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Covariant techniques for computation of the heat kernel

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The heat kernel associated with an elliptic second-order partial differential operator of Laplace type acting on smooth sections of a vector bundle over a Riemannian manifold, is studied. A general manifestly covariant method for computation of the coefficients of the heat kernel asymptotic expansion is developed. The technique enables one to compute explicitly the diagonal values of the heat kernel coefficients, so called Hadamard-Minackshisundaram-De Witt-Seeley coefficients, as well as their derivatives. The elaborated technique is applicable for a manifold of arbitrary dimension and for a generic Riemannian metric of arbitrary signature. It is very algorithmic, and well suited to automated computation. The fourth heat kernel coefficient is computed explicitly for the first time.

The general structure of the heat kernel coefficients is investigated in detail. On the one hand, the leading derivative terms in all heat kernel coefficients are computed. On the other hand, the generating functions in closed covariant form for the covariantly constant terms and some low-derivative terms in the heat kernel coefficients are constructed by means of purely algebraic methods. This gives, in particular, the whole sequence of heat kernel coefficients for an arbitrary locally symmetric space.

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1 Introduction

In this talk I am going to report on recent progress on developing some computational methods for the heat kernel that turned out to be very powerful for carrying out explicit computations [1, 2, 3, 4, 7, 5, 6]. We will start with the definition of the heat kernel and, then, will try to explain the main ideas of our approach and present the main results without going much into details. Consequently, I will not be as formal as probably many matematicians would like me to be.

The heat kernel proved to be a very powerful tool in mathematical physics as well as in quantum field theory. It has been the subject of much investigation in recent years in mathematical as well as in physical literature. (See, for example, [15, 11, 13, 1, 2, 7] and references therein.) The study of the heat kernel is motivated, in particular, by the fact that it gives a general framework of covariant methods for investigating the quantum field theories with local gauge symmetries, such as quantum gravity and gauge theories [10].

1.1 Preliminaries

To define the heat kernel one has to remember some preliminary facts from the differential geometry [11]. Let (M, g) be a smooth Riemannian manifold of dimension d with a positive definite Riemannian metric g. To simplify the exposition we assume additionally that it is *compact* and *complete*, i.e. without boundary, $\partial M = \emptyset$.

Let TM and T^*M be the tangent and cotangent bundles of the manifold M. On the tangent bundle TM of a Riemannian manifold there is always a unique canonical connection, so called *Levi-Civita connection*, ∇^{TM} , which is torsion-free and compatible with the metric g.

Let V be a smooth vector bundle over the manifold M, $\operatorname{End}(V)$ be the bundle of all smooth endomorphisms of the vector bundle V, and $C^{\infty}(M,V)$ and $C^{\infty}(M,\operatorname{End}(V))$ be the spaces of all smooth sections of the vector bundles V and $\operatorname{End}(V)$.

Further, we will also assume that V is a Hermitian vector bundle, i.e. there is a Hermitian pointwise fibre scalar product $\langle \varphi, \psi \rangle$ for any two sections of the vector bundle $\varphi, \psi \in C^{\infty}(M, V)$. The dual vector bundle V^* is naturally identified with V, so that

$$\langle \varphi, \psi \rangle = \operatorname{tr}_{V}(\bar{\varphi} \otimes \psi),$$
 (1)

where $\psi \in C^{\infty}(M, V)$, and $\bar{\varphi} \in C^{\infty}(M, V^*)$ and tr_V is the fibre trace. Using the invariant Riemannian volume element $d\operatorname{vol}(x)$ on the manifold M we define a natural L^2 inner product

$$(\varphi, \psi) = \operatorname{Tr}_{L^2}(\bar{\varphi} \otimes \psi) = \int_M d\operatorname{vol}(x) < \varphi, \psi > = \int_M d\operatorname{vol}(x) \operatorname{tr}_V(\bar{\varphi} \otimes \psi). \tag{2}$$

The Hilbert space $L^2(M, V)$ is defined to be the completion of $C^{\infty}(M, V)$ in this norm. Let ∇^V be a connection, or covariant derivative, on the vector bundle V

$$\nabla^{V}: C^{\infty}(M, V) \to C^{\infty}(M, T^{*}M \otimes V), \tag{3}$$

which is compatible with the Hermitian metric on the vector bundle V, i.e.

$$\nabla < \varphi, \psi > = < \nabla^{V} \varphi, \psi > + < \varphi, \nabla^{V} \psi > . \tag{4}$$

On the tensor product bundle $T^*M\otimes V$ we define the tensor product connection by means of the Levi-Civita connection

$$\nabla^{T^*M\otimes V} = \nabla^{T^*M} \otimes 1 + 1 \otimes \nabla^V. \tag{5}$$

Similarly, we extend the connection ∇^V with the help of the Levi-Civita connection to $C^{\infty}(M,V)$ -valued tensors of all orders and denote it just by ∇ . Usually there is no ambiguity and the precise meaning of the covariant derivative is always clear from the nature of the object it is acting on.

The composition of two covariant derivatives is a mapping

$$\nabla^{T^*M\otimes V}\nabla^V:\ C^{\infty}(M,V)\to C^{\infty}(M,T^*M\otimes V)\to C^{\infty}(M,T^*M\otimes T^*M\otimes V). \tag{6}$$

Let, further, tr_g denote the contraction of sections of the bundle $T^*M \otimes T^*M \otimes V$ with the metric on the cotangent bundle

$$\operatorname{tr}_g = g \otimes 1 : C^{\infty}(M, T^*M \otimes T^*M \otimes V) \to C^{\infty}(M, V).$$
 (7)

Then we can define a second-order differential operator, called the *generalized Laplacian*, by

$$\Box = \operatorname{tr}_g \nabla^{T^*M \otimes V} \nabla^V \tag{8}$$

$$\square : C^{\infty}(M, V) \to C^{\infty}(M, T^*M \otimes V) \to C^{\infty}(M, T^*M \otimes T^*M \otimes V) \to C^{\infty}(M, V).$$
 (9)

Further, let Q be a smooth Hermitian section of the endomorphism bundle, $\operatorname{End}(V)$, i.e.

$$\langle \varphi, Q\psi \rangle = \langle Q\varphi, \psi \rangle.$$
 (10)

Finally, we define a Laplace type differential operator F as the sum of the generalized Laplacian and the endomorphism Q

$$F = -\Box + Q. \tag{11}$$

1.2 Laplace type operator in local coordinates

The generalized Laplacian can be easily expressed in local coordinates. Let x^{μ} , ($\mu = 1, 2, ..., d$), be a system of local coordinates and ∂_{μ} and dx^{μ} be the local coordinate frames for the tangent and the cotangent bundles. We adopt the notation that the Greek indices label the tensor components with respect to local coordinate frame and range from 1 through $d = \dim M$. Besides, a summation is always caried out over repeated indices. Let $g_{\mu\nu} = (\partial_{\mu}, \partial_{\nu})$ be the metric on the tangent bundle, $g^{\mu\nu} = (dx^{\mu}, dx^{\nu})$ be the metric on the cotangent bundle, $g = \det g_{\mu\nu}$, $\Gamma^{\mu}_{\nu\lambda}$ be the Levi-Civita connection and \mathcal{A}_{μ} be the connection 1–form of ∇^{V} .

Then it is not difficult to obtain for the generalized Laplacian

$$\Box = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} = g^{-1/2} (\partial_{\mu} + \mathcal{A}_{\mu}) g^{1/2} g^{\mu\nu} (\partial_{\nu} + \mathcal{A}_{\nu}). \tag{12}$$

Therefore, a Laplace type operator is a second-order partial differential operator of the form

$$F = -g^{\mu\nu}\partial_{\mu}\partial_{\nu} - 2a^{\mu}\partial_{\mu} + q, \tag{13}$$

where a^{μ} is a End (V)-valued vector

$$a^{\mu} = g^{\mu\nu} \mathcal{A}_{\nu} + \frac{1}{2} g^{-1/2} \partial_{\nu} (g^{1/2} g^{\nu\mu})$$
 (14)

and q is a section of the endomorphism bundle End (V)

$$q = Q - g^{\mu\nu} \mathcal{A}_{\mu} \mathcal{A}_{\nu} - g^{-1/2} \partial_{\mu} (g^{1/2} g^{\mu\nu} \mathcal{A}_{\nu}). \tag{15}$$

Thus, a Laplace type operator is constructed from the following three pieces of geometric data

- a metric g on M, which determines the second-order part;
- a connection 1-form \mathcal{A} on the vector bundle V, which determines the first-order part;
- an endomorphism Q of the vector bundle V, which determines the zeroth order part.

It is worth noting that every second-order differential operator with a scalar leading symbol given by the metric tensor is of Laplace type and can be put in this form by choosing the appropriate connection ∇^V and the endomorphism Q.

1.3 Self-adjoint operators

Using the L^2 inner product we define the adjoint F^* of a differential operator F by

$$(F^*\varphi,\psi) = (\varphi, F\psi). \tag{16}$$

It is not difficult to prove that if the connection ∇ is compatible with the Hermitian metric on the vector bundle V and the boundary of the manifold M is empty, then the generalized Laplacian \square , and, obviously, any Laplace type operator F, is an *elliptic symmetric* differential operator

$$(\Box \varphi, \psi) = (\varphi, \Box \psi), \qquad (F\varphi, \psi) = (\varphi, F\psi), \tag{17}$$

with a positive principal symbol. Moreover, the operator F is essentially self-adjoint, i.e. there is a unique self-adjoint extension \bar{F} of the operator F. We will not be very careful about distinguishing between the operator F and its closure \bar{F} , and will simply say that the operator F is elliptic and self-adjoint.

Spectral theorem. There is a well known theorem about the spectrum of any elliptic self-adjoint differential operator F acting on smooth sections of a vector bundle V over a compact manifold $M, F : C^{\infty}(M, V) \to C^{\infty}(M, V)$, with a positive definite principal symbol [15]. Namely,

• the operator F has a discrete real spectrum, λ_n , (n = 1, 2, ...,), bounded from below

$$\lambda_n > -C,\tag{18}$$

with some real sonstant C,

• all eigenspaces of the operator F are finite-dimensional and the eigenvectors, φ_n , of the operator F,

$$F\varphi_n = \lambda_n \varphi_n,\tag{19}$$

are smooth sections of the vector bundle V, which form a complete orthonormal basis in $L^2(M,V)$.

$$(\varphi_n, \varphi_m) = \delta_{mn}. \tag{20}$$

In the following we wil assume that the endomorphism Q is bounded from below by a sufficiently large constant, so that the Laplace type operator F is *strictly positive*. This does not influence all the conclusions but simplifies significantly the technical details needed to treat the negative and zero modes of the operator F. This can be always done as long as we study only asymptotic properties of the spectrum for large eigenvalues but not the structure and the dimension of the null space and related cohomolgical and topological questions.

1.4 Heat kernel

Thus all eigenfunctions of the Laplace type operator F are smooth sections of the vector bundle V and, if the manifold M is compact, F has a unique self-adjoint extension, which we denote by the same symbol F.

Then the operator $U(t) = \exp(-tF)$ for t > 0 is well defined as a bounded operator on the Hilbert space of square integrable sections of the vector bundle V. These operators form a one-parameter semi-group. The kernel U(t|x,x') of this operator is defined by

$$U(t|x,x') = \exp(-tF)\delta(x,x') = \sum_{n} e^{-t\lambda_n} \varphi_n(x) \otimes \bar{\varphi}_n(x'), \tag{21}$$

where $\delta(x, x')$ is the covariant Dirac distribution along the diagonal of $M \times M$, and each eigenvalue is counted with multiplicities. It can be regarded as an endomorphism from the fiber of V over x' to the fiber of V over x.

The kernel U(t|x,x') of the operator $\exp(-tF)$ satisfies the heat equation

$$(\partial_t + F)U(t|x, x') = 0 (22)$$

with the initial condition

$$U(0^{+}|x,x') = \delta(x,x'). \tag{23}$$

That is why, it is called the *heat kernel*. It can be proved that there is a unique smooth solution, called the *fundamental solution*, of the heat equation satisfying that initial condition. Thus, the heat kernel is the *fundamental solution* of the heat equation. For t > 0 the heat kernel is a smooth section of the external tensor product of the vector bundles $V \boxtimes V^*$ over the tensor product manifold $M \times M$: $U(t|x,x') \in C^{\infty}(\mathbb{R}_+ \times M \times M, V \boxtimes V^*)$.

It is not difficult to prove that for a positive elliptic operator F the inverse operator, called also Green operator, $G = F^{-1}$, is a bounded operator with the kernel given by

$$G(x,x') = \int_{0}^{\infty} dt \, U(t|x,x'). \tag{24}$$

1.5 Trace of the heat kernel and the spectral functions

As we already said above, for all t > 0 the heat semi-group $U(t) = \exp(-tF)$ of a Laplace type operator F on a compact manifold M is a bounded operator on the Hilbert space $L^2(M, V)$ and is trace-class, with a well defined trace given by the formula

$$\operatorname{Tr}_{L^{2}} \exp(-tF) = \int_{M} d\operatorname{vol}(x) \operatorname{tr}_{V} U(t|x,x) = \sum_{n} e^{-t\lambda_{n}}.$$
 (25)

The trace of the heat kernel is obviously a spectral invariant of the operator F. It determines all other spectral functions by integral transforms.

1. The distribution function, $N(\lambda)$, defined as the number of the eigenvectors with the eigenvalues less than λ , is given by

$$N(\lambda) = \#\{\varphi_n | \lambda_n < \lambda\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dt}{t} e^{t\lambda} \operatorname{Tr}_{L^2} \exp(-tF), \tag{26}$$

where c is a positive constant.

2. The density function, $\rho(\lambda)$, is defined by derivative of the distribution function and is obviously

$$\rho(\lambda) = \frac{d}{d\lambda} N(\lambda) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt e^{t\lambda} \operatorname{Tr}_{L^2} \exp(-tF).$$
 (27)

3. The zeta-function, $\zeta(s)$, defined as the trace of the complex power of the operator F, is given by

$$\zeta(s) = \operatorname{Tr}_{L^2} F^{-s} = \int d\lambda \rho(\lambda) \lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \operatorname{Tr}_{L^2} \exp(-tF), \qquad (28)$$

where s is a complex variable with $\operatorname{Re} s > d/2$.

4. Finally, the trace of the powers of the resolvent operator is also expressed in terms of the heat kernel

$$R_k(z) = \operatorname{Tr}_{L^2}(F - z)^{-k} = \int d\lambda \rho(\lambda)(\lambda - z)^{-k} = \frac{1}{(k - 1)!} \int_0^\infty dt t^{k - 1} e^{tz} \operatorname{Tr}_{L^2} \exp(-tF),$$
(29)

where z is a complex parameter with Re z < 0 and k is a positive integer satisfying the condition k > d/2.

These spectral functions are very useful tools in studying the spectrum of the operator F. The zeta function enables one to define, in particular, the regularized determinant of the operator F,

$$\zeta'(0) = -\log \operatorname{Det} F, \tag{30}$$

which determines the one-loop effective action in quantum field theory. All these functions are, in principle, equivalent to each other. However, the heat kernel is a smooth function whereas the distribution and especially the density function are extremely singular. That is why the heat kernel seems to be more convenient for practical purposes.

2 Asymptotic expansion of the heat kernel

In the following we are going to study the heat kernel only locally, i.e. in the neighbourhood of the diagonal of $M \times M$, when the points x and x' are close to each other. The exposition will follow mainly our papers [1, 2]. We will keep a point x' of the manifold fixed and consider a small geodesic ball, i.e. a small neighbourhood of the point x': $B_{x'} = \{x \in M | r(x, x') < \varepsilon\}$, r(x, x') being the geodesic distance between the points x and x'. We will take the radius of the ball sufficiently small, so that each point x of the ball of this neighbourhood can be connected by a unique geodesic with the point x'. This can be always done if the size of the ball is smaller than the injectivity radius of the manifold at x', $\varepsilon < r_{\rm inj}(x')$.

Let $\sigma(x, x')$ be the geodetic interval, also called world function, defined as one half the square of the length of the geodesic connecting the points x and x'

$$\sigma(x, x') = \frac{1}{2}r^2(x, x'). \tag{31}$$

The first derivatives of this function with respect to x and x' define tangent vector fields to the geodesic at the points x and x'

$$u^{\mu} = g^{\mu\nu} \nabla_{\nu} \sigma, \qquad u^{\mu'} = g^{\mu'\nu'} \nabla'_{\nu'} \sigma. \tag{32}$$

and the determinant of the mixed second derivatives defines a so called *Van Vleck-Morette* determinant

$$\Delta(x, x') = g^{-1/2}(x)\det(-\nabla_{\mu}\nabla'_{\nu'}\sigma(x, x'))g^{-1/2}(x'). \tag{33}$$

Let, finally, $\mathcal{P}(x, x')$ denote the parallel transport operator along the geodesic from the point x' to the point x. It is a section of the external tensor product of the vector bundle

 $V \boxtimes V^*$ over $M \times M$, or, in other words, it is an endomorphism from the fiber of V over x' to the fiber of V over x.

Near the diagonal of $M \times M$ all these two-point functions are smooth single-valued functions of the coordinates of the points x and x'. To simplify the consideration, we will assume that these functions are analytic.

Then, the function

$$U_0(t|x,x') = (4\pi t)^{-d/2} \Delta(x,x') \exp\left(-\frac{1}{2t}\sigma(x,x')\right) \mathcal{P}(x,x')$$
(34)

satisfies the initial condition

$$U_0(0^+|x,x') = \delta(x,x'). \tag{35}$$

Moreover, locally it satisfies also the heat equation in the free case, when the Riemannian curvature of the manifold Riem, the curvature of the bundle connection \mathcal{R} and the endomorphism Q vanish: Riem = $\mathcal{R} = Q = 0$. Therefore, $U_0(t|x,x')$ is the exact heat kernel for a pure generalized Laplacian in flat Euclidean space with a flat trivial bundle connection and without the endomorphism Q.

This function gives a good framework for the approximate solution in the general case. Namely, by factorizing out this free factor we get an ansatz

$$U(t|x,x') = (4\pi t)^{-d/2} \Delta(x,x') \exp\left(-\frac{1}{2t}\sigma(x,x')\right) \mathcal{P}(x,x')\Omega(t|x,x'). \tag{36}$$

The function $\Omega(t|x,x')$, called the transport function, is a section of the endomorphism vector bundle End (V) over the point x'. Using the definition of the functions $\sigma(x,x')$, $\Delta(x,x')$ and $\mathcal{P}(x,x')$ it is not difficult to find that the transport function satisfies a transport equation

$$\left(\partial_t + \frac{1}{t}D + L\right)\Omega(t) = 0,\tag{37}$$

where D is the radial vector field, i.e. operator of differentiation along the geodesic, defined by

$$D = \nabla_u = u^{\mu} \nabla_{\mu},\tag{38}$$

and L is a second-order differential operator defined by

$$L = \mathcal{P}^{-1} \Delta^{-1/2} F \Delta^{1/2} \mathcal{P}. \tag{39}$$

The initial condition for the transpot function is obviously

$$\Omega(t|x,x') = I, (40)$$

where I is the identity endomorphism of the vector bundle V over x'.

One can prove that when the operator F is positive the function $\Omega(t)$ satisfies the following asymptotic conditions

$$\lim_{t \to \infty, 0} t^{\alpha} \partial_t^N \Omega(t) = 0 \quad \text{for any } \alpha > 0, \ N \ge 0.$$
 (41)

In other words, as $t \to \infty$ the function $\Omega(t)$ and all its derivatives decreases faster than any power of t, actually it decreases exponentially, and as $t \to 0$ the product of $\Omega(t)$ with any positive power of t vanishes.

Now, let us consider a slightly modified version of the Mellin transform of the function $\Omega(t)$ introduced in [2]

$$b_q = \frac{1}{\Gamma(-q)} \int_0^\infty dt t^{-q-1} \Omega(t). \tag{42}$$

This integral converges for $\operatorname{Re} q < 0$. By integrating by parts N times and using the asymptotic conditions (41) we get also

$$b_q = \frac{1}{\Gamma(-q+N)} \int_0^\infty dt t^{-q-1+N} (-\partial_t)^N \Omega(t). \tag{43}$$

This integral converges for $\operatorname{Re} q < N - 1$.

Using this representtion one can prove that [2]

- the function b_q is analytic everywhere, i.e. it is an entire function,
- the values of the function b_q at the integer positive points are given by

$$b_k = (-\partial_t)^k \Omega(t) \Big|_{t=0}, \tag{44}$$

• b_q satisfies an asymptotic condition

$$\lim_{|q| \to \infty, \text{ Re } q < N} \Gamma(-q + N)b_q = 0, \quad \text{for any } N > 0.$$
 (45)

By inverting the Mellin transform we obtain a new ansatz for the transport function and, hence, for the heat kernel

$$\Omega(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \, t^q \, \Gamma(-q) b_q \tag{46}$$

where c < 0 is a negative constant.

Substituting this ansatz into the transport equation we get a functional eqution for the function b_q

$$\left(1 + \frac{1}{q}D\right)b_q = L\,b_{q-1}.
\tag{47}$$

The initial condition for the transport function is translated into

$$b_0 = I. (48)$$

Thus, we have reduced the problem of solving the heat equation to the following problem: one has to find an entire function $b_q(x, x')$ that satisfies the functional equation (47) with the initial condition (48) and the asymptotic condition (45).

The function b_q turns out to be extremely useful in computing the heat kernel, the resolvent kernel, the zeta-function and the determinant of the operator F. It contains the same the information about the manifold as the heat kernel. In some cases the function b_q can be constructed just by analytical continuation from the integer positive values b_k [2].

Now we are going to do the usual trick, namely, to move the contour of integration over q to the right. Due to the presence of the gamma function $\Gamma(-q)$ the integrand has simple poles at the non-negative integer points $q = 0, 1, 2 \dots$, which contribute to the integral while moving the contour. So, we get

$$\Omega(t) = \sum_{k=0}^{N-1} \frac{(-t)^k}{k!} b_k + R_N(t), \tag{49}$$

where

$$R_N(t) = \frac{1}{2\pi i} \int_{c_N - i\infty}^{c_N + i\infty} dq \, t^q \, \Gamma(-q) b_q \tag{50}$$

with c_N is a constant satisfying the condition $N-1 < c_N < N$. As $t \to 0$ the rest term $R_N(t)$ behaves like $O(t^N)$, so we obtain an asymptotic expansion as $t \to 0$

$$\Omega(t|x,x') \sim \sum_{k>0} \frac{(-t)^k}{k!} b_k(x,x').$$
 (51)

Using our ansatz (36) we find immediately the trace of the heat kernel

Tr
$$_{L^2} \exp(-tF) = (4\pi t)^{-d/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \, t^q \, \Gamma(-q) B_q,$$
 (52)

where

$$B_q = \operatorname{Tr}_{L^2} b_q = \int_M d\operatorname{vol}(x) \operatorname{tr}_V b_q(x, x).$$
(53)

The trace of the heat kernel has an analogous asymptotic expansion as $t \to 0$

$$\operatorname{Tr}_{L^2} \exp(-tF) \sim (4\pi t)^{-d/2} \sum_{k\geq 0} \frac{(-t)^k}{k!} B_k.$$
 (54)

This is the famous Minackshisundaram-Pleijel asymptotic expansion. The physicists call it the Schwinger-De Witt expansion [10]. Its coefficients B_k are also called sometimes Hadamard-Minackshisundaram-De Witt-Seeley (HMDS) coefficients. This expansion is of great importance in differential geometry, spectral geometry, quantum field theory and other areas of mathematical physics, such as theory of Huygence' principle, heat kernel proofs of the index theorems, Korteveg-De Vries hierarchy, Brownian motion etc. (see, forexample, [16]).

For integer q = k = 1, 2, ... the functional equation (47) becomes a recursion system that, together with the initial condition (48), determines all the HMDS-coefficients b_k .

One should stress, however, that this series does not converge, in general. In that sense our ansatz (49) in form of a Mellin transform of an entire function is much better since it is exact and gives an explicit formula for the rest term.

Let us apply our ansatz for computation of the complex power of the operator F defined by

$$G^{p} = F^{-p} = \frac{1}{\Gamma(p)} \int_{0}^{\infty} dt \, t^{p-1} \, U(t).$$
 (55)

Using our ansatz for the heat kernel we obtain

$$G^{p} = (4\pi)^{-d/2} \Delta^{1/2} \mathcal{P} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \, \frac{\Gamma(-q)\Gamma(-q-p+d/2)}{\Gamma(p)} \left(\frac{\sigma}{2}\right)^{q+p-d/2} b_{q} \tag{56}$$

where c < -Re p + d/2.

Outside the digonal, i.e. for $\sigma \neq 0$, this integral converges for any p and defines an entire function of p. The integrand in this formula is a meromorphic function of p with some simple and maybe some double poles. If we move the contour of integration to the right, we get contributions from the simple poles in form of powers of σ and a logarithmic part due to the double poles (if any). This gives the complete structure of diagonal singularities of the complex power of the opertor F, $G^p(x, x')$. Thus the function b_q turns out to be very useful to study the diagonal singularities.

In particular case p = 1 we recover in this way the singularity structure of the Green function

$$G = (4\pi)^{-d/2} \Delta^{1/2} \mathcal{P} \left(\Phi + \Psi \log \frac{\sigma}{2} \right) + G_{\text{reg}}, \tag{57}$$

where

$$\Phi = \sum_{k=0}^{d/2-1} \frac{(-1)^k}{k!} \Gamma(d/2 - 1 - k) \left(\frac{2}{\sigma}\right)^{d/2 - 1 - k} b_k \tag{58}$$

$$\Psi = \begin{cases} 0, & \text{for odd } d\\ \frac{(-1)^{d/2}}{\Gamma(d/2)} b_{d/2-1} & \text{for even } d \end{cases}$$
 (59)

$$G_{\text{reg}} = (4\pi)^{-d/2} \Delta^{1/2} \mathcal{P} \frac{1}{2\pi i} \int_{q-i\infty}^{q+i\infty} dq \, \Gamma(-q) \Gamma(-q-1+d/2) \left(\frac{\sigma}{2}\right)^{q+1-d/2} b_q, \qquad (60)$$

where $[d/2] - 1 < \alpha < [d/2] - 1/2$. We see that due to the absence of the double poles in the integrand there is no logarithmic singularity in odd dimensions. Thus, all the singularities of the Green function are determined by the HMDS-coefficients b_k and the regular part is determined by the function b_q as well.

Now, let us consider the diagional limit of G^p . By taking the limit $\sigma \to 0$ we obtain a very simple formula in terms of the function b_q

$$G^{p}(x,x) = (4\pi)^{-d/2} \frac{\Gamma(p-d/2)}{\Gamma(p)} b_{d/2-p}(x,x).$$
(61)

This gives automatically the zeta-function of the operator F [2]

$$\zeta(p) = (4\pi)^{-d/2} \frac{\Gamma(p - d/2)}{\Gamma(p)} B_{d/2-p}.$$
(62)

Herefrom we see that both $G^p(x,x)$ and $\zeta(p)$ are meromorphic functions with simple poles at the points p=[d/2]+1/2-k, $(k=0,1,2,\ldots)$ and $p=1,2,\ldots,[d/2]$. In particular, the zeta-function is analytic at the origin. Its value at the origin is given by

$$\zeta(0) = \begin{cases} 0 & \text{for odd } d\\ (4\pi)^{-d/2} \frac{(-1)^{d/2}}{\Gamma(d/2+1)} B_{d/2} & \text{for even } d \end{cases}$$
 (63)

This gives the regularized number of all modes of the operator F since formally

$$\zeta(0) = \operatorname{Tr}_{L^2} \mathbf{1} = \sum_n 1.$$
 (64)

Moreover, the derivative of the zeta-function at the origin is also well defined. As we already mentioned above it determines the regularized determinant of the operator F since formally

$$\log \operatorname{Det} F = \operatorname{Tr}_{L^2} \log F = \sum_{n} \log \lambda_n = -\zeta'(0). \tag{65}$$

Thus we obtain for the determinant

log Det
$$F = -(4\pi)^{-d/2} \frac{\pi(-1)^{(d+1)/2}}{\Gamma(d/2+1)} B_{d/2}$$
 for odd d (66)

and

log Det
$$F = (4\pi)^{-d/2} \frac{(-1)^{d/2}}{\Gamma(d/2+1)} \left\{ B'_{d/2} - [\Psi(d/2+1) + \mathbf{C}] B_{d/2} \right\}$$
 for even d . (67)

Here $\Psi(z) = (d/dz)\log\Gamma(z)$ is the psi-function, $\mathbf{C} = -\Psi(1)$ is the Euler constant, and

$$B'_{d/2} = \frac{d}{dq} B_q \bigg|_{q=d/2}$$
 (68)

3 Non-recursive solution of the recursion system

The main problem we are solving is to compute the HMDS-coefficients, not only the integrated ones $B_k = \int_M d\text{vol}(x) \text{tr}_V b_k(x, x)$, which are determined by the diagonal values of $b_k(x, x)$, but rather the off-diagonal coefficients $b_k(x, x')$. They are determined by a recursion system which is obtained simply by restricting the complex variable q in the eq. (47) to positive integer values $q = 1, 2, \ldots$ This problem was solved in [1, 2] where

a systhematic technique for calculation of b_k was developed. The formal solution of this recursion system is

$$b_k = \left(1 + \frac{1}{k}D\right)^{-1}L\left(1 + \frac{1}{k-1}D\right)^{-1}L\cdots\left(1 + \frac{1}{1}D\right)^{-1}L\cdot I.$$
 (69)

So, the problem is to give a precise practical meaning to this formal operator solution. To do this one has, first of all, to define the inverse operator $(1 + D/k)^{-1}$. This can be done by constructing the complete set of eigenvectors of the operator D. However, first we introduce some auxiliary notions from the theory of symmetric tensors.

3.1 Algebra of symmetric tensors

Let ω^a and e_a be the basises in the cotangent T^*M and tangent TM bundles, $S^n(M)$ be the bundle of symmetric contravariant tensors of rank n, $S_n(M)$ be the bundle of symmetric n-forms and $S_m^n(M) = S_m(M) \otimes S^n(M)$ be the bundle of symmetric tensors of type (m, n) with the basis

$$s_{b_1...b_n}^{a_1...a_m} = \omega^{(a_1} \otimes \cdots \otimes \omega^{a_m)} \otimes e_{(b_1} \otimes \cdots \otimes e_{b_n)}. \tag{70}$$

where the parenthesis mean the symmetrization over all indices included.

In the space S_n^n there is a natural unity symmetric tensor

$$I_{(n)} = s_{1...n}^{1...n}, (71)$$

which is an identical endomorphism of the vector bundles S^n and S_n .

We define the following binary operations on symmetric tensors:

a) the exterior symmetric tensor product \vee

$$\vee: S_m^n \times S_i^i \to S_{m+i}^{n+i} \tag{72}$$

by

$$A \vee B = A_{(a_1...a_m}^{(b_1...b_n)} B_{a_{m+1}...a_{m+j}}^{b_{n+1}...b_{n+i}}) s_{b_1...b_{n+i}}^{a_1...a_{m+j}},$$

$$(73)$$

b) and an inner product \star

$$\star: S_m^n \times S_n^i \to S_m^i, \tag{74}$$

by:

$$A \star B = A_{a_1...a_m}^{c_1...c_n} B_{c_1...c_n}^{b_1...b_i} s_{b_1...b_i}^{a_1...a_m}.$$
 (75)

Further, we define also an $exterior\ symmetric\ covariant\ derivative\ s\nabla$ on symmetric tensors

$$s\nabla: S_n^m \to S_{n+1}^m \tag{76}$$

by

$$s\nabla A = \nabla_{(a_1} A^{b_1 \dots b_m}_{a_2 \dots a_{n+1})} s^{a_1 \dots a_{n+1}}_{b_1 \dots b_m}.$$
 (77)

Everything said above remains true if we consider End (V)-valued symmetric tensors, i.e. sections of the vector bundle $S_m^n \otimes \text{End}(V)$, for some vector bundle V over M. The product operations include then the usual endomorphism (matrix) inner product as well.

3.2 Covariant Taylor basis

Let us consider the space $\mathcal{L}(B_{x'}) = \{|f> \equiv f(x,x')| \ x \in B_{x'}\}$ of smooth analytic twopoint functions in a small neighbourhod B'_x of the diagonal x = x'. Here we denote the elements of this space by |f>. Let us define a special set of such functions $|n> \in \mathcal{L}(B_{x'})$ labeled by a natural number $n \in \mathbb{N}$ by

$$|0> = 1$$

 $|n> = \frac{(-1)^n}{n!} \vee^n u', \qquad (n=1,2,\ldots),$ (78)

where u' is the tangent vector field to the geodesic conecting the points x and x' at the point x' given by the first derivative of the geodetic interval σ

$$u' = (g'^{ab}\nabla_b'\sigma)e_a', \tag{79}$$

where prime ' denotes the objects and operations at the point x'. The functions $|n\rangle$ are two-point geometric objects, which are scalars at the point x and symmetric contravariant tensors at the point x', more precisely, they are sections of the vector bundle S^n over the point x'.

Let us define also the dual space of linear functionals

$$\mathcal{L}^*(B_{x'}) = \{ \langle f | : \mathcal{L}(B_{x'}) \to \mathbf{C} \}, \tag{80}$$

with the basis $\langle n|$ dual to the basis $|n\rangle$. The values of the dual basis functionals on the two-point functions are sections of the vector bundle of symmetric forms S_n defined to be the diagonal values of the symmetric exterior covariant derivative $s\nabla$

$$\langle n|f\rangle = [(s\nabla)^n f],$$
 (81)

where the square brackets mean restriction to the diagonal x = x'.

The basis $\langle n|$ is dual to $|m\rangle$ in the sense that

$$\langle n|m\rangle = \delta_{mn}I_{(n)}. \tag{82}$$

Using this notation the covariant Taylor series for an analytic function $|f\rangle$ can be written in the form

$$|f> = \sum_{n>0} |n> \star < n|f>$$
. (83)

Now it is almost obvious that our set of functions |n> forms a complete basis in $\mathcal{L}(B_{x'})$ due to the fact that there does not exist any nontrivial analytic function which is 'orthogonal' to all of the eigenfunctions |n>. In other words, an analytic function that is equal to zero together with all symmetrized derivatives at the point x=x' is, in fact, identically equal to zero in $B_{x'}$.

It is easy to show that these functions satisfy the equation

$$D|n> = n|n> \tag{84}$$

and, hence, are the eigenfunctions of the operator D with positive integer eigenvalues. Note, however, that the space of analytical functions $\mathcal{L}(B_{x'})$ is not a Hilbert space with a scalar product $\langle f|g\rangle$ defined above since there are a lot of analytic functions for which the norm $\langle f|f\rangle$ diverges. If we restrict ourselves to polynomial functions of some order then this problem does not appear. Thus the space of polynomials is a Hilbert space with the inner product defined above.

3.3 Covariant Taylor series for HMDS-coefficients b_k

The complete set of eigenfunctions $|n\rangle$ can be employed to present an arbitrary linear differential operator L in the form

$$L = \sum_{m,n>0} |m > \star < m|L|n > \star < n|, \tag{85}$$

where < m|L|n > are the 'matrix elements' of the operator L that are just End (V)-valued symmetric tensors, i.e. sections of the vector bundle $S_m^n(M) \otimes \text{End}(V)$. We will not study the question of convergency of the expansion (85). It can be regarded just as a formal series. When acting on an analytic function, this series is nothing but the Taylor series and converges in a sufficiently small region $B_{x'}$.

Now it should be clear that the inverse operator $(1 + \frac{1}{k}D)^{-1}$ can be defined by

$$\left(1 + \frac{1}{k}D\right)^{-1} = \sum_{n \ge 0} \frac{k}{k+n} |n > \star < n|. \tag{86}$$

Using this representation together with the analogous one for the operator L, (85), we obtain a covariant Taylor series for the coefficients b_k

$$b_k = \sum_{n \ge 0} |n > \star < n| b_k > \tag{87}$$

with the covariant Taylor coefficients $\langle n|b_k \rangle$ given by [1, 2]

$$\langle n|b_{k} \rangle = \sum_{n_{1},\dots,n_{k-1}\geq 0} \frac{k}{k+n} \cdot \frac{k-1}{k-1+n_{k-1}} \cdots \frac{1}{1+n_{1}}$$

$$\times \langle n|L|n_{k-1} \rangle \star \langle n_{k-1}|L|n_{k-2} \rangle \star \cdots \star \langle n_{1}|L|0 \rangle, \tag{88}$$

where $\langle m|L|n\rangle$ are the matrix elements of the operator L (39).

It is not difficult to show that for a differential operator L of second order, the matrix elements < m|L|n > do not vanish only for $n \le m + 2$. Therefore, the sum (88) always contains only a *finite* number of terms, i.e., the summation over n_i is limited from above

$$n_1 \ge 0, \qquad n_i \le n_{i+1} + 2, \qquad (i = 1, \dots, k-1; \ n_k \equiv n).$$
 (89)

3.4 Matrix elements < m|L|n>

Thus we reduced the problem of computation of the HMDS-coefficients b_k to the computation of the matrix elements of the operator L. The matrix elements < m|L|n> are symmetric tensors of the type (m, n), i.e. sections of the vector bundle $S_m^n(M)$.

The matrix elements < n|L|m > of a Laplace type operator have been computed in our papers [1, 2]. They have the following general form

$$< m|L|m + 2 >= g^{-1} \lor I_{(m)}$$
 (90)

$$\langle m|L|m+1 \rangle = 0 \tag{91}$$

$$\langle m|L|n\rangle = \binom{m}{n} I_{(n)} \vee Z_{(m-n)} - \binom{m}{n-1} I_{(n-1)} \vee Y_{(m-n+1)}$$

$$+\binom{m}{n-2}I_{(n-2)} \vee X_{(m-n+2)},$$
 (92)

where g^{-1} is the metric on the cotangent bundle, $Z_{(n)}$ is a section of the vector bundle $S_n(M) \otimes \text{End}(V)$, $Y_{(n)}$ is a section of the vector bundle $S_n^1(M) \otimes \text{End}(V)$ and $X_{(n)}$ is a section of the vector bundle $S_n^2(M)$. Here it is meant also that the binomial coefficient $\binom{n}{k}$ is equal to zero if k < 0 or n < k.

We will not present here explicit formulas for the objects $Z_{(n)}$, $Y_{(n)}$, and $X_{(n)}$, (they have been computed for arbitrary n in the our papers [1, 2]), but note that all these quantities are expressed polynomially in terms of three sorts of geometric data:

• symmetric tensors of type (2,n), i.e. sections of the vector bundle $S_n^2(M)$

$$K_{(n)} = (s\nabla)^{n-2} \text{Riem}, \tag{93}$$

where Riem is the symmetrized Riemann tensor

$$Riem = R^{(c}{}_{(a}{}^{d)}{}_{b)}s^{ab}_{cd}, \tag{94}$$

• sections of the vector bundle $\operatorname{End}(V) \otimes S_n^1(M)$

$$\mathcal{R}_{(n)} = (s\nabla)^{n-1}\mathcal{R},\tag{95}$$

where \mathcal{R} is the curvature of the connection on the vector bundle V in the form

$$\mathcal{R} = \mathcal{R}^a{}_b s^a_b, \tag{96}$$

• End (V)-valued symmetric forms, i.e. sections of the vector bundle End $(V) \otimes S_n(M)$, constructed from the symmetrized covariant derivatives of the endomorphism Q of the vector bundle V

$$Q_{(n)} = (s\nabla)^n Q. (97)$$

From the dimensional arguments it is obvious that the matrix elements < n|L|n> are expressed in terms of the Riemann curvature tensor, Riem, the bundle curvature, \mathcal{R} , and the endomorphism Q; the matrix elements < n+1|L|n>— in terms of the quantities ∇Riem , $\nabla \mathcal{R}$ and ∇Q ; the elements < n+2|L|n>— in terms of the quantities of the form $\nabla \nabla \text{Riem}$, Riem · Riem, etc.

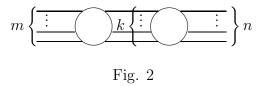
3.5 Diagramatic technique

In the computation of the HMDS-coefficients by means of the matrix algorithm a "diagrammatic" technique, i.e., a graphic method for enumerating the different terms of the sum (88), turns out to be very convenient and pictorial [1].

The matrix elements < m|L|n> are presented by some blocks with m lines coming in from the left and n lines going out to the right (Fig. 1),

$$m\left\{\begin{array}{c|c} \vdots & \vdots \\ \hline \end{array}\right\}n$$
Fig. 1

and the product of the matrix elements $< m|L|k> \star < k|L|n>$ by two blocks connected by k intermediate lines (Fig. 2),



that represents the contractions of the corresponding tensor indices (the inner product).

To obtain the coefficient $\langle n|b_k\rangle$ one should draw, first, all possible diagrams which have n lines incoming from the left and which are constructed from k blocks connected in all possible ways by any number of intermediate lines. When doing this, one should keep in mind that the number of the lines, going out of any block, cannot be greater than the number of the lines, coming in, by more than two and by exactly one. Then one should sum up all diagrams with the weight determined for each diagram by the number of intermediate lines from the analytical formula (88). Drawing of such diagrams is of no difficulties. This helps to keep under control the whole variety of different terms. Therefore, the main problem is reduced to the computation of some standard blocks, which can be computed once and for all.

For example, the diagrams for the diagonal values of the HMDS-coefficients $[b_k] = <0|b_k>$ have the form,

$$[b_1] = \bigcirc \tag{98}$$

$$[b_2] = \bigcirc \bigcirc + \frac{1}{3} \bigcirc \bigcirc \bigcirc$$
 (99)

$$[b_3] = \bigcirc \bigcirc \bigcirc \bigcirc + \frac{1}{3} \bigcirc \bigcirc \bigcirc \bigcirc + \frac{2}{4} \bigcirc \bigcirc \bigcirc$$

$$+ \frac{2}{4} \cdot \frac{1}{2} \bigcirc \bigcirc \bigcirc \bigcirc + \frac{2}{4} \cdot \frac{1}{3} \bigcirc \bigcirc \bigcirc \bigcirc + \frac{2}{4} \cdot \frac{1}{5} \bigcirc \bigcirc \bigcirc \bigcirc$$

$$(100)$$

3.6 Remarks

Let us make some remarks about the elaborated technique.

- this technique is applicable for a generic Riemannian manifold M and for a generic vector bundle V of arbitrary dimensions,
- this technique is manifestly *covariant*, which is an inestimable advantage in quantum field theory, especially in quantum gravity and gauge theories,
- since it is purely local, it is also valid for manifolds with boundary and noncompact manifolds, provided one considers the local HMDS-coefficients $b_k(x, x')$ in a small neighbourhood B of the diagonal of $M \times M$ that does not intersect with the boundary, $B \cap \partial M = \emptyset$,
- moreover, this technique works also in the case of pseudo-Riemannian manifolds and *hyperbolic* differential operators,
- this method is *direct*—it does not need to use any sophisticated functorial properties of the integrated coefficients B_k ,
- it gives not only the diagonal values of the HMDS-coefficients $[b_k]$ but also the diagonal values of all their derivatives; thus it gives immediately the asymptotics of the trace of derivatives of the heat kernel

$$\operatorname{Tr}_{L^2} P \exp(-tF), \tag{101}$$

where P is a differential operator,

- due to use of *symmetric* forms and *symmetric* covariant derivatives the famous 'combinatorial explosion' in the complexity of the HMDS-coefficients is avoided,
- the developed technique is very algorithmic and well suited to *automated computation* there are a number of usual algebraic operations on symmetric tensors that seems to be easily programmed, the needed input, i.e. the matrix elements $\langle n|L|m \rangle$, is computed in advance analytically and is already known,

- the developed method is *very powerful*; it enabled us to compute for the first time the diagonal value of the fourth HMDS-coefficient $[b_4]$ [1, 2]. (The third coefficient $[b_3]$ was computed first by GILKEY [14].)
- last, this technique enables one not only to carry out explicit computations, but also to analyse the general structure of the HMDS-coefficients b_k for all orders k.

4 Covariant approximation schemes for the heat kernel

4.1 General structure of HMDS-coefficients

Now we are going to investigate the general structure of the HMDS-coefficients. We will follow mainly our papers [1, 2, 3, 4, 5, 6] (see also our review papers [7, 8, 9]). Our analysis will be again purely local. Since locally one can always expand the metric, the connection and the endomorphism Q in the covariant Taylor series, they are completely characterized by their Taylor coefficients, i.e. the covariant derivatives of the curvatures, more precisely by the objects $K_{(n)}$, $\mathcal{R}_{(n)}$ and $Q_{(n)}$ introduced above. We introduce the following notation for all of them

$$\Re_{(n)} = \{ K_{(n+2)}, \mathcal{R}_{(n+1)}, Q_{(n)} \}, \qquad (n = 0, 1, 2, \ldots),$$
(102)

and call these objects *covariant jets*. n will be called the *order* of a jet $\Re_{(n)}$. Further we introduce an infinite set of covariant jets of all orders

$$\mathcal{J} = \{\Re_{(n)}; \ (n = 0, 1, 2, \ldots)\}. \tag{103}$$

The first two HMDS-coefficients have a very well known form [14, 2]

$$B_0 = \int_M d\text{vol}(x) \text{tr}_V I.$$
 (104)

$$B_1 = \int_M d\text{vol}(x) \text{tr}_V \left(Q - \frac{1}{6}R \right), \tag{105}$$

where R is the scalar curvature.

As far as the higher order coefficients B_k , $(k \ge 2)$, are concerned they are integrals of local invariants which are polynomial in the jets [15]. One can classify all the terms in them according to the number of the jets and their order. The terms linear in the jets in higher order coefficients B_k , $(k \ge 2)$, are given by integrals of total derivatives, symbolically $\int_M d\text{vol}(x) \operatorname{tr}_V \square^{k-1} \Re$. They are calculated explicitly in [1]. Since the total derivative do not contribute to an integral over a complete compact manifold, it is clear that the linear terms vanish. Thus B_k , $(k = 2, 3, \ldots)$, begin with the terms quadratic in the jets. These terms contain the jets of highest order (or the leading derivatives of the curvatures) and can be shown to be of the form $\int_M d\text{vol}(x) \operatorname{tr}_V \Re \square^{k-2} \Re$. Then it follows a

class of terms cubic in the jets etc.. The last class of terms does not contain any covariant derivatives at all but only the powers of the curvatures. In other words, the higher order HMDS-coefficients have a general structure, which can be presented symbolically in the form

$$B_{k} = \int_{M} d\operatorname{vol}(x) \operatorname{tr}_{V} \left\{ \Re \square^{k-2} \Re + \sum_{0 \leq i \leq 2k-6} \Re \nabla^{i} \Re \nabla^{2k-6-i} \Re + \dots + \sum_{0 \leq i \leq k-3} \Re^{i} (\nabla \Re) \Re^{k-i-3} (\nabla \Re) + \Re^{k} \right\}.$$

$$(106)$$

Leading derivatives in heat kernel asymptotics 4.2

More precisely, all quadratic terms can be reduced to a finite number of invariant structures, viz. [1, 2]

$$B_{k,2} = \frac{k!(k-2)!}{2(2k-3)!} \int_{M} d\text{vol}(x) \operatorname{tr}_{V} \left\{ f_{k}^{(1)} Q \square^{k-2} Q + 2f_{k}^{(2)} \mathcal{R}^{\beta\mu} \nabla_{\beta} \square^{k-3} \nabla_{\alpha} \mathcal{R}^{\alpha}{}_{\mu} + f_{k}^{(3)} Q \square^{k-2} R + f_{k}^{(4)} R_{\mu\nu} \square^{k-2} R^{\mu\nu} + f_{k}^{(5)} R \square^{k-2} R \right\},$$

$$(107)$$

where $R_{\mu\nu}$ is the Ricci tensor and $f_k^{(i)}$ are some numerical coefficients. These numerical coefficients can be computed by the technique developed in the previous section. From the formula (88) we have for the diagonal coefficients $[b_k]$ up to cubic terms in the jets

$$[b_k] = \langle 0|b_k \rangle = \frac{(-1)^{k-1}}{\binom{2k-1}{k}} \langle 0; k-1|L|0 \rangle$$

$$+ (-1)^k \sum_{i=1}^{k-1} \sum_{n_i=0}^{2(k-i-1)} \frac{\binom{2k-1}{i}}{\binom{2k-1}{k}\binom{2i+n_i-1}{i}} \langle 0; k-i-1|L|n_i \rangle \langle n_i; i-1|L|0 \rangle$$

$$+ O(\Re^3), \tag{108}$$

where

$$< n; k|L|m> = (\vee^k g^{-1}) \star < n|L|m>$$
 (109)

and $O(\Re^3)$ denote terms of third order in the jets.

By computing the matrix elements in the second order in the jets and integrating over M one obtains [1, 2]

$$f_k^{(1)} = 1 (110)$$

$$f_k^{(1)} = 1$$
 (110)
 $f_k^{(2)} = \frac{1}{2(2k-1)}$ (111)

$$f_k^{(3)} = \frac{k-1}{2(2k-1)} \tag{112}$$

$$f_k^{(4)} = \frac{1}{2(4k^2 - 1)} \tag{113}$$

$$f_k^{(5)} = \frac{k^2 - k - 1}{4(4k^2 - 1)}. (114)$$

One should note that the same results were obtained by a completely different method by Branson, Gilkey and Ørsted [12].

4.3 'Summation' of asymptotic expansion

Let us consider the situation when the curvatures are small but rapidly varying, i.e. the derivatives of the curvatures are more important than the powers of them. Then the leading derivative terms in the heat kernel are the largest ones. Thus the trace of the heat kernel has the form

Tr
$$_{L^2} \exp(-tF) \sim (4\pi t)^{-d/2} \left\{ B_0 - tB_1 + \frac{t^2}{2} H_2(t) \right\} + O(\Re^3),$$
 (115)

where $H_2(t)$ is some complicated nonlocal functional that has the following asymptotic expansion as $t \to 0$

$$H_2(t) \sim 2 \sum_{k>2} \frac{(-t)^{k-2}}{k!} B_{k,2} + O(\Re^3).$$
 (116)

Using the results for $B_{k,2}$ one can easily construct such a functional H_2 just by a formal summing the leading derivatives

$$H_{2} = \int_{M} d\text{vol}(x) \operatorname{tr}_{V} \left\{ Q \gamma^{(1)}(-t \,\Box) Q + 2\mathcal{R}_{\alpha\mu} \nabla^{\alpha} \frac{1}{\Box} \gamma^{(2)}(-t \,\Box) \nabla_{\nu} \mathcal{R}^{\nu\mu} - 2Q \gamma^{(3)}(-t \,\Box) R + R_{\mu\nu} \gamma^{(4)}(-t \,\Box) R^{\mu\nu} + R \gamma^{(5)}(-t \,\Box) R \right\},$$
(117)

where $\gamma^{(i)}(z)$ are entire functions defined by [1, 2]

$$\gamma^{(i)}(z) = \sum_{k \ge 0} \frac{k!}{(2k+1)!} f_k^{(i)} z^k = \int_0^1 d\xi \, f^{(i)}(\xi) \exp\left(-\frac{1-\xi^2}{4}z\right)$$
 (118)

where

$$f^{(1)}(\xi) = 1 (119)$$

$$f^{(2)}(\xi) = \frac{1}{2}\xi^2 \tag{120}$$

$$f^{(3)}(\xi) = \frac{1}{4}(1-\xi^2) \tag{121}$$

$$f^{(4)}(\xi) = \frac{1}{6}\xi^4 \tag{122}$$

$$f^{(5)}(\xi) = \frac{1}{48}(3 - 6\xi^2 - \xi^4). \tag{123}$$

Therefore, $H_2(t)$ can be regarded as generating functional for quadratic terms $B_{k,2}$ (leading derivative terms) in all HMDS-coefficients B_k . It plays also a very important role in investigating the nonlocal structure of the effective action in quantum field theory in high-energy approximation [2].

Let us note also that the function B_q introduced in the Sect. 1 can be obtained just by analytical continuation of the formula for B_k from integer points k to a complex plane q.

4.4 Covariantly constant background

Let us consider now the opposite case, when the curvatures are strong but slowly varying, i.e. the powers of the curvatures are more important than the derivatives of them. The main terms in this approximation are the terms without any covariant derivatives of the curvatures, i.e. the lowest order jets. We will consider mostly the zeroth order of this approximation which corresponds simply to covariantly constant background curvatures

$$\nabla \text{Riem} = 0, \qquad \nabla \mathcal{R} = 0, \qquad \nabla Q = 0.$$
 (124)

The asymptotic expansion of the trace of the heat kernel

$$\operatorname{Tr}_{L^2} \exp(-tF) \sim (4\pi t)^{-d/2} \sum_{k>0} \frac{(-t)^k}{k!} B_k.$$
 (125)

determines then *all* the terms without covariant derivatives (highest order terms in the jets), $B_{k,k}$, in all HMDS-coefficients B_k . These terms do not contain any covariant derivatives and are just polynomials in the curvatures and the endomorphism Q. Thus the trace of the heat kernel is a generating functional for all HMDS-coefficients for a covariantly constant background, in particular, for all symmetric spaces. Thus the problem is to calculate the trace of the heat kernel for covariantly constant background.

4.4.1 Algebraic approach

There exist a very elegant indirect way to construct the heat kernel without solving the heat equation but using only the commutation relations of some covariant first order differential operators [3, 4, 5, 6]. The main idea is in a generalization of the usual Fourier transform to the case of operators and consists in the following.

Let us consider for a moment a trivial case, where the curvatures vanish but not the potential term:

Riem = 0,
$$\mathcal{R} = 0$$
, $\nabla Q = 0$. (126)

In this case the operators of covariant derivatives obviously commute and form together with the potential term an Abelian Lie algebra

$$[\nabla_{\mu}, \nabla_{\nu}] = 0, \qquad [\nabla_{\mu}, Q] = 0. \tag{127}$$

It is easy to show that the heat semigroup operator can be presented in the form

$$\exp(-tF) = (4\pi t)^{-d/2} \exp(-tQ) \int_{\mathbb{R}^d} d\text{vol}(k) \exp\left(-\frac{1}{4t} < k, gk > +k \cdot \nabla\right), \tag{128}$$

where $\langle k, gk \rangle = k^{\mu}g_{\mu\nu}k^{\nu}$, $k \cdot \nabla = k^{\mu}\nabla_{\mu}$. Here, of course, it is assumed that the covariant derivatives commute also with the metric

$$[\nabla, g] = 0. \tag{129}$$

Acting with this operator on the Dirac distribution and using the obvious relation

$$\exp(k \cdot \nabla)\delta(x, x')|_{x=x'} = \delta(k), \tag{130}$$

one integrates easily over k and obtains the trace of the heat kernel

$$\operatorname{Tr}_{L^{2}} \exp(-tF) = (4\pi t)^{-d/2} \int_{M} d\operatorname{vol}(x) \operatorname{tr}_{V} \exp(-tQ). \tag{131}$$

In fact, the covariant differential operators ∇ do not commute, their commutators being proportional to the curvatures \Re . The commutators of covariant derivatives ∇ with the curvatures \Re give the first derivatives of the curvatures, i.e. the jets $\Re_{(1)}$, the commutators of covariant derivatives with $\Re_{(1)}$ give the second jets $\Re_{(2)}$, etc. Thus the operators ∇ together with the whole set of the jets \mathcal{J} form an *infinite* dimensional Lie algebra $\mathcal{G} = {\nabla, \Re_{(i)}; (i = 1, 2, ...)}$.

To evaluate the heat kernel in the considered (low-energy) approximation one can take into account a *finite* number of low-order jets, i.e. the low-order covariant derivatives of the background fields, $\{\Re_{(i)}; (i \leq N)\}$, and neglect all the higher order jets, i.e. the covariant derivatives of higher orders, i.e. put $\Re_{(i)} = 0$ for i > N. Then one can show that there exist a set of covariant differential operators that together with the background fields and their low-order derivatives generate a *finite* dimensional Lie algebra $\mathcal{G}_N = \{\nabla, \Re_{(i)}; (i = 1, 2, ..., N)\}$ [7, 8, 9].

Thus one can try to generalize the above idea in such a way that (128) would be the zeroth approximation in the commutators of the covariant derivatives, i.e. in the curvatures. Roughly speaking, we are going to find a representation of the heat semigroup operator in the form

$$\exp(-tF) = \int_{\mathbb{R}^D} dk \,\Phi(t,k) \exp\left(-\frac{1}{4t} < k, \Psi(t)k > +k \cdot T\right) \tag{132}$$

where $\langle k, \Psi(t)k \rangle = k^A \Psi_{AB}(t) k^B$, $k \cdot T = k^A T_A$, (A = 1, 2, ..., D), $T_A = X_A^{\mu} \nabla_{\mu} + Y_A$ are some first order differential operators and the functions $\Psi(t)$ and $\Phi(t, k)$ are expressed in terms of commutators of these operators— i.e., in terms of the curvatures.

In general, the operators T_A do not form a closed finite dimensional algebra because at each step taking more commutators there appear more and more derivatives of the curvatures. It is the *low-energy reduction* $\mathcal{G} \to \mathcal{G}_N$, i.e. the restriction to the low-order jets, that actually closes the algebra \mathcal{G} of the operators T_A and the background jets, i.e. makes it finite dimensional.

Using this representation one can, as above, act with $\exp(k \cdot T)$ on the Dirac distribution to get the heat kernel. The main point of this idea is that it is much easier to calculate the action of the exponential of the *first* order operator $k \cdot T$ on the Dirac distribution than that of the exponential of the second order operator \square .

4.4.2 Covariantly constant bundle curvature and covariantly constant endomorphsim Q in flat space

Let us consider now the more complicated case of nontrivial covariantly constant curvature of the connection on the vector bundle V in flat space:

Riem = 0,
$$\nabla \mathcal{R} = 0$$
, $\nabla Q = 0$. (133)

Using the condition of covariant constancy of the curvatures one can show that in this case the covariant derivatives form a *nilpotent* Lie algebra [3]

$$[\nabla_{\mu}, \nabla_{\nu}] = \mathcal{R}_{\mu\nu},\tag{134}$$

$$[\nabla_{\mu}, \mathcal{R}_{\alpha\beta}] = [\nabla_{\mu}, Q] = 0, \tag{135}$$

$$[\mathcal{R}_{\mu\nu}, \mathcal{R}_{\alpha\beta}] = [\mathcal{R}_{\mu\nu}, Q] = 0. \tag{136}$$

For this algebra one can prove a theorem expressing the heat semigroup operator in terms of an average over the corresponding Lie group [3]

$$\exp(-tF) = (4\pi t)^{-d/2} \exp(-tQ) \det_{\text{End}(TM)}^{1/2} \left(\frac{t\mathcal{R}}{\sinh(t\mathcal{R})}\right)$$
(137)

$$\times \int_{\mathbb{R}^d} d\text{vol}(k) \exp\left(-\frac{1}{4t} < k, gt\mathcal{R}\coth(t\mathcal{R})k > +k \cdot \nabla\right), \quad (138)$$

where $k \cdot \nabla = k^{\mu} \nabla_{\mu}$. Here functions of the curvatures \mathcal{R} are understood as functions of sections of the bundle End $(TM) \otimes \text{End}(V)$, and the determinant $\det_{\text{End}(TM)}$ is taken with respect to End (TM) indices, End (V) indices being intact.

It is not difficult to show that also in this case we have

$$\exp(k \cdot \nabla)\delta(x, x')|_{x=x'} = \delta(k). \tag{139}$$

Subsequently, the integral over k^{μ} becomes trivial and we obtain immediately the trace of the heat kernel [3]

$$\operatorname{Tr}_{L^{2}} \exp(-tF) = (4\pi t)^{-d/2} \int_{M} d\operatorname{vol}(x) \operatorname{tr}_{V} \exp(-tQ) \det^{1/2}_{\operatorname{End}(TM)} \left(\frac{t\mathcal{R}}{\sinh(t\mathcal{R})} \right). \tag{140}$$

Expanding it in a power series in t one can find all covariantly constant terms in all HMDS-coefficients B_k .

As we have seen the contribution of the bundle curvature $\mathcal{R}_{\mu\nu}$ is not as trivial as that of the potential term. However, the algebraic approach does work in this case too. It is a good example how one can get the heat kernel without solving any differential equations but using only the algebraic properties of the covariant derivatives.

4.4.3 Contribution of two first derivatives of the endomorphism Q

In fact, in flat space it is possible to do a bit more, i.e. to calculate the contribution of the first and the second derivatives of the potential term Q [4]. That is we consider the case when the derivatives of the endomorphism Q vanish only starting from the *third* order, i.e.

Riem = 0,
$$\nabla \mathcal{R} = 0$$
, $\nabla \nabla \nabla Q = 0$. (141)

Besides we assume the background to be *Abelian*, i.e. all the nonvanishing background quantities, $\mathcal{R}_{\alpha\beta}$, Q, $Q_{;\mu} \equiv \nabla_{\mu}Q$ and $Q_{;\nu\mu} \equiv \nabla_{\nu}\nabla_{\mu}Q$, commute with each other. Thus we have again a nilpotent Lie algebra

$$[\nabla_{\mu}, \nabla_{\nu}] = \mathcal{R}_{\mu\nu} \tag{142}$$

$$[\nabla_{\mu}, Q] = Q_{:\mu} \tag{143}$$

$$[\nabla_{\mu}, Q_{;\nu}] = Q_{;\nu\mu} \tag{144}$$

all other commutators being zero.

Now, let us represent the endomorphism Q in the form

$$Q = \Omega - \alpha^{ik} N_i N_k, \tag{145}$$

where $(i = 1, ..., q; q \le d)$, α^{ik} is some constant symmetric nondegenerate $q \times q$ matrix, Ω is a covariantly constant endomorphism and N_i are some endomorphisms with vanishing second covariant derivative:

$$\nabla \Omega = 0, \qquad \nabla \nabla N_i = 0. \tag{146}$$

Next, let us introduce the operators $X_A = (\nabla_{\mu}, N_i), (A = 1, \dots, d + q)$ and the matrix

$$(\mathcal{F}_{AB}) = \begin{pmatrix} \mathcal{R}_{\mu\nu} & N_{i;\mu} \\ -N_{k;\nu} & 0 \end{pmatrix}, \tag{147}$$

with $N_{i;\mu} \equiv \nabla_{\mu} N_i$.

The operator F can now be written in the form

$$F = -G^{AB}X_AX_B + \Omega, (148)$$

where

$$(G^{AB}) = \begin{pmatrix} g^{\mu\nu} & 0\\ 0 & \alpha^{ik} \end{pmatrix}. \tag{149}$$

and the commutation relations (144) take a more compact form

$$[X_A, X_B] = \mathcal{F}_{AB} \tag{150}$$

all other commutators being zero.

This algebra is again a nilpotent Lie algebra. Thus one can apply the previous theorem in this case too to get [4]

$$\exp(-tF) = (4\pi t)^{-(d+q)/2} \exp(-t\Omega) \det^{1/2} \left(\frac{t\mathcal{F}}{\sinh(t\mathcal{F})} \right)$$

$$\times \int_{\mathbb{R}^{d+q}} dk G^{1/2} \exp\left(-\frac{1}{4t} < k, Gt\mathcal{F}\coth(t\mathcal{F})k > +k \cdot X \right), \quad (151)$$

where $G = \det G_{AB}$ and $k \cdot X = k^A X_A$.

Thus we have expressed the heat semigroup operator in terms of the operator $\exp(k \cdot X)$. The integration over k is Gaussian except for the noncommutative part. Splitting the integration variables $(k^A) = (q^\mu, \omega^i)$ and using the Campbell-Hausdorf formula we obtain [4]

$$\exp(k \cdot X)\delta(x, x')\Big|_{x=x'} = \exp(\omega \cdot N)\delta(q), \tag{152}$$

where $\omega \cdot N = \omega^i N_i$.

Further, after taking off the trivial integration over q and a Gaussian integral over ω , we obtain the trace of the heat kernel in a very simple form [4]

$$\operatorname{Tr}_{L^{2}} \exp(-tF) = (4\pi t)^{-d/2} \int_{M} d\operatorname{vol}(x) \operatorname{tr}_{V} \Phi(t) \exp\left[-tQ + \frac{1}{4}t^{3} < \nabla Q, \Psi(t)g^{-1} \nabla Q > \right],$$
(153)

where $\langle \nabla Q, \Psi(t)g^{-1}\nabla Q \rangle = \nabla_{\mu}Q\Psi^{\mu}_{\nu}(t)g^{\nu\lambda}\nabla_{\lambda}Q$,

$$\Phi(t) = \det_{\text{End}(TM)}^{-1/2} K(t) \det_{\text{End}(TM)}^{-1/2} \left\{ 1 + t^2 [E(t) - S(t)K^{-1}(t)S(t)]P \right\}$$

$$\times \det_{\text{End}(TM)}^{-1/2} \left[1 + t^2 C(t)P \right], \qquad (154)$$

$$\Psi(t) = \{\Psi^{\mu}_{\nu}(t)\} = \left[1 + t^2 C(t)P\right]^{-1} C(t), \tag{155}$$

P is the matrix determined by second derivatives of the potential term,

$$P = \{P^{\mu}_{\nu}\}, \qquad P^{\mu}_{\nu} = \frac{1}{2}g^{\mu\lambda}\nabla_{\nu}\nabla_{\lambda}Q, \tag{156}$$

and the matrices $C(t) = \{C^{\mu}_{\ \nu}(t)\},\ K(t) = \{K^{\mu}_{\ \nu}(t)\}\ S(t) = \{S^{\mu}_{\ \nu}(t)\}\$ and $E(t) = \{E^{\mu}_{\ \nu}(t)\}$ are defined by

$$C(t) = \oint_C \frac{dz}{2\pi i} t \coth(tz^{-1}) (1 - z\mathcal{R} - z^2 P)^{-1}, \tag{157}$$

$$K(t) = \oint_C \frac{dz}{2\pi i} \frac{t}{z^2} \sinh(tz^{-1}) (1 - z\mathcal{R} - z^2 P)^{-1}, \tag{158}$$

$$S(t) = \oint_C \frac{dz}{2\pi i} \frac{t}{z} \sinh(tz^{-1}) (1 - z\mathcal{R} - z^2 P)^{-1}, \tag{159}$$

$$E(t) = \oint_C \frac{dz}{2\pi i} t \sinh(tz^{-1}) (1 - z\mathcal{R} - z^2 P)^{-1}, \tag{160}$$

where the integral is taken along a sufficiently small closed contour C that encircles the origin counter-clockwise, so that $F(z) = (1 - zR - z^2P)^{-1}$ is analytic inside this contour.

The formula (153) exhibits the general structure of the trace of the heat kernel. Namely, one sees immediately how the endomorphism Q and its first derivatives ∇Q enter the result. The nontrivial information is contained only in a scalar, $\Phi(t)$, and a tensor, $\Psi_{\mu\nu}(t)$, functions which are constructed purely from the curvature $\mathcal{R}_{\mu\nu}$ and the second derivatives of the endomorphism Q, $\nabla\nabla Q$.

So, we conclude that the HMDS-coefficients B_k are constructed from three different types of scalar (connected) blocks, Q, $\Phi_{(n)}(\mathcal{R}, \nabla \nabla Q)$ and $\nabla_{\mu} Q \Psi^{\mu\nu}_{(n)}(\mathcal{R}, \nabla \nabla Q) \nabla_{\nu} Q$. They are listed explicitly up to B_8 in [7].

4.4.4 Symmetric spaces

Let us now generalize the algebraic approach to the case of the *curved* manifolds with covariantly constant Riemann curvature and the trivial bundle connection [5, 6]

$$\nabla \text{Riem} = 0, \qquad \mathcal{R} = 0, \qquad \nabla Q = 0.$$
 (161)

First of all, we give some definitions. The condition (161) defines, as we already said above, the geometry of locally symmetric spaces. A Riemannian locally symmetric space which is simply connected and complete is globally symmetric space (or, simply, symmetric space). A symmetric space is said to be of compact, noncompact or Euclidean type if all sectional curvatures $K(u, v) = R_{abcd}u^av^bu^cv^d$ are positive, negative or zero. A direct product of symmetric spaces of compact and noncompact types is called semisimple symmetric space. A generic complete simply connected Riemannian symmetric space is a direct product of a flat space and a semisimple symmetric space.

It should be noted that our analysis in this paper is purely *local*. We are looking for a *universal* local function of the curvature invariants, that reproduces adequately the asymptotic expansion of the trace of the heat kernel. This function should give *all* the terms without covariant derivatives of the curvature in the asymptotic expansion of the heat kernel, i.e. in other words *all* HMDS-coefficients B_k for *any* locally symmetric space.

It is well known that the HMDS-coefficients have a *universal* structure, i.e. they are polynomials in the background jets (just in curvatures in case of symmetric spaces) with the numerical coefficients that do not depend on the global properties of the manifold, on the dimension, on the signature of the metric etc. It is this universal structure we are going to study.

It is obvious that any flat subspaces do not contribute to the HMDS-coefficients B_k . Therefore, to find this universal structure it is sufficient to consider only semisimple symmetric spaces. Moreover, since HMDS-coefficients are analytic in the curvatures, one can restrict oneself only to symmetric spaces of *compact* type. Using the factorization property of the heat kernel and the duality between compact and noncompact symmetric spaces one can obtain then the results for the general case by analytical continuation.

That is why we consider only the case of *compact* symmetric spaces when the sectional curvatures and the metric are *positive* definite.

Let e_a be a basis in the tangent bundle which is *covariantly constant (parallel)* along the geodesic. The frame components of the curvature tensor of a symmetric space are, obviously, constant and can be presented in the form

$$R_{abcd} = \beta_{ik} E^i_{ab} E^k_{cd}, \tag{162}$$

where E_{ab}^i , $(i = 1, ..., p; p \le d(d-1)/2)$, is some set of antisymmetric matrices and β_{ik} is some symmetric nondegenerate $p \times p$ matrix. The traceless matrices $D_i = \{D_{ib}^a\}$ defined by

$$D^{a}_{ib} = -\beta_{ik} E^{k}_{cb} g^{ca} = -D^{a}_{bi} \tag{163}$$

are known to be the generators of the holonomy algebra \mathcal{H}

$$[D_i, D_k] = F^j_{ik} D_j, \tag{164}$$

where F^{j}_{ik} are the structure constants.

In symmetric spaces a much richer algebraic structure exists. Indeed, let us define the quantities $C_{BC}^{A} = -C_{CB}^{A}$, (A = 1, ..., D; D = d + p):

$$C^{i}_{ab} = E^{i}_{ab}, \quad C^{a}_{ib} = D^{a}_{ib}, \quad C^{i}_{kl} = F^{i}_{kl},$$
 (165)

$$C^{a}_{bc} = C^{i}_{ka} = C^{a}_{ik} = 0, (166)$$

and the matrices $C_A = \{C_{AC}^B\} = (C_a, C_i)$:

$$C_a = \begin{pmatrix} 0 & D^b_{ai} \\ E^j_{ac} & 0 \end{pmatrix}, \qquad C_i = \begin{pmatrix} D^b_{ia} & 0 \\ 0 & F^j_{ik} \end{pmatrix}. \tag{167}$$

One can show that they satisfy the Jacobi identities [5, 6]

$$[C_A, C_B] = C_{AB}^C C_C \tag{168}$$

and, hence, define a Lie algebra \mathcal{G} of dimension D with the structure constants C_{BC}^A , the matrices C_A being the generators of adjoint representation.

In symmetric spaces one can find explicitly the generators of the infinitesimal isometries, i.e. the Killing vector fields ξ_A , and show that they form a Lie algebra of isometries that is (in case of semisimple symmetric space) isomorphic to the Lie algebra \mathcal{G} , viz.

$$[\xi_A, \xi_B] = C^C_{AB} \xi_C. \tag{169}$$

Moreover, introducing a symmetric nondegenerate $D \times D$ matrix

$$\gamma_{AB} = \begin{pmatrix} g_{ab} & 0\\ 0 & \beta_{ik} \end{pmatrix}, \tag{170}$$

that plays the role of the metric on the algebra \mathcal{G} , one can express the operator F in semisimple symmetric spaces in terms of the generators of isometries

$$F = -\gamma^{AB}\xi_A\xi_B + Q,\tag{171}$$

where $\gamma^{AB} = (\gamma_{AB})^{-1}$.

Using this representation one can prove a theorem that presents the heat semigroup operator in terms of some average over the group of isometries G [5, 6]

$$\exp(-tF) = (4\pi t)^{-D/2} \exp\left[-t\left(Q - \frac{1}{6}R_G\right)\right]$$
 (172)

$$\times \int_{\mathbb{R}^D} dk \gamma^{1/2} \det {}^{1/2}_{\mathrm{Ad}(\mathcal{G})} \left(\frac{\sinh \left(k \cdot C/2 \right)}{k \cdot C/2} \right) \exp \left(-\frac{1}{4t} < k, \gamma k > + k \cdot \xi \right)$$

where $\gamma = \det \gamma_{AB}$, $k \cdot C = k^A C_A$, $k \cdot \xi = k^A \xi_A$, and R_G is the scalar curvature of the group of isometries G

$$R_G = -\frac{1}{4} \gamma^{AB} C^C_{\ AD} C^D_{\ BC}. \tag{173}$$

Acting with this operator on the Dirac distribution $\delta(x, x')$ one can, in principle, evaluate the off-diagonal heat kernel $\exp(-tF)\delta(x, x')$, i.e. for non-coinciding points $x \neq x'$ (see [6]). To calculate the trace of the heat kernel, it is sufficient to compute only the coincidence limit x = x'. Splitting the integration variables $k^A = (q^a, \omega^i)$ and solving the equations of characteristics one can obtain the action of the isometries on the Dirac distribution [5, 6]

$$\exp(k \cdot \xi) \, \delta(x, x') \Big|_{x=x'} = \det_{\text{End}(TM)}^{-1} \left(\frac{\sinh(\omega \cdot D/2)}{\omega \cdot D/2} \right) \delta(q). \tag{174}$$

where $\omega \cdot D = \omega^i D_i$.

Using this result one can easily integrate over q in (172) to get the heat kernel diagonal. After changing the integration variables $\omega \to \sqrt{t}\omega$ it takes the form [5, 6]

$$[U(t)] = (4\pi t)^{-d/2} \exp\left[-t\left(Q - \frac{1}{8}R - \frac{1}{6}R_H\right)\right]$$

$$\times (4\pi)^{-p/2} \int_{\mathbb{R}^p} d\omega \, \beta^{1/2} \exp\left(-\frac{1}{4} < \omega, \beta\omega > \right)$$

$$\times \det^{1/2}_{\mathrm{Ad}(\mathcal{H})} \left(\frac{\sinh\left(\sqrt{t}\omega \cdot F/2\right)}{\sqrt{t}\omega \cdot F/2}\right) \det^{-1/2}_{\mathrm{End}(TM)} \left(\frac{\sinh\left(\sqrt{t}\omega \cdot D/2\right)}{\sqrt{t}\omega \cdot D/2}\right), \quad (175)$$

where $\omega \cdot F = \omega^i F_i$, $F_i = \{F^j_{ik}\}$ are the generators of the holonomy algebra \mathcal{H} in adjoint representation and

$$R_H = -\frac{1}{4}\beta^{ik} F^m_{il} F^l_{km} \tag{176}$$

is the scalar curvature of the holonomy group.

The remaining integration over ω in (175) can be done in a rather formal way [8, 9]. Let a_i^* and a_k be operators acting on a Hilbert space, that form the following Lie algebra

$$[a^j, a_k^*] = \delta_k^j, \tag{177}$$

$$[a^i, a^k] = [a_i^*, a_k^*] = 0. (178)$$

Let $|0\rangle$ be the 'vacuum vector' in the Hilbert space, i.e.

$$<0|0>=1,$$
 (179)

$$a^{i}|0> = 0, \qquad <0|a_{k}^{*} = 0.$$
 (180)

Then the heat kernel (175) can be presented in an formal algebraic form without any integration

$$[U(t)] = (4\pi t)^{-d/2} \exp\left[-t\left(Q - \frac{1}{8}R - \frac{1}{6}R_H\right)\right]$$

$$\times \left\langle 0 \left| \det^{1/2}_{\operatorname{Ad}(\mathcal{H})} \left(\frac{\sinh\left(\sqrt{t}a \cdot F/2\right)}{\sqrt{t}a \cdot F/2}\right) \det^{-1/2}_{\operatorname{End}(TM)} \left(\frac{\sinh\left(\sqrt{t}a \cdot D/2\right)}{\sqrt{t}a \cdot D/2}\right)\right.\right.$$

$$\times \exp\left(\langle a^*, \beta^{-1}a^* \rangle\right) \left| 0 \right\rangle. \tag{181}$$

where $a \cdot F = a^k F_k$ and $a \cdot D = a^k D_k$. This formal solution should be understood as a power series in the operators a^k and a_k^* and determines a well defined asymptotic expansion in $t \to 0$.

Let us stress that these formulas are manifestly covariant because they are expressed in terms of the invariants of the holonomy group H, i.e. the invariants of the Riemann curvature tensor. They can be used now to generate all HMDS-coefficients $[b_k]$ for any locally symmetric space, i.e. for any manifold with covariantly constant curvature, simply by expanding it in an asymptotic power series as $t \to 0$. Thereby one finds all covariantly constant terms in all HMDS-coefficients in a manifestly covariant way. This gives a very nontrivial example how the heat kernel can be constructed using only the Lie algebra of isometries of the symmetric space.

5 Conclusion

In present paper we have presented recent results in studying the heat kernel obtained in our papers [1, 2, 3, 4, 5, 6]. We discussed some ideas connected with the problem of developing consistent covariant approximation schemes for calculating the heat kernel. Especial attention is payed to the low-energy approximation. It is shown that in the local analysis there exists an algebraic structure (the Lie algebra of background jets) that turns out to be extremely useful for the study of the low-energy approximation. Based on the background jets algebra we have proposed a new promising approach for calculating the low-energy heat kernel.

Within this framework we have obtained closed formulas for the heat kernel diagonal in the case of covariantly constant background. Besides, we were able to take into account the first and second derivatives of the endomorphism Q in flat space. The obtained formulas are manifestly covariant and applicable for a generic covariantly constant background. This enables to treat the results as the generating functions for the whole set of

the Hadamard-Minakshisundaram-De Witt-Seeley-coefficients. In other words, we have calculated *all* covariantly constant terms in *all* HMDS-coefficients.

Needless to say that the investigation of the low-energy effective action is of great importance in quantum gravity and gauge theories because it describes the dynamics of the vacuum state of the theory.

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