The Asymptotic Safety Scenario for Quantum Gravity

Ultraviolette Fixpunkte für eine Theorie der Quantengravitation

Bachelor Thesis

vorgelegt von
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1. Introduction

1.1. Abstract

In this thesis I will explain the basics concepts of general relativity and quantum field theory that are needed to understand the concept of asymptotic safety in quantum gravity. I will then make use of a functional renormalization group equation to derive the beta functions for the gravitational and the cosmological constant using two different cutoff schemes and discuss their high and low energy behavior in a renormalization group flow diagram.

1.2. Motivation

The first physical theory that was known to mankind was the theory of gravitation. From early on people thought about gravity because it influenced their every day life directly. Galileo first found out that every object falls at the same rate of acceleration, later Newton put this in a mathematical framework and realized that gravity not only causes apples to fall to the ground but also that the planets and the moons elliptical orbits were due to the same force. In the late 19th century Maxwells theory of electromagnetism came into play which caused a man called Einstein to realize that the classical theory of gravity simply does not combine well with it. This was a huge problem, since both theories described sensible physics. He therefore derived his special and general theory of relativity. The latter being a new theory of gravity which revolutionized the meaning of gravity as a force. This theory fit experiments so well that it became the new standard theory for gravity, recently confirmed again by the detection of gravitational waves.

It was all good until the theory of quantum mechanics was invented. This theory explained many microscopic phenomena in the lab that were just incompatible with the classical theory. Combining it with the special theory of relativity gave rise to the most accurate theory of the microscopic world which is the theory of quantum fields. Every microscopic phenomenon was explainable by a quantum field theory, so the theory was called the standard model of particle physics, while every macroscopic phenomenon involving huge masses was explained by general relativity. The only systems in which problems arose were systems involving tiny objects with huge masses so that the laws of quantum field theory and the laws of gravity had to be applied at the same time, for example in the center of black holes or the beginning of the universe. To put this at a scale a combination of the fundamental constants G the gravitational constant, c the speed of light and $\hbar$ the fundamental constant of quantum mechanics gives rise to so called Planck units [1]

$$L_{pl} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\, m \tag{1.1}$$

$$E_{pl} = \sqrt{\frac{\hbar c^5}{G}} \approx 1.22 \times 10^{19}\, GeV. \tag{1.2}$$

It is expected that at length and energy scales, comparable to the Planck scale, a theory of quantum gravity is needed to explain whats going on. It is therefore very difficult to reproduce quantum gravity effects in the laboratory because currently available energy scales tested at the Large Hadron Collider do not exceed $13 \times 10^3\ GeV$ [2] which is far less than the energy needed to probe these effects. The standard approach of quantizing gravity in a perturbative quantum field theoretical framework using Feynman graphs and the spin two graviton as its force carrier is plagued by many non renormalizable ultraviolet divergences. It is however possible to treat general relativity as an effective field theory that is only valid for low energies far below the Planck scale like in [3]. Many other approaches came up to study gravity in all energy scales. Some involve introducing new degrees of freedom that were just not visible in the low energy limit but have to be introduced in the ultraviolet limit. This was for example the case in the theory of weak interactions where the ultraviolet limit included the W and Z bosons that rendered the theory renormalizable. Another idea that is put forward is the discretization of spacetime itself called quantum loop gravity where general relativity and its continuous spacetime are recovered in a low energy limit. An introduction to this can be found in [4]. One exceptional theory that automatically includes gravity is string theory. Here every particle is represented by a certain vibrational mode of a one dimensional string. The spin two graviton is easily included.

In this thesis, however, I will focus on implementing gravity in a field theoretical approach in which it is not quantized using perturbative methods but rather in a non perturbative manner that makes use of a renormalization group flow. If the flow has a finite ultraviolet limit a theory of quantum gravity in the sense of asymptotic safety can be derived from it.

In the following I will first explain some of the basics of general relativity and quantum field theory. Then I will introduce a functional renormalization group equation with which it is possible to calculate the renormalization group flow of an action functional from low to high energies. In the end I will use this renormalization group equation to study the fixed point behavior of the standard action of general relativity the Einstein Hilbert action in which I will not include the presence of mass so that only the pure gravitational field is under investigation. Throughout this thesis I will use the natural units in which $c = \hbar = 1$ which simplifies most of the equations and makes a simple relation between length, time, momentum and energy scales possible.

2. Concepts of general relativity

In this chapter I will explain the most important concepts of general relativity and curved spacetime geometry. More aspects of the topic can be found in [5],[6] and [7]. As the name general relativity suggests it is a generalization of special relativity which describes a flat spacetime in inertial reference frames i.e. frames that are at constant velocity to each other. The spacetime of general relativity locally reduces to Minkowsky space but globally it looks curved. Curvature in a mathematical sense will be explained later. The fact that it is flat in small regions comes from the equivalence principle. It

states that an observer in a closed small box can not distinguish between two situations. In the first he is sitting on a planet and being attracted by its gravitational field and in the second he is in free space in the absence of gravitation but his box is accelerated. It is really just the same situation because the 'gravitational force' he is experiencing on the planet is just the fact that the planet stands in his path of free fall. This means that gravity is just falling freely with no acceleration involved. So if no acceleration is present the laws of physics in the box just look like the ones from special relativity which means that spacetime in the small box looks like Minkowsky space as mentioned earlier. In the beginning I said that this is only true locally which here means in infinitesimal small regions. Thinking again of the small box sitting on the planet there is a way to detect the effects of gravitation which is by looking at a larger box that is not infinitesimally small. Two objects horizontally separated will fall towards the center of the planet and not just straight downwards. So for larger regions of spacetime the laws of special relativity are not valid which means that globally spacetime is not flat Minkowsky space but a more general curved manifold. It is necessary to introduce the notion of manifolds, a metric, tensors, geodesics, curvature and so on, so that in the end I can describe how energy causes curvature in spacetime through the Einstein field equations.
An n dimensional manifold M can be thought of as a space that looks locally like Euclidean space which means that every point of the manifold can be related to $\mathbb{R}^n$. This correspondence is defined in terms of coordinate maps $\phi_i : M \supset U_i \to \mathbb{R}^n$ where U_i is an open subset of the manifold. In general it is not possible to write this correspondence in terms of a single map but rather as an assembly of maps that is called an atlas if it covers the whole manifold i.e. $\bigcup_i U_i = M$. If two of those maps overlap they can be related by a coordinate transformation $\tau_{ij} : \phi_i \circ \phi_j^{-1}$ and if the map is smooth i.e. continuous and everywhere differentiable it is called a differomorphism. These maps correspond to the coordinates x^μ used later.
The next thing that has to be established is the notion of a vector on the manifold since vectors are crucial to understanding physics. For this purpose it is convenient to introduce a tangent space to each point of the manifold. These tangent spaces then contain every possible vector at that point, it should therefore be spanned by n basis vectors. The set that is the disjointed union of all tangent spaces is defined in $2n$ dimensions because for each point of the manifold there is an n dimensional tangent space. It is called the tangent bundle. Each vector V at a point p can be represented by a directional derivative acting on a function f so that $\frac{\partial f(p+Vt)}{\partial t}\big|_{t=0} = V^\mu \partial_\mu f(p)$ so in a sense $\partial_\mu = \frac{\partial}{\partial x^\mu}$ can be seen as a basis for the tangent space T_p. With this definition of the tangent space, vectors can be represented by $V = V^\mu \partial_\mu$ which should be independent of coordinates. From the chain rule follows that $\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu$ which means that the vector components V^μ transform like

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu. \tag{2.1}$$

Components that transform in this way are called contravariant. Now that vectors at each point p on the manifold can be constructed, linear mappings from vectors to real numbers would be desirable. To do that, another vector space called the dual space T_p^*

is introduced by the requirement that it includes all linear mappings $W : T_p \rightarrow \mathbb{R}$ and its basis vectors fulfill $e^\mu(e_\nu) = \delta^\mu_\nu$ where e_ν is the basis of T_p, and δ^μ_ν is the Kronecker delta. With these definitions linear maps $W(V)$ become $W_\nu V^\mu \delta^\nu_\mu = W_\mu V^\mu$. By the requirement that this mapping is also independent of coordinates the transformation properties of the covariant components W_μ can directly be extracted:

$$W^{'}_{\mu} = \frac{\partial x^\nu}{\partial x'^\mu} W_\nu \tag{2.2}$$

The concept of linear mappings from the vector space and its dual space can be generalized to a multilinear mapping which introduces a tensor of type (r, s):

$$T : T^*_p \times ...T^*_p \times T_p \times ...T_p \rightarrow \mathbb{R} \tag{2.3}$$

with r factors of T^*_p and s factors of T_p. Its components transform just as expected:

$$T'^{\mu_1 ... \mu_r}{}_{\nu_1 ... \nu_s} = \frac{\partial x'^{\mu_1}}{\partial x^{\sigma_1}} ... \frac{\partial x'^{\mu_r}}{\partial x^{\sigma_r}} \frac{\partial x^{\rho_1}}{\partial x'^{\nu_1}} ... \frac{\partial x^{\rho_s}}{\partial x'^{\nu_s}} T^{\sigma_1 ... \sigma_r}{}_{\rho_1 ... \rho_s} \,. \tag{2.4}$$

There are several operations that can be done to tensors so that the result is still a tensor. Addition and multiplication by scalars is straightforward. Another operation is the tensor product, where the components of the new tensor are every possible multiplication of components of the multiplied tensors for example $T^i_{\ j} = A^i B_j$. The last operation is the contraction which means setting an upper and a lower index equal and summing over it.
Probably the most important tensor in general relativity is the metric tensor with its components $g_{\mu\nu}$, which is used to measure lengths on a manifold. It is a bilinear mapping from $T_p \times T_p$ to the reals and its meaning is the inner product of two vectors.

$$g(A, B) = g_{\mu\nu} A^\mu B^\nu \tag{2.5}$$

Also it defines the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. So in fact it characterizes the geometry of the space which will be identified with the gravitational potential. So a solution to the field equation will be a metric tensor from which particle paths in the spacetime can be calculated. Its inverse is defined by $\delta^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu}$. Contracting an index of a tensor with the metric introduces a new tensor that generally receives the same name, for example $T^{\mu\nu} g_{\mu\sigma} = T_\sigma{}^\nu$. With this convention the inner product can also be expressed by

$$g(A, B) = g_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu = g^{\mu\nu} A_\mu B_\nu = A_\mu B^\mu, \tag{2.6}$$

since $g(A, B)$ is the inner product $g(A, A) = A^\mu A_\mu$ is just the length of the vector A. In spacetime with one time coordinate and three space coordinates and a metric signature of $(+ - --)$ it is possible for the inner product to be negative. The metric is then called indefinite because vectors can be perpendicular to them selves while not being zero. Another interesting concept that arises from the indefiniteness of the metric is that there are forbidden worldlines in the spacetime. Calculating the length l of a tangent vector to a world line there are three possible outcomes. $l > 0$ corresponds to timelike paths where the velocity of the particle is less than the speed of light, $l = 0$ is the worldline of light which is also called a lightlike path and the forbidden worldlines have $l < 0$ which corresponds to spacelike paths with a velocity that would exceed the speed of light. This concept is the same for special and general relativity. The placement of the indices on tensors is important in general, except when symmetries are involved. The metric tensor for example is symmetric in its indices so that $g_{\mu\nu} = g_{\nu\mu}$. As mentioned earlier it is the curvature of the space that is responsible for deviations from special relativity. To be able to make statements about curvature and also how geodesics i.e. shortest paths in a curved space are defined it is necessary to establish a notion of differentiation. For simple scalar functions there is no problem with the regular partial derivative. But when differentiating vector or tensor components it is not directly obvious what is even meant by that because vectors at different points still live in different tangent spaces. Since differentiation involves comparing vectors at nearby points there is clearly a problem. This problem manifests in the transformation properties for example for the partial derivative of a vector

$$\partial'_{\mu} A'^{\nu} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \partial_{\sigma} \left(\frac{\partial x'^{\nu}}{\partial x^{\rho}} A^{\rho} \right) = \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} \partial_{\sigma} A^{\rho} + \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\nu}}{\partial x^{\sigma} \partial x^{\rho}} A^{\rho} \tag{2.7}$$

which is definitely not the correct transformation for a type $(1, 1)$ tensor. The problem of comparing two vectors at points p and q can be solved by introducing a vector $A^{\nu}(q) = A^{\nu}(p) + \delta A^{\nu}$ at q that is in some way parallel to the vector at p but lives in the tangent space of q. The following limit then defines a covariant derivative:

$$\nabla_{\mu} A^{\nu} = \lim_{\delta x^{\mu} \to 0} \frac{1}{\delta x^{\mu}} \left(A^{\nu}(p) + \delta x^{\sigma} \partial_{\sigma} A^{\nu}(p) - A^{\nu}(p) + \delta A^{\nu}(p) \right) \tag{2.8}$$

Since $\delta A^{\nu}(p)$ should be linear in $\delta x^n u$ and $A^{\nu}(p)$ it can be written as $\delta A^{\nu} = -\Gamma^{\nu}{}_{\sigma\rho} A^{\sigma} \delta x^{\rho}$ which means that the covariant derivative reads

$$\nabla_{\mu} A^{\nu} = \partial_{\mu} A^{\nu} + \Gamma^{\nu}{}_{\sigma\mu} A^{\sigma}. \tag{2.9}$$

The connection coefficients are also called Christoffel symbols if they are chosen in such a way that the covariant derivative of the metric vanishes, which will be useful later as it preserves the inner product along any curve on the manifold. The Christoffel symbols are then given in terms of the first partial derivative of the metric.

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\beta\delta} \left(\partial_{\beta} g_{\gamma\delta} + \partial_{\gamma} g_{\beta\delta} - \partial_{\delta} g_{\beta\gamma} \right) \tag{2.10}$$

It should be noted that the Christoffel symbols do not transform like tensors, hence the name symbol.

Using the metric together with these Christoffel symbols it is possible to identify paths of shortest distance in spacetime also called geodesics. These are the paths that particles take if no external force is acting on them. The line element can therefore be written as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\mu}}{d\tau} d\tau^2 \tag{2.11}$$

with an arbitrary parameter τ that parameterizes the path. The length of this path is then given by

$$\int ds = \int d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\mu}{d\tau}}. \tag{2.12}$$

Remembering the principle of least action and Lagrangian mechanics, which will also be used in the next section, it is possible to identify the shortest path length by applying the Euler Lagrange equations to the Lagrangian $L\left(x, \frac{dx}{d\tau}\right) = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\mu}{d\tau}$.

$$\Rightarrow \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0 \tag{2.13}$$

The resulting equations are the geodesic equations

$$\ddot{x}^\mu + \Gamma^\mu{}_{\sigma\rho} \dot{x}^\sigma \dot{x}^\rho = 0 \tag{2.14}$$

where the dot indicates the differentiation with respect to τ. If τ is interpreted as a time variable along the path, the similarities to Lorentz spacetime are obvious. Here the term including the Christoffel symbol would vanish and the equation would just be describing a particle moving in a straight line. This means that the Christoffel symbols play the role of a 'force' that is acting on the particle diverting it from its straight path. This is of course not true because the deviation from the straight line is simply due to the fact that the space is curved.
To quantify the curvature of space I will define the Riemann curvature tensor

$$R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\epsilon_{\beta\delta} \Gamma^\alpha_{\epsilon\gamma} - \Gamma^\epsilon_{\beta\gamma} \Gamma^\alpha_{\epsilon\delta} \tag{2.15}$$

which is just the commutator of two covariant derivatives acting on a vector field i.e.

$$[\nabla_\mu, \nabla_\nu] A^\sigma = R^\sigma{}_{\rho\mu\nu} A^\rho. \tag{2.16}$$

If this tensor is zero everywhere, then it is possible to find a coordinate system in which the Christoffel symbols and also their first derivatives vanish identically. This means that it is possible to find a metric that is constant everywhere and describes a flat spacetime. So a nonzero curvature tensor implies a curved space. Also from the commutator formulation it becomes clear that the curvature tensor vanishes in flat space because the covariant derivatives become partial derivatives that commute.
The last important aspect of curved spaces is the volume integration. Integration is a continuous form of summing up little volume elements. As seen earlier the sum of tensors at different points introduces problems because of different tangent spaces in which the tensors are defined. So the only way to properly define an integral is to sum up scalars that are invariant under coordinate transformation. From multivariable calculus in $\mathbb{R}^n$ it is known that a transformation of the volume element introduces a factor of the determinant of the Jacobian $d^n x' = |\frac{\partial x'^\mu}{\partial x^\nu}| d^n x = J d^n x$, hence it transforms

like a scalar density of weight 1. Tensor densities of weight w are objects that on top of the usual tensor transformation come with a factor of J^w. The determinant of the metric g is also a scalar density but with weight -2 so $\sqrt{-g'} = J^{-1}\sqrt{-g}$. The minus sign was introduced because in the signature convention $(+---)$ the determinant will be negative so writing a minus in front of it will make the square root real. Now adding a factor of $\sqrt{-g}$ to the volume element will make it a scalar and therefore makes the integral invariant. This can be seen in a simple example in two dimensions. Consider an integral

$$\int f(x,y)\sqrt{g}\; dx\; dy \tag{2.17}$$

where $f(x,y)$ is a scalar function and (x,y) are Euclidean coordinates so the determinant g is equal to one. This integral expressed in polar coordinates produces canceling factors of the Jacobian coming from $\sqrt{-g}$ and from $dx;dy$ and leaves behind

$$\int f(r,\phi)\sqrt{g'}dr\; d\phi = \int f(r,\phi) r\; dr\; d\phi \tag{2.18}$$

with the correct volume element $r\; dr\; d\phi$. So after all to write volume integrals the invariant volume element is $\sqrt{-g}d^n x$.
Now all necessary pieces are collected to write down an action that describes the dynamics of the spacetime in four dimensions. It has to include the invariant volume element and some scalar Lagrangian

$$S = \int d^4x\sqrt{-g}\mathcal{L}. \tag{2.19}$$

Since the geometry of spacetime is given in terms of the metric the Lagrangian will include the metric itself and derivatives of it. The simplest scalar that is constructed from the metric and its derivative is the curvature scalar or Ricci scalar $R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}{}_{\mu\alpha\nu}$. It is an contraction of the Ricci Tensor $R_{\mu\nu}$ which itself is a contraction of the Riemann Tensor $R^{\mu}{}_{\nu\sigma\rho}$. Including the possibility for a constant Term and an overall multiplicative constant the simplest action called the Einstein Hilbert action reads

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}(R-2\Lambda). \tag{2.20}$$

The constant G_N is the gravitational constant that also appears in Newtons gravitational force law while Λ is called the cosmological constant and is interpreted as a vacuum energy density. From this action functional through the principle of least action it is possible to derive the Einstein field equations. They are a set of ten non linear differential equations for the components of the metric. Only including the Einstein Hilbert action as the action yields the vacuum field equations in the absence of matter and energy. Adding a further term including matter fields to the action reproduces the full Einstein field equations including a coupling to matter as seen in the next section.

2.1. A derivation of the vacuum field equations from the Einstein-Hilbert action

In this section I will use the Einstein Hilbert action to derive the field equations for a theory with and without matter content. For any given classical theory the principle of least action gives exact equations of motion. For a simple particle with kinetic energy T and potential energy V the Lagrangian is defined as the function $L = T - V$ and the action functional is then

$$S[q(t)] = \int dt L(\dot{q(t)}, q(t), t). \tag{2.21}$$

The principle of least action then says that the variation of $S[q(t)]$ should vanish along the physical path, so that the action is stationary. Especially at the boundary the action is kept fixed so that any boundary terms vanish. Doing the variation yields the Euler Lagrange equations of motion. In a quantum theory the classical path is the most likely path but also every other path is taken. This phenomenon is manifested in the path integral formulation of quantum mechanics.
To derive the Einstein field equations, consider the Einstein-Hilbert action including matter terms. In this action as explained before, the metric itself is the function that the action depends on.

$$\int d^4x\mathcal{L} = \int d^4x\sqrt{-g}\left(\frac{R-2\Lambda}{16\pi G} + \mathcal{L}_M\right). \tag{2.22}$$

Every type of matter or energy content is contained in the unspecified Lagrangian $\mathcal{L}_M$. Doing the variation gives

$$\begin{aligned} 0 = \delta S = \int \big[\delta\sqrt{-g}&\left(\frac{R-2\Lambda}{16\pi G} + \mathcal{L}_M\right) \\ &+ \sqrt{-g}\left(\frac{\delta R}{16\pi G} + \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}\delta g_{\mu\nu}\right)\big]. \end{aligned} \tag{2.23}$$

With the variations calculated in appendix A the field equations can be written down. Because the second part of δR involves total derivatives it can be reexpressed by a boundary integral for which the variation vanishes by construction. The variation then reads

$$\delta S = \int d^4x\sqrt{-g}\left[\frac{g^{\mu\nu}}{2}(R-2\Lambda) - R^{\mu\nu} + \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_M}{\delta g_{\mu\nu}}\right]\delta g_{\mu\nu} \tag{2.24}$$

which should be valid for all variations $\delta g_{\mu\nu}$.

$$\Rightarrow R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + g^{\mu\nu}\Lambda = 8\pi G T^{\mu\nu} \tag{2.25}$$

Thus the energy momentum tensor is defined by $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta g_{\mu\nu}}$.
Because of the tensor character of these equations, they are valid in all coordinate systems. This is necessary because for a given mass distribution the job is to figure out how the spacetime predicted by Einstein's equations looks like, which would be rather unpractical if one was constrained to some special coordinate system.

3. Generating functionals and a first approach to quantum gravity

3.1. The partition function and the effective action

In this section I will give a meaning to some quantum field theoretical objects following the conventions of [8]. These will be needed later in the thesis. I will explain the quantities using the example of a simple scalar theory. Later I will restrict to an exemplary Lagrangian including a kinetic term $(\partial\phi^2)$, a mass term $m^2\phi^2$ and an interaction term $\frac{\lambda}{4!}\phi^4$.

As seen in the previous section in classical physics performing the variation of the action $S = \int d^4x \mathcal{L}(x)$ gives rise to exact equations of motion. This is not the case in quantum mechanical physics, where only probabilities can be described by a theory. The observables that can actually be measured in experiments are scattering cross sections σ that can be computed through matrix elements $\mathcal{M}$ of the time evolution operator. These matrix elements are related to $n-$point functions G_n by the LSZ reduction formula [9]. These $n-$point functions can in the path integral formalism be written as

$$G_n(x_1, ..., x_n) = \frac{\int \mathcal{D}\phi \phi(x_1)...\phi(x_n) e^{-S[\phi]}}{\int \mathcal{D}\phi e^{-S[\phi]}}. \tag{3.1}$$

The $\int \mathcal{D}\phi$ denotes a functional integral over all possible field configurations which is weighted by the factor $e^{-S[\phi]}$ where $S[\phi]$ is the classical action. This notion is closely related to the partition function for N particles in statistical mechanics in the canonical ensemble

$$Z = \frac{1}{(2\pi\hbar)^{3N}} \int d^{3N}p \; d^{3N}q e^{-\beta H(p,q)}. \tag{3.2}$$

There the integral is performed with a weight factor of $e^{-\beta H(p,q)}$ where $H(p,q)$ is the Hamiltonian and β the inverse temperature. In statistical mechanics it is sufficient to know the partition function to derive all interesting properties of the regarded system which is just the same in quantum field theory. Introducing a source field $J(x)$ to the path integral makes it possible to write the $n-$ point correlation functions in terms of functional derivatives of the partition function

$$Z[J] = \int \mathcal{D}\phi e^{-S[\phi] + \int d^4x J(x)\phi(x)} \tag{3.3}$$

$$\Rightarrow G_n(x_1...x_n) = \frac{1}{Z[0]} \prod_{i=1}^{N} \frac{\delta}{\delta J(x_i)} Z[J]\big|_{J=0}. \tag{3.4}$$

$Z[J]$ is also called the generating functional of correlation functions. As an example consider the above mentioned ϕ^4 theory. The generating functional reads

$$Z[J] = \int \mathcal{D}\phi e^{-\int d^4x \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right) - \frac{\lambda}{4!}\phi^4 + J\phi} \tag{3.5}$$

For a small coupling constant λ an expansion of the following kind can be applied

$$e^{-S[\phi]} = e^{-\int d^4x \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right)}(1 + \int d^4x \frac{\lambda}{4!}\phi^4 + ...). \tag{3.6}$$

So the partition function can be written in terms of correlation functions of the free field with no coupling term proportional to λ in the exponent. The correlation functions can then again be rewritten as a sum over Feynman propagators which then can be represented in Feynman diagrams. To go the other way around each of these diagrams can be evaluated by the following set of rules. First the diagram has to be drawn and every line in it must be labeled with its corresponding momentum. Then for each internal line that does not have an unconnected end write the propagator $\frac{1}{k^2+m^2}$ where k is its momentum. Each interaction vertex then receives a power of the coupling λ and a delta function that forces momentum conservation. Finally the whole expression is integrated by a factor of $d^4k/(2\pi)^4$ for each internal momentum. In ϕ^4 theory only vertices where four lines meet are allowed, but this procedure can be done for every theory that can be handled perturbatively. The result of these Feynman diagrams is then proportional to the scattering amplitude of the process.
The generating functional $Z[J]$ contains all possible diagrams i.e. all correlation functions. But this also includes diagrams that are split up into several parts which are not connected and only add to the energy of the vacuum. The correlation functions are called dressed because the vacuum energy is included in them. Since these disconnected parts are not relevant for scattering experiments it would be nice to have a generating functional that only generates connected diagrams that are partially dressed by their self energy generated through self interaction. This can be achieved by the free energy functional which is defined in an analogous way to the free energy in statistical mechanics

$$W[J] = \ln Z[J] \tag{3.7}$$

$$\tag{3.8}$$

Taking variational derivatives with respect to the source gives only the connected correlation functions

$$G_n(x_1, ..., x_n)_{conn.} = \prod_{i=1}^{N} \frac{\delta}{\delta J(x_i)} W[J]\big|_{J=0} \tag{3.9}$$

and especially the first derivative gives the vacuum expectation value φ of the field ϕ and the second derivative gives the full propagator $D(x, y)$

$$\begin{aligned} \frac{\delta W[J]}{\delta J(x)}\big|_{J=0} &= \frac{1}{Z[0]} \int \mathcal{D}\phi \phi(x) e^{-S[\phi]} \\ &= \langle \phi(x) \rangle = \varphi \end{aligned} \tag{3.10}$$

$$\begin{aligned} \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}\big|_{J=0} &= \frac{\delta}{\delta J(x)} \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(y)} \\ &= -\frac{1}{Z[J]^2} \frac{\delta Z[J]}{\delta J(x)} \frac{\delta Z[J]}{\delta J(y)} + \frac{1}{Z} \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} \\ &= \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle = D(x, y). \end{aligned} \tag{3.11}$$

As mentioned earlier in classical mechanics the variation of the action S produces exact equations of motion. There is a formulation in which a quantum effective action $\Gamma[\varphi]$ can be defined that also produces equations of motion in this case for the vacuum expectation value of the field. This means that quantum corrections are automatically included. So what is needed is a change of variables from the source J to the expectation value φ which can be achieved by a Legendre transformation. The transformation reads

$$\Gamma[\varphi] = -W[J] + \int d^4y J(y)\varphi(y) \tag{3.12}$$

and differentiation of $\Gamma[\varphi]$ with respect to the new variable $\varphi(x)$ gives back the old variable $J(x)$

$$\begin{aligned}\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} &= -\int d^4y \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\varphi(x)} + \int d^4y \frac{\delta J(y)}{\delta\varphi(x)}\varphi(y) + J(x) \\ &= J(x).\end{aligned} \tag{3.13}$$

Also this last formula $\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = J(x)$ corresponds to exact quantum equations of motion for the field φ. Differentiating twice with respect to φ is the inverse propagator

$$\left(\frac{\delta^2 W}{\delta J(y)\delta J(x)}\right)^{-1} = \frac{\delta^2\Gamma}{\delta\varphi(y)\delta\varphi(x)} = D^{-1}(x,y) \tag{3.14}$$

which will be used later on. In this formulation a loop expansion becomes possible where the zeroth order corresponds to the classical action and higher orders represent quantum loop corrections. Rewriting the Legendre transformation 3.12 and shifting the integration variable from ϕ to $\phi - \varphi$ yields the following expression

$$\exp(-\Gamma) = \int \mathcal{D}\phi e^{-S[\phi+\varphi]+\int d^4x J(x)\phi(x)}. \tag{3.15}$$

Expanding the exponent in powers of ϕ makes it possible write

$$\Gamma[\varphi] = -\log\int\mathcal{D}\phi e^{S[\varphi]+\int d^4x(J-\frac{\delta S}{\delta\varphi})\big|_{\phi=\varphi}\phi - \frac{1}{2}\int d^4x\phi\frac{\delta^2 S}{\delta\varphi^2}\big|_{\phi=\varphi}\phi+...} \tag{3.16}$$

$$= S[\varphi] + \frac{1}{2}\log\det(\frac{\delta^2 S}{\delta\varphi^2}\big|_{\phi=\varphi}) + ... \tag{3.17}$$

$$= S[\varphi] + \frac{1}{2}\operatorname{Tr}\log(\frac{\delta^2 S}{\delta\varphi^2}\big|_{\phi=\varphi}) + ... \tag{3.18}$$

3.2. Divergences and renormalization

When applying the rules as mentioned above problems with virtual particle loops will arise. For example consider a diagram where a particle is coming in with momentum

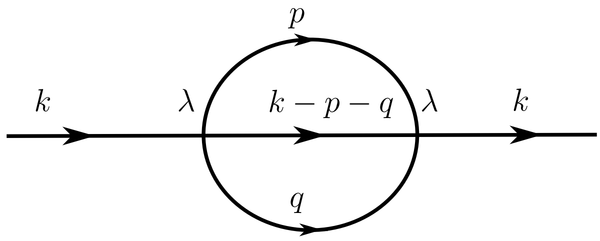

Figure 1: example of an divergent diagram, momenta are labeled with p, q and k, the coupling is labeled by λ

k then emitting two virtual particles and finally absorbing them again. This process is displayed in figure 1 and the corresponding amplitude is proportional to

$$\mathcal{M} \sim \lambda^2 \int d^4p d^4q \frac{1}{(k-p-q)^2+m^2} \frac{1}{q^2+m^2} \frac{1}{p^2+m^2}. \tag{3.19}$$

It is obvious that the integral will diverge since there are eight powers of momentum in the numerator and only six in the denominator. This is a common problem that arises in many field theories. The degree of divergence can be written in the form $D = 4L - 2P$ where L is the number of loops that add four powers of momentum in the numerator and P the number of propagators that add two powers in the denominator. This can be rewritten like $D = 4 - [\lambda]V - N$ where V is the number of vertices, N the number of external lines and $[\lambda]$ is the mass dimension of the coupling when $\hbar = c = 1$. In ϕ^4 theory the coupling is dimensionless so the diagrams only diverge if they have less than four external lines which makes the theory renormalizable. For a ϕ^5 theory on the other hand the coupling has dimension -1 which means that for a fixed number of external lines higher loop corrections always diverge because only the number of vertices V is getting bigger so the degree of divergence rises and the theory is non-renormalizable.
To get around the divergent diagrams it is convenient to think about the parameters coming up in the Lagrangian as the bare mass and the bare coupling that are not actually measured by experiments, since the measured quantities depend on the self energy and loop corrections to the vertices. Then rescaling the field by the wave function renormalization factor $\phi = Z\phi_r$ and eliminating the bare parameters will produce counter terms that can cancel the divergent parts of the calculations. The couplings are renormalized to measurable couplings that will depend on the energy scale of the process. This procedure however can not be used if every order in perturbation theory diverges since infinitely many counter terms were needed that had to be specified which makes the theory non predictive.

3.3. Why general relativity is perturbatively non-renormalizable

The naive approach to quantizing gravity is to just apply the methods explained in the end of the previous section i.e. trying to calculate the partition function via perturbative

Feynman diagrams. It is simplest to consider the case where the metric is build up from a fixed background metric that is just flat $\eta_{\mu\nu}$ and a fluctuation field h which I will call the graviton field. The graviton field will be the dynamic variable integrated over in the path integral. In general the background will not be flat but for the sake of this argument it is sufficient. To use similar methods as before the Einstein-Hilbert action can be rendered in terms of kinetic and self interacting terms by expanding in powers of the graviton field. For simplicity the explicit indices are suppressed and the cosmological constant is neglected.

$$S[g^{\mu\nu}] \sim \int d^4x \left(\mathcal{L}\big|_{h=0} + \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}\big|_{h=0} h_{\mu\nu} + \frac{\delta^2 \mathcal{L}}{\delta g_{\mu\nu} \delta g_{\sigma\rho}}\big|_{h=0} h_{\mu\nu} h_{\sigma\rho} ... \right) \tag{3.20}$$

$$S[g^{\mu\nu}] \sim \frac{1}{16\pi G} \int d^4x \left((\nabla h)^2 + h(\nabla h)^2 + h^2(\nabla h)^2 + ... \right). \tag{3.21}$$

The term $\int d^4x \mathcal{L}\big|_{h=0}$ equates to zero because of the assumption that the background is flat and so the curvature scalar vanishes for $h = 0$. The next term $\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}\big|_{h=0} h_{\mu\nu}$ is the term that produces the Einstein field equations which are set to be zero. The third term in the expansion is the first that actually has a non vanishing contribution to the expansion. Its form can be seen by looking at the second variation of R in Appendix A where all terms can be written in the form $(\nabla h)^2 +$ *total derivative*. The total derivatives vanish by the same argument as before in section 2. The following terms behave like $h^n(\nabla h)^2$ and since they come out of a Taylor expansion there are infinite of them in contrast to the simple ϕ^4 action from before. These infinite terms in the action correspond to higher and higher incoming lines in the vertices. It is enough to consider the three graviton interaction to see the problem that arises.

To get the standard notion of a coupling the graviton field can be rescaled by $h \to \sqrt{16\pi G} h$ so that

$$S[g^{\mu\nu}] \sim \int d^4x \left((\nabla h)^2 + \sqrt{16\pi G} h(\nabla h)^2 + 16\pi G h^2(\nabla h)^2 + ... \right). \tag{3.22}$$

Now consider a simple diagram where a graviton emits another graviton and absorbs it afterwards. This corresponds to the one loop self energy correction. Because of the fact that the self interaction contains two derivatives the amplitude will contain a quadratic factor of momentum in the numerator for each vertex. Together with the two propagators the amplitude diverges to forth power

$$\mathcal{M} \sim (16\pi G)^2 \int d^4p \frac{p^2 p^2}{p^2 p^2} = (16\pi G)^2 \int d^4p. \tag{3.23}$$

This is of cause linked to the dimensonality of the gravitational constant as explained in the previous section. It has negative mass dimension which produces more and more divergent terms that need fixing. By introducing a cutoff k up to which the momentum integration will be performed it becomes visible that for momenta of order $k \approx \frac{1}{\sqrt{16\pi G}} \approx M_{planck}$ the amplitude is larger than one which is a hint that new physics that can not be described perturbatively have to be introduced in this scale. In the following I will introduce a non perturbative ansatz to quantum gravity that can be valid up to arbitrary high energies.

4. Renormalization group flow

In this section I will review a non perturbative method, that makes use of the renormalization group (RG) flow. Since it is not possible to quantize gravity in a perturbative approach it seems plausible that the Einstein-Hilbert action itself can not be regarded as fundamental but rather as an effective theory valid at some small momentum scale k. Averaging over fluctuations with momenta larger than k in the effective action leaves a coarse grained action functional valid at the scale k. Lowering the cutoff k step by step down to the infrared results in the full quantum effective action Γ described before. The behavior of the scale dependent effective action, also called the effective average action Γ_k, is determined by the RG flow.
Since the RG flow should be well behaved from $k = 0$ to $k = \infty$ the flow should end in a fixed point that describes the microscopic action where no fluctuations are integrated out. The general ansatz for the effective average action is a series of all infinitely many operators consistent with the theory that are multiplied by coupling constants g_i. The flow can then be parameterized by the running of the coupling constants which is induced by the fact that not the observables but rather the effective coupling that describes the processes should change during the RG transformation. So if there exists an ultra violet fixed point in the flow of couplings the high energy limit is well behaved and if this fixed point is attractive for a finite number of couplings a predictive theory can be derived.
For the general ansatz the flow is determined by the beta function for each coupling $\beta_i = k\frac{\partial g_i}{\partial k} = \partial_t g_i$ with the RG time $t = \ln k$. So what needs to be established is a functional RG equation that captures effects of an infinitesimal change of k in the effective average action Γ_k from which the beta functions can be extracted. This can be implemented in different ways resulting in different flow equations. In this thesis I will focus on the Wetterich equation which is described in the next section.
The general idea behind the renormalization group flow can be understood by a simple example. Imagine a chain of spins that are either spin up or spin down and a coupling between neighboring spins. A renormalization group transformation would then be the process of averaging over neighboring spins and defining a coarse grained spin chain with greater separation between them so that the coupling between the new spins will be different from the old. Applying this transformation over and over again causes a flow of the coupling that depends on the spacing just like a change in energy scale induces a flow for the couplings of the effective average action. The limit in which the original system is recovered, is the small distance limit or the high energy limit while in the low energy limit or the large distance limit all the fluctuations are absorbed in redefinitions of the coupling.
Another example for running coupling constants can be seen in electrodynamics. Imagine a charged particle in vacuum that is surrounded by pairs of charged virtual particles. The effective charge that is measured is then dependent on the distance to the charge because the further away the measurement takes place, the more virtual particles will screen the charge. A measurement right next to the charge will almost include no virtual particles in the screening effect and the measured charge is nearly the bare charge.
For gravity the low energy limit should look like general relativity while the high en-

ergy limit is unknown. So the renormalization group transformations have to be applied backwards to recover the fundamental theory of quantum gravity. This is the reason why the ultraviolet behavior of the renormalization group flow is interesting in particular and should stay finite for a reasonable theory. Naively the gravitational coupling will grow when going to smaller distances because the energy involved in the process will be larger and energy couples to gravity directly. For a refined RG flow a differential equation that describes the change of the effective average action for infinitesimal RG transformations is needed. In this thesis it will be implemented in terms of the Wetterich equation but there are several other approaches using different functional RG equations.

4.1. Wetterich Equation

The implementation of the Wetterich equation requires a modification of the effective action that makes it scale dependent and contain the above mentioned limits $\Gamma_0 = \Gamma$ and $\Gamma_\infty = S$. To do that a k dependent IR cutoff action ΔS_k will be added to the bare action S and subtracted from the effective action Γ

$$\Gamma_k[\varphi] = -W_k[J] + \int d^4x J(x)\varphi(x) - \Delta S_k[\varphi] \tag{4.1}$$

$$Z[J] = \int \phi e^{-S[\phi]+\int d^4x J(x)\phi(x) - \Delta S_k[\phi]} \tag{4.2}$$

The cutoff action takes the following form

$$\Delta S_k = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4}\phi(q)R_k(p,q)\phi(p). \tag{4.3}$$

To make this cutoff action similar to a momentum dependent mass term the regulator function $R_k(p,q)$ is taken to be diagonal

$$R_k(q,p) = R_k(p)\delta(p-q)(2\pi)^4 \tag{4.4}$$

$$\tag{4.5}$$

so that the cutoff action takes the form

$$\Rightarrow \Delta S_k = \frac{1}{2}\int \frac{d^4p}{\sqrt{2\pi}^4}\phi(p)R_k(p)\phi(p). \tag{4.6}$$

The regulator $R_k(p)$ has to satisfy a set of conditions so that its implementation renders the described behavior for the action functional. It has to vanish for $p >> k$ so that modes with momenta larger than k are integrated out normally. Also it should diverge for $k \to \infty$ since in this classical limit no fluctuations should be regarded. Finally it should stay finite for smaller and smaller momenta p so that infrared divergences are avoided. The so called optimized cutoff function realizes all those conditions

$$R_k(p) = (k^2 - p^2)\Theta(k^2 - p^2). \tag{4.7}$$

It is optimal in the sense that it ensures the fastest convergence for the flow equation as shown in [10].
Under these conditions the bare action is recovered for $k \to \infty$

$$\begin{aligned}\exp(-\Gamma_k[\varphi]) = \int \mathcal{D}\phi \exp(-S[\phi] + \int d^4x J(x)\,(\phi(x) - \varphi(x)) \\ -\frac{1}{2}\int \frac{d^4p}{\sqrt{2\pi}^4} R_k(\phi^2(p) - \varphi^2(p))).\end{aligned} \tag{4.8}$$

Since $R_k \to \infty$ in the classical limit, the exponential term $\exp(-\frac{1}{2}\int \frac{d^4p}{\sqrt{2\pi}^4} R_k(\phi^2(p) - \varphi^2(p)))$ behaves like a delta function $\delta(\phi - \varphi)$ which leads to the limit $\Gamma_k = S$.
To get to the FRGE from here it is just differentiating Γ_k with respect to RG time t

$$\partial_t\Gamma_k = -\partial_t W_k - \int d^4x \frac{\delta W_k}{\delta J_k}\partial_t J_k + \int d^4x \varphi \partial_t J_k - \partial_t \Delta S_k \tag{4.9}$$

$$= \langle \partial_t \Delta S_k \rangle - \partial_t \Delta S_k. \tag{4.10}$$

Rewriting the expectation value $\langle \partial_t \Delta S_k \rangle$ yields

$$\langle \partial_t \Delta S_k \rangle = \frac{1}{2\sqrt{2\pi}^4}\left\langle \int d^4p \varphi \partial_t R_k \varphi \right\rangle \tag{4.11}$$

$$= \frac{1}{2\sqrt{2\pi}^4}\int d^4p \left((\partial_t R_k) D_k + \varphi \partial_t R_k \varphi\right). \tag{4.12}$$

In the second line the connected two point function $D_k(p,q) = \langle \phi(p)\phi(q)\rangle - \langle \phi(p)\rangle \langle \phi(q)\rangle$ has been used. A look back at **??** makes it possible to write

$$\partial_t \Gamma_k = \frac{1}{2}\int \frac{d^4p}{\sqrt{2\pi}^4}(\Gamma^{(2)} + R_k)^{-1}\partial_t R_k = \frac{1}{2}\,\mathrm{Tr}\,\frac{\partial_t R_k}{\Gamma^{(2)} + R_k} \tag{4.13}$$

where $\Gamma^{(2)} = \frac{\delta^2 \Gamma_k}{\delta\phi(x)\delta\phi(y)}$. This equation is called the Wetterich equation. The trace stands for a sum over all internal indices on top of the momentum integral.

4.2. Beta functions for the Einstein Hilbert truncation

There arises an obvious problem when trying to solve the Wetterich equation. The exact solution has to start with the most general ansatz of infinitely many couplings but it is impossible to solve infinitely many partial differential equations. Some proper approximations have to be made to get meaningful results. The first would be to truncate the ansatz and only take into account a finite number of couplings. With this truncation it is only possible to study a subspace of the infinite dimensional theory space. The remaining couplings then automatically decouple from the infinitely many others so their effects will not be regarded in the resulting beta functions. To get a feeling for the quality of the truncation one can compare the results to higher order truncations and look for differences in the flow. A problem that arises from the truncation however

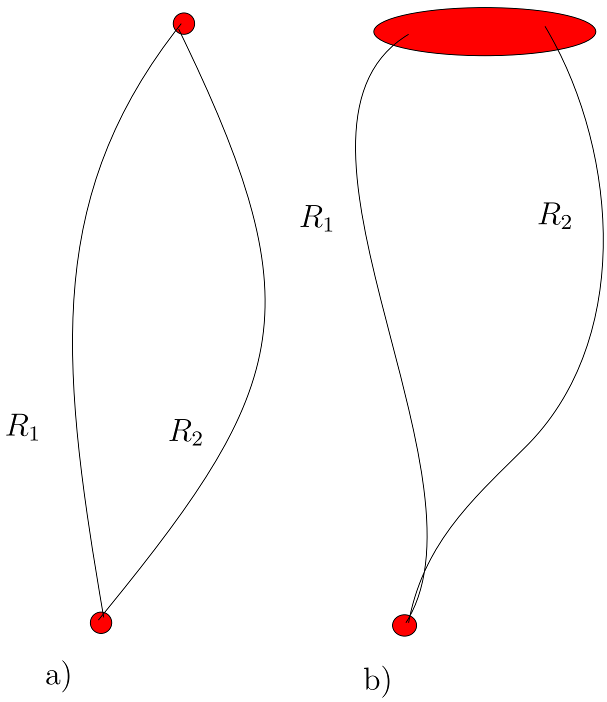

Figure 2: a) for an the general ansatz of infinite couplings the flow may be different but the fixed point is scheme independent. b) for a truncated flow the fixed point will be scheme dependent

is that the results become scheme dependent, which means that the choice of regulator and the way it is implemented has effects on the flow. As depicted in figure In this thesis I will follow the steps described in [11] and use the simplest truncation which is just the Einstein-Hilbert action which is parametrized by the dimensionless gravitational and cosmological constants $g_i = \{\bar{G}_k, \bar{\Lambda}_k\} = \{G_k k^2, \Lambda_k k^{-2}\}$. Also I will use the optimized regulator function $\mathcal{R}_k(p^2) = (k^2 - p^2)\Theta(k^2 - p^2)$ to derive one set of beta functions. Later I will compare the results to higher order truncations and different cutoff schemes. The ansatz for the effective average action is

$$\Gamma_k = Z_k \int d^4x\sqrt{-g}(R - 2\Lambda_k)$$
$$Z_k = \frac{1}{16\pi G_k}.$$

On the left hand side of the Wetterich equation there is a simple scale derivative of the average effective action. The right hand side will later be expanded in powers of curvature and the coefficients of $\sqrt{-g}$ and $\sqrt{-g}R$ will be compared so it makes sense to write the left side in powers of curvature as well

$$\partial_t\Gamma_k = (\partial_t Z_k)\int d^4x\sqrt{-g}R - 2\partial_t(Z_k\Lambda_k)\int d^4x\sqrt{-g} \tag{4.14}$$

$$\partial_t\Gamma_k = \partial_t(k^2\bar{Z}_k)\int d^4x\sqrt{-g}R - 2\partial_t(k^4\bar{Z}_k\bar{\Lambda}_k)\int d^4x\sqrt{-g}. \tag{4.15}$$

The right hand side is a little more complicated. It involves a trace over the modified propagator $D_k = \frac{1}{\Gamma_k^{(2)}+R_k}$. Writing the metric in terms of a background field $g_{\mu\nu}$ and a fluctuation field $h_{\mu\nu}$ makes it possible to distinguish momentum modes in terms of the eigenvalues of a differential operator constructed from the background metric and acting on the fluctuation metric. This is necessary because the implementation of the cutoff function needs a notion of high and low momenta. Depending on the choice of this differential operator different beta functions will result because the eigenvalues will differ and with this the point from which the momenta are cut off. The three types of cutoff schemes are type I, where the differential operator is just chosen to be the Laplacian $\Delta_I = -\nabla^2$, type II with $\Delta_{II} = -\nabla^2 + E$ where E is an additional multiplicative factor that doesn't contain couplings and type III where the differential operator is the full inverse propagator $\Delta_{III} = \Gamma^{(2)}$. In case of a type III cutoff scheme the regulator R_k will also contain couplings in its argument so the eigenvalues will change for different energy scales. This is called spectrally adjusted.
The Einstein-Hilbert action involves a gauge freedom which is the choice of a coordinate system. By fixing this freedom in a convenient way, it is possible to get rid of single covariant derivatives in the variation $\Gamma^{(2)}$. Introducing this gauge fixing always includes introducing a ghost action. I will not explain the whole procedure but just go with the results of [11]. Fixing the gauge like

$$S_{gf} = \frac{Z_k}{2\alpha}\int d^4x\sqrt{-g}\chi_\mu g^{\mu\nu}\chi_n u \tag{4.16}$$

$$\chi_\mu = \nabla^\nu h_{\mu\nu} - \frac{1+\rho}{4}\nabla_\mu h \tag{4.17}$$

$$\tag{4.18}$$

will result in ghost fields C_μ and $\bar{C}_\mu$ with their action

$$S_{gh} = -\int d^4x\sqrt{-g}\bar{C}_\mu\left(-\delta^\mu_\nu\nabla^2 - R^\mu_\nu\right)C^\nu \tag{4.19}$$

that will also contribute to the beta functions. The second variation of this action excluding the ghosts is partly computed in appendix A and reads

$$\left(\Gamma_k^{(2)}\right)^{\mu\nu}{}_{\alpha\beta} = Z_k\left(K^{\mu\nu}{}_{\sigma\rho}\left(-\nabla^2 - 2\Lambda_k\right) + U^{\mu\nu}{}_{\sigma\rho}\right) \tag{4.20}$$

with the following definitions

$$K^{\mu\nu}{}_{\sigma\rho} = \frac{1}{2}\left(\delta^{\mu\nu}{}_{\sigma\rho} - \frac{1}{2}g^{\mu\nu}g_{\sigma\rho}\right) = \frac{1}{2}\left(\left(\delta^{\mu\nu}{}_{\sigma\rho} - P^{\mu\nu}{}_{\sigma\rho}\right) - P^{\mu\nu}{}_{\sigma\rho}\right) \tag{4.21}$$

$$P^{\mu\nu}{}_{\sigma\rho} = \frac{1}{4}g^{\mu\nu}g_{\sigma\rho} \tag{4.22}$$

$$\delta^{\mu\nu}{}_{\sigma\rho} = \frac{1}{2}\left(\delta^\mu_\sigma\delta^\nu_\rho + \delta^\mu_\rho\delta^\nu_\sigma\right) \tag{4.23}$$

$$U^{\mu\nu}{}_{\sigma\rho} = K^{\mu\nu}{}_{\sigma\rho}R + \frac{1}{2}\left(g^{\mu\nu}R_{\sigma\rho} + g_{\sigma\rho}R^{\mu\nu}\right) - \delta^{(\mu}_{(\sigma}R^{\nu)}_{\rho)} - R^{(\mu}{}_{(\sigma}{}^{\nu)}{}_{\rho)} \tag{4.24}$$

Since the metric is symmetric $P^{\mu\nu}{}_{\sigma\rho}$ is a projector onto the trace part, while $\delta^{\mu\nu}{}_{\sigma\rho}$ is the identity. So $K^{\mu\nu}{}_{\sigma\rho}$ can be written as a projector on the trace free part minus the projector on the trace $\mathbb{K} = \frac{1}{2}\left((\mathbb{I}-\mathbb{P})-\mathbb{P}\right)$. Using this the inverse propagator can be written as

$$\Gamma_k^{(2)} = \frac{Z_k}{2}\left((\mathbb{I}-\mathbb{P})\left(-\nabla^2 - 2\Lambda_k + 2U\right) - \mathbb{P}\left(-\nabla^2 - 2\Lambda_k - 2U\right)\right). \tag{4.25}$$

The resulting beta functions are independent of the background metric, so it can be fixed to the simplest and maximally symmetric background, a sphere where curvature tensors take the form

$$R_{\mu\nu} = g_{\mu\nu}\frac{R}{4} \tag{4.26}$$

$$R_{\mu\nu\sigma\rho} = \frac{R}{12}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right). \tag{4.27}$$

With this simplification the tensor $U^{\mu\nu}_{\sigma\rho}$ can be rewritten

$$U = \frac{1}{3}\left(\mathbb{I}-\mathbb{P}\right)R. \tag{4.28}$$

Once the inverse propagator is written in this way it has to be modified by the regulator function. I will focus on a type I cutoff scheme so that

$$(R_k)^{\mu\nu}{}_{\sigma\rho} = Z_k K^{\mu\nu}{}_{\sigma\rho}\mathcal{R}_k(-\nabla^2) \tag{4.29}$$

$$R_k = \frac{Z_k}{2}(\mathbb{I}-\mathbb{P})\mathcal{R}_k(-\nabla^2) - \frac{Z_k}{2}\mathbb{P}\mathcal{R}_k(-\nabla^2). \tag{4.30}$$

The factor of $Z_k K^{\mu\nu}{}_{\sigma\rho}$ was introduced here so that $R_k(-\nabla^2)$ properly adapts to the kinetic $-\nabla^2$ term in the propagator. In this way it can properly cut of momentum modes at scale k. The propagator is then obtained by inverting $\Gamma_k^{(2)} + R_k$ by using the fact that the projectors $\mathbb{P}$ and $\mathbb{I}-\mathbb{P}$ are orthogonal to each other. Note that $\Gamma_k^{(2)} + R_k$ is of the form

$$\left(\mathbb{I}-\mathbb{P}\right)x - \mathbb{P}y. \tag{4.31}$$

This can trivially be inverted by

$$\begin{aligned}&\left(\left(\mathbb{I}-\mathbb{P}\right)x - \mathbb{P}y\right)\left(\left(\mathbb{I}-\mathbb{P}\right)\frac{1}{x} - \mathbb{P}\frac{1}{y}\right)\\ &= \left(\mathbb{I}-\mathbb{P}\right)^2 + \mathbb{P}^2 - \left(\mathbb{I}-\mathbb{P}\right)\mathbb{P}\frac{x}{y} - \mathbb{P}\left(\mathbb{I}-\mathbb{P}\right)\frac{y}{x}\\ &= \mathbb{I}.\end{aligned} \tag{4.32}$$

So the full regularized propagator reads

$$\begin{aligned}(\Gamma_k^{(2)} + R_k)^{-1} = \frac{2}{Z_k}\big[&(\mathbb{I}-\mathbb{P})\frac{1}{-\nabla^2 + \mathcal{R}_k(-\nabla^2) - 2\Lambda_k + \frac{2}{3}R}\\ &- \mathbb{P}\frac{1}{-\nabla^2 + \mathcal{R}_k(-\nabla^2) - 2\Lambda_k}\big].\end{aligned} \tag{4.33}$$

The ghost propagator can be derived using the same procedure

$$(\Gamma_k^{(2)} + R_k)^{-1} = \mathbb{I}\frac{1}{-\nabla^2 + \mathcal{R}_k + \frac{R}{4}} \tag{4.34}$$

$$(R_k)^\mu{}_\nu(-\nabla^2) = \delta^\mu{}_\nu \mathcal{R}_k(-\nabla^2). \tag{4.35}$$

The trace over this propagator can be evaluated using heat kernel techniques. The heat kernel $K(x,y,s) = \langle y|e^{-s\Delta}|x\rangle = e^{-s\Delta}\delta(x-y)$ is a solution to the heat equation $(\partial_s + \Delta)K(x,y,s) = 0$. The integral over its diagonal elements is the heat trace in this case with the Laplace operator $\Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu$. It can be expressed by the expansion

$$\operatorname{Tr} e^{-s\Delta} = \int d^4x\sqrt{-g}\,\langle x|e^{-s\Delta}|x\rangle \tag{4.36}$$

$$= \int d^4x\sqrt{-g}\frac{1}{(4\pi s)^2}\sum_n a_n s^n \tag{4.37}$$

$$= \int d^4x\sqrt{-g}\frac{1}{(4\pi s)^2}(1 + \frac{R}{6}s + \mathcal{O}(R^2)). \tag{4.38}$$

Only the first two orders of this expansion are relevant because only those coefficients have to be compared to the left hand side. Higher orders involve higher powers of curvature which do not appear in the Einstein-Hilbert action.
In the Wetterich equation the trace is performed over a function of the Laplace operator $f(\Delta)$. In this general case a Laplace transform is convenient to write the trace in the form of 4.38.

$$f(\Delta) = \int_0^\infty ds e^{-s\Delta}\tilde{f}(s) \tag{4.39}$$

With this the trace $\operatorname{Tr} f(\Delta)$ takes the form

$$\operatorname{Tr} f(\Delta) = \int ds \int d^4x\sqrt{-g}\frac{\tilde{f}(s)}{(4\pi)^2}(\mathbb{I}s^{-2} + \frac{R}{6}\mathbb{I}s^{-1} + \mathcal{O}(R^2)) \tag{4.40}$$

$$= \int d^4x\sqrt{-g}\frac{1}{(4\pi)^2}(Q_2[f] + \frac{R}{6}Q_1[f] + \mathcal{O}(R^2)) \tag{4.41}$$

Here the functionals

$$Q_n[f] = \int_0^\infty ds s^{-n}\tilde{f}(s) \tag{4.42}$$

have been defined. They can be expressed in terms of the original function $f(z)$ by

$$Q_n[f] = \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1} f(z). \tag{4.43}$$

After some manipulations it can be seen that this is equal to the original Q-functionals. Simply plug in the Laplace transform 4.39

$$Q_n[f] = \frac{1}{\Gamma(n)} \int_0^\infty dz \int_0^\infty ds z^{n-1} e^{-sz} \tilde{f}(s) \tag{4.44}$$
$$\tag{4.45}$$

and make the substitution $x = sz$. With the definition of the Γ function it is clear that this coincides with the original $Q_n(f)$ functionals 4.42

$$Q_n[f] = \frac{1}{\Gamma(n)} \int_0^\infty dx \int_0^\infty ds s^{-n} x^{n-1} e^{-x} \tilde{f}(s) \tag{4.46}$$
$$= \int_0^\infty ds s^{-n} \tilde{f}(s). \tag{4.47}$$

So from now on I will use 4.43 for $Q_n[f]$.
Putting everything together and expanding the propagators up to linear order in curvature yields the following expression for the Wetterich equation

$$\begin{aligned}\partial_t \Gamma_k = \frac{-1}{(4\pi)^2} \int d^4x \sqrt{-g} \big[& 5Q_2 \left(\frac{\partial_t \mathcal{R}_k + \frac{\mathcal{R}_k}{Z_k} \partial_t Z_k}{z + \mathcal{R}_k - 2\Lambda_k} \right) - 4Q_2 \left(\frac{\partial_t \mathcal{R}_k}{z + \mathcal{R}_k} \right) \\ & + \frac{5}{6} Q_1 \left(\frac{\partial_t \mathcal{R}_k + \frac{\mathcal{R}_k}{Z_k} \partial_t Z_k}{z + \mathcal{R}_k - 2\Lambda_k} \right) R - 3Q_2 \left(\frac{\partial_t \mathcal{R}_k + \frac{\mathcal{R}_k}{Z_k} \partial_t Z_k}{(z + \mathcal{R}_k - 2\Lambda_k)^2} \right) R \\ & - \frac{2}{3} Q_1 \left(\frac{\partial_t \mathcal{R}_k}{z + \mathcal{R}_k} \right) R - Q_2 \left(\frac{\partial_t \mathcal{R}_k}{(z + \mathcal{R}_k)^2} \right) R + \mathcal{O}(R^2) \big].\end{aligned} \tag{4.48}$$

The traces over internal indices were evaluated by

$$\operatorname{tr} \mathbb{P} : \operatorname{tr}(P^{\mu\nu}_{\sigma\rho}) = \frac{1}{4} g^{\mu\nu} g_{\mu\nu} = \frac{1}{4} \delta^\mu_\mu = 1 \tag{4.49}$$
$$\operatorname{tr} \mathbb{I} : \operatorname{tr}(\delta^{\mu\nu}_{\sigma\rho}) = \frac{1}{2} \left(\delta^\mu_\mu \delta^\nu_\nu + \delta^\mu_\nu \delta^\nu_\mu \right) = \frac{1}{2} (4 \cdot 4 + 4) = 10 \tag{4.50}$$
$$\operatorname{tr} \delta^\mu_\nu = 4. \tag{4.51}$$

Now all that is left to do is to calculate the Q-functionals and then compare the coefficients to extract the beta functions. The Q-functionals are evaluated using the optimized cutoff

$$\mathcal{R}_k(z) = (k^2 - z)\Theta(k^2 - z) \tag{4.52}$$
$$\tag{4.53}$$

with its scale derivative

$$\partial_t \mathcal{R}_k(z) = 2k^2 \Theta(k^2 - z) + 2k^2 (k^2 - z) \delta(k^2 - z). \tag{4.54}$$

The term with the delta function will cancel in the integration since it replaces z by k^2 which cancels the factor $k^2 - z$. The Q-functionals read

$$
\begin{aligned}
Q_1\left(\frac{\partial_t \mathcal{R}_k + \frac{\mathcal{R}_k}{Z_k}\partial_t Z_k}{z + \mathcal{R}_k - 2\Lambda_k}\right) &= \int_0^\infty \frac{2k^2 + (k^2 - z)\partial_t \ln(Z_k)}{z + (k^2 - z)\Theta(k^2 - z) - 2\Lambda}\Theta(k^2 - z)dz \\
&= \int_0^{k^2} \frac{2k^2 + (k^2 - z)\partial_t \ln(Z_k)}{k^2 - 2\Lambda}dz \\
&= \frac{4 + \partial_t \ln(Z_k)}{2(1 - 2\bar{\Lambda})}k^2 \qquad (4.55)\\
Q_2\left(\frac{\partial_t \mathcal{R}_k + \frac{\mathcal{R}_k}{Z_k}\partial_t Z_k}{z + \mathcal{R}_k - 2\Lambda_k}\right) &= \frac{6 + \partial_t \ln(Z_k)}{6(1 - 2\bar{\Lambda})}k^4 \qquad (4.56)\\
Q_2\left(\frac{\partial_t \mathcal{R}_k + \frac{\mathcal{R}_k}{Z_k}\partial_t Z_k}{(z + \mathcal{R}_k - 2\Lambda_k)^2}\right) &= \frac{6 + \partial_t \ln(Z_k)}{6(1 - 2\bar{\Lambda})^2}k^2 \qquad (4.57)\\
Q_1\left(\frac{\partial_t \mathcal{R}_k}{z + \mathcal{R}_k}\right) &= 2k^2 \qquad (4.58)\\
Q_2\left(\frac{\partial_t \mathcal{R}_k}{z + \mathcal{R}_k}\right) &= k^4 \qquad (4.59)\\
Q_2\left(\frac{\partial_t \mathcal{R}_k}{(z + \mathcal{R}_k)^2}\right) &= k^2. \qquad (4.60)
\end{aligned}
$$

Now putting everything together and comparing the coefficients for $\int d^4x\sqrt{-g}$ and $\int d^4x\sqrt{-g}R$ the optimized beta functions are

$$
\begin{aligned}
\beta_{\bar{\Lambda}} &= -2\bar{\Lambda} + \frac{\bar{G}}{6\pi}\frac{3 - 4\bar{\Lambda} - 12\bar{\Lambda}^2 - 56\bar{\Lambda}^3 + \frac{107 - 20\bar{\Lambda}}{12\pi}\bar{G}}{(1 - 2\bar{\Lambda})^2 - \frac{1 + \bar{\Lambda}}{12\pi}\bar{G}} \qquad (4.61)\\
\beta_{\bar{G}} &= 2\bar{G} - \frac{\bar{G}^2}{3\pi}\frac{11 - 18\bar{\Lambda} + 28\bar{\Lambda}^2}{(1 - 2\bar{\Lambda})^2 - \frac{1 + \bar{\Lambda}}{12\pi}\bar{G}}. \qquad (4.62)
\end{aligned}
$$

One nice feature of the optimized regulator is that the $Q-$functionals can be evaluated analytically. Choosing a different regulator function most often leads to the need of numeric evaluation. I will give an example of this so that the different flows can be compared in the end. The first thing that needs to be done is splitting the $Q-$functionals in so called threshold functions

$$
\begin{aligned}
\tilde{\phi}_n^m(x) &= \frac{1}{\Gamma(n)}\int_0^\infty dz \frac{\partial_t \mathcal{R}_k \, z^{n-1}}{(z + \mathcal{R}_k + k^2 x)^m} \qquad (4.63)\\
\tilde{\psi}_n^m(x) &= \frac{1}{\Gamma(n)}\int_0^\infty dz \frac{\mathcal{R}_k \, z^{n-1}}{(z + \mathcal{R}_k + k^2 x)^m}. \qquad (4.64)
\end{aligned}
$$

I will use a simple regulator of the form

$$\mathcal{R}_k(z) = z\left(\frac{z}{k^2}\right)^{-3} \tag{4.65}$$

$$\Rightarrow \partial_t \mathcal{R}_k(z) = 6k^2\left(\frac{z}{k^2}\right)^{-2} \tag{4.66}$$

or in terms of the dimensionless variable $y = \frac{z}{k^2}$

$$\mathcal{R}_k(y) = k^2 y^{-2} \tag{4.67}$$

$$\partial_t R_k(y) = 6k^2 y^{-2}. \tag{4.68}$$

The dimensionless threshold functions are then

$$\phi_n^m(x) = k^{2(m-n)}\tilde{\phi}_n^m(x) = \frac{1}{\Gamma(n)}\int dy \frac{6y^{n-3}}{(y+y^{-2}+x)^m} \tag{4.69}$$

$$\psi_n^m(x) = k^{2(m-n)}\tilde{\psi}_n^m(x) = \frac{1}{\Gamma(n)}\int dy \frac{y^{n-3}}{(y+y^{-2}+x)^m}. \tag{4.70}$$

Rewriting 4.48 and solving for the beta functions yields the following complicated system

$$\beta_G = \bar{G}(5\bar{G}\phi_1^1(-2\bar{\Lambda}) - 2(-6\pi + 2\bar{G}\phi_1^1(0) + 3\bar{G}\phi_2^2(0) \tag{4.71}$$

$$+ 9\bar{G}\phi_2^2(-2\bar{\Lambda}) - 5\bar{G}\psi_1^1(-2\bar{\Lambda}) + 18\bar{G}\psi(-2\Lambda)))/ \tag{4.72}$$

$$(5\bar{G}\psi_1^1(-2\bar{\Lambda}) + 6(\pi - 3\bar{G}\psi_2^2(-2\bar{\Lambda})) \tag{4.73}$$

$$\beta_\Lambda = (-5\bar{G}\phi_1^1(-2\bar{\Lambda})(-2\bar{\Lambda}\pi + 5\bar{G}\psi_2^1(-2\bar{\Lambda})) \tag{4.74}$$

$$- 2(12\bar{\Lambda}\pi^2 + 4\bar{G}\bar{\Lambda}\pi\phi_1^1(0) + 12\bar{G}\pi\phi_2^1(0) \tag{4.75}$$

$$+ 6\bar{G}\bar{\Lambda}\pi\phi_2^2(0) + 10\bar{G}(\bar{\Lambda}\pi + \bar{G}\phi_2^1(0))\psi_1^1(-2\bar{\Lambda}) \tag{4.76}$$

$$- 10\bar{G}^2\phi_1^1(0)\psi_2^1(-2\bar{\Lambda}) - 15\bar{G}^2\phi_2^2(0)\psi_2^1(-2\bar{\Lambda}) - 9\bar{G}\phi_2^2(-2\bar{\Lambda}) \tag{4.77}$$

$$(-2\bar{\Lambda}\pi + 5\bar{G}\psi_2^1(-2\bar{\Lambda})) - 36\bar{G}\bar{\Lambda}\pi\psi_2^2(-2\bar{\Lambda}) - 36\bar{G}^2\phi_2^1(0)\psi_2^2(-2\bar{\Lambda}) \tag{4.78}$$

$$+ 5\bar{G}\phi_2^1(-2\bar{\Lambda})(5\bar{G}\psi_1^1(-2\bar{\Lambda}) + 6(\pi - 3\bar{G}\psi_2^2(-2\bar{\Lambda}))))/ \tag{4.79}$$

$$(2\pi(5\bar{G}\psi_1^1(-2\bar{\Lambda}) + 6(\pi - 3\bar{G}\psi_2^2(-2\bar{\Lambda}))) \tag{4.80}$$

Since the threshold functions can not be evaluated for an analytically for the arbitrary variable $-2\bar{\Lambda}$ they have to be integrated at each step during the solving process of the flow for the corresponding value of $\bar{\Lambda}$. The flow and the corresponding fixed points are discussed in the upcoming section.

4.3. RG flow and fixed points

As mentioned earlier the Einstein-Hilbert action is only a truncation of an ansatz for the action functional that otherwise contains infinitely many coupling constants. A theory represented by a RG trajectory can only have a finite ultraviolet limit if the trajectory ends in a fixed point. A fixed point is characterized by the fact that exactly at the fixed point the beta functions are identically zero. In the vicinity of the fixed point

there are two possibilities for each coupling. It can either be attracted towards the fixed point or repelled from it. If there exists a UV fixed point, it is also necessary that at least one coupling is attracted towards the fixed point so that the trajectory stops in the fixed point. In general if there are finitely many attractive couplings, measuring these determines the value of every other coupling, because for a theory to be asymptotically safe i.e. to end in a fixed point, it can only start in the UV critical surface that contains every point in theory space that is attracted towards the fixed point. The dimension of the UV critical surface is equal to the number of attractive couplings. For example if there is only one attractive direction the critical surface would only contain one trajectory so by measuring the attractive coupling every other coupling is determined by the condition to lie on that trajectory. So the dimension of the UV critical surface sets the predictivity of the theory.
The UV critical surface around a fixed point can be studied by linearizing the flow

$$\beta_i = \beta_i\big|_{g^*} + \frac{\partial \beta_i}{\partial g_j}\big|_{g^*}(g_j - g_j^*) \tag{4.81}$$

Diagonalizing the matrix $\frac{\partial \beta_i}{\partial g_j}\big|_{g^*}$ and determining its eigenvalues λ_i provides information about whether a coupling is UV attractive or repulsive. Solutions to the linearized flow then are exponentials with the eigenvalue in the exponent. This means that positive eigenvalues correspond to repelled couplings and negative eigenvalues correspond to attracted couplings. So calculating the number of negative eigenvalues gives the dimension of the UV critical surface which is then spanned by the corresponding eigenvectors. Eigenvalues multiplied by minus one are called critical exponents $\theta_i = -\lambda_i$, so positive critical exponents are attractive.
From the derived beta functions it is directly possible to see several fixed points. For instance the Gaussian fixed point is in the origin where both couplings vanish. Also setting Newtons constant to zero shows a fixed point for its corresponding beta function. This means that trajectories approaching the axis $\bar{G} = 0$ can never cross it because the flow stops there. It is also directly visible from the beta functions that they become singular if $(1 - 2\bar{\Lambda})^2 - \frac{1+\bar{\Lambda}}{12\pi}\bar{G} = 0$. This leads to very large values for the beta functions near that line so that the couplings become infinite when approaching it. The flow for the Einstein-Hilbert truncation is displayed in figure 3. From this it can immediately be seen that there exists a UV fixed point with spiraling trajectories flowing into it which means that there will be two attractive directions. It is the non trivial fixed point that controls the UV behavior of all trajectories that start with $\bar{G} > 0$. Reading the beta functions as a vector field gives rise to the flow diagram 4. The arrows flow from the ultraviolet to the infrared in order to study the infrared behavior of the trajectories first. The one trajectory connecting the two fixed points, called the separatrix, is marked in red. It separates trajectories with negative and positive cosmological constant in the infrared. Trajectories on the upper half plane that do not flow exactly towards the Gaussian fixed point for $k \to 0$ can behave in three different ways. One possibility is that they flow towards the $\bar{G} = 0$ axis for $\bar{\Lambda} \to -\infty$ or they flow towards $\bar{G} \to \infty$. These two possible outcomes are distinguished by the line $\bar{G} = \frac{12\pi\left(4\bar{\Lambda}^2 - 4\bar{\Lambda} + 1\right)}{56\bar{\Lambda}^2 - 26\bar{\Lambda} + 23}$ on which the beta function for

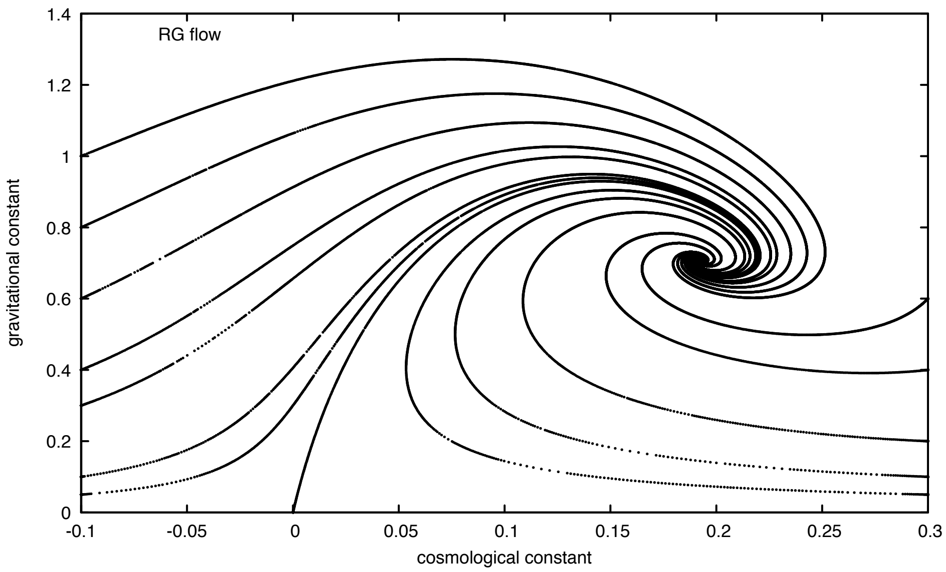

Figure 3: RG flow with integrated trajectories

$\bar{G}$ vanishes. Trajectories starting far above this line can flow towards larger $\bar{G}$ implying a strong gravitational coupling in the infrared which is not observed experimentally, while those below will flow towards $\bar{G} = 0$ for $k \to 0$. Trajectories following this behavior of vanishing $\bar{G}$ define a possible infrared limit. The seperation line is seen in 5. The last possible class of trajectories is distinguished by the seperatrix line in the sense that trajectories to the left of it follow one of the two possibilities described earlier and the ones on the right of it flow into the forbidden region marked in gray in 3. Trajectories can not be integrated to the other side of its boundary because at the boundary the beta functions become infinite as mentioned above. Also inside the gray area the sign of the second term in β_G and β_Λ flips so if the trajectory would be continued through the boundary it would change direction as it passes making it impossible for the trajectory to escape. Trajectories following this behavior can also not be used to define the infrared limit. There is one possible way to get beyond the gray area but this again includes setting the gravitational constant to zero and only taking trajectories on the $\bar{\Lambda}$ axis into account which can not be the right thing either since $\bar{G}$ is definitely not zero.
From now on I will invert the flow so that the trajectories start at low energy and go up to high energies so that the ultraviolet behavior can be studied. The numerical values of the fixed points can be calculated by setting $\vec{\beta} = 0$

$$\textit{trivial fixed point}: \left(\bar{\Lambda}_0, \bar{G}_0\right) = (0, 0) \tag{4.82}$$

$$\textit{non-trivial fixed point}: \left(\bar{\Lambda}^*, \bar{G}^*\right) = (0.193201, 0.707321)\,. \tag{4.83}$$

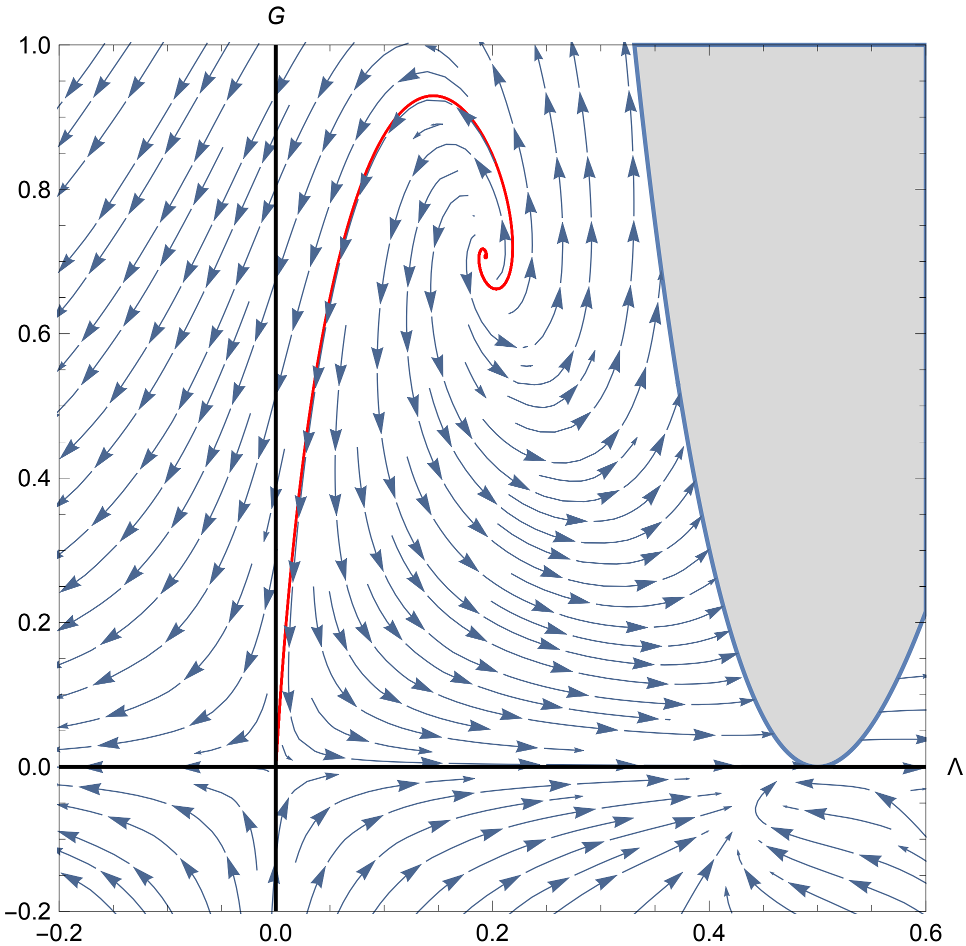

Figure 4: vector field $\vec{\beta}$, separatrix in red and forbidden area in gray

Linearizing the flow around the trivial Gaussian fixed point yields

$$\vec{\beta}_0 = \vec{\beta}|_{\bar{G}_0,\bar{\Lambda}_0} + \begin{pmatrix} \frac{\partial\beta_G}{\partial\bar{G}} & \frac{\partial\beta_G}{\partial\bar{\Lambda}} \\ \frac{\partial\beta_\Lambda}{\partial\bar{G}} & \frac{\partial\beta_\Lambda}{\partial\bar{\Lambda}} \end{pmatrix} |_{\bar{G}_0,\bar{\Lambda}_0} \begin{pmatrix} \bar{G} \\ \bar{\Lambda} \end{pmatrix} \tag{4.84}$$

$$= \begin{pmatrix} 2 & 0 \\ \frac{1}{2\pi} & -2 \end{pmatrix} \begin{pmatrix} \bar{G} \\ \bar{\Lambda} \end{pmatrix}. \tag{4.85}$$

Diagonalizing this system of equations gives the following system for the new coordinates g and λ

$$\vec{\beta} = \begin{pmatrix} \beta_g \\ \beta_\lambda \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} g \\ \lambda \end{pmatrix} \tag{4.86}$$

$$\Rightarrow \begin{pmatrix} g \\ \lambda \end{pmatrix} = \begin{pmatrix} g_0\ e^{2t} \\ \lambda_0\ e^{-2t} \end{pmatrix}. \tag{4.87}$$

The critical exponents $\theta_g = -2$ and $\theta_\lambda = 2$ are equivalent to the mass dimensions of the couplings which means that only for vanishing gravitational constant $\bar{G} = 0$ the Gaussian

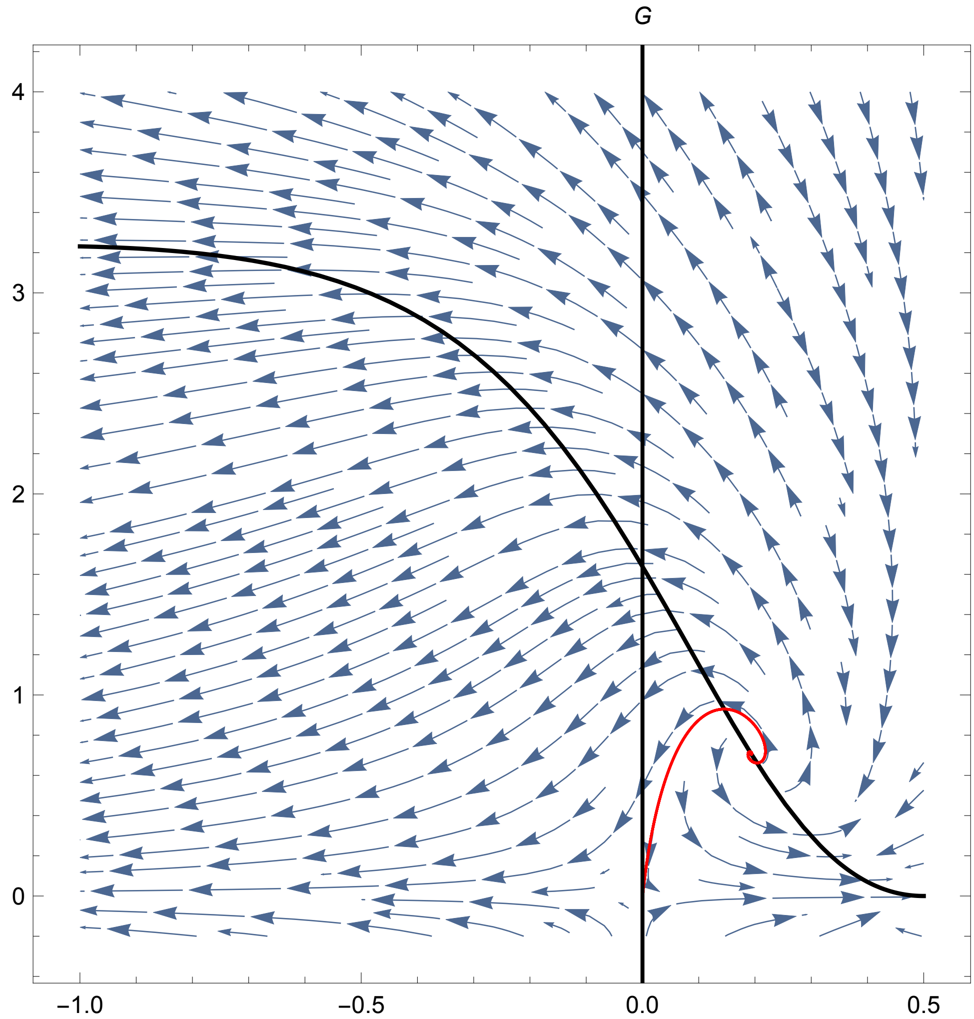

Figure 5: the line in black is the separation line between diverging $\bar{G}$ trajectories above it and $\bar{G} \to 0$ trajectories below it, the red line is again the separatrix connecting the two fixed points.

fixed point is UV stable so it can not be used as a high energy limit for the theory. As soon as $\bar{G}$ takes a small finite value it blows up for higher energies making the gaussian fixed point unstable. The cosmological constant however is attracted towards the fixed point. A linearization of the flow around the UV fixed point provides information about the relevant couplings in the ultraviolet. As mentioned earlier it is suspected that both the gravitational and the cosmological constant are attracted to the fixed point and therefore span the critical surface. The linearization reads

$$\vec{\beta}^* = \vec{\beta}\big|_{\bar{G}^*,\bar{\Lambda}^*} + \begin{pmatrix} \frac{\partial \beta_G}{\partial \bar{G}} & \frac{\partial \beta_G}{\partial \bar{\Lambda}} \\ \frac{\partial \beta_\Lambda}{\partial \bar{G}} & \frac{\partial \beta_\Lambda}{\partial \bar{\Lambda}} \end{pmatrix} \big|_{\bar{G}^*,\bar{\Lambda}^*} \begin{pmatrix} \bar{G} - \bar{G}^* \\ \bar{\Lambda} - \bar{\Lambda}^* \end{pmatrix} \tag{4.88}$$

$$= \begin{pmatrix} -2.34222 & -10.4398 \\ 0.959082 & -0.608382 \end{pmatrix} \begin{pmatrix} \bar{G} - \bar{G}^* \\ \bar{\Lambda} - \bar{\Lambda}^* \end{pmatrix} \tag{4.89}$$

and with the diagonalization

$$\begin{pmatrix} \beta_g \\ \beta_\lambda \end{pmatrix} = \begin{pmatrix} -1.4753 + 3.04321i & 0 \\ 0 & -1.4753 - 3.04321i \end{pmatrix} \begin{pmatrix} g \\ \lambda \end{pmatrix}. \tag{4.90}$$

The fact that the eigenvalues are complex reflects the spiraling behavior of the trajectories around the fixed point. It is still possible to identify attractive directions by looking at the real parts of the eigenvalues. Both real parts are negative which means that both directions are attractive and with that relevant. The UV critical surface is therefore indeed two dimensional so without setting couplings to zero it can be used as an ultraviolet completion of the theory.

In 6 the evolution of the gravitational constant is displayed. The Trajectory is chosen

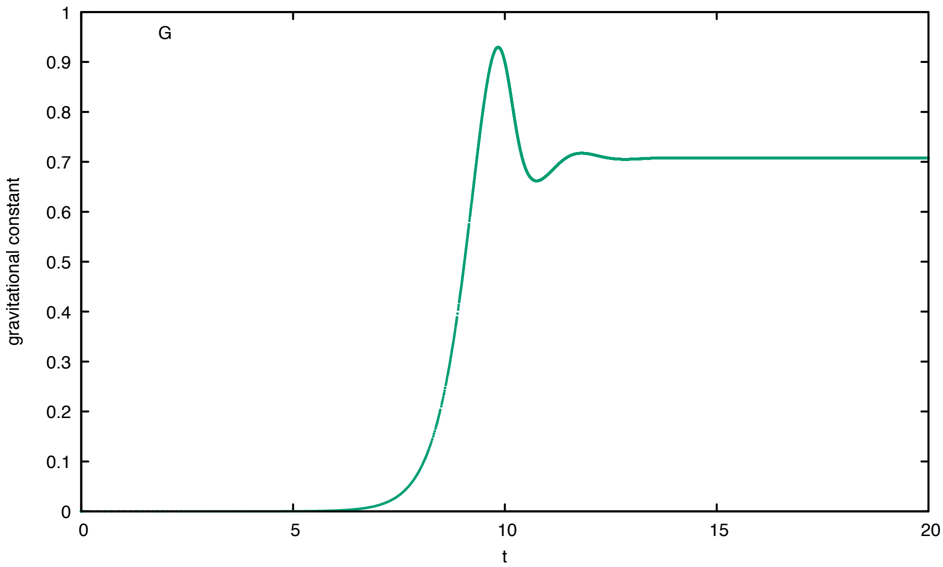

Figure 6: evolution of $\bar{G}$ along the separatrix

to be the separatrix starting at the Gaussian and ending at the UV fixed point. It is visible that for low energies the coupling stays relatively small. Then renormalization effects kick in which leads to an oscillating behavior until the the coupling stabilizes at its ultraviolet fixed point. So the initial thought that the gravitational constant grows without bound is not true when taking into account renormalization effects due to the cosmological constant. Taking into account more couplings can possibly change this behavior so higher truncations are necessary to check this result.

Numerically integrating the flow generated by the second regulator choice $\mathcal{R}_k = z\left(\frac{z}{k^2}\right)^{-3}$ can be seen in 7.

It shows qualitatively the same behavior as the first flow. There is also an ultraviolet fixed point with a two dimensional UV critical surface. And of course the trivial fixed point. The difference is that the fixed point values differ as expected. But the important fact is that the fixed point exists and is not just generated by a specific cutoff choice.

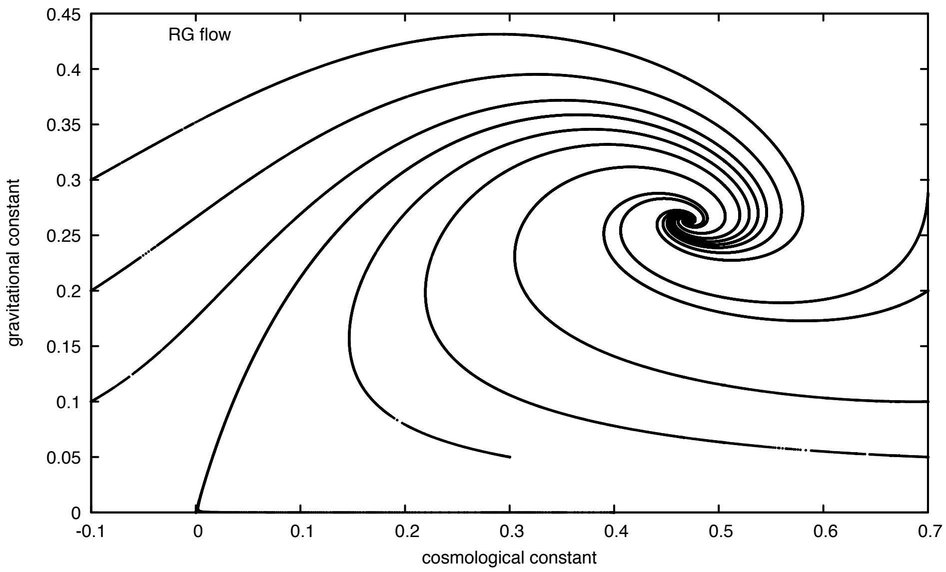

Figure 7: flow generated by the second regulator, it shows the same spiraling behavior around the fixed point as the first flow

5. Conclusion

To summarize this thesis, there was a non perturbative method chosen to make sense of quantum gravity in the framework of quantum field theory. It consists of a functional renormalization group equation for the effective average action parameterized by its coupling constants. I used a Type I cutoff together with the optimal regulator and a power law regulator function. Also I chose a specific type of gauge fixing that had the advantage that only Laplace like operators appeared in the regulated propagator which then could easily be treated with heat kernel methods. For these choices and a simple Einstein Hilbert truncation a UV fixed point with a two dimensional UV critical surface was found. This procedure was generalized to other regulators for example in [12] a sharp and an exponential cutoff are studied and the fixed point always appears. The inclusion of higher order curvature terms was pushed to R^6 in [13]. In every order a fixed point was found and not only this but also the critical exponents seem to stabilize and not vary too much from order to order. Also the critical surface turned out to have dimension three since the critical exponents beginning with the R^3 term were negative i.e. irrelevant. This behavior at higher orders is evidence that there exists a fixed point for arbitrary high orders in R^n but of course no proof. There can in principle all sorts of higher derivative operators be included, as an example see [14].
Further studies should include different gauge fixings and higher order truncations involving more complicated operators or even an infinite number of terms in a $f(R)$ truncation while disregarding for example Riemann squared terms. With these calculations the reliability of lower order truncations can be checked and verified if they are sufficient in describing quantum gravity at different scales. What is also missing is the coupling of gravity to matter fields. So an appropriate choice of matter fields should also be

included to check if the requirements for asymptotic safety are still met.
All in all there is much evidence that an asymptotic safety scenario could in fact be realized as a predictive theory of quantum gravity.

A. Variations

In chapter 4 the resulting beta functions are shown. To derive those one has to compute the second variation of the effective average action. To make clear what is meant by that, some variations are explicitly computed in this section. Some calculations can also be found in [15].
Since it is a priori not defined what is meant by the coarse graining in momentum contributions to the path integral that should be achieved by the regulator function $R_k(p,q)$, one has to split up the metric field in a background metric $g_{\mu\nu}$ and a fluctuation metric $\delta g_{\mu\nu} = h_{\mu\nu}$. From the background metric one can construct the Laplacian with a fixed spectrum and fixed eigenfunctions. If the eigenvalues are then higher than the momentum scale k, this corresponds to high momentum contributions and if it is smaller it corresponds to low momentum contributions. This splitting is called the background field method.
The needed variation is

$$\Gamma_k^{(2)} = \frac{\delta^2 \Gamma_k[g(x)]}{\delta g_{\mu\nu}(y) \delta g_{\sigma\rho}(z)}$$

Since $\Gamma_k[g_(x)]$ is defined as an integral and the functional derivatives are calculated at different spacetime points, the first derivative will produce a delta function that annihilates the integral and the second derivative will produce another delta function $\delta(x-z)$. Apart from that one can calculate the variations as if they were evaluated at the same points.
First of all one can calculate the inverse variation of the metric

$$\delta g^{\sigma\rho} = -g^{\sigma\mu} g^{\rho\nu} \delta g_{\mu\nu} = -h^{\sigma\rho} \tag{A.1}$$

since

$$\delta(g^{\mu\nu} g_{\nu\sigma}) = g^{\nu\sigma} \delta g^{\mu\nu} + g^{\mu\nu} \delta g_{\nu\sigma} = 0. \tag{A.2}$$

Now with the help of $\ln g = tr \ln g_{\mu\nu}$ one can calculate the first and second variation of $\sqrt{-g}$ where $h = g_{\mu\nu} h^{\mu\nu}$ is the trace.

$$
\begin{aligned}
\delta\sqrt{-g} &= \frac{-\delta g}{2\sqrt{-g}} = \frac{-1}{2\sqrt{-g}}\delta e^{\ln g} = \frac{-g}{2\sqrt{-g}} tr g^{\mu\nu}\delta g_{\alpha\beta} \\
&= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{-g}h^{\mu}_{\mu} = \frac{1}{2}\sqrt{-g}h \qquad \text{(A.3)} \\
\delta^2\sqrt{-g} &= \delta\left(\frac{1}{2}\sqrt{-g}g^{\mu\mu}h_{\mu\nu}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\sqrt{-g}g^{\sigma\rho}h_{\sigma\rho}g^{\mu\nu}h_{\mu\nu} - \sqrt{-g}h^{\mu\nu}h_{\mu\nu}\right) \\
&= \sqrt{-g}\left(\frac{1}{4}h^2 - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right) \qquad \text{(A.4)}
\end{aligned}
$$

The variation of the Christoffel symbol can be written in terms of covariant derivatives

$$
\delta\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\nabla_\beta h_{\gamma\delta} + \nabla_\gamma h_{\beta\delta} - \nabla_\delta h_{\alpha\beta}\right). \qquad \text{(A.5)}
$$

With those variations calculated it is easy to write down the variation of the Riemann tensor and its contractions

$$
\begin{aligned}
\delta R^{\alpha}{}_{\beta\mu\nu} &= \partial_\mu\delta\Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu\delta\Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\gamma}{}_{\beta\nu}\delta\Gamma^{\alpha}{}_{\gamma\mu} + \Gamma^{\alpha}{}_{\gamma\mu}\delta\Gamma^{\gamma}{}_{\beta\nu} \\
&\quad - \Gamma^{\gamma}{}_{\beta\mu}\delta\Gamma^{\alpha}{}_{\gamma\nu} - \Gamma^{\alpha}{}_{\gamma\nu}\delta\Gamma^{\gamma}{}_{\beta\mu} \\
&= \nabla_\mu\delta\Gamma^{\alpha}{}_{\beta\nu} - \nabla_\nu\delta\Gamma^{\alpha}{}_{\beta\mu} \qquad \text{(A.6)} \\
\delta R_{\beta\nu} = \delta R^{\alpha}{}_{\beta\alpha\nu} &= \frac{1}{2}g^{\alpha\gamma}(\nabla_\alpha(\nabla_\beta h_{\gamma_\nu} + \nabla_\nu h_{\gamma\beta} - \nabla_\gamma h_{\beta\nu}) \\
&\quad - \nabla_\nu(\nabla_\beta h_{\gamma\alpha} + \nabla_\alpha h_{\gamma\beta} + \nabla_\gamma h_{\beta\alpha})) \\
&= \frac{1}{2}\left(-\nabla^2 h_{\beta\nu} - \nabla_\nu\nabla_\beta h + \nabla_\alpha\nabla_\beta h^{\alpha}_{\nu} + \nabla_\alpha\nabla_\nu h^{\alpha}_{\beta}\right) \qquad \text{(A.7)} \\
\delta R &= -h_{\mu\nu}R^{\mu\nu} - \nabla^2 h + \nabla_\mu\nabla_\nu h^{\mu\nu}. \qquad \text{(A.8)}
\end{aligned}
$$

The second variations $\delta^2 R = R_{\mu\nu}\delta^2 g^{\mu\nu} + \delta g^{\mu\nu}\delta R_{\mu\nu} + g^{\mu\nu}\delta^2 R_{\mu\nu}$ are computed completely analogous

$$
\begin{aligned}
\delta^2 R &= R_{\mu\nu}\delta^2 g^{\mu\nu} + 2\delta g^{\mu\nu}\delta R_{\mu\nu} + g^{\mu\nu}\delta^2 R_{\mu\nu} \qquad \text{(A.9)} \\
&= 2R_{\mu\nu}h^{\mu\alpha}h^{\nu}_{\alpha} - h^{\mu\nu}\left(-\nabla^2 h_{\mu\nu} - \nabla_\nu\nabla_\mu h + \nabla_\alpha\nabla_\mu h^{\alpha}_{\nu} + \nabla_\alpha\nabla_\nu h^{\alpha}_{\mu}\right) \qquad \text{(A.10)} \\
&+ \nabla_\alpha\delta^2\Gamma^{\alpha}{}_{\mu\nu} - \nabla_\nu\delta^2\Gamma^{\alpha}{}_{\mu\alpha} + (\nabla_\mu h^{\mu\nu})(\nabla_\nu h) - \frac{1}{2}(\nabla_\mu h)(\nabla^\mu h) \qquad \text{(A.11)} \\
&+ \frac{1}{2}(\nabla^\alpha h_{\mu\nu})(\nabla_\alpha h^{\mu\nu}) - \nabla^\alpha h^{\mu\nu}\nabla_\mu h_{\nu\alpha}. \qquad \text{(A.12)}
\end{aligned}
$$

The final results are discussed in chapter 4. With the help of these first variations it is already possible to derive the Einstein field equations from the Einstein Hilbert action as done in section 2.1.

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Hiermit versichere ich, die vorgelegte Thesis selbstständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt zu haben, die ich in der Thesis angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Thesis erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der ‚Satzung der Justus-Liebig-Universität zur Sicherung guter wissenschaftlicher Praxis' niedergelegt sind, eingehalten. Gemäß § 25 Abs. 6 der Allgemeinen Bestimmungen für modularisierte Studiengänge dulde ich eine Überprüfung der Thesis mittels Anti-Plagiatssoftware.

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