# Stommel's Box Model of the Thermohaline Circulation

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PART 01

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  - Vulnerabilities
- The Main Physical Processes
- The model
  - Set up
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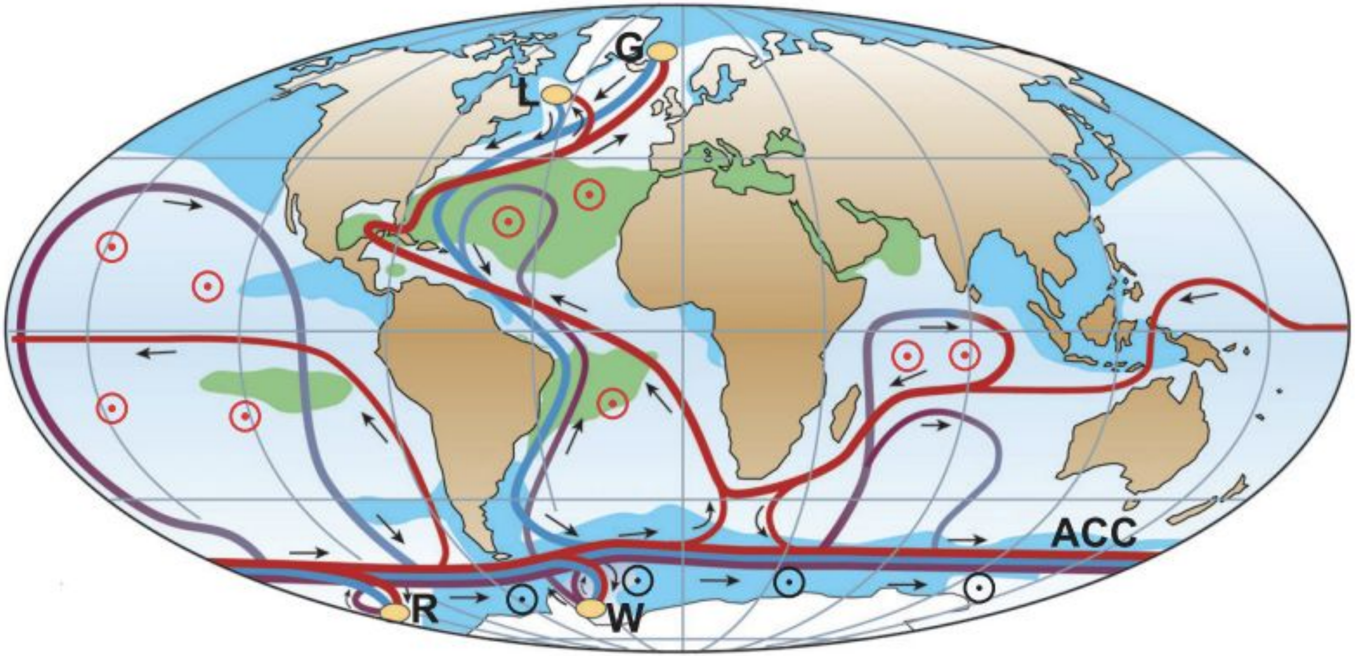
- Equilibrium states
  - fixed points
  - Stability analysis
- Simulations
- Stommel's Model and Climate Change

# The Thermohaline Circulation

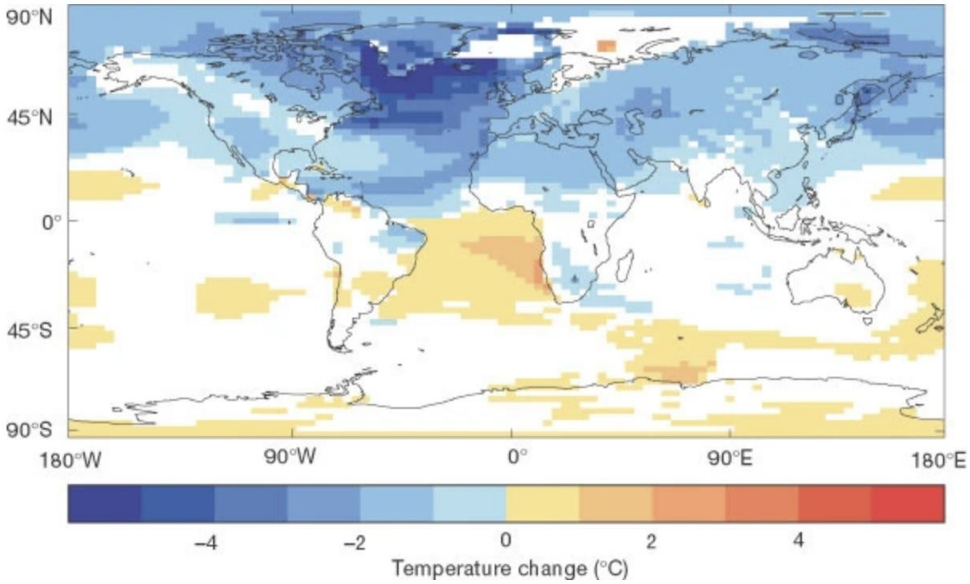



| | | | | | |
|---|---|---|---|---|---|
| | Surface flow | | Wind-driven upwelling | L | Labrador Sea |
| | Deep flow | | Mixing-driven upwelling | G | Greenland Sea |
| | Bottom flow | | Salinity > 36 ‰ | W | Weddell Sea |
| | Deep Water Formation | | Salinity < 34 ‰ | R | Ross Sea |

# Vulnerabilities

The thermohaline circulation is a very vulnerable system

Changes in salinity/ temperature → Changes in water density → Slow-down/shutdown of the circulation




## 0 1 . Climate and temperatures

Decrease of temperatures in the Northern hemisphere, increase in the Tropics

## 0 2 . Deep sea organisms

The deep water formation in the Labrador sea provides ¾ of the oxygen in the deep Atlantic ocean

## 0 3 . Sea level

Increase of sea level by up to 1 meter in the North Atlantic

# Has it happened in the past?

Yes

# Can it happen now?

Yes

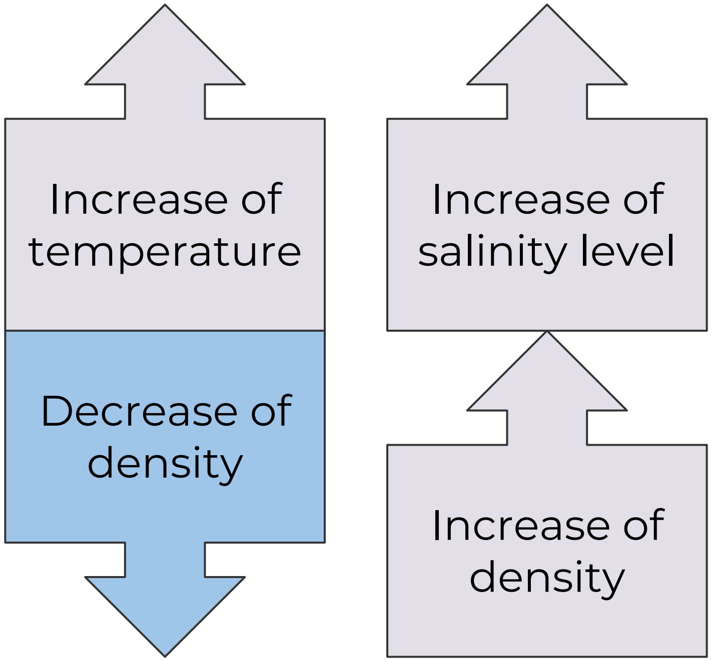
# Is it already happening?

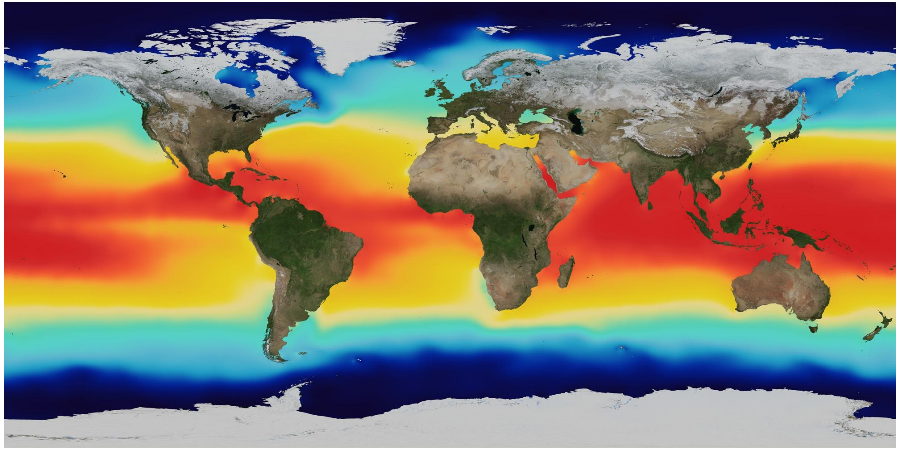
Maybe

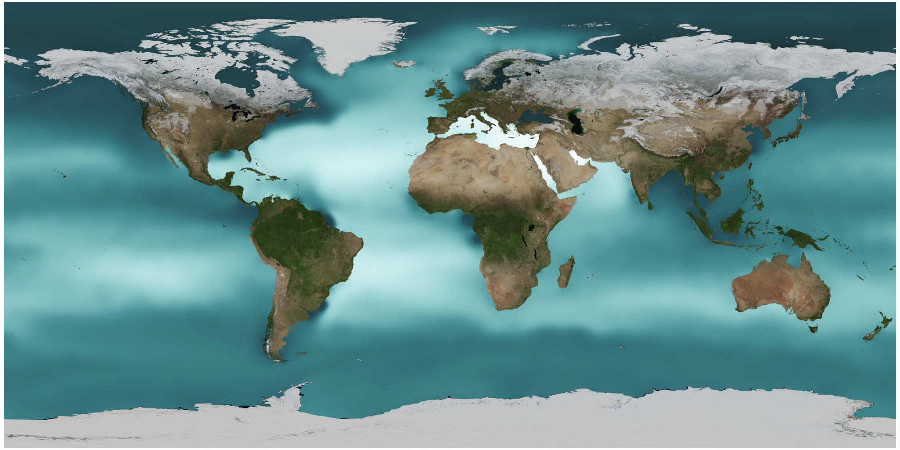
# Main Physical Processes

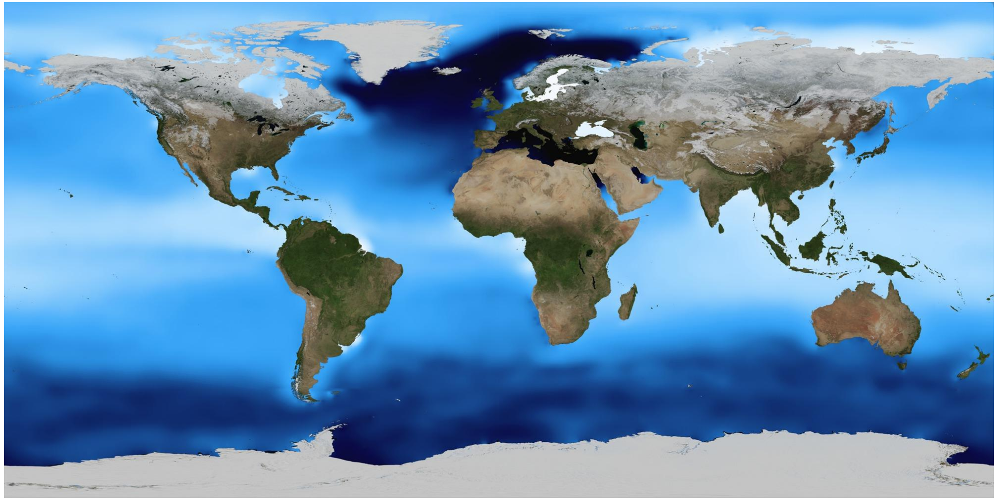
Water density is influenced by 2 main processes

- heating / cooling affect temperature
- evaporation / precipitation affect salinity level

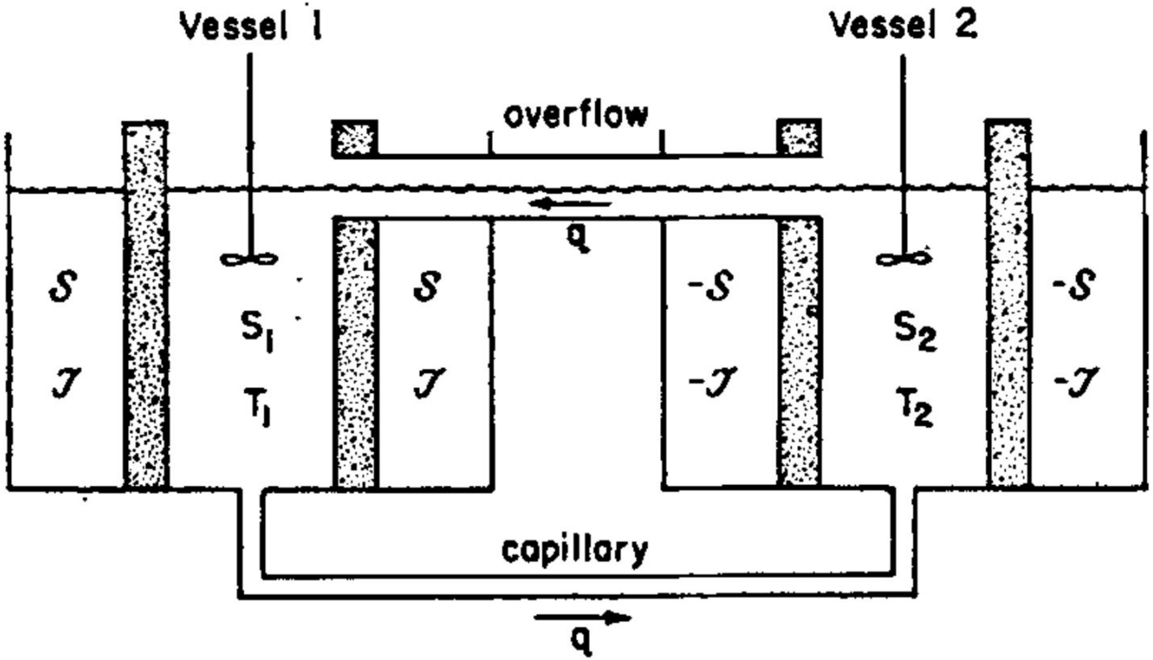








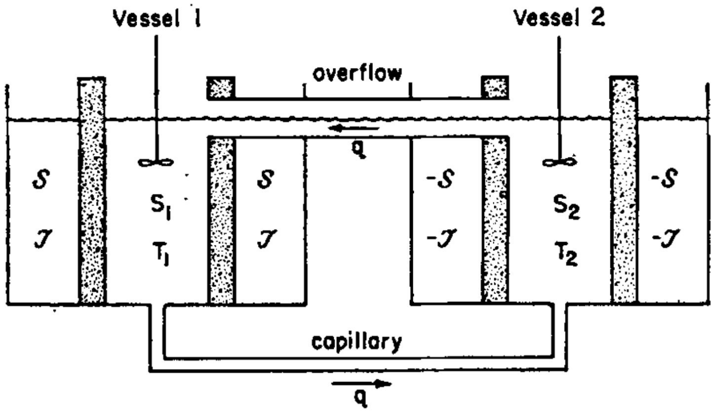
# The Model: Set Up



# The Model: Basic Equations

Laws for conservation of temperature and salinity:

$$
\begin{aligned}
T &= T_1 = -T_2 \\
S &= S_1 = -S_2
\end{aligned}
$$



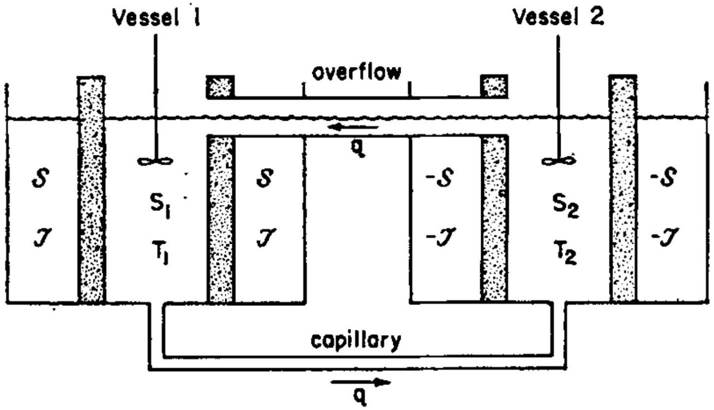

Transfer of heat and salt and flow rate:

$$
\begin{cases}
\dfrac{\mathrm{d}T}{\mathrm{d}t} = c(\boldsymbol{\mathcal{T}} - T) - |2q|T \\
\dfrac{\mathrm{d}S}{\mathrm{d}t} = d(\boldsymbol{\mathcal{S}} - S) - |2q|S
\end{cases}
$$

Non-dimensionalization: $\tau = ct \quad \delta = \dfrac{d}{c} \quad y = \dfrac{T}{\boldsymbol{\mathcal{T}}} \quad x = \dfrac{S}{\boldsymbol{\mathcal{S}}} \quad f = \dfrac{2q}{c}$

$$
\begin{cases}
\dfrac{\mathrm{d}y}{\mathrm{d}\tau} = 1 - y - |f|y \\
\dfrac{\mathrm{d}x}{\mathrm{d}\tau} = \delta(1 - x) - |f|x
\end{cases}
$$

# The Model: Flow Rate




Flow rate depends upon density difference between the 2 reservoirs:

$$kq = \rho_1 - \rho_2$$

Equation of state:

$$\rho = \rho_0(1 - \alpha T + \beta S)$$
$$\rho = \rho_0(1 + \alpha \boldsymbol{\mathcal{T}}(-y + Rx))$$

$$\Longrightarrow \quad \rho_1 = \rho_0(1 + \alpha \boldsymbol{\mathcal{T}}(-y + Rx)) \quad \rho_2 = \rho_0(1 - \alpha \boldsymbol{\mathcal{T}}(-y + Rx)) \quad \Longrightarrow \quad \rho_1 - \rho_2 = 2\alpha\rho_0 \boldsymbol{\mathcal{T}}(-y + Rx)$$

Get an equation for the flux:

$$\lambda = \left(\frac{c}{4\rho_0\alpha\boldsymbol{\mathcal{T}}}\right)k \quad \Longrightarrow \quad \lambda f = -y + Rx \quad \Longrightarrow \quad \boxed{\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}\tau} = 1 - y - \frac{y}{\lambda}|-y + Rx| \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} = \delta(1 - x) - \frac{x}{\lambda}|-y + Rx| \end{cases}}$$

# The Model: Fixed Points

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}\tau} = 1 - y - \frac{y}{\lambda}|-y + Rx| \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} = \delta(1 - x) - \frac{x}{\lambda}|-y + Rx| \end{cases}$$

$$\Longrightarrow \quad y = \frac{1}{1 + |f|}$$

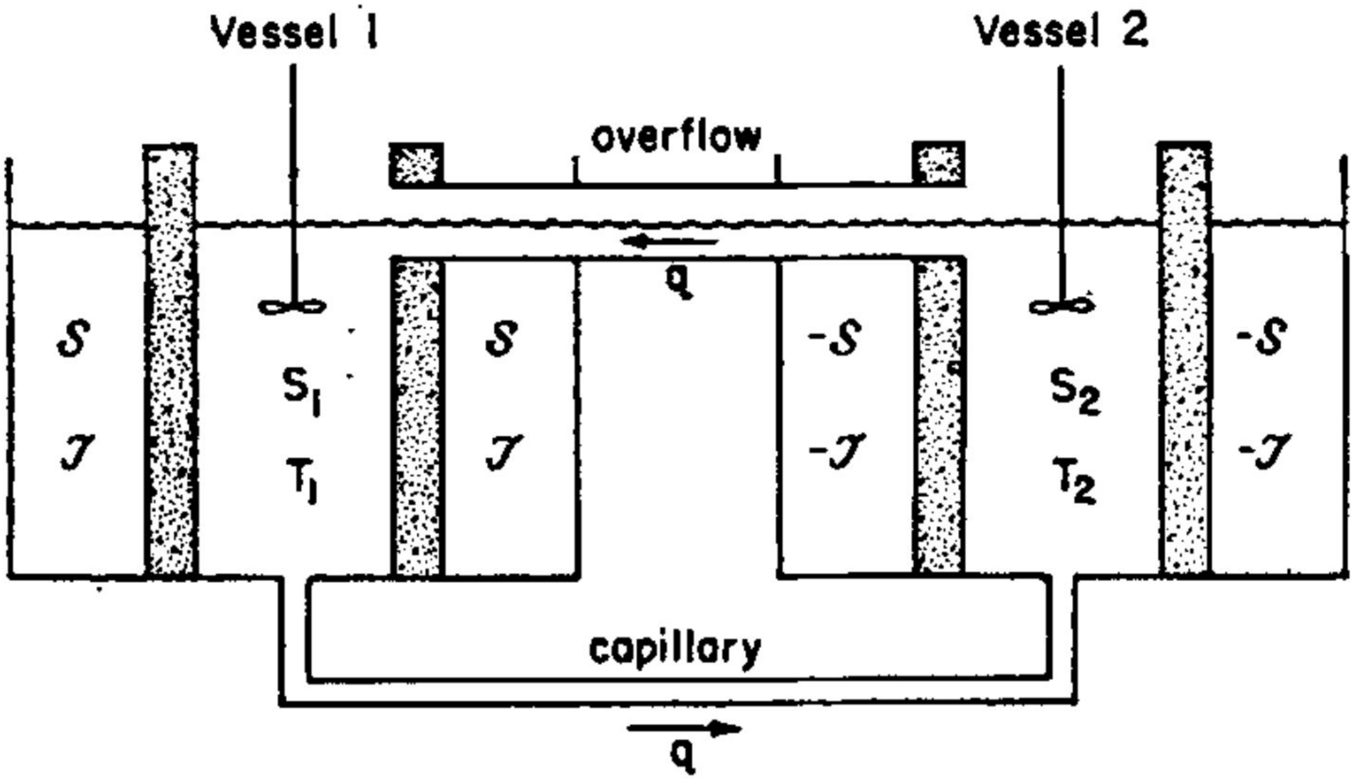
$$\Longrightarrow \quad x = \frac{1}{1 + \frac{|f|}{\delta}}$$

$$\lambda f = -y + Rx$$

$$\Longrightarrow \quad \boxed{\lambda f = -\frac{1}{1 + |f|} + \frac{R}{1 + \frac{|f|}{\delta}}}$$

**Stability analysis:**

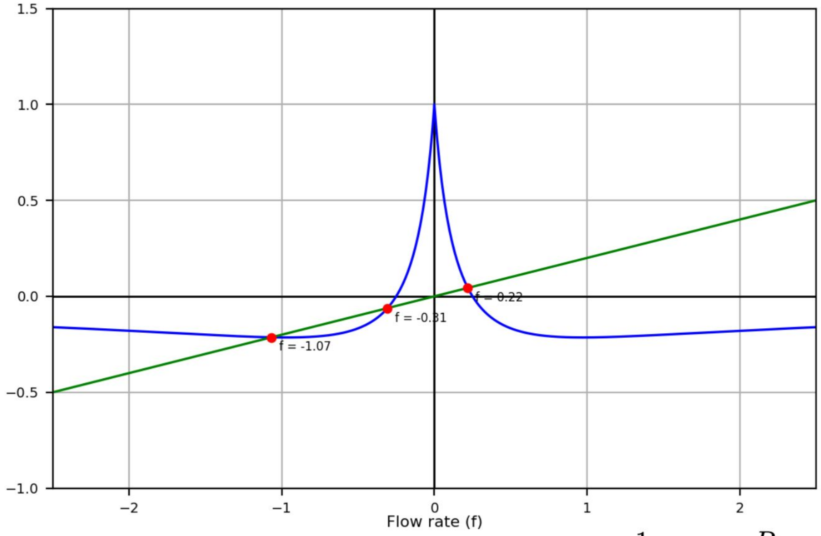
- find flow rate *f* at equilibrium => each flow rate defines a fixed point
- deduce values for *x* and *y* (salinity and temperature) for each fixed point
- linearize the 2D system around each fixed point
- find trace and determinant
- deduce stability (stable, undstable) and type (node, saddle, spiral) of each fixed point


Vessel 1
Vessel 2
overflow
q
$\mathcal{S}$
$\mathcal{T}$
$S_1$
$T_1$
$\mathcal{S}$
$\mathcal{T}$
$-\mathcal{S}$
$-\mathcal{T}$
$S_2$
$T_2$
$-\mathcal{S}$
$-\mathcal{T}$
capillary
q

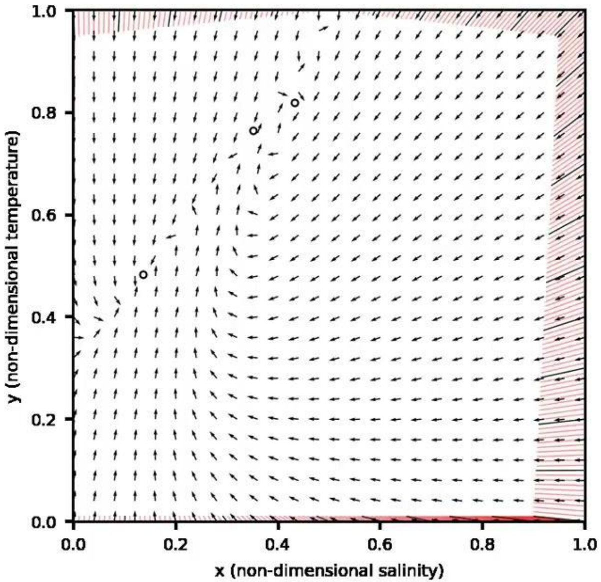
# Simulation System 1

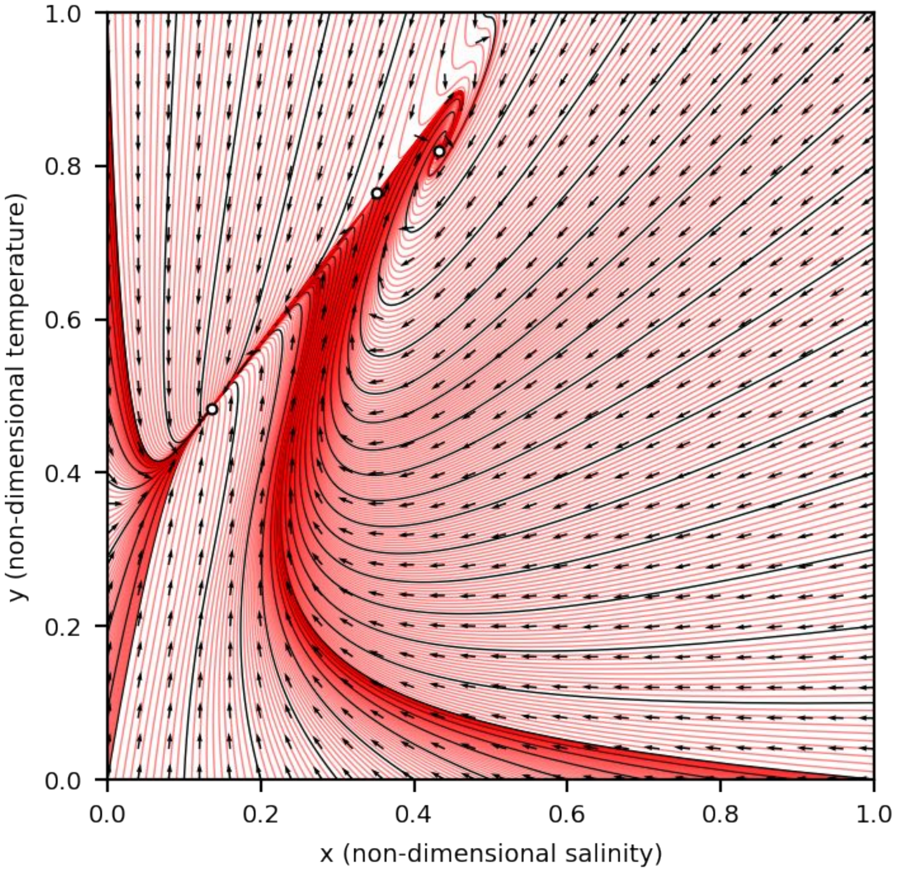
R = 2, λ = 1/5, and δ = 1/6

| Fixed Point | f | x | y | Trace | Determinant | Stability & Type |
|---|---|---|---|---|---|---|
| 1 | −1.07 | 0.14 | 0.48 | −4.37 | 2.75 | stable node |
| 2 | −0.31 | 0.35 | 0.76 | −2.10 | −2.15 | saddle node (unstable) |
| 3 | 0.22 | 0.43 | 0.82 | −1.83 | 4.16 | stable spiral |




$$\lambda f = -\frac{1}{1+|f|} + \frac{R}{1+\frac{|f|}{\delta}}$$

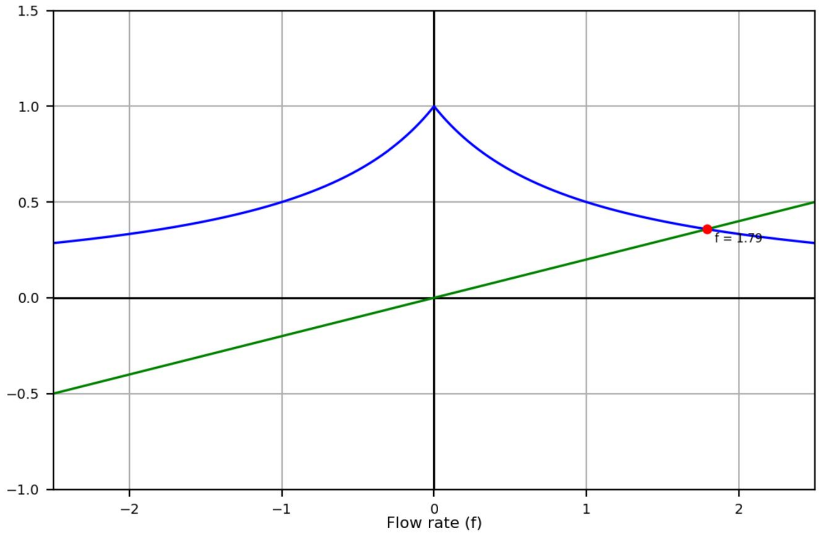


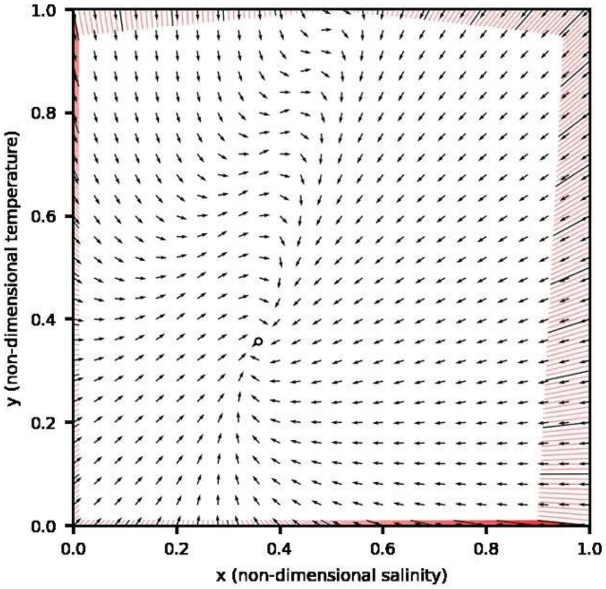

1.0
0.8
0.6
0.4
0.2
0.0
y (non-dimensional temperature)
0.0
0.2
0.4
0.6
0.8
1.0
x (non-dimensional salinity)

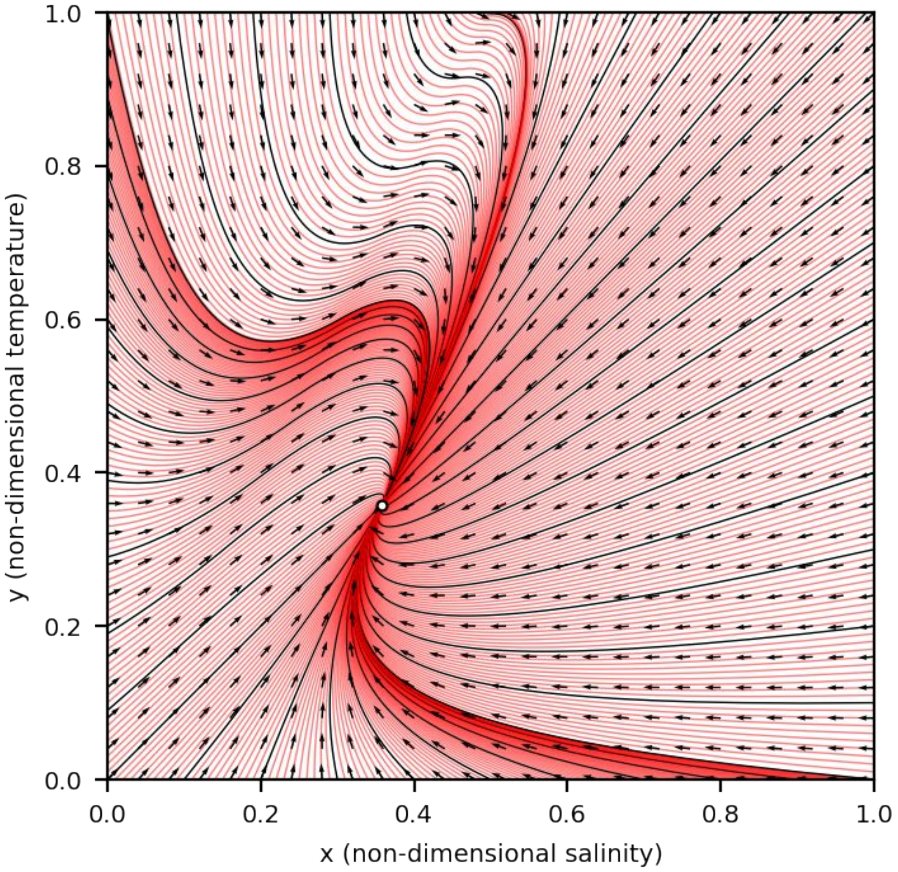
# Simulation System 2

R = 2, λ = 1/5, and **δ = 1**

| Fixed Point | f | x | y | Trace | Determinant | Stability & Type |
|---|---|---|---|---|---|---|
| 1 | 1.79 | 0.36 | 0.36 | −7.37 | 12.79 | stable node |



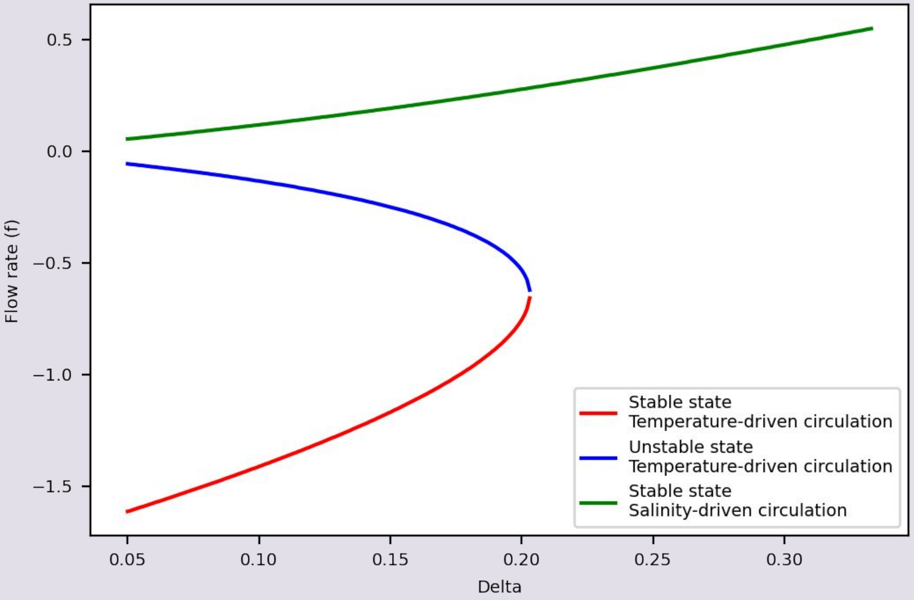

1.0
0.8
0.6
0.4
0.2
0.0
y (non-dimensional temperature)
0.0
0.2
0.4
0.6
0.8
1.0
x (non-dimensional salinity)

# Stommel's Model and Climate Change

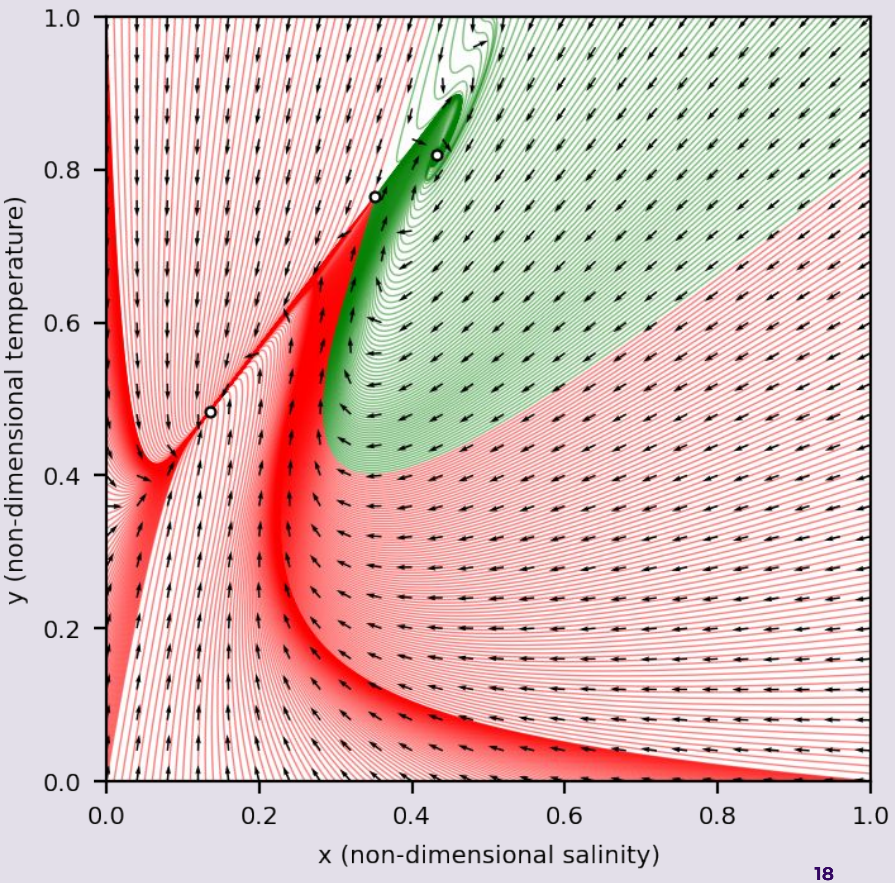
Bifurcation diagram for delta

c = temperature transfer coefficient
d = salinity transfer coefficient

$$\delta = \frac{d}{c}$$

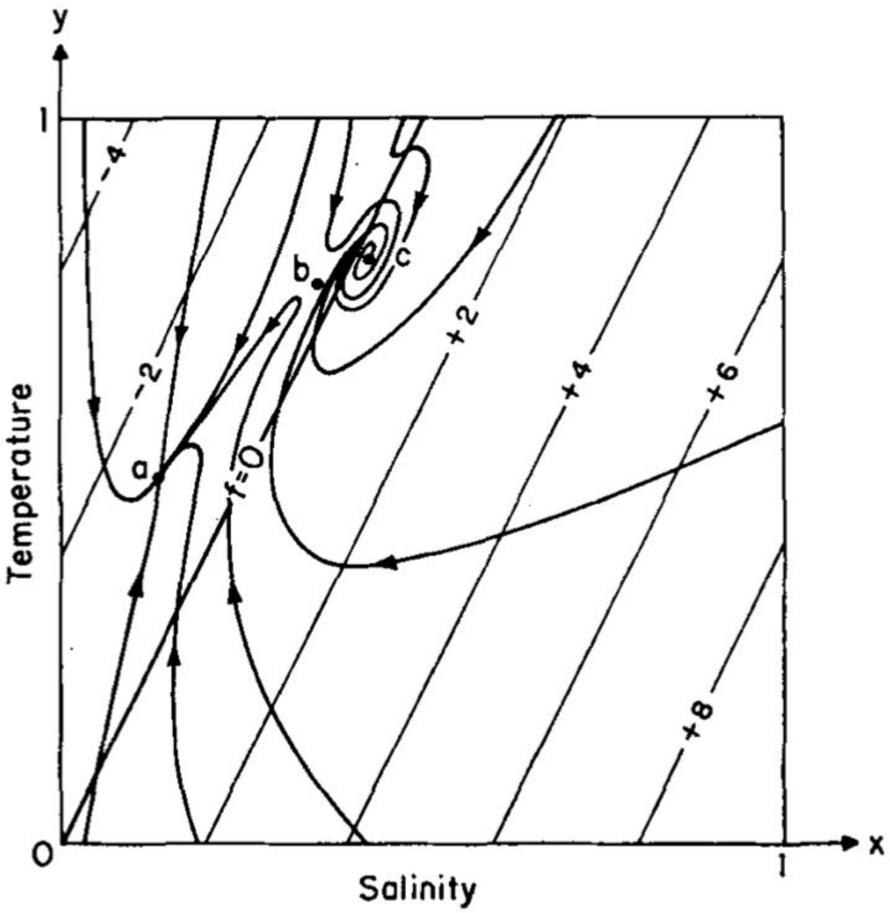


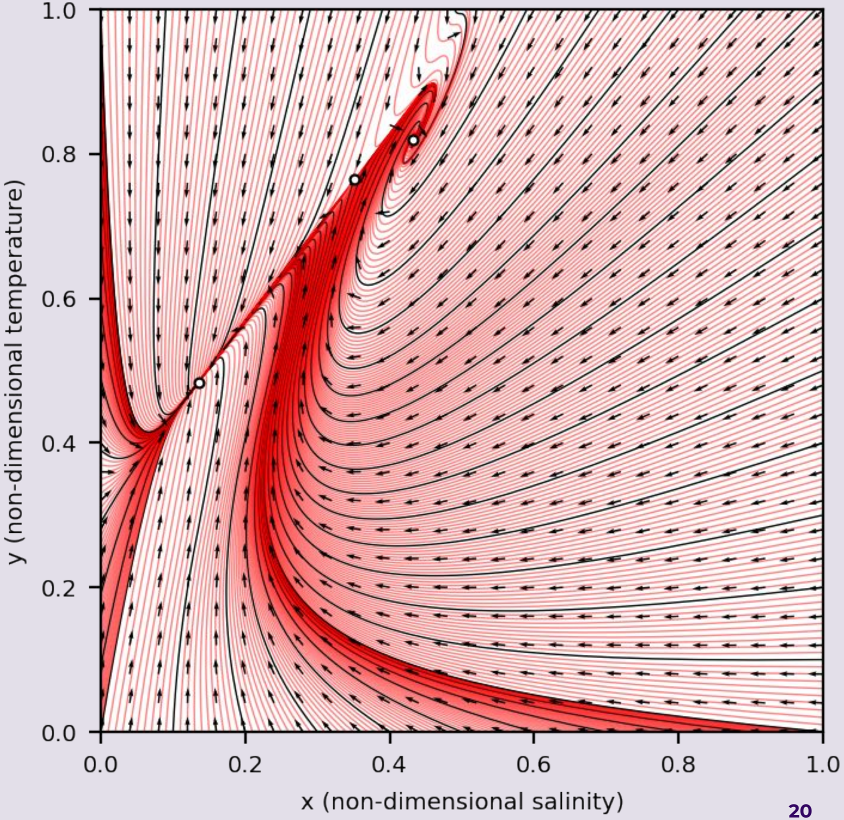
# Stommel's Model and Climate Change



# Conclusion

- Stommel models a flow rate based on density difference between 2 reservoirs
- density is affected by temperature or salinity
- the model can lead to 2 stable states with a circulation either thermally controlled or salinity driven
- the model shows that a big amount of freshwater in the cold zone (melt of Greenland ice sheet) can slow down or reverse the circulation: it is simulated by a change of parameters in the model
- this would have damaging effects on climate, biodiversity, and humans
- this fact been confirmed by numerous advanced models today


y
1
-4
-2
+2
+4
+6
+8
a
b
c
f=0
Temperature
0
Salinity
1
x


1.0
0.8
0.6
0.4
0.2
0.0
y (non-dimensional temperature)
0.0
0.2
0.4
0.6
0.8
1.0
x (non-dimensional salinity)