) $Q = -1$, respectively. (Adapted from [205].)

These features are illustrated in Figure 8.2, where we apply the map $(g, \lambda) \mapsto (\breve{g}, \check{\lambda})$ for different values of Q to a set of representative effective RG trajectories on the half-plane $g > 0$.

We emphasize again that all choices of Q are physically equivalent. Varying Q simply amounts to shifting contributions back and forth between the action and the measure.

(4) It is instructive to determine the linearized flow near the two "bare" fixed points and to determine the corresponding critical exponents, if such exponents can be defined.

Both the "effective" and the "bare" NGFP are inner points of the corresponding coupling constant space. The flow in the vicinity of one is the diffeomorphic image of the flow near the other. The RG running of the respective scaling fields is $\propto k^{-\theta}$ and $\propto \Lambda^{-\theta}$, respectively, *with the same critical exponents* θ.

The "effective" GFP shows the familiar pure power-law scaling (5.102). But the "bare" GFP is located on the line $\breve{g} = 0$, i.e., on the boundary of the domain on which the map from the effective to the bare couplings is defined. In its vicinity (on the half-plane with $\check{g} > 0$) the "bare" running is characterized by *logarithmically corrected power laws.* This is seen by expanding the relations (8.27) near the GFP:

$$\begin{aligned} \check{g} &= g + O(g^2, \lambda^2), \\ \check{\lambda} &= \lambda - \frac{g}{4\pi}\Big(Q - 5\ln g\Big) + \frac{g\lambda}{6\pi}\Big[3 - Q + 5\ln g\Big] + O(g^2, \lambda^2). \end{aligned} \tag{8.28}$$

These are the first few terms of a power-log series. Thus, contrary to the parameters in the EAA, the running of the bare parameters follows logarithmically corrected power laws when $\Lambda \to \infty$.

8.5 Further Remarks

We have only discussed some first steps toward solving the reconstruction problem for asymptotically safe theories. Yet it has become clear that, after specifying a UV-regularization scheme and a measure, every solution of the flow equation for the Effective Average Action without a UV cutoff gives rise to a regularized path integral with a well-defined limit $\Lambda \to \infty$. We close with a number of remarks.

(1) The basic idea behind the reconstruction step is to complete the Asymptotic Safety program by identifying a classical system, which, in a hitherto hidden way, got quantized by picking a certain RG trajectory of the EAA which extends from the NGFP in the UV to the IR.

The advantage of the strategy we adopted, selecting a theory on the basis of its effective rather than bare action, is that in this manner it is easier to see that it has indeed an "asymptotically safe" short-distance behavior since the effective action is already closer to the observables. The consequence is that extra work and further structural input is needed to obtain the eligible microscopic systems.

(2) Once we have a (regularized) functional with a well-defined continuum limit in our hands, involving a particular bare action, we can attempt a kind of "Legendre transformation" to find appropriate phase space variables, a microscopic Hamiltonian, and thus a canonical description of the bare theory. Only at this level we can identify the degrees of freedom that got quantized as well as their fundamental interactions.

Since the Hamiltonian is unlikely to turn out quadratic in the momenta, the "Legendre transformation" involved is to be understood as a generalized, i.e., quantum mechanical one. In the simplest case it consists in reformulating a given configuration space path integral $\int \mathcal{D}\widehat{\Phi} \exp\big(iS[\widehat{\Phi}]\big)$ as a phase space integral $\int \mathcal{D}\widehat{\Phi} \int \mathcal{D}\widehat{\Pi} \exp\big(i \int \widehat{\Pi}\dot{\widehat{\Phi}} - H[\widehat{\Pi}, \widehat{\Phi}]\big)$. In a way, we must undo the integration over the momenta.

However, given the complexity of the fixed point action which most probably contains higher derivative terms, a generalized Ostrogradsky-type phase space formalism [245, 246] will presumably emerge. In the same spirit, non-localities also suggest introducing additional auxiliary fields that render the action local, thereby further enlarging the phase space.

(3) Being interested in a canonical description of the bare fixed point action, one might wonder if there exists an alternative to the EAA approach that would deal directly with the RG flow of Hamiltonians rather than Lagrangians. Here the problem is that, if we apply a coarse-graining step to an action which contains only, say, first derivatives of the field, the result will also contain higher derivatives in general. This poses no special problem in a Lagrangian setting, but for the Hamiltonian formalism it implies that new momentum variables must be introduced. As a result, the coarse-grained Hamiltonian "lives" on a different phase space (in the sense of Ostrogradsky's method) than the original one. Therefore, there exists no obvious Hamiltonian analog of the flow on the space of actions.

A way to bypass this obstacle could be to abandon the spacetime averaging in favor of a purely spatial averaging. In this way Hamiltonian RG flows that do not generate higher-order time derivatives are conceivable.

9

Alternative Field Variables

In the previous chapters the field carrying the gravitational degrees of freedom was chosen to be a symmetric tensor $g_{\mu\nu}$. However, it is well known that General Relativity admits many reformulations which employ different field variables, such as tetrads and/or spin connections, for example. Every such reformulation is characterized by a specific set of fields and a certain covariance group acting on them. This suggests considering a theory space comprised of functionals which depend on this new set of field variables and are invariant under the corresponding group of transformations. Since classical equivalence of theories does not imply their quantum equivalence, the RG flow on the new space may or may not be physically equivalent to that of metric gravity. Searching for asymptotically safe theories of gravity it is therefore important to explore such alternative settings as well. This is the topic of the present chapter.

Furthermore, elementary field variables different from those considered so far can even occur within the metric approach when particular parametrizations of $g_{\mu\nu}$ are invoked. The example we are going to discuss here is the ADM decomposition of the metric, which suggests considering invariant functionals of a three-metric together with the lapse and shift functions.

9.1 Quantum Einstein–Cartan Gravity

In General Relativity there exists a rich variety of variational principles which give rise to Einstein's equation, or equations equivalent to it but expressed in terms of different field variables [247–249]. The best-known examples employ the Einstein–Hilbert action expressed in terms of the metric, $S_{\text{EH}}[g_{\mu\nu}]$, or the tetrad, respectively, $S_{\text{EH}}[e^a{}_\mu]$. The latter action functional is obtained by inserting the representation of the metric in terms of vielbeins, $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, into the former.[1]

[1] In this chapter coordinate (frame) indices are denoted by Greek (Latin) letters and η_{ab} is the frame metric. In the Euclidean setting, $\eta_{ab} = \delta_{ab}$, the four-dimensional Lorentz group is replaced with O(4).

Another formulation which is classically equivalent, at least in the absence of spinning matter, is provided by the first-order Hilbert–Palatini action $S_{\text{HP}}[e^a{}_\mu, \omega^{ab}{}_\mu]$, which, besides the tetrad, depends on the spin connection $\omega^{ab}{}_\mu$ assuming values in the Lie algebra of $\mathrm{O}(1,3)$ or $\mathrm{O}(4)$, respectively [247, 248, 250]. The variation of S_{HP} with respect to $\omega^{ab}{}_\mu$ leads, in vacuo, to an equation of motion stating that this connection has vanishing torsion. It can be solved algebraically as $\omega = \omega(e)$, which, when inserted into S_{HP}, brings us back to $S_{\text{EH}}[e] \equiv S_{\text{HP}}[e, \omega(e)]$.

Still another equivalent formulation is based on the self-dual Hilbert–Palatini action $S^{\text{sd}}_{\text{HP}}[e^a{}_\mu, \omega^{(+)}{}_\mu{}^{ab}]$, which only depends on the self-dual projection of the spin connection, $\omega^{(+)}{}_\mu{}^{ab}$ [16, 36–39]. This action in turn is closely related to the Plebański action [251], containing additional two-form fields, and to the Capovilla–Dell–Jacobson action [252, 253] which involves essentially only a self-dual connection. Similarly, Krasnov's diffeomorphism-invariant Yang–Mills theories [254, 255] allow for a "pure connection" reformulation of General Relativity.

The above variational principles are Lagrangian in nature; the fields employed provide a parametrization of configuration space. The corresponding Hamiltonian description in which the "carrier fields" of the gravitational interaction parametrize a phase space is obtained by applying a Legendre transform. In this way the ADM-Hamiltonian [256] and, most importantly, Ashtekar's Hamiltonian [257], make their appearance.

Regarding our search for an asymptotically safe quantum theory of gravity this multitude of classical formalisms offers many equally plausible possibilities to explore. Each of the classical systems mentioned above "inspires" a particular space of functionals and group of transformations under which they are invariant. In principle we can calculate the RG flow on these theory spaces without any further input data, in particular without selecting a bare action.

We are going to discuss the functional RG approach to "quantum Einstein–Cartan gravity" as an example here, following the original works [258–261]. Throughout the discussion we set $d = 4$.

In Einstein–Cartan gravity the field variables are the tetrad, or vielbein $e^a{}_\mu$ and a spin connection $\omega^{ab}{}_\mu$, the latter assuming values in the Lie algebra of the (Euclidean) "Lorentz" group, here taken to be $\mathrm{O}(4)$. This entails an enlargement of the group of gauge transformations from the diffeomorphisms, $\text{Diff}(\mathcal{M})$, to $\text{Diff}(\mathcal{M}) \ltimes \mathrm{O}(4)_{\text{loc}}$, so that gauge-invariant functionals $A[e, \omega]$ constitute a new "universality class" replacing that of metric gravity with its diffeomorphism-invariant functionals $A[g]$. By augmenting the number of field components from 10 in the case of $g_{\mu\nu}$, to 40 for a pair $(e^a{}_\mu, \omega^{ab}{}_\mu)$, Einstein–Cartan gravity, in fact, generalizes metric gravity: In particular, $\omega^{ab}{}_\mu$ allows for the possibility of *spacetime torsion* [247, 248, 262].

Classically, the dynamics of pure Einstein–Cartan gravity are determined by the *Hilbert–Palatini action* $S_{\text{HP}}[e, \omega]$. Since the resulting equations of motion give rise to vanishing torsion, this functional can be regarded as the counterpart of the Einstein–Hilbert action of metric gravity. However, quantum mechanically, generic configurations (e, ω) contributing to the functional integral will have non-zero torsion even if torsion should happen to vanish at the classical level: Hence, classical equivalence does by no means imply quantum equivalence. Therefore, the additional fields of Einstein–Cartan gravity can be expected to crucially affect the renormalization.

The Hilbert–Palatini action is often generalized to the so-called *Holst action* $S_{\text{Ho}}[e, \omega]$ which contains an additional term that only exists in four dimensions [263, 264]; its prefactor is the dimensionless *Immirzi parameter*, γ [265, 266]. Since this monomial vanishes for vanishing torsion, it is absent in the metric approach. While the classical field equations are independent of γ, the corresponding quantum theory is expected to depend on it. In this respect, γ can be compared to the θ-parameter of QCD; even though the latter multiplies a topological term and therefore does not affect the classical equations of motion, quantum observables may depend on it.

The Holst action underlies several approaches to the quantization of gravity, including canonical quantum gravity with Ashtekar's variables [36–38], Loop Quantum Gravity (LQG) [39], spin-foam models [267], and group field theory [268, 269]. Within LQG, for instance, γ enters the spectrum of area and volume operators as well as the entropy formula for black holes [38], which exemplifies the quantum significance of γ stated above. Furthermore, when coupled to fermions in a non-minimal way, the presence of the Immirzi term induces a CP violating four-fermion interaction that might be interesting for phenomenological reasons, e.g., in the early universe [270].

While in LQG, for example, γ is a constant parameter that labels distinct quantum theories, a consistent application of the functional RG in the context of Einstein–Cartan gravity should treat γ as a running coupling, associated with a certain monomial in the action functional, that is subject to renormalization.

From the perspective of the non-perturbative renormalization group the crucial question is whether the RG flow on the theory space $\mathcal{T}$ made up by all functionals of $e^a{}_\mu$ and $\omega^{ab}{}_\mu$ (and the necessary ghosts) that respect a background-type realization of the new gauge invariance possesses zero, one, or even several fixed points suitable for the Asymptotic Safety construction.

As we will see, the first non-perturbative studies indeed suggest the existence of two NGFPs, which both seem suitable for defining a quantum field theory of Einstein–Cartan gravity.

Since the underlying "universality class" of flows is different from that of metric gravity, there is no general principle implying that the corresponding quantum field theories are equivalent to their metric counterpart QEG.

In particular, even if we analyze the fixed points of Einstein–Cartan and metric gravity, respectively, using truncations that happen to be equivalent classically there are no conceptual reasons or mathematical mechanisms that would enforce equivalent results. In fact, the mere existence of fixed points in the $(e,\,\omega)$ universality class already constitutes a novel result, completely independent of the analogous findings obtained in metric gravity.

Logically, *Asymptotic Safety of metric gravity is neither necessary nor sufficient for Asymptotic Safety on the Einstein–Cartan theory space*, which may well probe different universality classes of microscopic theories.

9.2 Implementations of Gauge Invariance

When setting up an Effective Average Action for the Einstein–Cartan theory space, many of the relevant considerations parallel those in metric gravity. However, additional difficulties arise due to the more complicated structure of the gauge algebra. In the following we will therefore mainly focus on those aspects.

The formal starting point is the integral $Z = \int \mathcal{D}\hat{e}^a{}_\mu\, \mathcal{D}\hat{\omega}^{ab}{}_\mu \exp\big\{-S[\hat{e},\hat{\omega}]\big\}$, where the quantum fields $\hat{e}^a{}_\mu$ and $\hat{\omega}^{ab}{}_\mu$ are defined on a fixed differentiable manifold without boundary, $\mathcal{M}$, and the bare action S is invariant both under diffeomorphisms, $\mathrm{Diff}(\mathcal{M})$, and local Lorentz rotations. We consider the Euclidean version of the theory, so that the relevant group of gauge transformations is the semidirect product $\mathbf{G} = \mathrm{Diff}(\mathcal{M}) \ltimes \mathrm{O}(4)_{\mathrm{loc}}$.

(1) The pairs $\big(\hat{e}^a{}_\mu, \hat{\omega}^{ab}{}_\mu\big)$ determine an O(4)-covariant derivative

$$\hat{\nabla}_\mu \equiv \partial_\mu + \frac{1}{2}\hat{\omega}^{ab}{}_\mu M_{ab}, \tag{9.1}$$

where M_{ab} are the generators in the corresponding representation. The associated curvature and torsion tensors read

$$\begin{aligned} \hat{F}^{ab}{}_{\mu\nu} &\equiv \partial_\mu \hat{\omega}^{ab}{}_\nu + \hat{\omega}^a{}_{c\mu}\hat{\omega}^{cb}{}_\nu - (\mu \leftrightarrow \nu), \\ \hat{T}^a{}_{\mu\nu} &\equiv \partial_\mu \hat{e}^a{}_\nu + \hat{\omega}^a{}_{c\mu}\hat{e}^c{}_\nu - (\mu \leftrightarrow \nu). \end{aligned} \tag{9.2}$$

(2) Under the $\mathrm{O}(4)_{\mathrm{loc}}$ transformations $\boldsymbol{\delta}_{\mathrm{L}}$ with parameters $\lambda^a{}_b$ we have

$$\begin{aligned} \boldsymbol{\delta}_{\mathrm{L}}(\lambda)\hat{e}^a{}_\mu &= \lambda^a{}_b \hat{e}^b{}_\mu, \\ \boldsymbol{\delta}_{\mathrm{L}}(\lambda)\hat{\omega}^{ab}{}_\mu &= -\partial_\mu\lambda^{ab} + \lambda^a{}_c\hat{\omega}^{cb}{}_\mu + \lambda^b{}_c\hat{\omega}^{ac}{}_\mu \equiv -\hat{\nabla}_\mu\lambda^{ab}. \end{aligned} \tag{9.3}$$

The spacetime diffeomorphisms $\boldsymbol{\delta}_{\mathrm{D}}$ act as

$$\boldsymbol{\delta}_{\mathrm{D}}(w)\hat{e}^a{}_\mu = \mathcal{L}_w\, \hat{e}^a{}_\mu, \quad \boldsymbol{\delta}_{\mathrm{D}}(w)\hat{\omega}^{ab}{}_\mu = \mathcal{L}_w\, \hat{\omega}^{ab}{}_\mu, \tag{9.4}$$

where $\mathcal{L}_w$ denotes the Lie derivative along the generating vector field w.

Upon gauge fixing we also need to consider diffeomorphism ghosts and antighosts, $\mathcal{C}^\mu$ and $\bar{\mathcal{C}}_\mu$, respectively, and likewise Faddeev–Popov ghosts Σ^{ab} and $\bar{\Sigma}_{ab}$ for the local O(4) transformations. We require all ghost and antighost fields to transform under $\mathrm{Diff}(\mathcal{M})$ and $\mathrm{O}(4)_{\mathrm{loc}}$ as tensors of the corresponding type.

It then follows that the gauge transformations satisfy the Lie algebra relations

$$\begin{aligned}
[\boldsymbol{\delta}_{\mathrm{D}}(w_1), \boldsymbol{\delta}_{\mathrm{D}}(w_2)]\,\Phi &= \boldsymbol{\delta}_{\mathrm{D}}\left([w_1, w_2]\right)\Phi \\
[\boldsymbol{\delta}_{\mathrm{L}}(\lambda_1), \boldsymbol{\delta}_{\mathrm{L}}(\lambda_2)]\,\Phi &= \boldsymbol{\delta}_{\mathrm{L}}\left([\lambda_1, \lambda_2]\right)\Phi \\
[\boldsymbol{\delta}_{\mathrm{D}}(w), \boldsymbol{\delta}_{\mathrm{L}}(\lambda)]\,\Phi &= \boldsymbol{\delta}_{\mathrm{L}}\left(\mathcal{L}_w \lambda\right)\Phi
\end{aligned} \tag{9.5}$$

where Φ is any field from the set $\{\hat{e}^a{}_\mu, \hat{\omega}^{ab}{}_\mu, \mathcal{C}^\mu, \bar{\mathcal{C}}_\mu, \Sigma^{ab}, \Sigma_{ab}\}$. Furthermore $[w_1, w_2]$ denotes the Lie bracket of the vector fields w_1 and w_2, and $[\lambda_1, \lambda_2]$ is the commutator of two matrices. The algebra constituted by (9.5) is a semidirect product, $\mathrm{Diff}(\mathcal{M}) \ltimes \mathrm{O}(4)_{\mathrm{loc}}$, with the local Lorentz transformations playing the role of the invariant subalgebra.

In order to implement the gauge transformations on the theory space comprising functionals $A[\hat{e}^a{}_\mu, \hat{\omega}^{ab}{}_\mu, \mathcal{C}^\mu, \bar{\mathcal{C}}_\mu, \Sigma^{ab}, \bar{\Sigma}_{ab}]$ it is convenient to introduce *Ward operators* $\mathcal{W}_{\mathrm{D}}$ and $\mathcal{W}_{\mathrm{L}}$ such that $\boldsymbol{\delta}_{\mathrm{D,L}}\, A = -\mathcal{W}_{\mathrm{D,L}}\, A$ to linear order in the transformation parameters. Explicitly,

$$\begin{aligned}
\mathcal{W}_{\mathrm{D}}(w) = -\int \mathrm{d}^4x \Big(& \boldsymbol{\delta}_{\mathrm{D}}(w)\hat{e}^a{}_\mu(x)\frac{\delta}{\delta \hat{e}^a{}_\mu(x)} + \boldsymbol{\delta}_{\mathrm{D}}(w)\hat{\omega}^{ab}{}_\mu(x)\frac{\delta}{\delta\hat{\omega}^{ab}{}_\mu(x)} \\
& + \boldsymbol{\delta}_{\mathrm{D}}(w)\mathcal{C}^\mu(x)\frac{\delta}{\delta\mathcal{C}^\mu(x)} + \boldsymbol{\delta}_{\mathrm{D}}(w)\bar{\mathcal{C}}_\mu(x)\frac{\delta}{\delta\bar{\mathcal{C}}_\mu(x)} \\
& + \boldsymbol{\delta}_{\mathrm{D}}(w)\Sigma^{ab}(x)\frac{\delta}{\delta\Sigma^{ab}(x)} + \boldsymbol{\delta}_{\mathrm{D}}(w)\bar{\Sigma}_{ab}(x)\frac{\delta}{\delta\bar{\Sigma}_{ab}(x)}\Big)
\end{aligned} \tag{9.6}$$

and analogously for $\mathcal{W}_{\mathrm{L}}$. The Ward operators satisfy

$$\boxed{\begin{aligned}
[\mathcal{W}_{\mathrm{D}}(w_1), \mathcal{W}_{\mathrm{D}}(w_2)] &= \mathcal{W}_{\mathrm{D}}([w_1, w_2]), \\
[\mathcal{W}_{\mathrm{L}}(\lambda_1), \mathcal{W}_{\mathrm{L}}(\lambda_2)] &= \mathcal{W}_{\mathrm{L}}([\lambda_1, \lambda_2]), \\
[\mathcal{W}_{\mathrm{D}}(w), \mathcal{W}_{\mathrm{L}}(\lambda)] &= \mathcal{W}_{\mathrm{L}}(\mathcal{L}_w\lambda).
\end{aligned}} \tag{9.7}$$

Gauge-invariant functionals $A[\hat{e}, \hat{\omega}, \mathcal{C}, \bar{\mathcal{C}}, \Sigma, \bar{\Sigma}]$ are characterized by the conditions $\mathcal{W}_{\mathrm{D}}(w)\, A = 0 = \mathcal{W}_{\mathrm{L}}(\lambda)\, A$ for all w and λ.

(3) In order to construct a flow equation with the desired invariance properties it is important to notice that *the (ordinary) diffeomorphisms* $\boldsymbol{\delta}_{\mathrm{D}}(w)$ *are not covariant under* $\mathrm{O}(4)_{\mathrm{loc}}$. This is obvious from the fact that the Lie derivative involves partial rather than O(4)-covariant derivatives.

As a remedy we covariantize the diffeomorphisms by combining them with an appropriate O(4) transformation. Namely, introducing

$$\boxed{\widetilde{\boldsymbol{\delta}_{\mathrm{D}}}(w) \equiv \boldsymbol{\delta}_{\mathrm{D}}(w) + \boldsymbol{\delta}_{\mathrm{L}}(w\cdot\hat{\omega}),} \tag{9.8}$$

where $(w \cdot \hat{\omega})^{ab} \equiv w^\mu \hat{\omega}^{ab}{}_\mu$, it can be shown that these *modified diffeomorphisms* $\widetilde{\boldsymbol{\delta}_{\mathrm{D}}}$ act on all fields by covariant derivatives $\hat{\nabla}_\mu$ only:

$$\begin{aligned}
\widetilde{\boldsymbol{\delta}_{\mathrm{D}}}(w)\hat{e}^a{}_\mu &= w^\rho \hat{\nabla}_\rho \hat{e}^a{}_\mu + (\hat{\nabla}_\mu w^\rho)\hat{e}^a{}_\rho, \\
\widetilde{\boldsymbol{\delta}_{\mathrm{D}}}(w)\hat{\omega}^{ab}{}_\mu &= -\hat{F}^{ab}{}_{\mu\rho} w^\rho, \\
\widetilde{\boldsymbol{\delta}_{\mathrm{D}}}(w)\mathcal{C}^\mu &= w^\rho \hat{\nabla}_\rho \mathcal{C}^\mu - (\hat{\nabla}_\rho w^\mu)\mathcal{C}^\rho = w^\rho \partial_\rho \mathcal{C}^\mu - (\partial_\rho w^\mu)\mathcal{C}^\rho, \\
\widetilde{\boldsymbol{\delta}_{\mathrm{D}}}(w)\bar{\mathcal{C}}_\mu &= w^\rho \hat{\nabla}_\rho \bar{\mathcal{C}}_\mu + (\hat{\nabla}_\mu w^\rho)\bar{\mathcal{C}}_\rho, \\
\widetilde{\boldsymbol{\delta}_{\mathrm{D}}}(w)\Sigma^{ab} &= w^\rho \hat{\nabla}_\rho \Sigma^{ab}, \\
\widetilde{\boldsymbol{\delta}_{\mathrm{D}}}(w)\bar{\Sigma}_{ab} &= w^\rho \hat{\nabla}_\rho \bar{\Sigma}_{ab}.
\end{aligned} \tag{9.9}$$

Associating as above Ward operators $\widetilde{\mathcal{W}_{\mathrm{D}}}(w)$ to the modified diffeomorphisms leads to the following covariantized form of the gauge algebra:

$$\boxed{\begin{aligned}
\big[\widetilde{\mathcal{W}_{\mathrm{D}}}(w_1), \widetilde{\mathcal{W}_{\mathrm{D}}}(w_2)\big] &= \widetilde{\mathcal{W}_{\mathrm{D}}}([w_1, w_2]) - \mathcal{W}_{\mathrm{L}}(w_1 w_2 \cdot \hat{F}), \\
\big[\mathcal{W}_{\mathrm{L}}(\lambda_1), \mathcal{W}_{\mathrm{L}}(\lambda_2)\big] &= \mathcal{W}_{\mathrm{L}}([\lambda_1, \lambda_2]), \\
\big[\widetilde{\mathcal{W}_{\mathrm{D}}}(w), \mathcal{W}_{\mathrm{L}}(\lambda)\big] &= 0.
\end{aligned}} \tag{9.10}$$

Here $(w_1 w_2 \cdot \hat{F})^{ab} \equiv w_1^\mu w_2^\nu \, \hat{F}^{ab}{}_{\mu\nu}$. Note that while the modified diffeomorphisms commute with local Lorentz transformations, they no longer close among themselves; their commutator contains an $\mathrm{O}(4)_{\mathrm{loc}}$ transformation whose parameter involves $\hat{F}$, the curvature of $\hat{\omega}$.

Note that gauge-invariant functionals A can now also be characterized by the conditions $\widetilde{\mathcal{W}_{\mathrm{D}}}(w)\, A = 0 = \mathcal{W}_{\mathrm{L}}(\lambda)\, A$ for all w and λ.

(4) As in metric gravity, we employ the background field technique to cope with the requirement of Background Independence. We introduce arbitrary background fields $\bar{e}^a{}_\mu$ and $\bar{\omega}^{ab}{}_\mu$, whereby the background vielbein $\bar{e}^a{}_\mu$ is assumed to be *non-degenerate*. As a result, it gives rise to a well-defined inverse $(\bar{e}_a{}^\mu) \equiv (\bar{e}^a{}_\mu)^{-1}$, to a non-degenerate background metric $\bar{g}_{\mu\nu} \equiv \bar{e}^a{}_\mu \bar{e}^b{}_\nu \delta_{ab}$, and to the completely covariant derivative $\bar{D} \equiv \partial + \bar{\omega} + \bar{\Gamma} \equiv \bar{\nabla} + \bar{\Gamma}$ where $\bar{\Gamma} \equiv \bar{\Gamma}(\bar{e}, \bar{\omega})$ is fixed by the requirement $\bar{D}_\mu \bar{e}^a{}_\nu = 0$ [248].

We decompose the variables of integration as $\hat{e}^a{}_\mu \equiv \bar{e}^a{}_\mu + \hat{\varepsilon}^a{}_\mu$, $\hat{\omega}^{ab}{}_\mu \equiv \bar{\omega}^{ab}{}_\mu + \hat{\tau}^{ab}{}_\mu$, and then apply the Faddeev–Popov method in order to gauge fix the functional integral:

$$\begin{aligned}
Z = &\int \mathcal{D}\hat{\varepsilon}^a{}_\mu \mathcal{D}\hat{\tau}^{ab}{}_\mu \, \exp\big\{-S[\bar{e} + \hat{\varepsilon}, \bar{\omega} + \hat{\tau}] - S_{\mathrm{gf}}[\hat{\varepsilon}, \hat{\tau}; \bar{e}, \bar{\omega}]\big\} \\
&\times \int \mathcal{D}\mathcal{C}^\mu \mathcal{D}\bar{\mathcal{C}}_\mu \mathcal{D}\Sigma^{ab} \mathcal{D}\bar{\Sigma}_{ab} \, \exp\big\{-S_{\mathrm{gh}}\big\}.
\end{aligned} \tag{9.11}$$

Here $S_{\rm gf}$ and $S_{\rm gh}$ denote the gauge-fixing and ghost actions, respectively, $\mathcal{C}^\mu$ and $\bar{\mathcal{C}}_\mu$ are the diffeomorphism ghosts, and Σ^{ab} and $\bar{\Sigma}_{ab}$ those related to the local O(4). The gauge fixing is of the form

$$S_{\rm gf} = \frac{1}{2\alpha_{\rm D} \cdot 16\pi G} \int \mathrm{d}^4x\, \bar{e}\, \bar{g}^{\mu\nu} \mathcal{F}_\mu \mathcal{F}_\nu + \frac{1}{2\alpha_{\rm L}} \int \mathrm{d}^4x\, \bar{e}\, \mathcal{G}^{ab} \mathcal{G}_{ab} \tag{9.12}$$

where $\mathcal{F}_\mu$ and $\mathcal{G}^{ab}$ break the $\mathrm{Diff}(\mathcal{M})$ and $\mathrm{O}(4)_{\rm loc}$ gauge invariance, respectively. In order to render Γ_k manifestly $\mathrm{Diff}(\mathcal{M}) \ltimes \mathrm{O}(4)_{\rm loc}$ invariant we employ gauge conditions of the "background type" for which $S_{\rm gf}[\hat{\varepsilon}, \hat{\tau}; \bar{e}, \bar{\omega}]$ is invariant under the combined background gauge transformations $\boldsymbol{\delta}^{\rm B}_{\rm D,L}$ acting on both $(\hat{\varepsilon},\, \hat{\tau})$ and $(\bar{e},\, \bar{\omega})$ while, of course, it is not invariant under the "true" (or "quantum") gauge transformations, denoted by $\boldsymbol{\delta}^{\rm Q}_{\rm D}$ and $\boldsymbol{\delta}^{\rm Q}_{\rm L}$, respectively.

(5) The true diffeomorphisms act on the decomposed fields according to

$$\begin{aligned}
\boldsymbol{\delta}^{\rm Q}_{\rm D}(w)\bar{e}^a{}_\mu &= 0, \\
\boldsymbol{\delta}^{\rm Q}_{\rm D}(w)\hat{\varepsilon}^a{}_\mu &= \mathcal{L}_w(\bar{e}^a{}_\mu + \hat{\varepsilon}^a{}_\mu), \\
\boldsymbol{\delta}^{\rm Q}_{\rm D}(w)\bar{\omega}^{ab}{}_\mu &= 0, \\
\boldsymbol{\delta}^{\rm Q}_{\rm D}(w)\hat{\tau}^{ab}{}_\mu &= \mathcal{L}_w(\bar{\omega}^{ab}{}_\mu + \hat{\tau}^{ab}{}_\mu)
\end{aligned} \tag{9.13}$$

and the true O(4) gauge transformations are given by

$$\begin{aligned}
\boldsymbol{\delta}^{\rm Q}_{\rm L}(\lambda)\bar{e}^a{}_\mu &= 0, \\
\boldsymbol{\delta}^{\rm Q}_{\rm L}(\lambda)\hat{\varepsilon}^a{}_\mu &= \lambda^a{}_b(\bar{e}^b{}_\mu + \hat{\varepsilon}^b{}_\mu), \\
\boldsymbol{\delta}^{\rm Q}_{\rm L}(\lambda)\bar{\omega}^{ab}{}_\mu &= 0, \\
\boldsymbol{\delta}^{\rm Q}_{\rm L}(\lambda)\hat{\tau}^{ab}{}_\mu &= -\partial_\mu\lambda^{ab} + \lambda^a{}_c(\bar{\omega}^{cb}{}_\mu + \hat{\tau}^{cb}{}_\mu) + \lambda^b{}_c(\bar{\omega}^{ac}{}_\mu + \hat{\tau}^{ac}{}_\mu).
\end{aligned} \tag{9.14}$$

On the other hand, the background diffeomorphisms act as

$$\begin{aligned}
\boldsymbol{\delta}^{\rm B}_{\rm D}(w)\bar{e}^a{}_\mu &= \mathcal{L}_w\bar{e}^a{}_\mu, \\
\boldsymbol{\delta}^{\rm B}_{\rm D}(w)\hat{\varepsilon}^a{}_\mu &= \mathcal{L}_w\hat{\varepsilon}^a{}_\mu, \\
\boldsymbol{\delta}^{\rm B}_{\rm D}(w)\bar{\omega}^{ab}{}_\mu &= \mathcal{L}_w\bar{\omega}^{ab}{}_\mu, \\
\boldsymbol{\delta}^{\rm B}_{\rm D}(w)\hat{\tau}^{ab}{}_\mu &= \mathcal{L}_w\hat{\tau}^{ab}{}_\mu
\end{aligned} \tag{9.15}$$

and the background O(4) gauge transformations are

$$\begin{aligned}
\boldsymbol{\delta}^{\rm B}_{\rm L}(\lambda)\bar{e}^a{}_\mu &= \lambda^a{}_b\bar{e}^b{}_\mu, \\
\boldsymbol{\delta}^{\rm B}_{\rm L}(\lambda)\hat{\varepsilon}^a{}_\mu &= \lambda^a{}_b\hat{\varepsilon}^b{}_\mu, \\
\boldsymbol{\delta}^{\rm B}_{\rm L}(\lambda)\bar{\omega}^{ab}{}_\mu &= -\partial_\mu\lambda^{ab} + \lambda^a{}_c\bar{\omega}^{cb}{}_\mu + \lambda^b{}_c\bar{\omega}^{ac}{}_\mu \equiv -\bar{\nabla}_\mu\lambda^{ab}, \\
\boldsymbol{\delta}^{\rm B}_{\rm L}(\lambda)\hat{\tau}^{ab}{}_\mu &= \lambda^a{}_c\hat{\tau}^{cb}{}_\mu + \lambda^b{}_c\hat{\tau}^{ac}{}_\mu,
\end{aligned} \tag{9.16}$$

where $\bar{\nabla}$ denotes the O(4) covariant derivative (9.1) constructed from $\bar{\omega}^{ab}{}_\mu$.

Since no background split is introduced for the ghost fields, their true and background gauge transformations coincide. We require that under $\boldsymbol{\delta}^{\mathrm{Q}}_{\mathrm{D,L}}$ and $\boldsymbol{\delta}^{\mathrm{B}}_{\mathrm{D,L}}$ they transform tensorially according to their respective index structure.

Introducing Ward operators $\mathcal{W}^{\mathrm{B}}_{\mathrm{D}}$, $\mathcal{W}^{\mathrm{B}}_{\mathrm{L}}$ for the background gauge transformations, and $\mathcal{W}^{\mathrm{Q}}_{\mathrm{D}}$, $\mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}$ for the "quantum" or "true" ones, we can verify that the former satisfy the algebra

$$\boxed{\begin{aligned} \big[\mathcal{W}^{\mathrm{B}}_{\mathrm{D}}(w_1), \mathcal{W}^{\mathrm{B}}_{\mathrm{D}}(w_2)\big] &= \mathcal{W}^{\mathrm{B}}_{\mathrm{D}}([w_1, w_2]), \\ \big[\mathcal{W}^{\mathrm{B}}_{\mathrm{L}}(\lambda_1), \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}(\lambda_2)\big] &= \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}([\lambda_1, \lambda_2]), \\ \big[\mathcal{W}^{\mathrm{B}}_{\mathrm{D}}(w), \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}(\lambda)\big] &= \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}(\mathcal{L}_w \lambda), \end{aligned}} \tag{9.17}$$

while the latter obey the relations

$$\boxed{\begin{aligned} \big[\mathcal{W}^{\mathrm{Q}}_{\mathrm{D}}(w_1), \mathcal{W}^{\mathrm{Q}}_{\mathrm{D}}(w_2)\big] &= \mathcal{W}^{\mathrm{Q}}_{\mathrm{D}}([w_1, w_2]), \\ \big[\mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(\lambda_1), \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(\lambda_2)\big] &= \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}([\lambda_1, \lambda_2]), \\ \big[\mathcal{W}^{\mathrm{Q}}_{\mathrm{D}}(w), \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(\lambda)\big] &= \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(\mathcal{L}_w \lambda). \end{aligned}} \tag{9.18}$$

Like their precursors before the background split, these commutation relations are not $\mathrm{O}(4)_{\mathrm{loc}}$ covariant. This time within the background field setting, we therefore define modified diffeomorphisms:

$$\begin{aligned} \widetilde{\widetilde{\boldsymbol{\delta}}}{}^{\mathrm{B}}_{\mathrm{D}}(w) &\equiv \boldsymbol{\delta}^{\mathrm{B}}_{\mathrm{D}}(w) + \boldsymbol{\delta}^{\mathrm{B}}_{\mathrm{L}}(w \cdot \bar{\omega}), \\ \widetilde{\widetilde{\boldsymbol{\delta}}}{}^{\mathrm{Q}}_{\mathrm{D}}(w) &\equiv \boldsymbol{\delta}^{\mathrm{Q}}_{\mathrm{D}}(w) + \boldsymbol{\delta}^{\mathrm{Q}}_{\mathrm{L}}(w \cdot \bar{\omega}). \end{aligned} \tag{9.19}$$

In terms of their Ward operators, the modified background diffeomorphisms are then seen to satisfy the commutation relations

$$\boxed{\begin{aligned} \big[\widetilde{\widetilde{\mathcal{W}}}{}^{\mathrm{B}}_{\mathrm{D}}(w_1), \widetilde{\widetilde{\mathcal{W}}}{}^{\mathrm{B}}_{\mathrm{D}}(w_2)\big] &= \widetilde{\widetilde{\mathcal{W}}}{}^{\mathrm{B}}_{\mathrm{D}}([w_1, w_2]) - \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}(w_1 w_2 \cdot \bar{F}), \\ \big[\mathcal{W}^{\mathrm{B}}_{\mathrm{L}}(\lambda_1), \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}(\lambda_2)\big] &= \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}([\lambda_1, \lambda_2]), \\ \big[\widetilde{\widetilde{\mathcal{W}}}{}^{\mathrm{B}}_{\mathrm{D}}(w), \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}(\lambda)\big] &= 0. \end{aligned}} \tag{9.20}$$

Their "quantum" counterparts have the following Lie algebra relations:

$$\boxed{\begin{aligned} \big[\widetilde{\widetilde{\mathcal{W}}}{}^{\mathrm{Q}}_{\mathrm{D}}(w_1), \widetilde{\widetilde{\mathcal{W}}}{}^{\mathrm{Q}}_{\mathrm{D}}(w_2)\big] &= \widetilde{\widetilde{\mathcal{W}}}{}^{\mathrm{Q}}_{\mathrm{D}}([w_1, w_2]) + \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(w_1 w_2 \cdot \bar{F}), \\ \big[\mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(\lambda_1), \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(\lambda_2)\big] &= \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}([\lambda_1, \lambda_2]), \\ \big[\widetilde{\widetilde{\mathcal{W}}}{}^{\mathrm{Q}}_{\mathrm{D}}(w), \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(\lambda)\big] &= \mathcal{W}^{\mathrm{Q}}_{\mathrm{L}}(w \cdot \bar{\nabla} \lambda). \end{aligned}} \tag{9.21}$$

Both algebras, (9.20) and (9.21), are of immediate relevance to the functional RG equation: The "background" transformations constrain the theory space on which the flow takes place, while the algebra of the "quantum" transformations determines the ghost action [271].

(6) Let us choose the following convenient gauge conditions which are linear in $\hat{\varepsilon}^a{}_\mu$ and independent of $\hat{\tau}^{ab}{}_\mu$ [272]:

$$\begin{aligned}\mathcal{F}_\mu &= \bar{e}_a{}^\nu \big[\bar{D}_\nu \hat{\varepsilon}^a{}_\mu + \beta_\mathrm{D} \bar{D}_\mu \hat{\varepsilon}^a{}_\nu\big],\\ \mathcal{G}^{ab} &= \frac{1}{2}\bar{g}^{\mu\nu}\big[\hat{\varepsilon}^a{}_\mu \bar{e}^b{}_\nu - \hat{\varepsilon}^b{}_\mu \bar{e}^a{}_\nu\big] \equiv \hat{\varepsilon}^{[ab]}.\end{aligned} \tag{9.22}$$

With the new parameter β_D there is a total of three gauge-fixing parameters now: α_D, α_L, and β_D. Using (9.22) in (9.12) we can verify that $S_\mathrm{gf}[\hat{\varepsilon}, \hat{\tau}; \bar{e}, \bar{\omega}]$ is indeed background gauge invariant:

$$\mathcal{W}^\mathrm{B}_\mathrm{L}\, S_\mathrm{gf} = 0 = \widetilde{\widetilde{\mathcal{W}^\mathrm{B}_\mathrm{D}}}\, S_\mathrm{gf} \quad \Leftrightarrow \quad \mathcal{W}^\mathrm{B}_\mathrm{L}\, S_\mathrm{gf} = 0 = \mathcal{W}^\mathrm{B}_\mathrm{D}\, S_\mathrm{gf}. \tag{9.23}$$

However, the ghost sector requires some care. We would like the ghost action $S_\mathrm{gh}[\hat{\varepsilon}, \hat{\tau}, \mathcal{C}, \bar{\mathcal{C}}, \Sigma, \bar{\Sigma}; \bar{e}, \bar{\omega}]$ to be background gauge invariant, too. However, straightforwardly applying the Faddeev–Popov procedure to the original transformations

$$\boldsymbol{\delta}^\mathrm{Q} = \begin{pmatrix} \boldsymbol{\delta}^\mathrm{Q}_\mathrm{D}(w) \\ \boldsymbol{\delta}^\mathrm{Q}_\mathrm{L}(\lambda) \end{pmatrix} \tag{9.24}$$

we obtain, in the $\bar{\Sigma} - \mathcal{C}$-sector, the ghost action[2]

$$S^{\bar{\Sigma}-\mathcal{C}}_\mathrm{gf}[\mathcal{C}, \bar{\Sigma}; \bar{e}, \bar{\omega}] = -\int \mathrm{d}^4x\, \bar{e}\, \left(\bar{\Sigma}_{ab} \frac{\partial \mathcal{G}^{ab}}{\partial \hat{\varepsilon}^c{}_\nu} \boldsymbol{\delta}^\mathrm{Q}_\mathrm{D}(\mathcal{C}) \hat{\varepsilon}^c{}_\nu \right)\Bigg|_{\hat{\varepsilon}=0}, \tag{9.25}$$

which, with (9.22), evaluates to

$$S^{\bar{\Sigma}-\mathcal{C}}_\mathrm{gf}[\mathcal{C}, \bar{\Sigma}; \bar{e}, \bar{\omega}] = -\int \mathrm{d}^4x\, \bar{e}\, \bar{\Sigma}_{ab}\, \bar{e}^{b\mu} \mathcal{L}_\mathcal{C} \bar{e}^a{}_\mu. \tag{9.26}$$

While this latter functional is invariant under background diffeomorphisms, as it should be, it fails to be invariant under the $\mathrm{O}(4)_\mathrm{loc}$ transformations $\boldsymbol{\delta}^\mathrm{B}_\mathrm{L}(\lambda)$. The reason is that the Lie derivative of an O(4) tensor does not define an O(4) tensor. Rather, we have $\mathcal{L}_\mathcal{C}(\lambda^a{}_b\, \bar{e}^b{}_\mu) \neq \lambda^a{}_b\, \mathcal{L}_\mathcal{C} \bar{e}^b{}_\mu$, since $\lambda^a{}_b(x)$ is a spacetime scalar that transforms non-trivially under diffeomorphisms. Stated differently, $\mathrm{O}(4)_\mathrm{loc}$ transformations and (ordinary) diffeomorphisms do not commute. This is exactly what the above algebra relations express.

[2] Here we already specialized to the case $\hat{\varepsilon}^a{}_\mu = 0$, which will be sufficient for all truncations of "single-field" type [258].

The way out of this difficulty consists in applying the Faddeev–Popov procedure to the $\mathrm{O}(4)_{\mathrm{loc}}$-*covariantized* (true) gauge transformations rather than the original ones:

$$\widetilde{\widetilde{\boldsymbol{\delta}}}^{\mathrm{Q}} = \begin{pmatrix} \widetilde{\widetilde{\boldsymbol{\delta}}}^{\mathrm{Q}}_{\mathrm{D}}(w) \\ \boldsymbol{\delta}^{\mathrm{Q}}_{\mathrm{L}}(\lambda) \end{pmatrix}. \tag{9.27}$$

They are gauge fixed by the 10 gauge conditions

$$\begin{pmatrix} \mathcal{F}_\mu \\ \mathcal{G}^{ab} \end{pmatrix} \equiv \left(\mathcal{Q}^I\right) \tag{9.28}$$

for which we use a uniform notation here, with $\left(\mathcal{Q}^I\right) \equiv \left(\mathcal{F}_\mu\right)$ for $I = 1, \cdots, 4$ and $\left(\mathcal{Q}^I\right) \equiv \left(\mathcal{G}^{ab}\right)$ for $I = 5, \cdots, 10$. Denoting, in the same fashion, the ten parameters of the gauge transformations as $\left(\Lambda^I\right) = \left(w^\mu, \lambda^a{}_b\right)$, the Faddeev–Popov determinant reads

$$\det\left(\frac{\delta \mathcal{Q}^I(x)}{\delta \Lambda^J(y)}\right)\bigg|_{\Lambda=0}. \tag{9.29}$$

Exponentiating it we obtain a ghost action which has the structure

$$-\int \mathrm{d}^4x\, \bar{e} \begin{pmatrix} \bar{\mathcal{C}}_\mu \\ \bar{\Sigma}_{ab} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \Omega^\mu{}_\nu & \Omega^\mu{}_{cd} \\ \Omega^{ab}{}_\nu & \Omega^{ab}{}_{cd} \end{pmatrix} \begin{pmatrix} \mathcal{C}^\nu \\ \Sigma^{cd} \end{pmatrix}. \tag{9.30}$$

The Faddeev–Popov operator Ω appearing in (9.30) is rather complicated; we must refer to [258] for its explicit form. However, it is important to note that one can now check that the ghost action (9.30) is indeed invariant under background gauge transformations:

$$\mathcal{W}^{\mathrm{B}}_{\mathrm{L}}\, S_{\mathrm{gh}} = 0 = \widetilde{\widetilde{\mathcal{W}}}^{\mathrm{B}}_{\mathrm{D}}\, S_{\mathrm{gh}} \quad \Leftrightarrow \quad \mathcal{W}^{\mathrm{B}}_{\mathrm{L}}\, S_{\mathrm{gh}} = 0 = \mathcal{W}^{\mathrm{B}}_{\mathrm{D}}\, S_{\mathrm{gh}}. \tag{9.31}$$

This property is a central prerequisite for ultimately arriving at a background gauge-invariant Effective Average Action.

(7) The functional integral (9.11) gives rise to the associated Effective Average Action [14] in the by now familiar way: one adds a $\boldsymbol{\delta}^{\mathrm{B}}$-invariant mode cutoff to the bare action, $\Delta S_k \propto \int \mathrm{d}^4x\, \bar{e}\, (\hat{\varepsilon}, \hat{\tau})\, \mathcal{R}_k\, (\hat{\varepsilon}, \hat{\tau})^{\mathrm{T}}$, couples $\hat{\varepsilon}$ and $\hat{\tau}$ to sources, Legendre transforms the resulting generating functional $\ln Z_k$, and finally subtracts ΔS_k for the expectation value fields in order to arrive at the running action:

$$\Gamma_k[\varepsilon, \tau, \xi, \bar{\xi}, \Upsilon, \bar{\Upsilon}; \bar{e}, \bar{\omega}] \equiv \Gamma_k[e, \omega, \bar{e}, \bar{\omega}, \xi, \bar{\xi}, \Upsilon, \bar{\Upsilon}]. \tag{9.32}$$

In the list of arguments the expectation value fields are $\varepsilon \equiv \langle \hat{\varepsilon} \rangle$, $\tau \equiv \langle \hat{\tau} \rangle$, and for the ghosts

$$\xi^\mu \equiv \langle \mathcal{C}^\mu \rangle,\ \ \bar{\xi}_\mu \equiv \langle \bar{\mathcal{C}}_\mu \rangle,\ \ \Upsilon^{ab} \equiv \langle \Sigma^{ab} \rangle,\ \ \bar{\Upsilon}_{ab} \equiv \langle \bar{\Sigma}_{ab} \rangle. \tag{9.33}$$

In addition,

$$e^a{}_\mu \equiv \langle \hat{e}^a{}_\mu \rangle = \bar{e}^a{}_\mu + \varepsilon^a{}_\mu, \quad \omega^{ab}{}_\mu \equiv \langle \hat{\omega}^{ab}{}_\mu \rangle = \bar{\omega}^{ab}{}_\mu + \tau^{ab}{}_\mu. \tag{9.34}$$

The Effective Average Action Γ_k may be considered a functional of either the fluctuations $\varepsilon^a{}_\mu$ and $\tau^{ab}{}_\mu$ or the complete classical fields $e^a{}_\mu$ and $\omega^{ab}{}_\mu$.

Obviously the action Γ_k is defined on a rather complicated theory space, $\mathcal{T}$. It consists of functionals depending on two independent vielbein variables $(e, \bar{e})$, two spin connections $(\omega, \bar{\omega})$, as well as on the diffeomorphism and O(4) ghosts and antighosts, respectively. The functionals $A[\cdot]$ in $\mathcal{T}$ are constrained by the requirement of background gauge invariance: $\mathcal{W}^{\rm B}_{\rm D}(w)\, A = 0 \wedge \mathcal{W}^{\rm B}_{\rm L}(\lambda)\, A = 0, \forall w^\mu, \lambda^a{}_b$. Additionally they must satisfy Ward identities for BRST and split-symmetry transformations, in analogy with the metric case.

(8) Given this theory space and the above definition of Γ_k, it is now straightforward to derive its FRGE. Again, it has the same structure as the previous flow equation:

$$k\partial_k \Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[(\Gamma^{(2)}_k + \mathcal{R}_k)^{-1} k\partial_k \mathcal{R}_k\right]. \tag{9.35}$$

With $\mathcal{R}_k[\bar{e}, \bar{\omega}]$ specified appropriately, the equation indeed defines a flow on $\mathcal{T}$, i.e., it does not generate background gauge invariance violating terms.

9.3 Truncations of Hilbert–Palatini Type

As a first concrete example [260, 261], the flow equation for $\Gamma_k[e, \omega, \cdots]$ has been solved on a three-dimensional truncated theory space spanned by actions of the generalized Hilbert–Palatini, i.e., Holst type:

$$\boxed{\begin{aligned}\Gamma_k[e,\omega,\cdots] = -\frac{1}{16\pi G_k}\int \mathrm{d}^4x\, e\,\Big[e_a{}^\mu e_b{}^\nu \Big(F^{ab}{}_{\mu\nu} - \frac{1}{\gamma_k}\star F^{ab}{}_{\mu\nu}\Big)\\ - 2\Lambda_k\Big] + S_{\rm gf} + S_{\rm gh}.\end{aligned}} \tag{9.36}$$

This ansatz consists of the Hilbert–Palatini action known from classical Einstein–Cartan gravity, plus the Immirzi term which only exists in four dimensions; in fact, $\star F^{ab}{}_{\mu\nu} \equiv \frac{1}{2}\varepsilon^{ab}{}_{cd} F^{cd}{}_{\mu\nu}$ is the dual of the curvature of ω, $F \equiv F(\omega)$, with respect to the frame indices.

Besides G_k, (9.36) contains two more running parameters: the cosmological constant Λ_k and the dimensionless Immirzi parameter γ_k. The gauge fixing and ghost terms are assumed to retain their classical form for all k, and the parameters $\alpha_{\rm D}$, $\alpha_{\rm L}$ and $\beta_{\rm D}$ are treated as constants. Thus locally the truncated theory space $\mathcal{T}$ can be coordinatized by triples (g, λ, γ) where $g_k \equiv k^2 G_k$ and $\lambda_k \equiv k^{-2}\Lambda_k$.

(1) The non-perturbative beta functions related to the ansatz (9.36) have been obtained for two different projection schemes in [260, 261]. The two computations differed also with respect to certain additional approximations. They have been applied on top of the truncation in order to cope with the formidable algebra behind the functional traces that must be evaluated.[3]

Focusing on the first scheme from now on, we can discuss its results only briefly here; for a detailed discussion we refer to [260], and to [258, 259] for shorter accounts.

The projected RG equations are of the form $\partial_t g_k = \beta_g \equiv (2+\eta_{\mathrm{N}})g_k$, $\partial_t \lambda_k = \beta_\lambda$, $\partial_t \gamma_k = \beta_\gamma$ with the anomalous dimension of Newton's constant and the other two beta functions given by:

$$\begin{aligned}
\eta_{\mathrm{N}}(g,\lambda,\gamma) &= 16\pi\, g\, f_+(\lambda,\gamma),\\
\beta_\gamma(g,\lambda,\gamma) &= 16\pi\, g\,\gamma\Big[\gamma\, f_-(\lambda,\gamma) - f_+(\lambda,\gamma)\Big],\\
\beta_\lambda(g,\lambda,\gamma) &= -2\,\lambda + 8\pi\, g\Big[2\,\lambda\, f_+(\lambda,\gamma) + f_3(\lambda,\gamma)\Big].
\end{aligned} \tag{9.37}$$

The functions $f_\pm$ and f_3 are extremely complicated and we must refer to [260] for their explicit form. Besides λ and γ, they depend parametrically also on the three gauge-fixing constants and an additional mass parameter $\bar{\mu}$ which is needed to give a uniform dimension to $\bar{\varepsilon}^a{}_\mu$ and $\bar{\tau}^{ab}{}_\mu$. Indeed, only after rescaling $\bar{\varepsilon}^a{}_\mu \to \bar{\mu}^{\frac{1}{2}}\,\bar{\varepsilon}^a{}_\mu$, $\bar{\tau}^{ab}{}_\mu \to \bar{\mu}^{-\frac{1}{2}}\,\bar{\tau}^{ab}{}_\mu$ the inverse propagator $\Gamma_k^{(2)}$ constitutes an operator with well-defined spectrum and trace.[4] The resulting RG flow is reflection symmetric under $\gamma \to -\gamma$.

(2) Being particularly interested in the regions near $\gamma = 0$ and $\gamma = \pm\infty$, we are led to coordinatize $\mathcal{T}$ by an *atlas consisting of two charts*.

In order to cover the neighborhood of the submanifold $\gamma = \pm\infty$ in $\mathcal{T}$, we introduce a new coordinate $\hat{\gamma}$. In the overlap $|\gamma| \in\,]0, +\infty[$ of the (g,λ,γ)- and the $(g,\lambda,\hat{\gamma})$-chart, the coordinates γ and $\hat{\gamma}$ are related by the *transition function* $\hat{\gamma}(\gamma) = \gamma^{-1}$. With $\beta_{\hat{\gamma}}(g,\lambda,\hat{\gamma}) = -\hat{\gamma}^2\,\beta_\gamma(g,\lambda,\hat{\gamma}^{-1})$, the flow equation in the $\hat{\gamma}$-chart has the structure

$$\begin{aligned}
\eta_{\mathrm{N}}(g,\lambda,\gamma) &= 16\pi\, g\, f_+(\lambda,\hat{\gamma}^{-1}),\\
\beta_{\hat{\gamma}}(g,\lambda,\hat{\gamma}) &= 16\pi\, g\,\hat{\gamma}\Big[f_+(\lambda,\hat{\gamma}^{-1}) - \hat{\gamma}^{-1}\, f_-(\lambda,\hat{\gamma}^{-1})\Big],\\
\beta_\lambda(g,\lambda,\hat{\gamma}) &= -2\,\lambda + 8\pi\, g\Big[2\,\lambda\, f_+(\lambda,\hat{\gamma}^{-1}) + f_3(\lambda,\hat{\gamma}^{-1})\Big].
\end{aligned} \tag{9.38}$$

[3] In [260] the proper time approximation of the FRGE has been used (as in [273] for metric gravity), while [261] employed a novel "Wegner-Houghton-like" functional RG equation which follows from (9.35) under certain assumptions.

[4] At a more abstract level, $\bar{\mu}$ characterizes a Riemannian metric in the *space of fields* which is necessary to set up the FRGE [274].

(3) The beta functions β_g, β_γ, $\beta_{\hat{\gamma}}$, and β_λ have simple poles at $\gamma = \hat{\gamma} = \pm 1$, presumably artifacts of the approximations employed. In fact, the analysis shows that for γ not too close to ± 1 the functions $f_\pm$ and f_3 are actually *independent* of γ. For such values of γ it is a rather precise approximation to replace them by functions $\tilde{f}_\pm$ and $\tilde{f}_3$ that only depend on λ:

$$
\begin{aligned}
\partial_t g_k &= \Big[2 + 16\pi\, g_k\, \tilde{f}_+(\lambda_k)\Big] g_k, \\
\partial_t \gamma_k &= 16\pi\, g_k\, \gamma_k \Big[\gamma_k\, \tilde{f}_-(\lambda_k) - \tilde{f}_+(\lambda_k)\Big], \\
\partial_t \lambda_k &= -2\,\lambda_k + 8\pi\, g_k \Big[2\,\lambda_k\, \tilde{f}_+(\lambda_k) + \tilde{f}_3(\lambda_k)\Big].
\end{aligned}
\tag{9.39}
$$

and likewise for the $\hat{\gamma}$-chart. While the equations (9.39) are manifestly equivalent to (9.37) when $|\gamma| \not\approx 1$, a detailed analysis [260] indicates that, somewhat surprisingly, for $|\gamma| \to 1$, too, the regular beta functions (9.39) rather than those of (9.37) are likely to apply.

(4) The singularities at $\gamma = \pm 1$ are a consequence of the fact that for these values of the Immirzi parameter the Holst action involves the *projection operator on (anti-)selfdual field strength tensors*, $P^\pm = \frac{1}{2}(1 \pm \star)$. It can be shown that the (anti-)selfdual projection of the field-strength tensor of a generic spin connection equals the field strength tensor of the (anti-)selfdual part of this spin connection: $(P^\pm F)^{ab}(\omega) = F^{ab}(P^\pm \omega)$. Hence, if $\gamma_k = \pm 1$, the (anti-)selfdual part of ω completely drops out from the action (9.36). However, the underlying functional integral still includes the integration over the decoupled projection. It is this divergence which is mirrored by the flow equation.

If one is interested in "chiral gravity" mediated by an (anti-)selfdual spin connection $\omega^\pm$ an independent calculation should be set up, beginning with a theory space of functionals $A[e, \omega^\pm]$ which depend on either a selfdual or an anti-selfdual connection. There is no functional integration over the discarded set of fields then, and the resulting RG equations and beta functions are well behaved [275].

(5) The system (9.39) and its analogue in the $\hat{\gamma}$-chart imply $\beta_\gamma = 0$ and $\beta_{\hat{\gamma}} = 0$ for $\gamma^* = 0$ and $\hat{\gamma}^* = 0$, respectively. For each of the two sets of equations we find a fixed point $\mathbf{NGFP_0} \equiv (g_0^*, \lambda_0^*, \gamma^*)$ and $\mathbf{NGFP_\infty} \equiv (g_\infty^*, \lambda_\infty^*, \hat{\gamma}^*)$ of (9.37), (9.38) with $g_{0,\infty}^* > 0$, but $\lambda_{0,\infty}^* < 0$ and $g_0^* \neq g_\infty^*$, $\lambda_0^* \neq \lambda_\infty^*$. Both of them seem eligible for the Asymptotic Safety construction.

At either fixed point, the g and λ directions are to a very good approximation eigendirections of the linearized flow on $\mathcal{T}$. This is even exactly true for the γ- and $\hat{\gamma}$-directions, respectively. At $\mathbf{NGFP_0}$ and $\mathbf{NGFP_\infty}$, both the g and λ directions are relevant scaling fields, i.e., their associated critical exponents θ_1 and θ_2 are real and positive. In contrast, at $\mathbf{NGFP_0}$ the Immirzi parameter γ is irrelevant ($\theta_\gamma < 0$), whereas at $\mathbf{NGFP_\infty}$ its inverse $\hat{\gamma}$ is relevant ($\theta_{\hat{\gamma}} > 0$).

(6) The existence of the two non-Gaussian fixed points in the new "universality class" based on the elementary fields e and ω is a significant result clearly, a first hint at the viability of the Asymptotic Safety program in Einstein–Cartan gravity.

As we pointed out already, their existence is logically and computationally independent of, and not implied by, any known properties of the metric theory.

Defining $\mathcal{I} \equiv \frac{1}{16\pi G_k} \int \mathrm{d}^4x \, e \, \varepsilon^{\mu\nu\rho\sigma} T^a{}_{\mu\nu} T^b{}_{\rho\sigma} \delta_{ab}$, the Immirzi term appears in the functional integral as $\exp\bigl\{-\frac{1}{\gamma} \cdot \mathcal{I} + \text{surface term}\bigr\}$. Therefore, for $\gamma \to 0^+$, configurations with $\mathcal{I} > 0$ get strongly suppressed whereas those with $\mathcal{I} < 0$ are enhanced. For $\gamma \to 0^-$, the situation is just reversed. These two cases are related by parity, and neither of them leads to a *complete* suppression of torsion. Hence there is no general reason for metric gravity to be recovered at any value of γ.

(7) By letting $\lambda = 0$ we obtain the (g, γ)- *and* $(g, \hat{\gamma})$*-subtruncation*, respectively, with β_g, β_γ, and $\beta_{\hat{\gamma}}$ given by (9.37) and (9.38), but with the functions $f_\pm$ evaluated at $\lambda = 0$. In this case the results are compatible with $\beta_\gamma = 0 = \beta_{\hat{\gamma}} \Leftrightarrow f_+(0,\gamma)|_{\gamma=0,\,\pm\infty} = \bigl(\gamma \, f_-(0,\gamma)\bigr)|_{\gamma=0,\,\pm\infty}$. As a consequence, the renormalization of the Immirzi term is only due to the running of the overall prefactor $1/G_k$. This result needs to be corroborated by a more precise treatment. If correct, the Immirzi parameter owes its RG running to a non-zero cosmological constant in an essential way.

(8) Within the $\lambda = 0$-subtruncations there exists an intriguing *duality relating small and large values of the Immirzi parameter* [260]. The RG flow is then invariant under the mapping $\gamma \mapsto 1/\gamma$. A non-zero value of the cosmological constant breaks this invariance, however.

(9) Summarizing the results obtained with both the first [260] and the second projection scheme [261] it can be said that there is indeed non-trivial evidence for the Asymptotic Safety of pure gravity in the Einstein–Cartan setting. There seem to exist at least two NGFPs, located at $\gamma = 0$ and $\gamma = \pm\infty$, respectively, which are suitable for taking the continuum limit there. Interestingly, no reliable fixed point with a finite, non-zero value of γ is found.

The stability and robustness properties of the Einstein–Cartan results turned out somewhat less pronounced than those of comparable truncations in metric gravity. The reason is probably the enlargement of the gauge sector by the local Lorentz transformations. In fact, similar observations were made in "tetrad-only gravity" whose theory space consists of functionals $A[e^a{}_\mu]$, which are also $\mathrm{Diff}(\mathcal{M}) \ltimes \mathrm{O}(4)_{\mathrm{loc}}$ invariant [276].[5]

[5] See also [277, 278] for a perturbative investigation of Einstein–Cartan gravity in the special case $\lambda = 0$.

(10) Besides the space of Einstein–Cartan-type actions $A[e, \omega]$ a number of related theory spaces, based on other field variables, have been scrutinized for the possibility of asymptotically safe theories.

This includes in particular *(anti-)selfdual gravity*. It employs actions $A[e, \omega^{(\pm)}]$ depending on the vierbein field along with an O(4)-valued spin connection, which is required to be self-dual (or alternatively, anti-selfdual) with respect to its frame indices [275].

This theory space is closely related to the Euclidean analog of *Ashtekar's variables* [36, 37, 257, 279].[6] In fact, Ashtekar showed that if one starts from the classical Einstein–Cartan theory and performs a suitably chosen canonical transformation to new variables, the pertinent spin connection can be chosen as self-dual (or anti-selfdual). For the Lorentzian signature and structure group $\mathrm{O}(1,3)$ self-dual connections are necessarily complex, which complicates their quantization. In the Euclidean case they are real, however, and the condition of self-duality precisely halves the number of the connection's independent components.

Within the Euclidean setting, a variant of the calculations described above led to the conclusion that *self-dual gravity is also likely to be asymptotically safe* [275].

Similar results were also obtained in studies where the metric and the torsion tensor $T^{\lambda}{}_{\mu\nu}$ were considered the independent field variables of the "Holst truncation" [274], and in a more general truncation, comprising all invariants quadratic in the torsion and non-metricity tensor [281].

9.4 Quantum Arnowitt–Deser–Misner Gravity

An alternative set of fields carrying the gravitational degrees of freedom arises from the Arnowitt–Deser–Misner (ADM) or $(3+1)$-formulation of gravity [256, 282], see [283] for a pedagogical introduction.

This construction equips spacetime with a *foliation structure* by welding together spatial slices on which the time coordinate τ is constant. This introduces a preferred direction which may be interpreted as an (Euclidean) time direction and provides a causal structure. As a byproduct it gives access to interesting observables comprising, e.g., the expectation value for spatial volumes $V_3(\tau)$ as a function of time. This facilitates the direct comparison with results obtained from the Causal Dynamical Triangulations program [244] which evaluates the gravitational partition sum by Monte Carlo methods.

In this section we review the construction of the corresponding FRGE and highlight the connections to the covariant formulation of Quantum Einstein Gravity.

[6] See also [280] for the relation between self-duality and helicity.

(1) At the classical level, the ADM formulation of gravity starts from a d-dimensional (Euclidean) spacetime $\mathcal{M}$ with metric $g_{\mu\nu}$ carrying coordinates x^μ. Subsequently, the construction introduces a time function $\tau(x)$ which assigns a specific time to each spacetime point. The points with the same value of $\tau(x)$ form D-dimensional spatial slices $\Sigma_\tau \equiv \{x : \tau(x) = \tau\}$. Here $D \equiv d-1$.

The gradient of the time function, $n_\mu \equiv N\partial_\mu \tau(x)$, defines a vector n^μ normal to the spatial slices. The *lapse function* $N(x)$ ensures that $g_{\mu\nu}n^\mu n^\nu = 1$. One then introduces a new coordinate system $x^\mu \mapsto (\tau, y^i)$, $i = 1, \cdots, D$ where the y^i provide coordinates on Σ_τ.[7] Defining the vector field t^μ through the relation $t^\mu \partial_\mu \tau = 1$, the coordinate systems on neighboring spatial slices are related by requiring that the coordinates y^i are constant along the integral curves of t^μ.

The tangent space at a point in $\mathcal{M}$ can then be decomposed into a subspace spanned by vectors tangent to Σ_τ and its complement. The corresponding basis vectors are constructed from the Jacobians

$$t^\mu = \left.\frac{\partial x^\mu}{\partial \tau}\right|_{y^i}, \quad {e_i}^\mu = \left.\frac{\partial x^\mu}{\partial y^i}\right|_\tau. \tag{9.40}$$

The normal vector thus satisfies $g_{\mu\nu}\, n^\mu\, {e_i}^\nu = 0$, and the metric induced on the spatial slices is given by

$$\sigma_{ij}(\tau, y) \equiv {e_i}^\mu\, {e_j}^\nu\, g_{\mu\nu}. \tag{9.41}$$

In general t^μ is neither tangent nor normal to the spatial slices. The Jacobians (9.40) imply that its decomposition into components normal and tangent to Σ is given by

$$t^\mu = N\, n^\mu + N^i\, {e_i}^\mu, \tag{9.42}$$

where $N^i(\tau, y)$ is called the *shift vector*. Furthermore, (9.40) implies that the coordinate one-forms in the two coordinate systems are related by

$$\mathrm{d}x^\mu = t^\mu \mathrm{d}\tau + {e_i}^\mu \mathrm{d}y^i = N n^\mu \mathrm{d}\tau + {e_i}^\mu\, (\mathrm{d}y^i + N^i \mathrm{d}\tau). \tag{9.43}$$

Combining the relations (9.42) and (9.43) with the normal property of n^μ and the definition of σ_{ij}, the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ can be recast in terms of the *ADM fields* $\{N, N_i, \sigma_{ij}\}$:

$$\boxed{\mathrm{d}s^2 = N^2 \mathrm{d}\tau^2 + \sigma_{ij}\, (\mathrm{d}y^i + N^i \mathrm{d}\tau)(\mathrm{d}y^j + N^j \mathrm{d}\tau).} \tag{9.44}$$

For the components of the metric tensor the decomposition (9.44) implies the relations

$$g_{\mu\nu} = \begin{pmatrix} N^2 + N_i N^i & N_j \\ N_i & \sigma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^j}{N^2} \\ -\frac{N^i}{N^2} & \sigma^{ij} + \frac{N^i N^j}{N^2} \end{pmatrix}, \tag{9.45}$$

[7] In this section Greek letters are used for spacetime indices while tangent indices with respect to the spatial slices are denoted by Latin letters $i, j, \cdots$.

where spatial indices i, j are raised and lowered with σ_{ij}. This entails that the relation between $g_{\mu\nu}$ and the ADM fields is actually non-linear.

(2) An infinitesimal coordinate transformation $v^\mu(\tau, y)$ acting on the metric can be expressed in terms of the Lie derivative, $\delta g_{\mu\nu} = \mathcal{L}_v\, g_{\mu\nu}$. Decomposing $v^\mu \equiv \big(f(\tau, y), \zeta^i(\tau, y)\big)$ into its temporal and spatial parts, this transformation determines the transformation properties of the component fields under $\mathrm{Diff}(\mathcal{M})$:

$$\begin{aligned}
\delta N &= \partial_\tau(fN) + \zeta^k \partial_k N - N N^i \partial_i f, \\
\delta N_i &= \partial_\tau(N_i f) + \zeta^k \partial_k N_i + N_k \partial_i \zeta^k + \sigma_{ki} \partial_\tau \zeta^k + N_k N^k \partial_i f + N^2 \partial_i f, \\
\delta \sigma_{ij} &= f \partial_\tau \sigma_{ij} + \zeta^k \partial_k \sigma_{ij} + \sigma_{jk} \partial_i \zeta^k + \sigma_{ik} \partial_j \zeta^k + N_j \partial_i f + N_i \partial_j f.
\end{aligned} \tag{9.46}$$

Thus, at the level of the component fields the action of $\mathrm{Diff}(\mathcal{M})$ is non-linear. Notably, $\mathrm{Diff}(\mathcal{M})$ contains the subgroup of *foliation preserving diffeomorphisms*, $\mathrm{Diff}(\mathcal{M}, \Sigma)$, where $f = f(\tau)$ is restricted to be independent of the spatial coordinates. Inspecting (9.46), one observes that the restriction $f(\tau, y) \to f(\tau)$ eliminates all non-linear terms from the transformation laws so that the component fields transform linearly with respect to this subgroup.

(3) The construction of the FRGE tailored to the ADM formalism again builds on the background field formalism. In this case the gravitational degrees of freedom are encoded in the lapse function $N(\tau, y)$, the shift vector $N_i(\tau, y)$, and the metric on the spatial slices $\sigma_{ij}(\tau, y)$.

Following the metric construction, these fields are decomposed into fixed but arbitrary backgrounds (marked with bars) and fluctuations (marked with hats). The *linear ADM split* uses

$$N = \bar{N} + \widehat{N}, \quad N_i = \bar{N}_i + \widehat{N}_i, \quad \sigma_{ij} = \bar{\sigma}_{ij} + \widehat{\sigma}_{ij}. \tag{9.47}$$

For convenience, we denote the collection of physical fields, background fields, and the fluctuations by χ, $\bar{\chi}$, and $\widehat{\chi}$, respectively, and set $\widehat{\sigma} \equiv \bar{\sigma}^{ij} \widehat{\sigma}_{ij}$. The linear split of σ_{ij} can be replaced by

$$\sigma_{ij} = \bar{\sigma}_{ik} \left[e^{\widehat{\sigma}} \right]^k{}_j, \tag{9.48}$$

which defines the *exponential ADM split.*

The properties of these choices are conveniently discussed by writing the determinant of the spacetime metric in terms of the ADM fields:

$$\det g = N^2 \, \det \sigma. \tag{9.49}$$

Combining this relation with the decomposition (9.47) one concludes that the fluctuations can not change the signature in the (Euclidean) time direction. The exponential ADM split furthermore fixes the signature of the metric on the spatial slices.

(4) At this stage it is useful to note that there exists a local map relating the fluctuation fields of the covariant and the ADM formulation. This map allows us to understand the ADM construction as a particular background-dependent redefinition of the fluctuation fields.

The explicit relation between the two settings is found by starting from the linear split in the metric setting, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, and carrying out an ADM decomposition of both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ according to (9.45). The result allows us to express the components of $h_{\mu\nu}$ in terms of $\widehat{N}, \widehat{N}_i$, and $\widehat{\sigma}_{ij}$ and their background values:

$$\begin{aligned} h_{00} &= 2\bar{N}\widehat{N} + \widehat{N}^2 + \sigma^{ij}(\bar{N}_i + \widehat{N}_i)(\bar{N}_j + \widehat{N}_j) - \bar{\sigma}^{ij}\bar{N}_i\bar{N}_j, \\ h_{0i} &= \widehat{N}_i, \\ h_{ij} &= \widehat{\sigma}_{ij}. \end{aligned} \tag{9.50}$$

Note that the resulting map is again non-linear. Due to the presence of σ^{ij} in h_{00} it involves the spatial fluctuations $\widehat{\sigma}_{ij}$ to arbitrary high orders. At the linear level the map (9.50) boils down to

$$h_{00} \approx 2\bar{N}\widehat{N} - \widehat{\sigma}^{ij}\bar{N}_i\bar{N}_j + 2\,\bar{\sigma}^{ij}\widehat{N}_i\bar{N}_j, \quad h_{0i} = \widehat{N}_i, \quad h_{ij} = \widehat{\sigma}_{ij}. \tag{9.51}$$

(5) The construction of the FRGE for the Effective Average Action Γ_k then proceeds along the lines of Chapter 2 and gives [284]

$$\partial_t \Gamma_k[\widehat{\chi};\bar{\chi}] = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\partial_t \mathcal{R}_k\right]. \tag{9.52}$$

The Hessian $\Gamma_k^{(2)}$ now comprises the second functional derivative of Γ_k with respect to all fluctuation fields appearing in the ADM formulation.

A further difference is that the background gauge transformations preserved by the flow equation (9.52) are actually not the full background diffeomorphism group *but its foliation preserving subgroup only.*

This feature results from the requirement that the k-dependent mass term ΔS_k related to the regulator $\mathcal{R}_k$ must be quadratic in the fluctuation fields in order to obtain a FRGE of the standard form. Thus, the regulator term is invariant only under linear transformations acting on the fluctuation fields. At the same time (9.46) indicates that the ADM fields transform non-linearly under diffeomorphisms. Therefore, (9.52) is invariant under foliation preserving background transformations only.[8]

[8] In principle the map (9.50) may be used to construct a regulator ΔS_k preserving full background diffeomorphism invariance. At the level of the ADM fields this leads to the infinite-order problem discussed in Section 4.2.

9.5 Quantum ADM Gravity in the Einstein–Hilbert Truncation

We illustrate the working of the FRGE by approximating the gravitational part of Γ_k by the (Euclidean) Einstein–Hilbert action. In terms of the ADM fields this implies

$$\Gamma_k^{\rm grav} = \frac{1}{16\pi G_k} \int {\rm d}\tau {\rm d}^D y \, N\sqrt{\sigma} \left[K_{ij}K^{ij} - K^2 - {}^{(D)}R + 2\Lambda_k \right], \tag{9.53}$$

where the extrinsic curvature is defined as

$$K_{ij} \equiv \frac{1}{2N}\Big(\partial_\tau \sigma_{ij} - D_i N_j - D_j N_i\Big), \quad K \equiv \sigma^{ij} K_{ij}, \tag{9.54}$$

and ${}^{(D)}R$ and D_i are, respectively, the Ricci scalar and covariant derivative constructed from σ_{ij}. In order to lighten the notation we omit the prescript from the intrinsic curvature from now on, denoting $R \equiv {}^{(D)}R$.

The ansatz for $\Gamma_k^{\rm grav}$ comprises two scale-dependent couplings, the cosmological constant Λ_k and Newton's constant G_k. The beta functions encoding the scale dependence of these couplings may be obtained by substituting the ansatz into the FRGE and extracting suitable geometric terms on both sides. The form of (9.53) suggests evaluating the flow equation at the background level, i.e., setting $\widehat{\chi} = 0$ after taking the necessary functional derivatives. In this case, it then suffices to keep track of terms that are at most quadratic in the fluctuation fields as these are necessary to construct the Hessian $\Gamma_k^{(2)}$.

(1) Similarly to the metric computation in Chapter 5, the evaluation of the flow equation can be simplified by choosing a suitable background. In order to facilitate the comparison with the metric setting, it is useful to choose a class of backgrounds with topology $S^1 \times S^D$, with

$$\boxed{\bar{N}(\tau, y) = 1, \quad \bar{N}_i(\tau, y) = 0, \quad \bar{\sigma}_{ij}(\tau, y) = \bar{\sigma}_{ij}^{S^D}(y).} \tag{9.55}$$

Here $\bar{\sigma}_{ij}^{S^D}(y)$ denotes the metric on the D-sphere with radius r, which is taken to be independent of τ. At the geometric level this entails:

$$\bar{K}_{ij} = 0, \quad \bar{R}_{ijkl} = \tfrac{\bar{R}}{D(D-1)} \left(\bar{\sigma}_{ik}\bar{\sigma}_{jl} - \bar{\sigma}_{il}\bar{\sigma}_{jk}\right), \quad \bar{R}_{ij} = \tfrac{1}{D}\,\bar{\sigma}_{ij}\,\bar{R}. \tag{9.56}$$

The scale dependence of the cosmological constant and Newton's constant can be read off from the terms proportional to the background volume and the integrated intrinsic background curvature, respectively.

(2) A key obstacle in the actual construction of flows in the ADM formalism results from the lapse N and shift N_i which enter (9.53) as Lagrange multipliers. At the level of the FRGE, which is based on an off-shell formalism, this leads to a Hessian $\Gamma_k^{(2)}$ which is degenerate. Imposing the *proper-time gauge* that would fix N and N_i to their background values does not resolve this problem.

An interesting route for addressing this obstacle efficiently is to start from the background gauge-fixing procedure that we had used in the covariant formulation:

$$\Gamma_k^{\rm gf} = \frac{1}{32\pi G_k\,\alpha}\int {\rm d}^d x\sqrt{\bar g}\,\bar g^{\mu\nu}F_\mu F_\nu . \tag{9.57}$$

In the *generalized harmonic gauge* (4.54), F_μ is linear in the fluctuation field,

$$F_\mu = \bar D^\nu h_{\mu\nu} - \varpi\,\bar D_\mu h . \tag{9.58}$$

Here $h \equiv \bar g^{\mu\nu}h_{\mu\nu}$, and α, ϖ are free gauge parameters. The harmonic gauge is obtained by setting $\alpha = 1$, $\varpi = \frac{1}{2}$ while Landau-type gauges take the limit $\alpha \to 0$. By construction it is clear that all gauge-fixing terms in this class preserve background diffeomorphism invariance.

The analogous gauge conditions in the ADM formalism are obtained by substituting the map (9.50) into (9.58). At the linear level, and for the specific background (9.55), this results in

$$\boxed{\begin{aligned} F &= 2\partial_\tau \widehat N + \partial_i \widehat N^i - \varpi\,\partial_\tau\left(2\widehat N + \widehat\sigma\right), \\ F_i &= \partial_\tau \widehat N_i + \bar D^j\widehat\sigma_{ij} - \varpi\,\bar D_i\left(2\widehat N + \widehat\sigma\right), \end{aligned}} \tag{9.59}$$

where $F_\mu = (F, F_i)$ has been decomposed in a spatial and time component. The ghost action for this class of gauge fixings is again given by (4.55).

(3) At this stage it is instructive to combine the gravitational and gauge-fixing terms and write down the part of the action quadratic in the fluctuation fields on flat space, $\bar R = 0$. Abbreviating $\bar D^2 \equiv \bar\sigma^{ij}\bar D_i\bar D_j$ and setting $\alpha = 1$ this yields

$$\begin{aligned} &(32\pi G_k)\left(\tfrac{1}{2}\,\delta^2\Gamma_k^{\rm grav} + \Gamma_k^{\rm gf}\right) \\ &\quad = \int {\rm d}\tau {\rm d}^d y\,\sqrt{\bar\sigma}\,\Big\{\tfrac{1}{2}\widehat\sigma_{ij}\left[-\partial_\tau^2 - \bar D^2 - 2\Lambda_k\right]\widehat\sigma^{ij} + \widehat N^i\left[-\partial_\tau^2 - \bar D^2\right]\widehat N_i \\ &\qquad - \tfrac{1}{2}\widehat\sigma\left[(1-2\varpi^2)(-\partial_\tau^2 - \bar D^2) - \Lambda_k\right]\widehat\sigma \\ &\qquad + 4\widehat N\left[(1-\varpi)^2(-\partial_\tau^2) - \varpi^2\bar D^2\right]\widehat N \\ &\qquad - 2\widehat N\left[2\varpi(1-\varpi)(-\partial_\tau^2) - (1-2\varpi^2)\bar D^2 - \Lambda_k\right]\widehat\sigma \\ &\qquad - (1-2\varpi)(2\widehat N + \widehat\sigma)\left(\bar D_i\bar D_j\widehat\sigma^{ij} + 2\partial_\tau\bar D_i\widehat N^i\right)\Big\}. \end{aligned} \tag{9.60}$$

This shows that there is a *unique gauge choice*, corresponding to the harmonic gauge $\varpi = \frac{1}{2}$, where all derivatives combine into the d-dimensional Laplacian on flat space. We will adhere to this choice in the sequel.

(4) The beta functions governing the scale dependence of G_k and Λ_k are obtained by restoring the intrinsic curvature terms in (9.60) and evaluating the resulting operator traces via standard heat kernel techniques. The result has been obtained in [285] and is conveniently expressed in terms of the dimensionless couplings $\lambda_k \equiv \Lambda_k k^{-2}$ and $g_k = G_k k^{d-2}$. Restricting to $d = 3+1$ dimensions and the optimized regulator (5.120) it reads

$$\partial_t \lambda_k = \beta_\lambda(g_k, \lambda_k), \quad \partial_t g_k = \beta_g(g_k, \lambda_k), \tag{9.61}$$

where the beta functions are given by

$$\begin{aligned} \beta_g(g,\lambda) &= (2+\eta_{\rm N})\, g, \\ \beta_\lambda(g,\lambda) &= (\eta_{\rm N}-2)\,\lambda - \frac{g}{24\pi}\left(30 + 3\eta_{\rm N} - \frac{36-6\eta_{\rm N}}{1-2\lambda} - \frac{12-2\eta_{\rm N}}{2-3\lambda}\right), \end{aligned} \tag{9.62}$$

and the anomalous dimension of Newton's coupling is

$$\eta_{\rm N} = \frac{g B_1(\lambda)}{1 - g B_2(\lambda)}. \tag{9.63}$$

The functions B_1 and B_2 depend only on λ and read

$$\begin{aligned} B_1 &= \frac{1}{6\pi}\left(-9 - \frac{8}{1-2\lambda} + \frac{2}{2-3\lambda} - \frac{2}{(1-2\lambda)^2} - \frac{9}{(2-3\lambda)^2}\right), \\ B_2 &= -\frac{1}{72\pi}\left(11 - \frac{24}{1-2\lambda} + \frac{6}{2-3\lambda} - \frac{4}{(1-2\lambda)^2} - \frac{18}{(2-3\lambda)^2}\right). \end{aligned} \tag{9.64}$$

(5) Clearly, the most important feature of the beta functions are their fixed points where $\beta_\lambda(g_*, \lambda_*) = 0$, $\beta_g(g_*, \lambda_*) = 0$. The GFP is located at the origin, $(\lambda_*, g_*) = (0,0)$, and its stability coefficients are given by the classical mass dimension of the couplings. In addition, there exists a unique NGFP,

$$\text{NGFP}^{\text{ADM}}: \quad g_* = 0.901, \quad \lambda_* = 0.222, \quad g_*\lambda_* = 0.200. \tag{9.65}$$

which comes with a complex pair of critical exponents,

$$\text{NGFP}^{\text{ADM}}: \quad \theta_{1,2} = 1.432 \pm 2.586 i, \tag{9.66}$$

indicating that it acts as a spiraling UV attractor for the RG trajectories in its vicinity. (This is the same characteristic behavior of the NGFP found when evaluating the RG flow on foliated spacetimes using the Matsubara formalism [284, 286] and on Friedmann–Lemaître–Robertson–Walker backgrounds [287, 288].)

It is interesting to compare the properties of this NGFP to the one seen in QEG. Evaluating the fixed point equations based on (5.63) using the same gauge fixing and the same cutoff shape function gives

$$\text{NGFP}^{\text{QEG}}: \quad g_* = 0.707, \quad \lambda_* = 0.193, \quad g_*\lambda_* = 0.137, \tag{9.67}$$

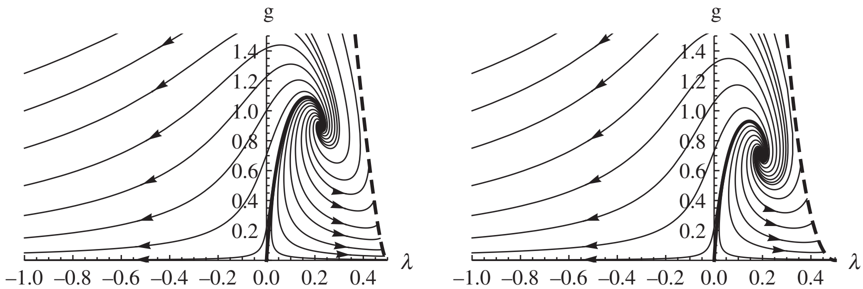


Figure 9.1. Phase portraits obtained by integrating the RG equations from Quantum ADM Gravity (left) and QEG (right) in the Einstein–Hilbert truncation using the optimized regulator and harmonic gauge.

and the critical exponents

$$\mathrm{NGFP}^{\mathrm{QEG}}: \quad \theta_{1,2} = 1.475 \pm 3.043i. \tag{9.68}$$

Thus, the fixed point properties in the two cases are strikingly similar. The difference arises from the non-linearity of the map (9.50), which gives rise to additional terms in $\Gamma_k^{(2)}$.

(6) The full phase diagram is obtained by integrating the system (9.61) numerically. It is displayed in the left diagram of Figure 9.1. The phase diagram resulting from the QEG beta functions using the same gauge fixing and regulator profile is shown in the right diagram for comparison.

In both cases the structure of the flow in the physically interesting region is determined by three features: A singular line associated with a divergence of the anomalous dimension η_{N} (thick dashed line) bounds the region to the right. The flow to the left of this line is governed by the interplay of the NGFP and the GFP. Flows in the vicinity of the GFP are characterized by $g_k \ll 1$ and exhibit classical properties in the sense that the dimensionful couplings G_k and Λ_k are (approximately) k-independent in this regime.

Depending on whether the trajectories flow to the left (right) of the separation line (bold) the value of Λ_k in this classical regime is negative (positive). Following the classification introduced in Section 5.3.4, these classes of solutions have been termed Type Ia and Type IIIa, respectively. The single trajectory ending at the GFP gives rise to a vanishing IR value of the cosmological constant and is referred to as Type IIa.

The diagram also illustrates that the NGFP of Quantum ADM Gravity is appropriate for providing the high-energy completion of the RG trajectories leaving the classical regime for increasing k.

(7) Two of the most important results provided by the FRGE tailored to the ADM formalism may be summarized as follows:

- **(i)** Initial works on functional renormalization group flows in the ADM formalism [284, 286] used a background $S^1 \times S^d$ and proper time gauge, fixing the lapse function and shift vector to their background values. The radius of S^d is taken to be time independent so that the (Euclidean) time direction associated with the S^1 constitutes a global Killing vector field. This feature allows us to analytically continue the flow equation to Lorentzian signature. The fluctuations along the time direction can then be taken into account through Matsubara sums, which, in contrast to the heat kernel computations, can be carried out *in both Euclidean and Lorentzian signature.* This shows that *the NGFP known from Euclidean signature computations persists for Lorentzian signature* [286].
- **(ii)** The analysis of ADM-based RG flows on flat Friedmann–Lemaître–Robertson–Walker backgrounds [287, 288] supports the existence of a NGFP suitable for Asymptotic Safety. Supplementing the gravitational sector by minimally coupled (non-selfinteracting) matter fields [287] showed that many gravity-matter systems, including the matter content of the standard model of particle physics, actually possess NGFPs as well. Depending on the details of the matter sector the spiraling UV attractor seen in the pure-gravity case is turned into a UV attractor with real critical exponents. The overall picture obtained from the ADM framework thereby resembles the one seen within the covariant approach [289–291] if the same approximations and coarse-graining schemes are employed.

9.6 Observables in Quantum ADM Gravity

A primary motivation for considering the FRGE in the presence of a foliation structure is its close relation to the Causal Dynamical Triangulation (CDT) program [244].

The latter regularizes the gravitational partition function by introducing piecewise linear building blocks and evaluates the resulting sum using Monte Carlo methods. The presence of a causal structure is thereby essential for obtaining structures resembling a macroscopic spacetime from these elementary building blocks [44, 46]. Since the simulations are capable of tracking a finite number of building blocks only, CDT geometries have periodic boundary conditions in the time direction and are necessarily compact. Their topology is fixed to either $S^1 \times S^3$ or $S^1 \times T^3$, where T^3 denotes the flat three-torus.

(1) A quantity that lends itself to a direct comparison of the FRG and CDT approach is the profile of the physical volumes, $V_3(\tau)$, as a function of Euclidean time τ. From the FRGE perspective, this information can be obtained by solving the equations of motion arising from the Effective Average Action Γ_k. Utilizing that the ansatz (9.53) is nothing but the Einstein–Hilbert action, the equations determining the self-consistent backgrounds are Einstein's equations in the presence of a k-dependent cosmological constant:

$$\left.\frac{\delta\Gamma_k[\hat{\chi};\bar{\chi}]}{\delta\hat{\chi}}\right|_{\hat{\chi}=0} = 0 \Longleftrightarrow R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda_k = 0. \tag{9.69}$$

(2) We first focus on the solutions of this equation for RG trajectories of **Type IIIa** where $\Lambda_k > 0$ throughout. In this case, the solutions are given by four-spheres:

$$\mathrm{d}s^2 = r^2\, \mathrm{d}\tau^2 + r^2 \sin^2(\tau) \left(\mathrm{d}\psi^2 + \sin^2\psi\, \mathrm{d}\theta^2 + \sin^2\psi\, \sin^2\theta\, \mathrm{d}\phi^2\right), \tag{9.70}$$

with radius $r^2 = \frac{3}{\Lambda_k}$. The time variable τ associated with the foliation takes values $\tau \in [0, \pi)$, mimicking the periodic boundary conditions encountered in the CDT framework.

The profile for the spatial volumes $V_3(\tau)$ is obtained by integrating (9.70) over the compact slices where τ is constant,

$$\boxed{V_3(\tau) = 2\pi^2 \left(\frac{3}{\Lambda_k}\right)^{3/2} \sin^3(\tau).} \tag{9.71}$$

Thus $V_3(\tau)$ possesses the $\sin^3(\tau)$-profile characteristic for Euclidean de Sitter space. In the classical regime, where $\Lambda_k \simeq \Lambda_0$ is k-independent, $V_3(\tau)$ inherits this property. The volume is then set by Λ_0 and can be used to identify the underlying RG trajectory.

Typical volume profiles obtained as a self-consistent solution of the Effective Average Action in the classical regime and from the CDT program are shown in Figure 9.2. In both cases the τ-dependence of the volume distribution follows the $\sin^3(\tau)$-profile.

(3) The RG trajectory of **Type IIa**, connecting the NGFP to the GFP, plays a special role in the phase diagram shown in Figure 9.1. In this case the value of the cosmological constant vanishes in the infrared, $\Lambda_0 = 0$. The self-consistent (compact) solutions of (9.69) then have the topology $S^1 \times T^3$ and the spatial slices are given by three-tori with arbitrary, time-independent volume $V_3(\tau) = V_3 = \text{const}$.

This solution together with the volume profiles measured on CDT configurations with topology $S^1 \times T^3$ are also shown in Figure 9.2. While in CDT the total volume of the configuration is fixed by the number of simplices, the value V_3 determined from the Effective Average Action is a free parameter which can be adjusted to match the CDT result.

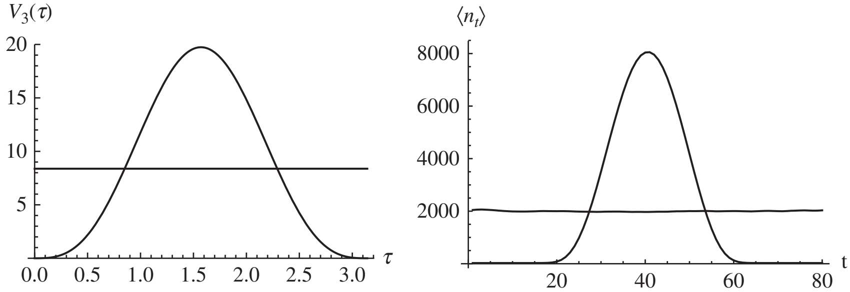


Figure 9.2. Volume profiles obtained from solving the self-consistency conditions of the Effective Average Action (left) and from the CDT simulations [292–294] (right). The CDT results are obtained from simulations using topologies $S^1 \times T^3$ (horizontal line) and $S^1 \times S^3$ (peaked line) and show the expectation value $\langle n_t \rangle$ of the number of simplices found in a spatial slice. Their FRGE counterparts result from selecting trajectories of Type IIa (horizontal line) and Type IIIa (peaked line), respectively.

Following up on the comparison of spectral dimensions [295], the matching of the long-distance properties arising from the microscopic quantum descriptions of spacetime constitutes a prototypical example for connecting the FRG and CDT frameworks in a systematic way.

10

Matter Coupled to Quantum Gravity

The previous chapters focused on Asymptotic Safety in the context of pure gravity. But a realistic description of our world should also include matter degrees of freedom. At the level of the FRGE these can easily be incorporated by supplementing the theory space of pure gravity by the fields carrying the matter degrees of freedom. The UV completion based on a non-Gaussian fixed point is not necessarily limited to gravity, and it is conceivable that the Asymptotic Safety mechanism is also realized in suitable gravity-matter theories. Initiated in [289, 290, 296] the systematic study of asymptotically safe gravity-matter systems is still in its infancy and we limit the discussion to the basic ideas and refer to [297] for a recent overview.

10.1 Free Matter Fields in the Einstein–Hilbert Truncation

A natural starting point studies the effect of adding free matter fields on the flow of Newton's constant and the cosmological constant. In this case the gravitational sector of the Effective Average Action is provided by the Einstein–Hilbert truncation (5.18). The truncation ansatz is supplemented by $N_{\rm S}$ scalar fields ϕ^i, $N_{\rm V}$ abelian vector fields A^i_μ, and $N_{\rm D}$ Dirac spinors ψ^i, all of them non-interacting among themselves but with a minimal coupling to gravity via the scale-independent action

$$S^{\rm matter} = S^{\rm scalar} + S^{\rm vector} + S^{\rm fermion}. \tag{10.1}$$

Here the scalar and fermionic contributions are

$$\begin{aligned} S^{\rm scalar} &= \frac{1}{2} \sum_{i=1}^{N_{\rm S}} \int {\rm d}^d x \sqrt{g}\, \phi^i \, (-D^2)\, \phi^i, \\ S^{\rm fermion} &= i \sum_{i=1}^{N_{\rm D}} \int {\rm d}^d x \sqrt{g}\, \bar{\psi}^i \, \not{\nabla}\, \psi^i. \end{aligned} \tag{10.2}$$

The covariant derivative in the Dirac action is given by the O(d) covariant derivative (9.1) specialized to the case of Dirac fermions: $\nabla_\mu = \partial_\mu + \frac{1}{8}[\gamma_a, \gamma_b]\, \omega^{ab}{}_\mu$. Here the spin connection $\omega^{ab}{}_\mu$ is determined in terms of the vielbein $e_a{}^\mu$ in the usual way,

$$\omega^{ab}{}_\mu = e_\nu{}^a\, \Gamma^\nu{}_{\sigma\mu}\, e^{\sigma b} + e_\nu{}^a\, \partial_\mu\, e^{\nu b}, \tag{10.3}$$

with $\Gamma^\alpha{}_{\mu\nu}$ being the (torsionless) Christoffel symbol.[1]

The action for the vector fields comprises N_V copies of the gauge-invariant Maxwell action and the corresponding gauge-fixing and ghost contributions:

$$\begin{aligned} S^{\text{vector}} = &\frac{1}{4}\sum_{i=1}^{N_\mathrm{V}} \int \mathrm{d}^d x \sqrt{g}\, g^{\mu\nu} g^{\alpha\beta} F^i_{\mu\alpha} F^i_{\nu\beta} \\ &+ \sum_{i=1}^{N_\mathrm{V}} \int \mathrm{d}^d x \sqrt{\bar{g}} \left[\frac{1}{2\xi}\left(\bar{g}^{\mu\nu} \bar{D}_\mu A^i_\nu\right)^2 + \bar{C}^i\,(-\bar{D}^2)\, C^i\right]. \end{aligned} \tag{10.4}$$

The field-strength tensor for this abelian $\mathrm{U}(1)^{N_\mathrm{V}}$ gauge theory reads $F^i_{\mu\nu} \equiv \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$, and the kinetic term for the vector fields is supplemented by a gauge-fixing action and scalar ghost fields $\bar{C}, C$ exponentiating the ($\bar{g}$-dependent!) Faddeev–Popov determinant. We adopt the Feynman gauge, setting $\xi = 1$ in the sequel.

The beta functions of this truncation can be obtained following the computation outlined in Section 5.2. The coupled RG equations controlling the scale dependence of the dimensionless Newton constant g_k and the cosmological constant λ_k have the structure (5.63),

$$\begin{aligned} \partial_t g_k &= \beta_g(g_k, \lambda_k) \equiv \left[d - 2 + \eta_\mathrm{N}(g_k, \lambda_k)\right] g_k \\ \partial_t \lambda_k &= \beta_\lambda(g_k, \lambda_k). \end{aligned} \tag{10.5}$$

The modified functions $\beta_\lambda(g, \lambda)$ and $\eta_\mathrm{N}(g, \lambda)$ depend parametrically on the number of matter fields N_S, N_V, and N_D now. Explicitly,

$$\begin{aligned} \beta_\lambda = &-(2 - \eta_\mathrm{N})\lambda_k + \frac{1}{2} g_k\, (4\pi)^{1-d/2} \\ &\times \Big[2d(d+1)\Phi^1_{d/2}(-2\lambda_k) - d(d+1)\eta_\mathrm{N}\widetilde{\Phi}^1_{d/2}(-2\lambda_k) \\ &- 8d\Phi^1_{d/2}(0) + 4\left(N_\mathrm{S} + (d-2)N_\mathrm{V} - 2^{d/2} N_\mathrm{D}\right)\Phi^1_{d/2}(0)\Big], \end{aligned} \tag{10.6}$$

while the anomalous dimension of Newton's constant is given by

$$\eta_\mathrm{N}(g_k, \lambda_k) = \frac{g_k B_1(\lambda_k)}{1 - g_k B_2(\lambda_k)} \tag{10.7}$$

[1] For a more detailed discussion of structural aspects related to the inclusion of fermions in the gravitational Effective Average Action we refer to [298, 299].

with the following functions of the cosmological constant:

$$
\begin{aligned}
B_1(\lambda_k) \equiv & \frac{1}{3}(4\pi)^{1-d/2} \\
& \times \Big[d(d+1)\Phi^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\Phi^2_{d/2}(-2\lambda_k) \\
& - 4d\Phi^1_{d/2-1}(0) - 24\Phi^2_{d/2}(0) \\
& + 2\left(N_\mathrm{S} + (d-2)N_\mathrm{V} - 2^{d/2}N_\mathrm{D}\right)\Phi^1_{d/2-1}(0) \\
& - 3\left(4N_\mathrm{V} - 2^{d/2}N_\mathrm{D}\right)\Phi^2_{d/2}(0)\Big], \\
B_2(\lambda_k) \equiv & -\frac{1}{6}(4\pi)^{1-d/2} \\
& \times \Big[d(d+1)\widetilde{\Phi}^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\widetilde{\Phi}^2_{d/2}(-2\lambda_k)\Big].
\end{aligned}
\tag{10.8}
$$

Since the contribution of minimally coupled matter fields does not give rise to terms proportional to the anomalous dimension of Newton's constant, $B_2(\lambda_k)$ is identical to the one in the Einstein–Hilbert truncation of pure gravity, (5.59).

Thus, including free matter fields leads to a two-parameter "deformation" of the beta functions governing the scale dependence of Newton's constant and the cosmological constant in the pure gravity case. Introducing the ratio of the threshold functions,

$$
\Phi_\mathrm{M} \equiv \frac{\Phi^2_{d/2}(0)}{\Phi^1_{d/2-1}(0)},
\tag{10.9}
$$

the corresponding contributions to β_λ and β_g, i.e., to B_1, actually can be encoded in two "deformation parameters," namely[2]

$$
\begin{aligned}
d_\lambda &= N_\mathrm{S} + (d-2)N_\mathrm{V} - 2^{d/2}N_\mathrm{D}, \\
d_g &= N_\mathrm{S} + (d-2)N_\mathrm{V} - 2^{d/2}N_\mathrm{D} - 3\left(2\,N_\mathrm{V} - 2^{d/2-1}\,N_\mathrm{D}\right)\Phi_\mathrm{M}.
\end{aligned}
\tag{10.10}
$$

Specifying to $d=4$ and the optimized cutoff shape function (E.14) these relations reduce to

$$
\begin{aligned}
d_\lambda &= N_\mathrm{S} + 2N_\mathrm{V} - 4N_\mathrm{D}, \\
d_g &= N_\mathrm{S} - N_\mathrm{V} - N_\mathrm{D}.
\end{aligned}
\tag{10.11}
$$

For any choice of the shape function the map (10.10) assigns a particular set of "coordinates" (d_g, d_λ) to a given matter system. For example, the matter content

[2] The relation between the number of matter fields and the parameters d_λ and d_g actually depends on the choice of the coarse-graining operator. Following the terminology introduced in [187], one may also resort to coarse-graining operators of Type II, which, besides the Laplacian, also include specific endomorphism terms proportional to the Ricci scalar. In this case one obtains $d_g = N_\mathrm{S} + (d-8)\,N_\mathrm{V} + 2^{d/2-1}\,N_\mathrm{D}$ while d_λ is unaffected. More details on the fixed point structure of gravity-matter systems implementing the Type II regularization can be found in [187, 291].

of the standard model of particle physics contains 12 vector fields (eight gluons, three gauge bosons for the weak interactions, and the photon), 45/2 fermionic degrees of freedom (three generations of fermionic matter fields with the lightest containing one Dirac field for electron, 1/2 Dirac field for left-handed neutrino, three Dirac fields for the up quark (three colors), and three Dirac fields for the down quarks), and four scalar fields. Evaluating the relations (10.11) thus assigns the coordinates $(d_g, d_\lambda) = (-61/2, -62)$ to the standard model field content.

At this point it is instructive to study the fixed point structure of the beta functions (10.5) as a function of the deformation parameters (d_g, d_λ).

By solving the equation $\eta_{\mathrm{N}}(g_*^{\mathrm{NGFP}}, \lambda_*^{\mathrm{NGFP}}) = -2$ for g_*^{NGFP} and substituting the resulting expression into $\beta_\lambda(g_*^{\mathrm{NGFP}}, \lambda_*^{\mathrm{NGFP}}) = 0$ it follows that the beta functions admit *at most three* NGFPs. Moreover, the resulting equation for $\lambda_*^{\mathrm{NGFP}}(d_g, d_\lambda)$ possesses a singular locus at $d_g = 14$ so that there are no fixed points for this particular value.

The number of physically interesting NGFPs, i.e., fixed points situated at $g_*^{\mathrm{NGFP}} > 0$ and $\lambda_*^{\mathrm{NGFP}} < 1/2$, is shown in Figure 10.1. We see that for a generic gravity-matter system the occurrence of zero or one NGFP is rather common. Matter configurations supporting more than one NGFP appear in a compact region close to the origin of the d_g-d_λ plane only.

Testing whether a NGFP may provide a suitable flow of Newton's constant and the cosmological constant includes determining its stability properties. A NGFP will be denoted as UV-FP (IR-FP) if it attracts the (projected) RG flow in its vicinity as $k \to \infty$ ($k \to 0$). In addition the NGFP can be a saddle point coming with one UV-attractive and one UV-repulsive eigendirection.

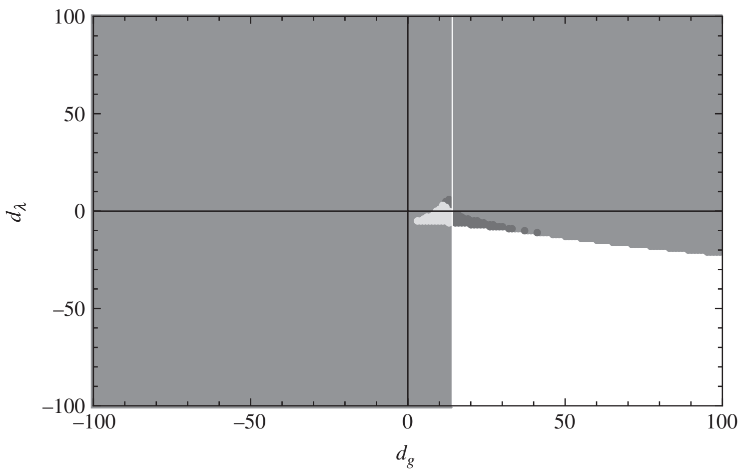


Figure 10.1. Number of NGFP solutions of the system (10.5) as a function of the deformation parameters d_g and d_λ. The system supports regions with zero (white area), one (gray area), two (dark gray area), and three (light gray area) NGFPs. There are no fixed point solutions on the line $d_g = 14$.

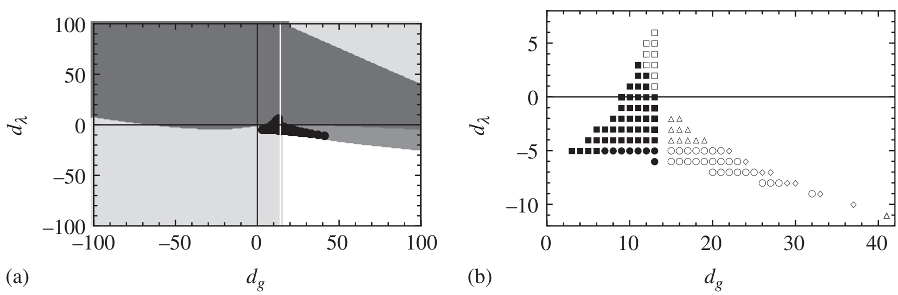
The stability properties complementing the classification of the fixed points are shown in Figure 10.2. The stability properties of the zero and one NGFP solutions are classified in the left diagram. The dark-gray area extending from the upper-left quadrant indicates a UV-FP with complex critical exponents, exhibiting the spiraling attractor behavior typical for gravity without additional matter fields. The two disjoint light-gray regions in the lower-left and upper-right corner support UV-FPs with two real critical exponents. Saddle points or IR-FPs occur within the gray wedge paralleling the d_g-axis only.

The special values of d_g, d_λ giving rise to more than one NGFP are detailed in the right diagram of Figure 10.2.

Notably, the classification provided in Figure 10.2 only relies on the numerical value of the deformation parameters d_g, d_λ. As a consequence it is independent of the cutoff shape functions used in the matter sector since changes in these quantities can be absorbed in a redefinition of d_g and d_λ. The location of a given matter model within the d_g-d_λ-plane does depend on the specific regularization scheme.

For illustrative purposes, we use the map (10.11) to study the fixed point structure arising for the matter field content of the standard model together with its most common extensions. The results are summarized in Table 10.1.

We see that all matter systems considered are located in regions of the d_g-d_λ-plane which support *a single* UV-attractive NGFP. In particular, the matter content of the standard model (and its minimal extensions) is located in the lower-left quadrant. As a consequence the stability coefficients of the single UV-FP are real. Note that the position and stability coefficients of the NGFPs appearing in this sector are remarkably insensitive to the precise matter field content of the model.



(a) (b)

Figure 10.2. Stability properties of the NGFPs appearing in Figure 10.1. The left diagram covers the cases of zero (white area) and one NGFP (gray shaded regions). The shading indicates the occurrence of a UV-FP with complex critical exponents (dark-gray area), a UV-FP with real critical exponents (light gray area), or the existence of a single saddle point or IR-FP (gray area). The right diagram magnifies the black region supporting two (open symbols) or three NGFPs (filled symbols). The stability properties of the fixed points associated with the filled squares are, respectively, UV/UV/saddle, while the filled circles represent points with characteristics UV/saddle/IR. The stability properties of the areas supporting two fixed points are UV/IR (open diamond), UV/saddle (open triangle), UV/UV (open square), and saddle/IR (open circles).

Table 10.1 *Non-Gaussian fixed points of selected gravity-matter models as obtained with the Type I map. All models, apart from the minimally supersymmetric standard model (MSSM) and the grand unified theories (GUT), sit in the lower-left quadrant of Figure 10.2. All matter configurations, apart from the* SO*(10) GUT, possess a single UV-attractive NGFP with real critical exponents. Except for the MSSM, the Type II map gives rise to the same qualitative picture.*

Matter sector	N_{S}	N_{D}	N_{V}	d_g	d_λ	g_*^{NGFP}	$\lambda_*^{\mathrm{NGFP}}$	θ_1	θ_2
Pure gravity	0	0	0	0	0	0.707	0.193	$1.475 \pm 3.043i$	
Quantum electrodynamics	0	1	1	-2	-2	0.836	0.157	$1.226 \pm 2.531i$	
Standard model (SM)	4	$\frac{45}{2}$	12	$-\frac{61}{2}$	-62	0.894	-1.174	3.831	1.962
SM + dark matter scalar (DM)	5	$\frac{45}{2}$	12	$-\frac{59}{2}$	-61	0.916	-1.185	3.829	1.963
SM + 3 right-handed neutrinos	4	24	12	-32	-68	0.863	-1.239	3.848	1.969
SM + 3 neutrinos + DM + axion	6	24	12	-30	-66	0.905	-1.264	3.845	1.970
Minimal supersymmetric standard model	49	$\frac{61}{2}$	12	$\frac{13}{2}$	-49	5.763	-6.424	3.933	2.135
SU(5) grand unified theory	124	24	24	76	76	0.156	0.371	12.51	6.972
SO(10) grand unified theory	97	24	45	28	91	0.139	0.368	$8.131 \pm 1.210i$	

At this stage the following remark is in order. At the time of writing the investigation of Asymptotic Safety in gravity-matter systems is still in the early stages. Nevertheless, the rather generic occurrence of UV-FPs is a strong indication that the Asymptotic Safety mechanism is not limited to the gravitational interactions and may also be realized within gravity-matter systems. This result is also supported by complementary and significantly more complex projections [291, 300].

As we saw, the inclusion of minimally coupled matter fields leads to new NGFPs with different stability characteristics. Conversely, it is expected that the gravitational interactions at the NGFP turn on interactions in the matter sector [301]. On this basis the fixed point solutions discussed in this section should be seen as the projection of NGFPs on higher dimensional theory spaces including such interactions onto the subspace where the matter-interactions are switched off.

10.2 Interacting Gravity-Matter Fixed Points

A rather remarkable feature of interacting gravity-matter fixed points is that their UV-critical surface may have a lower dimensionality than the one of the corresponding Gaussian matter fixed point in the absence of gravity, thus giving rise to an *enhanced degree of predictivity.*

In this section we illustrate this mechanism by considering standard Quantum Electrodynamics (QED) coupled to asymptotically safe gravity, QEG. Following the analyses in [271, 302, 303] we present evidence for the existence of an asymptotically safe combined theory in which the fine-structure constant α is computable from first principles.

10.2.1 QED on Minkowski Space

With the advent of perturbative renormalization theory in the late 1940s of the last century QED matured to a physical theory of unprecedented predictive power. With only two input parameters, the electron's mass and charge, it can explain a wealth of experimental data, often with spectacular precision. While QED is considered the prototype of a perturbatively renormalizable theory, it is natural to ask about its status beyond perturbation theory. Does it exist as a fundamental theory? And more concretely, is there a non-Gaussian RG fixed point with respect to which it is asymptotically safe?

Already in the early years of QED a non-trivial UV fixed point was speculated about, albeit for a somewhat different reason [304]. The well-known one loop formula for the renormalized charge,

$$\frac{1}{e_{\text{ren}}^2} - \frac{1}{e_\Lambda^2} = \frac{1}{6\pi^2} \ln\left(\frac{\Lambda}{m_{\text{ren}}}\right), \tag{10.12}$$

suggests that it might be difficult to obtain an *interacting* theory in the limit when the UV cutoff Λ is removed: Keeping in (10.12) the value e_Λ of the bare charge fixed and sending Λ to infinity one finds that the renormalized charge e_{ren} vanishes, so one is left with a *trivial* theory. Conversely, keeping e_{ren} fixed, it is impossible to let $\Lambda \to \infty$ since e_Λ diverges at a finite value of Λ, the *Landau pole*.

Clearly, if the exact version of (10.12) displays a UV fixed point such that $e_\Lambda \to e_*$ for $\Lambda \to \infty$ the prospects for an interacting, cutoff-free theory are much better. However, presumably no such fixed point exists. Comprehensive lattice simulations [305–307] and studies using non-perturbative FRGE methods [308] suggest that QED is very likely to be a trivial theory in four dimensions.

As we shall see, the situation changes when QED gets coupled to quantum gravity: The combined theory QED + QEG appears to have significantly better chances to exist non-perturbatively than QED on its own.

10.2.2 QED Coupled to QEG

Following [302] we describe the non-perturbative RG behavior of QED coupled to QEG in terms of a three-dimensional truncation of theory space, including the charge $e(k)$, or equivalently the fine-structure constant $\alpha(k) \equiv e(k)^2/(4\pi)$, along with Newton's constant and the cosmological constant as running quantities. The truncated EAA reads in the simplest case,

$$\Gamma_k = \frac{1}{4e(k)^2} \int \mathrm{d}^d x\, \sqrt{g}\, F_{\mu\nu} F^{\mu\nu} + \frac{1}{16\pi G_k} \int \mathrm{d}^d x\, \sqrt{g}\, (-R + 2\bar{\lambda}_k) + \cdots\,, \tag{10.13}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the field-strength tensor of the abelian gauge field A_μ, the photon. The dots in (10.13) represent the gauge-fixing and ghost terms for both general coordinate and U(1) invariance, as well as the Dirac action for the electron which, by assumption, contains no running parameters.

According to [302] the ensuing dimensionless RG equations can be approximated by the following simplified system:

$$\begin{aligned} \partial_t g &= \beta_g \equiv \big[2 + \eta_{\mathrm{N}}(g,\lambda)\big] g, \\ \partial_t \lambda &= \beta_\lambda(g,\lambda), \\ \partial_t \alpha &= \beta_\alpha \equiv \Big[A\, h_2(\alpha) - \frac{6}{\pi} \Phi^1_1(0)\, g\Big] \alpha, \end{aligned} \tag{10.14}$$

with the coefficient

$$A \equiv \frac{2}{3\pi} n_F. \tag{10.15}$$

We consider a variant of quantum electrodynamics with n_F "flavors" of electrons here, and for simplicity, we stick to $d = 4$ dimensions. Several remarks are in order at this point.

(1) The first two equations in (10.14) are taken to be those of pure gravity in the Einstein–Hilbert truncation [14] which we discussed in Section 5.2.2. Neglecting the backreaction of the matter fields on the renormalization in the gravity sector is at least partially justified by the above investigations of free matter fields coupled to gravity. They show that a Maxwell field and one or a few Dirac fields do not qualitatively alter the RG flow of g and λ.

(2) Following (5.65), the anomalous dimension η_{N} can be expanded for small g

$$\eta_{\text{N}} \equiv B_1(\lambda)\, h_1(\lambda, g) = B_1(\lambda)\Big(g + B_2(\lambda)g^2 + \cdots\Big), \tag{10.16}$$

with the coefficient functions B_1 and B_2 given in (5.59). From the analysis there we know that $B_1(\lambda) < 0$ for all λ and that $g(k)$ does not change very much if one approximates $\lambda(k) \approx 0$, $B_1(\lambda) \approx B_1(0)$, $B_2(\lambda) \approx 0$, whence

$$\eta_{\text{N}}(g, \lambda) \approx B_1(0) g < 0. \tag{10.17}$$

(3) The beta function for α in the third equation of (10.14) involves a pure matter part, written as $A\, h_2(\alpha)\, \alpha$, and as the most important new ingredient the quantum gravity contribution $\propto g$ obtained in [271]. The former part is well known from perturbation theory, its first two terms being (for $n_F = 1$)

$$\beta_\alpha(\alpha)\big|_{g=0} \equiv A\, h_2(\alpha)\, \alpha = \alpha \left[\frac{2}{3}\left(\frac{\alpha}{\pi}\right) + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 + O(\alpha^3)\right]. \tag{10.18}$$

For a qualitative understanding it will be sufficient to employ the one-loop approximation, i.e.,

$$h_2(\alpha) = \alpha. \tag{10.19}$$

Indeed, the lattice and flow equation studies mentioned above indicate that there exists no non-trivial continuum limit for QED (without gravity), and this means that $\beta_\alpha(\alpha)\big|_{g=0}$ has no zero at any $\alpha > 0$. Therefore, $h_2(\alpha) = \beta_\alpha/(A\alpha)$ starts out as $h_2(\alpha) = \alpha$ in the perturbative regime $\alpha \lesssim 1$, and for larger α it is still known to be an increasing function: $h_2'(\alpha) > 0$. To be able to solve the RG equations analytically we set $h_2(\alpha) = \alpha$ for all values of α. This is a qualitatively reliable approximation since, as we will see, at most a zero of $h_2(\alpha)$ could change the general picture.

(4) As it stands, β_α applies only above the threshold due to the mass of the electron at $k = m_e$. At low scales $k \lesssim m_e$ the fermion loops no longer renormalize α. In the full-fledged EAA this decoupling is described by a certain threshold function. Here a simplified description will be sufficient where we set $A = 0$ if $k < m_e$.

(5) The most interesting term in the RG equations is the contribution to β_α proportional to the product $g\, \alpha$ which describes the renormalization of the gauge coupling due to graviton loops. Remarkably enough, its sign is such that it counteracts the fermionic contribution, i.e., *it opposes the electromagnetic screening of charges*. It drives $\alpha(k)$ towards *smaller* values for increasing k, in particular when $k \to \infty$.

In the EAA framework the gravity contribution to β_α was first derived in [271, 303] where both abelian and non-abelian gauge fields were considered. In either case it was found that the pure gauge theory (no fermions) becomes *asymptotically free* when coupled to QEG, i.e., $\alpha(k) \to 0$ for $k \to \infty$.[3]

10.3 Two non-trivial Fixed Points

Let us start the analysis of the RG equation (10.14) by finding its fixed points $(g_*, \lambda_*, \alpha_*)$. First of all there exists an obvious Gaussian fixed point, GFP, situated at $g_* = \lambda_* = \alpha_* = 0$.

Furthermore we know that the subsystem of flow equations for pure gravity in the Einstein–Hilbert truncation, (5.63), admits for a NGFP at $(g_0^*, \lambda_0^*) \neq 0$. This fixed point lifts to a NGFP of the full system located at $(g_0^*, \lambda_0^*, \alpha_* = 0)$. We denote it by $\mathbf{NGFP_1}$. This fixed point is trivial from the QED perspective; the electromagnetic interaction is "switched off" there, while the gravitational selfinteraction is the same as in pure gravity.

There exists a second NGFP though, non-trivial also in the QED sense, provided the equation

$$h_2(\alpha) = \frac{6}{\pi} \frac{g_*}{A} \Phi_1^1(0) \tag{10.20}$$

admits a solution for some $\alpha = \alpha_* \neq 0$. We demonstrate below that this is indeed the case and that $\alpha_* > 0$. This fixed point will be called $\mathbf{NGFP_2}$.

(1) We are particularly interested in an Asymptotic Safety scenario at the second NGFP. In its vicinity the linearized flow is governed by

$$\partial_t u_i(k) = \sum_j B_{ij} \left(u_j(k) - u_j^* \right), \tag{10.21}$$

where $u \equiv (g, \lambda, \alpha)$, and the stability matrix is of the form

$$B = (B_{ij}) = \left. \begin{pmatrix} \partial_g \beta_g & \partial_\lambda \beta_g & 0 \\ \partial_g \beta_\lambda & \partial_\lambda \beta_\lambda & 0 \\ \partial_g \beta_\alpha & \partial_\lambda \beta_\alpha & \partial_\alpha \beta_\alpha \end{pmatrix} \right|_{u=u_*} . \tag{10.22}$$

[3] The same behavior occurs also in an earlier perturbative calculation [309]; see also [310–318] for a controversial discussion of different perturbative schemes.

Two of its eigenvalues coincide with those of pure gravity in the Einstein–Hilbert truncation, giving rise to the familiar two UV-attractive directions [160, 173]. The third eigenvalue is given by

$$\partial_\alpha \beta_\alpha(u_*) = \left(A h_2(\alpha_*) - \frac{6}{\pi} \Phi_1^1(0) g_* \right) + A h_2'(\alpha_*) \alpha_* = A h_2'(\alpha_*) \alpha_* . \tag{10.23}$$

As $\alpha_* > 0$, the sign of $\partial_\alpha \beta_\alpha(u_*)$ agrees with the sign of $h_2'(\alpha_*)$.

At this point we exploit the information from lattice and flow equation studies which tried to find a non-trivial continuum limit of QED without gravity. We saw that their negative results suggest that $h_2'(\alpha_*) > 0$ holds true even beyond perturbation theory.

As a consequence, the third eigenvalue $\partial_\alpha \beta_\alpha(u_*)$, corresponding to the α direction in g-λ-α-theory space, is UV repulsive. With two UV-attractive and one repulsive direction the UV-critical hypersurface $\mathscr{S}_{\mathrm{UV}}$ pertaining to $\mathbf{NGFP_2}$ is a two-dimensional surface in a three-dimensional space, i.e., $\dim \mathscr{S}_{\mathrm{UV}}(\mathbf{NGFP_2}) = 2$.

(2) Let us also determine the critical exponents of the other fixed point $\mathbf{NGFP_1}$. As β_g and β_λ do not depend on α in our approximation the first two of them remain the same as for $\mathbf{NGFP_2}$. However, for the third eigenvalue we now obtain

$$\partial_\alpha \beta_\alpha(u_*) = \left(A h_2(\alpha_*) - \frac{6}{\pi} \Phi_1^1(0) g_* \right) + A h_2'(\alpha_*) \alpha_* \overset{\alpha_*=0}{=} -\frac{6}{\pi} \Phi_1^1(0) g_* < 0. \tag{10.24}$$

Hence, the third direction is UV attractive as well, and $\mathbf{NGFP_1}$ has a three-dimensional UV-critical hypersurface, $\dim \mathscr{S}_{\mathrm{UV}}(\mathbf{NGFP_1}) = 3$.

We conclude that $\dim \mathscr{S}_{\mathrm{UV}}(\mathbf{NGFP_2}) < \dim \mathscr{S}_{\mathrm{UV}}(\mathbf{NGFP_1})$. This reflects an *enhanced predictivity of the Asymptotic Safety construction with respect to* $\mathbf{NGFP_2}$ *as compared to* $\mathbf{NGFP_1}$.

10.4 Approximate Analytical Solution

Let us now analyze the flow in a simple analytically tractable approximation. For that we expand the functions h_1 and h_2 to first order in g and α, respectively:

$$h_1(g) = g + O(g^2) \qquad \text{and} \qquad h_2(\alpha) = \alpha + O(\alpha^2). \tag{10.25}$$

Furthermore, we neglect the running of the cosmological constant and fix $\lambda = \lambda_0$ to a constant value. The remaining system of flow equations reads

$$\begin{aligned} \partial_t g &= \big[2 + B_1(\lambda_0)\, g \big] g, \\ \partial_t \alpha &= \left(A\alpha - \frac{6}{\pi} \Phi_1^1(0)\, g \right) \alpha. \end{aligned} \tag{10.26}$$

In this approximation the existence of the $\mathbf{NGFP_2}$ is obvious and we can read off its coordinates:

$$g_* = -\frac{2}{B_1(\lambda_0)} > 0 \qquad \text{and} \qquad \alpha_* = \frac{6}{\pi}\frac{g_*}{A}\Phi_1^1(0) > 0. \tag{10.27}$$

Note that as we already anticipated α_* indeed turns out positive.

In the sequel we eliminate the constant $B_1(\lambda_0)$ in favor of g_*, setting $B_1(\lambda_0) = -2/g_*$ everywhere.

The solution to the decoupled first equation in (10.26) then reads:

$$g(k) = \frac{G_0 k^2}{1 + \frac{G_0 k^2}{g_*}}\,. \tag{10.28}$$

The constant of integration $G_0 \equiv \lim_{k\to 0} g(k)/k^2$ can be interpreted as the IR value of the running Newton constant. The solution (10.28) for g shares a crucial feature with any asymptotically safe trajectory of the exact system for pure gravity, namely that it connects the classical regime $g(k) \approx G_0 k^2$ for $k \ll m_{\mathrm{Pl}} \equiv G_0^{-1/2}$ and the fixed point regime $g(k) \approx g_*$ for $k \gg m_{\mathrm{Pl}}$.

The simplified second RG equation (10.26) for α is a differential equation of the Riccati type which can be solved in closed form without specifying the function $g(k)$. Its general solution reads, with $\Phi_1^1 \equiv \Phi_1^1(0)$,

$$\frac{1}{\alpha(k)} = \frac{1}{\alpha_0}\exp\left(\frac{6}{\pi}\Phi_1^1\int_{k_0}^{k}\frac{g(k')}{k'}\mathrm{d}k'\right) - A\int_{k_0}^{k}\exp\left(\frac{6}{\pi}\Phi_1^1\int_{k'}^{k}\frac{g(k'')}{k''}\mathrm{d}k''\right)\frac{\mathrm{d}k'}{k'}, \tag{10.29}$$

where $\alpha_0 = \alpha(k_0)$ is the value of the fine-structure constant at a fixed reference scale k_0. If we insert the function $g(k)$ of (10.28) we can perform the integrations in (10.29) and find

$$\boxed{\begin{aligned}\frac{1}{\alpha(k)} &= \left(\tfrac{g_*+G_0k^2}{g_*+G_0k_0^2}\right)^{\frac{3}{\pi}\Phi_1^1 g_*}\\ &\quad\times\left[\tfrac{1}{\alpha_0} - \tfrac{1}{\alpha_*}\left(1+\tfrac{g_*}{G_0k_0^2}\right){}_2F_1\left(1,1,1+\tfrac{3}{\pi}\Phi_1^1 g_*; -\tfrac{g_*}{G_0k_0^2}\right)\right]\\ &\quad+\tfrac{1}{\alpha_*}\left(1+\tfrac{g_*}{G_0k^2}\right){}_2F_1\left(1,1,1+\tfrac{3}{\pi}\Phi_1^1 g_*; -\tfrac{g_*}{G_0k^2}\right),\end{aligned}} \tag{10.30}$$

where ${}_2F_1(a,b,c;z)$ denotes the (ordinary) hypergeometric function.

From (10.30) we infer that there exist three kinds of possible UV behaviors for $\alpha(k)$. The (k-independent) value of the terms inside the square brackets $[\cdots]$ on the right-hand side of (10.30) determines which one of them is realized.

(1) For a strictly positive value $[\cdots] > 0$ we see that $\lim_{k\to\infty}\alpha(k) = 0$ as the prefactor of $[\cdots]$ diverges proportional to $k^{\frac{6}{\pi}\Phi_1^1 g_*}$ when $k \to \infty$. This corresponds to an asymptotically free fine-structure constant. The corresponding RG trajectories of the full system are asymptotically safe with respect to $\mathbf{NGFP_1}$.

(2) For a negative value $[\cdots] < 0$ there exists a scale k_L at which the two terms on the right-hand side of (10.30) cancel, so that α diverges at the finite scale $k = k_L$. This amounts to a Landau type singularity.

(3) A third type of limiting behavior is obtained for the case that the bracket vanishes exactly: $[\cdots] = 0$. We are then left with the second term on the right-hand side of (10.30), and since ${}_2F_1(a, b, c; 0) = 1$, we find $\lim_{k\to\infty} \alpha(k) = \alpha_*$, corresponding to an asymptotically safe trajectory with a non-zero gauge coupling at $\mathbf{NGFP_2}$. This is precisely the behavior to be expected due to the UV-repulsive direction of the fixed point. Since for $k \to \infty$ the trajectory will only flow into $\mathbf{NGFP_2}$ for one specific value of α_0, this value of α_0, and hence the whole trajectory $\alpha(k)$, can be predicted under the assumption of Asymptotic Safety with respect to $\mathbf{NGFP_2}$.

The situation is illustrated by the g-α-phase portrait in Figure 10.3. Bearing in mind that the arrows always point toward the IR, we see that $\mathbf{NGFP_1}$ is IR repulsive in both directions shown, while $\mathbf{NGFP_2}$ is IR attractive in one direction. This is consistent with our earlier discussion which showed that in the three-dimensional g-λ-α-space $\mathbf{NGFP_1}$ has three and $\mathbf{NGFP_2}$ has only two IR repulsive (or equivalently, UV attractive) eigendirections.

In Figure 10.3, the trajectories inside the $\mathbf{GFP}$-$\mathbf{NGFP_1}$-$\mathbf{NGFP_2}$ triangle are those corresponding to the case $[\cdots] > 0$ above; they are asymptotically safe with respect to $\mathbf{NGFP_1}$. The $\mathbf{NGFP_2} \to \mathbf{GFP}$ boundary of this triangle is the unique

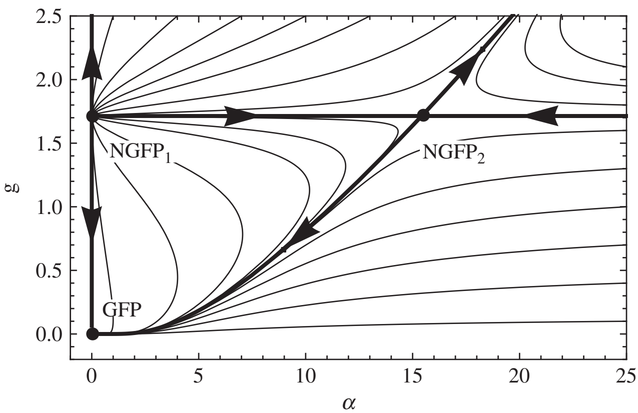


Figure 10.3. The RG flow on the g-α-plane implied by the simplified equations (10.26). It is dominated by two non-Gaussian fixed points. (Adapted from [302].)

trajectory (heading toward smaller g and α values), which is asymptotically safe with respect to the second non-trivial fixed point, $\mathbf{NGFP_2}$.

The diagram in Figure 10.3 corresponds to a massless electron for which A keeps its non-zero value at arbitrarily small scales. In reality the α-running due to the fermions stops near m_e, of course.

10.5 Asymptotic Safety Construction at $\mathbf{NGFP_2}$

Let us investigate the unique asymptotically safe trajectory emanating from $\mathbf{NGFP_2}$ in more detail. We note that the condition of a vanishing bracket $[\cdots]$ in (10.30) is self-consistent in the sense that the resulting function $\alpha_0(k_0)$ is of identical form as the remaining function $\alpha(k)$:

$$\boxed{\frac{1}{\alpha(k)} = \frac{1}{\alpha_*}\left(1 + \frac{g_*}{G_0 k^2}\right)\, {}_2F_1\left(1, 1, 1 + \frac{3}{\pi}\Phi^1_1 g_*; -\frac{g_*}{G_0 k^2}\right).} \tag{10.31}$$

(1) Let us approximate this function for scales $k \ll m_{\rm Pl}$ much below the Planck scale $m_{\rm Pl} \equiv G_0^{-1/2}$. Later on we will need it at $k = m_e$, for instance, where m_e is the mass of the electron. Then the argument $\frac{g_*}{G_0 m_e^2} = g_* \left(\frac{m_{\rm Pl}}{m_e}\right)^2 \approx 10^{44}$ is extremely large and this will be an excellent approximation. Hence, we may safely truncate the general series expansion of the hypergeometric function,

$$\begin{aligned} {}_2F_1(a, a, c; z) = &\ \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}(-z)^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (1-c+a)_n}{(n!)^2} z^{-n} \\ &\times \Big(\ln(-z) + 2\psi(n+1) - \psi(a+n) - \psi(c-a-n)\Big), \end{aligned}$$

after its first term, and approximate the resulting factor $1 + G_0 k^2/g_* \approx 1$, such that our final result for scales $k \ll m_{\rm Pl}$ reads:

$$\frac{1}{\alpha(k)} = \frac{g_*}{\alpha_*} \cdot \frac{3}{\pi}\Phi^1_1 \cdot \left[\ln\left(\frac{g_*}{G_0 k^2}\right) - \gamma - \psi\left(\frac{3}{\pi}\Phi^1_1 g_*\right)\right]. \tag{10.32}$$

Here ψ denotes the Digamma function and γ is Euler's constant. Using (10.27) in order to re-express g_*/α_* in terms of $A \equiv (\frac{2}{3\pi}) n_F$ we can also write (10.32) in the following form:

$$\boxed{\frac{1}{\alpha(k)} = \frac{A}{2}\left[\ln\left(\frac{g_*}{G_0 k^2}\right) - \gamma - \psi\left(\frac{3}{\pi}\Phi^1_1 g_*\right)\right].} \tag{10.33}$$

For $k \ll m_{\rm Pl}$, we recover the logarithmic running $\alpha(k)^{-1} = -A \ln k + \text{const}$ familiar from pure QED.

(2) For k comparable to, or even larger than, the Planck mass the gravity corrections set in, stop the logarithmic running of $\alpha(k)$ and cause the coupling to freeze at a finite value $\alpha(k \to \infty) = \alpha_*$. Obviously, along this RG trajectory no Landau pole singularity is encountered!

(3) Equation (10.33) shows that $\alpha(k) \propto 1/n_F$ at all scales. As a consequence, if we consider a toy model with a large number of electron flavors, all α-values that appear along the RG trajectory can be made arbitrarily small, which renders perturbation theory in α increasingly precise. At the fixed point we have, for instance,

$$\alpha_* = 9\, \Phi_1^1(0)\, \frac{g_*}{n_F}. \tag{10.34}$$

(4) In an Asymptotic Safety scenario based on the fixed point $\mathbf{NGFP_2}$, *the infrared value of the fine-structure constant* $\alpha_{\text{IR}} \equiv \lim_{k\to 0} \alpha(k)$ *is a computable number.*

Using (10.33) to calculate α_{IR} we must remember however that as it stands it holds true only for $k \gtrsim m_e$. When k drops below the electron mass the standard QED contribution to the running of $\alpha(k)$ goes to zero, and the gravity corrections are negligible there anyway. Hence, approximately, $\partial_t \alpha(k) = 0$ for $0 \le k \lesssim m_e$.

Thus (10.33) leads to the following prediction for $\alpha_{\text{IR}} \approx \alpha(m_e)$:

$$\boxed{\frac{1}{\alpha_{\text{IR}}} = \frac{A}{2}\left[2\ln\left(\frac{m_{\text{Pl}}}{m_e}\right) + \ln(g_*) - \gamma - \psi\left(\frac{3}{\pi}\Phi_1^1 g_*\right)\right].} \tag{10.35}$$

As the fixed point coordinates are an output of the RG equations, the only input parameter needed to predict α_{IR} in this approximation is the electron mass expressed in Planck units, m_e/m_{Pl}.

(5) On the other hand, if QED + QEG is not asymptotically safe with respect to $\mathbf{NGFP_2}$ then its RG trajectory is one of those inside the $\mathbf{GFP}$-$\mathbf{NGFP_1}$-$\mathbf{NGFP_2}$ triangle in Figure 10.3. In this case it is asymptotically safe with respect to the other non-trivial fixed point, $\mathbf{NGFP_1}$. The resulting quantum field theory is also free from divergences and valid at all energies. In this case the U(1) coupling is not a prediction but an experimental input, though.

10.6 Numerical Solution

The analytically accessible RG equations solved above follow from the original systems of equations (10.14) by a number of additional approximations. Staying within the one-loop approximation of β_α, it has been checked with numerical methods that these additional approximations cause only minor changes of the flow. The analytic solution correctly describes all qualitative properties of the RG flow, and is precise even at a "semiquantitative" level [302].

The numerical solution confirms the existence of the two non-Gaussian fixed points $\mathbf{NGFP_1}$ and $\mathbf{NGFP_2}$ as well as their stability properties. In accord

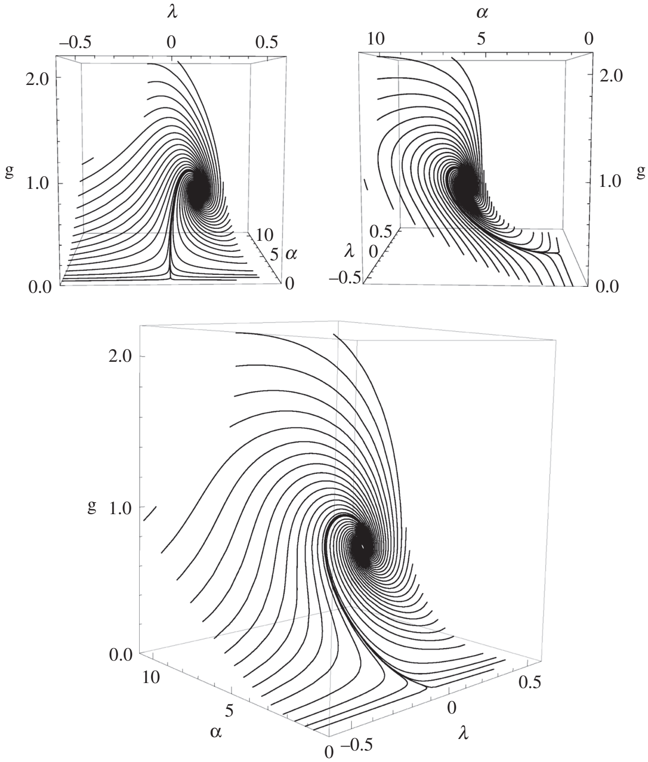


Figure 10.4. The diagrams show three projections of the same family of trajectories running inside the UV-critical surface $\mathscr{S}_{\mathrm{UV}}$ of $\mathbf{NGFP_2}$ in g-λ-α-space. They are seen to form a two-dimensional surface in a three-dimensional theory space. (Taken from [302].)

with the discussion above, $\mathbf{NGFP_2}$ is seen to have two UV-attractive and one repulsive direction. All RG trajectories approaching $\mathbf{NGFP_2}$ for $k \to \infty$ lie in its two-dimensional UV-critical surface $\mathscr{S}_{\mathrm{UV}}$. This surface is visualized in Figure 10.4 by a family of trajectories starting on $\mathscr{S}_{\mathrm{UV}}$ close to the FP, which were traced down to lower scales k.

10.7 Can We Compute the Fine-Structure Constant?

(1) To summarize: We coupled Quantum Electrodynamics to quantized gravity and explored the possibility of an asymptotically safe UV limit of the combined system. Using a three-parameter truncation we found evidence indicating that this is indeed possible. There exist two non-trivial fixed points which lend themselves for an Asymptotic Safety construction. Using the first one, $\mathbf{NGFP_1}$, the ultraviolet value of the fine-structure constant is zero. Its infrared value α_{IR} is a free parameter which is not fixed by the theory itself but has to be taken from experiment. Basing the theory on the second non-Gaussian fixed point $\mathbf{NGFP_2}$ instead, the fixed point value α_* is non-zero, and the ("renormalized") low-energy value of the fine-structure constant α_{IR} can be predicted in terms of the electron mass in Planck units.

(2) In either case the coupled theory QED + QEG is well behaved in the ultraviolet. In particular, there is *no Landau pole singularity*, and *no triviality problem*, i.e., the continuum limit is an interacting theory.

We see that its coupling to quantum gravity cures the notorious UV problems standard QED is suffering from despite its perturbative renormalizability.

(3) On theory spaces with more than one eligible fixed point a comparison of predictions to experimental data can reveal in principle which of them is the actual UV fixed point. It is therefore tempting to insert numbers into our prediction (10.35) for the low-energy value of α.

With $m_e = 5.11 \times 10^{-4}$ GeV and $m_{\mathrm{Pl}} = 1.22 \times 10^{19}$ GeV one finds $m_e/m_{\mathrm{Pl}} = 4.19 \times 10^{-23}$, and the cutoff (E.13), for example, yields $\Phi^1_1 = 1$. The coordinate $g_* = -2/B_1(\lambda_0)$ depends on the value chosen for λ_0. For $\lambda_0 = 0$ or the fixed point value of λ in the Einstein–Hilbert truncation, $\lambda_0 = \lambda_* \approx 0.193$, we get $g_* \approx 1.71$ or $g_* \approx 0.83$, respectively. From that we obtain the result

$$\frac{1}{\alpha_{\mathrm{IR}}} \overset{\lambda_0=0}{\approx} 10.91\, n_F \qquad \text{or} \qquad \frac{1}{\alpha_{\mathrm{IR}}} \overset{\lambda_0=\lambda_*}{\approx} 10.96\, n_F. \tag{10.36}$$

It is remarkably insensitive to the value of g_* and/or Φ^1_1.

For $n_F = 1$, this estimate differs from the fine-structure constant measured in real nature, $\alpha \approx 1/137$, by a factor of roughly 13.

It is clear though that even within the limits of our crude approximation (10.25) a serious comparison with experiment must include the renormalization effects due to the other particles besides the electron, all those of the standard model, and possibly beyond. Within the "n_F flavor QED" considered here we could mimic their effect by appropriately choosing n_F. We would then conclude that the observed α_{IR} is consistent with Asymptotic Safety at $\mathbf{NGFP_2}$ if $n_F = 13$.

(4) Let us turn to the full standard model finally. Applying the above discussion to the *weak hypercharge* rather than the electromagnetic U(1) one again has a one-loop equation of the type $\partial_t \alpha_1 = A\alpha_1^2$, but this time with $A = 41/(20\pi)$ and $\alpha_1 \equiv 5\alpha/(3\cos^2\theta_W)$, where θ_W is the Weinberg angle [319].

It is convenient to compare the prediction of Asymptotic Safety to the experimental value for α at the Z mass. From (10.35) with the new value of A, and m_e replaced by M_Z, we obtain (with $\lambda_0 = 0$)

$$\alpha_1^{\mathrm{SM}}(M_Z) \approx 1/25.7. \tag{10.37}$$

This prediction differs from the experimental value $\alpha_1^{\rm exp}(M_Z) \approx 1/59.5$ by only about a factor of two.

The similarity of the two numbers is certainly quite impressive if one recalls that the rather simple RG equations we employed here must bridge more than 22 orders of magnitude in scale.

Nevertheless, we are not yet in the position to say whether the gravity-matter system observed in nature is actually asymptotically safe with respect to a generalization of $\mathbf{NGFP_1}$ or $\mathbf{NGFP_2}$. It goes without saying that in order to address this question much more accurate RG calculations for more general field systems are needed.

(5) The above discussion of the coupled theory QED + QEG also sheds light on another puzzle related to the four fundamental interactions, namely the *hierarchy problem*, i.e., the question of why the typical energy scales of particle physics to which the standard model applies are so much smaller than the Planck scale, $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV [320]. Within the present setting, for example, it is natural to ask why in real nature there is the huge hierarchy between the electron mass m_e and the Planck scale.

If we adopt the point of view that the low-energy quantities $\alpha_{\rm IR} \approx 1/137$ and m_e are known from the experiment, then (10.33) or (10.35) can be used to predict the value of $m_{\rm Pl}/m_e$. Using only non-gravitational input data, $\alpha_{\rm IR}$ and m_e, this leads without further ado to a ratio $m_{\rm Pl}/m_e$ which is *naturally large*, as a consequence of the logarithm in (10.35). The number we find is not too different from the experimental value $m_{\rm Pl}/m_e \approx 2.39 \times 10^{22}$, and comes even closer when the argument is applied to the hypercharge.

(6) In conclusion it can be said that the present example provides a proof of principle which indicates that, by coupling the standard model to asymptotically safe gravity, it might well be possible to compute some of its otherwise undetermined free parameters from first principles and to understand the scales it displays.

Moreover, according to the analysis in [321] also its Higgs sector acquires a higher degree of predictivity and this allowed for an Asymptotic Safety-based prediction of the Higgs mass.

11

Towards Phenomenology

In this chapter we review the first steps undertaken in order to extract physical information directly from the RG flow. Focusing on how the quantum gravity effects might modify the classical picture of spacetime, the investigations are restricted to the level of expectation values. They involve further approximations on top of the truncations of theory space and therefore come with a higher degree of uncertainty presumably. For the time being much less is known about the observable predictions of QEG than about its RG flows. Thus, the results presented here should still be considered preliminary.

11.1 Scale-Dependent Riemannian Structure

Let us fix an RG trajectory for the EAA as well as a manifold $\mathcal{M}$ with a given topological and smooth structure to model spacetime. Then the self-consistent metrics that can persist on this manifold are determined by the system of tadpole equations (4.105). If we focus on the sector with vanishing ghosts only the simpler equation (4.106) needs to be solved. Let us assume there is a solution $\bar{g}_k^{\text{sc}}$ which depends smoothly on k, either for a certain range of scales only, or even for all $k \in [0, \infty)$. This solution promotes $\mathcal{M}$ to an entire *family of Riemannian manifolds whose members are labeled by the scale:* $(\mathcal{M}, \bar{g}_k^{\text{sc}})$. If it is based on a complete asymptotically safe RG trajectory defined for all $k \in [0, \infty)$, this family, as a whole, can be regarded as the metric structure of a "quantum spacetime" at the mean field level.

Heuristically, one would try to interpret the family $(\mathcal{M}, \bar{g}_k^{\text{sc}})$ in the following way. Assuming that Γ_k defines an effective field theory valid near typical scales of order k, the infinitely many metrics $\bar{g}_k^{\text{sc}}$ all refer *to the same physical system*, the "quantum spacetime," but describe its effective metric structure on different scales. An observer using a "microscope" with a resolving power $\ell \approx k^{-1}$ will perceive the universe to be a Riemannian manifold with a particular metric expectation value, namely $g_{\mu\nu} = \bar{g}_{\mu\nu} \equiv \bar{g}^{\text{sc}}_{k\,\mu\nu}$. Even though at every fixed k, $\bar{g}_k^{\text{sc}}$

is a smooth classical metric the quantum spacetime as a whole can acquire very non-classical and in particular fractal-like features due to the scale dependence of the metric.

The family of metrics $\bar{g}^{\rm sc}_{k\,\mu\nu}$ can be determined from the tadpole equation resulting from the EAA. For bi-metric truncations this equation boils down to (5.11), and for the special case of single-metric truncations it reads simply

$$\boxed{\frac{\delta}{\delta g_{\mu\nu}(x)}\bar{\Gamma}_k[g]\Big|_{g=\bar{g}^{\rm sc}_k}=0.} \tag{11.1}$$

Becoming even more specific, for the single-metric and bi-metric Einstein–Hilbert truncations, defined by (5.21) and (6.10), respectively, the corresponding tadpole equations have the form of Einstein's equation in classical General Relativity, with a k-dependent Newton's coupling and cosmological constant though:

$$\boxed{R_{\mu\nu}(\bar{g}^{\rm sc}_k)-\frac{1}{2}\,R(\bar{g}^{\rm sc}_k)\,\bar{g}^{\rm sc}_{k\,\mu\nu}=-\Lambda_k\,\bar{g}^{\rm sc}_{k\,\mu\nu}+8\pi G_k\,\langle T_{\mu\nu}\rangle_k.} \tag{11.2}$$

Here Λ_k stands for $\bar{\lambda}_k$ and $\Lambda^{(1)}_k$ in the single- and the bi-metric Einstein–Hilbert truncation, respectively. Moreover, $\langle T_{\mu\nu}\rangle_k$ is the energy momentum tensor describing the matter system, if any. Supplementing (11.2) by the equation of motion for the matter fields, also implied by the EAA, leads to a closed system of equations which allows to determine $\{\bar{g}^{\rm sc}_{k\,\mu\nu}\}$.

11.2 Cutoff Identifications

Having some family of metrics $\{\bar{g}^{\rm sc}_{k\,\mu\nu}\}$ at our disposal there are various ways to proceed. The first one tries to extract information directly from the full stack of metrics, without ascribing any specific meaning to the parameter which labels them, k.

A second possibility is the approach of the *cutoff identifications* which tries to identify k with a certain geometrical or dynamical scale. The idea is the same as in Section 2.3 where we saw how sometimes decoupling arguments can be used to "guess" difficult terms in $\Gamma_{k=0}$ from a simpler truncation provided we know something about the relevant threshold, if any. Analogously it may be possible to find the metric expectation value proper, $\langle\widehat{g}(x)\rangle\equiv\bar{g}^{\rm sc}_{k=0}(x)$, by inserting an appropriate function $k=k(x)$ into the family: $\bar{g}^{\rm sc}_{k=k(x)}(x)=\langle\widehat{g}(x)\rangle$.

The "resolving power" ℓ of the microscope is in general a complicated function of k, and the challenge is to express ℓ and the cutoff by the "correct" function on spacetime:

$$x^\mu\mapsto k(x^\mu). \tag{11.3}$$

On general grounds this function should behave as a scalar under a change of coordinates. The details of this map strongly depend on the particular physical situation under consideration. Explicit maps may be obtained from symmetry considerations, spectral arguments [322], or imposing conditions like the conservation of the stress-energy tensor [323, 324]. Typical choices employed in the literature relate the scale k to the proper distance between two points $d(x,x')$, computed from a classical reference metric, physical quantities like the Hubble scale $H(t)$, or a scalar quantity constructed from curvature invariants:

$$\text{Type I:} \qquad k^2 = \xi^2\, d(x,x')^{-2}, \tag{11.4}$$

$$\text{Type II:} \qquad k^2 = \xi^2\, H(t)^2, \tag{11.5}$$

$$\text{Type III:} \qquad k^2 = \xi^2\, \sqrt{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}}. \tag{11.6}$$

Here ξ is a free constant parameter of order one. Substituting the identification $k \equiv k(x)$ into the family $\bar{g}^{\mathrm{sc}}_{k\,\mu\nu}$ results in a *single* effective metric for the spacetime geometry. At scales where Λ_k and G_k are (approximately) scale-independent this effective metric reduces to the one of General Relativity. The running of the couplings induces distinct modifications controlled by the beta functions and the choice of $k(x)$. Figure 5.2 indicates that these corrections will set in when $k^2 \gtrsim m_{\mathrm{Pl}}^2$ generically, which is a natural scale for quantum gravity effects.

An alternative to the RG improvement of the *equations of motion* described above can be borrowed from QED and QCD [325–327]. It consists in carrying out the cutoff identification $k \to k(x)$ at the level of the *running action* and deriving the equations of motion thereafter [328–330]. Instead of calculating complicated terms in the effective action directly, a shortcut is sometimes available. Symmetry and decoupling arguments as in Section 2.3 may be used then to obtain those terms by inserting a certain *field dependent* cutoff identification $k(x)$ into a comparatively simple k-dependent action, typically comprising only the perturbatively relevant terms with the corresponding scale dependence. The leading-log approximation of QED and QCD in strong slowly varying fields, for example, can be obtained in this manner, letting $k^4 \to F_{\mu\nu}(x)F^{\mu\nu}(x)$ [326, 327].

Applying the same approach to gravity allows to construct improved actions by promoting k to an external field on spacetime directly, $k = k(x^\mu)$, or by identifying it with a function of the field strength. The latter corresponds to a cutoff identification of Type III then. This procedure leads to additional terms in the equations of motion. A more detailed discussion of the approach can be found in [328–333].

In view of the somewhat heuristic character of these strategies it needs to be emphasized that in principle *cutoff identifications can be avoided completely*, albeit at the expense of by far harder explicit calculations.

11.3 Structure of "Quantum Spacetime"

We start by discussing the implications of (11.2) for the case of an empty spacetime without matter. For concreteness we base the discussion in this chapter on the single-metric Einstein–Hilbert truncation. Occasionally the resolving power of the microscope is approximated by $\ell \approx 1/k$, implementing (11.4) on not too strongly curved spacetimes.

(1) Example: RG trajectories of Type IIIa. Considering solutions of (11.2) that exist on all scales from the UV down to $k=0$, the simplest situation arises when all metrics $\bar{g}_k^{\rm sc}$, $k \in [0, \infty)$, can be put on the same manifold $\mathcal{M}$. Then the "running spacetime" can have the same topological structure on all scales and the delicate issue of a topology change does not arise. This situation is realized along the RG trajectories of the Type IIIa. Along them, the cosmological constant is always positive, $\Lambda_k > 0$, $k \in [0, \infty)$; see Section 5.3.4.

Under those circumstances any fixed metric $\mathring{g}_{\mu\nu}$ on $\mathcal{M}$ that solves Einstein's equation with cosmological constant $\mathring{\Lambda}$, i.e., $G_{\mu\nu}(\mathring{g}) = -\mathring{\Lambda}\, \mathring{g}_{\mu\nu}$, immediately implies a family of self-consistent backgrounds [322]:

$$\boxed{\bar{g}^{\rm sc}_{k\,\mu\nu} = \frac{\mathring{\Lambda}}{\Lambda_k} \mathring{g}_{\mu\nu}.} \tag{11.7}$$

Taking advantage of the identity $R^{\mu}{}_{\nu}(cg) = (\frac{1}{c}) R^{\mu}{}_{\nu}(g)$, valid for any constant $c > 0$, we see that the rescaled metrics (11.7) indeed solve the tadpole equation (11.2).

The Type IIIa trajectories feature a classical regime at low scales where the dimensionful couplings do not run. Henceforth we identify $\mathring{\Lambda}$ with the value of the cosmological constant in this regime, and interpret $\mathring{g}_{\mu\nu}$ as the metric of the classical macroscopic world. When we increase k from the IR towards higher scales, Λ_k increases monotonically. As a consequence, the running self-consistent metric "shrinks" relative to the classical one. While $\Lambda_k \propto k^4$ in the semiclassical regime, near $k \approx m_{\rm Pl}$ the trajectory enters the fixed point regime where

$$\Lambda_k = k^2 \lambda_*, \quad \text{and} \quad \bar{g}^{\rm sc}_{k\,\mu\nu} = \frac{1}{k^2} \frac{\mathring{\Lambda}}{\lambda_*} \mathring{g}_{\mu\nu}. \tag{11.8}$$

The behavior (11.8) applies to *all* asymptotically safe RG trajectories near the NGFP.

(2) Self-similarity in the fixed point regime [163]. Let us consider a reference spacetime $(\mathcal{M}, \mathring{g})$ which displays a certain feature (radius of curvature, etc.) with a characteristic length L_0. We consider L_0 a proper distance with respect to $\mathring{g}$, and assume that it is determined by the only dimensionful quantity in Einstein's equation, $\mathring{\Lambda}$. Hence, up to factors of order unity, $L_0 \approx 1/\sqrt{\mathring{\Lambda}}$. Then, by

(11.7) the very same feature, but this time measured with the metric $\bar{g}_k^{\mathrm{sc}}$ has the proper size:

$$L(k) = \sqrt{\frac{\mathring{\Lambda}}{\Lambda_k}} L_0 \approx \sqrt{\frac{1}{\Lambda_k}}. \tag{11.9}$$

If we explore spacetime at a certain length scale, ℓ, characteristic of the experiment or observation performed, the action Γ_k at, roughly, $k \approx 1/\ell$, should provide the corresponding effective field theory description.

So, the characteristic length of the spacetime observed, measured with the optimally suited metric $\bar{g}_k^{\mathrm{sc}}\big|_{k\approx\ell^{-1}}$, is perceived to be $L(\ell) \equiv L(k = \ell^{-1}) \approx (\Lambda_{k=\ell^{-1}})^{-1/2}$.

In the special case of scales belonging to the fixed point regime where $\Lambda_k \propto k^2 \propto 1/\ell^2$ we thus have

$$\boxed{L(\ell) \propto \ell.} \tag{11.10}$$

This is a remarkable result: when we observe a given structure in spacetime with a "microscope" of variable resolution $\ell \ll \ell_{\mathrm{Pl}}$, this structure seems to possess a characteristic length which is always *of the same order as the experiment's resolution*, no matter to which level of precision we tune the microscope.

If we try to see finer details in the picture, we decrease ℓ by increasing k. This increases Λ_k, and the larger cosmological constant in turn leads to a shorter proper length $L(k) \propto 1/\sqrt{\Lambda_k}$. In the fixed point regime, the two opposing trends "increasing resolving power for higher k" and "shrinking detail of a shrinking universe due to larger $\Lambda_k \propto k^2$" cancel precisely, and so the picture we see in the microscope does not become more detailed when we turn its adjusting screw.

The scaling relation (11.10) suggests that the QEG spacetime when observed below the Planck length displays a self-similar structure and more generally may share certain properties with a *fractal*.

On sub-Planckian length scales the QEG spacetime has no intrinsic scale because in the fractal regime, i.e., when the RG trajectory is still close to the NGFP, the parameters which usually set the scales of the gravitational interaction, G_k and $\bar{\lambda}_k$, are not yet "frozen out." This happens only later on, somewhere half way between the non-Gaussian and the Gaussian fixed point, at a characteristic scale of the order of m_{Pl}.[1]

Below the Planck scale, G_k and Λ_k stop running and, as a result, the best-suited metric $\bar{g}_k^{\mathrm{sc}}$ becomes independent of k so that $L(\ell) = \mathrm{const}$ for $\ell \gg \ell_{\mathrm{Pl}}$. In this regime spacetime is well described by a single classical Riemannian manifold and we are in the familiar realm of General Relativity.

[1] Recall that the Planck mass and length are defined in terms of Newton's constant in the classical regime, G_{class}, i.e., $m_{\mathrm{Pl}} \equiv \ell_{\mathrm{Pl}}^{-1} \equiv G_{\mathrm{class}}^{-1/2}$.

(3) Fundamental limitation on the distinguishability of spacetime points [230]. It has been argued (in the example $\mathcal{M} = S^4$, but the conclusions probably hold much more generally) that there exists a fundamental obstruction which makes it impossible to physically distinguish spacetime points that are too close [230].

The underlying mechanism is the same which led to the self-similarity above: In the Planck regime, the (familiar) effect that a more energetic probe yields a better resolution is precisely compensated for by the backreaction of the spacetime.

Roughly speaking, the more energetic the probe is, the larger the relevant value of the cosmological constant, and the smaller the overall scale of the universe. This renders the S^4 "fuzzy" in the sense that there is a *minimal angular separation* below which two points cannot be resolved by any experiment.

But the interpretation of this effect is subtle, though [230, 334]. In particular, care is required in associating a *smallest possible proper length* to the indistinguishably close spacetime points. As it turns out, there does indeed exist a non-zero lower bound on the proper distances computed with the fixed metric of the classical regime; however, there is no such bound for proper distances referring to the family of scale-dependent metrics.

Possible implications of this effect for black holes and cosmological spacetimes have been investigated in [231]. See also [335] for a related discussion.

(4) Dimensional reduction and graviton propagator [173]. A further argument indicating that in QEG the spacetime may have properties reminiscent of a fractal, namely a non-integer effective dimensionality is based on the anomalous dimension $\eta_{\mathrm{N}} \equiv \partial_t \ln G_k$.

The RG trajectories of the Einstein–Hilbert truncation (within its domain of validity) have $\eta_{\mathrm{N}} \approx 0$ for $k \to 0$ and $\eta_{\mathrm{N}} \approx -2$ for $k \to \infty$, the smooth change by two units occurring near $k \approx m_{\mathrm{Pl}}$. As a consequence, the *effective dimensionality* defined as $d_{\mathrm{eff}} \equiv d + \eta_{\mathrm{N}}$ equals 4 for $\ell \gg \ell_{\mathrm{Pl}}$ and 2 for $\ell \ll \ell_{\mathrm{Pl}}$.

We know that the UV fixed point has $\eta_{\mathrm{N}}(g_*, \lambda_*) = -2$ and can use this information in order to determine the momentum dependence of the dressed graviton propagator for momenta $p^2 \gg m_{\mathrm{pl}}^2$. Expanding Γ_k about flat space and omitting the standard tensor structures [173] we find the inverse running propagator $\widetilde{\mathcal{G}}_k(p)^{-1} \propto G_k^{-1} p^2$. The conventional dressed propagator $\widetilde{\mathcal{G}}(p)$ contained in $\Gamma \equiv \Gamma_{k=0}$ is obtained from $\widetilde{\mathcal{G}}_k$ in the limit $k \to 0$.

For $p^2 > k^2 \gg m_{\mathrm{Pl}}^2$ the relevant cutoff scale is the physical momentum p^2 itself. Hence, the k-evolution of $\widetilde{\mathcal{G}}_k(p)$ stops at the threshold $k = \sqrt{p^2}$. Therefore,

$$\widetilde{\mathcal{G}}(p)^{-1} \propto p^2 \, G_k^{-1} \Big|_{k=\sqrt{p^2}} \propto (p^2)^{1-\frac{\eta}{2}} \tag{11.11}$$

because $G_k^{-1} \propto k^{-\eta}$ when $\eta \equiv \eta_{\mathrm{N}}(g, \lambda)$ is approximately constant.

In flat spacetimes of d dimensions, and for $\eta \neq 2-d$, the Fourier transform of $\widetilde{\mathcal{G}}(p) \propto 1/(p^2)^{1-\eta/2}$ yields the following propagator in position space:

$$\mathcal{G}(x;y) \propto \frac{1}{|x-y|^{d_{\text{eff}}-2}} \quad \text{with} \quad d_{\text{eff}} \equiv d+\eta. \tag{11.12}$$

This form of the two-point function is well known from the theory of critical phenomena, for instance. (In the latter case it applies to large distances.)

Equation (11.12) is not valid directly at the NGFP. For, say, $d=4$ with $\eta=-2$ the dressed propagator is $\widetilde{\mathcal{G}}(p)=1/p^4$, and it has the following representation in position space:

$$\mathcal{G}(x;y) = -\frac{1}{8\pi^2} \ln(\mu|x-y|)\,. \tag{11.13}$$

Here μ is an arbitrary constant with the dimension of a mass. Obviously (11.13) has the same structure as a $1/p^2$*-propagator in two dimensions.*

Slightly away from the NGFP, before other physical scales intervene, the propagator is of the form (11.12). This demonstrates that the quantity η_{N} has indeed the interpretation of an anomalous contribution to the spacetime dimension: fluctuation effects modify the decay properties of $\mathcal{G}$ so as to correspond to a spacetime of effective dimensionality $d_{\text{eff}}=4+\eta_{\text{N}}$.

Thus the general properties of the RG trajectories imply a remarkable *dimensional reduction*: Spacetime, probed by a "graviton" with $p^2 \ll m_{\text{Pl}}^2$ is four-dimensional, but it appears to be two-dimensional for a graviton with $p^2 \gg m_{\text{Pl}}^2$ [173].

More generally, when the classical spacetime is d-dimensional, the anomalous dimension at the fixed point is $\eta_{\text{N}}^*=2-d$. Therefore, for any d, the effective dimensionality as implied by the RG running of Newton's constant is $d_{\text{eff}}=d+\eta_{\text{N}}^*=2$ [163, 173].

(5) Spectral dimension [295, 322]. Standard fractals admit various, in general inequivalent, notions which generalize the classical concept of a dimension [336, 337]. One of them, the *spectral dimension*, is particularly well suited for an application to quantum spacetimes. It has been evaluated both within the EAA-based continuum approach to Asymptotic Safety [322] and by Monte Carlo simulations of Causal Dynamical Triangulations [46, 47], and many other approaches to quantum gravity [338].

To define the spectral dimension we consider first the diffusion process where a spinless test particle performs a Brownian random walk on an ordinary Riemannian manifold with a fixed classical metric $g_{\mu\nu}(x)$. This process is described by the heat kernel $K_g(x,x';T)$ discussed in Appendix D. The heat kernel gives the probability density for a transition of the particle from x to x' during the fictitious time T. It satisfies the heat equation

$$\partial_T K_g(x,x';T) = -\Delta_g K_g(x,x';T), \tag{11.14}$$

where $\Delta_g \equiv -D^2$. In flat space, this equation is solved by

$$K_g(x,x';T) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{ip\cdot(x-x')}\, e^{-p^2 T}. \tag{11.15}$$

In general, the heat kernel is a matrix element of the operator $\exp(-T\Delta_g)$. In the random walk picture its trace per unit volume,

$$P_g(T) = V^{-1} \int \mathrm{d}^d x\, \sqrt{g(x)}\, K_g(x,x;T) \equiv V^{-1}\, \mathrm{Tr}\, \exp(-T\Delta_g), \tag{11.16}$$

with $V \equiv \int \mathrm{d}^d x \sqrt{g(x)}$, has the interpretation of an average return probability.

We know that P_g possesses an asymptotic early time expansion (for $T \to 0$) of the form $P_g(T) = (4\pi T)^{-d/2} \sum_{n=0}^{\infty} A_n T^n$, with A_n denoting the Seeley–DeWitt coefficients. This expansion motivates the definition of a scale-dependent spectral dimension $D_s(T)$ as the logarithmic derivative

$$D_s(T) \equiv -2 \frac{d \ln P_g(T)}{d \ln T}. \tag{11.17}$$

The standard definition of the spectral dimension is obtained from this expression in the limit $d_s \equiv \lim_{T\to 0} D_s(T)$. On classical manifolds, where the early time expansion of $P_g(T)$ is valid, d_s agrees with the topological dimension d of the manifold.

Coming back to quantum gravity where we integrate over all metrics on a given manifold, the idea is to define a spectral dimension for quantum spacetimes by replacing the return probability $P_g(T)$ in (11.17) with its expectation value:

$$P(T) = \langle P_{\widehat{g}}(T) \rangle \equiv \int \mathcal{D}\widehat{g}\mathcal{D}C\mathcal{D}\bar{C}\; P_{\widehat{g}}(T)\, \exp\big(-S_{\text{bare}}[\widehat{g}, C, \bar{C}]\big)\,. \tag{11.18}$$

Here $\widehat{g}_{\mu\nu}$ denotes the microscopic metric, and S_{bare} is the bare action related to the UV fixed point with the gauge-fixing and ghost terms included.

In [322] the expectation value (11.18) has been evaluated by mean field techniques thereby making essential use of the family of metrics $\{\bar{g}_k^{\text{sc}}\}$. The result was that along the RG trajectory the spectral dimension interpolates smoothly between its macroscopic value $D_s(T) = d$ in the classical regime, and

$$D_s(T) = \frac{1}{2}\, d \qquad \text{(NGFP regime)} \tag{11.19}$$

near the non-Gaussian fixed point. For the derivation of this result we refer to [322].

It is remarkable that the *spectral* dimension shows the same dimensional reduction from four macroscopic to two microscopic dimensions which was found on the basis of the logically independent *effective* dimension $d_{\text{eff}} \equiv d + \eta_{\text{N}}$. It is perhaps even more remarkable that d_{eff} and d_s agree in the NGFP regime if, and only if, $d = 4$.

A more detailed discussion of the fractal-like properties of the QEG spacetime can be found in [295, 322]. See also [191] and [339] for analyses using higher derivative and bi-metric truncations, respectively.

Furthermore, it is intriguing that the numerical simulations in the CDT approach also find evidence for a similar dimensional reduction from four to about two dimensions [46, 47]. See [295, 340] for a detailed comparison.

(6) The NGFP: entrance gate to a new phase of gravity? In the fixed point regime, the relation (11.8) implies that proper lengths computed with respect to $\bar{g}^{\rm sc}_{k\,\mu\nu}$ and $\mathring{g}_{\mu\nu}$, respectively, differ by a constant factor. It is proportional to $1/k$ and vanishes in the limit $k\to\infty$ therefore: $\bar{g}^{\rm sc}_{k\,\mu\nu} \propto \mathring{g}_{\mu\nu}/k^2 \to 0$.

With all due caution, one may speculate that the vanishing of all running lengths measured with $\bar{g}^{\rm sc}_{k\,\mu\nu}$ relative to the corresponding classical lengths given by $\mathring{g}_{\mu\nu}$ reflects that, at asymptotically high scales, a new phase of gravity is approached in which the metric has a vanishing expectation value. This hints at the possibility of a considerable symmetry enhancement at the fixed point as presumably diffeomorphism invariance is unbroken in the new phase.

11.4 Black Holes

In this section we specialize to a particularly interesting class of effective spacetimes, namely black holes. We follow the work in [157] and consider the case of static, spherically symmetric spacetimes with a point mass M situated at the origin. The self-consistent metrics obtained in this case are given by a one-parameter family of Schwarzschild metrics:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{11.20}$$

where the lapse function contains the scale-dependent Newton's constant:

$$f(r) = 1 - \frac{2\,G_k\,M}{r}. \tag{11.21}$$

The flow of G_k depicted in Figure 5.2 then implies that the one-parameter family of self-consistent metrics interpolates between the Schwarzschild solution in the classical regime and flat Minkowski space as $k\to\infty$. This feature has drastic consequences for the structure of the quantum spacetime close to the central singularity at $r=0$.

In order to understand these consequences, it is plausible to apply the cutoff identification (11.4). Furthermore, following [157] we choose $d(x,x')$ as the proper distance between the origin and the point x, given by the classical line element, along a purely radial curve $\mathcal{C}_r$. The resulting function $d(r)$ is given by

$$d(r)|_{r<2G_0M} = 2G_0M\arctan\sqrt{\frac{r}{2G_0M-r}} - \sqrt{r(2G_0M-r)} \tag{11.22}$$

$$d(r)|_{r>2G_0M} = \pi G_0 M + 2G_0 M \ln\left(\sqrt{\frac{r}{2G_0M}} + \sqrt{\frac{r}{2G_0M} - 1}\right) + \sqrt{r(r-2G_0M)} \tag{11.23}$$

for points inside and outside the classical horizon, respectively. The two expressions match at $r = 2G_0M$. For $r \ll 2G_0M$, (11.22) possesses the expansion

$$d(r)|_{r\ll 2G_0M} \approx \frac{2}{3}\frac{1}{\sqrt{2G_0M}} r^{3/2} + O(r^{5/2}), \tag{11.24}$$

while the asymptotic behavior for large r resulting from (11.23) is

$$d(r)|_{r\gg 2G_0M} \approx r + O(r^0). \tag{11.25}$$

For analytic calculations it is convenient to approximate the distance function by

$$d(r) = \sqrt{\frac{2r^3}{2r + 9G_0M}}. \tag{11.26}$$

This simple formula has the correct qualitative properties, and in particular, interpolates smoothly between the two scaling regimes obtained as $r \to 0$ and $r \to \infty$, respectively.

Substituting the interpolation function (11.26) into the cutoff identification $k(r) = \xi/d(r)$ gives

$$k(r) = \xi\sqrt{\frac{2r + 9G_0M}{2r^3}}. \tag{11.27}$$

We now insert this cutoff identification into the solution (11.21). It would be straightforward to carry out this improvement on the basis of the scale dependence encoded in the beta functions (5.108) using numerical methods. In order to get analytic access to the system, one can approximate the scale dependence of Newton's constant by the one-loop result (5.82). Combining it with (11.27) then yields the improved radial function:

$$\boxed{f(r) = 1 - \frac{4G_0Mr^2}{2r^3 + \tilde{\omega}G_0(2r + 9G_0M)}.} \tag{11.28}$$

Here we abbreviated $\tilde{\omega} \equiv \omega\xi^2$. This lapse function determines all properties of the quantum-improved black hole such as its horizon structure, asymptotic behavior, and thermodynamics.

Figure 11.1 shows the improved radial function for various values of M. The analysis leads to the following main conclusions:

(1) Asymptotic behavior. For large r the improved lapse function approaches the classical one,

$$f(r) = 1 - \frac{2G_0M}{r}\left(1 - \tilde{\omega}\frac{G_0}{r^2}\right) + O\left(\frac{1}{r^4}\right). \tag{11.29}$$

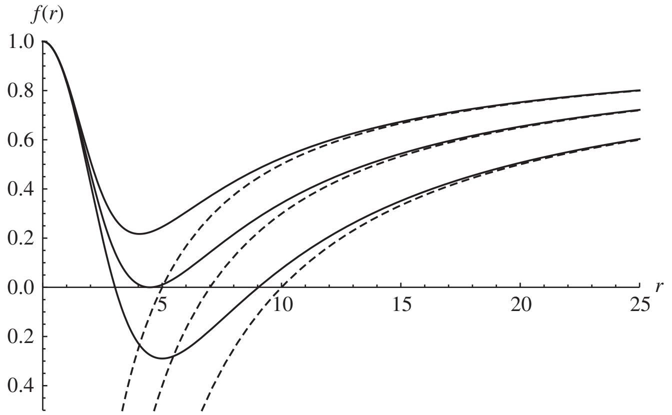


Figure 11.1. The radial function (11.29) for $G_0 = 1$, $\tilde{\omega} = \frac{118}{15\pi}$, and different values of the mass, $M = 2.5, 3.5027, 5$ (top, middle, bottom). The classical behavior represented by the dashed lines is shown for comparison.

This can also be seen in Figure 11.1. Furthermore, (11.29) allows us to read off the leading correction to Newton's potential. For $\tilde{\omega} = 118/15\pi$ this correction is in agreement with the perturbative result obtained in [3]. This agreement can be used to fix the hitherto undetermined parameter ξ.

(2) Regularity. The classical Schwarzschild solution possesses a curvature singularity at $r = 0$ where the square of the Riemann tensor (Kretschmann scalar) diverges. For the improved spacetime the fate of this singularity can be investigated by expanding (11.28) for small r:

$$f(r) \approx 1 - \frac{4r^2}{9\tilde{\omega}G_0} + O(r^3). \tag{11.30}$$

We see that the improved radial function is actually well-behaved at $r = 0$ and has the form of the de Sitter metric with an effective cosmological constant $\Lambda_{\text{eff}} = 4/(3\tilde{\omega}G_0) > 0$. Assuming that the RG improvement is reliable even for $r \to 0$, this feature would amount to a *de Sitter core* of the black hole [157]. The regularity of the improved spacetime is confirmed by calculating the square of its Riemann tensor. With $m_{\text{Pl}} \equiv G_0^{-1/2}$,

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 2\,(4/3)^3\,\tilde{\omega}^2 m_{\text{Pl}}^4 + O(r). \tag{11.31}$$

In contrast to the classical Schwarzschild black hole, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is finite, pointing at the resolution of the classical black hole singularity.

(3) Horizon structure. The horizon structure encoded in the zeros of the improved radial function (11.28) turns out to be richer than in the classical case. As it can be seen from Figure 11.1 the number of horizons, N_{H}, depends on whether M is larger or smaller than a certain *critical mass* $M_{\rm cr}$:

$$N_{\rm H} = \begin{cases} 0 & \text{for } M < M_{\rm cr} \\ 1(\text{double}) & \text{for } M = M_{\rm cr} \\ 2 & \text{for } M > M_{\rm cr} \end{cases} \quad . \tag{11.32}$$

For $M > M_{\rm cr}$ there exists both an outer and an inner horizon, situated at r_+ and r_-, respectively. If the mass of the black hole equals the critical mass, the two horizons coincide, while for $M < M_{\rm cr}$ there is no horizon at all. For the interpolation (11.26) with $\tilde{\omega} = \frac{118}{15\pi}$ the critical mass is found to be $M_{\rm cr} \approx 3.5\, m_{\rm Pl}$.

(4) Causal structure. Apart from its regularity at $r = 0$, the causal structure of the spacetime with the improved radial function is very similar to the one of a classical Reissner–Nordström black hole [341]. For $M > M_{\rm cr}$, one can distinguish five main regions:

$$\begin{aligned} \text{I and V}: &\quad r_+ < r < \infty, \\ \text{II and IV}: &\quad r_- < r < r_+, \\ \text{III and III}': &\quad 0 < r < r_-. \end{aligned} \tag{11.33}$$

It is straightforward to calculate the geodesics of this spacetime. A typical geodesic is depicted in the Penrose diagram of Figure 11.2. Starting in region I ($r < \infty$) at rest, the test mass will fall into the black hole, pass the horizon r_+, and transit the region II with inverted time direction. When passing region III, the test mass is repelled outwards, back to regions with larger r, namely IV and V. This is in full analogy to the Reissner–Nordström case. However, one difference is that the improved lapse function allows a particle with sufficient energy (positive kinetic energy at infinity) to actually reach the now non-singular point $r = 0$.

(5) Thermodynamics. Following the standard discussion of black hole thermodynamics in Euclidean time and imposing regularity at the cone of the resulting $R^2 \times S^2$ topology, the temperature of a horizon in the spacetime with line element (11.20) is given by [342]

$$T_{\rm BH} = \frac{1}{4\pi} \partial_r f(r) \Big|_{r = r_{\rm H}} \tag{11.34}$$

For the improved lapse function (11.28) this relation yields the following temperature of the outer black hole horizon at $r_+ \equiv r_+(M)$:

$$T_{\rm BH} = \frac{1}{2\pi} M G_0 \frac{\left(r_+^2 - \tilde{\omega}\, G_0 \left(1 + 9\,\tilde{\omega}\, \frac{G_0 M}{r_+}\right)\right)}{\left(r_+^2 + \tilde{\omega}\, G_0 (1 + \frac{9 G_0 M}{2\, r_+})\right)^2} \quad . \tag{11.35}$$

Since also the radius r_+ is a function of M, we plot the resulting temperature $T_{\rm BH}(M)$ in Figure 11.3. One observes that for masses $M > M_{\rm cr}$ the temperature

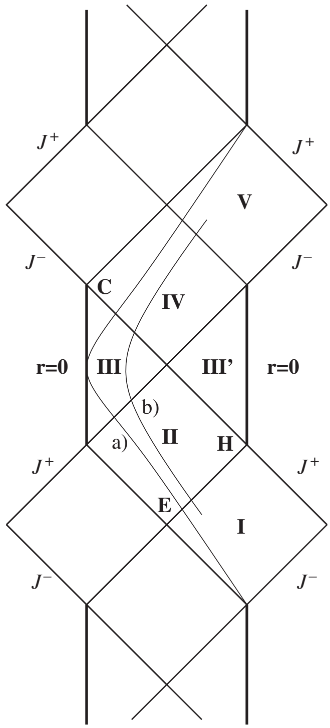


Figure 11.2. Penrose diagram for the improved Schwarzschild spacetime [157]. Depicted are two geodesics of radially infalling particles, one of a particle with positive kinetic energy at infinity, and another one of a particle starting at rest at finite r. One observes that the latter does not reach the origin at $r = 0$.

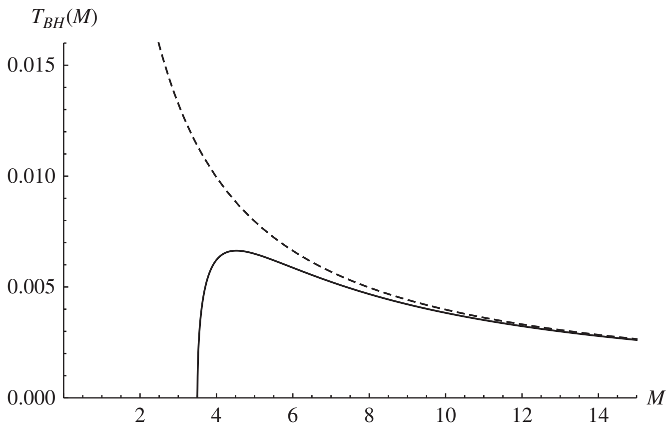


Figure 11.3. Mass dependence of the Hawking temperature for the improved Schwarzschild black hole in Planck units (solid line) for $\tilde{\omega} = \frac{118}{15\pi}$. For comparison the standard semiclassical temperature is also shown (dashed line).

follows the same $1/M$ dependence as in the semiclassical case [342]. However, when M approaches $M_{\rm cr}$, the temperature of the improved black hole drops to zero. This might indicate the formation of a stable black hole remnant as the endpoint of the evaporation process.

The original analysis of RG improved black holes [157] which we outlined here has been extended in various ways; see [343] for a detailed review.

In particular, the dynamics of black hole evaporation has been studied on the basis of the Vaidya metric in [344], while the RG improved Kerr metric has been constructed and analyzed in [345, 346]. The structure of the Cauchy horizon singularity of a black hole formed in a generic collapse was studied in [347]. In [348] the results of [157] have been reanalyzed from a thermodynamics perspective, and the role of higher derivative terms in the Effective Average Action has been explored in [349]. Furthermore, RG improvements including scale-dependent surface terms have been carried out in [182] and an equation highlighting the state-counting properties of Γ_k has been proposed. Moreover, the effect of a scale-dependent cosmological constant has been studied in [350, 351]. Further aspects of RG improved black hole geometries have been explored in [352–357]. Quite surprisingly, it has been found [358] that the improved Schwarzschild spacetime shares many properties of the "Planck stars" found in the context of Loop Quantum Gravity.

In general, these works showed that quantum effects originating in the vicinity of the nontrivial UV fixed point lead to drastic modifications of the classical picture at short distances or high momenta, while the long-distance properties of the improved black hole spacetime essentially agree with classical General Relativity.

11.5 Cosmology

As a last example, let us discuss the possible implications of QEG in the Friedmann–Lemaître cosmological models [323, 359, 360].

A natural starting point for investigating the imprints of Asymptotic Safety in cosmology studies the self-consistent solutions of (11.2) for the homogeneous and isotropic, spatially flat Robertson–Walker metrics:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]. \tag{11.36}$$

This ansatz is supplemented by a stress-energy tensor corresponding to a perfect fluid,

$$T_\mu{}^\nu = \mathrm{diag}[-\rho, p, p, p], \tag{11.37}$$

satisfying the equation of state $p = w\rho$. The effect of scale-dependent couplings is then captured by the dynamics of the scale factor $a(t)$.

The complete cosmic histories resulting from this framework have been studied in a series of works [323, 331, 359–361]. This section briefly summarizes the main results of these investigations. A more detailed review can be found in [362]; see also [363, 364].

Substituting the ansatz for the metric, (11.36), and the stress-energy tensor (11.37) into (11.2), results in the standard Friedmann equations with Newton's coupling and the cosmological constant depending on the scale k. Promoting k to a function of the cosmic time, $k \to k(t)$, the dynamics is governed by the improved Friedmann and continuity equations:

$$\boxed{\begin{aligned} &H^2 = \frac{8\pi}{3} G(t)\rho + \frac{1}{3}\Lambda(t), \\ &\dot{\rho} + 3H(\rho + p) = -\frac{\dot{\Lambda} + 8\pi\rho\,\dot{G}}{8\pi G(t)}. \end{aligned}} \tag{11.38}$$

The second equation arises by combining the Bianchi identity and Einstein's equation, $D^\mu[\Lambda(t)\, g_{\mu\nu} - 8\pi G(t)\, T_{\mu\nu}] = 0$. The extra term on its right-hand side has the interpretation of an energy transfer between the gravitational degrees of freedom and matter. Introducing the critical density $\rho_{\mathrm{crit}} \equiv 3H(t)^2/(8\pi G(t))$ and defining the relative densities $\Omega_{\mathrm{matter}} = \rho/\rho_{\mathrm{crit}}$ and $\Omega_\Lambda = \rho_\Lambda/\rho_{\mathrm{crit}}$ the first equation is equivalent to $\Omega_{\mathrm{matter}} + \Omega_\Lambda = 1$.

(1) Cosmology in the fixed point regime. We first focus on the very early part of the cosmological evolution where the RG running of G_k and Λ_k is controlled by the NGFP:

$$G_k = g_*\, k^{-2}, \qquad \Lambda_k = \lambda_*\, k^2. \tag{11.39}$$

Identifying the scale k with the Hubble parameter, as suggested by (11.5), the system (11.38) has the analytic solution

$$H(t) = \frac{\alpha}{t}, \qquad a(t) = A\, t^\alpha, \qquad \alpha = \left[\tfrac{3}{2}(1+w)(1-\Omega_\Lambda^*)\right]^{-1}, \tag{11.40}$$

together with

$$\rho(t) = \widehat{\rho}\, t^{-4}, \quad G(t) = \widehat{G}\, t^2, \quad \Lambda(t) = \widehat{\Lambda}\, t^{-2}. \tag{11.41}$$

Here A is a positive parameter. The constants $\widehat{G}$, $\widehat{\Lambda}$, and $\widehat{\rho}$ are fixed by the coordinates of the NGFP and the free parameter ξ,

$$\widehat{\rho} = \frac{3}{8\pi}\frac{\xi^2\alpha^4}{g_*}\left(1 - \frac{1}{3}\lambda_*\xi^2\right), \quad \widehat{G} = \frac{g_*}{\xi^2\alpha^2}, \quad \widehat{\Lambda} = \lambda_*\, \xi^2\, \alpha^2. \tag{11.42}$$

The vacuum energy density in the fixed point regime,

$$\Omega_\Lambda^* = \tfrac{1}{3}\, \lambda_*\, \xi^2, \tag{11.43}$$

takes values in the interval $0 < \Omega_\Lambda^* < 1$. For radiation dominance, $w = 1/3$, and the fixed point (5.95), one typically has $1/2 < \alpha < 2$, but α may be made arbitrarily large by choosing the value of ξ such that $\Omega_\Lambda^* \simeq 1$.

The cosmological solutions (11.40) possess *no particle horizon* if $\alpha \geq 1$, while for $\alpha < 1$ there is a horizon of radius $r_{\rm H} = t/(1-\alpha)$.

Moreover, the universe described by the fixed point solution is in a phase of *power law inflation* if $\alpha > 1$. Assuming radiation dominance, $w = 1/3$, this requires $\Omega_\Lambda^* > 1/2$. For the NGFP coordinates (5.95) this corresponds to $2.13 < \xi < 3.02$.

Remarkably, the asymptotic behavior of the solution for $t \to 0$[2] is actually independent of the chosen improvement scheme: all choices entail $k \propto t^{-1} +$ *subleading*, corroborating the robustness of the improvement procedure.

(2) Quantum gravity-driven inflation. Realizing an inflationary phase during the fixed point regime by having $\Omega_\Lambda^* \geq 1/2$ is a rather attractive scenario. Being driven by quantum gravity effects, more precisely by the originally large cosmological constant, *inflation ends automatically* at a certain transition time $t_{\rm tr}$ when the RG flow enters into the classical regime and $\Lambda(t)$ is sufficiently small. For $t > t_{\rm tr}$ the evolution is then given by a classical Friedmann–Lemaître–Robertson–Walker universe.

Thus the NGFP-driven inflationary phase occurs in a completely natural way and does not require any extra ingredients like an inflaton or even a specific inflaton potential.

Quite remarkably, the NGFP-driven inflation may leave observable imprints in the cosmic fluctuation spectrum.

The transition time $t_{\rm tr}$ is determined by the scale k where the underlying RG trajectory enters into the classical regime when $k \lesssim m_{\rm Pl}$, $H(t_{\rm tr}) \approx m_{\rm Pl}$. Since $\xi = O(1)$ the relation $H(t) = \alpha/t$ then leads to the estimate

$$t_{\rm tr} = \alpha\, t_{\rm Pl}. \tag{11.44}$$

If Ω_Λ^* is very close to one, i.e., if $\alpha \gg 1$, the cosmic time $t_{\rm tr}$ when the Hubble parameter is of order $m_{\rm Pl}$ can be much larger than the Planck time. The latter is then located well within the NGFP regime.

We now consider the evolution of a fluctuation with comoving length Δx. The corresponding proper length is $L(t) = a(t)\Delta x$. During the NGFP regime, $L(t)$ is related to the proper length at the transition time $t_{\rm tr}$ via $L(t) = (t/t_{\rm tr})^\alpha\, L(t_{\rm tr})$. The ratio of $L(t)$ and the Hubble radius $\ell_{\rm H}(t)$ then evolves as

$$\frac{L(t)}{\ell_{\rm H}(t)} = \left(\frac{t}{t_{\rm tr}}\right)^{\alpha-1} \frac{L(t_{\rm tr})}{\ell_{\rm H}(t_{\rm tr})}. \tag{11.45}$$

[2] In [365, 366] an analogous fixed point cosmology due to a hypothetical IR fixed point of the RG flow has been used to describe the acceleration of the universe at late times ($t \to \infty$). It turned out to be consistent with all observational data available [367].
See also [368] for a general analysis of cosmologies with a time-dependent cosmological constant.

For $\alpha > 1$ the proper length of the object grows faster than the Hubble radius. Fluctuations which are of sub-Hubble size at early times can therefore cross the Hubble radius and become "super-Hubble"-size at later times.

For definiteness, let us consider a fluctuation which, at the transition time $t_{\rm tr}$, is e^N times larger than the Hubble radius. For this fluctuation, (11.45) implies

$$\frac{L(t)}{\ell_{\rm H}(t)} = e^N \left(\frac{t}{t_{\rm tr}}\right)^{\alpha-1} . \tag{11.46}$$

The time t_N where this fluctuation crosses the Hubble radius, $L(t_N) = \ell_{\rm H}(t_N)$, is given by

$$t_N = t_{\rm tr} \exp\left(-\frac{N}{\alpha - 1}\right) . \tag{11.47}$$

Thus, even for moderate values of α, NGFP-driven inflation easily magnifies fluctuations to a size many orders of magnitude larger than the Hubble radius.

Interestingly, the cosmological structures we observe today may indeed have crossed the Hubble radius during the NGFP regime. Considering the largest structures visible today and using the classical evolution equations to backtrace them in time to the point where $H = m_{\rm Pl}$, we find that their size back then was about $e^{60}\ell_{\rm Pl}$. Setting $N = 60$ the time t_{60} when these structures crossed the horizon can be estimated from (11.47). For $\alpha = 25$, $t_{60} = t_{\rm tr}/12.2 = 2.05 t_{\rm Pl}$. Thus t_{60} is one order of magnitude smaller than $t_{\rm tr}$. Hence, in this setting structures observed today may have their origin in the quantum gravity regime controlled by the NGFP.

(3) Scale-free fluctuation spectrum. The NGFP also offers a natural mechanism for generating a *nearly scale-free spectrum of primordial fluctuations* [359, 360]. This can be seen as follows: due to the anomalous dimension at the NGFP, $\eta_{\rm N} = -2$, the associated effective graviton propagator (at the background level) has a characteristic $1/p^4$ dependence. This implies a logarithmic dependence for the two-point graviton correlator in configuration space, $\langle h_{\mu\nu}(x) h_{\rho\sigma}(y)\rangle \sim \ln(x-y)^2$. As a consequence, curvature fluctuations $\delta\mathbf{R} \propto \partial^2 h$ (where $\mathbf{R}$ stands for any component of the Riemann tensor) must behave as, symbolically, $\langle \delta\mathbf{R}(\mathbf{x},t)\delta\mathbf{R}(\mathbf{y},t)\rangle \propto 1/|\mathbf{x}-\mathbf{y}|^4$.

Now we make the natural assumption that the primordial matter density fluctuations $\delta\rho$ originate from the *fluctuations of the geometry itself.*

The classical Einstein equations then provide the relation $\delta\rho \propto \delta\mathbf{R}$. As a consequence, the correlation function $\xi(\mathbf{x}) \equiv \langle \delta(\mathbf{x})\delta(0)\rangle$ of the density contrast $\delta(\mathbf{x}) \equiv \delta\rho(\mathbf{x})/\rho$ behaves as [359, 360]

$$\xi(\mathbf{x}) \propto \frac{1}{|\mathbf{x}|^4} , \tag{11.48}$$

provided the physical distance $a(t)|\mathbf{x}|$ is smaller than the Planck length. From the three-dimensional Fourier transform of (11.48) we immediately get $|\delta_k|^2 \propto |\mathbf{k}|$, which amounts to a scale-invariant power spectrum, with spectral index $n = 1$ [369].

Clearly, small deviations from $n = 1$ are to be expected as the prediction of an exactly scale-free spectrum holds only directly at the NGFP. Since the NGFP is supposed to govern the dynamics of the very early universe this entails, in particular, that the power spectrum can acquire non-trivial corrections during the later cosmological epochs.

(4) Including anisotropy. In [370] the framework of isotropic Friedmann–Lemaître–Robertson–Walker cosmologies has been generalized to anisotropic Bianchi IX spacetimes. It was found that the chaotic approach to the initial singularity is removed by the scale-dependent couplings: the series of classical, chaotic Kasner oscillations ceases, and the solutions uniformly approach a point singularity as $t \to 0$. The transition between the chaotic Kasner oscillations and the non-chaotic approach to the singularity is triggered by quantum effects dominating over the classical dynamics when the value of the running cosmological constant becomes large.

(5) Toward realistic cosmologies. An interesting alternative to the analysis of the self-consistent solutions arising from (11.2) carries out a cutoff identification of Type III, (11.6), at the level of the action rather than a specific solution. For $k^2(x) = \xi^2 R$ with R being the Ricci scalar, this procedure results in quantum gravity inspired actions of the form $S[g] = \int \mathrm{d}^4x \sqrt{-g}\, f(R)$. In [333] it was found that the resulting modified gravity actions give rise to fluctuation spectra compatible with the data published by the Planck collaboration in 2015 [371].

12
Miscellanea

In the previous chapters of this monograph we introduced the basic ideas and methods underlying the "Asymptotic Safety program," which aims at establishing a concrete realization of Weinberg's scenario on the basis of the functional renormalization group.

Being a rapidly growing direction of research, it is clear that we could not cover all developments here. This concerns in particular the large and steadily increasing number of truncated RG flows that have been computed and analyzed explicitly. However, regarding the viability of the Asymptotic Safety program, the developments confirm the picture we have drawn here without any exception.

In this final chapter we list and briefly summarize a number of further, disconnected developments and investigations. A detailed, pedagogical account is likewise well beyond the scope of this book, and the reader is invited to consult the literature for further details.

(1) Infinite-dimensional truncations of $f(R)$-type. In Section 5.3.5 we discussed finite-dimensional truncation subspaces where the gravitational part of the Effective Average Action has the form

$$\bar{\Gamma}_k[g] = \int \mathrm{d}^d x \sqrt{g}\, f_k(R), \tag{12.1}$$

with $f_k(R)$ being a *polynomial* in the Ricci scalar. The results there rest on the fact that one can derive a partial differential equation (PDE) that governs the scale dependence of the function $f_k(R)$ without making any assumption on its functional form [185, 186]. The polynomial approximation of $f_k(R)$ can be thought of as an expansion approximating the flow in a regime where the dimensionless background curvature $r \equiv \bar{R}/k^2$ is small. In order to establish the consistency of the NGFP seen in these expansions, it is interesting to understand under which conditions these polynomial expansions can be extended to a *global* *solution of the PDE*, which is well-defined for all values of r. Since the resulting functions $f_k(R)$ and in particular the *fixed function* $f_*(R)$ carry information about an infinite number of coupling constants, these computations are said to employ an *infinite-dimensional truncation*.

Starting from [372], the properties of this class of truncated flow equations has been studied extensively in both $d=3$ [373–375] and in $d=4$ dimensions [376–383].

In general, the PDE governing the flow of $f_k(R)$ is not unique. It depends parametrically on the choices for the gauge-fixing term and coarse-graining operator. All constructions studied to date evaluate the flow on a maximally symmetric background satisfying (5.29). Furthermore, they employ a transverse-traceless decomposition [384, 385] of the fluctuation field:

$$h_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu} + \bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu + 2\bar{D}_\mu \bar{D}_\nu \sigma - \frac{2}{d}\bar{g}_{\mu\nu}\bar{D}^2\sigma + \frac{1}{d}\,\bar{g}_{\mu\nu}\,\phi. \tag{12.2}$$

The term *transverse-traceless decomposition* indicates that the component fields are transverse with respect to the covariant derivative constructed from the background metric and satisfy the differential constraints

$$\bar{D}^\mu h^{\mathrm{T}}_{\mu\nu} = 0, \qquad \bar{g}^{\mu\nu} h^{\mathrm{T}}_{\mu\nu} = 0, \qquad \bar{D}_\mu \xi^\mu = 0, \qquad \bar{g}^{\mu\nu} h_{\mu\nu} = \phi\,. \tag{12.3}$$

In practice the transverse-traceless decomposition has the highly welcome consequence that all terms containing a contraction of a covariant derivative with a fluctuation field vanish. In combination with a maximally symmetric background this ensures that all differential operators combine into Laplacians $\bar{D}^2 \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$. The operators under the traces on the right-hand side of the FRGE are then functions of $\bar{D}^2$ and $\bar{R}$. They can be evaluated using heat kernel techniques or as spectral sums.

In order to describe the new ideas and structures appearing in the context of the $f(R)$-type truncations, we focus here on the specific setting introduced in [376, 377]. This construction uses the freedom in choosing a gauge-fixing and coarse-graining scheme in such a way that the contributions of the gauge degrees of freedom, the ghost sector, and the Jacobians coming from the transverse-traceless decomposition precisely cancel. In other words, this setting is *minimal* in the sense that $\partial_t \Gamma_k$ receives non-zero contributions from the transverse-traceless modes $h^{\mathrm{T}}_{\mu\nu}$ and the trace modes ϕ only. The two remaining traces are evaluated as a sum over eigenvalues weighted by the corresponding degeneracies. Due to the regulator $\mathcal{R}_k$ they receive contributions from a finite number of terms only and thus converge. Applying a smoothing procedure for the step functions results in a two-parameter family of smooth partial differential equations. It is conveniently expressed in terms of the dimensionless function $\varphi_k(r)$, defined through

$$f_k(\bar{R}) \equiv k^4 \varphi_k\left(\bar{R}k^{-2}\right), \qquad r \equiv \bar{R}\,k^{-2}. \tag{12.4}$$

Explicitly, the PDE for $\varphi_k(r)$ is given by [377]:

$$\boxed{\begin{aligned}\dot{\varphi}_k + 4\varphi_k - 2r\varphi_k' = &\frac{\tilde{c}_1\varphi_k' + \tilde{c}_2(\dot{\varphi}_k' - 2r\varphi_k'')}{3\varphi_k - ((3\alpha_2+1)\,r-3)\varphi_k'} \\ &+ \frac{c_1\varphi_k' + c_2\varphi_k'' + c_3\dot{\varphi}_k' + c_4\,(\dot{\varphi}_k'' - 2r\varphi_k''')}{2\varphi_k + (3-(3\alpha_0+2)\,r)\varphi_k' + (3\alpha_0\,r + r - 3)^2\varphi_k''}.\end{aligned}} \tag{12.5}$$

Here dots and primes denote derivatives with respect to the renormalization group time t and the dimensionless curvature r, respectively. The coefficients c_i and $\tilde{c}_i$ are certain polynomials in r. They read

$$\begin{aligned}
c_1 &= -\frac{((6\alpha_0-1)\,r-6)\left(\alpha_0(6\alpha_0-1)\,r^2 + (10-48\alpha_0)r + 42\right)}{2304\pi^2}, \\
c_2 &= -\frac{((6\alpha_0-1)\,r-6)\left(\alpha_0\left(54\alpha_0^2-3\alpha_0-1\right)r^3 + \left(270\alpha_0^2+42\alpha_0-35\right)r^2 - 39(18\alpha_0+1)r + 378\right)}{4608\pi^2}, \\
c_3 &= -\frac{(\alpha_0 r-1)((6\alpha_0-1)r-6)^2}{2304\pi^2}, \\
c_4 &= \frac{(\alpha_0 r-1)((6\alpha_0-1)r-6)^2((9\alpha_0+5)r-9)}{4608\pi^2},
\end{aligned} \tag{12.6}$$

and

$$\begin{aligned}
\tilde{c}_1 &= -\frac{5(6\alpha_2 r + r - 6)\left(\left(18\alpha_2^2+9\alpha_2-2\right)r^2 - 18(8\alpha_2+1)r + 126\right)}{6912\pi^2}, \\
\tilde{c}_2 &= -\frac{5(6\alpha_2 r + r - 6)((3\alpha_2+2)r-3)((6\alpha_2-1)r-6)}{6912\pi^2}.
\end{aligned} \tag{12.7}$$

Furthermore, the parameters α_0 and α_2 in (12.5) capture the freedom of including an endomorphism term proportional to the Ricci scalar in the coarse-graining operator.

The first and second term on the right-hand side of (12.5) are the contributions of the transverse-traceless and the conformal fluctuations, respectively. Hence, the order of the PDE is determined by the conformal sector. The conformally reduced version of the full equation is obtained by dropping the first term. Thus the conformally reduced and the full equation have the same structure, suggesting that the conformally reduced setting provides a good approximation to the full equation.

By definition, fixed functions $\varphi_*(r)$ are the k-stationary solutions of the evolution equation (12.5). In general, a proper fixed function should satisfy the following criteria:

1. $\varphi_*(r)$ *should be globally well-defined:* By the general discussion of Chapter 3, a fixed point corresponds to a special type of complete RG trajectory where $\Gamma_k = \Gamma_*$ for all $k \in [0, \infty)$. When working on a maximally symmetric

background with fixed, positive scalar curvature, $\bar{R} > 0$, the transition to the dimensionless curvature $r = \bar{R}k^{-2}$ implies that the fixed function $\varphi_*(r)$ should be regular on the entire half-line $r \in [0, \infty]$.

2. $\varphi_*(r)$ *should be an isolated solution:* A fixed point of the renormalization group corresponds to a single point in theory space. At the level of functional truncations, this feature is reflected by the condition that $\varphi_*(r)$ should not be part of a multi-dimensional family of solutions.

3. $\varphi_*(r)$ *should admit a discrete spectrum of globally well-defined deformations:* Once a suitable fixed function satisfying the conditions 1. and 2. has been constructed, one should verify that the solution gives rise to a discrete set of globally well-defined deformations $\chi_n(r)$. This can be studied by substituting the ansatz

$$\varphi_k(t, r) = \varphi_*(r) + \epsilon\, e^{-\theta_n t} \chi_n(r) \tag{12.8}$$

into the PDE (12.5) and looking for globally well-defined solutions $\chi_n(r)$ of the linearized equation. The condition that $\chi_n(r)$ is globally well-defined is expected to put conditions on the admissible values of θ_n such that the stability coefficients are bounded from above and discrete. Since $\varphi_*(r)$ is typically known only numerically, carrying out this calculation in practice is computationally quite challenging. The best indications that the θ_n indeed form a discrete spectrum is then provided by the polynomial expansion of $f_k(R)$ as displayed in Table 5.3.

We now illustrate the implementation of these conditions for the PDE (12.5). Imposing that $\varphi_k(r)$ is independent of k leads to an ordinary differential equation (ODE) for the fixed function $\varphi_*(r)$:

$$\boxed{\begin{aligned} 4\varphi - 2r\varphi' = & \frac{\tilde{c}_1\varphi' - 2\tilde{c}_2 r\varphi''}{3\varphi - (3\alpha_2 r + r - 3)\varphi'} \\ & + \frac{c_1\varphi' + c_2\varphi'' - 2c_4 r\varphi'''}{(3\alpha_0 r + r - 3)^2\varphi'' + (3 - (3\alpha_0 + 2)r)\varphi' + 2\varphi}. \end{aligned}} \tag{12.9}$$

The coefficients c_i and $\tilde{c}_i$ are again given by (12.7) and (12.6). Since there is no risk of confusion we omit the asterisk and denote the fixed function by $\varphi(r)$ from now on.

Equation (12.9) is a non-linear third-order equation for $\varphi(r)$. Thus, locally, there is a three-parameter family of solutions to this equation. The condition that a fixed function should be globally well-defined may reduce the number of these free parameters. In particular, one may expect isolated fixed functions if there is a balance between the order of the ODE and its singularity structure [383]:

$$\text{Degree of ODE} - \text{number of singularities} = 0. \tag{12.10}$$

Thus any singularity reduces the number of free parameters by one.

For *fixed singularities* this feature can be understood as follows: Let r_i^{sing} denote the position of a fixed singularity of order one. Casting the ODE (12.9) into normal form by solving for $\varphi'''(r)$, the right-hand side of the resulting equation admits a Laurent expansion,

$$\varphi'''(r) = \frac{e_i(\varphi''(r_i), \varphi'(r_i), \varphi(r_i), r_i)}{r - r_i^{\text{sing}}} + \text{regular terms}, \tag{12.11}$$

where the explicit form of the e_i can be derived from (12.9). Demanding that $\varphi'''(r)$ does not diverge at r_i^{sing} requires that the functions e_i must vanish at $r = r_i^{\text{sing}}$. This puts a non-trivial boundary condition on admissible solutions which reduces the number of free parameters.

In principle the LHS of (12.10) also takes into account *movable singularities*, which arise from a φ-dependent denominator developing a zero dynamically. Their occurrence requires studying explicit solutions. However, so far, only fixed singularities played a role in restricting the fixed function $\varphi(r)$.

The set of fixed singularities for (12.9) is found by casting the equation into normal form by solving for $\varphi'''(r)$. The resulting expression indicates that, in general, the third derivative will be infinite if the prefactor rc_4 multiplying $\varphi'''(r)$ vanishes. From (12.6) one finds that this prefactor is a fifth order polynomial with roots located at

$$r_1^{\text{sing}} = 0, \quad r_2^{\text{sing}} = \frac{9}{5 + 9\alpha_0}, \quad r_3^{\text{sing}} = \frac{1}{\alpha_0}, \quad r_{4,5}^{\text{sing}} = \frac{6}{6\alpha_0 - 1}. \tag{12.12}$$

Obviously the position of the fixed singularities r_i^{sing}, $i = 2, 3, 4, 5$, depends on the "endomorphism" parameter α_0 which characterizes $\mathcal{R}_k$. The fixed singularities reveal that there are two distinguished values for α_0: r_2^{sing} is infinite if $\alpha_0 = -\frac{5}{9}$, while the double root $r_{4,5}^{\text{sing}}$ is shifted to infinity for $\alpha_0 = \frac{1}{6}$. The latter choice reduces the number of fixed singularities from five to three so that the balance equation (12.10) is satisfied and one expects that the system admits isolated global solutions.

The value of α_2 is not fixed by the global structure of the ODE. A conceivable, but far less cogent, selection principle for this value is provided by the condition of equal lowest eigenvalues, i.e., one implements that the smallest eigenvalues of the coarse-graining operators in the transverse-traceless and conformal sector agree. These conditions fix the free parameters to

$$\alpha_0 = \tfrac{1}{6}, \qquad \alpha_2 = -\tfrac{1}{2}. \tag{12.13}$$

The set of fixed singularities for this choice is then

$$r_1^{\text{sing}} = 0, \qquad r_2^{\text{sing}} = \tfrac{18}{13}, \qquad r_3^{\text{sing}} = 6\,. \tag{12.14}$$

At this stage the task of finding admissible fixed functions has been reduced to searching the solution space of the ODE, verifying that there are indeed globally

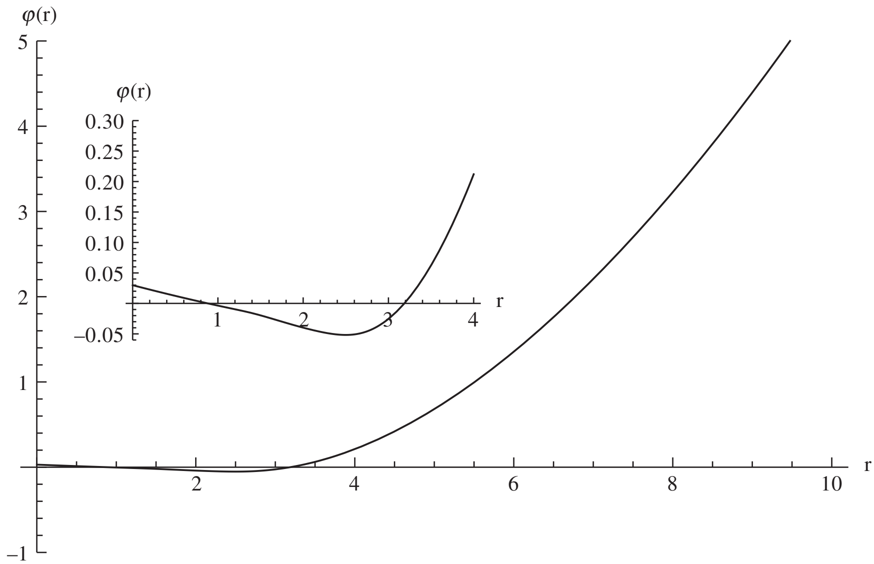


Figure 12.1. The isolated, globally well-defined fixed function $\varphi_*(r)$ constructed from imposing regularity at the three fixed singularities (12.14). The fixed function is bounded from below, comes with a positive cosmological constant λ_* and Newton's constant g_*, and displays a global minimum at $r_{\min} \approx 2.5$. Together with $\varphi_*(0) > 0$ this minimum ensures that the solution passes the redundancy test [382]. (From [377].)

well-defined solutions satisfying the non-local boundary conditions imposed by the fixed singularities.

Practically, this is done by applying a *shooting method*. The initial conditions for solving the ODE are set up at $r = 0$. The algorithm then uses analytic expansions around r_i^{sing} to move away from, or extrapolate a solution through, a fixed singularity. These expansions are connected by numerical solutions obtained on the intervals $[r_i^{\text{sing}} + \epsilon, r_{i+1}^{\text{sing}} - \epsilon]$, with $\epsilon = 10^{-3}$, say, where the ODE is regular.

This strategy indeed identifies an isolated, globally well-defined fixed function $\varphi_*(r)$. This solution is shown in Figure 12.1.

The expansion of the solution for $r \ll 1$ allows us to read off the values of the dimensionless cosmological constant and Newton's constant associated with the fixed function and the universal product $\tau_* = g_* \, \lambda_*$:

$$\lambda_* = 0.411, \qquad g_* = 0.547, \qquad \tau_* = 0.224. \tag{12.15}$$

Thus, $\varphi_*(r)$ gives rise to a positive fixed point value for λ_* and g_* in agreement with the results found in the Einstein–Hilbert truncation. The value of τ_* is in reasonable agreement with the results of the R^2-truncation (see Figure 5.5). Moreover, a stability analysis of the ODE (12.9) using a polynomial approximation of the fixed function shows that the stability coefficients obtained in this way are quite similar to the ones reported in Table 5.2. This provides a strong indication that the NGFP seen in finite-dimensional truncations indeed possesses an extension to a fixed function which includes an infinite number of coupling constants.

(2) The perturbative two-loop counterterm is asymptotically safe [386]. The perturbative quantization of the Einstein–Hilbert action gives rise to a divergence at two-loop order [2, 387, 388],

$$\Gamma^{\rm GS} = \frac{1}{\epsilon} \frac{209}{2880} \frac{1}{(16\pi^2)^2} \int \mathrm{d}^4x \sqrt{g}\, C_{\mu\nu}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\rho\sigma} C_{\rho\sigma}{}^{\mu\nu}, \tag{12.16}$$

with $C_{\mu\nu\rho\sigma}$ being the Weyl-tensor. This divergence is not of the form of the bare action and can thus not be absorbed into the bare coupling constants. It requires adding a new counterterm with an undetermined coefficient. In combination with power-counting arguments, this is taken as a signal that the perturbative quantization of the Einstein–Hilbert action requires fixing an infinite number of free parameters from experimental data, i.e., the Einstein–Hilbert action leads to a perturbatively non-renormalizable quantum field theory.

The occurrence of the two-loop divergence (12.16) triggered the investigation of a variety of alternative routes toward quantizing gravity, e.g., by modifying the quantization scheme or introducing new degrees of freedom and symmetries. Eventually, any theory of quantum gravity has to explain the fate of this divergence though.

For the Asymptotic Safety program, this question has been clarified in [386]. For this purpose the Einstein–Hilbert truncation, discussed in Section 5.2, is supplemented by the interaction term

$$\Gamma_k^{\rm GS} = \bar{\sigma}_k \int \mathrm{d}^4x \sqrt{g}\, C_{\mu\nu}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\rho\sigma} C_{\rho\sigma}{}^{\mu\nu}. \tag{12.17}$$

The beta functions $k\partial_k\, g_i \equiv \beta_{g_i}(\lambda, g, \sigma)$ for the three dimensionless couplings $g_i \equiv \{\lambda, g, \sigma\}$ can be constructed using the heat kernel techniques introduced in Appendix D. In particular, the computation of β_σ can be simplified by choosing a Ricci-flat background, $\bar{R}_{\mu\nu} = 0$.

The resulting system of beta functions exhibits two rather remarkable properties. Firstly, the new coupling σ_k does not enter the beta functions of the Einstein–Hilbert sector. The beta functions for g and λ are thus given by (5.63). For the optimized shape function (E.13) they read

$$\begin{aligned} \beta_g &= (2 + \eta_{\rm N})\; g, \\ \beta_\lambda &= (\eta_{\rm N} - 2)\; \lambda + \frac{g}{2\pi} \left(\frac{5}{1-2\lambda} - 4 - \frac{5}{6} \eta_{\rm N} \frac{1}{1-2\lambda} \right). \end{aligned} \tag{12.18}$$

The anomalous dimension of Newton's constant is of the usual form,

$$\eta_{\rm N} = \frac{g\, B_1}{1 - g\, B_2}, \tag{12.19}$$

with

$$B_1 = \frac{1}{3\pi}\left[\frac{5}{1-2\lambda} - \frac{9}{(1-2\lambda)^2} - 5\right], \; B_2 = -\frac{1}{6\pi}\left[\frac{5}{2}\frac{1}{(1-2\lambda)} - \frac{3}{(1-2\lambda)^2}\right]. \tag{12.20}$$

Secondly, β_σ is cubic in σ,

$$\beta_\sigma = c_0 + (2 + c_1)\,\sigma + c_2\,\sigma^2 + c_3\,\sigma^3, \tag{12.21}$$

with the following coefficients [386]:

$$\begin{aligned}
c_0 &= \frac{1}{64\pi^2(1-2\lambda)}\left[\frac{2-\eta_{\mathrm{N}}}{2(1-2\lambda)} + \frac{6-\eta_{\mathrm{N}}}{(1-2\lambda)^3} - \frac{5\eta_{\mathrm{N}}}{378}\right],\\
c_1 &= \frac{3g}{16\pi(1-2\lambda)^2}\left[5(6-\eta_{\mathrm{N}}) + \frac{23(8-\eta_{\mathrm{N}})}{8(1-2\lambda)} - \frac{7(10-\eta_{\mathrm{N}})}{10(1-2\lambda)^2}\right],\\
c_2 &= \frac{g^2}{2(1-2\lambda)^3}\left[\frac{233(12-\eta_{\mathrm{N}})}{10} - \frac{9(14-\eta_{\mathrm{N}})}{7(1-2\lambda)}\right],\\
c_3 &= \frac{6\pi g^3(18-\eta_{\mathrm{N}})}{(1-2\lambda)^4}.
\end{aligned} \tag{12.22}$$

As the coefficient $c_3(g,\lambda)$ is non-zero, the structure of (12.18) and (12.21) then guarantees that there exists always at least one NGFP which generalizes the NGFP of the Einstein–Hilbert truncation. For the optimized shape function, this fixed point is located at

$$\mathrm{NGFP}^{\mathrm{GS}}: \; \lambda_* = 0.193, \quad g_* = 0.707, \quad \sigma_* = -0.305, \tag{12.23}$$

and its stability coefficients are given by

$$\theta_{1,2} = 1.475 \pm 3.043\,i, \qquad \theta_3 = -79.39. \tag{12.24}$$

Thus, the inclusion of the two-loop counterterm in the truncation does not vanquish the NGFP responsible for Asymptotic Safety. Moreover, the negative stability coefficient indicates that there is no additional free parameter related to the enlarged truncation space. This explicit computation strongly supports the expectation that perturbative counterterms are actually harmless for the Asymptotic Safety mechanism.

(3) Paramagnetic dominance. One may ask if there is a simple physical property of the quantum dynamically interacting non-linear gravitons which is responsible for their inclination to form a NGFP. In [183] such a mechanism, referred to as paramagnetic dominance, has been identified, and intuitive analogies with better-understood physical phenomena were drawn (magnetism in solids, asymptotic freedom of gluons).

Quite generally, $\Gamma_k[h;g]$ gives rise to an inverse propagator for the $h_{\mu\nu}$-fluctuations which consists of a kinetic term involving covariant derivatives as well as potential terms built from the background Riemann tensor $\bar{R}_{\mu\nu\alpha\beta}$ and its contractions. For the Einstein–Hilbert truncation, say, we saw in (5.24) that the relevant terms read, symbolically and with suppressed indices, $\Gamma_k^{\text{quad}} \sim \int h\left[\bar{D}^2 + \bar{R}\right] h$. By analogy with elementary magnetic systems, $h\bar{D}^2 h$ and $h\bar{R}h$, respectively, describe dia- and paramagnetic-type interactions of the metric fluctuations $h_{\mu\nu}$ with a given background field $\bar{g}_{\mu\nu}$.

It was shown that in $d > 3$ the gravitational antiscreening and the formation of the NGFP in QEG is entirely due to a strong *dominance of the ultralocal paramagnetic interactions over the diamagnetic ones* that favor screening. (In $d < 3$ both the dia- and paramagnetic effects support antiscreening.) In [183, 389] the spacetimes of QEG are interpreted as a polarizable medium with a “paramagnetic” response to external perturbations. In this respect they show similarities with the vacuum state of Yang–Mills theory.[1]

(4) Composite operators. In applications of the EAA we may encounter *coevolving quantities*, $\mathcal{Q}_k$, which satisfy their own flow equation, with a beta functional depending on Γ_k though:

$$\partial_t \mathcal{Q}_k = \overline{\mathcal{B}}'_k\{\mathcal{Q}_k, \Gamma_k\} \quad \text{and} \quad \partial_t \Gamma_k = \overline{\mathcal{B}}_k\{\Gamma_k\}. \tag{12.25}$$

A typical example are composite operators that are not already included in Γ_k. In [243] a general EAA-based method was introduced to define and renormalize such operators.[2] In [229] it has been employed to compute the anomalous scaling dimensions of geometrically motivated observables (length and volume operators) appearing in quantum gravity.

The basic idea of this method is to repeat the derivation of the FRGE with the modified bare action $S + \varepsilon \cdot O$ which couples the composite operators of interest, $O_i(x)$, to sources $\varepsilon_i(x)$.[3] One may then take any number of functional derivatives of the FRGE with respect to the sources before setting $\varepsilon = 0$ at the end. For a single operator insertion, for example, the flow equation for $[O_k]_i \equiv \delta\Gamma_k/\delta\varepsilon_i$ then reads

$$\partial_t\big(\varepsilon \cdot [O_k]\big) = -\frac{1}{2}\,\mathrm{STr}\left[\Big(\Gamma_k^{(2)} + \mathcal{R}_k\Big)^{-1}\Big(\varepsilon \cdot [O_k]^{(2)}\Big)\Big(\Gamma_k^{(2)} + \mathcal{R}_k\Big)^{-1}\partial_t \mathcal{R}_k\right]. \tag{12.26}$$

To solve (12.26) one expands $[O_k]_i$ in a k-independent basis of operators, $[O_k]_i = \sum_j Z_{ij}(k) O_j$, so that (12.26) becomes a differential equation for the

[1] Along a different line of research further similarities (vacuum condensates) were found in [222, 390].
[2] For earlier work in a non-gravitational context also see [34, 391].
[3] The dot in expressions like $\varepsilon \cdot O$ symbolizes an integration over spacetime and a summation over any type of indices carried by ε and O.

mixing matrix $Z_{ij}(k)$. Letting $\gamma_{Z,ij} \equiv \left(Z^{-1}\partial_t Z\right)_{ij}$ it reads

$$\sum_j \gamma_{Z,ij} O_j(x) = -\frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\left(O_i^{(2)}(x)\right)\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}\partial_t \mathcal{R}_k\right]. \tag{12.27}$$

Projecting the right-hand side of (12.27) on the basis $\{O_j\}$ results in explicit expressions for $\gamma_{Z,ij}$ in terms of the couplings which parametrize Γ_k.

It can be shown [243] that the quantum-corrected scaling dimensions of the pertinent scaling fields are given by the eigenvalues of the matrix $d_i\delta_{ij} + \gamma_{Z,ij}$, with d_i denoting the classical mass dimension of O_i.

For a detailed discussion we refer to [243] where also the relation of those eigenvalues to the critical exponents θ_i is explained, and to [229] for applications in gravity.

(5) Gibbons–Hawking boundary term. On spacetime manifolds with a non-empty boundary, the action functionals for gravity and matter typically include certain boundary terms in order to admit a consistent classical variational principle [137]. The Einstein–Hilbert action, for example, is augmented by the Gibbons–Hawking surface term [392] with a prefactor which is fixed by that of the "bulk" term and therefore involves Newton's constant.[4] This surface term plays a crucial role in the Euclidean approach to black hole thermodynamics, for instance [342].

In [182], Quantum Einstein Gravity and its Asymptotic Safety were explored on spacetimes with a non-trivial boundary using the Einstein–Hilbert ansatz generalized by a Gibbons–Hawking term. The running prefactor of the latter determines a kind of *boundary Newton constant* which is allowed to differ from the familiar one in the bulk. Indeed, the two versions of Newton's constant were found to display *opposite* RG running. Invoking a scale-dependent variant of the ADM mass, it was argued that this is precisely the signature of gravitational anti-screening, and that the problem of the "bulk-boundary matching" [395] presents itself differently in the background field setting. In an application to black holes, indications were found that their specific heat capacity might turn positive at Planckian scales [182, 396].

(6) Counting of modes and a *C*-function. The gravitational EAA has natural mode counting and monotonicity properties. In [154] they were exploited to construct with the help of Γ_k a function $\mathscr{C}$, which has properties similar to Zamolodchikov's C-function [397, 398] but without being restricted to two dimensions though. This function $\mathscr{C}$ becomes stationary both at critical points and in classical regimes, and it decreases monotonically along RG trajectories

[4] For different boundary corrections see also [393, 394].

if split-symmetry is broken sufficiently weakly. The $\mathscr{C}$-function gives a precise meaning to the "number" of field modes, $\mathcal{N}$, that are integrated out between two scales k_1 and k_2, respectively: $\mathcal{N}_{k_1,k_2} \equiv \mathscr{C}_{k_2} - \mathscr{C}_{k_1}$.

For the particularly interesting case of generalized crossover trajectories connecting the NGFP in the UV to a classical regime with positive cosmological constant in the IR, the $\mathscr{C}$-function depends on a gravitational instanton which constitutes the background spacetime. In the case of four-dimensional de Sitter space, its Bekenstein–Hawking entropy $S_{\mathrm{BH}}^{\mathrm{dS}}$ was found to play a role analogous to the central charge in Zamolodchikov's theorem. As a direct implication of Asymptotic Safety the total number of modes turns out to be *finite*:

$$\mathcal{N} = \mathscr{C}^{\mathrm{UV}} - \mathscr{C}^{\mathrm{IR}} \approx 3\pi / G_{\mathrm{cr}} \Lambda_{\mathrm{cr}} = S_{\mathrm{BH}}^{\mathrm{dS}}. \tag{12.28}$$

This motivates interpreting the entropy of de Sitter space as the total number $\mathcal{N}$ of metric and ghost fluctuation modes integrated out along an asymptotically safe RG trajectory ending in a classical regime with parameters G_{cr} and Λ_{cr}, respectively.

In [154] the above result has also been compared to the so-called *N-bound hypothesis*[5] [399, 400] and a conjectured *Λ-N-connection*[6] [399–401] which had emerged from entirely different considerations, but pointing in a similar direction though.

For details the reader is referred to [154, 402].

(7) Finiteness of entanglement entropy. Asymptotic Safety can give finite values to certain quantities which "count" the number of active degrees of freedom or field modes and would be badly divergent in standard field theory. One example is $\mathcal{N}$ above, another are *entanglement entropies* of quantum fields on the dynamical spacetimes of QEG [403].

For matter fields living on a classical spacetime the entanglement entropy across a given surface of area A is typically of the form $S_{\mathrm{ent}} \propto A/\varepsilon^2$. Here ε is a short distance cutoff, and S_{ent} is quadratically divergent [404, 405]. Loosely speaking this divergence is caused by the "too many" degrees of freedom standard field theories have in the ultraviolet. In QEG instead it can be shown [403] that Asymptotic Safety renders the corresponding entropy *finite*.

This is yet another way of seeing that the spacetimes of asymptotically safe gravity theories host far less degrees of freedom at short distances than the familiar matter theories on Minkowski space.

[5] In its stronger form [399], the claim is that in any universe with a positive cosmological constant, containing arbitrary matter, the observable entropy S_{obs} is bounded by $S_{\mathrm{obs}} \leq 3\pi/G\Lambda \equiv N$.

[6] Arguments from string theory [399, 400] and Loop Quantum Gravity [401] motivated the conjecture that all universes with a positive cosmological constant are described by a microscopic quantum theory which has only a *finite* number of degrees of freedom, and that this number is determined by Λ.

Appendix A
Notation and Conventions

In this book, we follow the conventions used in [14] and the bulk of the Asymptotic Safety literature. The dimension of the spacetime manifold $\mathcal{M}$ is denoted by d. Our spacetime metric $g_{\mu\nu}$ uses mostly plus conventions

$$\text{Lorentzian metric}: \quad \text{sign}(g_{\mu\nu}) = (-,+,+,+,\ldots)\,, \tag{A.1}$$

$$\text{Euclidean metric}: \quad \text{sign}(g_{\mu\nu}) = (+,+,+,+,\ldots)\,. \tag{A.2}$$

As usual, the inverse metric is indicated by upper indices $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$. Symmetrization and anti-symmetrization are indicated by round and square brackets, respectively, and we normalize with unit strength, for example,

$$H_{\alpha\beta} = H_{(\alpha\beta)} + H_{[\alpha\beta]}. \tag{A.3}$$

Covariant quantities constructed from $g_{\mu\nu}$ use their usual symbol while covariant objects constructed from the background metric $\bar{g}_{\mu\nu}$ are indicated with a bar. The covariant derivative D_μ is defined by the Christoffel symbols, i.e.,

$$D_\mu\, H^\alpha{}_\beta = \partial_\mu H^\alpha{}_\beta + \Gamma_{\mu\sigma}{}^\alpha\, H^\sigma{}_\beta - \Gamma_{\mu\beta}{}^\sigma\, H^\alpha{}_\sigma \tag{A.4}$$

which are given by

$$\Gamma_{\mu\nu}{}^\lambda = \tfrac{1}{2}\, g^{\lambda\sigma}\, \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{A.5}$$

The Riemann tensor $R_{\mu\nu\rho}{}^\lambda$, Ricci tensor $R_{\mu\nu}$, and the Ricci scalar R, respectively, are defined as

$$R_{\mu\nu\rho}{}^\lambda \equiv \partial_\nu \Gamma_{\mu\rho}{}^\lambda - \partial_\mu \Gamma_{\nu\rho}{}^\lambda + \Gamma_{\mu\rho}{}^\sigma \Gamma_{\nu\sigma}{}^\lambda - \Gamma_{\nu\rho}{}^\sigma \Gamma_{\mu\sigma}{}^\lambda, \tag{A.6}$$

$$R_{\mu\nu} \equiv R_{\mu\lambda\nu}{}^\lambda, \tag{A.7}$$

$$R \equiv g^{\mu\nu}\, R_{\mu\nu}. \tag{A.8}$$

The curvature tensor satisfies the first and second Bianchi identity

$$R_{\mu[\nu\alpha\beta]} = 0, \qquad R_{\mu\nu[\alpha\beta;\rho]} = 0. \tag{A.9}$$

The second Bianchi identity implies

$$R_{\mu\nu\alpha\beta}{}^{;\beta} = R_{\mu\alpha;\nu} - R_{\nu\alpha;\mu}, \quad R_{\mu\nu}{}^{;\nu} = \tfrac{1}{2} R_{;\mu}. \tag{A.10}$$

The commutator of two covariant derivatives satisfies

$$\begin{aligned} [D_\mu, D_\nu]\, H_\lambda &= R_{\mu\nu\lambda}{}^{\rho} H_\rho, \\ [D_\mu, D_\nu]\, H_{\lambda\sigma} &= R_{\mu\nu\lambda}{}^{\rho} H_{\rho\sigma} + R_{\mu\nu\sigma}{}^{\rho} H_{\lambda\rho}. \end{aligned} \tag{A.11}$$

The Weyl tensor $C_{\mu\nu\rho\sigma}$ is related to the Riemann tensor by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \tfrac{2}{d-2}\left(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}\right) + \tfrac{2}{(d-1)(d-2)}\, R\, g_{\mu[\rho} g_{\sigma]\nu}. \tag{A.12}$$

Occasionally we write

$$\Delta \equiv -g^{\mu\nu} D_\mu D_\nu \equiv -D^2 \tag{A.13}$$

for the (negative) tensor Laplacian.

On a d-dimensional Euclidean spacetime $(\mathbb{R}^d, \delta_{\mu\nu})$ it is often convenient to expand fields in plane waves. Our convention regarding Fourier transforms is

$$\phi_p = \int \mathrm{d}^d x \phi(x) e^{-ipx} \tag{A.14}$$

implying the inverse transformation

$$\phi(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \phi_p e^{ipx}. \tag{A.15}$$

Throughout the work quantum fields like $\widehat{\phi}$ appearing as an integration variable in a partition function will be denoted with a caret. The corresponding classical fields given by their expectation values use the same symbol without the caret, $\phi = \langle \widehat{\phi} \rangle$. Background fields like the background metric $\bar{g}_{\mu\nu}$ are distinguished by a bar and tensors carrying a bar like $\bar{R}_{\mu\nu}$ are understood to be constructed from the corresponding background quantities.

Appendix B
Organizing the Derivative Expansion

A natural ordering principle for the interaction terms entering into the Effective Average Action is provided by their number of derivatives. Due to the Bianchi identities satisfied by the curvature tensors, constructing a basis of invariants at each order in the derivative expansion is nontrivial. Such a basis, constructed from the Riemann tensor, its contractions and covariant derivatives, may be obtained based on the representation theory of the symmetric, general linear, and orthogonal groups [161].

B.1 An Explicit Basis

On a manifold without boundary (or equivalently neglecting total derivatives) this basis contains *one* element with, respectively, zero and two covariant derivatives, *three* elements with four derivatives, and *ten* monomials built from six derivatives.

For practical purposes it is convenient to trade the Riemann tensor for the Weyl tensor via (A.12). An explicit basis containing all curvature monomials with up to six derivatives is then given by the following set:

$$
\begin{aligned}
&\mathcal{R}^0 = 1\,, && \\
&\mathcal{R}^1 = R\,, && \\
&\mathcal{R}^2_1 = R^2\,, && \mathcal{R}^2_2 = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} \\
&\mathcal{R}^2_3 = E && \\
&\mathcal{R}^3_1 = R\Delta R\,, && \mathcal{R}^3_2 = R_{\mu\nu}\Delta R^{\mu\nu}\,, \\
&\mathcal{R}^3_3 = R^3\,, && \mathcal{R}^3_4 = R R_{\mu\nu} R^{\mu\nu}\,, \\
&\mathcal{R}^3_5 = R_\mu{}^\nu R_\nu{}^\alpha R_\alpha{}^\mu\,, && \mathcal{R}^3_6 = R_{\mu\nu}R_{\alpha\beta}C^{\mu\alpha\nu\beta}\,, \\
&\mathcal{R}^3_7 = R C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}\,, && \mathcal{R}^3_8 = R_{\mu\nu}C^{\mu\alpha\beta\gamma}C^\nu{}_{\alpha\beta\gamma}\,, \\
&\mathcal{R}^3_9 = C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\alpha\beta}C_{\alpha\beta}{}^{\mu\nu}\,, && \mathcal{R}^3_{10} = C^\alpha{}_\mu{}^\beta{}_\nu C^\mu{}_\rho{}^\nu{}_\sigma C^\rho{}_\alpha{}^\sigma{}_\beta\,.
\end{aligned}
\tag{B.1}
$$

In this basis the integrand of the Gauss-Bonnet term, constituting a topological invariant if $d=4$, is denoted by

$$E = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\,. \tag{B.2}$$

In the context of the FRGE, other Riemann monomials involving four or six derivatives such as $R^{\mu\nu\rho\sigma}R_{\rho\nu\mu\sigma}$, $R^{\mu\nu}D_\sigma D_\nu R_\mu{}^\sigma$, $R^{\alpha\rho\beta\sigma}R_{\alpha\beta\mu\nu}R_{\rho\sigma}{}^{\mu\nu}$, $R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}R_{\rho\mu\sigma\nu}$, and $R_{\alpha\rho\beta\sigma}R^{\alpha\mu\beta\nu}R^\rho{}_\nu{}^\sigma{}_\mu$ can be encountered. These are not part of the basis (B.1), but can be expressed in terms of the basis elements by exploiting the following geometrical identities:

$$R^{\mu\nu\rho\sigma}R_{\rho\nu\mu\sigma} = \tfrac{1}{2}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\,, \tag{B.3}$$

$$R^{\mu\nu}D_\sigma D_\nu R_\mu{}^\sigma = \tfrac{1}{2}R^{\mu\nu}D_\mu D_\nu R + R_\mu{}^\nu R_\nu{}^\rho R_\rho{}^\mu - R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}\,, \tag{B.4}$$

$$R^{\alpha\rho\beta\sigma}R_{\alpha\beta\mu\nu}R_{\rho\sigma}{}^{\mu\nu} = \frac{1}{2}R_{\alpha\beta\rho\sigma}R^{\alpha\beta\mu\nu}R^{\rho\sigma}{}_{\mu\nu}\,, \tag{B.5}$$

$$R_{\alpha\beta}{}^{\rho\sigma}R^{\alpha\mu\beta\nu}R_{\rho\mu\sigma\nu} = \frac{1}{4}R_{\alpha\beta\rho\sigma}R^{\alpha\beta\mu\nu}R^{\rho\sigma}{}_{\mu\nu}\,, \tag{B.6}$$

$$R_{\alpha\rho\beta\sigma}R^{\alpha\mu\beta\nu}R^\rho{}_\nu{}^\sigma{}_\mu = R_{\alpha\rho\beta\sigma}R^\alpha{}_\mu{}^\beta{}_\nu R^{\rho\mu\sigma\nu} - \frac{1}{4}R_{\alpha\beta\rho\sigma}R^{\alpha\beta\mu\nu}R^{\rho\sigma}{}_{\mu\nu}. \tag{B.7}$$

For manifolds with sufficiently low dimensions some of the basis elements become linearly dependent, or even vanish.

If $\mathbf{d<6}$ the contraction of three Riemann tensors, antisymmetrized over all lower indices, satisfies

$$R_{[\alpha\beta}{}^{\alpha\beta}\,R_{\rho\sigma}{}^{\rho\sigma}\,R_{\mu\nu]}{}^{\mu\nu} = 0\,. \tag{B.8}$$

This identity allows to express $\mathcal{R}^3_{10}$ in terms of the remaining basis elements.

For $\mathbf{d=4}$ the contraction of the Weyl tensor furthermore satisfies

$$C_{\mu\nu\rho\alpha}\,C^{\mu\nu\rho\beta} = \tfrac{1}{4}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\,\delta^\beta_\alpha\,. \tag{B.9}$$

Thus in four dimensions $\mathcal{R}^3_8$ can be expressed in terms of $\mathcal{R}^3_7$. As a consequence, the derivative expansion of the gravitational Effective Average Action in $d=4$, at sixth order, contains only *eight* of the ten basis elements listed in (B.1).

For $\mathbf{d\leq 3}$ the set of basis elements is reduced further due to the vanishing of the Weyl tensor. Finally, two-dimensional manifolds are Einstein spaces such that the Ricci tensor can be expressed in terms of the Ricci scalar and the metric.

B.2 Prototypical Background Spaces

Expanding an arbitrary functional of $g_{\mu\nu}$ in the basis (B.1) the corresponding expansion coefficients can be found by evaluating this functional on a number of inequivalent metrics. Those metrics will give certain known values to the basis elements and their linear combinations. Hence, if we know the value of the functional for a large enough set of such "backgrounds," we are able to deduce the functional's expansion coefficients. Next, we list a number of typical spaces employed for this purpose.

(1) The S^4 background. Most of the computations carried out within the Asymptotic Safety program utilize the one-parameter family of four-spheres as a background. Since these backgrounds are maximally symmetric, using, in $d = 4$,

$$\mathrm{d}s^2 = a^2 \left[\mathrm{d}\theta_1^2 + \sin^2\theta_1 \left(\mathrm{d}\theta_2^2 + \sin^2\theta_2 \left(\mathrm{d}\theta_3^2 + \sin^2\theta_3\, \mathrm{d}\varphi^2\right)\right)\right] \tag{B.10}$$

leads to significant simplifications. In particular, the Riemann tensor and Ricci tensor satisfy

$$R_{\mu\nu} = \frac{1}{4}\, g_{\mu\nu}\, R\,, \qquad R_{\mu\nu\rho\sigma} = \frac{R}{12}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\,. \tag{B.11}$$

Hence, $R = 12/a^2$, and the Riemann tensor is covariantly constant in this case, $D_\alpha R_{\mu\nu\rho\sigma} = 0$. Moreover, $C_{\mu\nu\rho\sigma} = 0$.

(2) The CP^2 background. In order to include the effects stemming from the Weyl tensor, non-spherical backgrounds are needed. One class of such backgrounds is the complex projective space CP^2 equipped with the one-parameter family of (scaled) Fubini-Study metrics:

$$\mathrm{d}s^2 = \frac{a^2}{(1 + z_i \bar{z}^i)^2}\left[(1 + z_i\bar{z}^i)\mathrm{d}z_j \mathrm{d}\bar{z}^j - \bar{z}^j z_i \mathrm{d}z_j \mathrm{d}\bar{z}^i\right]\,, \quad i,j = 1,2\,. \tag{B.12}$$

This background is still Einstein but has a nontrivial, covariantly constant Weyl tensor:

$$R_{\mu\nu} = \frac{1}{4}\, g_{\mu\nu}\, R\,, \qquad C_{\mu\nu\rho\sigma} \neq 0\,, \qquad D_\alpha C_{\mu\nu\rho\sigma} = 0\,. \tag{B.13}$$

Thus, on CP^2 the Goroff-Sagnotti term [2, 387, 388] is given a non-vanishing value.[1]

For completeness we evaluate the basis elements (B.1) for the particular backgrounds (B.10) and (B.12). The result is summarized in Table B.1.

(3) Ricci-flat backgrounds. The computation [388] is carried out on a Ricci-flat background, implying that the curvature tensors satisfy

$$R = 0\,, \quad R_{\mu\nu} = 0\,, \quad C_{\mu\nu\rho\sigma} \neq 0\,. \tag{B.14}$$

An example for a *compact* space satisfying these properties is a $K3$ manifold.[2] Unfortunately, it is very hard to write down the explicit form of the metric in

[1] The RG flow of the Goroff-Sagnotti term was explored in the Asymptotic Safety context in [386].

[2] A *non-compact* example is a pp-wave background.

Table B.1 *Evaluation of the basis elements on the four-sphere, the complex projective space CP^2, the $K3$ manifold, a generic Einstein manifold, and the non-Einstein background $S^2 \times S^2$.*

	S^4	CP^2	$K3$	Einstein space	$S^2 \times S^2$
Vol	$\frac{8\pi^2 a^4}{3}$	$\frac{\pi^2 a^4}{2}$	Vol	Vol	$16\pi^2 a_1^2 a_2^2$
$\mathcal{R}^1$	$\frac{12}{a^2}$	$\frac{24}{a^2}$	0	$\mathcal{R}^1$	$\frac{2}{a_1^2 a_2^2}\left(a_1^2 + a_2^2\right)$
$\mathcal{R}_1^2$	$\frac{144}{a^4}$	$\frac{576}{a^4}$	0	$\mathcal{R}_1^2$	$\frac{4}{a_1^4 a_2^4}\left(a_1^2 + a_2^2\right)^2$
$\mathcal{R}_2^2$	0	$\frac{96}{a^4}$	$\mathcal{R}_2^2$	$\mathcal{R}_2^2$	$\frac{4}{3a_1^4 a_2^4}\left(a_1^2 + a_2^2\right)^2$
$\mathcal{R}_3^2$	$\frac{24}{a^4}$	$\frac{192}{a^4}$	$\mathcal{R}_2^2$	$\mathcal{R}_2^2 + \frac{1}{6}\mathcal{R}_1^2$	$\frac{8}{a_1^2 a_2^2}$
$\mathcal{R}_1^3$	0	0	0	0	0
$\mathcal{R}_2^3$	0	0	0	0	0
$\mathcal{R}_3^3$	$\frac{1728}{a^6}$	$\frac{13824}{a^6}$	0	$\mathcal{R}_3^3$	$\frac{8}{a_1^6 a_2^6}\left(a_1^2 + a_2^2\right)^3$
$\mathcal{R}_4^3$	$\frac{432}{a^6}$	$\frac{3456}{a^6}$	0	$\frac{1}{4}\mathcal{R}_3^3$	$\frac{4}{a_1^6 a_2^6}\left(a_1^2 + a_2^2\right)\left(a_1^4 + a_2^4\right)$
$\mathcal{R}_5^3$	$\frac{108}{a^6}$	$\frac{864}{a^6}$	0	$\frac{1}{16}\mathcal{R}_3^3$	$\frac{2}{a_1^6 a_2^6}\left(a_1^6 + a_2^6\right)$
$\mathcal{R}_6^3$	0	0	0	0	$\frac{2}{3a_1^6 a_2^6}\left(a_1^2 - a_2^2\right)^2\left(a_1^2 + a_2^2\right)$
$\mathcal{R}_7^3$	0	$\frac{2304}{a^6}$	0	$\mathcal{R}_7^3$	$\frac{8}{3a_1^6 a_2^6}\left(a_1^2 + a_2^2\right)^3$
$\mathcal{R}_9^3$	0	$\frac{384}{a^6}$	$\mathcal{R}_9^3$	$\mathcal{R}_9^3$	$\frac{4}{9a_1^6 a_2^6}\left(a_1^2 + a_2^2\right)^3$
$\mathcal{R}_8^3$	0	$\frac{576}{a^6}$	0	$\frac{1}{4}\mathcal{R}_7^3$	$\frac{2}{3a_1^6 a_2^6}\left(a_1^2 + a_2^2\right)^3$
$\mathcal{R}_{10}^3$	0	$\frac{192}{a^6}$	$\frac{1}{2}\mathcal{R}_9^3$	$\mathcal{R}_{10}^3$	$\frac{2}{9a_1^6 a_2^6}\left(a_1^2 + a_2^2\right)^3$

this case, so one is forced to manipulate the curvature monomials in an abstract way.

Important information is thereby contained in the identity

$$\begin{aligned} R_{\mu\nu\rho\sigma} D^2 R^{\mu\nu\rho\sigma} = & \, 4 R^{\mu\nu\rho\sigma} D_\mu D_\rho R_{\nu\sigma} + 2 R_{\mu\nu} R^\mu{}_{\rho\sigma\lambda} R^{\nu\rho\sigma\lambda} \\ & - R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu} - 4 R^\alpha{}_\mu{}^\beta{}_\nu R^\mu{}_\rho{}^\nu{}_\sigma R^\rho{}_\alpha{}^\sigma{}_\beta \, . \end{aligned} \tag{B.15}$$

On a Ricci-flat manifold of dimension $d < 6$ this identity reduces to

$$C_{\mu\nu\rho\sigma} D^2 C^{\mu\nu\rho\sigma} = -3 C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\alpha\beta} C_{\alpha\beta}{}^{\mu\nu} \, . \tag{B.16}$$

Thus, we conclude that a non-vanishing value for the invariant $\mathcal{R}_9^3$ requires that $D_\alpha C_{\mu\nu\rho\sigma} \neq 0$. A computation on a Ricci-flat background therefore entails that $C_{\mu\nu\rho\sigma}$ does not commute with covariant derivatives and one has to keep track of all terms containing two powers of the Weyl tensor and two covariant derivatives acting on them.

(4) Generic Einstein background. By definition, the Ricci tensor of an Einstein space is proportional to the metric,

$$R_{\mu\nu} = \frac{1}{d} \, R \, g_{\mu\nu} \, . \tag{B.17}$$

By virtue of the Bianchi identities (A.10) this entails that

$$D_\mu R = 0 \, , \qquad D^\sigma C_{\mu\nu\rho\sigma} = 0 \, . \tag{B.18}$$

Thus, evaluating the flow equation on a generic Einstein background allows us to distinguish two of the three fourth-order invariants, as well as four out of the ten sixth-order invariants.

(5) The background $S^2 \times S^2$. In order to extract complementary information on the RG flow non-Einstein backgrounds are needed. The simplest class of such backgrounds is given by the direct product of two two-spheres with different radii:

$$\mathrm{d}s^2 = a_1^2 \left(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \, \mathrm{d}\theta_2^2 \right) + a_2^2 \left(\mathrm{d}\theta_3^2 + \sin^2\theta_3 \, \mathrm{d}\theta_4^2 \right) \, . \tag{B.19}$$

For $a_1 \neq a_2$ this metric does not satisfy the Einstein condition. But the resulting Weyl tensor is covariantly constant though:

$$D_\alpha C_{\mu\nu\rho\sigma} = 0 \, . \tag{B.20}$$

This class of backgrounds allows us to distinguish two of the three fourth-order invariants, as well as three out of the ten sixth-order invariants.

Notably, the information content of this projection is complementary to the information obtained on a generic Einstein background. Carrying out analogous computations using both the background classes S^4 and $S^2 \times S^2$ allows to determine the RG flow of all fourth-order couplings and three sixth-order couplings.

Appendix C
Metric Variations

The construction of the flow equation typically requires the functional differentiation, or equivalently variation of covariant objects constructed from $g_{\mu\nu}$ and its derivatives. The behavior under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ of various basic quantities is summarized in Table C.1.

The variations of typical invariants building up Γ_k can be obtained from these basic variations via the product rule. For example

$$\delta^2 \left(\sqrt{g} R^2\right) = R^2 \delta^2 \sqrt{g} + 4R \left(\delta\sqrt{g}\right)\left(\delta R\right) + 2\sqrt{g}\left(\delta R\right)^2 + 2\sqrt{g} R \delta^2 R. \tag{C.1}$$

Once the complexity of the monomials increases, it is economical to construct the variations of the more complicated invariants using suitable computer algebra packages. In this case the Table C.1 can serve as a benchmark which allows to test the implementation.

Table C.1 *Behavior of selected covariant objects under variation of the metric,* $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. *Indices are raised with* $g^{\mu\nu}$.

$$\begin{aligned}
\delta g_{\mu\nu} &\equiv h_{\mu\nu} \\
\delta g^{\mu\nu} &= -h^{\mu\nu} \\
\delta \Gamma_{\mu\nu}{}^{\lambda} &= \tfrac{1}{2} g^{\lambda\sigma}\big(D_\mu h_{\nu\sigma} + D_\nu h_{\mu\sigma} - D_\sigma h_{\mu\nu}\big) \\
\delta\sqrt{g} &= \tfrac{1}{2}\sqrt{g}\, g^{\mu\nu}\, h_{\mu\nu} \\
\delta R_{\mu\nu\rho}{}^{\lambda} &= \tfrac{1}{2}\big(-D_\mu D_\rho h_\nu{}^\lambda + D_\nu D_\rho h_\mu{}^\lambda + D_\mu D^\lambda h_{\nu\rho} - D_\nu D^\lambda h_{\mu\rho} - R_{\mu\nu\rho}{}^{\sigma}\, h_\sigma{}^\lambda + R_{\mu\nu\sigma}{}^{\lambda} h_\rho{}^\sigma\big) \\
\delta R_{\mu\nu} &= \tfrac{1}{2}\big(R_{\mu\sigma} h_\nu{}^\sigma + R_\nu{}^\sigma h_{\sigma\mu} + 2R_{\sigma\mu\nu\lambda} h^{\sigma\lambda} + D_\nu D_\sigma h^\sigma{}_\mu \\
&\quad - D_\mu D_\nu h_\alpha{}^\alpha - D_\lambda D^\lambda h_{\mu\nu} + D_\mu D^\sigma h_{\sigma\nu}\big) \\
\delta R &= -R^{\mu\nu} h_{\mu\nu} + D_\beta\big(D_\alpha h^{\alpha\beta} - D^\beta h_\alpha{}^\alpha\big) \\
\delta^2\sqrt{g} &= \tfrac{1}{2}\sqrt{g}\big(\tfrac{1}{2} g^{\mu\nu} g^{\rho\sigma} h_{\mu\nu} h_{\rho\sigma} - h^{\mu\nu} h_{\mu\nu}\big) \\
\delta^2 R &= R_{\beta\mu} h^{\beta\gamma} h_\gamma{}^\mu - R_{\alpha\beta\gamma\rho} h^{\beta\gamma} h^{\alpha\rho} - 3h^{\beta\gamma} D_\gamma D_\alpha h^\alpha{}_\beta + 2h^{\beta\gamma} D_\beta D_\gamma h^\alpha{}_\alpha + 2h_{\beta\gamma} D_\lambda D^\lambda h^{\beta\gamma} \\
&\quad - h^{\beta\gamma} D_\alpha D_\beta h^\alpha{}_\gamma - (D_\alpha h^{\beta\gamma})(D_\beta h^\alpha{}_\gamma) + \tfrac{3}{2}(D_\lambda h_{\beta\gamma})(D^\lambda h^{\beta\gamma}) - 2(D_\gamma h^{\beta\gamma})(D_\alpha h^\alpha{}_\beta) \\
&\quad + 2(D_\beta h^{\beta\gamma})(D_\gamma h^\alpha{}_\alpha) - \tfrac{1}{2}(D_\lambda h_\gamma{}^\gamma)(D^\lambda h_\alpha{}^\alpha)
\end{aligned}$$

Appendix D
Heat Kernel Techniques

Structurally the right-hand side of the functional renormalization group equation is given by a trace over an operator-valued function. Typically the underlying differential operators are quite involved. For specific choices of the background geometry and gauge-fixing conditions they may simplify to differential operators whose spectral properties have already been studied in the mathematical physics literature.

A typical example are differential operators of second order,

$$\Delta \equiv -g^{\mu\nu}\nabla_\mu\nabla_\nu + \mathbf{E}, \tag{D.1}$$

where $\nabla_\mu \equiv D_\mu + \mathbf{A}_\mu$ is a covariant derivative which may include a Yang–Mills connection $\mathbf{A}_\mu$ in addition to the Levi–Civita connection implicit in D_μ. The endomorphism $\mathbf{E}$ is a linear map acting on spacetime and internal indices carried by the field.

In such cases, a powerful tool for evaluating the flow equations are heat kernel techniques [406–409]. The central idea is to introduce the heat kernel

$$K(s;x,y;\Delta) \equiv \langle x|e^{-s\Delta}|y\rangle. \tag{D.2}$$

The heat kernel satisfies the heat equation

$$(\partial_s + \Delta_x)\, K(s;x,y;\Delta) = 0, \tag{D.3}$$

subject to the initial condition

$$\lim_{s\to 0} K(s;x,y;\Delta) = \delta(x,y). \tag{D.4}$$

For example, on a flat manifold $\mathbb{R}^d$ with $\mathbf{A}_\mu = 0$ and the endomorphism $\mathbf{E} = m^2$ given by the mass of a scalar field the heat kernel is known exactly:

$$K(s;x,y;\Delta) = \frac{1}{(4\pi s)^{d/2}}\, \exp\left(-\frac{(x-y)^2}{4s} - s\,m^2\right). \tag{D.5}$$

The trace of the operator $e^{-s\Delta}$ can be expressed in terms of the heat kernel evaluated at coincident points:

$$\mathrm{Tr}\left[e^{-s\Delta}\right] = \int \mathrm{d}^d x \, \sqrt{g} \, K(s; x, x; \Delta) \, . \tag{D.6}$$

The right-hand side of this equation can then be studied by applying various complementary approximation schemes to K, like the early or the late time expansion.

D.1 Early Time Expansion

The early time expansion organizes the operator trace (D.6) in terms of an asymptotic series in s:

$$\mathrm{Tr}\left[e^{-s\Delta}\right] = \frac{1}{(4\pi s)^{d/2}} \int \mathrm{d}^d x \, \sqrt{g} \sum_{n=0}^{\infty} \mathrm{tr}\,(\mathbf{a}_n)\, s^n \, . \tag{D.7}$$

Here tr denotes a trace over the internal space on which $\mathbf{A}_\mu$ and $\mathbf{E}$ act as matrices. Comparing with (D.6) indicates that the early-time expansion of $K(s; x, x; \Delta)$ is given by

$$K(s; x, x; \Delta) = \frac{1}{(4\pi s)^{d/2}} \sum_{n=0}^{\infty} \mathrm{tr}\,(\mathbf{a}_n)\, s^n . \tag{D.8}$$

Dimensional analysis shows that the mass dimension of the expansion coefficients $\mathbf{a}_n$ is given by $[\mathbf{a}_n] = 2n$. This makes the early time expansion particularly suitable for evaluating the gravitational FRGE in a derivative expansion since terms built from $2n$ derivatives appear in the coefficient $\mathbf{a}_n$ only.

The $\mathbf{a}_n$ can be expressed in terms of the curvature tensors constructed from $g_{\mu\nu}$, $\mathbf{A}_\mu$, and the endomorphism $\mathbf{E}$. For $n = 0, 1, 2$ their explicit form can be found in [408].[1] For the operator (D.1) they are given by

$$\mathbf{a}_0 = \mathbf{1}, \tag{D.9}$$

$$\mathbf{a}_1 = \tfrac{1}{6}\, R\, \mathbf{1} - \mathbf{E}, \tag{D.10}$$

$$\begin{aligned} \mathbf{a}_2 = \; & \tfrac{1}{180}\left(R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - R^{\mu\nu} R_{\mu\nu} + \tfrac{5}{2} R^2 + 6\nabla^2 R\right) \mathbf{1} \\ & + \tfrac{1}{12} \mathbf{\Omega}_{\mu\nu} \mathbf{\Omega}^{\mu\nu} - \tfrac{1}{6} R\, \mathbf{E} + \tfrac{1}{2} \mathbf{E}^2 - \tfrac{1}{6} \nabla^2 \mathbf{E}. \end{aligned} \tag{D.11}$$

[1] Our conventions differ from the ones used in [408] by a relative minus sign in the endomorphism $\mathbf{E}$ and the Riemann tensor. The Ricci tensor and Ricci scalar agree. Results given in [409] can be converted to ours by applying the replacement rules $is \mapsto s$, $\mathcal{R}_{\mu\nu} \mapsto \mathbf{\Omega}_{\mu\nu}$ and a rescaling $a_k \mapsto k!\, a_k$. The conventions for the curvature tensors agree in this case.

Here $\mathbf{\Omega}_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$ is the total curvature of the connection and $\mathbf{1}$ denotes the unit on the internal space. For scalars (S), vectors (V), and symmetric second-rank tensors (ST) one has, for instance,

$$\mathbf{1}_S = 1, \quad [\mathbf{1}_V]_\mu{}^\nu = \delta^\nu_\mu, \quad [\mathbf{1}_{ST}]_{\mu\nu}{}^{\alpha\beta} = \tfrac{1}{2}\left(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu\right). \tag{D.12}$$

The $\mathbf{\Omega}$-dependent terms capture the dependence of the heat kernel coefficients on the matrix properties of the fields. For any fixed set of internal indices, they give rise to specific matrix-valued contributions built from contractions of the curvature tensors and their covariant derivatives. Their explicit expressions can be obtained by evaluating the commutators on a "test" field carrying the appropriate set of internal indices.

We illustrate the role of Ω by various examples. On flat space one has

$$\Omega_{\mu\nu}\Omega^{\mu\nu} = \mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} \tag{D.13}$$

with $\mathbf{F}_{\mu\nu}$ the field strength associated with the Yang-Mills connection. In this case the trace tr is over matrices constructed from the generators of the gauge group.

In curved space and for $\mathbf{A}_\mu = 0$ we see that Ω vanishes when the heat kernel is evaluated for scalar fields. For a vector field ξ_α one has instead

$$\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu}\,\xi_\alpha = -R_{\mu\nu\rho\alpha}R^{\mu\nu\rho\beta}\,\xi_\beta, \tag{D.14}$$

where we used the relations (A.11) for converting the commutators to Riemann curvature tensors. From (D.14) one then reads off the matrix

$$[\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu}]_\alpha{}^\beta = -R_{\mu\nu\rho\alpha}R^{\mu\nu\rho\beta}. \tag{D.15}$$

The analogous computation for symmetric second-rank tensors yields

$$[\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu}]_{\alpha\beta}{}^{\rho\sigma} = -R_{\mu\nu\gamma\alpha}R^{\mu\nu\gamma\rho}\delta^\sigma_\beta - R_{\mu\nu\gamma\beta}R^{\mu\nu\gamma\sigma}\delta^\rho_\alpha + 2R_{\mu\nu\alpha}{}^\rho R^{\mu\nu}{}_\beta{}^\sigma. \tag{D.16}$$

The expansion (D.7) contains the trace of matrix valued quantities whereby the outer indices are contracted. For the unit matrix, these traces count the number of independent components:

$$\mathrm{tr}_S\,\mathbf{1} = 1, \quad \mathrm{tr}_V\,\mathbf{1} = d, \quad \mathrm{tr}_{ST}\,\mathbf{1} = \tfrac{1}{2}d(d+1). \tag{D.17}$$

Tracing the outer indices in (D.15) and (D.16), taking into account the appropriate symmetrizations, furthermore yields

$$\begin{aligned} \mathrm{tr}_V\,[\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu}] &= -R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \\ \mathrm{tr}_{ST}\,[\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu}] &= -(d+2)\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \end{aligned} \tag{D.18}$$

For completeness, we also display the coefficient $\mathbf{a}_3$. Its general expression is given in [408]. Upon dropping surface terms, integrating by parts, and using the Bianchi identities, the result reads [187, 409]:

$$
\begin{aligned}
\mathbf{a}_3 = \tfrac{1}{6}\Big[& \left(\tfrac{1}{6}R\,\mathbf{1} - \mathbf{E}\right)^3 + \tfrac{1}{2}\left(\tfrac{1}{6}R\,\mathbf{1} - \mathbf{E}\right)\nabla^2\left(\tfrac{1}{6}R\,\mathbf{1} - \mathbf{E}\right) \\
& + \tfrac{1}{30}\left(\tfrac{1}{6}R\,\mathbf{1} - \mathbf{E}\right)\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - R^{\mu\nu}R_{\mu\nu} + \nabla^2 R\right) \\
& + \tfrac{1}{2}\left(\tfrac{1}{6}R\,\mathbf{1} - \mathbf{E}\right)\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}^{\mu\nu} - \tfrac{1}{10}\left(\nabla^\mu\mathbf{\Omega}_{\mu\alpha}\right)\left(\nabla_\nu\mathbf{\Omega}^{\nu\alpha}\right) \\
& + \tfrac{1}{30}\left(2\mathbf{\Omega}^\mu{}_\nu\mathbf{\Omega}^\nu{}_\rho\mathbf{\Omega}^\rho{}_\mu - 2R^\mu{}_\nu\mathbf{\Omega}^\nu{}_\rho\mathbf{\Omega}^\rho{}_\mu + R^{\mu\nu\rho\sigma}\mathbf{\Omega}_{\mu\nu}\mathbf{\Omega}_{\rho\sigma}\right) \\
& - \tfrac{1}{630}R\,\nabla^2\,R\,\mathbf{1} + \tfrac{1}{140}R_{\mu\nu}\,\nabla^2\,R^{\mu\nu}\mathbf{1} \\
& + \tfrac{1}{7560}\big(48R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} - 64R^\mu{}_\nu R^\nu{}_\rho R^\rho{}_\mu + 6R_{\mu\nu}R^\mu{}_{\alpha\beta\gamma}R^{\nu\alpha\beta\gamma} \\
& + 17R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} - 28R^\alpha{}_\mu{}^\beta{}_\nu R^\mu{}_\rho{}^\nu{}_\sigma R^\rho{}_\alpha{}^\sigma{}_\beta\big)\mathbf{1}\Big].
\end{aligned}
\tag{D.19}
$$

D.2 The Off-Diagonal Heat Kernel

The early-time expansion of the heat kernel, discussed in the previous section, provides a powerful tool for evaluating operator traces where the trace argument is given by a minimal second-order differential operator. The off-diagonal heat kernel techniques constitute an important generalization of these methods.

First introduced by DeWitt [50, 410] as a technique to compute the standard heat kernel coefficients appearing in (D.8), the methods are also useful when evaluating FRGEs [411]. Their key benefit is that they allow to construct the early-time expansion of operator traces of the form

$$
\mathrm{Tr}\left[\mathcal{O}\,e^{-s\Delta}\right] = \mathrm{Tr}\left[\mathcal{O}^{\mu_1\cdots\mu_a}\,D_{\mu_1}\cdots D_{\mu_a}\,e^{-s\Delta}\right]. \tag{D.20}
$$

Here $\mathcal{O}$ is an operator insertion which may be matrix-valued in field space. Typically the covariant derivatives will be with respect to the background metric and capture derivatives that do not organize themselves into Laplacian type operators. A frequent example is the insertion $\mathcal{O} = \bar{R}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$.

Without loss of generality, one can assume that the tensors $\mathcal{O}^{\mu_1\cdots\mu_a}$ are totally symmetric under all permutations of indices. Any antisymmetric contribution in $\mathcal{O}^{\mu_1\cdots\mu_a}$ gives rise to commutators of covariant derivatives which can be re-expressed in terms of curvature tensors via the standard commutator relations (A.11). The resulting terms are again of the form (D.20) with the resulting components $\mathcal{O}^{\mu_1\cdots\mu_a}$ having a lower rank.

Writing the RHS of (D.20) in terms of matrix elements,

$$
\mathrm{Tr}\left[\mathcal{O}\,e^{-s\Delta}\right] = \mathrm{tr}\int \mathrm{d}^d x\sqrt{g(x)}\int \mathrm{d}^d y\sqrt{g(y)}\langle\, y\,|\,\mathcal{O}\,|\,x\,\rangle\langle\, x\,|e^{-s\Delta}|\,y\,\rangle, \tag{D.21}
$$

relates the traces with operator insertions $\mathcal{O}$ to the heat kernel $K(s,x,y)$ at non-coincident points. Precisely, (D.21) entails the master formula

$$\mathrm{Tr}\left[\mathcal{O}\, e^{-s\Delta}\right] = \mathrm{tr}\int \mathrm{d}^d x\sqrt{g}\ \mathcal{O}^{\mu_1\ldots\mu_a}\, K_{\mu_1\ldots\mu_a}, \tag{D.22}$$

where tr denotes a trace over internal field indices.

In order to ease our notation, we indicate expressions evaluated in the coincidence limit $y\to x$ by an overline. The tensors $K_{\mu_1\ldots\mu_a}$ are covariant derivatives of the off-diagonal heat kernel $K(s;x,y)$ with respect to its argument x evaluated at the coincidence point $y=x$:

$$K_{\mu_1\ldots\mu_a} \equiv \overline{D_{(\mu_1}\ldots D_{\mu_a)}\, K(s;x,y)}. \tag{D.23}$$

The definition (D.23) involves the heat kernel at non-coincidence points. Therefore, the evaluation of operator traces via the master formula (D.22) is commonly referred to as the *off-diagonal heat kernel technique*.

The tensors $K_{\mu_1\ldots\mu_a}$ possess an early-time expansion which has the same structure as the one encountered for the diagonal heat kernel $K(s;x,x)$ in (D.8). The coefficients appearing in this expansion can be obtained by noting that the off-diagonal heat kernel has the asymptotic expansion

$$K(s;x,y) = (4\pi s)^{-d/2}\, e^{-\frac{\sigma(x,y)}{2s}} \sum_{n=0}^{\infty} s^n\, \mathbf{A}_n(x,y). \tag{D.24}$$

subject to the boundary condition

$$\overline{\mathbf{A}_0(x,y)} = \mathbf{1}. \tag{D.25}$$

The *world function* $\sigma(x,y)$ which appears in (D.24) is given by half the geodesic distance between x and y and satisfies

$$\tfrac{1}{2}\sigma_{;\mu}\sigma^{;\mu} = \sigma, \quad \overline{\sigma(x,y)} = \overline{\sigma(x,y)_{;\mu}} = 0, \quad \overline{\sigma(x,y)_{;\mu(\alpha_1\ldots\alpha_a)}} = 0,\ a\geq 2. \tag{D.26}$$

A non-trivial contribution from $\sigma(x,y)$ arises when exactly two covariant derivatives act on it in a symmetric way:

$$\overline{\sigma(x,y)_{;(\mu\nu)}} = g_{\mu\nu}(x). \tag{D.27}$$

The properties of the world function allow to express the tensors $K_{\mu_1\ldots\mu_a}$ in terms of the metric and symmetrized covariant derivatives of the off-diagonal heat kernel coefficients $\mathbf{A}_n$ evaluated at the coincidence point. For the tensors with rank up to four, the resulting combinatorics leads to the following formulae:

$$\begin{aligned}
K &= \frac{1}{(4\pi s)^{d/2}} \sum_{n\geq 0} s^n\, \overline{\mathbf{A}_n}, \\
K_\mu &= \frac{1}{(4\pi s)^{d/2}} \sum_{n\geq 0} s^n\, \overline{D_\mu \mathbf{A}_n}, \\
K_{(\mu\nu)} &= \frac{1}{(4\pi s)^{d/2}} \sum_{n\geq 0} s^{n-1} \left(-\frac{1}{2} g_{\mu\nu} \overline{\mathbf{A}_n} + \overline{D_{(\mu} D_{\nu)} \mathbf{A}_{n-1}} \right), \\
K_{(\mu\nu\rho)} &= \frac{1}{(4\pi s)^{d/2}} \sum_{n\geq 0} s^{n-1} \left(-\frac{3}{2} g_{(\mu\nu} \overline{D_{\rho)} \mathbf{A}_n} + \overline{D_{(\mu} D_\nu D_{\rho)} \mathbf{A}_{n-1}} \right), \\
K_{(\mu\nu\rho\sigma)} &= \frac{1}{(4\pi s)^{d/2}} \sum_{n\geq 0} s^{n-2} \left(\frac{3}{4} g_{(\mu\nu} g_{\rho\sigma)} \overline{\mathbf{A}_n} - 3 g_{(\mu\nu} \overline{D_\rho D_{\sigma)} \mathbf{A}_{n-1}} \right. \\
&\qquad \left. + \overline{D_{(\mu} D_\nu D_\rho D_{\sigma)} \mathbf{A}_{n-2}} \right).
\end{aligned} \tag{D.28}$$

It is also useful to consider the tensors $K_{(\alpha_1 \ldots \alpha_{2a})}$ in the limit where all curvatures and the endomorphism piece vanish. Then the only non-vanishing contribution in the formulas (D.28) comes from the $\overline{\mathbf{A}_0}$-term without any covariant derivatives. In this limit the combinatorics can be worked out in full generality, giving the "flat space" result

$$K_{(\alpha_1 \ldots \alpha_{2a})} = \frac{1}{(4\pi s)^{d/2}} \left(-\frac{1}{2s} \right)^a \left[g_{\alpha_1 \alpha_2} \cdots g_{\alpha_{(2a-1)} \alpha_{2a}} + \cdots \right]. \tag{D.29}$$

Here the dots indicate the remaining $\left(\frac{(2a)!}{2^a a!} - 1 \right)$ index permutations of the first term. The validity of this result can be verified by applying $2a$ partial derivatives to the exact off-diagonal heat kernel on flat space (D.5) and setting $m = 0$.

Evaluating the expressions (D.28) requires the computation of derivatives of the off-diagonal heat kernel coefficients at coincident points. They can be obtained systematically as follows.

Substituting the ansatz (D.24) into the heat equation (D.2) leads to a recursion relation for the coefficients $\mathbf{A}_n$ and their covariant derivatives:

$$\left(n - \tfrac{d}{2} + \tfrac{1}{2} \sigma_{;\mu}{}^{\mu} \right) \mathbf{A}_n + \sigma^{;\mu} \mathbf{A}_{n;\mu} - \mathbf{A}_{n-1;\mu}{}^{\mu} + \mathbf{E}\, \mathbf{A}_{n-1} = 0, \; n \geq 0. \tag{D.30}$$

Equipping this relation with the boundary condition $\mathbf{A}_{-1} = 0$, and taking suitably symmetrized covariant derivatives of it allows to compute the coincidence limit of the coefficients $\mathbf{A}_n$ and their covariant derivatives. The results for the terms including up to four covariant derivatives are thus found to be given by

$$\begin{aligned}
\overline{\mathbf{A}_0} &= \mathbf{1}, \\
\overline{D_\mu \mathbf{A}_0} &= 0, \\
\overline{D_{(\nu} D_{\mu)} \mathbf{A}_0} &= \tfrac{1}{6} R_{\mu\nu}, \\
\overline{D_{(\alpha} D_\nu D_{\mu)} \mathbf{A}_0} &= \tfrac{1}{4} R_{(\mu\nu;\alpha)}, \\
\overline{D_{(\beta} D_\alpha D_\nu D_{\mu)} \mathbf{A}_0} &= \tfrac{3}{10} R_{(\mu\nu;\alpha\beta)} + \tfrac{1}{12} R_{(\alpha\beta} R_{\mu\nu)} + \tfrac{1}{15} R_{\gamma(\beta|\delta|\alpha} R^\gamma{}_\mu{}^\delta{}_{\nu)},
\end{aligned} \tag{D.31}$$

together with

$$\begin{aligned}
\overline{\mathbf{A}_1} &= \tfrac{1}{6} R - \mathbf{E}, \\
\overline{D_\mu \mathbf{A}_1} &= \tfrac{1}{12} R_{;\mu} - \tfrac{1}{6} \mathbf{\Omega}_{\nu\mu}{}^{;\nu} - \tfrac{1}{2} \mathbf{E}_{;\mu}, \\
\overline{D_{(\nu} D_{\mu)} \mathbf{A}_1} &= \tfrac{1}{90} R^{\alpha\beta\gamma}{}_\mu R_{\alpha\beta\gamma\nu} + \tfrac{1}{90} R^{\alpha\beta} R_{\alpha\mu\beta\nu} - \tfrac{1}{45} R_{\mu\alpha} R^\alpha{}_\nu + \tfrac{1}{36} R R_{\mu\nu} \\
&\quad - \tfrac{1}{60} \Delta R_{\mu\nu} + \tfrac{1}{20} R_{;\mu\nu} + \tfrac{1}{6} \mathbf{\Omega}_{\alpha(\mu} \mathbf{\Omega}^\alpha{}_{\nu)} - \tfrac{1}{6} \mathbf{\Omega}_{\alpha(\mu;}{}^\alpha{}_{\nu)} \\
&\quad - \tfrac{1}{6} R_{\mu\nu} \mathbf{E} - \tfrac{1}{3} \mathbf{E}_{;(\mu\nu)},
\end{aligned} \tag{D.32}$$

and

$$\begin{aligned}
\overline{\mathbf{A}_2} \;=\; & \tfrac{1}{180} \left(R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - R^{\mu\nu} R_{\mu\nu} + \tfrac{5}{2} R^2 + 6 \nabla^2 R \right) \mathbf{1} \\
& + \tfrac{1}{12} \mathbf{\Omega}_{\mu\nu} \mathbf{\Omega}^{\mu\nu} - \tfrac{1}{6} R\, \mathbf{E} + \tfrac{1}{2} \mathbf{E}^2 - \tfrac{1}{6} \nabla^2 \mathbf{E}.
\end{aligned} \tag{D.33}$$

Notably, the coefficients $\overline{\mathbf{A}_0}$, $\overline{\mathbf{A}_1}$, and $\overline{\mathbf{A}_2}$ coincide with the standard early-time heat kernel coefficients (D.9), (D.10), and (D.11).

The tensors $K_{\mu_1 \ldots \mu_a}$ with $a \le 6$ as well as the coefficients $\overline{D_{\mu_1} \ldots D_{\mu_a} \mathbf{A}_n}$ with up to six covariant derivatives can be found in [412, 413]. The complexity of the coefficients increases rather quickly for increasing a and n [414]. Since they are typically manipulated within a computer algebra program we refrain from giving explicit expressions for them.

D.3 Integral Transforms

The FRGE for the Effective Average Action contains functional traces over functions $W(\Delta)$ of certain differential operators:

$$\mathrm{Tr}\,[W(\Delta)]\,. \tag{D.34}$$

Typically, the differential operator, Δ, is constructed from covariant derivatives built from the background metric. If the theory contains additional gauge fields, the connection may contain these fields as well. In many cases Δ reduces to an operator of the type considered above, i.e. a second order, Laplace-type operator, $\Delta = -\nabla^2 \mathbf{1} + \mathbf{E}$, where the endomorphism $\mathbf{E}$ is a linear map acting on the spacetime and internal indices carried by the fields.

(1) The derivative expansion of the operator trace (D.34) can be obtained from the early-time expansion of the heat kernel (D.7). Expressing $W(z)$ by its Fourier transform, as in Chapter 5, or alternatively through its inverse Laplace transform $\widetilde{W}(s)$, according to,

$$W(z) \equiv \int_0^\infty \mathrm{d}s\, \widetilde{W}(s)\, e^{-sz}, \tag{D.35}$$

the trace (D.34) can be related to the traced heat kernel:

$$\mathrm{Tr}\left[W(\Delta)\right] = \int_0^\infty \mathrm{d}s\, \widetilde{W}(s)\ \mathrm{Tr}\left[e^{-s\Delta}\right]. \tag{D.36}$$

Substituting the early-time expansion of the heat kernel (D.7) yields

$$\boxed{\mathrm{Tr}\left[W(\Delta)\right] = \frac{1}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} Q_{d/2-k}\left[W\right] \int \mathrm{d}^d x\, \sqrt{g}\ \mathrm{tr}(\mathbf{a}_k)} \tag{D.37}$$

with the "Q-functionals" defined analogous to (5.47):

$$Q_n\left[W\right] \equiv \int_0^\infty \mathrm{d}s\, s^{-n}\, \widetilde{W}(s). \tag{D.38}$$

The factor on the right-hand side of (D.37) involving the spacetime integral carries the information about the various field monomials, while the Q-functionals encode the properties of the function $W(z)$.

For positive n, the Q-functional (D.38) is related to $W(z)$ through a Mellin-transform:

$$\boxed{Q_n\left[W\right] = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1}\, W(z), \quad n > 0.} \tag{D.39}$$

The equivalence of (D.38) and (D.39) is proven by substituting (D.35) into (D.39) and using the integral representation of the gamma function.

The index n can be continued to negative values. Denoting the k-th derivative of $W(z)$ by $W^{(k)}(z)$, (D.35) implies

$$W^{(k)}(z) = (-1)^k \int \mathrm{d}s\, s^k\, \widetilde{W}(s)\, e^{-sz}. \tag{D.40}$$

As a consequence,

$$Q_n\left[W\right] = (-1)^k\ Q_{n+k}\left[W^{(k)}\right]. \tag{D.41}$$

For negative values $n \leq 0$, one can then chose an integer k such that $n+k>0$. Following the proof for positive n leads to the general formula

$$\boxed{Q_n\,[W] = \frac{(-1)^k}{\Gamma(n+k)} \int_0^\infty \mathrm{d}z\, z^{n+k-1}\, W^{(k)}(z)\,.} \tag{D.42}$$

Given a specific function $W(z)$, (D.39) and (D.42) determine the moments $Q_n[W]$ in the expansion (D.37) to any order.

(2) In many practical computations, the relevant values of n are either integers or half-integers, depending on whether the dimension of spacetime is even or odd. In these cases the general formula (D.42) simplifies considerably. Evaluating (D.35) and its derivatives with respect to z at $z=0$ establishes that for $m=0,1,2,\cdots$:

$$Q_{-m}\,[W] = (-1)^m\; W^{(m)}(0). \tag{D.43}$$

If $n=-(2m+1)/2$ is a negative half-integer, it is convenient to choose $k=m+1$, so that

$$Q_{-m}\,[W] = \frac{(-1)^{m+1}}{\sqrt{\pi}} \int_0^\infty \mathrm{d}z\, z^{-1/2}\, W^{(m+1)}(z)\,, \quad m \in \mathbb{N}_0. \tag{D.44}$$

(3) In typical computations the cutoff operator $\mathcal{R}_k$ is often proportional to the identity on the internal space, and the endomorphism $\mathbf{E}$ is proportional to the identity matrix in field space: $\mathbf{E}=q\mathbf{1}$. Then, with $\Delta=(-\nabla^2+q)\mathbf{1}$, the argument W of the Q_n functionals reduces to a scalar function of $-\nabla^2+q$, or its eigenvalues p^2+q actually.[2]

The most common and prototypical form of $W(z)$ is therefore as follows:

$$\boxed{W(z) \equiv W(\tfrac{p^2}{k^2}) = (Z_k)^{-1}\;(P_k+q)^{-p}\;\partial_t\,(Z_k\mathcal{R}_k)\,.} \tag{D.45}$$

It contains a normalization constant Z_k, a certain power of the regularized inverse propagator $P_k(p^2) \equiv p^2 + k^2\, R^{(0)}(p^2/k^2)$, and the scale derivative of the cutoff operator

$$\mathcal{R}_k(p^2) = k^2 R^{(0)}\left(\frac{p^2}{k^2}\right). \tag{D.46}$$

Here $R^{(0)}(z)$ is a dimensionless shape function capturing the freedom of selecting a largely arbitrary cutoff (see Appendix E).

It is convenient to employ the dimensionless variables

$$z \equiv \frac{p^2}{k^2}, \quad w \equiv \frac{q}{k^2}, \tag{D.47}$$

[2] Here and also often in the main text the suggestive notation p^2 is used for the eigenvalues of arbitrary Laplacians $-\nabla^2$. Note that no flat spacetime is implied.

and to express $Q_n[W]$ in terms of an appropriate basis set of *standard dimensionless threshold functions*. In the case at hand the following two families of such functions are sufficient [14]:

$$
\begin{aligned}
\Phi_n^p(w) &\equiv \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{\left[z + R^{(0)}(z) + w\right]^p}, \\
\widetilde{\Phi}_n^p(w) &\equiv \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} \frac{R^{(0)}(z)}{\left[z + R^{(0)}(z) + w\right]^p}.
\end{aligned}
\tag{D.48}
$$

Substituting (D.45) into (D.39) and rewriting the Q-functional in terms of the dimensionless variables (D.47) then leads to the useful relation:

$$
\boxed{Q_n\left[\frac{\partial_t\left(Z_k \mathcal{R}_k\right)}{Z_k\left(P_k + q\right)^p}\right] = k^{2n-2p+2}\left(2\Phi_n^p(w) - \eta\, \widetilde{\Phi}_n^p(w)\right).}
\tag{D.49}
$$

Here η is the "anomalous dimension" associated with the running normalization constant Z_k:

$$
\eta \equiv -\left(Z_k\right)^{-1} \partial_t Z_k.
\tag{D.50}
$$

(4) The threshold functions (D.48) encode the regulator-dependence of the beta functions, i.e., they are manifestly $R^{(0)}$-dependent in general. But certain combinations of threshold functions assume *universal values* though, i.e., they do not depend on the specific choice of shape function but only on the generic properties of the regulator (2.65).

An example for such a universal behavior are threshold functions where the integrand happens to be a total derivative:

$$
\Phi_n^{n+1}(0) = \frac{1}{\Gamma(n+1)} \int_0^\infty \mathrm{d}z\, \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^n}{\left(z + R^{(0)}(z)\right)^n}\right).
\tag{D.51}
$$

Using only that $\lim_{z\to\infty} R^{(0)}(z)$ is required to be finite, this implies that

$$
\boxed{\Phi_n^{n+1}(0) = \frac{1}{\Gamma(n+1)}, \qquad n \geq 0,}
\tag{D.52}
$$

independently of all local details of the function $R^{(0)}(z)$.

For $n = 0$ the result (D.52) is established by a direct evaluation of the Q-functional via (D.43).

The identity (D.52) serves as a prototypical example showing how renormalization scheme-independent quantities arise from the FRGE.

A second instructive example showing how universality emerges is the EAA-based computation of the trace anomaly in two dimensions, which involves the interplay of several threshold functions with complicated arguments; see [415] for the details.

Appendix E
Cutoff- and Threshold Functions

The explicit calculation of renormalization group trajectories makes it necessary to concretely specify the cutoff, or regulator function $\mathcal{R}_k(p^2) = \mathcal{Z}_k k^2 R^{(0)}(\frac{p^2}{k^2})$. One requires that it satisfies the following general properties:

(*i*) $\mathcal{R}_k(p^2) \to 0$ for $k \to 0$. This property ensures that $\lim_{k\to 0} \Gamma_k = \Gamma_0$ coincides with the usual effective action.

(*ii*) $\mathcal{R}_k(p^2)$ approaches zero for $p^2 \gg k^2$, entailing that the high-momentum modes with $p^2 \gg k^2$ are not affected by the regulator. At the level of the path integral the corresponding modes are integrated out without suppression factor.

(*iii*) $\mathcal{R}_k(p^2) \propto k^2$ for $p^2 \lesssim k^2$, i.e., the low-momentum modes acquire an effective mass of order k, suppressing their contribution in the flow equation. Moreover, this requirement ensures that the regulator diverges if k is sent to infinity, completely suppressing the propagation of all modes in this limit.

These general properties can be met in a number of ways. Quite often their precise implementation is tailored to the problem under investigation. The most frequently used cutoff shape functions $R^{(0)}(z)$ are discussed below and listed in Table E.1. For clarity the wave function factor $\mathcal{Z}_k$ is set to one in the sequel.

Table E.1. Cutoff shape functions $R^{(0)}(z)$ and their respective virtues

Regulator	$R^{(0)}(z)$	Analytical threshold functions	Smooth	Complex frequencies
Exponential (exp)	$\frac{z}{e^z-1}$		**X**	
Sharp (sc)	$\widehat{r}\,\Theta(1-z)$	**X**		
Optimized (opt)	$(1-z)\Theta(1-z)$	**X**		
Analytic poly	$\frac{1}{1+c_1 z+c_2 z^2+\cdots}$			**X**
Analytic exp	$\frac{z}{e^{z^2}-1}$		**X**	**X**
Mass	1			

E.1 Exponentially Decaying Cutoffs

The characteristic feature of this class of cutoffs is that they vanish exponentially for $p^2 \gg k^2$. The prototype of a cutoff shape function $R^{(0)}$ belonging to this class is

$$R^{(0)}(z)^{\exp} = \frac{z}{e^z - 1}, \tag{E.1}$$

and its one-parameter generalization

$$R^{(0)}(z;s)^{\exp} = \frac{sz}{e^{sz} - 1}, \quad s > 0. \tag{E.2}$$

The key advantage of these regulators is that they are smooth, infinitely differentiable functions. This makes them an attractive choice when computing derivative expansions to high order.

As a drawback the integrals appearing in the threshold functions can, in general, not be carried out analytically. The beta functions are then given by a set of coupled integrodifferential equations, entailing that computing RG trajectories may be numerically more expensive than for regulators with threshold functions that can be evaluated analytically.

For vanishing arguments the threshold functions related to the generalized exponential cutoff $R^{(0)}(z;s)^{\exp}$ in (E.2) can be obtained analytically. Using the integral representations for polylogarithms and the Riemann ζ-function, $\mathrm{Li}_\nu(s) = \left(\frac{1}{\Gamma(\nu)}\right) \int_0^\infty \mathrm{d}z \, \frac{s z^{\nu-1}}{e^z - s}$ and $\zeta(\nu) = \left(\frac{1}{\Gamma(\nu)}\right) \int_0^\infty \mathrm{d}z \, \frac{z^{\nu-1}}{e^z - 1}$ [416], respectively, one easily verifies that, for example,

$$\begin{aligned} \Phi^1_n(0;s)^{\exp} &= \frac{n}{s^n} \big\{ \zeta(n+1) - \mathrm{Li}_{n+1}(1-s) \big\}, \\ \Phi^2_n(0;s)^{\exp} &= \frac{1}{s^{n-2}(1-s)} \, \mathrm{Li}_{n-1}(1-s). \end{aligned} \tag{E.3}$$

For non-vanishing arguments no analytic solution to the integrals is known.

E.2 The Sharp Cutoff

The sharp cutoff (sc) is best described using dimensionful variables. It reads [160]

$$\mathcal{R}_k(q^2)^{\mathrm{sc}} = \widehat{R} \, \Theta\left(1 - \frac{q^2}{k^2}\right). \tag{E.4}$$

Here Θ denotes the Heaviside step function and the limit $\widehat{R} \to \infty$ is taken *after* the q^2-integration or the integration over $z \equiv q^2/k^2$ in the threshold functions.

This choice makes the coarse-graining procedure particularly transparent: the cutoff vanishes for the UV modes while the IR modes acquire an infinite mass term in the limit $\widehat{R} \to \infty$. A highly welcome feature of the sharp cutoff is that it permits an explicit analytical evaluation of the associated threshold functions (D.48) [160]:

$$\boxed{\begin{aligned} &\Phi^1_n(w)^{\mathrm{sc}} = -\frac{1}{\Gamma(n)}\ \ln(1+w) + \varphi_n, \\ &\Phi^p_n(w)^{\mathrm{sc}} = \frac{1}{\Gamma(n)}\frac{1}{(p-1)}\frac{1}{(1+w)^{p-1}}, \quad p>1, \\ &\widetilde{\Phi}^1_n(w)^{\mathrm{sc}} = \frac{1}{\Gamma(n+1)}, \\ &\widetilde{\Phi}^p_n(w)^{\mathrm{sc}} = 0, \qquad\qquad\qquad\qquad p>1. \end{aligned}} \tag{E.5}$$

Furthermore, an additional bonus of the sharp cutoff is that via the constants $\varphi_n \equiv \varphi_n(s)$, $s>0$, which appear in (E.5), this cutoff actually amounts to a family of shape functions labeled by a parameter $s>0$ similar to the generalized exponential cutoffs (E.2).

(1) It is instructive to understand why the s-independent function (E.4) actually gives rise to a *family* of threshold functions. To see this, recall that the z-integrals (5.51) for the threshold functions resulted from setting $\mathcal{R}_k(q^2) = k^2 R^{(0)}(q^2/k^2)$. Since (E.4) is not of this form we first reintroduce $\mathcal{R}_k(q^2)$ and rewrite (5.51) accordingly.[1] The evaluation of the integrals defining the threshold functions then proceeds as follows. With $z \equiv q^2/k^2$ and $w \equiv v/k^2$ they read

$$\Phi^p_n\left(\frac{v}{k^2}\right) = \frac{1}{\Gamma(n)}\,(k^2)^{p-n}\int_0^\infty \mathrm{d}q^2\,(q^2)^{n-1}\frac{R^{(0)}(q^2/k^2) - (q^2/k^2)R^{(0)\prime}(q^2/k^2)}{(q^2+\mathcal{R}_k(q^2)+v)^p}. \tag{E.6}$$

Here the prime denotes the derivative of $R^{(0)}$ with respect to its argument. The integrand in equation (E.6) can then be written as a derivative with respect to k, for any $p>1$:

$$\begin{aligned} \Phi^p_n\left(\frac{v}{k^2}\right) &= -\frac{(k^2)^{p-n-1}}{2\Gamma(n)(p-1)}\int_0^\infty \mathrm{d}q^2\,k\frac{D}{Dk}\,\frac{(q^2)^{n-1}}{(q^2+\mathcal{R}_k(q^2)+v)^{p-1}} \\ &= -\frac{(k^2)^{p-n-1}}{2\Gamma(n)(p-1)}\int_0^\infty \mathrm{d}q^2\,k\frac{\partial}{\partial k}\,\frac{(q^2)^{n-1}}{(q^2+\mathcal{R}_k(q^2)+\tilde{v})^{p-1}}\bigg|_{\tilde{v}=v}. \end{aligned} \tag{E.7}$$

Here the derivative D/Dk acts by definition only on the k-dependence of $\mathcal{R}_k$, but not on v. In order to rewrite D/Dk in terms of $\partial/\partial k$ we introduced the constant $\tilde{v}$, which is strictly independent of k. Only at the end of the calculation it is identified with $v \equiv wk^2$.

[1] Formally we have $R^{(0)}(z)^{\mathrm{sc}} = \widehat{r}\,\Theta(1-z)$ with the dimensionless parameter $\widehat{r} \equiv \widehat{R}/k^2$. However, note that the limits $\widehat{R}\to\infty$ and $\widehat{r}\to\infty$ are equivalent only for finite and non-zero values of k, which may require special care.

For $p=1$ the formula (E.7) breaks down since its right-hand side is no longer well defined. Assuming $p>1$, we interchange the q-integration and the derivative with respect to k. This is allowed since the integral in (E.7) is absolutely convergent. If one now substitutes the sharp cutoff $\mathcal{R}_k(q^2)^{\rm sc} \equiv \widehat{R}\,\Theta(1-q^2/k^2)$ one finds:

$$\Phi^p_n\left(\frac{v}{k^2}\right)^{\rm sc} = -\frac{(k^2)^{p-n-1}}{2\Gamma(n)(p-1)}\,k\frac{\partial}{\partial k}\int_0^\infty {\rm d}q^2\,\frac{(q^2)^{n-1}}{\left(q^2+\widehat{R}\,\Theta(1-\frac{q^2}{k^2})+\tilde{v}\right)^{p-1}}\Bigg|_{\tilde{v}=v}. \tag{E.8}$$

Taking the limit $\widehat{R}\to\infty$ restricts the momentum integration to $q^2>k^2$:

$$\Phi^p_n\left(\frac{v}{k^2}\right)^{\rm sc} = -\frac{(k^2)^{p-n-1}}{2\Gamma(n)(p-1)}\,k\frac{\partial}{\partial k}\int_{k^2}^\infty {\rm d}q^2\,\frac{(q^2)^{n-1}}{(q^2+\tilde{v})^{p-1}}\Bigg|_{\tilde{v}=v}. \tag{E.9}$$

The resulting integral is trivially evaluated by acting with the k-derivative on the lower integration limit. This yields our final result for the threshold functions with a sharp cutoff,

$$\Phi^p_n(w)^{\rm sc} = \frac{1}{\Gamma(n)}\,\frac{1}{(p-1)}\,\frac{1}{(1+w)^{p-1}} \quad \text{for} \quad p>1, \tag{E.10}$$

which is in accord with (E.5).

(2) One easily verifies that for $p>1$ the threshold functions (E.10) satisfy the recursion relation (5.97) even though the latter was derived for a smooth cutoff.

We now use this recursion relation in order to *define* what $\Phi^1_n(w)^{\rm sc}$ at $p=1$ is supposed to be. We insist that it solves the differential equation $\frac{\rm d}{{\rm d}w}\Phi^1_n(w)^{\rm sc} = -\frac{1}{\Gamma(n)}\frac{1}{1+w}$ that arises from substituting the known $\Phi^2_n(w)^{\rm sc}$ into (5.97). Its solution contains a free constant of integration:

$$\Phi^1_n(w)^{\rm sc} = -\frac{1}{\Gamma(n)}\,\ln(1+w) + \Phi^1_n(0)^{\rm sc}. \tag{E.11}$$

The integration constants $\Phi^1_n(0)^{\rm sc} \equiv \varphi_n$ are arbitrary numbers a priori. They parametrize a residual cutoff scheme dependence which is still present after having opted for a sharp cutoff.

To parametrize this residual cutoff scheme dependence it is often convenient to focus on a one-parameter family of numbers $\{\varphi_n\}$. In [160] it has been proposed to relate them to the threshold functions obtained with the s-dependent exponential cutoff functions (E.2). This is done by requiring that the threshold functions for the sharp and exponential cutoffs agree at $w=0$. For this value of the argument we may use the analytic formula (E.3) then:

$$\varphi_n(s) \equiv \Phi^1_n(0)^{{\rm sc}(s)} \overset{!}{=} \Phi^1_n(0;s)^{\rm exp}. \tag{E.12}$$

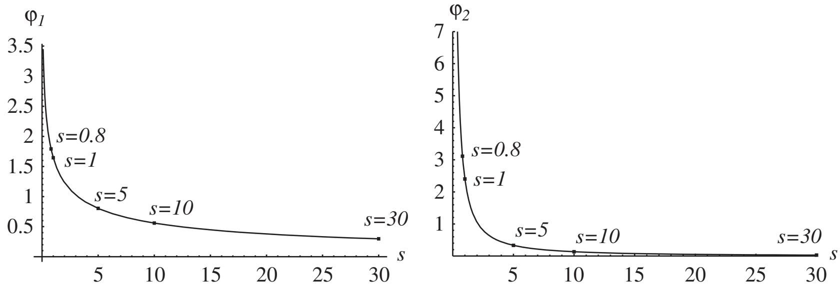


Figure E.1. Dependence of the constants of integration φ_1 and φ_2, (E.12), on the shape parameter s.

In particular, one has $\varphi_n(1) = n\zeta(n+1)$ with $\zeta(n)$ being the Riemann zeta function. The resulting functions $\varphi_1(s)$ and $\varphi_2(s)$, which incidentally are the only ones needed for the Einstein–Hilbert truncation in $d = 4$, are shown in Figure E.1.
(3) For all $\widetilde{\Phi}^p_n$-functions the integral appearing in (D.48) can be evaluated unambiguously after substituting the sharp cutoff. In a calculation similar to the one above we obtain the results given in (E.5).

Thanks to its explicitly computable threshold functions and the simple possibility of "deforming" it to a full family of cutoffs, the sharp cutoff represents a convenient tool for reliability tests of truncations using universal, i.e., scheme-independent quantities. (See [160] for further details.)

E.3 The Optimized Cutoff

Another convenient choice for the cutoff function is the so-called "optimized" shape function [64], of the form

$$R^{(0)}(z)^{\text{opt}} = (1-z)\,\Theta(1-z) \tag{E.13}$$

with Θ again denoting the Heaviside step function. In this case the integral defining the threshold functions (D.48) can be carried out analytically, yielding

$$\boxed{\begin{aligned} \Phi^p_n(w)^{\text{opt}} &= \frac{1}{\Gamma(n+1)}\,\frac{1}{(1+w)^p}, \\ \widetilde{\Phi}^p_n(w)^{\text{opt}} &= \frac{1}{\Gamma(n+2)}\,\frac{1}{(1+w)^p}. \end{aligned}} \tag{E.14}$$

In special cases it is possible to show that the regulator (E.13) is "optimal" in the sense that it minimizes the systematic error when computing a physical quantity within a given truncation; in this sense it "optimizes" the physics predictions [64]. Moreover, the Q-functionals appearing in (D.49) vanish identically when evaluated with $R^{(0)}(z)^{\text{opt}}$ if n is sufficiently negative [185, 186].

E.4 Regulators for Complex Frequencies

When computing real-time correlation functions the argument of the regulator may actually become complex. In this case it becomes important that one can control the residues induced by the regulator within the complex plane. In this context one may resort to regulator functions featuring an algebraic decay [417],

$$R^{(0)}(z) = \left[1 + c_1 z + c_2 z^2 + \cdots\right]^{-1}, \tag{E.15}$$

with c_i free constants, or an exponential decay [418]

$$R^{(0)}(z) = z\left[e^{z^2} - 1\right]^{-1}. \tag{E.16}$$

Regulators of this type are relevant in setting up flow equations on spacetimes with Lorentzian signature.

E.5 Mass-Type Cutoff

Strictly speaking, the mass-type cutoff defined by

$$\mathcal{R}_k(p^2) = k^2 \quad \Leftrightarrow \quad R^{(0)}(z) = 1 \tag{E.17}$$

is not a valid regulator function since it does not fall off for $p^2 \gg k^2$ and thus violates the general requirement (*ii*). Its main advantage is that it leads to very simple momentum integrals and therefore is often used for a first orientation. Its main disadvantage is that with (E.17) the factor $\partial_t \mathcal{R}_k$ under the supertrace in the FRGE is now p^2-independent and no longer renders the traces UV convergent. Therefore, if one uses (E.17), an explicit UV regularization of some sort must be introduced.

Appendix F
Field Decompositions

A key step in the evaluation of the (truncated) functional renormalization group equation (4.80) is the inversion of the Hessian $(\Gamma_k^{(2)} + \mathcal{R}_k)$. This step can often be facilitated by rewriting the fluctuation fields in terms of suitable component fields. This appendix introduces the two most commonly used decompositions.

A typical example for a field decomposition of the metric fluctuations $h_{\mu\nu}$ is the traceless decomposition encountered in (5.27). In this case $h_{\mu\nu}$ is expressed in terms of a traceless symmetric tensor $\mathring{h}_{\mu\nu}$ and its trace ϕ according to

$$h_{\mu\nu} = \mathring{h}_{\mu\nu} + \frac{1}{d}\bar{g}_{\mu\nu}\phi. \tag{F.1}$$

The component fields are then subject to the constraints

$$\bar{g}^{\mu\nu}\mathring{h}_{\mu\nu} = 0, \quad \phi = \bar{g}^{\mu\nu}h_{\mu\nu}. \tag{F.2}$$

Note that the Jacobian related to the transverse decomposition of the field is trivial.

The transverse-traceless (TT) decomposition [384, 385] provides a useful refinement of the transverse decomposition, setting

$$h_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu} + \bar{D}_\mu\xi_\nu + \bar{D}_\nu\xi_\mu + 2\bar{D}_\mu\bar{D}_\nu\,\sigma - \frac{2}{d}\bar{g}_{\mu\nu}\bar{D}^2\sigma + \frac{1}{d}\,\bar{g}_{\mu\nu}\,\phi. \tag{F.3}$$

In this case all component fields appearing on the right-hand side are transverse with respect to the background covariant derivative and satisfy the differential constraints

$$\bar{D}^\mu h^{\mathrm{T}}_{\mu\nu} = 0, \quad \bar{g}^{\mu\nu}h^{\mathrm{T}}_{\mu\nu} = 0, \quad \bar{D}_\mu\xi^\mu = 0, \quad \bar{g}^{\mu\nu}h_{\mu\nu} = \phi\,. \tag{F.4}$$

Typically the transverse-traceless (TT) decomposition of the metric fluctuations is accompanied by a transverse decomposition of the vector fields according to

$$C_\mu = C^{\mathrm{T}}_\mu + \bar{D}_\mu c, \quad \bar{D}^\mu C^{\mathrm{T}}_\mu = 0. \tag{F.5}$$

Using the component fields appearing in (F.3) and (F.5) has the advantage that all terms containing contractions of fluctuation fields with a background covariant derivative vanish. Typically this results in a drastic simplification of the differential operators contained in $(\Gamma_k^{(2)} + \mathcal{R}_k)$; see [411] for a detailed discussion.

Let us summarize the most important properties of the TT decomposition:

(1) Zero modes: From the structure of (F.3) and (F.5), it is clear that not all modes of the component fields contribute to the metric fluctuations $h_{\mu\nu}$ or the vector fields C_μ. In the metric decomposition, the constant mode of σ, vectors $\mathcal{C}_\mu \equiv \bar{D}_\mu \sigma$ satisfying the conformal Killing equation $\bar{D}_\mu \mathcal{C}_\nu + \bar{D}_\nu \mathcal{C}_\mu - \frac{2}{d}\bar{g}_{\mu\nu}\bar{D}_\alpha \mathcal{C}^\alpha = 0$, and transversal vectors solving the Killing equation $\bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu = 0$, do not contribute to $h_{\mu\nu}$. Analogously, the decomposition of a vector field is independent of a constant mode c. These modes are unphysical and must be excluded from the spectrum.

(2) Nontrivial Jacobian: The field decompositions (F.3) and (F.5) give rise to nontrivial Jacobi determinants

$$J_{\text{grav}} = \left(\det{}'_{(1T,0)}[M]\right)^{1/2}, \quad J_{\text{vector}} = \left(\det{}'_0[-\bar{D}^2]\right)^{1/2}. \tag{F.6}$$

Here

$$M^{(\mu,\nu)} = \begin{bmatrix} -2\left[\bar{g}^{\mu\nu}\bar{D}^2 + \bar{R}^{\mu\nu}\right] & -2\bar{R}^{\mu\lambda}\bar{D}_\lambda \\ 2\bar{D}_\lambda \bar{R}^{\lambda\nu} & \frac{d-1}{d}(\bar{D}^2)^2 + \bar{D}_\mu \bar{R}^{\mu\nu}\bar{D}_\nu \end{bmatrix} \tag{F.7}$$

is a $(d+1)\times(d+1)$-dimensional matrix in field space whose first d components act on transverse vectors and the last component on scalars, respectively. Furthermore, the prime indicates that the zero modes have been excluded from the determinants. The Jacobians may be taken into account by introducing suitable sets of auxiliary fields along the lines of the Faddeev–Popov trick. The TT decomposition is particularly powerful on backgrounds satisfying the Einstein condition $\bar{R}_{\mu\nu} = \left(\frac{1}{d}\right)\bar{g}_{\mu\nu}\bar{R}$ where the operator M is block diagonal.

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