Evolution of a Lagrangian particle through Refraction with Reflected Expansion structure

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1 Introduction

This document is aimed at clarifying the computation of the evolution of a Lagrangian particle when it travels through a Refraction structure with Reflected Expansion. Please read carefully the prerequisite section to make sure you understand well the several gas dynamics notions at stake in this problem.

The MATLAB program used at the beginning is inherited from a previous internship, made by Yann, and is suspected to have incoherence. My goal here was to make it coherent, commented and to correct it when needed.

2 Prerequisite

Let's consider a shock tube where an incident shock, normal to the tube, is propagating, to be consistent with Henderson's experiments [1] [2]. An oblique gas interface is set artificially, even if the membrane used to keep gases apart is neglected thereafter. As shown in figure 1, an angle of incidence ω_i is defined from the interface to the shock at the intersection point. At the interface, the incident shock is reflected through the first phase, represented by the subscript I, and is refracted through the second phase, represented by the subscript II. According to Yann's work (see figure 5.1 in [5]), setting $\omega_i = 18$ deg allows to get a regular Refraction with Reflected Expansion for all Mach numbers (note that $\chi = 1/\xi_i$ is directly related to the Mach number of the shock and to the heat ratio of phase I). That is why an expansion fan is observed in phase I when the incident shock is reflected and a transmitted shock in phase II : the three phenomena meet at a single point, situated at the gas interface for a regular refraction pattern, see figure 1 (b).

The goal here is to determine whether a detonation could occur in such conditions, with a stoichiometric hydrogen-oxygen phase for phase I and an inert helium phase for phase II. Thus, pressure evolution and temperature evolution have to be computed to run chemical calculus. A Lagrangian description of the problem is the best way to calculate pressure, temperature and specific volume (inverse of density) of a fluid particle along its path through the refraction structure. It is identified by the blue square on figure 1.

To make the calculus easier, let's assume both gases are calorically perfect, with constant heat ratios γ_I and γ_{II} and following an adiabatic evolution, which brings $PV^\gamma = constant$.

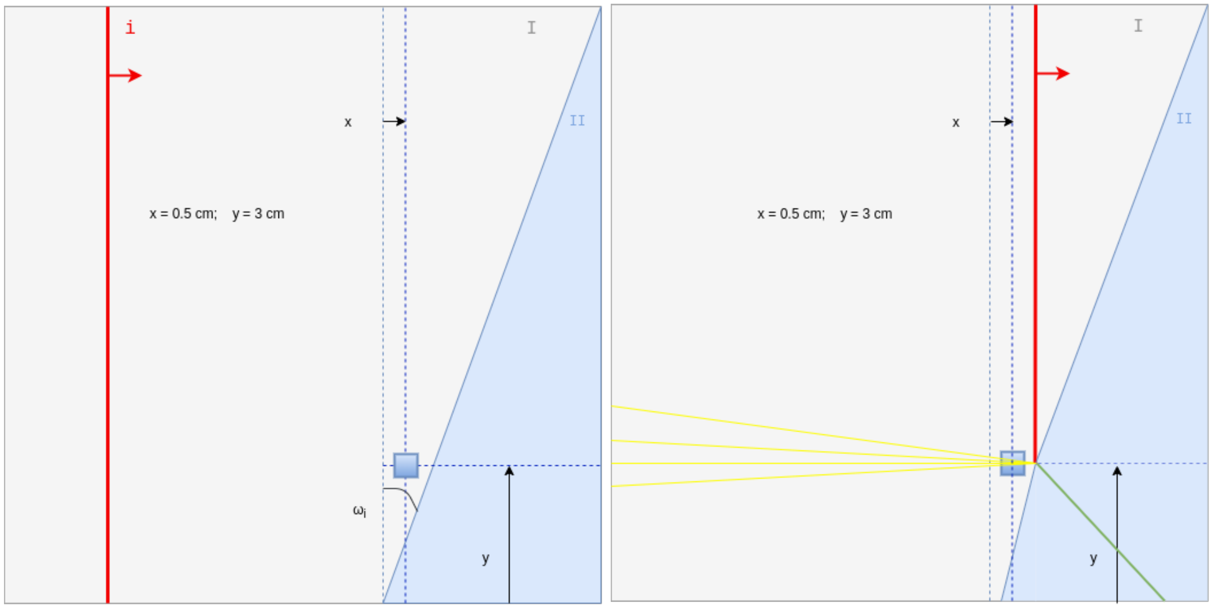
Three different gas dynamics notions are involved here : jump relations when an oblique shock is crossed, Prandtl-Meyer expansion equations and isentropic evolution relations. My method also involves shock polars to make the most of the membrane equilibrium, see [4] and [5] for further explanation. Jump relations are needed when our Lagrangian particle crosses the incident shock; Prandtl-Meyer theory allows to compute the isentropic evolution of our Lagrangian particle in the expansion fan. All useful formulae and explanations can be found in [3], chapter 9 *Oblique Shock and Expansion Waves.*

3 Oblique shock

3.1 Initial conditions before the incident shock

Initial conditions in zone `(0I)` are all given and reminded in table 1.

Notice that R_I is the gas constant of phase I and that a_{0I} is defined according to the perfect gases theory. For zone `(0II)`, membrane equilibrium gives $P0_I = P0_{II}$ and flows are parallel on both sides of the membrane. However, gas temperatures may be different and heat ratios are not equal thus $a_{0II} \neq a_{0I}$ and Mach numbers are different on both sides of the membrane. This remark will also be useful when treating zones `(2)` and `(t)`.



(a) Initial problem (b) Incident shock reaches gas interface

Figure 1: Illustration of the problem to be solved

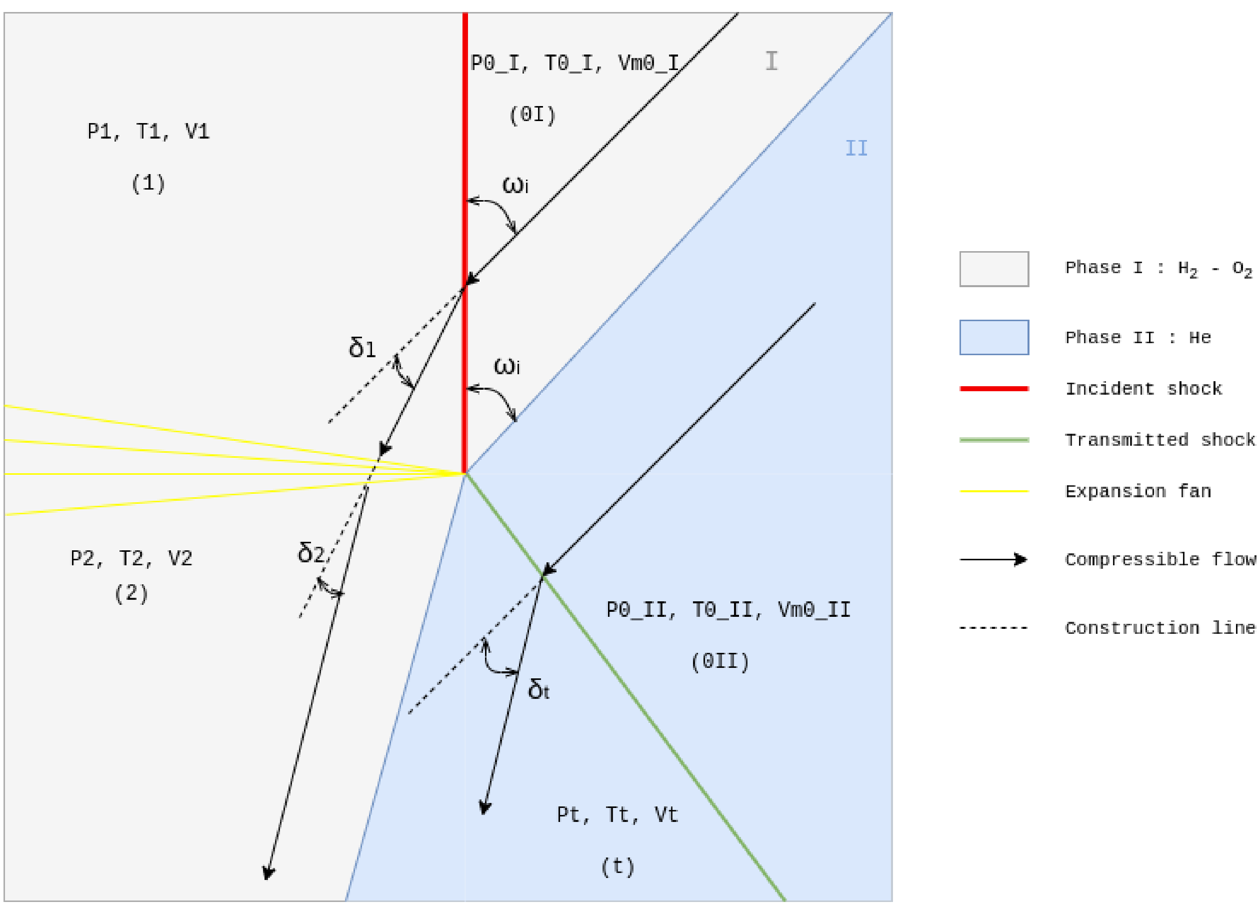


Figure 2: Symbols, zones and angles for section 3

Measure (unit)	Symbol	Value
Pressure (Pa)	$P0_I$	101 000
Temperature (K)	$T0_I$	600
Specific Volume	$Vm0_I$	$Vm0_I = R_I * T0_I/P0$
Mach number	M0	6
Angle of incidence (deg)	ω_i	18
Normal mach number	$M0_n$	$M0_n = M0 \sin(\omega_i)$
Speed of sound (m/s)	a_{0I}	$a_{0I} = \sqrt{\gamma_I R_I T0_I}$

Table 1: Initial conditions

3.2 Post shock conditions

Oblique shocks theory allows to calculate each measure in zone `(1)` : to make sure symbols are consistent all along this document, let's write equations `(9.13)` to `(9.18)` and equation `(9.23)` from [3] with the symbols set in figure 2.

$$M0_n = M0 \sin(\omega_i) \tag{1}$$

$$M1_n^2 = \frac{1 + \frac{\gamma_I - 1}{2} M0_n^2}{\gamma_I M0_n^2 - \frac{\gamma_I - 1}{2}} \tag{2}$$

$$\frac{V1}{Vm0_I} = \frac{(\gamma_I + 1) M0_n^2}{2 + (\gamma_I - 1) M0_n^2} \tag{3}$$

$$\frac{P1}{P0_I} = 1 + \frac{2\gamma_I}{\gamma_I + 1}(M0_n^2 - 1) \tag{4}$$

$$\frac{T1}{T0_I} = \frac{P1}{P0_I}\frac{V1}{Vm0_I} \tag{5}$$

$$M1 = \frac{M1_n}{\sin(\omega_i - \delta_1)} \tag{6}$$

where δ_1 is the angle of deflection behind the incident shock.

$$\tan(\delta_1) = 2\cot(\omega_i)\frac{M0_n^2 - 1}{M0^2(\gamma_I + \cos 2\omega_i) + 2} \tag{7}$$

With these equations, the flow in zone `(1)` is fully described and, assuming all measures keep constant in this zone, it is now possible to compute the evolution of a Lagrangian particle in zones `(0I)` and `(1)`. Some geometric considerations allow to know the time spent in zone `(0I)` if computation starts 2 cm before the bottom of the interface, and then the time spent in zone `(1)` depends on the beginning of the expansion fan.

4 Expansion fan

It is important to note that the evolution inside the expansion fan is continuous : it is schematized by a finite succession of jumps leading each to a finite deflection but keep in mind that those "jumps" are in fact Mach waves. That is why the evolution in the expansion fan is isentropic.

4.1 Post expansion conditions

Let's use the Prandtl-Meyer theory to compute the total deflection, δ_2, and the Mach number $M2$ behind the expansion. To do so, equations `(9.42)` and `(9.43)` from [3] can be highly useful. ν is the Prandtl-Meyer function, depending on the heat ratio of the gas at stake.

$$\delta_2 - \delta_1 = \nu(M2) - \nu(M1) \tag{8}$$

Reminding that $\delta_t = \delta_1 + \delta_2$ and that $P2 = Pt$ from the membrane equilibrium between zone `(2)` and zone `(t)`, it is now clear that Mach number can be found from equation 8.

To get the pressure jump and the deflection angle when crossing the transmitted jump, the method described in section 3.3 of [5] is applied. Thanks to Yann's work, the transmitted shock polar and the expansion polar are computed and their intersection indicates the pressure jump ξ_t through the transmitted shock and the deflection angle δ_t behind it.

Having now the Mach number $M2$, the other measures can be easily found, either with the isentropic relations or with the perfect gases law.

$$P2 = Pt = P0_{II} \times \xi_t \tag{9}$$

$$T2 = T1\left(\frac{P2}{P1}\right)^{\frac{\gamma_I - 1}{\gamma_I}} \tag{10}$$

$$V2 = R_I \frac{T2}{P2} \tag{11}$$

4.2 Evolution in the expansion fan

The evolution of our Lagrangian particle inside the expansion is led by isentropic relations, as said before. According to Yann's work, this evolution is computed with tiny steps of deflection. Once the number of points of computation n_{exp} is set, the deflection interval is defined by $\mathrm{d}\delta = \frac{\delta_2}{n_{exp}}$.

To initialise the computation, the speed of the flow before the expansion is needed, and it should be broken down into its normal and tangential components, as the tangential component of the speed keeps constant in the expansion. At each step of computation, square of the Mach number, pressure, temperature, specific volume, speed of the flow and position of the Lagrangian particle are updated according to isentropic equations pointed out in [5], section 5.2.4. Those equations, which are a little confusing at first sight, can be obtained by taking the logarithmic derivative of equations `(8.40)`, related to `(8.42)` and `(8.43)` thanks to `(8.41)`, in [3].

The final values of this computation are coherent with those found in the previous section.

5 Conclusion

As a brief conclusion, let's compare the results found by Yann and those found with the method described in this document. The first incoherence spotted in Yann's code was the calculus of the Mach number in zone `(0I)` : it was computed as if the shock was normal to the interface, which is not the case. The evolution of pressure, temperature and specific volume with this incoherence is plotted in blue on figure 3. After correction, the new evolution is plotted in yellow. The red dotted line represents the evolution of our Lagrangian particle following the method developed in this document.

The three methods lead to the same jump after the incident shock but have quite distinct results when one focuses on the time spent between the incident shock and the beginning of the expansion or on the time spent inside the expansion. This distinction maybe related to the Mach number obtained after the incident shock because it is the only measure that affects the evolution in the expansion but not pressure or temperature in zone `(1)`.

References

[1] Henderson L.F. Abd-el Fattah, A.M. Shock waves at a slow-fast gas interface. *Journal of Fluid Mechanics*, 1978.

[2] Henderson L.F. Lozzi A. Abd-el Fattah, A.M. Precursor shock waves at a slow-fast gas interface. *Journal of Fluid Mechanics*, 1976.

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[4] G. Ben-Dor. *Shock Wave Reflection Phenomena.* Springer, 1991.

[5] Y. de Gouvello. Detonation initiation by shock wave refraction. *ENSTA Paris - Tsinghua University*, 2019.

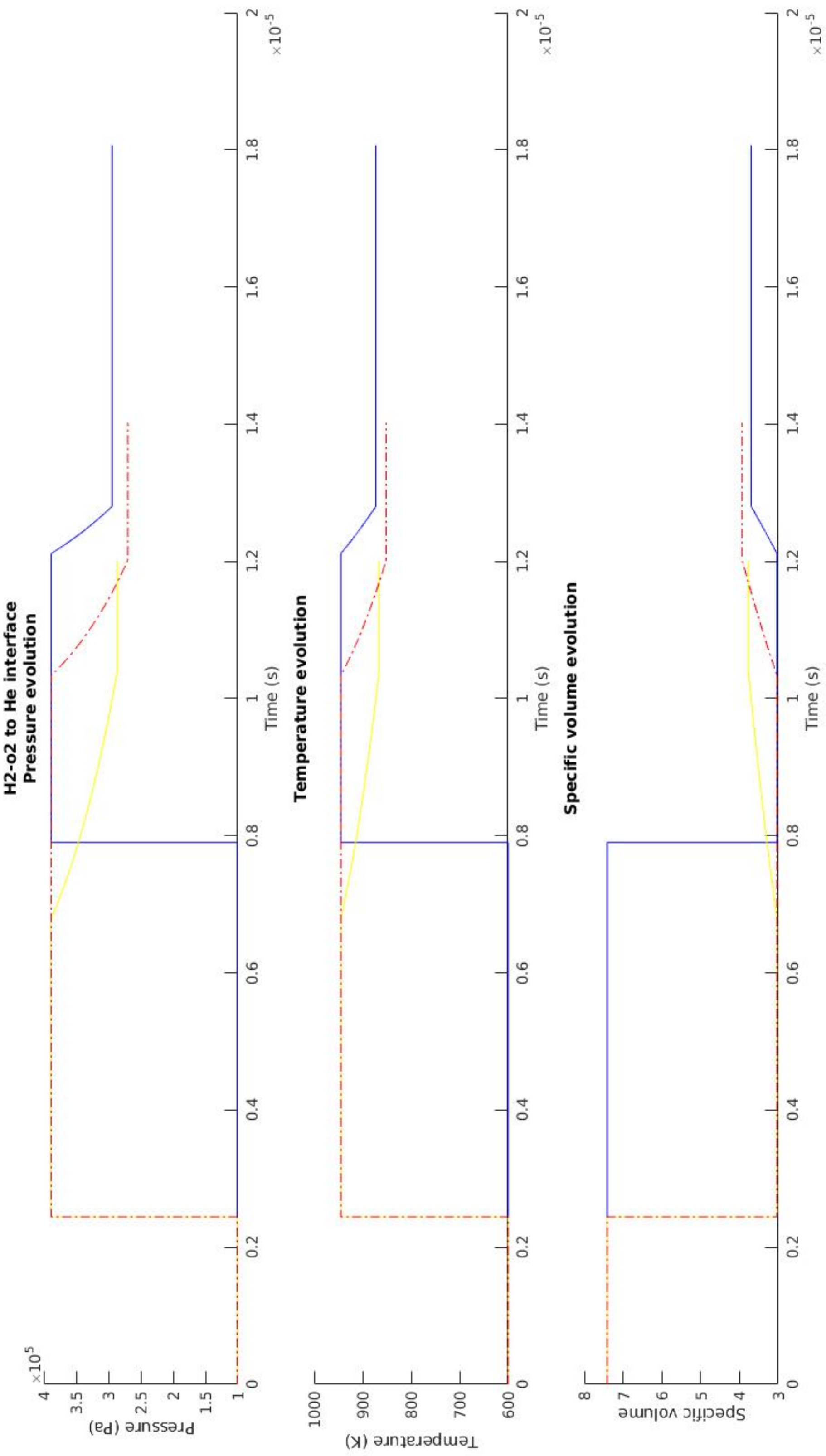

Figure 3: Evolution of the Lagrangian particle $x = 0.5, y = 3$ at $M0 = 6$ with the three methods