INF5620 -
First compulsory project

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1   mathematical problem

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t) \tag{1}$$

The way we solve this numerically is by discretization

$$\frac{\partial^2 u}{\partial t^2} \rightarrow \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} \tag{2}$$

$$b\frac{\partial u}{\partial t} \rightarrow b\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} \tag{3}$$

$$\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) \rightarrow \frac{\left(q(x,y)\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2},j} - \left(q(x,y)\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2},j}}{\Delta x}$$

$$= \frac{q(x,y)_{i+\frac{1}{2},j}^n \frac{u_{i+1,j}^n - u_{i,j}^n}{\Delta x} - q(x,y)_{i-\frac{1}{2},j}^n \frac{u_{i,j}^n - u_{i-1,j}^n}{\Delta x}}{\Delta x}$$

$$= \frac{1}{\Delta x^2}\left(q(x,y)_{i+\frac{1}{2},j}^n \left(u_{i+1,j}^n - u_{i,j}^n\right) - q(x,y)_{i-\frac{1}{2},j}^n \left(u_{i,j}^n - u_{i-1,j}^n\right)\right) \tag{4}$$

We can find a solution for $q(x,y)_{i+\frac{1}{2},j}^n$ and $q(x,y)_{i-\frac{1}{2},j}^n$ by taking the average between $q(x,y)_{i+1,j}^n$ and $q(x,y)_{i,j}^n$ and between $q(x,y)_{i,j}^n$ and $q(x,y)_{i-1,j}^n$.

$$q(x,y)_{i+\frac{1}{2},j}^n \approx \frac{q(x,y)_{i+1,j}^n + q(x,y)_{i,j}^n}{2} \tag{5a}$$

and

$$q(x,y)_{i-\frac{1}{2},j}^n \approx \frac{q(x,y)_{i,j}^n + q(x,y)_{i-1,j}^n}{2} \tag{6a}$$

This gives us

$$\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) \rightarrow \frac{1}{\Delta x^2}\left(\frac{q(x,y)_{i+1,j}^n + q(x,y)_{i,j}^n}{2}\left(u_{i+1,j}^n - u_{i,j}^n\right) - \frac{q(x,y)_{i,j}^n + q(x,y)_{i-1,j}^n}{2}\left(u_{i,j}^n - u_{i-1,j}^n\right)\right)$$

$$= \frac{1}{2\Delta x^2}\left(\left(q(x,y)_{i+1,j}^n + q(x,y)_{i,j}^n\right)\left(u_{i+1,j}^n - u_{i,j}^n\right) - \left(q(x,y)_{i,j}^n + q(x,y)_{i-1,j}^n\right)\left(u_{i,j}^n - u_{i-1,j}^n\right)\right) \tag{7}$$

By doing the same for $\left(q(x,y)\frac{\partial u}{\partial y}\right)$, we get

$$\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)$$

$$\rightarrow \frac{1}{2\Delta y^2}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)$$

(8)

The last part of eq.(1) becomes

$$f(x,y,t)\rightarrow f(x,y,t)^n_{i,j}$$

(9)

If we now combine our discretized equations we get a numerically solvable wave equation

$$\frac{u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}}{\Delta t^2}+b\frac{u^{n+1}_{i,j}-u^{n-1}_{i,j}}{2\Delta t}=$$

$$\frac{1}{2\Delta x^2}\left(\left(q(x,y)^n_{i+1,j}+q(x,y)^n_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)$$

$$+\frac{1}{2\Delta y^2}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)$$

$$+f(x,y,t)^n_{i,j}$$

We would now like to solve this equation for $u^{n+1}_{i,j}$

$$u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}+\frac{b\Delta t}{2}\left(u^{n+1}_{i,j}-u^{n-1}_{i,j}\right)=$$

$$\frac{\Delta t^2}{2\Delta x^2}\left(\left(q(x,y)^n_{i+1,j}+q(x,y)^n_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)$$

$$+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)$$

$$+\Delta t^2 f(x,y,t)^n_{i,j}$$

$$u^{n+1}_{i,j}\left(1+\frac{b\Delta t}{2}\right)=$$

$$\frac{\Delta t^2}{2\Delta x^2}\left(\left(q(x,y)^n_{i+1,j}+q(x,y)^n_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)$$

$$+\frac{\Delta t^2}{2\Delta y^2}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)$$

$$+\Delta t^2 f(x,y,t)^n_{i,j}+2u^n_{i,j}+u^{n-1}_{i,j}\left(\frac{b\Delta t}{2}-1\right)$$

We end up with the discretized equation

$$u^{n+1}_{i,j}=$$

$$\frac{\Delta t^2}{2\Delta x^2\left(1+\frac{b\Delta t}{2}\right)}\left(\left(q(x,y)^n_{i+1,j}+q(x,y)^n_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right)$$

$$+\frac{\Delta t^2}{2\Delta y^2\left(1+\frac{b\Delta t}{2}\right)}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right)$$

$$+\frac{\Delta t^2}{1+\frac{b\Delta t}{2}}f(x,y,t)^n_{i,j}+2u^n_{i,j}+u^{n-1}_{i,j}\frac{1+\frac{b\Delta t}{2}}{\frac{b\Delta t}{2}-1}$$

(10)

2 Boundary condition

If we take a look at out numerical equation, we can see that well get a problem when we get to the boundary $i = 0, \quad i = L_x, \quad j = 0$ and $j = L_y$

We have been given the boundary condition $\frac{\partial u}{\partial \eta} = 0$

$$\frac{\partial u}{\partial \eta} = 0 \rightarrow [D_{2x}u]_{i,j}^n = 0 \tag{11}$$

When we are on the boundary $x = 0 \Rightarrow i = 0$ we get

$$[D_{2x}u]_{i,j}^n = \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2\Delta x} = 0 \tag{12}$$

$$\frac{u_{1,j}^n - u_{-1,j}^n}{2\Delta x} = 0$$
$$\rightarrow u_{1,j}^n = u_{-1,j}^n$$

When we are on the boundary $y = 0 \Rightarrow j = 0$ we get

$$[D_{2x}u]_{i,j}^n = \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2\Delta x} = 0 \tag{13}$$

$$\frac{u_{1,j}^n - u_{-1,j}^n}{2\Delta x} = 0$$
$$\rightarrow u_{1,j}^n = u_{-1,j}^n$$

3 Initial condition

If we take a look at eq.(??) we can see that we are going to get a problem with $u_{i,j}^{n-1}$ in our first time step. We can solve this by looking at our initial condition. If we assume that the velocity at $t = 0$ is zero, $u'(x, y, 0) = 0$ then we get

$$\frac{\partial u}{\partial t} = 0 \rightarrow [D_t u]_{i,j}^n = 0 \tag{14}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} = 0 \tag{15}$$

For $t = 0 \rightarrow n = 0$ we get

$$\frac{u_{i,j}^1 - u_{i,j}^{-1}}{2\Delta t} = 0 \tag{16}$$

$$u_{i,j}^1 = u_{i,j}^{-1} \tag{17}$$

This solves our problem. If we now use $n = 0$ in eq.(??) we get

$$u_{i,j}^1 \left(1 + \frac{b\Delta t}{2}\right) =$$

$$\frac{\Delta t^2}{2\Delta x^2} \left((q(x,y)_{i+1,j}^0 + q(x,y)_{i,j}^0)(u_{i+1,j}^0 - u_{i,j}^0) - (q(x,y)_{i,j}^0 + q(x,y)_{i-1,j}^0)(u_{i,j}^0 - u_{i-1,j}^0) \right)$$

$$+ \frac{\Delta t^2}{2\Delta y^2} \left((q(x,y)_{i,j+1}^0 + q(x,y)_{i,j}^0)(u_{i,j+1}^0 - u_{i,j}^0) - (q(x,y)_{i,j}^0 + q(x,y)_{i,j-1}^0)(u_{i,j}^0 - u_{i,j-1}^0) \right)$$

$$+ \Delta t^2 f(x,y,t)_{i,j}^0 + 2u_{i,j}^0 + u_{i,j}^{-1}\left(\frac{b\Delta t}{2} - 1\right)$$

If we now use $u_{i,j}^1 = u_{i,j}^{-1}$ we get

$$u_{i,j}^1 =$$

$$\frac{\Delta t^2}{4\Delta x^2} \left(\left(q(x,y)_{i+1,j}^0 + q(x,y)_{i,j}^0 \right) \left(u_{i+1,j}^0 - u_{i,j}^0 \right) - \left(q(x,y)_{i,j}^0 + q(x,y)_{i-1,j}^0 \right) \left(u_{i,j}^0 - u_{i-1,j}^0 \right) \right)$$

$$+ \frac{\Delta t^2}{4\Delta y^2} \left(\left(q(x,y)_{i,j+1}^0 + q(x,y)_{i,j}^0 \right) \left(u_{i,j+1}^0 - u_{i,j}^0 \right) - \left(q(x,y)_{i,j}^0 + q(x,y)_{i,j-1}^0 \right) \left(u_{i,j}^0 - u_{i,j-1}^0 \right) \right)$$

$$+ \frac{\Delta t^2}{2} f(x,y,t)_{i,j}^0 + u_{i,j}^0$$

4 Stability criterion