10 - Excellent presentation and very well researched, very well written paper, which would be a useful introduction to this topic for any other math.mod. student.

Stommel's Box Model of the Thermohaline Circulation

Mathematical Modeling

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Abstract

In 1961, Stommel published the first mathematical model of the thermohaline circulation. He modeled the ocean with two vessels, one where the water is cold and relatively fresh, the other with warm and saltier water. The convection between the two interconnected reservoirs was defined to be proportional to density difference maintained by heat and salt transfer. Depending on the parameters, the model can lead to two stable regimes characterized by a circulation that is either fast and temperature-driven, or slower and salinity-controlled. This paper describes Stommel's model and analyzes its equilibrium states. We subsequently demonstrate that it helps understand the temperature-controlled circulation today. Finally, we show that the model explains how global warming, particularly the melt of the Greenland ice sheet, can slow down, or even worse, reverse the circulation, which would dramatically affect climate, biodiversity, and ultimately humans.

1 Background

Ocean ~~surface~~ currents have a profound impact on today's climate and biodiversity. In addition to redistributing heat, hence regulating weather in different parts of the world, they are critically important to sea life when they carry nutrients and organisms to new places. This circulation is driven by two main processes: wind and water density distribution. The later forcing mechanism regulates what is called the thermohaline circulation, also referred to as the "ocean conveyer belt", and was first modelled by the oceanographer Henry Stommel in 1961 [1].

The Thermohaline Circulation

The thermohaline circulation consists of fluxes of water forming a global network of interconnected currents where water masses alternate from being on the surface to traveling deep in the ocean.

Concretely, in the sea of Labrador and other places in the North Atlantic, cold and relatively salty water masses become dense enough to sink to depths between 1 and 4 km [2] and feed deep water currents traveling South. While a very small portion mixes with deep water cycling around Antarctica and reemerges there, most of the water eventually resurfaces in the Indian and Pacific ocean when mixed with warm water. They then travel back to the South Atlantic, carried by the Agulhas Current in the Indian Ocean, and reach the Caribbean. Too hot to sink, the salty tropical water is swept into the Gulf Stream and North Atlantic current, brought back to the Greenland and Nordic seas. Models have estimated the duration of a full cycle to be between 1000 and 2000 years [3].

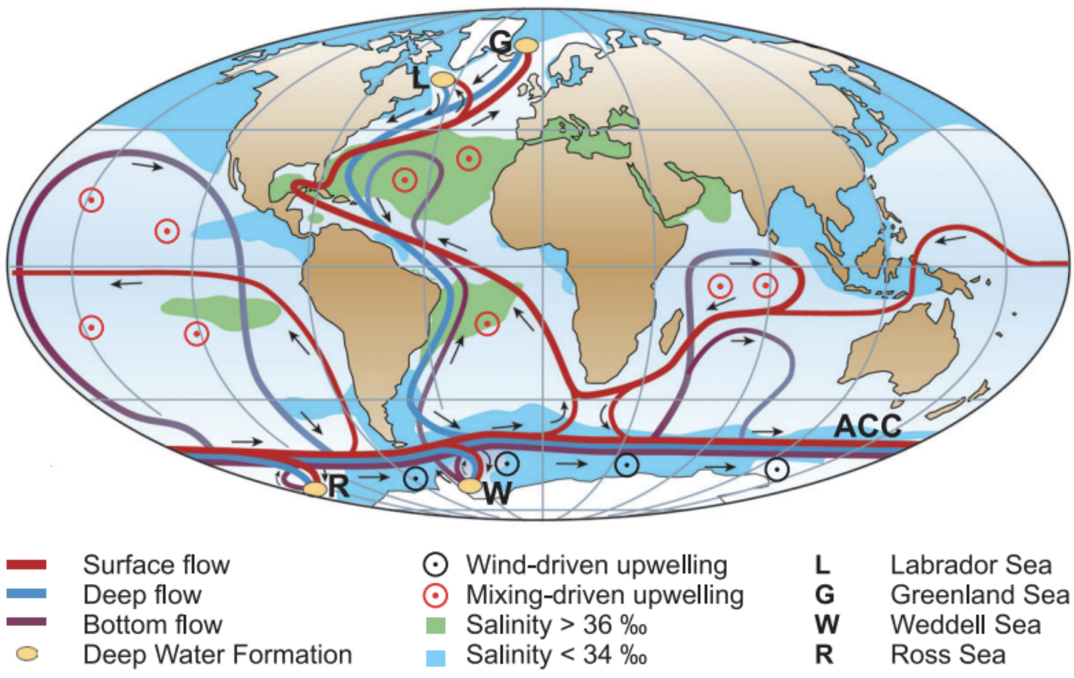

Figure 1: Meridional circulation. Surface currents are shown in red, deep water masses in light blue and bottom flows in purple. Yellow circles represent deep water formations. The image was extracted from [4].

Figure 1 displays a schematic representation of the thermohaline circulation and wind-driven currents (together forming the overturning meridional circulation). In addition to the two deep water formations in the North Atlantic (in the Greenland-Norwegian and Labrador seas), water masses also sink near Antarctica, in the Weddell and the Ross seas. Researchers highlight that this is a simplification and maps charting this circulation are still evolving today.

Vulnerabilities

The thermohaline circulation is very vulnerable. Changes in salinity or temperature in regions where deep water currents are formed can induce changes in water density resulting in a deceleration, or even worse, a shutdown of the circulation. This would have damaging effects on climate, biodiversity, and ultimately humans.

In the North Atlantic, the surface currents going North carry warm and salty tropical water, warming oceans around northern western Europe and Greenland. It increases air temperature in these regions and explains why, despite being located at high latitudes, Europe has reasonably high temperatures. For instance, we observe in Figure 2 that Paris is at the same latitude as Montreal but is 5 to $10°C$ warmer on average. In general, "over the three main deep water formation regions of the global oceans, air temperatures are warmer by up to $10°C$ compared to the latitudinal mean" [5]. A slowdown in the circulation could induce a drop of 5 to $10°C$ in northern Europe and Greenland, with a stronger effect in Winter [6]. Moreover, the heat transport is reduced, countries in the southern hemisphere would become warmer and rainfall patterns would change, increasing the probability of drought. This would have damaging effects on the agriculture sector in these regions. A map of the potential changes in temperature if the thermohaline circulation would stop is displayed in Figure 3.

In addition to temperatures, a slow-down of the circulation would have damaging effects on biodiversity. The "sinks" in the Labrador and Greenland seas bring oxygen down to the

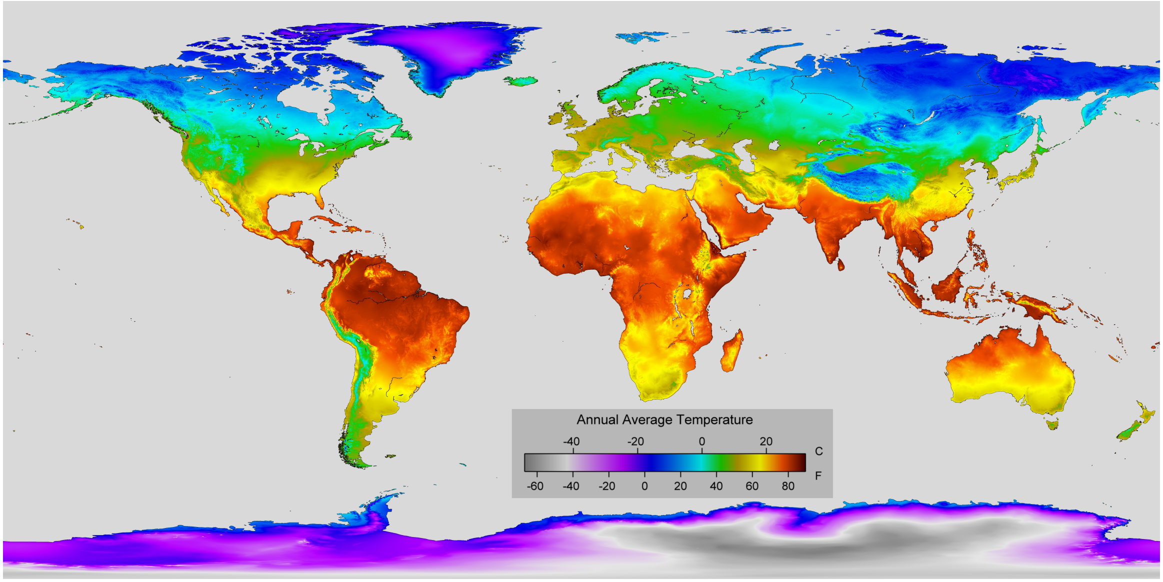

Figure 2: Map of annual average temperature downloaded from wikimedia.

deep ocean. Referred to as the "lungs of the ocean", these deep water formations are vital to all marine life, especially deep-sea organisms. The Labrador ~~see~~ sea has been estimated to provide 3/4 of the oxygen present in the deep Atlantic ocean. Furthermore, the same process takes place with carbon dioxide, reducing the amount of greenhouse gas in Earth's atmosphere. A shutdown of the circulation would therefore worsen the effects of global warming by reducing the amount of carbon dioxide the oceans absorb. Finally, the sea level in the northern Atlantic is almost a meter lower than in other comparable parts of the Pacific (where no deep water is formed) "due to geostrophic balance of sea currents and sea surface slopes" [7]. A change in the ocean currents would affect the sea surface topology and increase the sea level by up to 1 meter in the North Atlantic [7].

This should refer only to the coastal Atlantic near the North American East coast. Elsewhere the Atlantic is not any lower than the rest of the world Oceans. The Pacific is also lower by about the same amount along the Japanese east coast. In both cases the reason is the presence of a western boundary current (Gulf stream / Kuroshio)

There is strong evidence that such a shutdown has happened in the past. Eleven thousand years ago, in central Canada, the dams holding Lake Agassiz collapsed, leading to a flood of fresh water in the northern Atlantic and a slow-down of the thermohaline circulation. While it is still debated today, it is hypothesized that this flood started a cold event called the Younger Dryas [8]. Today, scientists suspect that the melt of the Greenland ice cap due to global warming could release a big enough amount of fresh water to slow down the circulation. The question of whether it is already happening today is subject to debate in the scientific community. While NASA's measurements suggest that the circulation might even have sped slightly, other mathematical models claim that it has been slowing down after 1975 [9].

Considering the damaging effects a deceleration or shutdown of water currents could have, the thermohaline circulation is extensively studied today. Researchers attempt to model this complicated phenomenon with numerous mathematical models. Henry Stommel was one of the first to come up with a model for the thermohaline circulation, which we explain and analyze in this paper.

2 The Main Physical Processes

In general, ocean water undergoes 2 processes:

- Heating and cooling affect temperature.
- Precipitation and evaporation affect salinity levels.

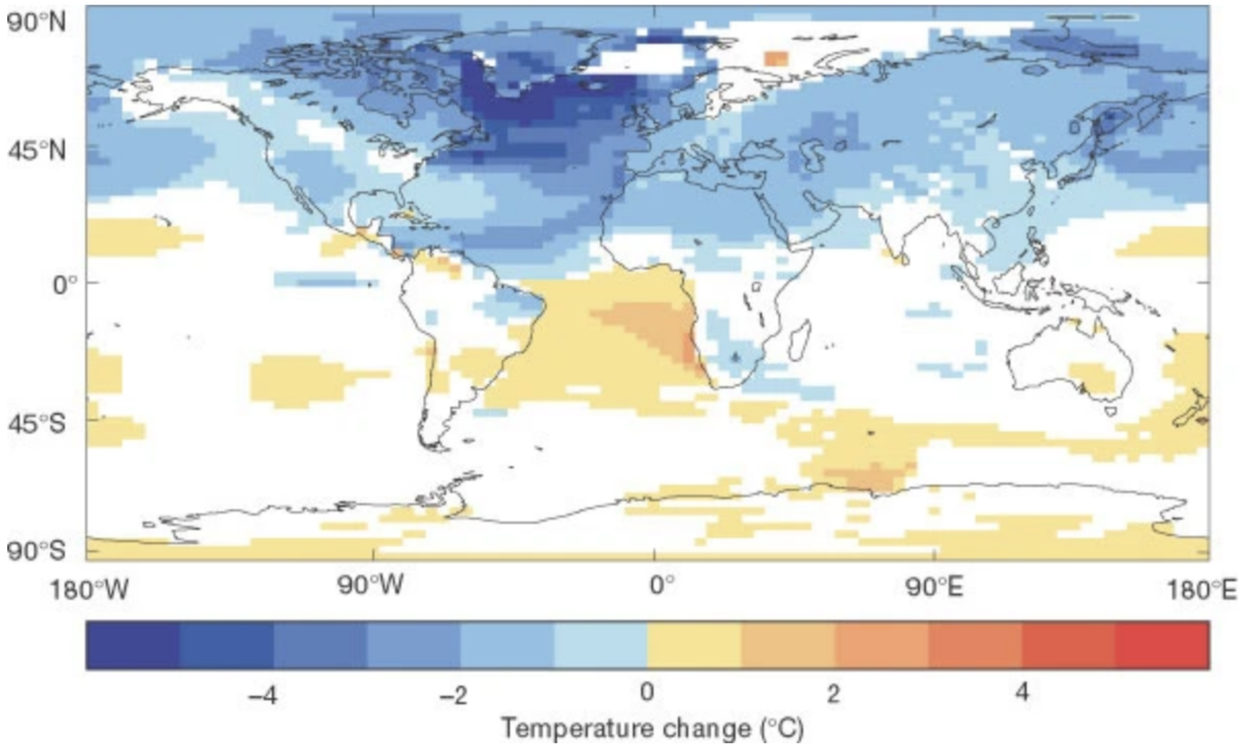

Figure 3: Change in surface air temperature caused by a shutdown of the North Atlantic deep water formation in a current ocean–atmosphere circulation model. Image extracted from [6].

On the one side, salinity level is proportional to density, meaning that an increase in salinity causes an increase in water density. On the other side, temperature is inversely proportional to density as warm water is less dense than cold water.

These two density-modifying processes usually work contrarily to one another. In tropical places, the high temperature pushes the water to not be too dense. However, intense evaporation in the tropics counterbalances and increases the salinity level, thus the water's density. In places at very high latitudes, precipitations and the lack of evaporation decreases the density, but the water is also cooled down due to the cold climate, which eventually increases water density.

Today, temperature has a predominant influence on the density field. It is the reason why cold water masses sink and feed deep currents near the Greenland. Figure 4 displays maps of oceans' temperature, salinity level, and density. We notice the deep water formation regions with dense water in the North Atlantic. It corresponds to cold water in the temperature map and moderately salty water in the salinity map. The fact that the water salinity level in the deep water formation regions is not too low compared to other places helps explain why the water sinks: the water becomes denser despite a medium level of salinity because temperature dominates on the density field. In places such as the Mediterranean and other semi-enclosed areas, we observe that the water is very dense, salty, and warm. This means that, as Stommel mentioned in his paper, "salinity can have a predominant influence on the density field" [1] in these regions because of high evaporation and low freshwater runoff. It is the reason why one might feel lighter when swimming in these seas.

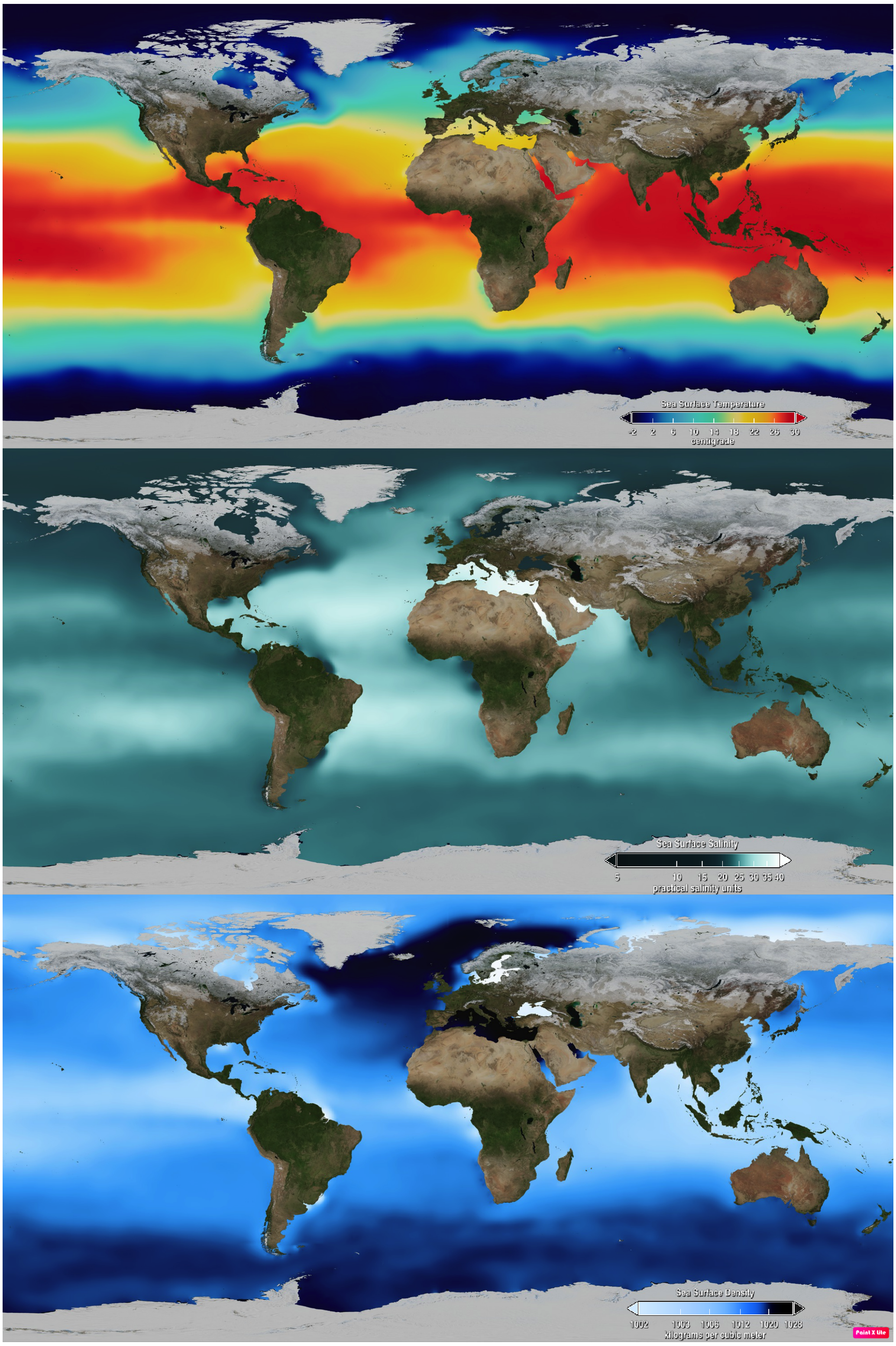

Figure 4: Maps of ocean water surface temperature (top), salinity (middle) and density (bottom), extracted from NASA's scientific visualization studio.

3 The Model

A Two-box Model

The thermohaline circulation is a difficult problem to model. As Rahmstorf et. all write, it is a "highly non-linear system with multiple equilibrium states" [10]. The advection of heat and salt is central to the problem and hence cannot be neglected. Moreover, the non-linear equation of state for sea water and the different boundary conditions for heat and salt add to the difficulty. Therefore, Stommel introduced a conceptual model, characterizing the ocean with two boxes connected by an open channel in the upper part and a pipe at the bottom.

Model Set Up

As shown in Figure 5, the model consists of two vessels of water stirred as to maintain uniform temperatures T_1 and T_2 and uniform salinity S_1 and S_2. The walls enable transfer of heat and salt. Outside of the walls, the temperature and salinity are maintained at constant values, positive on the left and negative on the right, acting as thermostats for the two vessels. This model was conceived to simulate two regions with different climates. The left box represents an equatorial zone where the thermostat causes a temperature and salinity growth $(\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{S}} > 0)$ while the right box models high latitude regions, where the thermostat forces temperature and salinity to diminish $(-\boldsymbol{\mathcal{T}}, -\boldsymbol{\mathcal{S}} < 0)$. The connection between the two reservoirs is made by a capillary tube with resistance k. On the surface, the two vessels are connected by a large overflow so that the water level is the same everywhere.

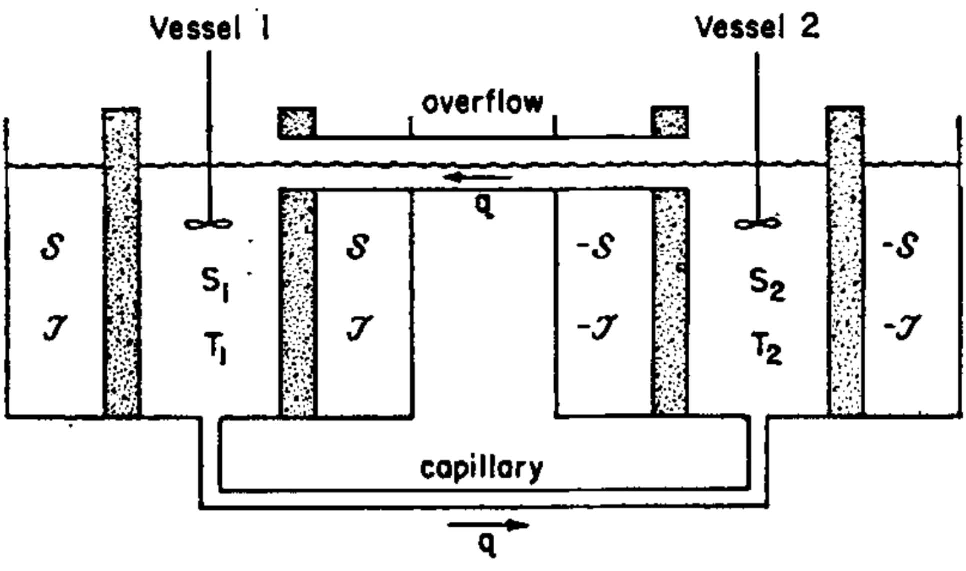

Figure 5: Diagram of Stommel's model extracted from his paper [1].

Creation of the Model

The solutions must be of a symmetric form so that the laws for the conservation of temperature and salinity apply. Thus, Stommel defines: $T = T_1 = -T_2$ and $S = S_1 = -S_2$. The porous walls around the two vessels enable transfer of heat and salt in a simple linear way:

$$\begin{cases} \frac{\mathrm{d}T}{\mathrm{d}t} = c(\boldsymbol{\mathcal{T}} - T) \\ \frac{\mathrm{d}S}{\mathrm{d}t} = d(\boldsymbol{\mathcal{S}} - S) \end{cases}$$

Where the relaxation constant c is the heat transfer coefficient and d is the salinity transfer coefficient.

To add convection, Stommel defines a flow rate q in the capillary defined positive if directed from the left to the right reservoir:

$$\begin{cases} \frac{\mathrm{d}T}{\mathrm{d}t} = c(\boldsymbol{\mathcal{T}} - T) - |2q|T \\ \frac{\mathrm{d}S}{\mathrm{d}t} = d(\boldsymbol{\mathcal{S}} - S) - |2q|S \end{cases}$$

Notice the term $-|2q|$. When water flows from one reservoir to another in the capillary, the same amount flows back on the surface. Since we defined $T = T_1 = -T_2$, temperature decreases by $|q|T$ (capillary) and increases by $|q|(-T)$ (surface), which adds up to a decrease of $|2q|T$. The same reasoning can be applied to salinity.

The next step consists of non-dimentionalization. Define:

$$\tau = ct \qquad \delta = \frac{d}{c} \qquad y = \frac{T}{\boldsymbol{\mathcal{T}}} \qquad x = \frac{S}{\boldsymbol{\mathcal{S}}} \qquad f = \frac{2q}{c} \tag{1}$$

Where f is the non-dimensional flow rate and x and y are the non-dimensional salinity and temperature variables. This leads to the following non-dimensional system of equations:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}\tau} = 1 - y - |f|y \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} = \delta(1 - x) - |f|x \end{cases} \tag{2}$$

Regarding flow rate, Stommel models it so that it depends upon the density difference maintained by heat and salt transfer between the two reservoirs. It is also retarded by friction in the pipes (with a resistance k) and is assumed to be an instantaneous balance. Based on these principles, he defines:

$$kq = \rho_1 - \rho_2 \tag{3}$$

where ρ_1 is the density in the left reservoir and ρ_2 in the right reservoir. Then, he assumes the equation of state:

$$\rho = \rho_0(1 - \alpha T + \beta S)$$

where α is the thermal expansion coefficient and β is the haline contraction coefficient. Expressed in terms of the non-dimensional quantities x and y, it translates to:

$$\rho = \rho_0(1 + \alpha\boldsymbol{\mathcal{T}}(-y + Rx))$$

with $R = \frac{\beta\boldsymbol{\mathcal{S}}}{\alpha\boldsymbol{\mathcal{T}}}$, "a measure of the ratio of the effect of salinity and temperature on the density in the final equilibrium states" [1]. Using this equation of state:

$$\rho_1 = \rho_0(1 + \alpha\boldsymbol{\mathcal{T}}(-y + Rx))$$

$$\rho_2 = \rho_0(1 - \alpha\boldsymbol{\mathcal{T}}(-y + Rx))$$

$$\rho_1 - \rho_2 = 2\alpha\rho_0\boldsymbol{\mathcal{T}}(-y + Rx) \tag{4}$$

Finally, grouping (1), (3), and (4) and defining $\lambda = (\frac{c}{4\rho_0\alpha\boldsymbol{\mathcal{T}}})k$, it gives the following equation for the flow rate:

$$\lambda f = -y + Rx \tag{5}$$

Replacing this result in (2), we get:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}\tau} = 1 - y - \frac{y}{\lambda}|-y + Rx| \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} = \delta(1-x) - \frac{x}{\lambda}|-y + Rx| \end{cases} \tag{6}$$

4 Equilibrium States

Fixed Points

The equilibrium/fixed points refer to ways in which the convection can occur between the coupled vessels without change in time. To find these points, we solve $\frac{\mathrm{d}y}{\mathrm{d}\tau} = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}\tau} = 0$ simultaneously. This leads to a cubic for y in terms of x, with 1 or 3 pairs $(x,\, y)$ as solutions. For a specific flow rate $f = \frac{-y+Rx}{\lambda}$, the corresponding values of x and y at equilibrium can be deduced from (2) as:

$$x = \frac{1}{1 + \frac{|f|}{\delta}} \tag{7}$$

$$y = \frac{1}{1 + |f|} \tag{8}$$

Grouping (5), (7) and (8) together, the flow rate f at equilibrium is the solution of :

$$\lambda f = -\frac{1}{1+|f|} + \frac{R}{1 + \frac{|f|}{\delta}} \tag{9}$$

Stability Analysis

Once the fixed points with non-dimensional flow rate f, non-dimensional salinity x and non dimensional temperature y have been found, we perform a stability analysis. In order to study the stability of a fixed point (x, y), we do a linearization of (6) around (x, y). This entails calculation the Jacobian matrix of (6) at (x, y).

If $f > 0$, the system (6) becomes:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}\tau} = 1 - y + \frac{y^2}{\lambda} - \frac{Rxy}{\lambda} = F(x,y) \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} = \delta(1-x) + \frac{xy}{\lambda} - \frac{Rx^2}{\lambda} = G(x,y) \end{cases}$$

which gives the Jacobian matrix $J_{f>0}$:

$$J_{f>0} = \begin{bmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial x} \\ \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial x} \end{bmatrix} = \begin{bmatrix} -1 + \frac{2y}{\lambda} - \frac{Rx}{\lambda} & -\frac{Ry}{\lambda} \\ \\ \frac{x}{\lambda} & -\delta + \frac{y}{\lambda} - \frac{2Rx}{\lambda} \end{bmatrix}$$

with trace $Tr_{f>0}$ and determinant $det_{f>0}$:

$$Tr_{f>0} = (-1-\delta) + \frac{3}{\lambda}(y - Rx)$$
$$det_{f>0} = \delta + \frac{1}{\lambda}(-y(1+2\delta) + Rx(2+\delta) + \frac{2}{\lambda}(-y+Rx)^2)$$

If $f < 0$, the system(6) becomes:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}\tau} = 1 - y - \frac{y^2}{\lambda} + \frac{Rxy}{\lambda} = H(x,y) \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} = \delta(1-x) - \frac{xy}{\lambda} + \frac{Rx^2}{\lambda} = I(x,y) \end{cases}$$

which gives the Jacobian matrix $J_{f<0}$:

$$J_{f<0} = \begin{bmatrix} \frac{\partial H}{\partial y} & \frac{\partial H}{\partial x} \\ \\ \frac{\partial I}{\partial y} & \frac{\partial I}{\partial x} \end{bmatrix} = \begin{bmatrix} -1 - \frac{2y}{\lambda} + \frac{Rx}{\lambda} & \frac{Ry}{\lambda} \\ \\ -\frac{x}{\lambda} & -\delta - \frac{y}{\lambda} + \frac{2Rx}{\lambda} \end{bmatrix}$$

with trace $Tr_{f<0}$ and determinant $det_{f<0}$:

$$Tr_{f<0} = (-1-\delta) - \frac{3}{\lambda}(y - Rx)$$
$$det_{f<0} = \delta + \frac{1}{\lambda}(y(1+2\delta) - Rx(2+\delta) + \frac{2}{\lambda}(-y+Rx)^2)$$

Using the Poincaré diagram displayed in Figure 6, then can finally deduce the stability of the fixed point.

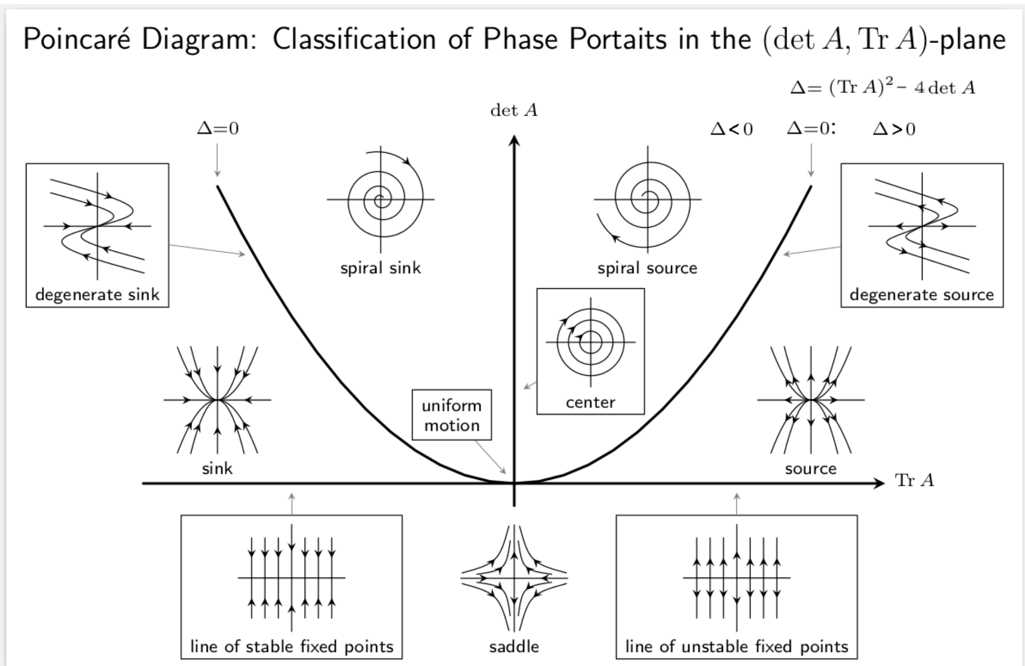

Figure 6: Poincaré diagram. It is used to analyze the stability of a fixed point based on its corresponding Jacobian matrix A (calculated after linearizing the system around the fixed point). The image was extracted from wikimedia.

5 Simulations

In his paper, Stommel gives two examples of systems with 1 and 3 equilibrium points. Using Python and jupyter notebooks, we ran simulations to study these systems. For each set of parameters, the simulation performs the following:

1. Find all possible values for the flow rate f at equilibrium by locating the intersection of the two curves defined by the left and right parts of equation (9). Each flow rate value defines a fixed point.

2. Deduce x and y at each fixed point.

3. Linearize the system at each fixed point. Calculate the Jacobian matrix, its trace and determinant, and establish the fixed point stability (stable, unstable) and type (node, spiral, saddle).

4. Plot the phase portrait of the system on the x-y phase plane (with trajectories, fixed points and vector field). Trajectories are drawn using the Euler integration technique (numerical method)

5. Create animations.

Simulation 1: System With 3 Fixed Points

The first system is defined by $R = 2$, $\lambda = \frac{1}{5}$, and $\delta = \frac{1}{6}$. Figure 7 displays the process of finding values for the flow rate at equilibrium using (9). The simulation results are gathered in Table 1 and the phase portrait can be visualized in Figure 8.

Fixed Point	f	x	y	Trace	Determinant	Stability & Type
1	-1.07	0.14	0.48	-4.37	2.75	stable node
2	-0.31	0.35	0.76	-2.10	-2.15	saddle node (unstable)
3	0.22	0.43	0.82	-1.83	4.16	stable spiral

Table 1: Results of simulation 1. Each row gathers information about a fixed point. f is the non-dimensional flow rate at equilibrium, x and y the corresponding non-dimensional salinity and temperature values. Trace and Determinant are used to evaluate the stability and type of the fixed point using the method from Section 4.

Simulation 2: System With 1 Fixed Point

The second system is defined by $R = 2$, $\lambda = \frac{1}{5}$, and $\delta = 1$. Figure 9 displays the process of finding flow rate values at equilibrium using equation (9). The simulation results are gathered in Table 2 and the x-y phase portrait can be visualized in Figure 10.

Fixed Point	f	x	y	Trace	Determinant	Stability & Type
1	1.79	0.36	0.36	-7.37	12.79	stable node

Table 2: Results of simulation 2. f is the non-dimensional flow rate at equilibrium, x and y the corresponding non-dimensional salinity and temperature values. Trace and Determinant are used to evaluate the stability and type of the fixed point using the method from Section 4.

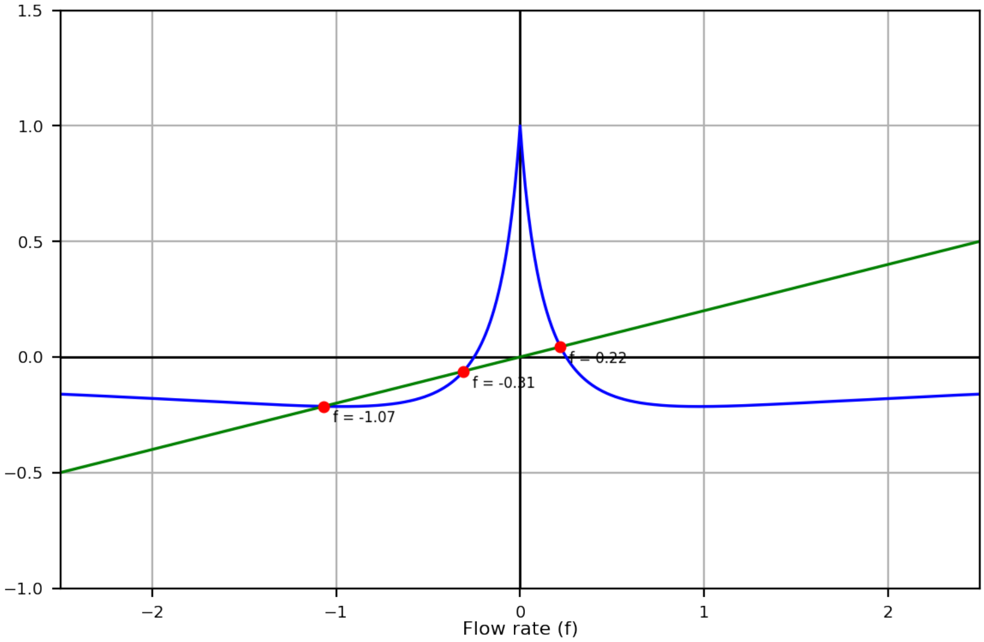

Figure 7: Flow rate values (red circles) at equilibrium (simulation 1). The green curve represents the function λf and the blue curve $-\frac{1}{1+|f|} + \frac{R}{1+\frac{|f|}{\delta}}$ (from equation (9)).

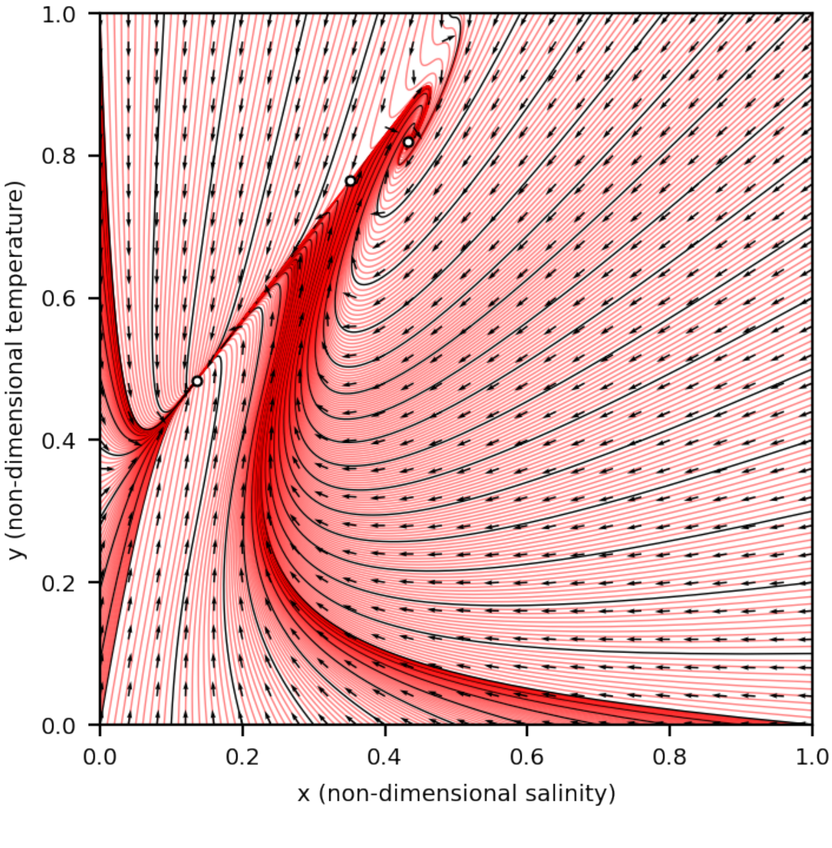

Figure 8: Phase portrait on the x-y plane (simulation 1). The fixed points are represented by white circles, trajectories by red and black curves and vector field by black arrows.

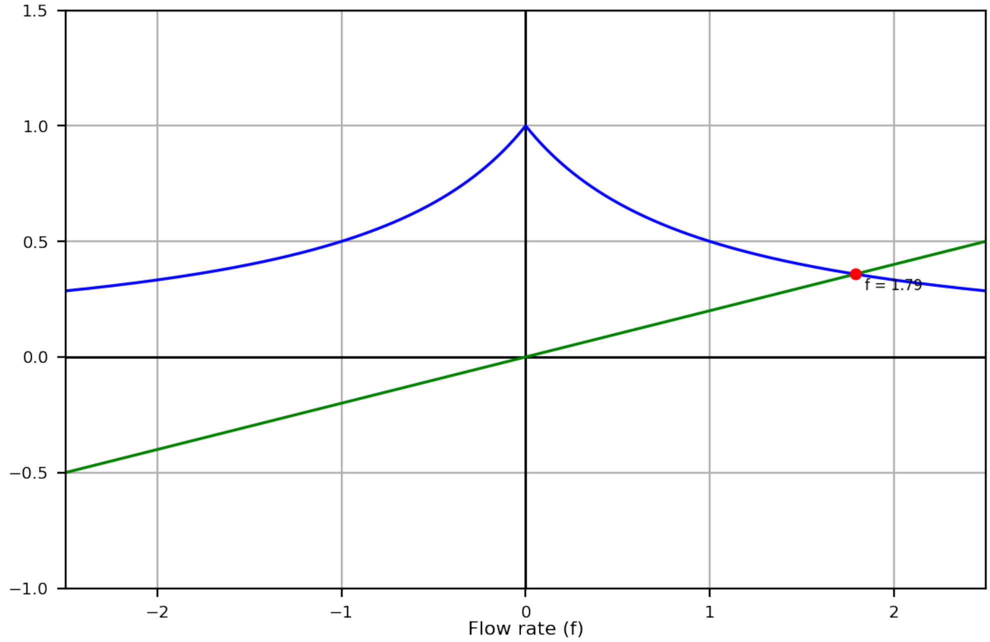

Figure 9: Flow rate value (red circle) at equilibrium (simulation 2). The green curve represents the function λf and the blue curve $-\frac{1}{1+|f|} + \frac{R}{1+\frac{|f|}{\delta}}$ (from equation (9)).

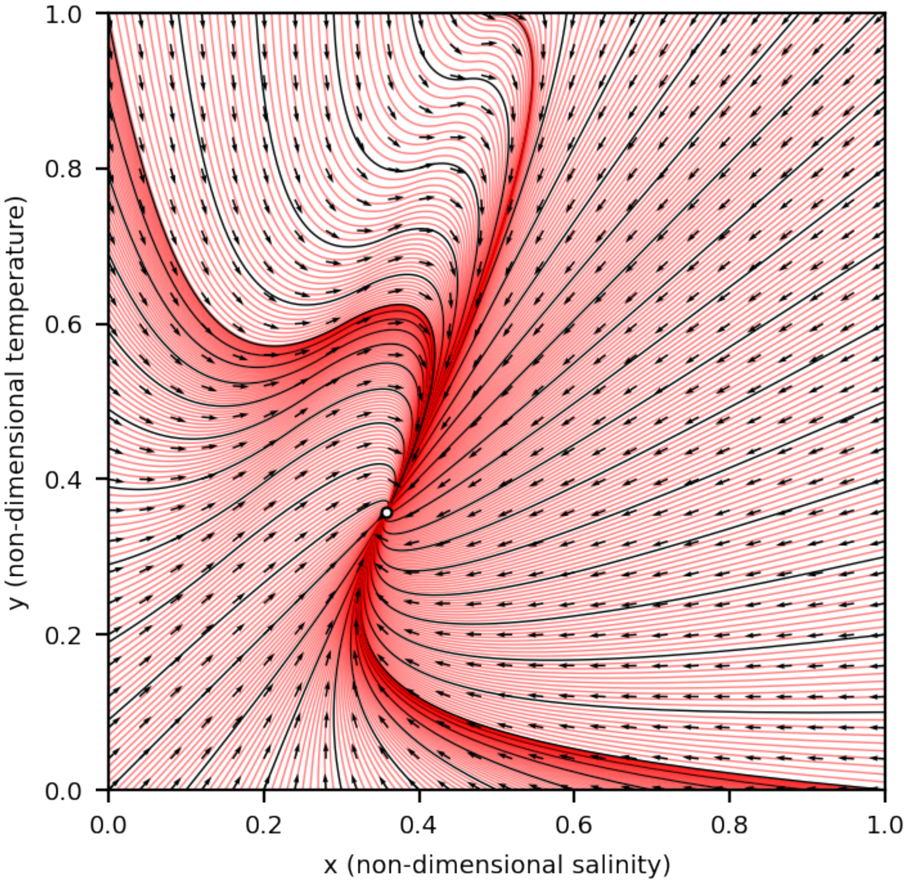

Figure 10: Phase portrait on the x-y plane (simulation 2). The fixed point is represented by a white circle, trajectories by red and black curves and vector field by black arrows.

Interpretation of Results

The type of circulation at equilibrium is indicated by the flow rate sign. If $f < 0$, then $q < 0$ ($f = \frac{2q}{c}$ and $c > 0$). Therefore, as defined in Figure 5 the flow is directed from the right box to the left box. This means that the fluid in the right box is denser than in the left box. As the right box contains a cold and fresh fluid, we deduce that temperature caused the density increase (fresh water is less dense than salty water). Similarly, if $f > 0$, then $q > 0$, the flux goes from the left box (with warm and salty water) to the right box. In this case, the water is denser in the left box because of a high salinity level. **In summary, a negative flow rate indicates a temperature-driven circulation, while a positive flow rate suggests salinity-driven currents.**

We deduce circulation types for the 3 equilibriums in simulation 1 (summarized in Table 1). The first fixed point describes a stable state that is thermally controlled, with a relatively fast circulation ($|f| = 1.07$). Fixed point 2 models an unstable state that is thermally controlled. Then, follows a stable state that is salinity-controlled (fixed point 3), with a quite slow circulation ($|f| = 0.22$).

The difference between flow rates for salinity-driven and temperature-driven circulations was pointed out by Stommel as he writes: "the heat transfer mechanism has a more rapid effect on the density of a newly arriving parcel of water, but [..] in the long run, given sufficient time, evaporation can reverse the thermal influence" [1]. In a salinity-driven circulation, a slower circulation is necessary to allow for salinity difference to built up enough (with evaporation) to eventually dominate over temperature. On the other side, a temperature-driven circulation must be fast enough to shorten evaporation time.

In simulation 1, we chose a specific value $\delta = \frac{d}{c} = \frac{1}{5}$. The temperature transfer coefficient c is 5 times bigger than the salinity transfer coefficient d, which makes a temperature-driven circulation a possibility. However, in simulation 2, there is only one stable state and it is salinity-controlled. In this case, $\delta = 1$ so $d = c$. The salinity transfer coefficient is big enough to counterbalance and let salinity always dominate over temperature on the density field.

6 Stommel's Model and Climate Change

Today, the thermohaline circulation is thermally controlled. Cold water sinks in the North Atlantic and resurfaces in the Indian and Pacific Oceans when mixing with warmer water. However, as seen in the Background section, global warming and the melt of the Greenland ice sheet might slow down this circulation. In this section, we show how Stommel's model considers global warming.

Parameters change

In his paper, Stommel writes that the circulation might be impacted by a "slight change in the parameters" [1]. We plot the bifurcation diagrams for δ and λ to understand this process, using the parameters from simulation 1 (with 3 equilibrium states).

Change of δ

Figure 11 displays the bifurcation diagram for δ.

- $\delta \gtrapprox \frac{1}{5}$: the circulation can only be salinity-driven. $\delta \approx \frac{1}{5}$ is a threshold, known as the Stommel's bifurcation point, where a temperature-driven circulation (blue and red curves) would reverse to salinity driven currents. This change would be irreversible, even if δ comes back to its original value, as the salinity driven circulation is characterized by a stable fixed point.

- $\delta \lessapprox \frac{1}{5}$: the system allows for 3 equilibrium points. We notice that the larger δ is, the slower the temperature driven circulation is (red curve, the absolute value of the flow rate f decreases).

In the real world, the melt of the Greenland ice sheet can be simulated by increasing the value of δ. Indeed, as $\delta = \frac{d}{c}$ increases, the salinity transfer coefficient d has more weights in the ratio with the heat transfer coefficient c, meaning that salinity decreases very fast in the right box representing the deep water formation regions of today. This is exactly what ice does when it melts, releasing a huge amount of fresh water in the ocean. Therefore, we conclude from our observations that the melt of the Greenland ice sheet, simulated by increasing the value of δ, causes a slow-down of a temperature driven-circulation. Eventually, it reaches a threshold, the Stommel bifurcation point, where the circulation stops and reverses to a salinity-driven circulation that flows in the opposite way.

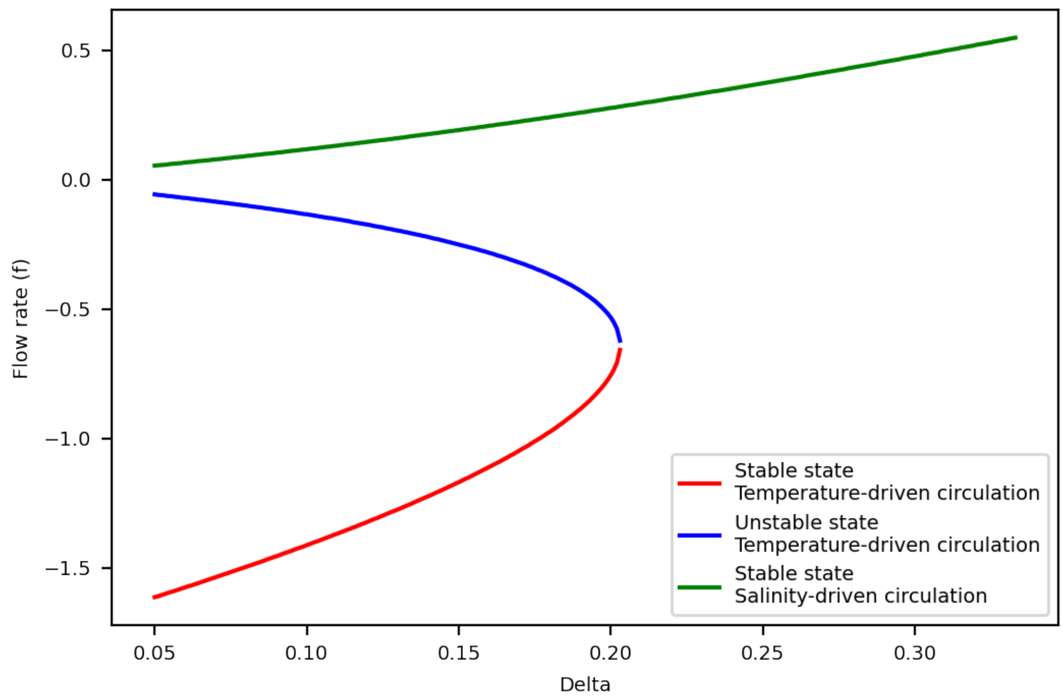

Figure 11: Bifurcation diagram for delta (δ).

Change of λ

The same analysis can be done for λ. The bifurcation diagram is plotted in Figure 12. Remind the definition of λ as:

$$\lambda = (\frac{c}{4\rho_0 \alpha \boldsymbol{\mathcal{T}}})k$$

The behaviour is very similar as for δ except that the bifurcation point is at $\lambda \approx \frac{1}{4}$. Here, an increase in λ can for example indicate a decrease of $\boldsymbol{\mathcal{T}}$. In this case, it means that the water is not cooled as much as it was before in the deep water formation regions. This is also a consequence of global warming, as ocean waters get warmer. As Stommel writes in the paper, "a slight change in lambda could cause the temperature-dominated circulation at a to jump to the reverse salinity circulation at c, and it would then stay there even when λ was restored to its original value" [1]. Climate change is not reversible!

It is a hysteresis cycle. You mention it later, should have been made more clear here.
After all, the cooling of the Younger Dryas wasn't permanent.

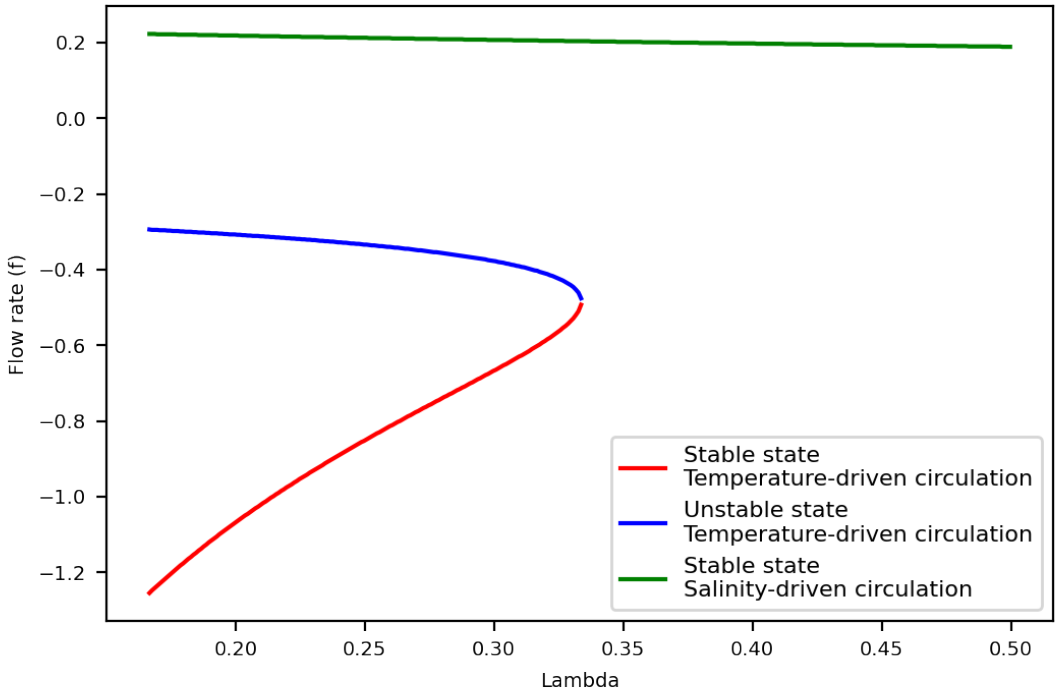

Figure 12: Bifurcation diagram for lambda (λ).

Perturbations in the momentary state of the system

Another way Stommel describes potential changes in the circulation is in the context of perturbations "in the momentary state of the system" [1]. Figure 13 displays the phase portrait from simulation 1. In green, the trajectories tend to fixed point 3, a stable state with a salinity-driven circulation. The red curves tend to fixed point 1, a state state with a temperature-driven circulation. We observe that a slight change in salinity x or temperature y can change the fixed point the trajectory tends to, hence change the circulation type, especially if located at the boundary between the red and the green curves.

7 Conclusion

We showed that Stommel's simplified box model of the thermohaline circulation helps understand the structure of water currents today. He assumed a flow rate based on the density difference between two regions, leading to systems with multiple equilibrium states characterized by temperature or salinity-driven circulations. We observed that a slight change in the parameters or in the momentary state of the system can slow down or reverse a circulation and that the melt of the Greenland ice sheet and global warming in general is part of these perturbations.

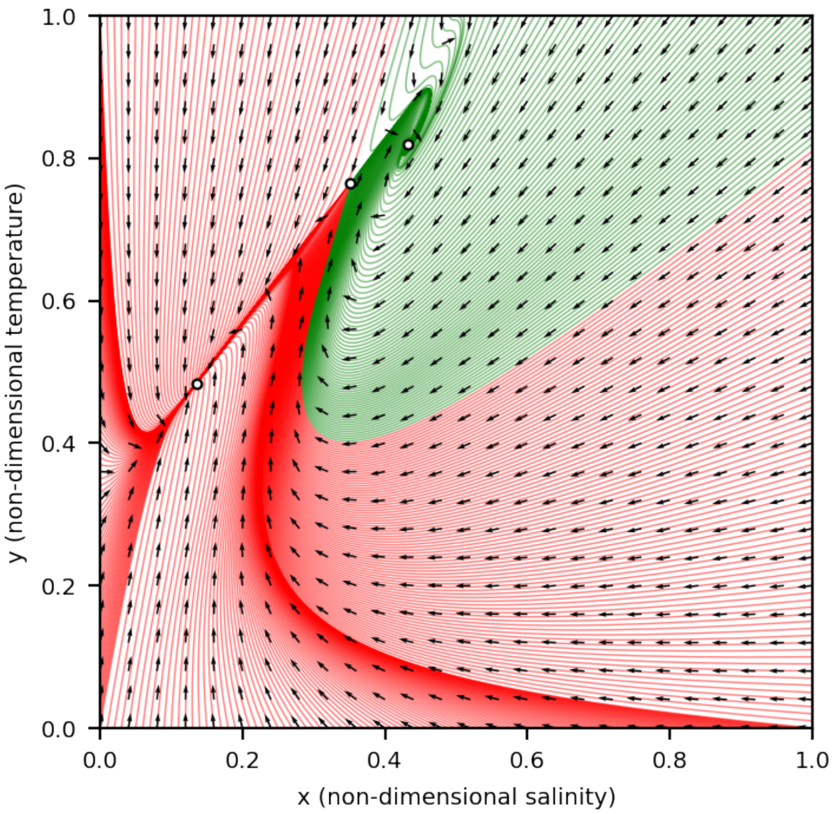

Figure 13: Phase portrait (simulation 1). The fixed points are displayed with white circles. Green trajectories represent systems leading to a salinity-driven circulation. Red trajectories tend to a temperature-driven circulation.

Since Stommel published his paper, numerous more advanced mathematical models have been developed. In particular, scientists study how close we currently are from Stommel's bifurcation point, the threshold beyond which no deep water formation in the North Atlantic can be sustained. For example, Rahmstorf et all write that "The proximity of the present-day climate to the Stommel bifurcation point [...] varies from less than 0.1 Sv to over 0.5 Sv" [10], where the Sv unit ~~models change of~~ freshwater input. Comparing 11 climate models, they conclude that change of freshwater forcing, or melting of the ice cap, can indeed induce a hysteresis behavior of the thermohaline circulation, which "supports the validity of Stommel's classic feedback" [10].

(1 million cubic meters per second) quantifies the

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