Quark Spectral Functions from Dyson-Schwinger Equations with Spectral Renormalization

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Outline

1. Introduction: Dyson-Schwinger Equations, Spectral Representations
2. Methodology: Spectral Renormalization Scheme
3. (Preliminary) Results

Outline

Real-time Formulation of QFT

- Application of non-perturbative techniques (FRG, DSE, lattice): Usually Euclidean expressions
- If we want to access dynamical properties $\rightarrow$ Need real-time quantities!

Is there a (simple) possibility to get access to the respective real-time quantities?

- Problem: Map from $\mathbb{R}^4$ to Minkowski space is non-trivial!
- Need Numerical reconstruction techniques or access to algebraic momentum structure!

Dyson-Schwinger Equations

- From shift symmetry of the path integral measure, the master DSE can be derived:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \frac{\delta S[\phi]}{\delta\varphi(x)}\left[\varphi = G\cdot\frac{\delta}{\delta\phi} + \phi\right] \tag{1}$$

- Higher correlation functions are obtained by taking functional derivatives: $\left(\frac{\delta}{\delta\phi}\right)^n$

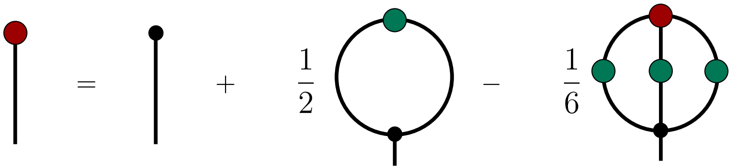


Figure: Diagrammatic representation of the master DSE in scalar theory

Källén-Lehmann Spectral Representation of Correlation Functions

- Spectral representation of the (scalar) propagator:

$$G(p) = \int_0^\infty \frac{\mathrm{d}\lambda^2}{\pi} \frac{\rho(\lambda^2)}{p^2 + \lambda^2} . \tag{2}$$

- Inverse relation between spectral function ρ and (retarded) propagator:

$$\rho(\omega, |\mathbf{p}|) = 2\,\mathrm{Im}\left[G\left(-i(\omega + i0^+), |\mathbf{p}|\right)\right] . \tag{3}$$

- Spectral reps. for higher order correlators should exist from axiomatic viewpoint.
  Here: Classical vertex approximation!

Example: Scalar Theory

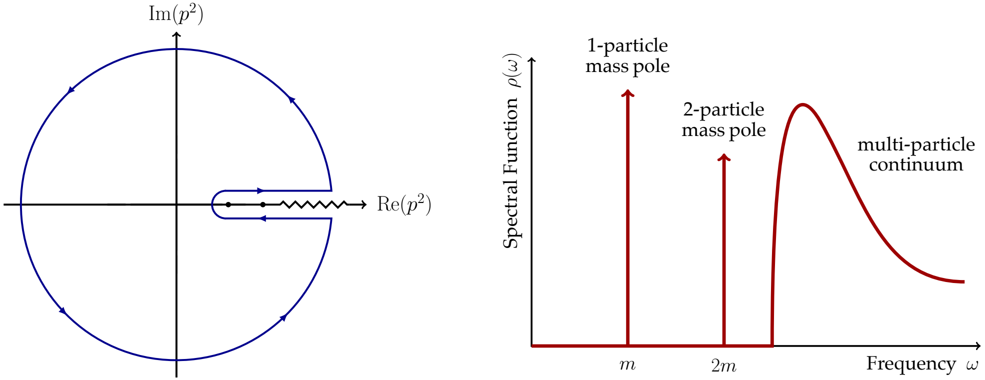


Figure: *Left:* Analytic Structure of a scalar propagator with respective integration contour in the complex plane. *Right:* Shape of the associated spectral function.

Relevant DSEs for QCD Propagators

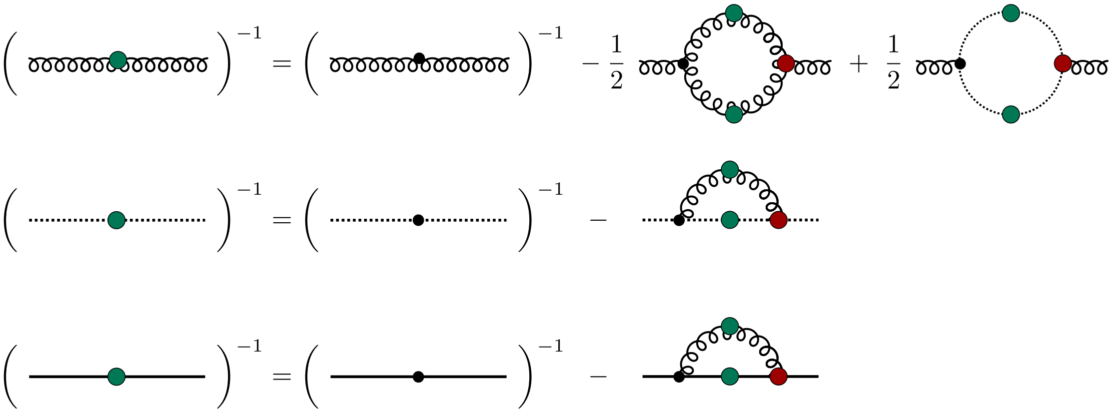
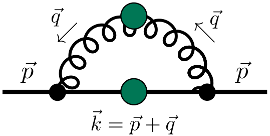


Figure: *Top row:* Gluon DSE, truncated at one-loop. *Middle row:* Ghost DSE. *Bottom row:* Quark DSE.

Quark Self Energy Diagram

- Assuming classical vertices, the nontrivial contribution to the quark DSE reads:



$$:= \Sigma_q(p) = (-ig)^2 \delta^{ab} C_f \int_q \Pi_\perp^{\mu\nu}(q) G_A(q) \gamma_\mu G_q(p+q) \gamma_\nu . \tag{4}$$

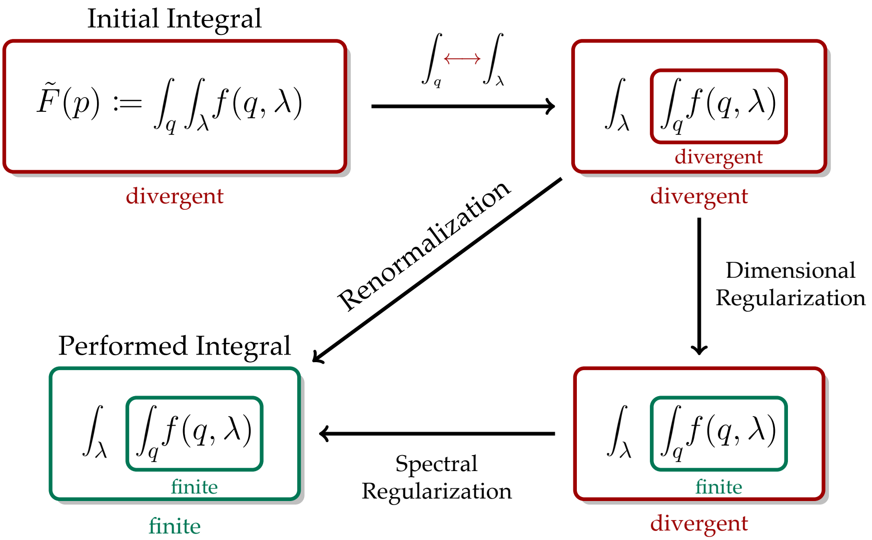
- Insert spectral representations for gluon and quark propagators

$$G_A(q) = \int_{\lambda_1} \frac{\lambda_1 \rho_A(\lambda_1)}{q^2 + \lambda_1^2}, \tag{5}$$

$$\begin{aligned} G_q(p+q) &= -i(\not{p} + \not{q}) \left(\frac{Z_q(p)}{p^2 + M_q^2(p)} \right) + \left(\frac{Z_q(p) M_q(p)}{p^2 + M_q^2(p)} \right) \\ &= (\not{p} + \not{q}) \int_{\lambda_2} \frac{\lambda_2 \rho_q^D(\lambda_2)}{(p+q)^2 + \lambda_2^2} + \int_{\lambda_2} \frac{\lambda_2 \rho_q^M(\lambda_2)}{(p+q)^2 + \lambda_2^2} . \end{aligned} \tag{6}$$

Outline

Spectral Renormalization Scheme: Overview



BPHZ-type Subtraction Scheme for the Spectral Integrands

- Remnant of swapping integration order: we are not done after Dimensional Regularization!

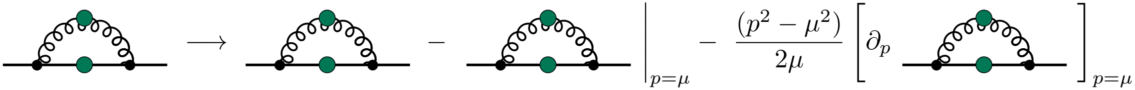


Figure: Schematic spectral BPHZ-renormalization procedure at the example of the quark self energy diagram.

- The diagram is subtracted by the first two terms of its own Taylor expansion around the RG scale μ in order to cancel quadratic and logarithmic divergences of the spectral integrands.

- This results in finite spectral integrands in the limit $\varepsilon \to 0$.

Analytic Continuation of the finite Integrands

- Technical Detail: Split calculation of the diagram in Dirac vector and scalar part.

$$\Sigma_q(p) = \not{p} \cdot \Sigma_q^D(p) + \Sigma_q^M(p), \tag{7}$$

- We are left with finite spectral integrals:

$$\Sigma_q^D(p) = (-ig)^2 \delta^{ab} C_f \int\limits_{\{\lambda_1,\lambda_2\}} \lambda_1 \lambda_2 \rho_A(\lambda_1) \rho_q^D(\lambda_2) \cdot I_q^D\left(p, \lambda_1, \lambda_2, x\right), \tag{8}$$

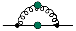
$$\Sigma_q^M(p) = (-ig)^2 \delta^{ab} C_f \int\limits_{\{\lambda_1,\lambda_2\}} \lambda_1 \lambda_2 \rho_A(\lambda_1) \rho_q^M(\lambda_2) \cdot I_q^M\left(p, \lambda_1, \lambda_2, x\right). \tag{9}$$

- Analytic continuation to real frequencies according to

$$I_q^{(D/M)}\left(\omega, \lambda_1, \lambda_2\right) := I_q^{(D/M)}\left(-i(\omega + i0^+)\right). \tag{10}$$

Iterative Solution of the Quark Propagator DSE

1. Choose appropriate initial spectral functions $\rho_{q,\mathrm{init}}^{(D/M)}$

2. Compute  employing the updated spectral functions

3. Access $\Gamma_q^{(2)}$ and the propagator dressings M_q and Z_q via the DSE

4. Extract $\rho_q^{(D/M)}$ from the respective propagator dressings

Feed back into the DSE

Iterate until convergence

Initial Setup: Gluon Spectral Function

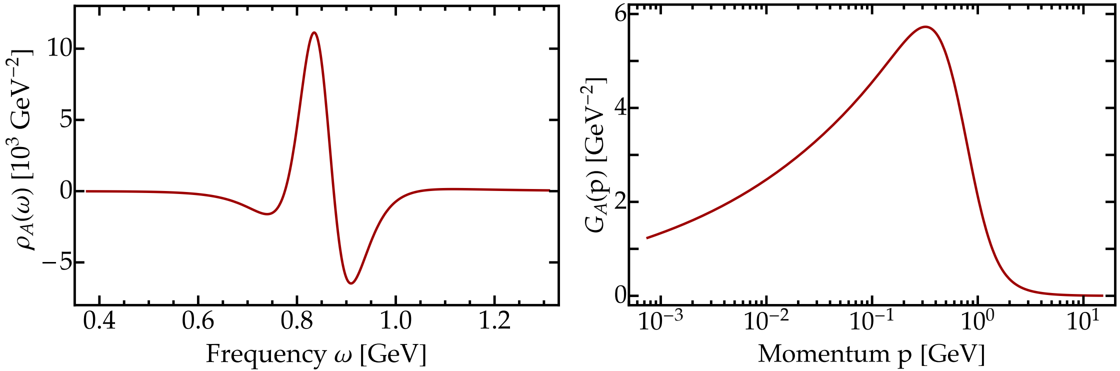


Figure: Input gluon spectral function and corresponding propagator. Taken from arXiv:1804.00945.

Outline

Dressing Functions

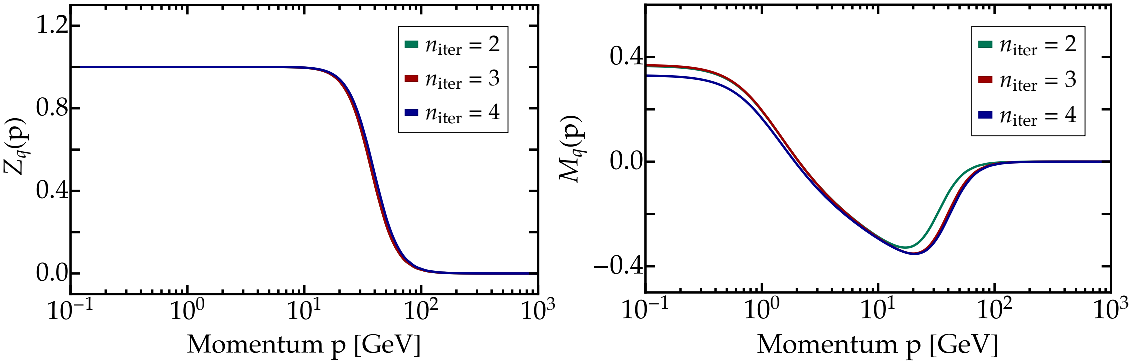


Figure: Computed quark propagator dressings $Z_q(p)$ and $M_q(p)$ for different iteration steps.

Quark Spectral Functions

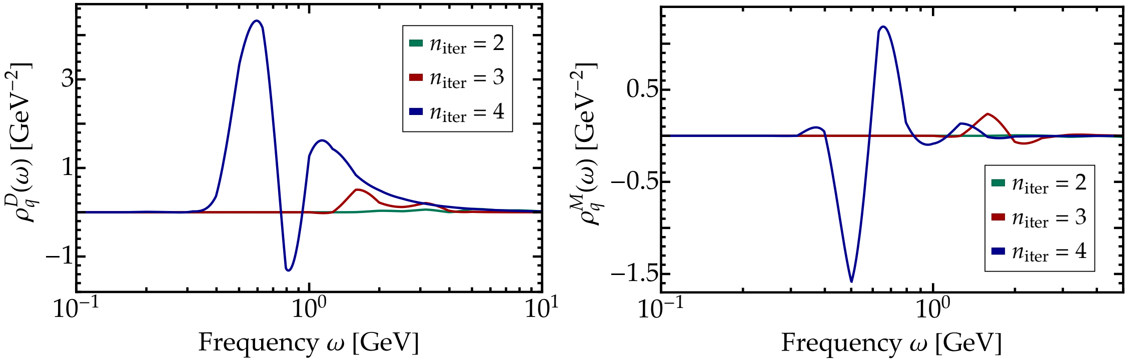


Figure: Computed quark spectral functions $\rho_q^D(\omega)$ and $\rho_q^M(\omega)$ for different numbers of iterations.

Problem: Mismatch in the Comparison of the Euclidean Propagators

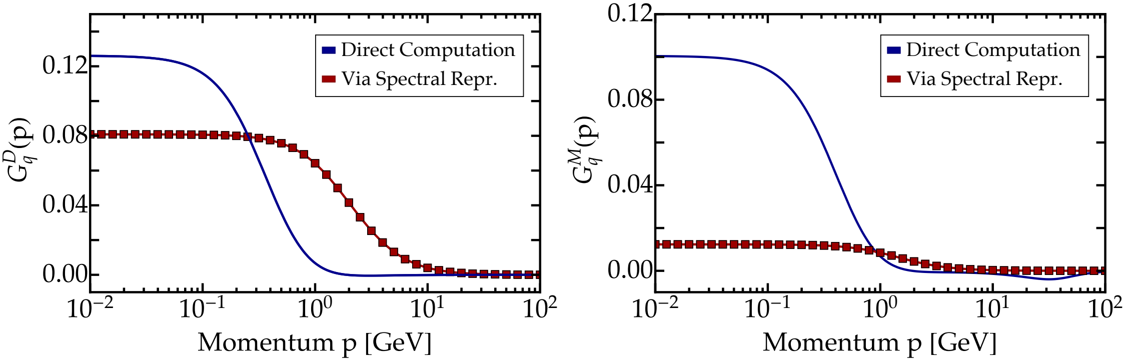


Figure: Comparison of the respective parts of the Euclidean quark propagator with the result obtained from the spectral representation for the last iteration step $n_{\text{iter}} = 4$.

Summary and Outlook

Current State of the Art:

- Spectral renormalization scheme allows us to access (quark) spectral functions from the respective DSE
- Need to resolve mismatch in benchmark tests (Euclidean Propagators, Dressings, Perturbative expression for the Diagram ..)

Next steps:

- Update Numerics: Avoid double counting of pole contribution, test different initial conditions, discretization ...

On longer terms:

- Full access to all QCD spectral functions ($\rightarrow$ Jan H., Nicolas, Julian et al.)
- Extend results to finite temperatures and densities ($\rightarrow$ Joschka)
- Access real-time dynamics of QCD

Backup I: Dirac Structure of the Quark Propagator DSE

The quark propagator is obtained by inverting the full two-point function:

$$\begin{aligned}\Gamma_q^{(2)}(p) = G_q^{-1}(p) &= G_{q,0}^{-1}(p) - \Sigma_q(p) \\ &= i\not{p}A(p) + B(p) \\ &= \frac{1}{Z_q(p)}\Big(i\not{p} + M_q(p)\Big)\,. \end{aligned} \tag{11}$$

Here we introduced the dressing functions $Z_q(p)$ and $M_q(p)$ via

$$Z_q(p) = \frac{1}{A(p)} \qquad \text{and} \qquad M_q(p) = \frac{B(p)}{A(p)}. \tag{12}$$

Backup I: Dirac Structure of the Quark Propagator DSE

In terms of these dressings, the quark propagator reads:

$$
\begin{aligned}
G_q(p) &= Z_q(p)\left(\frac{1}{i\not{p} + M_q(p)}\right) \\
&= Z_q(p)\left(\frac{-i\not{p} + M_q(p)}{(i\not{p} + M_q(p))(-i\not{p} + M_q(p))}\right) \\
&= Z_q(p)\left(\frac{-i\not{p} + M_q(p)}{p^2 + M_q^2(p)}\right) \\
&= -i\not{p}\left(\frac{Z_q(p)}{p^2 + M_q^2(p)}\right) + \left(\frac{Z_q(p)M_q(p)}{p^2 + M_q^2(p)}\right) \\
&\equiv -i\not{p} \cdot G_q^D(p) + G_q^M(p).
\end{aligned}
\tag{13}
$$

Backup I: Dirac Structure of the Quark Propagator DSE

Now we can identity the different parts of this results with the spectral representation of the quark propagator:

$$\begin{aligned} G_q(p) &= -i\not{p} \cdot G_q^D(p) + G_q^M(p) \\ &\equiv \not{p} \int_\lambda \frac{\lambda \rho_q^D(\lambda)}{p^2 + \lambda^2} + \int_\lambda \frac{\lambda \rho_q^M(\lambda)}{p^2 + \lambda^2}. \end{aligned} \tag{14}$$

But how can we now extract the spectral functions from the propagator?

Backup II: Extracting the Spectral Function from the Retarded Propagator

Consider explicitly the "analytic continuation" $p \to -i\,(\omega + i\varepsilon)$ in the limit $\epsilon \to 0^+$:

$$
\begin{aligned}
\lim_{\epsilon \to 0^+} \frac{1}{-(\omega + i\epsilon)^2 + \lambda^2} &= \lim_{\epsilon \to 0^+} \frac{1}{-\omega^2 + \epsilon^2 + \lambda^2 - 2i\epsilon\omega} \\
&= \lim_{\epsilon \to 0^+} \frac{-\omega^2 + \epsilon^2 + \lambda^2 + 2i\epsilon\omega}{(\epsilon^2 + \lambda^2 - \omega^2)^2 + 4\omega^2\epsilon^2}.
\end{aligned}
\tag{15}
$$

Backup II: Extracting the Spectral Function from the Retarded Propagator

Focusing only on the imaginary part yields

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \mathrm{Im}\left[\frac{1}{-(\omega+i\epsilon)^2+\lambda^2}\right] &= \lim_{\epsilon\to 0^+} \frac{2\epsilon\omega}{(\epsilon^2+\lambda^2-\omega^2)^2+4\omega^2\epsilon^2} \\
&= \frac{1}{2\omega}\lim_{\epsilon\to 0^+} \frac{\epsilon}{\left(\frac{\epsilon^2+\lambda^2-\omega^2}{2\omega}\right)^2+\epsilon^2} \\
&= \frac{\pi}{2\omega}\delta\left(\frac{\lambda^2-\omega^2}{2\omega}\right) \\
&= \frac{\pi}{2\omega}\left(\left.\frac{\partial\left(\frac{\lambda^2-\omega^2}{2\omega}\right)}{\partial\lambda}\right|_{\lambda=\omega}\right)^{-1}\delta\left(\lambda-\omega\right) \\
&= \frac{\pi}{2\omega}\delta\left(\lambda-\omega\right).
\end{aligned} \tag{16}$$

Backup II: Extracting the Spectral Function from the Retarded Propagator

This leaves us with

$$G(p) = \int_\lambda \frac{\lambda\rho(\lambda)}{p^2+\lambda^2} \;\longrightarrow\; \int_\lambda \frac{\lambda\rho(\lambda)}{-\omega^2+\lambda^2} + i\frac{\rho(\omega)}{2} = G\left(-i(\omega+i0^+)\right), \tag{17}$$

and explains why the spectral function can be obtained from the imaginary part of the retarded propagator:

$$\rho(\omega) = 2\,\mathrm{Im}\left[G\left(-i(\omega+i0^+)\right)\right]. \tag{18}$$

For our specific case of the quark propagator we conclude that:

$$\rho_q^D(\omega) = 2\,\mathrm{Im}\left[\Pi_{\not{p}} G_q\left(-i(\omega+i0^+)\right)\right] = 2\,\mathrm{Im}\left[-iG_q^D(-i(\omega+i0^+))\right] \tag{19}$$

$$\rho_q^M(\omega) = 2\,\mathrm{Im}\left[\Pi_{\mathbb{1}} G_q\left(-i(\omega+i0^+)\right)\right] = 2\,\mathrm{Im}\left[G_q^M(-i(\omega+i0^+))\right], \tag{20}$$