UNIVERSITÄT
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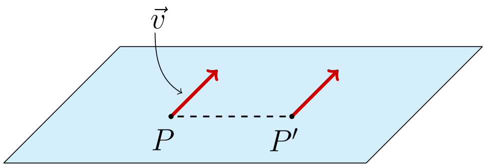
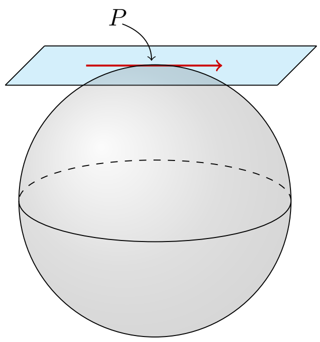
SEIT 1386

$\mathbb{R}^n$





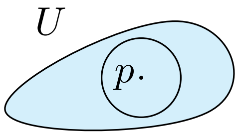
$\mathbb{R}^n$	$\mathcal{M}$

$M \subseteq \mathbb{R}^n \forall p \in U \exists \varepsilon > 0 B_\varepsilon(p) \subset U$

$\emptyset, \mathbb{R}^n$

$U, V \subset \mathbb{R}^n \Rightarrow U \cap V \mathbb{R}^n$

$U_i, i \in \mathbb{R}^n \Rightarrow \bigcup_{i \in} U_i \subset \mathbb{R}^n$



$X \mathcal{O} \subset \mathcal{P}(X)$

$\emptyset, X \in \mathcal{O}$

$U, V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$

$U_i \in \mathcal{O} \Rightarrow \bigcup_{i \in} U_i \in \mathcal{O}$

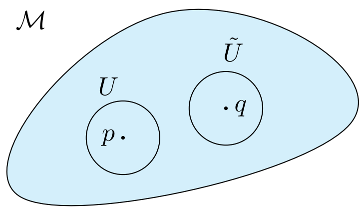
$(X, \mathcal{O} = \mathcal{P}(x))$

$N \subset X (N, \mathcal{O}_1) \mathcal{O}_1$

$$V \in \mathcal{O}_1 \Leftrightarrow \exists U \in \mathcal{O} V = N \cap U$$

$\mathcal{M} n$

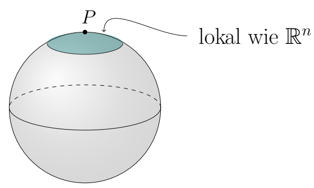
$\mathcal{M} \forall p, q \in \mathcal{M} p \neq q \exists U \in p V \in q U, V \in \mathcal{O}$



$\mathcal{M}\{U_1, \dots, U_k, \dots\} U_i \in \mathcal{O} \forall p \in \mathcal{M} U p K p \in U_k \subseteq U$

$\mathcal{M}\mathbb{R}^n \forall p \in \mathcal{M} U p X : U \to V \subseteq \mathbb{R}^n$

$(X, U)\mathcal{M}p$
$\mathcal{A} = \{(x_\alpha, U_\alpha)_{\alpha \in \mathcal{A}}\}\mathcal{M}$

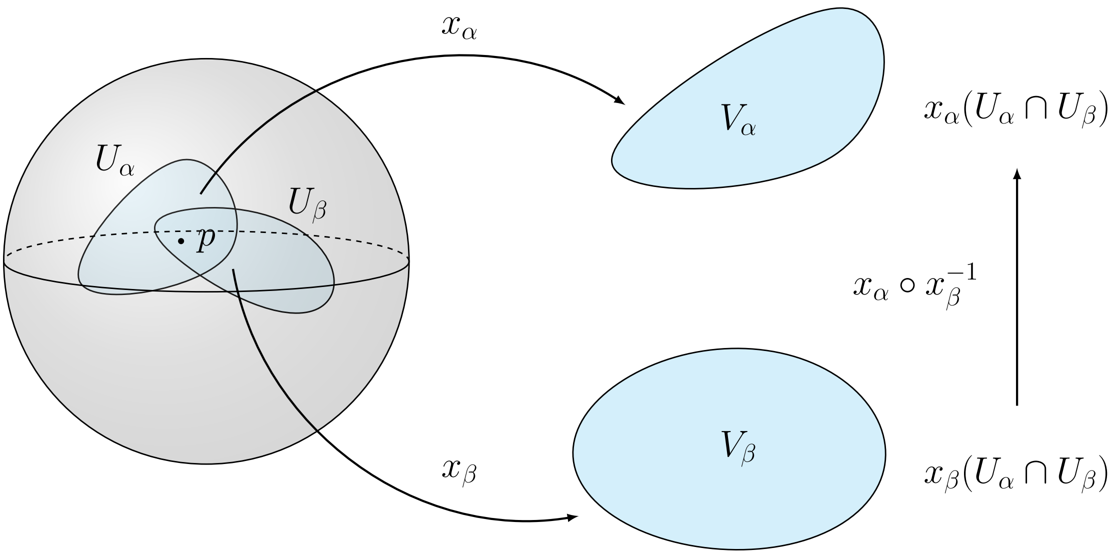


$$\bigcup_{\alpha \in \mathcal{A}} = \mathcal{M}$$

$\mathbb{R}^n$

$x_\alpha x_\beta$

$$x_\alpha \circ x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \to x_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$$



$x_\alpha \circ x_\beta^{-1}$

$\mathcal{M}$

> $\mathcal{A} = \{(x_\alpha, U_\alpha)\}\mathcal{M}C^\infty x_\alpha \circ x_\beta^{-1}\alpha, \beta \in AC^\infty$
>
> $\mathcal{A}C^\infty\mathcal{M}$
> $(x, U)\mathcal{A}x \circ x^{-1}C^\infty$

$C^\infty C^\infty$

C^∞

$\Rightarrow$

$(V_j)(U_j)\forall V_j \exists U_j V_j \subseteq U_j$
$\forall p \in X \exists U U_i$
$\Rightarrow \exists$

$$f_i : V_i \subseteq X \to [0, 1], \sum_{i \in I} f_i(x) = 1$$

$\mathbb{R}^n \mathcal{A} = \{(, \mathbb{R}^n)\}$

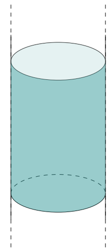
$VBB = \{v_1, \cdots, v_n\} \mathcal{A} = \{(\chi_B, V)\}$

$$\chi_B : V \to \mathbb{R}^n$$
$$v = \sum_{i=1}^{n} a_i v_i \mapsto \sum_{i=1}^{n} a_i e_i$$

$(e_1, \cdots, e_n)$

$M \subseteq \mathbb{R}^n, (\chi_U, U) \chi_U = |_U, V \subseteq M^n, M\mathcal{A} = \{(\chi_X, U)\} M$
$\mathcal{A}_V = \{(\chi_V, U_V\}(\chi_V, U_V) = (\chi_{U \cap V}, U \cap V)$

$M_1 = S^1, M_2 = \mathbb{R}, M_1 \times M_2 =$
$M_1^{n_1}, M_1^{n_2} M_1 \times M_2 n_1 + n_2$
$\mathcal{A} = \{(x \times y, U \times V)\}$



$$(x, U) = M_1$$
$$(y, V) = M_2$$

$(x \times y)(p_1, p_2) = (x(p_1), y(p_2))$

$S_R^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i^2 = R^2\} \subset \mathbb{R}^{n+1}$

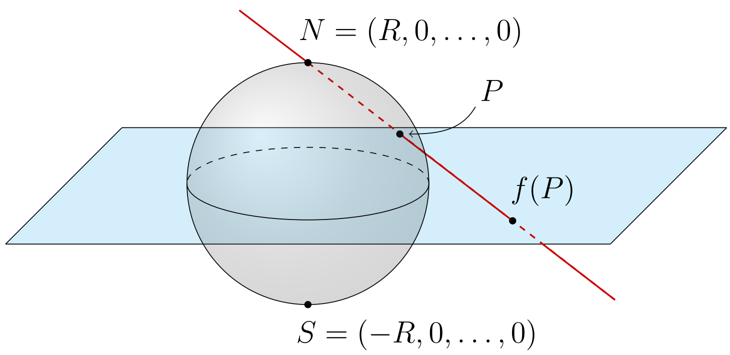
$U \subset S_R^n \Leftrightarrow \exists U' \subset \mathbb{R}^{n+1} U = U' \cap S_R^n$

N

$$f_N : S_R^n \backslash \{N\} \to \mathbb{R}^n$$
$$f_N(x_0, \ldots, x_n) = \frac{R}{R - x_0}(x_1, \ldots, x_n)$$

$$f_S : \mathcal{M}_s \to \mathbb{R}^n$$

$\to f_N f_s$



$N\mathcal{A}_N = \{(\chi|_U, U \cap N)\}$

M, N

$$f : N \to M$$

$f(N) \subset M$

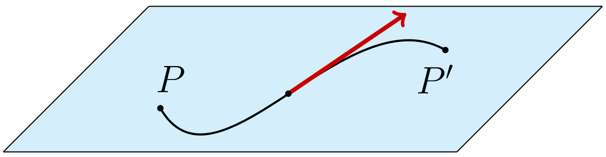
$f : N \to f(N)$

$Mp \in M\mathbb{R}$

$$v : \mathcal{F}(M) \to \mathbb{R}$$

$v(fg) = v(f)g(p) + f(p)v(g)$

$MpMpT_pM$
T_pM



$U \subseteq Mp \in U \exists \varphi \in \mathcal{F}(M)$

$\varphi \subseteq U$

$\varphi U' \subset Up$

$(x, U)\varphi\varepsilon > 0 B_{2\varepsilon}(x(p)) \subset V \subset \mathbb{R}^n \psi : \mathbb{R}^n \to \mathbb{R}$

$$\left.\begin{array}{r}(\varphi) \subset B_{2\varepsilon}(x(p)) \\ \varphi = 1 B_\varepsilon\end{array}\right\}$$

$\square$

$$\varphi(q) = \left\{\begin{array}{l}\psi(x(q)) q \in U \\ 0\end{array}\right.$$

$v \in T_pM$

$v() = 0$

$f = gpv(f) = v(g)$

$\varphi U\varphi f = \varphi gU$

$$
\begin{aligned}
v(\varphi f) &= v(\varphi)f(p) + \varphi(p)v(f) \\
&= v(\varphi)f(p) + v(f) \\
v(\varphi g) &= v(\varphi)g(p) + v(g)
\end{aligned} \qquad \square
$$

$v(\varphi f) = v(\varphi g) \Leftrightarrow v(f) = v(g)$

$v(\lambda f) = \lambda v(f), \lambda \in \mathbb{R}, f \in \mathcal{F}(\mathbb{R})$
$v(\lambda) = 0v(\lambda) = \lambda v(1)v(1) = 0$

$$v(1) = v(1 * 1) = 1v(1) + v(1)1 = 2v(1) \Rightarrow v(1) = 0 \qquad \square$$

$T_p\mathcal{M}$

$(x, U)\mathcal{M}p\frac{\partial}{\partial x_i}\big|_p i = 1, \ldots, m$

$$\frac{\partial}{\partial x_i}\Big|_p(f) := \partial_i(f \circ x^{-1})\Big|_{x(p)}$$

$\partial_i i$

$(\frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_m}\big|_p)T_p\mathcal{M}$

$$v = \sum_{i=1}^{m} v(x_i)\frac{\partial}{\partial x_i}\Big|_p = \sum_{i=1}^{m} \xi\frac{\partial}{\partial x_i}\Big|_p.$$

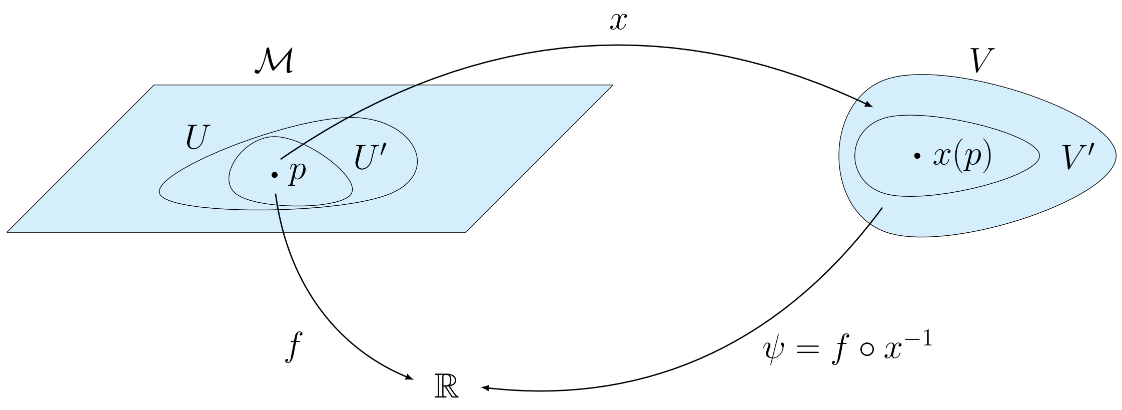
$$\frac{\partial}{\partial x_1}\Big|_p(x^j) = \delta_{ij} \qquad \square$$

$(\frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_m}\big|_p)T_p\mathcal{M}$

$f : U \subset \mathcal{M} \to \mathbb{R}U' \subset Upp \in U'f_i : U' \to \mathbb{R}$

$$f = f(p) + \sum_{i=1}^{m}(x_i - x_i(p))f_i.$$

$f_i(p) = \frac{\partial}{\partial x_i}\big|_p(f)$



$$\psi(U) - \psi(U_0) = \int_0^1 \frac{}{t}\psi(tU + (1-t)U_0)t$$

$U = x(q)q \in \mathcal{M}U_0 = x(p)$

$$\psi(U) - \psi(U_0) = \sum_i (U^i - U_0^i) \int_0^1 \underbrace{\frac{\psi}{U'}(tU + (1-t)U_0)t}_{:=\psi_i(U)}$$

$f_i = \psi_i \circ x : U \subset \mathcal{M} \to \mathbb{R} f_i$

$$\psi(U) - \psi(U_0) = \psi(x(q)) - \psi(x(p)) = f(q) - f(p)$$

$$U^i = x^i(q)$$

$$U_0^i = x^i(p)$$

$$\psi_i(U) = \psi_i(x(1)) = f_i(q)$$

$$f(q) - f(p) = \sum_{i=1}^{n} (x_i(q) - x_i(p)) f_i(q)$$

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Big|_p(f) &= \partial_i \underbrace{(f \circ x^{-1})}_{\psi}\Big|_{x(p)} \\
&= \partial_i \psi\Big|_{x(p)} \\
&= \psi_i(x(p)) = f_i(p)
\end{aligned}$$

$\psi(U) = \psi(U_0) + \sum_i (U^i - U_0^i)\psi_i(U)$

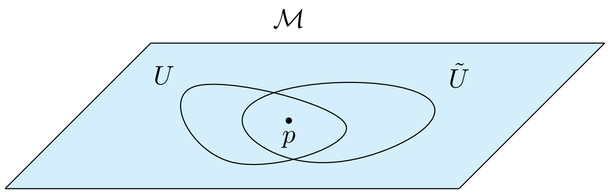
$$\Rightarrow \frac{\partial}{\partial U_i}\psi\big|_{x(p)} = \psi_i(U)\big|_x(p) = \psi(x(p)) \qquad \square$$

$f_i(p) = \frac{\partial}{\partial U_i}\big|_p(f)$

$$\begin{aligned} v(f) &= v(f(p) + \sum_i (x_i - x_i(p))f_i) \\ &= v(f(p)) + \sum_i v(x_i - x_i(p)f_i) \end{aligned}$$

$$\begin{aligned} v(f) &= \sum_i (\underbrace{(x_i(p) - x_i(p))v(f_i)}_{=0} + \underbrace{v(x_i - x_i(p))}_{=v(x_i)} f_i) \\ &= \sum_i v(x_i) f_i \\ &= \sum_i v(x_i) \frac{\partial}{\partial x_i}\big|_p f \qquad \square \end{aligned}$$

$(x, U)(\tilde{x}, \tilde{U})p \in \mathcal{M}$



$$\frac{\partial}{\partial \tilde{x}_i}\big|_p = \sum_j \underbrace{\frac{\partial}{\partial \tilde{x}_i}\big|_p(x_j)}_{\in\mathbb{R}} \frac{\partial}{\partial x_j}\big|_p$$

$\tilde{v}_i = \sum_j a_{ij} v_j$

$\mathcal{M}\mathcal{N} f : \mathcal{M} \to \mathcal{N} f p$

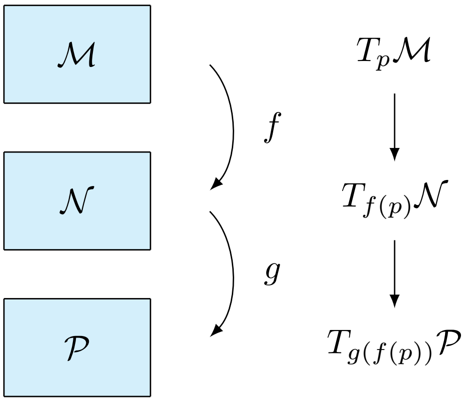
$$\begin{aligned} f\big|_p : T_p\mathcal{M} &\to T_{f(p)}\mathcal{N} \\ v &\mapsto f\big|_p(v), \end{aligned}$$

$$\underbrace{f\big|_p(v)}_{T_{f(p)}\mathcal{N}}\underbrace{(\phi)}_{\in\mathcal{F}(\mathcal{M})} = \underbrace{v(\phi\circ f)}_{\in\mathcal{F}(\mathcal{N})}, \forall p \in \mathcal{F}(\mathcal{N})$$

$f\big|_p$

$f:\mathcal{M}\to\mathcal{N} g:\mathcal{N}\to\mathcal{P}$

$$(g\circ f)\big|_p = g\big|_{f(p)}\circ f\big|_p$$



$$\begin{aligned}(g\circ f)\big|_p(v(\phi)) &= v(\phi\circ g\circ f)\\ &= f\big|_p(v)(\phi\circ g)\\ &= g\big|_{f(\phi)}\circ f\big|_p(v)(\phi)\end{aligned}$$ □

$f:\mathcal{M}\to\mathcal{N}(x,U)\mathcal{M}p(y,V)\mathcal{N}pf_j = y_j\circ f f_j:\mathcal{M}\to\mathbb{R}$

$$\underbrace{f\big|_p(\frac{\partial}{\partial x_i}\big|_p)}_{\in T_{f(p)}\mathcal{M}} = \sum_j \underbrace{\frac{\partial}{\partial x_i}\big|_p(f_j)}_{\in\mathbb{R}}\underbrace{\frac{\partial}{\partial y_j}\big|_{f(p)}}_{\in T_{f(p)}\mathcal{N}}$$

$f:\mathcal{M}\to\mathcal{N}\ \mathcal{M}=m\mathcal{N}=n$

$fp\, f\big|_p$

$p\in\mathcal{M}\in\mathcal{M}\Leftrightarrow\ f\big|_p = \mathcal{N}$

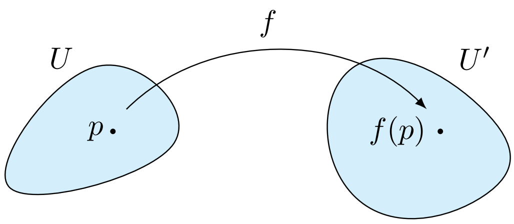
$q \in \mathcal{N} \in \mathcal{N} \Leftrightarrow \forall p \in f^{-1}(q)$

$f \Leftrightarrow f p \in \mathcal{M}$

$f \Leftrightarrow f\big|_p p \in \mathcal{M}$

$f : \mathcal{M} \to \mathcal{N} f\big|_p : T_p\mathcal{M} \to T_{f(p)}\mathcal{N} U p U' f(p)$

$$f\big|_U : U \to U'$$



$$\begin{tikzcd}
\mathcal{M} \arrow[r, "f"] \arrow[d, "x"'] & \mathcal{N} \arrow[d, "y"] \\
V \subseteq \mathbb{R}^m \arrow[r, "\phi"'] & V \subseteq \mathbb{R}^m
\end{tikzcd}$$

$(x, U)(y, U')\mathcal{M}\mathcal{N} p f(p) f(U) \subset U' \phi \phi\big|_{x(p)} \mathbb{R}^n \phi \hat{V} x(p) \hat{V}' y(f(p)) = \phi(x(p)) \hat{\phi}\big|_{\hat{V}}$

$$f\big|_{x^{-1}(\hat{V})} : x^{-1}(\hat{V}) \to y^{-1}(\hat{V}')$$

$$\phi = y \circ f \circ x^{-1} \Rightarrow f = y^{-1} \circ \phi \circ x \qquad \square$$

$f : \mathcal{M} \to \mathcal{N}\, \mathcal{M} = m\, \mathcal{N} = n$

$_p\, f = r(y, U')f(p)(x, U)p$

$$y \circ f \circ x^{-1}(U_1, \ldots, U_m) = (U_1, \ldots, U_r, \phi_{r+1}(U), \ldots, \phi_n(U)).$$

$y(f(p)) = 0xx(p) = 0\phi_j(0) = 0\forall j > r$

$f = rp(x, U)(y, U')$

$$y \circ f \circ x^{-1}(U_1, \ldots, U_m).$$

$(y, U')\mathcal{N} f(p)(\hat{x}, U)\mathcal{M}p\hat{x}(p) = 0$

$$\hat{A} = (\hat{A})_{ij} = (\partial_i \hat{\phi}_j),$$

$\phi \circ y \circ f \circ x^{-1}{}_p\, f = r$

$$\tilde{A} \neq 0.$$

$\tilde{A} = (\hat{A}_{ij})_{1 \leq i \leq r}$

$$x_i = \begin{cases} y_i \circ f & 1 \leq i \leq r \\ \hat{x}_i & r + 1 \leq i \leq n \end{cases}$$

$x(p) = 0$

$$\partial_i (x_j \circ \hat{x}^{-1})(0) = \begin{pmatrix} \partial_i \hat{\phi}_j(0) & \star \\ 0 & \mathbb{1} \end{pmatrix}.$$

$x = m = \mathcal{M}pxUpV0\mathbb{R}^n x : \mathcal{M} \to V$

$$\begin{aligned} \phi(U_1, \ldots, U_m) &= y \circ f \circ x^{-1}(U_1, \ldots, U_m) \\ &= (U_1, \ldots, U_r, \phi_{r+1}(U), \ldots, \phi_m(U)). \end{aligned}$$

$\phi_k U' \phi_i(0) = 0$

$$A_{ij} = (\partial_i \phi_j)_{ij} = \begin{pmatrix} \mathbb{1} & 0 \\ \star & \partial_i \phi_i \end{pmatrix}$$

$\phi = rU = 0$

$$\partial_i \phi_j \forall i, j > r. \qquad \square$$

$f : \mathcal{M} \to \mathcal{N}\, \mathcal{M} = m\, \mathcal{N} = n$

$q \in \mathcal{N}$

$$\mathcal{H} = f^{-1}(q) = \{p \in \mathcal{M}(f(p)) = q\}$$

$m - n$

$f\mathcal{H} = f^{-1}(q)r\mathcal{H}m - rT_p\mathcal{H}$

$$f\big|_p \subseteq T_p\mathcal{M}, \forall p \in \mathcal{H}.$$

$\mathcal{M}$

$$T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M} = \{(p,V)|p\in\mathcal{M}, v\in T_p\mathcal{M}\}$$

$C^\infty T\mathcal{M}$

$$\begin{aligned}\pi : T\mathcal{M} &\to \mathcal{M}\\ (p,V) &\mapsto p\end{aligned}$$

$(x,U)\mathcal{M}^m(\overline{x},\overline{U})T\mathcal{M}$

$$\begin{aligned}&\overline{U} = \pi^{-1}(U) = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}\\ &\overline{x} : \overline{U} \to x(U)\times\mathbb{R}^m \subset \mathbb{R}^{2m}\\ &(p,V) \mapsto (x(p),\xi)\end{aligned}$$

$\xi = (\xi^1,\ldots,\xi^m)\in\mathbb{R}^m$

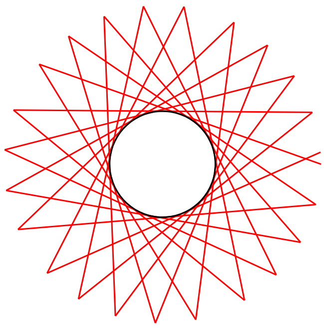
$$v = \sum_{i=1}^m \xi_i \frac{\partial}{\partial x_i}\big|_p, \forall p\in U.$$

$T\mathcal{M}\overline{x}$
$(\overline{x},\overline{U})(\overline{y},\overline{U}')$

$$\overline{y}\circ\overline{x}^{-1}\circ \underbrace{x(\overline{U}\cap\overline{U}')}_{x(U\cap U')\times\mathbb{R}^m} \to \underbrace{\overline{y}(\overline{U}\cap\overline{U}')}_{y(U\cap U')\times\mathbb{R}^m}$$

$$(x,\xi)\mapsto(y\circ x^{-1}(U),\eta)$$

$\eta = \left(y \circ x^{-1}\right)\big|_U \xi$
$y \circ x^{-1} \overline{y} \circ \overline{x}^{-1} T\mathcal{M} O \subset T\mathcal{M} \overline{x} (O \cap \overline{U}) V \times \mathbb{R}^m (x, U) \in \mathcal{A}_{\mathcal{M}} (\overline{x}, \overline{U}) \in \mathcal{A}_{T\mathcal{M}}$



$T\mathcal{M}\mathcal{A}_{T\mathcal{M}}$

$T\mathcal{M}$

$\mathcal{M}\mathbb{R}k\mathcal{M}$

$$\pi : E \to \mathcal{M},$$

$\forall p \in \mathcal{M} E_p := \pi^{-1}(\{p\}) \mathbb{R} k E_p E p$

$p\mathcal{M}U p\mathcal{M}$

$$\begin{tikzcd}
\phi : U \times \mathbb{R}^k \arrow[rr] \arrow[dr, "pr_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\
& U &
\end{tikzcd}$$

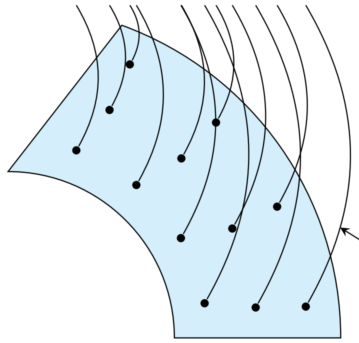
$\pi \circ \phi = pr_1$

$q \in U$

$$\phi\big|_q : \{q\} \times \mathbb{R}^k \to E_q$$
$$\{q, \xi\} \mapsto \phi_q(\xi) := \phi(q, \xi)$$

ϕE

$(\pi, E, \mathcal{M}) E E \mathcal{M}$



$$
\begin{aligned}
E = \mathcal{N} \times \mathbb{R}^k \to \mathcal{M} \\
(p, \xi) \mapsto p
\end{aligned}
$$

$$
\begin{aligned}
\pi : T\mathcal{M} \to \mathcal{M} \\
(p, V) \to V
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{M} = \mathbb{RP}^n \\
&E = \{(l, x) | l \in \mathbb{RP}^n, x \in l \subset \mathbb{R}^{n+1}\} \\
&\pi : E \to \mathcal{M} = \mathbb{RP}^n \\
&(l, x) \mapsto l
\end{aligned}
$$

$1 E_l$

$$
\begin{aligned}
(l, x) + (l, y) &:= (l, x + y) \\
k(l, x) &:= (l, kx)
\end{aligned}
$$

$E_p, p \in \mathcal{M} k$

$(U_\alpha)_{\alpha\in\mathcal{A}}\mathcal{M}$

$\forall\alpha\in\mathcal{A}p\in U_\alpha$

$$\phi_{\alpha,p}:\mathbb{R}^\alpha\to E_p$$

$$\begin{aligned}
&E=\cup_{p\in\mathcal{M}}E_p\\
&\pi:E\to\mathcal{M}\\
&(p,V)\mapsto p\\
&\phi_\alpha:U_\alpha\times\mathbb{R}^k\to E\big|_{U_\alpha}\\
&(p,\xi)\mapsto(p,\phi_{\alpha,p}(\xi)).
\end{aligned}$$

$(\pi,E,\mathcal{M})$

$\mathcal{M}E\pi:E\to\mathcal{M}\{U_\alpha\}\mathcal{M}$

$$\phi_\alpha^{-1}=\varphi:\pi^{-1}(U_\alpha)\to U_\alpha\times\mathbb{R}^\alpha,$$

$pr_1\circ\varphi_\alpha=\pi U_\alpha\cap U_\beta\neq\emptyset$

$$\varphi_\alpha\circ\varphi_\beta^{-1}\to(U_\alpha\cap U_\beta)\times\mathbb{R}^k,$$

$$(\varphi_\alpha\circ\varphi_\beta^{-1})(p,v)=(p,\tau(p)v)$$

$\tau:U_\alpha\cap U_\beta\to(k,\mathbb{R})k\mathcal{M}\varphi_\alpha^{-1}$

$p\in\mathcal{M}E_p:=\pi^{-1}(\{p\})p\in U_\alpha$

$$\varphi_\alpha\big|_p:E_p\to\{p\}\times\mathbb{R}^k.$$

$E_p\varphi_\alpha\big|_pU_\alpha U_\alpha\overline{U}_\alpha\subseteq\mathbb{R}^m\varphi_\alpha$

$$\pi^{-1}(U_\alpha)\to\overline{U}_\alpha\times\mathbb{R}^k.\qquad\square$$

EE

$(x,U)\mathcal{M}p\in Uv\in T_p\mathcal{M}$

$$v=\sum_{i=1}^m\xi_i\frac{\partial}{\partial x_i}\Big|_p$$

$$\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^m$$
$$v \mapsto (p, V)$$

$(x)(\overline{x})$

$$\frac{\partial}{\partial x_i}\Big|_p = (\frac{\partial \overline{x}_j}{\partial x_i})\frac{\partial}{\partial \overline{x}_j}\Big|$$
$$v = \sum_{j=1}^{m} \xi_j \frac{\partial}{\partial \overline{x}_j}\Big|_p = \sum_{ji1}^{m} \xi_i \frac{\partial}{\partial \overline{x}_i}\Big|_p$$
$$= \sum_{i,j}^{m} \xi_i \frac{\partial \overline{x}_j}{\partial x_i}\frac{\partial}{\partial \overline{x}_j}\Big|_p$$
$$\Rightarrow \bar{\xi}_j = \sum_i v_i \frac{\partial \overline{x}_j}{\partial x_i}$$

$$\varphi \circ \varphi^{-1}(x, v) = (x, \overline{v}) = (x, \tau(x), v)$$

$\tau(x)\frac{\partial \overline{x}_j}{\partial x_i}$

$$\pi : E \to \mathcal{M}$$
$$\pi' : E' \to \mathcal{M}'$$

$kk'(U_\alpha)_{\alpha\in A}\alpha \in Ap \in U_\alpha$

$$\phi_{\alpha,p} : \mathbb{R}^k \to E_p, g_{\alpha,\beta} : U_\alpha \cap U_\beta \to (k, \mathbb{R})$$
$$\phi'_{\alpha,p} : \mathbb{R}^{k'} \to E'_p, g'_{\alpha,\beta} : U_\alpha \cap U_\beta \to (k', \mathbb{R})$$

$$\mathcal{E}_p := E_p \oplus E_p$$
$$\mathcal{E} = \bigcup_{p\in\mathcal{M}} \mathcal{E}_p$$

$$\Phi_{\alpha,p} : \mathbb{R}^k \oplus \mathbb{R}^{k'} \to E_P \oplus E'_p$$
$$(v, w) \mapsto (\phi_{\alpha p}(v), \phi'_{\alpha p}(w))$$

$$G_{\alpha\beta} : U_\alpha \cap U_\beta \to (k + k', \mathbb{R})$$
$$p \mapsto \begin{pmatrix} g_{\alpha\beta}(p) & 0 \\ 0 & g'_{\alpha\beta}(p) \end{pmatrix}$$

$\mathcal{E}\mathcal{E}EE'$

$$\mathcal{E} = E \oplus E'.$$

$E'E''\mathcal{M}(U_\alpha)$

$$(E' \oplus E'')_p := E'_p \oplus E''_p$$
$$\phi_{\alpha p} : \mathbb{R}^{k'} \times \mathbb{R}^{k''} \to E'_p \oplus E''_p$$
$$(v, w) \mapsto \phi'_{\alpha p}(v) \oplus \phi''_{\alpha p}(w)$$

$$g_{\alpha\beta} = g'_{\alpha\beta}(p) \oplus g''_{\alpha\beta}(p)$$

$$_p := (E'_p, E''_p)$$

$$\phi_{\alpha p} : (\mathbb{R}^{k'}, \mathbb{R}^{k''}) \to (E'_p, E''_p)$$
$$f \mapsto \phi_{\alpha p} \circ f \circ (\phi'_{\alpha p})^{-1}$$

$(\pi, E, \mathcal{M})E^* = (E, \mathbb{R})\mathbb{R}1T^*\mathcal{M} = (T\mathcal{M}, \mathbb{R})T^*_p\mathcal{M}$
$f : \mathcal{M} \to \mathbb{R}$

$$f\big|_p : T_p\mathcal{M} \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$$

$f\big|_p \in T^*_p\mathcal{M} \subset T^*\mathcal{M}x : U \to x(U)$

$$x\big|_p : T_p\mathcal{M} \to \mathbb{R}^n$$

$\{x^1\big|_p, \ldots, x^n\big|_p\}T^*_p\mathcal{M}$

$x^i\big|_p$

$\frac{\partial}{\partial x^i}\big|_p$

$(x, U)(y, U')$

$$\frac{\partial}{\partial x^i}\big|_p = \sum_j a^j_i \frac{\partial}{\partial y^j}\big|_p, a^j_i = \frac{\partial y^j}{\partial x^i}$$
$$x^k = \sum b^k_l y^l\big|_p = \sum \frac{\partial x^k}{\partial y^l} y^l\big|_p$$

$$
\begin{aligned}
\wedge^m(E', E'')_p &:= \wedge^m(E'_p, E''_p) \\
&= \{f : \underbrace{E'_p \times \cdots \times E'_p}_{n-} \to E''_p\}
\end{aligned}
$$

f

$$
\begin{aligned}
&\phi_{\alpha p} : \wedge^n(\mathbb{R}^{k'}, \mathbb{R}^{k''}) \to \wedge^n(E'_p, E''_p) \\
&f \mapsto ((v_1, \ldots, v_n) \mapsto \phi''_{\alpha p}(f(\phi_{\alpha p})^{-1}(v_1), \ldots f(\phi_{\alpha p})^{-1}(v_n)))
\end{aligned}
$$

$g_{\alpha\beta}$

$$
\begin{gathered}
\wedge^1(E', E'') = (E', E'') \\
\wedge^1(T\mathcal{M}, \mathbb{R}) = T^*\mathcal{M}
\end{gathered}
$$

$(\pi, E, \mathcal{M})(\pi, E', \mathcal{M}')(f, L) f : \mathcal{M} \to \mathcal{M}' L : E \to E'$

$\pi' \circ L = f \circ \pi$

$L\big|_{E_p} \mathbb{R}$

$$
\begin{tikzcd}
E \arrow[r, "L"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\
\mathcal{M} \arrow[r, "f"'] & \mathcal{M}'
\end{tikzcd}
$$

$\mathcal{M}\mathcal{M}' f : \mathcal{M} \to \mathcal{M}'(f, f) T\mathcal{M} T\mathcal{M}'$

$(\pi, E, \mathcal{M}) k E' \subset E k'$

$$
\pi\big|_{E'} : E' \to \mathcal{M},
$$

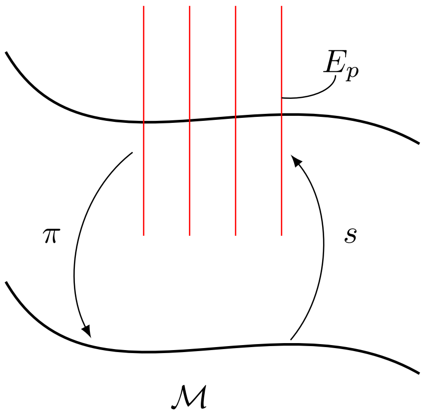
$S^n \subset \mathbb{R}^{n+1}$

$$TS^m \cong \{(p,x) \in S^n \times \mathbb{R}^{n+1} | x \perp p\} \subset \underbrace{S^n \times \mathbb{R}^{n+1}}$$

$\mathbb{RP}^n$

$$\{(l,x) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} | x \in l\} \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$$

$(\pi, E, \mathcal{M}) S : \mathcal{M} \to E E \pi \circ s = \big|_{\mathcal{M}} E \Gamma(E) U \subset \mathcal{M} E U s : U \to E \pi \circ s =_U$



$$S : \mathcal{M} \to E$$
$$p \mapsto 0 \in E_p$$

$T\mathcal{M} V : \mathcal{M} \to T\mathcal{M} \mathfrak{X}(\mathcal{M})$

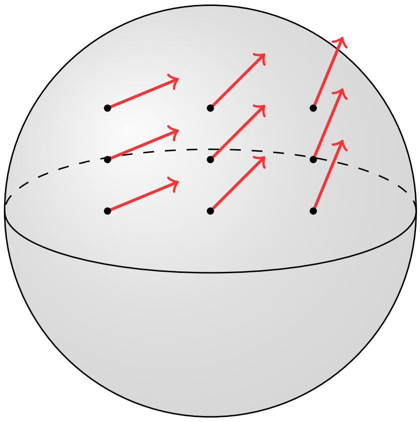
$\Gamma(E) \mathcal{F}(\mathcal{M})$

$s_1, s_2 \in \Gamma(E) s_1 + s_2 \in \Gamma(E)$

$$(s_1 + s_2)(p) := s_1(p) p s_2(p)$$

$\phi \in \mathcal{F}(\mathcal{M}), s \in \Gamma(E) \phi \circ s \in \Gamma(E)$

$$(\phi \circ s)(p) := \phi(p) s(p). \qquad \square$$



$(\pi, E, \mathcal{M}) p \in \mathcal{M} x \in E_p s \in \Gamma(E) s(p) = x$

$E W \ni p$

$$\phi : W \times \mathbb{R}^k \to \pi^{-1}(W) = E\big|_W$$

$\varphi \in \mathcal{F}(\mathcal{M}) \varphi(p) = 1(\varphi) \subset W \xi \in \mathbb{R}^k \phi(p, \xi) = x$

$$s(q) = \begin{cases} \phi(q, \varphi(q)\xi) & q \in W \\ 0_q & q \notin W \end{cases}$$

s

sW

$s 0 \mathcal{M} \setminus W$

$$s(p) = (\varphi(p), \varphi(p)\xi) = \varphi(p\xi) = x \qquad \square$$

$(\pi, E, \mathcal{M}) k U \subset \mathcal{M} E U k (s_1, \ldots, s_k) U s_i \in \Gamma_i(E) p \in U s_1(p), \ldots, s_k(p) E_p$

$(\pi, E, \mathcal{M}) k$

$(s_1, \ldots, s_k) U \subset \mathcal{M}$

$$\phi : U \times \mathbb{R}^k \to E\big|_U$$
$$(p, \xi) \to \sum_{i=1}^{k} \xi_i s_i(p),$$

$\phi : U \times \mathbb{R}^k \to E\big|_U (s_1, \ldots, s_k)$

$$s_i(p) = \phi(p, e_i).$$

$\{e_i\}\mathbb{R}^k$

$$\phi\big|_p : \{p\} \times \mathbb{R}^k \to E\big|_p$$

$$\phi : U \times \mathbb{R}^k \to E\big|_U,$$

$pUV \subset Up$

$$\psi_V : V \times \mathbb{R}^k \to E\big|_V.$$

$$\psi_V^{-1} \circ \phi(q, \xi) = (q, \underbrace{\psi_q^{-1} \circ \phi_q(\xi)})$$ □

$\psi_V^{-1} \circ \phi : V \times \mathbb{R}^k \to V \times \mathbb{R}^k \phi V U \phi$

ϕ_p

$(s_1, \ldots, s_k)\phi s \in \Gamma_U(E) U \subset \mathcal{M}$

$$\sigma : U \to \mathbb{R}^k,$$

$$s(p) = \sum_{i=1}^{k} \sigma_i(p) s_i(p)$$
$$\phi(p, \sigma(p)) = s(p).$$

$\sigma s \phi$

σs
$(t_1, \ldots t_k) V \psi U \cap V \neq \emptyset U \cap V$

$$s_i = \sum_j g_i^j t_j,$$

$g_i^j : U \cap V \to \mathbb{R} g(p) = (g_i^j(p))_{i,j=1}^k$

$$\begin{aligned} g(p)(t_1(p), \ldots, t_k(p)) = (s_1(p), \ldots, s_k(p)) \\ g : U \cap V \ni p \to g(p) \in (E\big|_p) \end{aligned}$$

$s \in \Gamma_{U\cap V}(E) \sigma_\phi \sigma_\psi$

$$\sigma_\phi^i = \sum_{j=1}^k g_i^j \sigma_\psi^j \sigma_\phi = g\sigma_\psi g : U \cap V \to (k, \mathbb{R})$$

$E \xrightarrow{\pi} \mathcal{M} f : \mathcal{N} \to \mathcal{M} E f f^* E$

$(f^*E)_{p\in\mathcal{N}} = \{(p, x) | x \in E_{f(p)}\}$

$\phi : U \times \mathbb{R}^k \to E\big|_U E$

$$\begin{aligned} f^*\phi : f^{-1}(U) \times \mathbb{R}^k \to (f^*E)\big|_{f^{-1}(U)} \\ (p, \xi) \mapsto (p, \phi(f(p), \xi)) \end{aligned}$$

$E f \delta : \mathcal{N} \to E \pi \circ s = f$

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) \\ [x, y]f := x(y(f)) - y(x(f)) \end{aligned}$$

$[x, y] \mathfrak{X}(\mathcal{M})$

$(\pi, E, \mathcal{M}) k E$

$$\begin{aligned} : \mathfrak{X}(\mathcal{M}) \times \Gamma(E) \to \Gamma(E) \\ (x, s) \mapsto (x, s) = {}_x s \end{aligned}$$

x

$$\begin{aligned} {}_{x_1+x_2} s = {}_{x_1} s + {}_{x_2} s \\ {}_{\phi x} s = \phi_x s \end{aligned}$$

$$\begin{aligned} {}_x(s_1 + s_2) = {}_x s_1 + {}_x s_2 \\ {}_x(\phi s) = x(\phi) s + \phi_x s \end{aligned}$$

$_x s s x$

$$E = T\mathcal{M}$$
$$: \underbrace{\mathfrak{X}(\mathcal{M})} \times \underbrace{\mathfrak{X}(\mathcal{M})} \to \mathfrak{X}(\mathcal{M})$$

$E = \mathcal{M} \times \mathbb{R}^k$

$$s : \mathcal{M} \to E$$
$$p \mapsto (p, \sigma(p))$$

$\sigma = (\sigma_1, \ldots, \sigma_k) \sigma_i \in \mathcal{F}(\mathcal{M})$

$$({}_x s)(p) = (p, x_p(\sigma_1), \ldots, x_p(\sigma_k))$$

${}_x s = x(\sigma)$

$x_1, x_2 \in \mathfrak{X}(\mathcal{M}) x_1(p) = x_2(p)$

$$({}_{x_1} s)(p) = ({}_{x_2} s)(p).$$

$s_1, s_2 \in \Gamma(\mathcal{M}) s_1 = s_2 p$

$$({}_x s_1)(p) = ({}_x s_2)(p).$$

$\phi \in \mathcal{F}(\mathcal{M})\ \phi \subseteq U \phi = 1 V \subset U$

$$\phi s_1 = \phi s_2$$
$${}_x(\phi s_1)(p) = {}_x(\phi s_2)(p)$$

$${}_x(\phi s_1)(p) = \underbrace{x(p)}_{=0} s_1(p) + \underbrace{\phi(p)}_{=1} {}_x s_1(p) = D_x s_1(p).$$

$$({}_x s_1)(p) = ({}_x s_2)(p). \qquad \square$$

$\mathcal{L} : \Gamma(E) \to \Gamma(E')$

$$\mathcal{L}(\phi s) = \phi \mathcal{L}(s), \forall \phi \in \mathcal{F}(\mathcal{M})$$

$p \in \mathcal{M} s, \tilde{s} \in \Gamma(E) s(p) = \tilde{s}(p)$

$$\mathcal{L}(s)(p) = \mathcal{L}(\tilde{s})(p)$$

$$
\begin{aligned}
&.s : \mathfrak{X}(\mathcal{M}) \to \Gamma(E) \\
&x \mapsto {}_x s
\end{aligned}
$$

$U p \phi = (s_1, \ldots, s_k) U \varphi \, \varphi \subset U \varphi(p) = 1$

$$s = \sum_{i=1}^{k} \sigma_i s_i, \tilde{s} = \sum_{i=1}^{k} \tilde{\sigma}_i s_i$$

$\sigma_i(p) = \tilde{\sigma}_i(p)$

$$
\begin{aligned}
\mathcal{L}(s)(p) &= \varphi^2(p)\mathcal{L}(s)(p) \\
&= \mathcal{L}(\varphi^2 s)(p) \\
&= \sum_{i=1}^{k} \mathcal{L}((\varphi\sigma_i)(\varphi s_i))(p) \\
&= \sum_{i=1}^{k} \varphi(p)\sigma_i(p)\mathcal{L}((\varphi s_i))(p)
\end{aligned}
$$

$\tilde{s}$

$$\mathcal{L}(\tilde{s})(p) = \sum_{i=1}^{k} \varphi(p)\tilde{\sigma}_i(p)\mathcal{L}((\varphi s_i))(p).$$

$\sigma_i = \tilde{\sigma}_i \varphi(p) = 1$

$$\mathcal{L}(s)(p) = \mathcal{L}(\tilde{s})(p)$$ □

(n, s)

$$T^n_s(\mathcal{M}) = \left(\bigotimes_{i=1}^{n} T\mathcal{M} \right) \otimes \left(\bigotimes_{i=1}^{s} T^{s*}\mathcal{M} \right).$$

(n, s)

$$B : \underbrace{\mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M})}_{s} to \underbrace{\mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M})}_{n},$$

B

n

$$t = \sum_{i=1}^{n} \xi_i \otimes \eta_i$$

V

$$V \times V \to \mathbb{R}$$

$$V^* \otimes V^* = (V \otimes V)^*$$

$$V \otimes V \to \mathbb{R}$$

$\xi, \eta \in V^* \xi, \eta : V \to \mathbb{R}$

$$(\xi \otimes \eta)(v, w) = \xi(v)\eta(w).$$

$$\left(\bigotimes^{n} V \right) \otimes \left(\bigotimes^{s} V^* \right)$$

$B(n, s) B p s$

$$\begin{aligned} &B_p : T_p\mathcal{M}^s \to T_p\mathcal{M}^n \\ &(v_1, \ldots, s) \mapsto B_p(v_1, \ldots, v_s) \end{aligned}$$

$$s(p) = (p, \sigma(p)), s \in \Gamma(\mathcal{M} \times \mathbb{R}^k),$$

$\sigma = (\sigma_1, \ldots, \sigma_k) \sigma_i \in \mathcal{F}(\mathcal{M})$

$${}_x s = (x(\sigma_1), \ldots, x(\sigma_k))$$

$\omega 1\mathcal{M}_{k\times k}(\mathbb{R})$

$$\begin{aligned}
&\omega \in \Gamma((T\mathcal{M}, {}_{k\times k}(\mathbb{R}))) \\
&\omega_p : T\mathcal{M} \to {}_{k\times k}(\mathbb{R}) \\
&\omega = (\omega ij)_{i,j=1}^{k}
\end{aligned}$$

$\omega_{ij} 1\mathcal{M}$

$$w_p^{ij} : T_p\mathcal{M} \to \mathbb{R}.$$

1

$$({}_x^{\omega} s)(p) = (p, x_p(\sigma) + \omega_p(x_p)\sigma(p))$$

$(E, \pi, \mathcal{M}) E \omega 1(E, E)$

$$D_x^{\omega} s = {}_x s + \omega(x) s,$$

E

$' E$

$$\omega(x) s = {}'_x s - {}_x s$$

$1(E, E)$

$E = \mathcal{M} \times \mathbb{R}^k$

$${}_x^{\omega} s = (p, x_(\sigma) + \omega_p(x_p)\sigma_p)$$

$cov d_x^{\omega} s E$

$$\phi : U \times \mathbb{R}^k \to E\big|_U.$$

$1\phi = (s_1, \ldots, s_k) E U x \in \mathfrak{X}(\mathcal{M})_x s_1, \ldots, {}_x s_k \in \Gamma(E)$

$${}_x s_i = \sum_{i=1}^{k} \omega_{ij}(x) s_j$$

$1\omega_{ij}(x) : U \to \mathbb{R} x s = \sum_i \sigma_i s_i$

$$\begin{aligned}
{}_x s &= \sum (x(\sigma_i) s_i + \sigma_i + \sigma_{ix} s_i) \\
&= \sum_j x(\sigma_j) s_j + \sum_{j=1}^{k} \sum_{i=1}^{k} \sigma_i \omega_{ij} s_j \\
&= \sum_j \left[x(\sigma_j) + \sum : i\sigma_i \omega_{ij}(x) \right] s_j
\end{aligned}$$