QUANTUM GRAVITY AT TWO LOOPS

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We show that the S matrix of pure Einstein gravity diverges at the two-loop order in four dimensions

Ultraviolet divergences have plagued quantum field theory since its early days. Quantum electrodynamics, the first theory to be studied in detail, was promptly recognized to have a divergent perturbation expansion. Nonetheless, the expansion is in this case *renormalizable*, and thus predictive, provided one restricts the dimensionality of space–time to be at most four. The divergences can be absorbed into the definitions of only two parameters, the electron mass and the electron charge, whose values can be extracted from experiment. The need for infinite renormalizations clearly leaves much to be desired, and has long been regarded by many physicists as at most a temporary remedy. Nonetheless, renormalizability is one of the main principles that underlie the construction of models based on Yang–Mills theory, which describe essentially all of particle physics as we know it today.

This rather spectacular success, however, is incomplete in two respects. The criterion of renormalizability is not sufficient to select a unique theory, and gravitational interactions as described by Einstein's theory do not allow this kind of treatment. Renormalizability is a consequence of gauge invariance and of the mass dimensionality of the couplings, which cannot be negative (for example, both the electron mass and the electron charge have non-negative dimension). On the other hand, the gravitational coupling is characterized by Newton's constant, which for three or more space–time dimensions, has the dimension of a negative power of mass. The ultraviolet behavior of Einstein's theory of gravity is thus more singular than that of renormalizable theories. Indeed, elementary power-counting implies that, were divergences to be present, they would correspond to *local* invariants of dimensionalities increasing with each order of the perturbation expansion. A predictive perturbation expansion for Einstein's theory would require that, order by order, ultraviolet divergences cancel out of its S matrix. Thus finiteness, rather than renormalizability, would appear to be the proper guiding principle in the search for a unified theory of all

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interactions, including gravity. It is encouraging that cancellations of divergences do take place in four dimensions at one loop, for Einstein gravity not coupled to any matter. However, the cancellation responsible for the one-loop finiteness is due to an accident peculiar to four dimensions [1], and it is not clear what this implies for higher orders. Only in this case the quantity

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma}{}^{\gamma\delta} = -4R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + 16R^{\mu\nu}R_{\mu\nu} - 4R^2 \tag{1}$$

is a total derivative which, when integrated over all space time, yields the Euler number. Consequently, all the possible one-loop divergences of pure gravity in four dimensions, which correspond to the invariants on the RHS of eq. (1), vanish when the field equation $R_{\mu\nu} = 0$ is used. The S matrix is necessarily one-loop finite, and the Green functions can be made finite by a suitable (gauge-dependent) *local* field redefinition. The cancellation is peculiar to four dimensions. For example, in six dimensions the theory diverges at one loop [2].

In ref. [1] it was also pointed out that the coupling of a scalar field to gravity introduces nonrenormalizable ultraviolet divergences at one loop. The same was later shown to hold true for a large number of matter couplings [3], and for some time was believed to be a general rule. It is now well known that the pure supergravity theories (i.e. supergravity theories built out of the gravity supermultiplets alone) are exceptions to this general pattern [4] [‡1]. Their irreducible supersymmetry turns the one-loop finiteness of the S matrix for pure gravity into the one-loop finiteness of the whole S matrix. This is interesting, as it suggests a selection rule for a set of spectra and couplings. However, the ultraviolet behavior of supergravity theories is similar to that of Einstein gravity, as they involve nonrenormalizable interactions, parametrized by Newton's constant. Moreover, it is not easy to connect the gauge theories of the strong, weak and electromagnetic interactions to the pure extended supergravities. On the other hand, abandoning the restriction to pure supergravities has the consequence, within a local field theory of point particles, of introducing ultraviolet divergences at one loop [‡2].

For pure supergravity theories, there are formal arguments that even exclude two-loop divergences [7] in four dimensions. Thus, here the problem first appears at three loops. It has also been suggested [8] that divergences may be excluded for a few more orders on the basis of superspace symmetry arguments. The maximal supergravity theory in four dimensions would then be six-loop finite. However, the result of ref. [8] is obtained with strong assumptions on the (unknown) formulations of the extended supergravity theories in terms of $N > 2$ superfields, basically that they lead to perturbation expansions qualitatively similar to the well-known case of $N = 1$ superfields. With similar assumptions one can also arrive at several other predictions [‡3]. Some of these are more accessible than the three-loop problem in supergravity, and still contain much useful information. In particular, one can study the divergences of $N = 4$ Yang–Mills [‡4] continued to $d > 4$, where the theory is not renormalizable [12].

The result of the analysis in ref. [12] is that the basic element of the power-counting of ref. [8], the postulated form of the on-shell effective action, is *violated* in the two-loop divergences of $N = 4$ Yang–Mills in seven dimensions. Thus, with the assumptions behind the theorem of ref. [8] called into question, one is led to expect that, in four dimensions, divergences appear in supergravity theories starting from three loops. This conclusion rests on the implicit assumption that supergravity theories do not entail any new symmetry principle beyond the manifest realization of a number of supersymmetries in superspace. Similarly, for pure gravity in four dimensions, one would expect that, barring some unknown cancellation mechanism, divergences would appear already at two loops.

‡1 For a review of supergravity see ref [5]

‡2 This was first pointed out in ref [6]

‡3 Among these is an elegant discussion of the finiteness of a number of renormalizable supersymmetric models See refs [8,9]. Recent reviews are given in ref. [10]

‡4 As for the extended ($N > 2$) supergravity theories, the possibility of formulating the $N = 4$ YM theory in terms of $N = 4$ superfields has been called into question See ref [11]

The subject of this work is the analysis of the two-loop divergences of pure Einstein gravity in four dimensions. This is, in many respects, a direct analogue of the more difficult case of extended supergravity at three loops. We find that Einstein gravity does diverge at the two-loop order. This, combined with the results of ref. [12], is a rather compelling indication that four-dimensional supergravity theories diverge as well, starting from the three-loop level.

We now briefly discuss the main points of our work. Further details will be presented elsewhere [13]. The lagrangian of Einstein gravity can be conveniently written, in natural units,

$$L = -2\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}, \tag{2}$$

where the signature for the space–time metric is chosen to be $(-+++)$, and the Ricci tensor is

$$\begin{aligned} R_{\mu\nu} &= \delta^{\sigma}_{\rho} R^{\rho}{}_{\mu\nu\sigma} \\ &= \partial_{\nu}\Gamma^{\rho}{}_{\mu\rho} - \partial_{\rho}\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} - \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\sigma}{}_{\rho\sigma}. \end{aligned} \tag{3}$$

De Witt's background field method [14] is a very convenient tool, as it leads to Green functions which are generally covariant off-shell, and minimizes the number of terms in single diagrams. Thus, we split the metric field into the background metric and the quantum field $h_{\mu\nu}$:

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}. \tag{4}$$

In the De Donder gauge, which leads to the simplest propagator, the gauge-fixing and ghost terms are

$$\begin{aligned} &-\sqrt{-g}\left(h_{\mu}{}^{\nu}{}_{,\nu} h^{\mu\rho}{}_{,\rho} - h^{\mu\rho}{}_{,\rho} h^{\nu}{}_{\nu,\mu} + \tfrac{1}{4} h^{\nu}{}_{\nu,\mu} h^{\rho}{}_{\rho}{}^{,\mu}\right) \\ &\quad + \sqrt{-g}\big(-\bar{c}^{\mu,\nu} c_{\mu,\nu} - R_{\mu\nu}\bar{c}^{\mu} c^{\nu} - \bar{c}^{\mu,\nu} c^{\rho}{}_{,\mu} h_{\nu\rho} \\ &\quad - \bar{c}^{\mu,\nu} c^{\rho}{}_{,\nu} h_{\mu\rho} - \bar{c}^{\mu,\nu} c^{\rho} h_{\mu\nu,\rho} + \bar{c}^{\mu}{}_{,\mu} c^{\rho,\nu} h_{\nu\rho} \\ &\quad + \tfrac{1}{2}\bar{c}^{\mu}{}_{,\mu} c^{\rho} h^{\nu}{}_{\nu,\rho}\big), \end{aligned} \tag{5}$$

where semicolons denote covariant differentiation with respect to the background metric.

The possible two-loop divergences of the theory in four dimensions correspond to *local* invariants of dimensionality six, built out of the Riemann tensor and of its derivatives. They are listed below, following ref. [15]:

$$R_{,\mu} R^{,\mu},\ R^{3},\ R_{\alpha\beta,\mu} R^{\alpha\beta,\mu},\ R R_{\alpha\beta} R^{\alpha\beta}, \tag{6a}$$

$$\begin{aligned} &R_{\alpha\gamma} R_{\beta\delta} R^{\alpha\beta\gamma\delta},\ R_{\alpha}{}^{\beta} R_{\beta}{}^{\gamma} R_{\gamma}{}^{\alpha},\ R R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}, \\ &R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\epsilon} R_{\delta}{}^{\epsilon}, \end{aligned} \tag{6b}$$

$$R^{\alpha\beta}{}_{\gamma\delta} R^{\gamma\delta}{}_{\epsilon\zeta} R^{\epsilon\zeta}{}_{\alpha\beta},\ R_{\alpha\beta\gamma\delta} R^{\alpha}{}_{\epsilon}{}^{\gamma}{}_{\zeta} R^{\beta\epsilon\delta\zeta}. \tag{6c}$$

Of these, the eight in eqs. (6a) and (6b) clearly vanish when the field equations are used. Therefore, they do not affect the S matrix. The remaining ones in eq. (6c) do not vanish when the field equations are used, and correspond to possible divergences of the theory. They are not linearly independent, however, once one requires that the full antisymmetrization of any five of the four-dimensional Lorentz indices in them vanish. Such identities are usually ambiguous in dimensionally regularized theories, but give no problem in this case of a purely bosonic theory. In fact, the indices are attached to the fields, and working in $(4-\epsilon)$ dimensions corresponds in this case to working in four dimensions, with fields having ϵ vanishing components. The divergences can thus be parametrized by

$$\sqrt{-g}\, R^{\alpha\beta}{}_{\gamma\delta} R^{\gamma\delta}{}_{\epsilon\zeta} R^{\epsilon\zeta}{}_{\alpha\beta}. \tag{7}$$

The two-loop finiteness of Einstein gravity is equivalent to the vanishing of the coefficient of the invariant of eq. (7) in the divergent part of the off-shell effective action or, equivalently to the finiteness of the on-shell effective action. This, in turn, is equivalent to the vanishing of the coefficient of the single pole ($1/\epsilon$ part) in dimensional regularization. The double pole ($1/\epsilon^2$ part) is essentially a one-loop effect, and vanishes in this case [16] ‡5. One then needs to select Green functions suitable to reveal the coefficient of the invariant in eq. (7). To this end, one expands the background metric as $g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$. It turns out that there is one term,

$$H^{\alpha\beta}\partial_{\alpha}\partial_{\epsilon}\partial_{\zeta} H^{\gamma\delta}\partial_{\beta}\partial_{\gamma}\partial_{\delta} H^{\epsilon\zeta}, \tag{8}$$

that can determine the coefficient of the invariant in eq. (7) at the cubic level, with all the external momenta satisfying the mass-shell condition,

‡5 More details on this point can be found in ref. [12].

though not collinear, and thus continued to complex values. Tracking the coefficient of the term in eq. (8) leads to the divergences of the whole S matrix in a particularly convenient way. One only needs to compute on-shell Green functions with three external legs and, for example, all structures containing the divergence or the trace of H can be ignored. Moreover, it can be seen that no contribution to the term in eq. (8) can come from propagator diagrams (in which one would use the nonlinear field equations). Thus, the whole problem is determined by the vertex correction at two loops.

Naively, one would need to consider 36 diagrams. A large number of these would contain a ghost field circulating in one of the loops, since the ghosts couple non-polynomially to the background metric field. However, it turns out that one can simplify matters considerably, and the whole problem can be reduced to computing only 8 diagrams. First, the ghosts and the metric field can be conveniently combined. To this end, one rewrites the ghost fields in terms of real fields, which are to be treated as *commuting*. Then, one notices that the expansion in terms of real fields proceeds via

$$\bar{c}_\mu c_\nu = \tfrac{1}{2}(1 - \sigma_2)_{ij} c_{i\mu} c_{j\nu}, \tag{9}$$

where σ_2 is the familiar Pauli matrix. The matrix in eq. (9) is a projection operator and satisfies, together with a few similar others, a simple commutative algebra. The two real ghosts can thus be regarded as the vectors originating from a six-dimensional metric field upon reduction à la Kaluza–Klein. The corresponding scalars can also be decoupled by means of suitable projection operators. Finally, the anticommuting nature of the ghosts only enters the algebra of the projectors and, for two-loop diagrams, can be enforced by altering the interaction vertex between two ghost fields and a quantum metric field by a factor i, thus endowing the ghost–ghost–quantum graviton vertex with an imaginary charge. With a single type of quantum line, there are only the 14 diagrams of figs. 1 and 2. Furthermore, it is easy to convince oneself that all the diagrams with two background fields emitted from the same point cannot contribute to the structure in eq. (8). Thus,

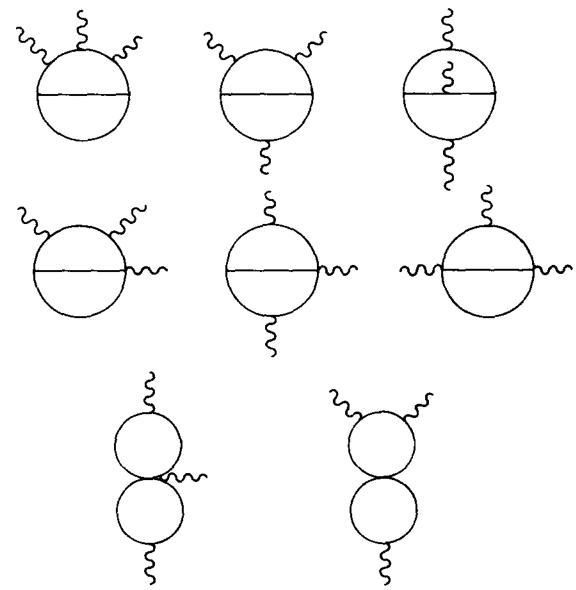

Fig 1 Graphs with a single external line from each vertex

in conclusion, the diagrams of fig. 1 determine all the two-loop divergences of Einstein's theory of gravity in four dimensions [‡6]. The resulting 8 diagrams, however, are still quite complicated, and dealing with this problem requires the use of sophisticated computer software, designed specifically for this application.

There remains the problem of extracting the divergent parts from the Feynman integrals. There are subtle technical issues encountered in calculations of ultraviolet divergences at more than one loop. These are connected in part with the nonlocal nature of the divergences of individual diagrams, the so-called overlapping divergences. However, it is well known that overlapping divergences are a consequence of the splitting of various contributions to the S matrix. The counterterms needed to remove lower-loop divergences eliminate them at any order of perturbation theory, with the consequence that the counterterms of any local field theory are *local*, i.e. polynomial in the fields and in their momenta [17].

[‡6] Actually, one can also show that the last diagram in fig 1 does not contribute to the on-shell divergences, while the one next to it only contributes to the double pole

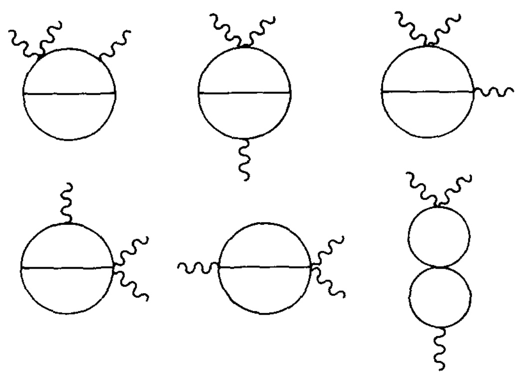

Fig 2 Graphs with two external lines from one vertex

This observation can be turned into a very convenient technique for computing divergent parts of diagrams, if one works with individual integrals where the subdivergent parts are subtracted out. Such integrals can be manipulated very conveniently, as their infinite parts are polynomial in the momenta, and the evaluation of the divergent part of a general l-loop integral can be reduced to the evaluation of a massless propagator integral (i.e. an integral depending on only one external momentum and no masses) at $(l-1)$ loops [18]. Moreover, no counterterm diagrams need be calculated explicitly, as their effect is included in the "subtracted" integrals. This approach clearly bypasses the problems in the background field method connected with the distinction between background and quantum lines. The integration technique is also well suited for being implemented on computers. The combination of these methods allows one to solve the whole problem, once the suitable software has been developed, in about three days of running time on a VAX 11/780.

The final result for the two-loop divergences of Einstein gravity in four dimensions is

$$\Gamma_\infty = \frac{209}{2880(4\pi)^4}\,\frac{1}{\epsilon}\int d^4x\sqrt{-g}\;R_{\alpha\beta}{}^{\gamma\delta}R_{\gamma\delta}{}^{\epsilon\zeta}R_{\epsilon\zeta}{}^{\alpha\beta}, \tag{10}$$

where Γ_∞ denotes the residual infinite part of the effective action, after the field equations are imposed. The double pole is absent, in agreement with the arguments of ref. [16].

A drawback of the simplified approach discussed above is that it leaves only one check on the result, the vanishing of the coefficient of the double pole. Of course, this is not sensitive to the subleties of the subtraction, which affect only the $1/\epsilon$ part. On the other hand, the proper way of performing the subtractions is well-defined and unambiguous. For completeness, we have also computed the vertex correction fully off-shell. In this case the requirement that the effective action be general coordinate invariant leads to an overdetermined system with many more structures than invariants, and provides several checks on the calculation.

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