# Examples for the Filtering Problem

## Example 6.2.9 (Noisy observations of a constant process)

This is the continuous analogue of Example 6.2.1: where you try to find the best linear estimate of independent real random variables.

Consider the case:

$dX_t = 0, i.e. X_t = X_0; E[X_0] = \hat{X}_0, E[X_0^2] = a^2$ (*system*)

$dZ_t = X_t dt + m dV_t; X_0 = 0$ (*observation*)

$H_t = \frac{dZ}{dt} = X_t + mW_t, W_t$ (*white noise*)

**The Riccati equation:**

Any first-order ODE that is quadratic in the unknown function. In other words, it is an equation of form

$y'(x) = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x)$

where $q_0(x) \neq 0$, $q_2(x) \neq 0$ and $q_i(x)$ are continuous functions of $x$.

It just makes things easier to be solved.

First we solve the corresponding Riccati equation for

$S(t) = E[(X_t - \hat{X}_t)^2]$

$\frac{dS}{dt} = -\frac{1}{m^2}S^2$

$S(0) = a^2$

i.e. $S(t) = \frac{a^2m^2}{m^2+a^2t}, t \geq 0$

This gives the following equation for $\hat{X}_t$:

$d\hat{X}_t = -\frac{a^2}{m^2+a^2t}\hat{X}_t dt + \frac{a^2}{m^2+a^2t}dZ_t$

$\hat{X}_0 = E[X_0] = 0$

or $d(\hat{X}_t exp(\int_0^t \frac{a^2}{m^2+a^2d}ds)) = exp(\int_0^t \frac{a^2}{m^2+a^2d}ds)\frac{a^2}{m^2+a^2t}dZ_t$

which gives

$\hat{X}_t = \frac{m^2}{m^2+a^2t}\hat{X}_0 + \frac{a^2}{m^2+a^2t}Z_t$

$t \geq 0$

## Example 6.2.10 (Noisy observations of a Brownian motion)

Modify the preceding example slightly such that

$dX_t = c dU_t; E[X_0] = \hat{X}_0; E[X_0^2] = a^2$ (*system*)

$dZ_t = X_t dt + m dV_t; X_0 = 0$ (*observation*)

$H_t = \frac{dZ}{dt} = X_t + mW_t, W_t$ (*white noise*)

The Riccati equation becomes

$\frac{dS}{dt} = -\frac{1}{m^2}S^2 + c^2, S(0) = a^2$

or $\frac{m^2 dS}{m^2c^2 - S^2} = dt, (S \neq mc)$

This gives

$|\frac{mc+s}{mc-s}| = Kexp(\frac{2ct}{m})$

$K = |\frac{mc+a^2}{mc-a^2}|$

or

$$S(t) = \begin{cases} mc\frac{K.exp(\frac{2ct}{m})-1}{K.exp(\frac{2ct}{m})+1}, & if S(0) < mc \\ mc(const), & if S(0) = mc \\ mc\frac{K.exp(\frac{2ct}{m})+1}{K.exp(\frac{2ct}{m})-1}, & if S(0) > mc \end{cases}$$

Thus, in all cases the mean square error S(t) tends to $mc$ as $t \to \infty$.

For simplicity, put $a = 0, m = c = 1$. Then

$$S(t) = \frac{exp(2t) - 1}{exp(2t) + 1} = tanh(t)$$

The equation for $\hat{X}_t$ is

$d\hat{X}_t = -tanh(t)\hat{X}_t dt + tanh(t)dZ_t$

$\hat{X}_0 = 0$

or $d(cosh(t)\hat{X}_t) = sinh(t)dZ_t$

So:

$\hat{X}_t = \frac{1}{cosh(t)}\int_0^t sinh(s)dZ_t$

If we return to the interpretation of $Z_t$:

$$Z_t = \int_0^t H_s ds$$

where $H_s$ are the "original" observations.

In the continuous version of the filtering problem we assume that the observations $H_t \in \mathbb{R}^m$ are performed continuously and are of the form

$H_t = c(t, X_t) + \gamma(t, X_t) \cdot \widetilde{W}_t$

where $\widetilde{W}_t$: $r$-dimensional white noise

and $\quad c : \mathbb{R}^{n+1} \to \mathbb{R}^m$

$\quad \gamma : \mathbb{R}^{n+1} \to \mathbb{R}^{m \times r}$

are functions satisfying Theorem 5.2.1: Existence and uniqueness theorem for stochastic differential equations.

We can write:

$$\hat{X}_t = \frac{1}{cosh(t)}\int_0^t sinh(s)H_s ds$$

so $\hat{X}_t$ is approximately (for large $t$) a weighted average of the observations $H_s$, with increasing emphasis on observations as time increases.

**Remark:**

It is interesting to compare formula above with established formulas in forecasting.

The *exponentially weighted moving average* $\hat{X}_n$: a first-order infinite impulse response filter that applies weighting factors which decrease exponentially. The weighting for each older datum decreases exponentially, never reaching zero.

suggested by C.C.Holt in 1985 is given by

$$\widetilde{X}_n = (1-\alpha)^n Z_0 + \alpha\sum_{k=1}^{n}(1-\alpha)^{n-k}Z_k$$

where $\alpha$ is some constant; $0 \le \alpha \le 1$.

This may be written

$$\widetilde{X}_n = \beta^{-n}Z_0 + (\beta-1)\beta^{-n-1}\sum_{k=1}^{n}\beta^k Z_k$$

where $\beta = \frac{1}{1-\alpha}$ (assuming $\alpha < 1$), which is the discrete version of what we just proven. Or more precisely - of the formula corresponding to it in the general case when $a \neq 0$ and $m, c$ are not necessarily equal to 1.

## Example 6.2.11 (Estimation of a parameter)

Suppose we want to estimate the value of a (constant) parameter $\theta$, based on observation $Z_t$ satisfying the model

$$dZ_t = \theta M(t)dt + N(t)dB_t$$

where $M(t), N(t)$ are known functions. In this case the stochastic differential equation for $\theta$ is of course $d\theta = 0$.

so the Riccati equation for $S(t) = E[(\theta - \hat{\theta}_t)^2]$ is

$$\frac{dS}{dt} = -(\frac{M(t)S(t)}{N(t)})^2$$

which gives

$$S(t) = (S_0^{-1} + \int_0^t M(s)^2 N(s)^{-2}ds)^{-1}$$

and the Kalman-Bucy filter is

$$d\hat{\theta}_t = \frac{M(t)S(t)}{N(t)^2}(dZ_t - M(t)\hat{\theta}_t dt)$$

This can be written

$$(S_0^{-1} + \int_0^t M(s)^2 N(s)^{-2}ds)d\hat{\theta}_t + M(t)^2 N(t)^{-2}\hat{\theta}_t dt = M(t)N(t)^{-2}dZ_t$$

Left hand side: $d((S_0^{-1} + \int_0^t M(s)^2 N(s)^{-2}ds)\hat{\theta}_t)$

So we obtain

$$\hat{\theta}_t = \frac{\hat{\theta}_0 S_0^{-1} + \int_0^t M(s)N(s)^{-2}dZ_s}{S_0^{-1} + \int_0^t M(s)^2 N(s)^{-2}ds}$$

This estimate coincides with the maximum likelihood estimate in classical estimation theory if $S_0^{-1} = 0$.

## Example 6.2.12 (Noisy observations of a population growth)

Consider a simple growth model ($r$ constant)

$dXt = rX_t dt$

$E[X_0] = b > 0$

$E[(X_0 - b)^2] = a^2$

with observations

$dZ_t = X_t dt + mdV_t$, $\qquad m$ constant

The corresponding Riccati equation

$$\frac{dS}{dt} = 2rS - \frac{1}{m^2}S^2$$

$S(0) = a^2$

gives the logistic curve

$S(t) = \frac{2rm^2}{1+Ke^{-2rt}}$

where $K = \frac{2rm^2}{a^2} - 1$

So the equation for $\hat{X}_t$ becomes

$$d\hat{X}_t = (r - \frac{X}{m^2})\hat{X}_t dt + \frac{S}{m^2}dZ_t$$

$\hat{X}_0 = E[X_0] = b$

For simplicity let us assume that $a^2 = 2rm^2$, so that $S(t) = 2rm^2$ for all $t$

(In the general case $S(t) \to 2rm^2$ as $t \to \infty$, so this is not an unreasonable approximation for large $t$). Then we get

$$d(\exp(rt)\hat{X}_t) = \exp(rt)2rdZ_t$$

$\hat{X}_0 = b$

or

$$\hat{X}_t = \exp(-rt)[\int_0^t 2r\exp(rs)dZ_s + b]$$

As in example 6.2.10 this may be written

$$\hat{X}_t = \exp(-rt)[\int_0^t 2r\exp(rs)H_s ds + b]$$

if $Z_t = \int_0^t H_s ds$

For example, assume that $H_s = \beta$ (const) for $0 \le s \le t$, i.e. that our observations (for some reason) give the same value $\beta$ for all times $s \le t$. Then, as as $t \to \infty$.

$$\hat{X}_t = 2\beta - (2\beta - b)\exp(-rt) \to 2\beta$$

If $H_s = \beta \cdot \exp(\alpha s)$, $s \ge 0$ ($\alpha$ const), for large $t$, we get

$$\hat{X}_t = \exp(-rt)[\frac{2r\beta}{r+\alpha}(\exp(r+\alpha)t - 1) + b] \approx \frac{2r\beta}{r+\alpha}\exp(\alpha t)$$

Thus, only if $\alpha = r$, i.e. $H_s = \beta\exp(rs)$; $s \ge 0$, does the filter "believe" the observations in the long run. And only if $\alpha = r$ and $\beta = b$, i.e. $H_s = b\exp(rs)$; $s \ge 0$ does the filter "believe" the observations at all times.

## Example 6.2.13 (Constant coefficients – general discussion)

Now consider the system

$dX_t = FX_t dt + CdU_t$;     constants $F, C \neq 0$

with observations

$dZ_t = GX_t dt + DdV_t$;     constants $G, D \neq 0$

The corresponding Ricatti equation

$S' = 2FS - \frac{G^2}{D^2}S^2$;          $S(0) = a^2$

has the solution

$$S(t) = \frac{\alpha_1 - K\alpha_2 \exp(\frac{(\alpha_2-\alpha_1)G^2 t}{D^2})}{1 - K\exp(\frac{(\alpha_2-\alpha_1)G^2 t}{D^2})}$$

where
$$\alpha_1 = G^{-2}(FD^2 - D\sqrt{F^2D^2 + G^2C^2})$$
$$\alpha_2 = G^{-2}(FD^2 + D\sqrt{F^2D^2 + G^2C^2})$$

$K = \frac{a^2 - \alpha_1}{a^2 - \alpha_2}$

This gives the solution for $\hat{X}_t$ of the form

$$\hat{X}_t = \exp(\int_0^t H(s)ds)\hat{X}_0 + \frac{G}{D^2}\int_0^t \exp(\int_s^t H(u)du)S(s)dZ_s$$

where

$$H(s) = F - \frac{G^2}{D^2}S(s)$$

For large $s$ we have $S(s) \approx \alpha_2$. This gives

$$\hat{X}_t \approx \hat{X}_0 \exp((F - \frac{G^2\alpha_2}{D^2})t) + \frac{G\alpha_2}{D^2}\int_0^t \exp((F - \frac{G^2\alpha_2}{D^2})(t-s))dZ_s = \hat{X}_0\exp(-\beta t) + \frac{G\alpha_2}{D^2}\exp(-\beta t)\int_0^t \exp(\beta s)d$$

where $\beta = D^{-1}\sqrt{F^2D^2 + G^2C^2}$. So we get approximately the same behavior as in the previous example.