Gravity-Matter Systems in Asymptotically Safe Quantum Gravity

Mathieu Kaltschmidt

Institute for Theoretical Physics
Heidelberg University

Bachelor Thesis Defence
July 18$^{\text{th}}$ 2019

Contents

General Relativity

Understanding of gravity is based on the concept of **curved spacetime**.

- Metric tensor, Riemann tensor + contractions

$$g_{\mu\nu}, \qquad R^{\alpha}{}_{\beta\gamma\delta}, \qquad R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}, \qquad \mathcal{R} = R^{\mu}{}_{\mu} \tag{1}$$

- Einstein-Hilbert action:

$$\mathcal{S}_{\text{EH}} = \frac{1}{16\pi G} \int_x \sqrt{g}\,(\mathcal{R} - 2\Lambda) + \text{matter} \tag{2}$$

- Equations of motion: Einstein's equations

$$\frac{1}{8\pi G}\left[G_{\mu\nu} + \Lambda g_{\mu\nu}\right] = T_{\mu\nu} \tag{3}$$

But is there a *quantum* theory of gravity?

A Path Integral for Gravity?

Naive approach: quantizing gravity via the path integral formalism.

- Introduce linear split in metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sqrt{G}\, h_{\mu\nu} \tag{4}$$

- Path integral representation of quantum gravity:

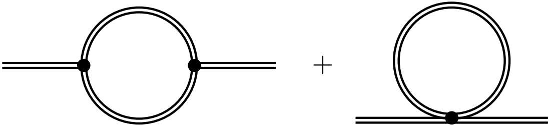
$$Z[J;\bar{g}] = \int_\Phi \mathrm{e}^{-\mathcal{S}_{\mathrm{grav}}\left[\bar{g}_{\mu\nu},\Phi\right] + \int_x \sqrt{\bar{g}}\, J\cdot\Phi}, \qquad \Phi = \left(h_{\mu\nu}, c_\mu, \bar{c}_{\mu\nu}\right) \tag{5}$$

- Analysis of canonical mass dimensions:

$$[G] = \left[\mathrm{d}^d x \sqrt{g}\, \mathcal{R}\right] = 2 - d, \qquad [\Lambda] = 2 \tag{6}$$

G **has negative mass dim. in** $d = 4$ **spacetime dimensions.**

Perturbative Non-Renormalizability



$\propto \quad G\,(p^2)^2$

Figure 1: Vacuum polarization diagrams up to 1-loop order.

- **t'Hooft & Veltman:** Theory renormalizable up to 1-loop order.
- **Goroff & Sagnotti:** Non-vanishing counterterms at 2-loop order:

$$S_{\mathrm{GS}} = \frac{1}{\varepsilon}\frac{209}{2880}\frac{1}{(4\pi)^4}\int_x \sqrt{g}\, C_{\mu\nu}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\rho\sigma} C_{\rho\sigma}{}^{\mu\nu} \tag{7}$$

Failure of perturbative quantization!

The Functional Renormalization Group

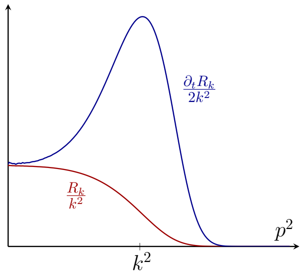


Figure 2: Shape of a typical exponential regulator function $R(p^2)$ and its derivative w. r. t. the RG time $t = -\ln\left(k/\Lambda\right)$.

- **Idea:** Shell-wise momentum integration to solve path integral
- **Realization:** Introducing IR cutoff scale k via **regulator**

The Flow Equation

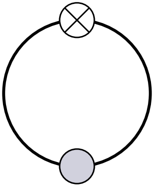
- Interpolation process between $\Gamma_{k\to\infty} \equiv \mathcal{S}$ and $\Gamma_{k\to 0} \equiv \Gamma$ governed by Wetterich's flow equation:

Wetterich equation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{STr}\Big[G_k\,\partial_t R_k\Big], \qquad G_k := \left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}_{ji} \tag{8}$$

- Diagrammatical representation as exact 1-loop equation:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}$$



Asymptotic Safety

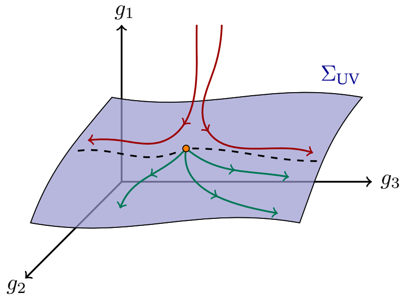


Figure 3: Visualization of a fixed point g^* and its corresponding UV hypersurface Σ_{UV}.

---

1. Existence of an UV-attractive Non-Gaussian Fixed Point.

2. Γ_k fixed by a finite amount of measurements:

$$\dim \Sigma_{\mathrm{UV}} < \infty.$$

Einstein-Hilbert Truncation

- Simple truncation, takes only Λ and $\mathcal{R}$ into account:

$$\Gamma_k = \frac{Z_{h,k}}{16\pi G} \int_x \sqrt{g}\,[-\mathcal{R} + 2\Lambda_k] + \mathcal{S}_{\text{gf}} + \mathcal{S}_{\text{gh}} \tag{9}$$

- Left hand side of the flow equation:

$$\partial_t \Gamma_k = \frac{Z_{h,k}}{16\pi G} \int_x \sqrt{g} \left\{ \eta_h \mathcal{R} + 2\left(k^2 (\partial_t \lambda_k) + \Lambda_k (2 - \eta_h) \right) \right\} \tag{10}$$

- **Idea:** Compute the functional trace on the r. h. s. of the flow equation and compare terms of different orders in $\mathcal{R}$ to obtain the beta functions.

Solving the Flow Equation

First. Background field method and *York decomposition* of the fluctuation field $h_{\mu\nu}$:

$$h_{\mu v} = h^{\mathrm{TT}}_{\mu v} + \bar{\nabla}_\mu \xi_v + \bar{\nabla}_v \xi_\mu + \left(\bar{\nabla}_\mu \bar{\nabla}_v - \frac{1}{d} \bar{g}_{\mu v} \bar{\Delta} \right) \sigma + \frac{1}{d} \bar{g}_{\mu v} h. \tag{11}$$

Then. Proceed as follows:

1. Choose a suitable regulator R_k
2. Determine the full propagator G_k
3. Compute the scale derivative of the chosen regulator $\partial_t R_k$
4. Make use of *heat-kernel techniques*

Beta Functions and Fixed Points

- Beta function for Newtons coupling constant:

$$\beta_g = \partial_t g_k = (2 + \eta_h)\, g_k. \tag{12}$$

- Graviton anomalous dimension:

$$\eta_h = -\frac{5 g_k}{3\pi} \left(\frac{1 - \frac{\eta_h}{4}}{1 - 2\lambda_k} + 2 \frac{1 - \frac{\eta_h}{6}}{(1 - 2\lambda_k)^2} \right). \tag{13}$$

- Beta function for the cosmological constant:

$$\beta_\lambda = \partial_t \lambda_k = -4\lambda_k + \frac{\lambda_k}{g_k} \partial_t g_k + \frac{5}{4\pi} g_k \frac{1 - \frac{\eta_h}{6}}{1 - 2\lambda_k}. \tag{14}$$

Phase Diagram

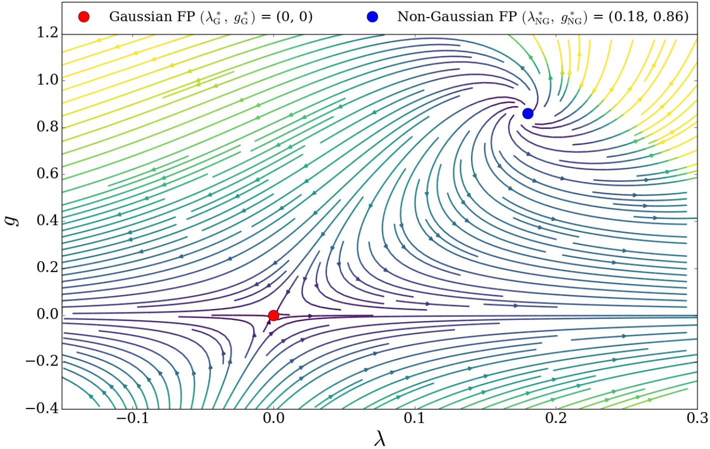


Figure 4: Flow diagram for the Einstein-Hilbert truncation in $h^{\rm TT}$ approximation. The flow points towards the infrared.

Inclusion of the other Graviton modes

- Gauge fixing choice $\beta = 0, \alpha \to 0$ leaves us with:

$$G_{k,hh} = \frac{32\pi}{Z_h} \begin{pmatrix} \frac{1}{\bar{\Delta}[1+r_k]-2\Lambda_k+\frac{2}{3}\mathcal{R}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-\frac{8}{3}}{\bar{\Delta}[1+r_k]-\frac{4}{3}\Lambda_k} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

- **Attention:** New heat-kernel coefficients due to modified Laplacian $\tilde{\Delta} = -\nabla^2 + \mathbf{E}$ occurring in the ghost term!

Gravity-Matter systems

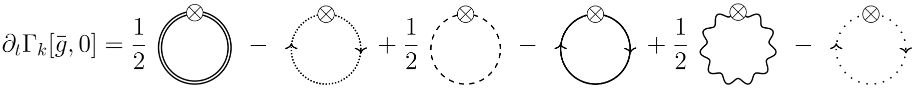


Figure 5: Flow equation (17) for Γ_k including different matter contributions in diagrammatic representation.

- Inclusion of matter straightforward:

$$\Gamma_k = \Gamma_{\text{EH}} + \mathcal{S}_{\text{gf}} + \mathcal{S}_{\text{gh}} + \overbrace{\mathcal{S}_S + \mathcal{S}_D + \mathcal{S}_V}^{=:\,\Gamma_{\text{matter}}} \tag{16}$$

- Resulting flow equation:

$$\begin{aligned}\partial_t\Gamma_k = \frac{1}{2}\,\mathrm{Tr}\Big[G_k\,\partial_t R_k\Big]_{hh} - \mathrm{Tr}\Big[G_k\,\partial_t R_k\Big]_{\bar{c}c} + \frac{1}{2}\,\mathrm{Tr}\Big[G_k\,\partial_t R_k\Big]_{\phi\phi} \\ - \mathrm{Tr}\Big[G_k\,\partial_t R_k\Big]_{\bar{\psi}\psi} + \frac{1}{2}\,\mathrm{Tr}\Big[G_k\,\partial_t R_k\Big]_{AA} - \mathrm{Tr}\Big[G_k\,\partial_t R_k\Big]_{\bar{C}C}\end{aligned} \tag{17}$$

Matter Contributions in BG Approximation

- Scalar contribution:

$$\mathcal{S}_S = \frac{Z_S}{2} \int_x \sqrt{\bar{g}} \sum_{i=1}^{N_S} \phi^i \left(-\bar{\nabla}^2\right) \phi^i + \mathcal{O}(h). \tag{18}$$

- Fermion contribution:

$$\mathcal{S}_D = Z_D \int_x \sqrt{\bar{g}} \sum_{i=1}^{N_D} \bar{\psi}^i \left(i\bar{\not{\nabla}}\right) \psi^i + \mathcal{O}(h). \tag{19}$$

- Gauge Field + Ghost contribution

$$\begin{aligned}\mathcal{S}_V = {} & \frac{Z_V}{2} \int_x \sqrt{\bar{g}} \sum_{i=1}^{N_V} A^i_\lambda \left[-\bar{g}^{\mu\lambda}\bar{\nabla}^2 + \bar{R}^{\mu\lambda}\right] A^i_\mu \\ & + \int_x \sqrt{\bar{g}} \sum_{i=1}^{N_V} \bar{C}_i(-\bar{\nabla}^2)C_i,\end{aligned} \tag{20}$$

Modified Beta-Functions I

- Modified beta function for Newtons coupling constant:

$$\partial_t g_k = 2g_k + \frac{g_k^2}{\pi}\left[\frac{2\left(1-\frac{\eta_h}{4}\right)}{1-2\lambda_k} - \frac{10}{3}\frac{1-\frac{\eta_h}{6}}{\left(1-2\lambda_k\right)^2} + \frac{1}{6}\frac{1-\frac{\eta_h}{4}}{1-\frac{4}{3}\lambda_k} - \frac{10}{3}\left(1-\frac{\eta_c}{4}\right)\right.$$
$$\left. + \frac{N_S}{6}\left(1-\frac{\eta_S}{4}\right) - \frac{5N_D}{12}\left(1-\frac{\eta_D}{2}\right) - 2N_V\left(\frac{1}{3}-\frac{\eta_V}{12}\right)\right] \tag{21}$$

- Graviton anomalous dimension:

$$\eta_h = \frac{g_k}{\pi}\left[\frac{2\left(1-\frac{\eta_h}{4}\right)}{1-2\lambda_k} - \frac{10}{3}\frac{1-\frac{\eta_h}{6}}{\left(1-2\lambda_k\right)^2} + \frac{1}{6}\frac{1-\frac{\eta_h}{4}}{1-\frac{4}{3}\lambda_k} - \frac{10}{3}\left(1-\frac{\eta_c}{4}\right)\right.$$
$$\left. + \frac{N_S}{6}\left(1-\frac{\eta_S}{4}\right) - \frac{5N_D}{12}\left(1-\frac{\eta_D}{2}\right) - 2N_V\left(\frac{1}{3}-\frac{\eta_V}{12}\right)\right] \tag{22}$$

Modified Beta-Functions II

- Modified beta function for the cosmological constant:

$$\partial_t \lambda_k = (\eta_h - 2)\,\lambda_k + \frac{g_k}{4\pi}\left[\left(\frac{5\left(1-\frac{\eta_h}{6}\right)}{1-2\lambda_k}\right) + \left(\frac{5\left(1-\frac{\eta_h}{6}\right)}{1-\frac{4}{3}\lambda_k}\right) - 8\left(1-\frac{\eta_c}{6}\right)\right.$$

$$\left. + N_S\left(1-\frac{\eta_S}{6}\right) - 4N_D\left(1-\frac{\eta_D}{3}\right) - N_V\left(1-\frac{\eta_V}{6}\right)\right] \tag{23}$$

- Perturbative Approximation:

$$\begin{aligned} g_k^* &= \frac{-12\pi}{N_S - \frac{5}{2}N_D - 4N_V - 27} \\ \lambda_k^* &= -\frac{3}{4}\,\frac{N_S - 4N_D - N_V + 2}{N_S - \frac{5}{2}N_D - 4N_V - 27} \end{aligned} \tag{24}$$

Impact on the NGFP: Perturbative Approximation I

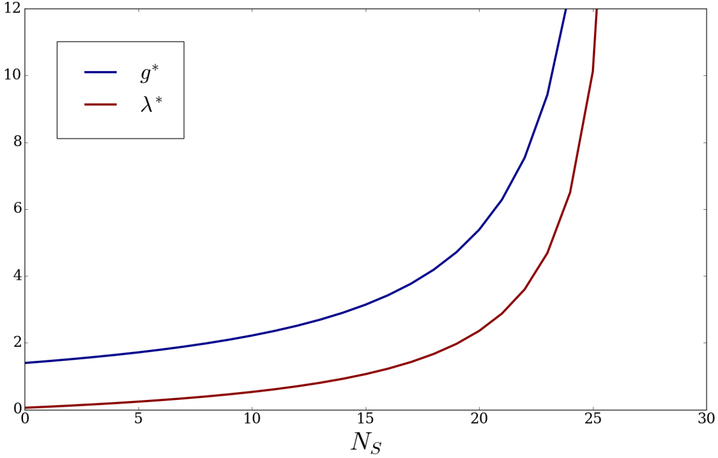


Figure 6: Impact of scalar fields on the NGFP.

Impact on the NGFP: Perturbative Approximation II

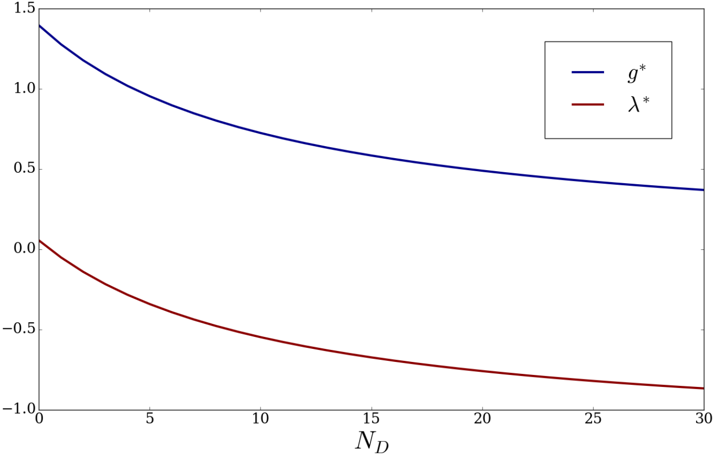


Figure 7: Impact of fermionic fields on the NGFP.

Impact on the NGFP: Perturbative Approximation III

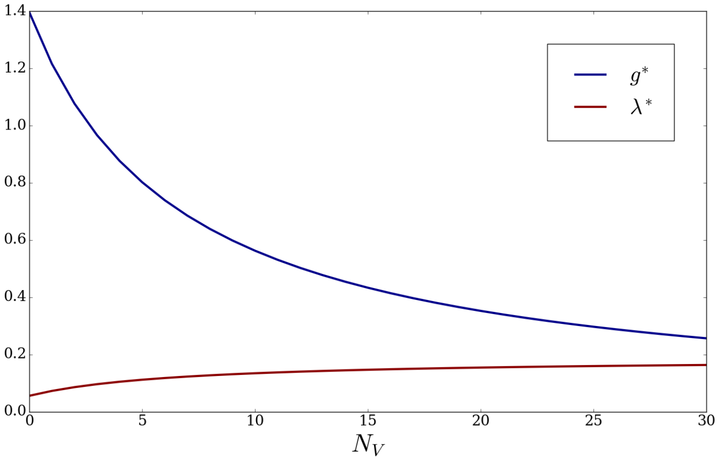


Figure 8: Impact of gauge fields on the NGFP.

Conclusion

- **Pure-Gravity:** We found a NGFP at $(g_k, \lambda_k) = (0.86, 0.18)$.
- **Gravity-Matter:** We analyzed the impact of different matter fields on the fixed point.
  Scalars tend to increase the fixed point values drastically after surpassing a certain amount, **fermions** decrease the values slightly and **gauge fields** leave the value of λ^* almost unchanged, whereas the value of g^* decreases rather strict.
- **Computation:** Background field approximation has to be treated with care! Violates *Nielsen Identities* and therefore *background independence.*
- **Outlook:** This work provides a suitable framework for further investigations. Next steps could involve inclusion of higher-cuvature terms or the choice of different regulators.

References

[1] Nicolai Christiansen et al. "Asymptotic safety of gravity with matter". In: *Phys. Rev.* D97.10 (2018), p. 106012. arXiv: 1710.04669 [hep-th].

[2] Pietro Donà, Astrid Eichhorn, and Roberto Percacci. "Matter matters in asymptotically safe quantum gravity". In: *Phys. Rev.* D89.8 (2014), p. 084035. arXiv: 1311.2898 [hep-th].

[3] Marc H. Goroff and Augusto Sagnotti. "Quantum Gravity at Two Loops". In: *Phys. Lett.* 160B (1985), pp. 81–86.

[4] Jan Meibohm, Jan M. Pawlowski, and Manuel Reichert. "Asymptotic safety of gravity-matter systems". In: *Phys. Rev.* D93.8 (2016), p. 084035. arXiv: 1510.07018 [hep-th].

[5] Jan. M. Pawlowski et al. *The Functional Renormalization Group & applications to gauge theories and gravity*. Lecture Notes (Access currently restricted to students). Heidelberg University. 2019.

[6] M. Reuter. "Nonperturbative evolution equation for quantum gravity". In: *Phys. Rev.* D57 (1998), pp. 971–985. arXiv: hep-th/9605030 [hep-th].

[7] Martin Reuter and Frank Saueressig. "Quantum Einstein Gravity". In: *New J. Phys.* 14 (2012), p. 055022. arXiv: 1202.2274 [hep-th].

[8] Gerard 't Hooft and M. J. G. Veltman. "One loop divergencies in the theory of gravitation". In: *Ann. Inst. H. Poincare Phys. Theor.* A20 (1974), pp. 69–94.

[9] Steven Weinberg. "Ultraviolet Divergences in Quantum Theories of Gravitation". In: *General Relativity: An Einstein Centenary Survey*. 1980, pp. 790–831.

[10] Christof Wetterich. "Exact evolution equation for the effective potential". In: *Phys. Lett.* B301 (1993), pp. 90–94. arXiv: 1710.05815 [hep-th].

[11] Kenneth G. Wilson. "Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture". In: *Phys. Rev. B* 4 (9 1971), pp. 3174–3183.