Mathilde Simoni
mps565

HW4 Computational Physics: Exploring Radial Velocity

1. A 2-Body system

I used the 4th order Runga-Kutta integrator for all simulations in this assignment.

Non-dimensionalization

Let

- $1AU = 1.496 \times 10^{8}$ km
- $1 year = 3.156 \times 10^{7}$ seconds
- $1 M_*$ (mass of sun) $= 1.989 \times 10^{30}$ kg

We need to non-dimentionalize the equations:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i & (1) \\ \dfrac{dv_i}{dt} = -\sum\limits_{i \neq j} G \dfrac{m_j (x_i - x_j)}{|\bar{x}_i - \bar{x}_j|^2} & (2) \end{cases}$$

We set:

$$\boxed{\begin{aligned} x &= 1AU\,\hat{x} \\ t &= 1year\,\hat{t} \\ m &= 1M_*\,\hat{m} \\ v &= \frac{1AU}{1year}\,\hat{v} \end{aligned}} \quad \left.\begin{aligned} &\text{with } x = [L] \text{ in km} \\ &\text{with } t = [T] \text{ in seconds (s)} \\ &\text{with } m = [M] \text{ in kg} \\ &\text{with } v = \left[\tfrac{L}{T}\right] \text{ in } \tfrac{km}{s} \end{aligned}\right\} \hat{x}, \hat{t}, \hat{m}, \hat{v} \text{ are non-dimensional}$$

Now, for (1): $\dfrac{dx_i}{dt} = v_i \iff \dfrac{d(1AU\,\hat{x}_i)}{d(1year\,\hat{t})} = \dfrac{1AU}{1year}\hat{v}_i$

$$\iff \frac{1AU}{1year}\frac{d(\hat{x}_i)}{d(\hat{t})} = \frac{1AU}{1year}\hat{v}_i$$

$$\iff \frac{d\hat{x}_i}{d\hat{t}} = \hat{v}_i$$

For (2): $\dfrac{dv_i}{dt} = -\sum\limits_{i \neq j} G \dfrac{m_j (x_i - x_j)}{|\bar{x}_i - \bar{x}_j|^3} \iff \dfrac{d(\frac{1AU}{1year}\hat{v}_i)}{d(1year\,\hat{t})} = -\sum\limits_{i \neq j} G \dfrac{M_* \hat{m}_j (1AU\,\hat{x}_i - 1AU\,\hat{x}_j)}{|1AU\,\hat{x}_i - 1AU\,\hat{x}_j|^3}$

$$\iff \frac{d\hat{v}_i}{d\hat{t}} \cdot \frac{1AU}{1year^2} = -\sum_{i \neq j} \frac{G M_* 1AU\;\hat{m}_j (\hat{x}_i - \hat{x}_j)}{1AU^3\,|\hat{x}_i - \hat{x}_j|^3}$$

$$\iff \frac{1AU}{1year^2}\frac{d\hat{v}_i}{d\hat{t}} = -\frac{GM_*}{1AU^2}\sum_{i \neq j}\frac{\hat{m}_j(\hat{x}_i - \hat{x}_j)}{|\hat{x}_i - \hat{x}_j|^3}$$

$$\iff \frac{d\hat{v}_i}{d\hat{t}} = -\sum_{i \neq j} cte\,\frac{\hat{m}_j(\hat{x}_i - \hat{x}_j)}{|\hat{x}_i - \hat{x}_j|^3} \quad \text{with } cte = \frac{GM_*\,year^2}{1AU^3}$$

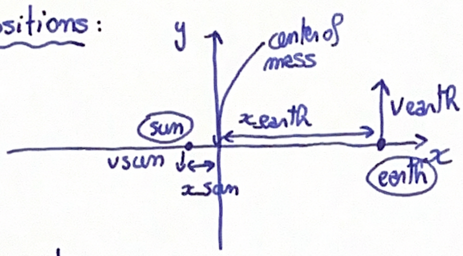
Conclusion We got:

$$\boxed{\begin{aligned} &\frac{d\hat{x}_i}{d\hat{t}} = \hat{v}_i \\ &\frac{d\hat{v}_i}{d\hat{t}} = -\sum_{i \neq j} cte\,\frac{\hat{m}_j(\hat{x}_i - \hat{x}_j)}{|\bar{x}_i - \bar{x}_j)^3} \\ &cte = \frac{GM_*\,year^2}{AU^3} \end{aligned}}$$

Initial Conditions

For this 2-body system, we simulate the earth around the sun
We assume that the earth has a circular orbit around the sun.

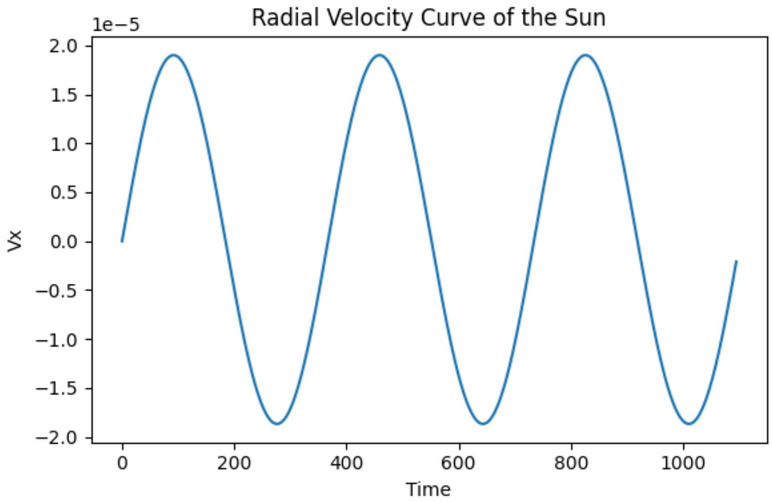
Initial positions:



* We set x_earth to 1AU
* then $x\_sun = \frac{m\_earth \times x\_earth}{m\_sun}$ m_earth mass of earth, m_sun mass of sun
* $v\_earth = \sqrt{\frac{cte \cdot m\_sun}{x\_sun + x\_earth}}$ cte = constant for non-dimensionalization, $cte = \frac{G \times M_* \times 1 year^2}{1AU^3}$
* $v\_sun = \sqrt{\frac{cte \times m\_earth}{x\_sun + x\_earth}}$
* z_sun, z_earth = 0

Analysis

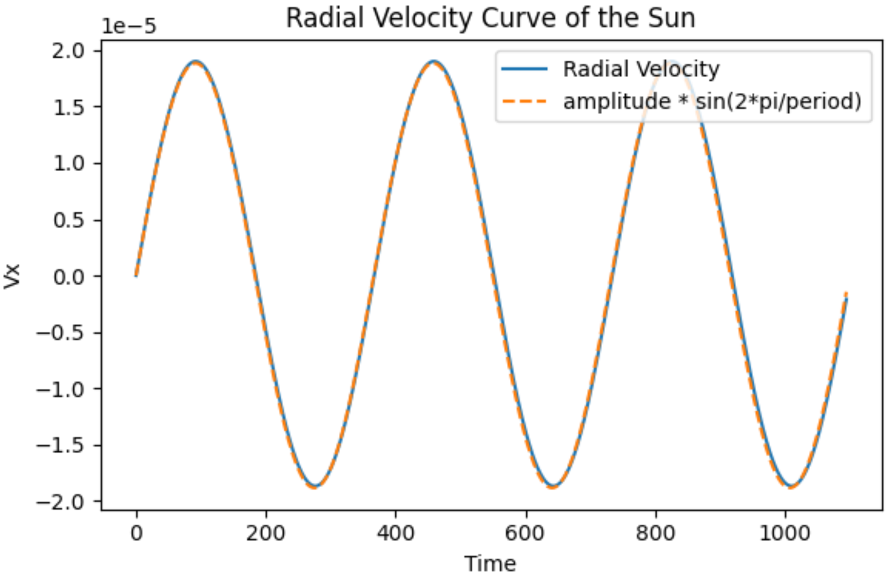
We chose to analyze the variation of Vx. However, the same results could have been obtained by analyzing Vy.



Based on this curve, we observe that the radial velocity as a function of time is a periodic function. We calculate its period and amplitude:

```
period = 366.5 days
amplitude = 1.883e-05 (non-dimensional units)
```

Then, we plot the function *amplitude * sin(2pi/period)* and obtain the following graph, which shows that the radial velocity of the star (here the sun) follows a sine function with appropriate parameters (orbit and amplitude).



<u>Derivation of Circular Velocity</u>

<u>Derivation of circular velocity:</u>

$$\begin{cases} a^3 = \dfrac{G(M_\star + M_p)T^2}{4\pi^2} & (1) \\ V_p = \sqrt{\dfrac{G(M_\star + M_p)}{a}} & (2) \\ V_\star = \dfrac{M_p V_p}{M_\star} & (3) \end{cases}$$

(2) and (3): $V_\star = \dfrac{M_p}{M_\star}\sqrt{\dfrac{G(M_\star+M_p)}{a}}$ (4)

(4) and (1):

$$V_\star = \frac{M_p}{M_\star}\sqrt{\frac{G(M_\star+M_p)(4\pi^2)^{1/3}}{G^{1/3}(M_\star+M_p)^{1/3}T^{2/3}}}$$

$$V_\star = \frac{M_p}{M_\star}\sqrt{\frac{G^{2/3}(M_\star+M_p)^{2/3}(4\pi^2)^{1/3}}{T^{2/3}}}$$

$$V_\star = \frac{M_p}{M_\star} \times G^{2/6}(M_\star+M_p)^{2/6} \times (4\pi^2)^{1/6}T^{-1/3}$$

$$V_\star = (4\pi^2)^{1/6} \times G^{1/3} \times \left(\frac{M_p}{M_\star}\right)(M_p+M_\star)^{1/3}T^{-1/3}$$

So $\boxed{V_\star \propto M_p(M_p+M_\star)^{1/3}}$

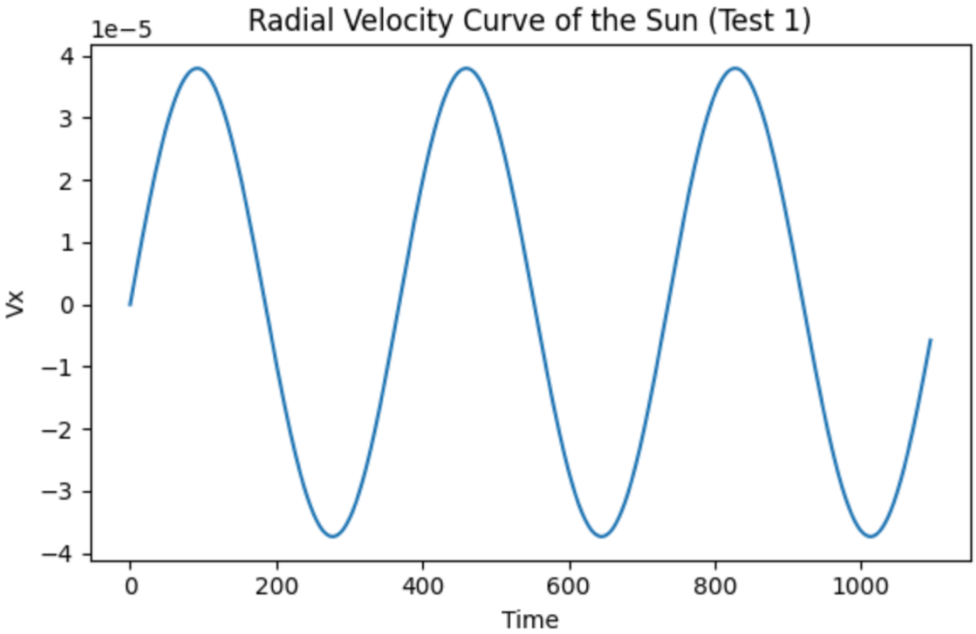
In our case, V* corresponds to the maximum radial velocity of the sun, which is the amplitude calculated in the previous step.

```
Estimate of ampliude from derivation: 1.887e-05
Amplitude from simulation: 1.883e-05
```

The difference between the estimation and the value from the simulation is 4e-8.

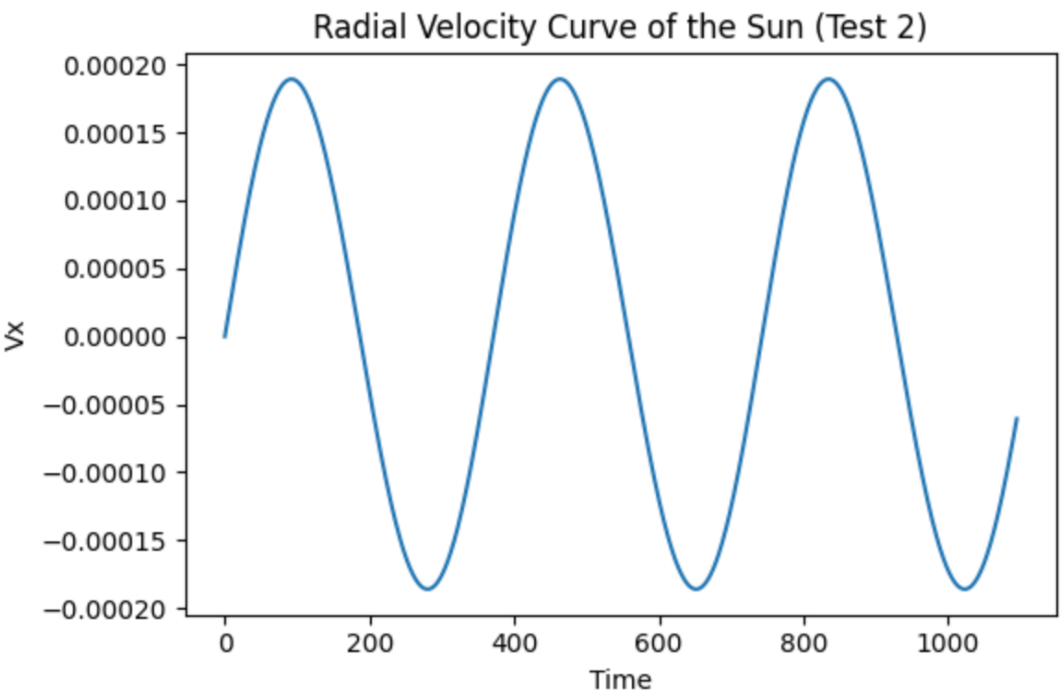
2. Simulations with different masses (for the planet)

Test 1: mass_planet = mass_earth * 2



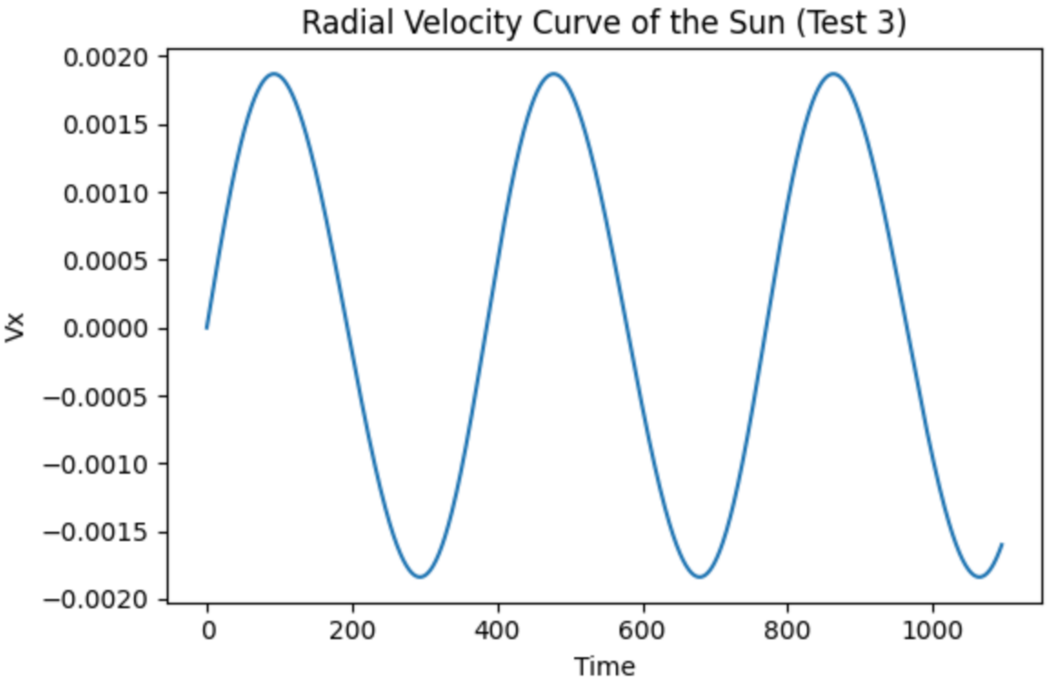

```
Estimate of Ampliude from derivation: 3.7732187248494797e-05
Amplitude from simulation: 3.764088739800562e-05
```

Test 2: mass_planet = mass_earth * 10




```
Estimate of Ampliude from derivation: 0.0001886624467734942
Amplitude from simulation: 0.00018763443711050326
```

Test 3: mass_planet = mass_earth * 100

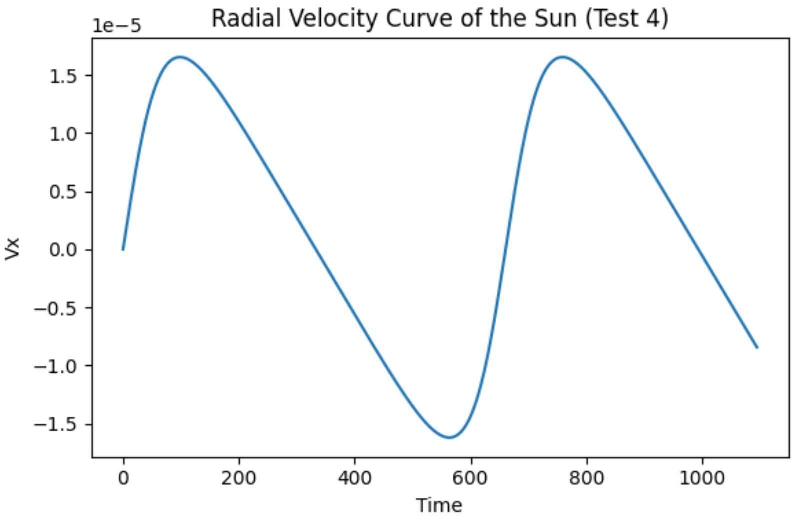


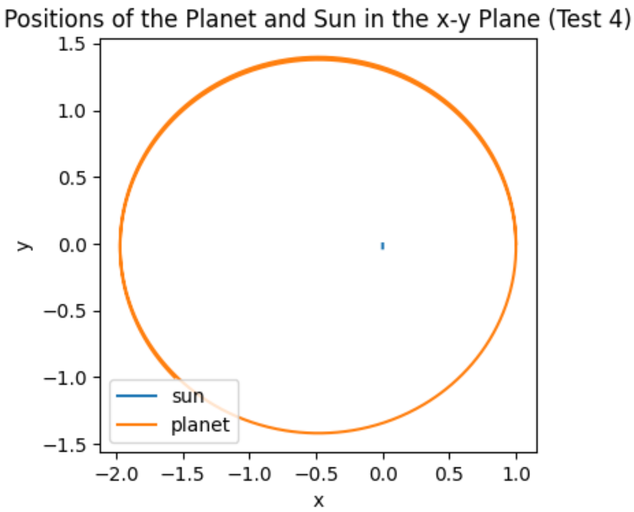

```
Estimate of Ampliude from derivation: 0.0018867943858100 74
Amplitude from simulation: 0.0018541807326232901
```

For the 3 tests, we observe that the expression derived for the amplitude of the radial velocity curve is accurately represented in the results.

3. Exploring the Effects of Eccentricity

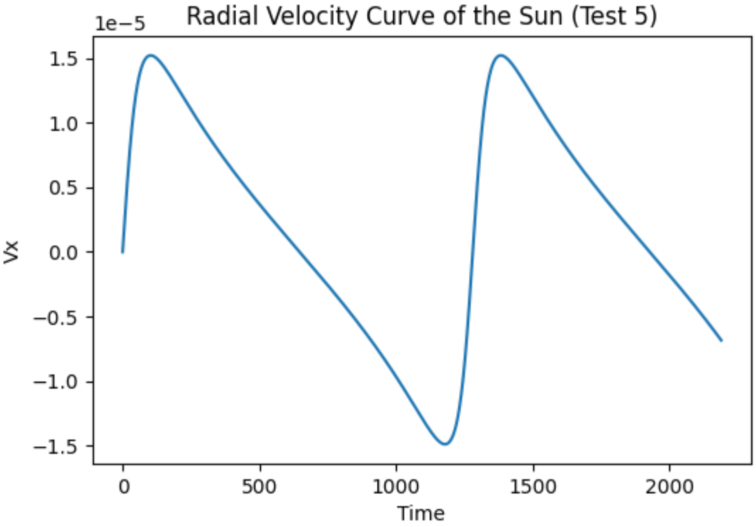
Test 4: initial_velocity_planet = initial_velocity_earth * 1.15

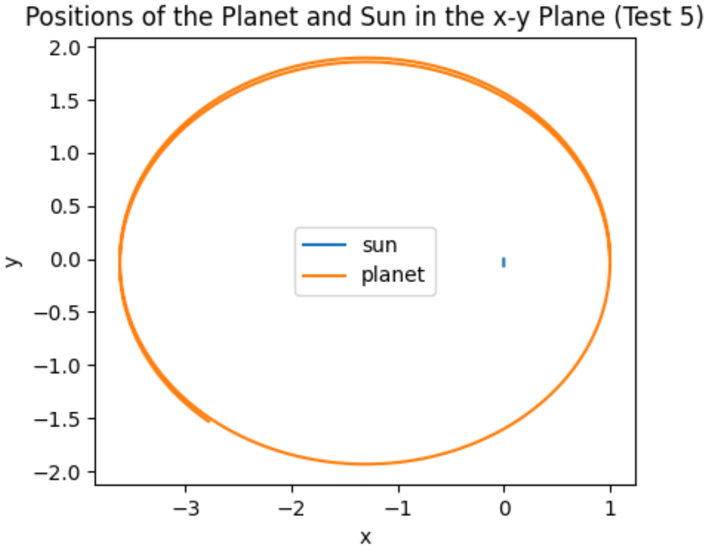




We observe a small eccentricity reflected in the radial velocity curve where the sine wave is a little bit tilted.

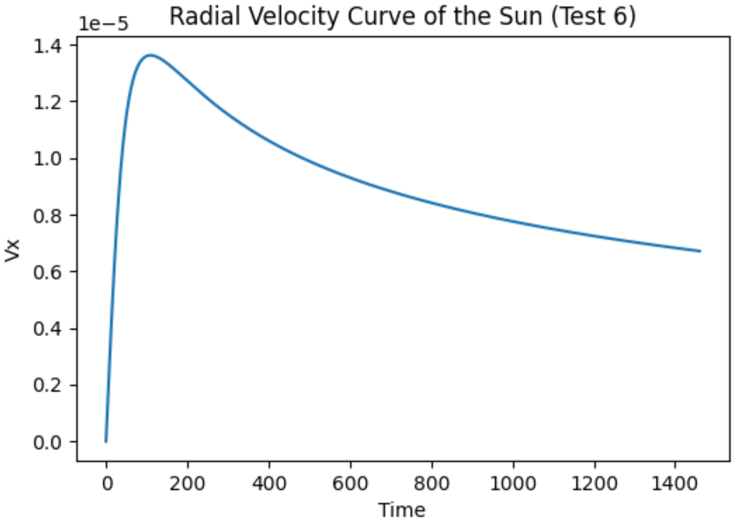
Test 5: initial_velocity_planet = initial_velocity_earth * 1.25

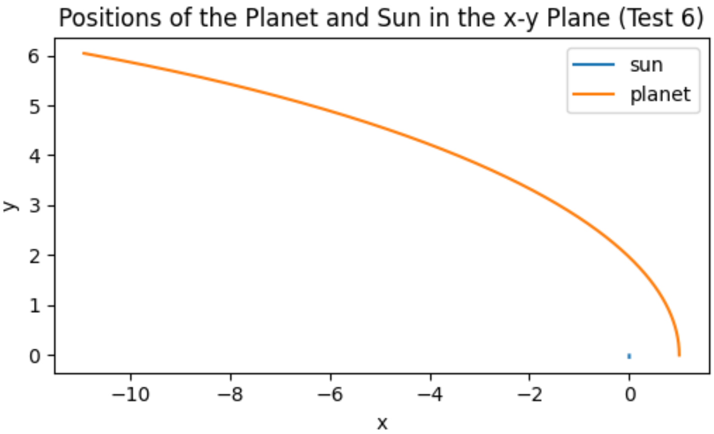




Here, there is a bigger eccentricity as the since wave is more tilted.

Test 6: initial_velocity_planet = initial_velocity_earth * 1.4





In this case, the mass of the planet becomes too large and the planet leaves its orbit around the sun (right graph). This is also reflected in the left graph: as time progresses, the radial velocity of the sun stabilizes because it is not influenced by the planet anymore.

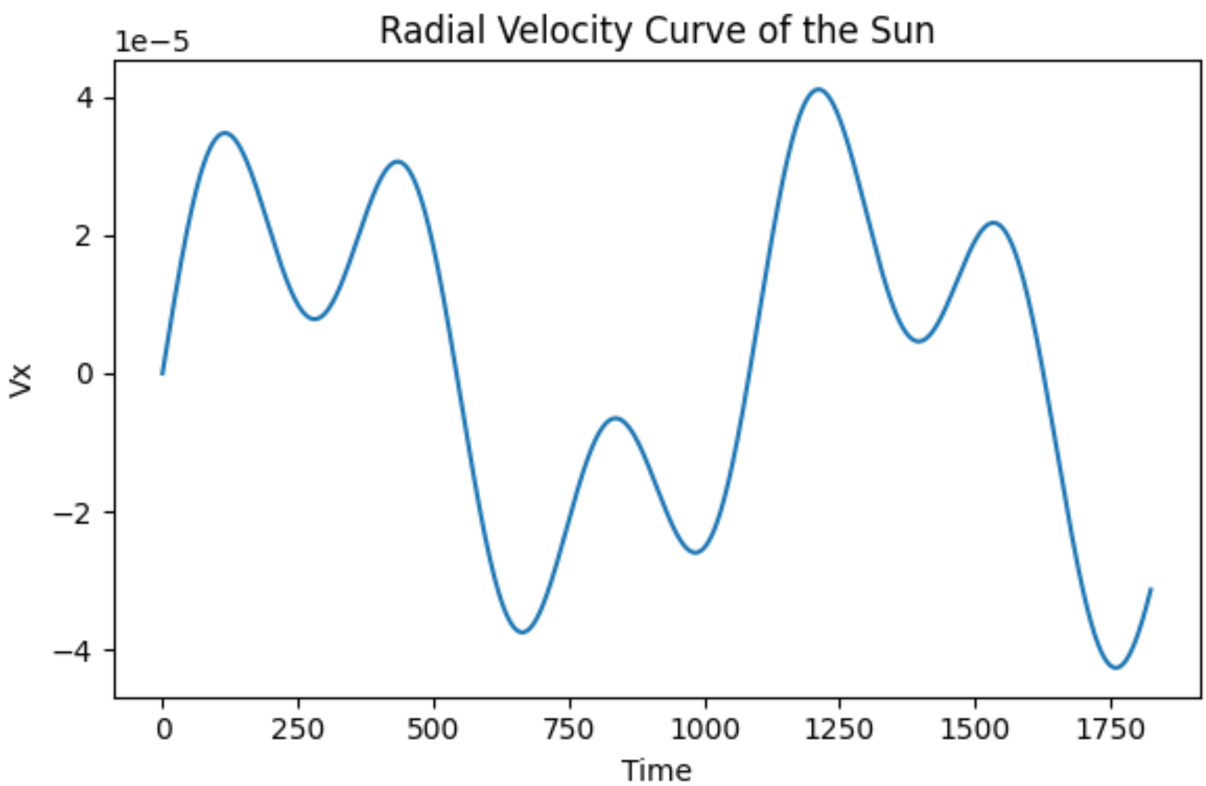
4. **A 3-Body system with 2 non-interactive planets**

In this simulation, I set up:

- 1 star (which has the mass of the sun)
- 1 planet initially located 1 AU from the sun on the x-axis, with mass = mass_earth
- 1 planet initially located 2 AU from the sun on the x-axis, with mass = mass_earth*2

<u>Analysis</u>

We obtain the following graph:



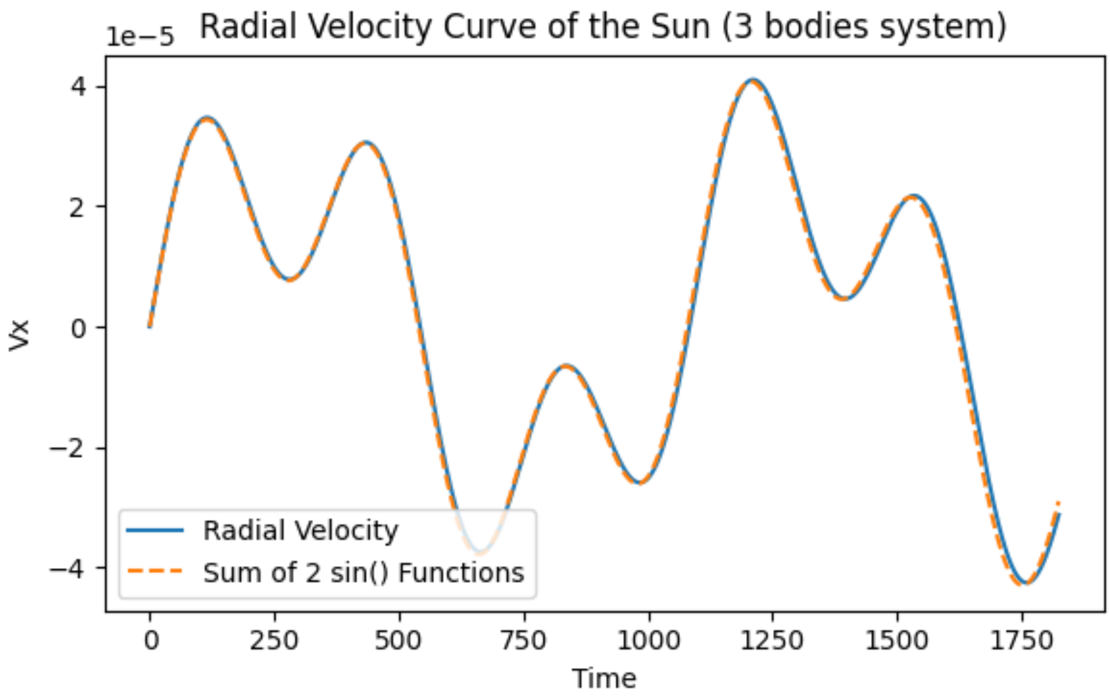
In order to get the theoretically expected curve, we calculate the amplitude and period of the sun's radial velocity curve with:

- Planet 1 only (part 1 of this assignment)
    - Period_1 = 366.5 days
    - Amplitude_1 = 1.883e-5 (non-dimensional units)
- Planet 2 only (code in the Jupyter notebook)
    - Period_2 = 1039.5 days
    - Amplitude_2 = 2.66e-5 (non-dimensional units)

Based on those results, we plot the following function:

*Amplitude_1 * sin(2pi/Period_1) + Amplitude_2 * sin(2pi/Period_2)*

We obtain the following graph:



Therefore, we conclude that the radial velocity curve of the sun in this system is the sum of 2 individual sinusoids with the appropriate periods and amplitudes.