Non-perturbative aspects of gauge theories
Exercise sheet 12 – Quantum Gravity in the Einstein-Hilbert truncation

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due date: 28 January 2019

Exercise 20: Quantum Gravity in the Einstein-Hilbert truncation

In this exercise we investigate quantum gravity in the Einstein-Hilbert truncation,

$$\Gamma_k = 2\kappa^2 Z_k \int \mathrm{d}^4x \sqrt{g}\, [-R + 2\Lambda_k] \,. \tag{1}$$

As a further simplification we only take contributions from the transverse-traceless spin-two mode of the graviton into account, i.e., we neglect the other graviton as well as the ghost modes. Due to this approximation, we never have to specify a gauge-fixing action. Start from transverse-traceless graviton two-point function

$$\Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}} = \frac{Z_k}{32\pi}\left(\bar{\Delta} - 2\Lambda_k + \frac{2}{3}\bar{R}\right) , \tag{2}$$

define the regulator as

$$R_k = \Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}}\Big|_{\Lambda_k=\bar{R}=0} \cdot r_k\left(\frac{\bar{\Delta}}{k^2}\right) , \tag{3}$$

with the Litim-type cutoff

$$r_k(x) = \left(\frac{1}{x} - 1\right)\Theta(1-x) \,. \tag{4}$$

Evaluate now the trace over the Laplace operator on the right-hand side of the Wetterich equation

$$\mathrm{Tr}\, \frac{1}{\Gamma^{(2)}_{h^{\mathrm{TT}}h^{\mathrm{TT}}} + R_k} \partial_t R_k \,, \tag{5}$$

with heat-kernel techniques, see next page as well as Appendix G.1 of the lecture notes for details.

Turn now to the left-hand side of the Wetterich equation and take a scale derivative of (1). Compare the terms proportional to $\sqrt{\bar{g}}$ and $\sqrt{\bar{g}}\bar{R}$ from the left-hand side with the result from the right-hand side, (5). Deduce from this the flow equations of the Newton coupling and the cosmological constant. The resulting flow equations are

$$\begin{aligned}
\partial_t g_k &= (2+\eta_g) g_k \,, \\
\eta_g &= -\frac{5}{6\pi} g_k \left(2 \frac{1-\frac{1}{6}\eta_g}{(1-2\lambda_k)^2} + \frac{1-\frac{1}{4}\eta_g}{1-2\lambda_k} \right) , \\
\partial_t \lambda_k &= -4\lambda_k + \frac{\lambda_k}{g_k}\partial_t g_k + \frac{5}{4\pi} g_k \frac{1-\frac{1}{6}\eta_g}{1-2\lambda_k} \,.
\end{aligned} \tag{6}$$

Bonus question 1:
Why is the spin-two approximation rather good in most cases?

Bonus question 2:
Use mathematica and find numerically the non-Gaussian fixed point of (6) and determine the eigenvalues of the stability matrix.

Heat-kernel techniques

Heat-kernel techniques are used to evaluate the trace of a function that depends on the Laplace operator on a curved background. You can use the formula

$$\mathrm{Tr}\, f(\Delta) = \frac{1}{(4\pi)^2} \Big[B_0(\Delta) Q_2[f(\Delta)] + B_2(\Delta) Q_1[f(\Delta)] \Big] + \mathcal{O}(R^2) \,, \tag{7}$$

with the definition

$$Q_n[f(x)] = \frac{1}{\Gamma(n)} \int \mathrm{d}x\, x^{n-1} f(x) \,. \tag{8}$$

The B_n are called heat-kernel coefficients and often written as

$$B_n(\Delta) = \int \mathrm{d}^4 x \sqrt{g}\ \mathrm{Tr}\ b_n(\Delta) \,. \tag{9}$$

The values of the heat-kernel coefficients depend on the field. For the transverse-traceless tensor (TT), transverse vectors (TV) and scalars (S) on a sphere, they are given by

	TT	TV	S
Tr b_0	5	3	1
Tr b_2	$-\frac{5}{6}R$	$\frac{1}{4}R$	$\frac{1}{6}R$

Solution

We need to evaluate

$$
\begin{aligned}
\mathrm{Tr}\; G_{h^{\mathrm{TT}}}\partial_t R_k &= \mathrm{Tr}\;\frac{\Delta(\partial_t r_k(\Delta)-\eta_g r_k(\Delta))}{\Delta(1+r_k(\Delta))-2\Lambda_k+\frac{2}{3}\bar{R}} \\
&= \mathrm{Tr}\;\frac{\Delta(-2\frac{\Delta}{k^2}r_k'(\Delta)-\eta_g r_k(\Delta))}{\Delta(1+r_k(\Delta))-2\Lambda_k}-\frac{2}{3}\bar{R}\;\mathrm{Tr}\;\frac{\Delta(-2\frac{\Delta}{k^2}r_k'(\Delta)-\eta_g r_k(\Delta))}{(\Delta(1+r_k(\Delta))-2\Lambda_k)^2}
\end{aligned}
\tag{10}
$$

With the heat-kernel formulas we obtain for example for the first term

$$
\begin{aligned}
\mathrm{Tr}\;\frac{\Delta(-2\frac{\Delta}{k^2}r_k'(\Delta)-\eta_g r_k(\Delta))}{\Delta(1+r_k(\Delta))-2\Lambda_k} &= \frac{1}{(4\pi)^2}\Bigg(B_0(\Delta)\,Q_2\left[\frac{\Delta(-2\frac{\Delta}{k^2}r_k'(\Delta)-\eta_g r_k(\Delta))}{\Delta(1+r_k(\Delta))-2\Lambda_k}\right] \\
&\qquad + B_2(\Delta)\,Q_1\left[\frac{\Delta(-2\frac{\Delta}{k^2}r_k'(\Delta)-\eta_g r_k(\Delta))}{\Delta(1+r_k(\Delta))-2\Lambda_k}\right]\Bigg) \\
&= \frac{1}{(4\pi)^2}\int \mathrm{d}^4x\sqrt{g}\,\left[5\,\Phi_2^1(-2\Lambda_k)-\frac{5}{6}\bar{R}\,\Phi_1^1(-2\Lambda_k)\right]
\end{aligned}
\tag{11}
$$

where the Φ_m^n are the threshold functions

$$
\begin{aligned}
\Phi_n^p(\omega) &= \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z z^{n-1}\frac{z(-2zr_k(z)-\eta_g r_k(z))}{(z(1+r_k(z))+\omega)^p} \\
&= \frac{1}{\Gamma(n)}\frac{1}{(1+\omega)^p}\left(\frac{2}{n}-\frac{\eta_g}{n(n+1)}\right).
\end{aligned}
\tag{12}
$$

In the last line we have evaluated the threshold functions for the Litim-type cutoff (4). So the term in (11) becomes

$$
5\,\Phi_2^1(-2\Lambda_k)-\frac{5}{6}\bar{R}\,\Phi_1^1(-2\Lambda_k) = 5\frac{1-\frac{\eta_g}{6}}{1-2\lambda_k}-\frac{5}{6}\bar{R}\frac{2-\frac{\eta_g}{2}}{1-2\lambda_k}\,. \tag{13}
$$

For the second term in (10)

$$
\begin{aligned}
-\frac{2}{3}\bar{R}\;\mathrm{Tr}\;\frac{\Delta(-2\frac{\Delta}{k^2}r_k'(\Delta)-\eta_g r_k(\Delta))}{(\Delta(1+r_k(\Delta))-2\Lambda_k)^2} &= -\frac{2}{3}\frac{\bar{R}}{(4\pi)^2}B_0(\Delta)\,Q_2\left[\frac{\Delta(-2\frac{\Delta}{k^2}r_k'(\Delta)-\eta_g r_k(\Delta))}{(\Delta(1+r_k(\Delta))-2\Lambda_k)^2}\right] \\
&= -\frac{2}{3}\frac{\bar{R}}{(4\pi)^2}\int \mathrm{d}^4x\,5\,\Phi_2^2(-2\Lambda_k) \\
&= -\frac{10}{3}\frac{\bar{R}}{(4\pi)^2}\int \mathrm{d}^4x\,\frac{1-\frac{\eta_g}{6}}{(1-2\lambda_k)^2}\,.
\end{aligned}
\tag{14}
$$

We have found all the terms proportional to $\sqrt{g}$ and $\sqrt{g}R$ and dropped all term of $\mathcal{O}(R^2)$.

From (1) we get

$$\partial_t \Gamma_k^{\text{grav}} = 2\kappa^2 \int \mathrm{d}^4x \sqrt{g} \left[- (\partial_t Z_k) R + 2 (\partial_t Z_k \Lambda_k) \right]$$

$$= 2 Z_k \kappa^2 \int \mathrm{d}^4x \sqrt{g} \left[\eta_g R + 2 \left(k^2 \partial_t \lambda_k + 2\Lambda_k - \eta_g \Lambda_k \right) \right] \,, \tag{15}$$

Further we know from $g_k = G_k k^2 = G k^2 / Z_k$ that

$$\beta_g = \partial_t g_k = (2 + \eta_g)\, g_k \,. \tag{16}$$

So the comparison of $\sqrt{g}$ terms leads to

$$\begin{aligned} \partial_t \lambda_k &= (-2 + \eta_g) \lambda_k + \frac{1}{4\kappa^2} \,(\text{right-hand side}) \\ &= -4\lambda_k + \frac{\lambda_k}{g_k} \partial_t g_k + 8\pi g_k \left(\frac{5}{(4\pi)^2} \frac{1 - \frac{\eta_g}{6}}{1 - 2\lambda_k} \right) \\ &= -4\lambda_k + \frac{\lambda_k}{g_k} \partial_t g_k + \frac{10}{4\pi} g_k \frac{1 - \frac{\eta_g}{6}}{1 - 2\lambda_k} \,. \end{aligned} \tag{17}$$

And the comparison of $\sqrt{g}R$ terms leads to

$$\begin{aligned} \eta_g &= \frac{1}{2\kappa^2} \,(\text{right-hand side}) \\ &= 16\pi \left(-\frac{5}{6} \frac{1}{(4\pi)^2} \frac{2 - \frac{\eta_g}{2}}{1 - 2\lambda_k} - \frac{10}{3} \frac{1}{(4\pi)^2} \frac{1 - \frac{\eta_g}{6}}{(1 - 2\lambda_k)^2} \right) \\ &= -\frac{5}{3\pi} \left(\frac{1 - \frac{\eta_g}{4}}{1 - 2\lambda_k} + 2 \frac{1 - \frac{\eta_g}{6}}{(1 - 2\lambda_k)^2} \right) \,. \end{aligned} \tag{18}$$

Bonus question 1:
Why is the spin-two approximation rather good in most cases?
The TT-mode contains 5 degrees of freedom. The graviton vector mode 3 and the two scalar graviton modes each 1. The ghost modes are -8 degrees of freedom. In Landau gauge $\alpha \to 0$, the graviton vector mode and one scalar are exactly cancelled by the corresponding ghost mode. In summary, one has $5 + 1 - 4$ degrees of freedom. Furthermore the graviton degrees of freedom are usually enhanced by the cosmological constant ($1 - 2\lambda_k$ in the denominator of the graviton propagator). So with the TT-approximation we are taking the most of the degrees of freedom into account.

Bonus question 2:
Use mathematica and find numerically the non-Gaussian fixed point of (6) and determine the eigenvalues of the stability matrix.
See 'Background-TT-equations.nb' in the git. The fixed-point values are

$$(g_k^*,\, \lambda_k^*) = (0.86,\, 0.18) \,. \tag{19}$$

And the critical exponents are given by

$$\theta_{1,2} = 2.9 \pm 2.6\, i\,. \tag{20}$$