INF5620 -
Finite Elements

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Project 1: 1D wave equation with finite elements

The 1D-wave equation:

$$u_{tt} = c^2 u_{xx} \tag{1}$$

for $x \in [0, L]$ and $t \in (0, T]$. The boundary and initial conditions are

$$u(0,t) = U_0(t), \quad u_x(L) = 0, \quad u(x,0) = I(0), \quad u_t(x,0) = V(x) \tag{2}$$

1a)

Use a finite difference method in time to formulate a sequence of spatial problems.

$$[D_t D_t u = c^2 u_{xx}]_i^n \tag{3}$$

$$\frac{u_i^{n+1} - 2u_i^n + u_i^[n-1}{\Delta t^2} = c^2 u_{xx}$$

$$u_i^{n+1} = \Delta t^2 c^2 u_{xx} + 2u_i^n - u_i^{n-1} \tag{4}$$

1b) and 1c)

Use a Galerkin method to derive a variational form of each of the spatial problems.
Galerkin method: multiply eq. (4) by v and integrate over the whole domian ($x \in [0, L]$).

$$\int_0^L u_i^{n+1} v dx = \int_0^L \Delta t^2 c^2 u_{xx} v dx + \int_0^n 2u_i^n v dx - \int_0^L u_i^{n-1} v dx \tag{5}$$

Let's take a look at the first integral on the right hand side, while we let $v = \phi_i$ for $i \in 0, 1, 2, 3, ...$
The notation u_{xx} means:

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}|_i^n$$

$$I_1 = \int_0^L \Delta t^2 c^2 u_{xx} \phi_i(x) dx = \Delta t^2 c^2 \left(\left[\frac{du}{dx} \phi_i(x) \right]_0^L - \int_0^L \frac{du}{dx} \frac{d\phi}{dx} dx \right)$$

We demand $\phi_i(0) = \phi_i(L) = 0$.

$$I_1 = -\Delta t^2 c^2 \int_0^L u_i^{n'} \phi_i' dx \tag{6}$$

Now we insert $u_i^n = \sum_j c_j \phi_j$ into eq. (5) and $u_i^{n'} = \sum_j c_j \phi_j'$ into (6).

$$I_1 = -\Delta t^2 c^2 \sum_j c_j^n \left(\int_0^L \phi_j' \phi_i' dx \right)$$

$$\sum_j c_j^{n+1} \left(\int_0^L \phi_j \phi_i \right) = -\Delta t^2 c^2 \left[\sum_j c_j^n \left(\int_0^L \phi_j' \phi_i' \right) \right] + \sum_j 2c_j^n \left(\int_0^L \phi_j \phi_i dx \right) - \sum_j c_j^{n-1} \left(\int_0^L \phi_j \phi_i dx \right) \tag{7}$$

for $i \in 0, 1, 2, 3, ...$ We get to integrals to calculate:

$$A_{ij} = \int_0^L \phi_i \phi_j dx$$
$$B_{ij} = \int_0^L \phi_i' \phi_j' dx \tag{8}$$

ϕ_i is the i'th basisfunction for P1 elements (hat functions). We know that the product $\phi_i \phi_j$ is nonzero only for $i = j$ og $j = i \pm 1$. Start with $A_{ii\pm1}$ and do a change of variable to a local variable X in element i. $x = x_m + X\frac{h}{2}$.

$$A_{ii\pm1} = \frac{h}{2} \int_{-1}^1 \frac{1}{2}(1-X)\frac{1}{2}(1+X)dX = \frac{h}{8}[X - \frac{X^3}{3}]_{-1}^1 = h\frac{1}{6} \tag{9}$$

Now let's look at A_{ii} with tha same change of variable. Now we have to integrate over 2 elements, becaus each of the basis functions is nonzero on to neighboring elements.

$$A_{ii} = \frac{h}{2} \left(\int_{-1}^1 \frac{1}{4}(1-X)^2 dX + \int_{-1}^1 \frac{1}{4}(1+X)^2 dX \right) = \frac{h}{8} \left(\int_{-1}^1 1^2 - 2X + X^2 dX + \int_{-1}^1 1^2 + 2X + X^2 dX \right)$$
$$= \frac{h}{8} \left(\int_{-1}^1 1 - 2X + X^2 + 1 + 2X + X^2 dX \right) = \frac{h}{8} \left(\int_{-1}^1 2 + 2X^2 dX \right) = \frac{1}{4} \left[X + \frac{X^3}{3} \right]_{-1}^1 = h\frac{2}{3}$$

For $i = 0$ and $i = N$ we have to divide by two because for these elements we only have half of a basis functions.
Good, now we have computed one of the integrals.

$$A_{ij} = \int_0^L \phi_i \phi_j dx = \delta_{ij\pm1}\frac{h}{6} + \delta_{ij}\frac{4h}{6} \text{if } i = j \neq 0, N$$
$$A_{0,0} = A_{N,N} = \frac{2h}{6}$$

Ok, let's calculate the other integral

$$B_{ij} = \int_0^L \phi_i' \phi_j' dx \tag{10}$$

We see (if we draw the functions ϕ_i') that the product $\phi_i' \phi_j'$ only gives nonzero values if $i = j$ or $i = j \pm 1$. $\phi_i' \phi_j'$ is a constant over each element, which makes the integral easy to solve. We do the same change of variable as before.

$$\begin{aligned}B_{ii\pm1} = \int_0^L \phi_i'\phi_{i\pm1}'dx &= \int_{-1}^1 \left(\frac{d\phi_i}{dX}\frac{dX}{dx}\right)\left(\frac{d\phi_{i\pm1}}{dX}\frac{dX}{dx}\right)\frac{dx}{dX}dX \\ &= \frac{2}{h}\int_{-1}^1\left(\frac{d\phi_i}{dX}\right)\left(\frac{d\phi_{i\pm1}}{dX}\right)dX = \frac{2}{h}\int_{-1}^1\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)dX \\ &= \frac{2}{h}\left[\frac{-X}{4}\right]_{-1}^1 = \frac{2}{h}\left[\frac{-X}{4}\right]_{-1}^1 = \frac{-1}{h}\end{aligned} \tag{11}$$

$$\begin{aligned}B_{ii} = \int_0^L \phi_i'\phi_i'dx &= \frac{2}{h}\int_{-1}^1\left(\frac{1}{2}\right)^2 dX + \frac{2}{h}\int_{-1}^1\left(\frac{1}{2}\right)^2 dX \\ &= \frac{4}{h}\int_{-1}^1\left(\frac{1}{4}\right)dX = \frac{2}{h}\end{aligned} \tag{12}$$

$$\begin{aligned}B_{ij} = \delta_{ij\pm1}\frac{-1}{h} + \delta_{ij}\frac{2}{h}\text{if } (i=j) \neq 0, N \\ B_{00} = B_{NN} = \frac{1}{h}\end{aligned} \tag{13}$$

Insert our result into eq. (7):

$$\begin{aligned}h\sum_j c_j^{n+1}\mathbf{M_{ij}} = \frac{\Delta t^2 c^2}{h}\left[\sum_j c_j^n \mathbf{K_{ij}}\right] \\ +h\sum_j 2c_j^n\mathbf{M_{ij}} - h\sum_j c_j^{n-1}\mathbf{M_{ij}}\end{aligned} \tag{14}$$

Divide by h

$$\begin{aligned}\sum_j c_j^{n+1}\mathbf{M_{ij}} = \frac{\Delta t^2 c^2}{h^2}\left[\sum_j c_j^n \mathbf{K_{ij}}\right] \\ +\sum_j 2c_j^n\mathbf{M_{ij}} - \sum_j c_j^{n-1}\mathbf{M_{ij}}\end{aligned} \tag{15}$$

where $M_{ij} = \delta_{ij}\frac{1}{6} + \delta_{ij\pm1}\frac{4}{6}$ and $K_{ij} = \delta_{ij} - \delta_{ij\pm1}2$
We can collect all the equations in a matrix equation.

$$\mathbf{Mc^{n+1}} = C^2\mathbf{Kc^n} + 2\mathbf{Mc^n} - \mathbf{Mc^{n-1}} \tag{16}$$

where $C = \frac{\Delta tc}{h}$ is the courant number.

1d)

Interpret equation number i in the linear system as a finite difference approximation of $u_{tt} = c^2u_{xx}$ using the following scheme:

$$[D_tD_t(u + \frac{1}{6}\Delta x^2 D_xD_xu) = c^2d_xD_xu]_i^n \tag{17}$$

or written with an f:

$$\begin{aligned}[D_tD_tf = c^2D_xD_x]_i^n \\ \frac{f_i^{n+1} - 2f_i^n + f_i^{n-1}}{\Delta t^2} = \frac{c^2}{\Delta x^2}[u_{i+1}^n - 2u_i^n + u_{i-1}^n] \\ f_i^{n+1} - 2f_i^n + f_i^{n-1} = C^2[u_{i+1}^n - 2u_i^n + u_{i-1}^n]\end{aligned} \tag{18}$$

where $C = c\frac{\Delta t}{\Delta x}$ and $f_i^n = u_i^n + \frac{1}{6}[D_x D_x u]_i^n = u_i^n + \frac{1}{6}[u_{i+1}^n - 2u_i^n + u_{i-1}^n]$.

The equation (18) becomes

$$\begin{aligned} u_i^{n+1} + \frac{1}{6}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] \\ -2[u_i^n + \frac{1}{6}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]] \\ +u_i^{n-1} + \frac{1}{6}\left[u_{i+1}^n - 2u_i^{n-1} + u_{i-1}^{n-1}\right] \\ = C^2[u_{i+1}^n - 2u_i^n + u_{i-1}^n] \end{aligned}$$

We put the u's for timestep n+1 on the left hand side.

$$\begin{aligned} \frac{1}{6}u_{i-1}^{n+1} + \frac{2}{3}u_i^{n+1} + \frac{1}{6}u_{i+1}^{n+1} \\ = 2\left[\frac{1}{6}u_{i-1}^n + \frac{2}{3}u_i^n + \frac{1}{6}u_{i+1}^n\right] \\ +C^2\left[u_{i-1}^n - 2u_i^n + u_{i+1}^n\right] \\ -\left[\frac{1}{6}u_{i-1}^{n-1} + \frac{2}{3}u_i^{n-1} + \frac{1}{6}u_{i+1}^{n-1}\right] \end{aligned}$$

This is for $i \in 0, 1, 2, 3, ...$, but we can collect all the equations (for different i) in a matrix equation:

$$\tilde{\mathbf{M}}\mathbf{u}^{n+1} = 2\tilde{\mathbf{M}}\mathbf{u}^n - \tilde{\mathbf{M}}\mathbf{u}^{n-1} + C^2\tilde{\mathbf{K}}\mathbf{u}^n \tag{19}$$

where

$$\tilde{\mathbf{M}} = \frac{1}{6}\begin{pmatrix} 4 & 1 & 0 & 0 & ... & 0 \\ 1 & 4 & 1 & 0 & ... & 0 \\ 0 & 1 & 4 & 1 & ... & 0 \\ 0 & 0 & 1 & 4 & 1... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 & 4 \end{pmatrix} \tag{20}$$

and

$$\tilde{\mathbf{K}} = \begin{pmatrix} -2 & 1 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & ... & 0 \\ 0 & 0 & 1 & -2 & 1... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 & -2 \end{pmatrix} \tag{21}$$

are tridiagonal matrices. We notice that eq. (19) is the same as the equation as our linear system in eq. (16), for $u = c$ and if we ignore M_{00}, M_{NN} and K_{00}, K_{NN}. I just realized that we should not have anything but zero around the frame (except for the first and last element. These should maybe be put to 1?) of the matrices because we have decided that our boundary condition is $u_0^n = u_N^n = 0$, and if our frame is nonzero we will change u_0.

1e)

Perform an analysis of the scheme in 1d) by investigating a Fourier component

$$u_p^n = \exp(ikp\Delta x)\exp(-i\tilde{\omega}n\Delta t) \tag{22}$$

. Show that the stability criterion is

$$C \leq \frac{1}{\sqrt{3}} \tag{23}$$

$$\begin{aligned} D_t D_t u &= \exp(ikp\Delta x)\left(\frac{\exp(-i\tilde{\omega}(n+1)\Delta t - 2\exp(-i\tilde{\omega}n\Delta t) + \exp(-i\tilde{\omega}(n-1)\Delta t)}{\Delta t^2}\right) \\ &= \frac{\exp(ikp\Delta x)\exp(-i\tilde{\omega}n\Delta t)}{\Delta t^2}\left(\exp(-i\tilde{\omega}\Delta t) + \exp(i\tilde{\omega}\Delta t) - 2\right) \\ &= \frac{u_p^n}{\Delta t^2}\left(2\cos(\tilde{\omega}\Delta t) - 2\right) = \frac{u_p^n}{\Delta t^2}\left(-4\sin^2(\frac{\tilde{\omega}\Delta t}{2})\right) \end{aligned} \tag{24}$$

$$D_x D_x u = \frac{u_p^n}{\Delta x^2}\left(-4\sin^2(\frac{k\Delta x}{2})\right) \tag{25}$$

$$\begin{aligned} D_t D_t(u + \frac{1}{6}\Delta x^2 D_x D_x u) &= D_t D_t\left(u - \frac{2}{3}u\sin^2(\frac{k\Delta x}{2})\right) \\ &= D_t D_t(u[1 - \frac{2}{3}\sin^2(\frac{k\Delta x}{2}]) = [1 - \frac{2}{3}\sin^2(\frac{k\Delta x}{2})]D_t D_t(u) \\ &= [1 - \frac{2}{3}\sin^2(\frac{k\Delta x}{2})]\frac{u_p^n}{\Delta t^2}\left(-4\sin^2(\frac{\tilde{\omega}\Delta t}{2})\right) \end{aligned} \tag{26}$$

Equation (17) becomes

$$\begin{aligned} [1 - \frac{2}{3}\sin^2(\frac{k\Delta x}{2})]\frac{u_p^n}{\Delta t^2}\left(-4\sin^2(\frac{\tilde{\omega}\Delta t}{2})\right) &= c^2\frac{u_p^n}{\Delta x^2}\left(-4\sin^2(\frac{k\Delta x}{2})\right) \\ [1 - \frac{2}{3}\sin^2(\frac{k\Delta x}{2})]\left(\sin^2(\frac{\tilde{\omega}\Delta t}{2})\right) &= C^2\sin^2(\frac{k\Delta x}{2}) \\ \sin^2(\frac{\tilde{\omega}\Delta t}{2}) &= \frac{C^2\sin^2(\frac{k\Delta x}{2})}{1 - \frac{2}{3}\sin^2(\frac{k\Delta x}{2})} \\ \sin^2(\frac{\tilde{\omega}\Delta t}{2}) &= \frac{C^2}{(\frac{1}{\sin^2(\frac{k\Delta x}{2})} - \frac{2}{3})} \end{aligned} \tag{27}$$

where C is the Courant number.
The left hand side can not be larger than 1, which means that the right hand side also has to be smaller or equal to 1 :

$$1 \geq \frac{C^2}{(\frac{1}{\sin^2(\frac{k\Delta x}{2})} - \frac{2}{3})}$$

We notice that the right hand side has a maximum when $\sin^2(\frac{k\Delta x}{2})$ has a maximum; 1. This means the right hand side can not be larger than

$$\frac{C^2}{(\frac{1}{\sin^2(\frac{k\Delta x}{2})_{max}} - \frac{2}{3})} = \frac{C^2}{(1 - \frac{2}{3})} = 3C^2$$

This gives us

$$\begin{aligned} 1 \geq \frac{C^2}{(\frac{1}{\sin^2(\frac{k\Delta x}{2})} - \frac{2}{3})}|_{max} &= 3C^2 \\ &\Rightarrow 1 \geq 3C^2 \\ &\Rightarrow C \leq \frac{1}{\sqrt{3}} \end{aligned} \tag{28}$$

1f)

Look at equation (27) and solve for $\tilde{\omega}$

$$\tilde{\omega} = \frac{2}{\Delta t} \arcsin \frac{C}{\sqrt{\left(\frac{1}{\sin^2\left(\frac{k\Delta x}{2}\right)} - \frac{2}{3}\right)}} \tag{29}$$

Plot the error in wave velocity:

$$\begin{aligned}\frac{\tilde{c}}{c} = \frac{\tilde{\omega}}{ck} &= \frac{2}{\Delta tck} \arcsin \frac{C}{\sqrt{\left(\frac{1}{\sin^2\left(\frac{k\Delta x}{2}\right)} - \frac{2}{3}\right)}} \\ &= \frac{\Delta x}{\Delta tc} \frac{2}{\Delta xk} \arcsin \frac{C}{\sqrt{\left(\frac{1}{\sin^2\left(\frac{k\Delta x}{2}\right)} - \frac{2}{3}\right)}}\end{aligned} \tag{30}$$

change of variable $p = \frac{2\Delta x}{2}$

$$\frac{\tilde{c}}{c} = \frac{1}{Cp} \arcsin \frac{C}{\sqrt{\left(\frac{1}{\sin^2(p)} - \frac{2}{3}\right)}} \tag{31}$$

Insert this expression in the program given in the exercise (for the funciton r(C,p)), see program wave1D_fem.py. The result is in figure (1).

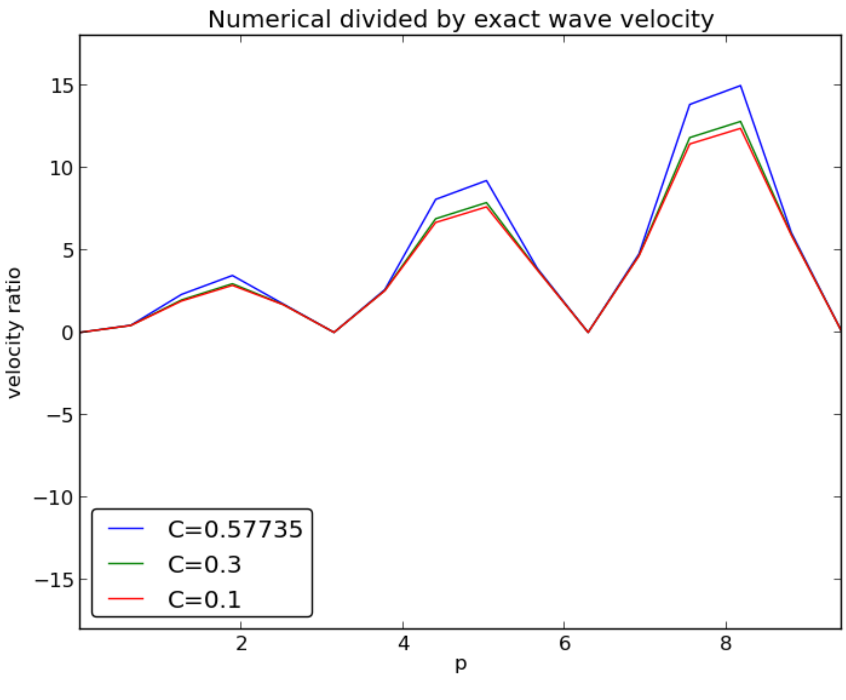

Figure 1: The error in wave velocity

We can never get the exact solution; there is no C which makes $\omega = \tilde{\omega}$. The blue curve in figure (1) is the error for maximum C.

1g)

One can replace M by diag(Me) where $e = (1, 1, ..., 1)$, and get the same result as if we did the integrals in M with the Trapezoidal rule. Show that this is true.

The trapezoidal rule:

$$\int_a^b f(x)dx \approx \frac{b-a}{N}\left[\frac{f(x_0)-f(x_N)}{2} + \sum_{k=1}^{N-1} f(x_k)\right] \tag{32}$$

The trapezoidal rule for finding the integrals in M:

$$M_{ij} = \int_0^L \phi_i\phi_j dx \approx \frac{L}{N}\left[\frac{\phi_i(x_0)\phi_j(x_0)-\phi_i(x_N)\phi_j(x_N)}{2} + \sum_{k=1}^{N-1} \phi_i(x_k)\phi_j(x_k)\right] \tag{33}$$

where x_i is our nodes.
We see that the only nonzero terms are M_{ii} because we only look at the nodes and ϕ_i is only nonzero for node i (x_i). This is great, because this means we get a diagonal matrix!
yey!

$$M_{ii} \approx h\left[\frac{\phi_i(x_0)\phi_i(x_0)-\phi_i(x_N)\phi_i(x_N)}{2} + \sum_{k=1}^{N-1} \phi_i(x_k)\phi_i(x_k)\right] \tag{34}$$

where I used that $\frac{L}{N} = h$.
For $i \neq 0, N$:

$$M_{ii} = h\phi_i(x_i)^2 = h \tag{35}$$

Now we will look at the matrix (M) we found in 1d and follow the approach given in the exercise.

$$\text{diag}Me_{ii} = Me_i = \frac{1}{6}[1+4+1] = 1 \tag{36}$$

But remember that when we found M in 1d), we divided by h after integrating. Therefore the two approaches give the same result.