INF5620 -
First compulsory project

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1 mathematical problem

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t) \tag{1}$$

The way we solve this numerically is by discretization

$$\frac{\partial^2 u}{\partial t^2} \to \frac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2} \tag{2}$$

$$b\frac{\partial u}{\partial t} \to b\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} \tag{3}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) &\to \frac{\left(q(x,y)\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2},j} - \left(q(x,y)\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2},j}}{\Delta x} \\ &= \frac{q(x,y)_{i+\frac{1}{2},j}^{n}\frac{u_{i+1,j}^{n}-u_{i,j}^{n}}{\Delta x} - q(x,y)_{i-\frac{1}{2},j}^{n}\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}}{\Delta x} \\ &= \frac{1}{\Delta x^2}\left(q(x,y)_{i+\frac{1}{2},j}^{n}\left(u_{i+1,j}^{n} - u_{i,j}^{n}\right) - q(x,y)_{i-\frac{1}{2},j}^{n}\left(u_{i,j}^{n} - u_{i-1,j}^{n}\right)\right)\end{aligned} \tag{4}$$

We can find a solution for $q(x,y)_{i+\frac{1}{2},j}^{n}$ and $q(x,y)_{i-\frac{1}{2},j}^{n}$ by taking the average between $q(x,y)_{i+1,j}^{n}$ and $q(x,y)_{i,j}^{n}$ and between $q(x,y)_{i,j}^{n}$ and $q(x,y)_{i-1,j}^{n}$.

$$q(x,y)_{i+\frac{1}{2},j}^{n} \approx \frac{q(x,y)_{i+1,j}^{n} + q(x,y)_{i,j}^{n}}{2} \tag{5a}$$

and

$$q(x,y)_{i-\frac{1}{2},j}^{n} \approx \frac{q(x,y)_{i,j}^{n} + q(x,y)_{i-1,j}^{n}}{2} \tag{6a}$$

This gives us

$$\begin{aligned}\frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) &\to \frac{1}{\Delta x^2}\left(\frac{q(x,y)_{i+1,j}^{n} + q(x,y)_{i,j}^{n}}{2}\left(u_{i+1,j}^{n} - u_{i,j}^{n}\right) - \frac{q(x,y)_{i,j}^{n} + q(x,y)_{i-1,j}^{n}}{2}\left(u_{i,j}^{n} - u_{i-1,j}^{n}\right)\right) \\ &= \frac{1}{2\Delta x^2}\left(\left(q(x,y)_{i+1,j}^{n} + q(x,y)_{i,j}^{n}\right)\left(u_{i+1,j}^{n} - u_{i,j}^{n}\right) - \left(q(x,y)_{i,j}^{n} + q(x,y)_{i-1,j}^{n}\right)\left(u_{i,j}^{n} - u_{i-1,j}^{n}\right)\right)\end{aligned} \tag{7}$$

By doing the same for $\left(q(x,y)\frac{\partial u}{\partial y}\right)$, we get

$$\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) \rightarrow \frac{1}{2\Delta y^2}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right) \tag{8}$$

The last part of eq.(1) becomes

$$f(x,y,t) \rightarrow f(x,y,t)^n_{i,j} \tag{9}$$

If we now combine our discretized equations we get a numerically solvable wave equation

$$\begin{aligned}
\frac{u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}}{\Delta t^2}+b\frac{u^{n+1}_{i,j}-u^{n-1}_{i,j}}{2\Delta t} = {} & \\
& \frac{1}{2\Delta x^2}\left(\left(q(x,y)^n_{i+1,j}+q(x,y)^n_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right) \\
& + \frac{1}{2\Delta y^2}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right) \\
& + f(x,y,t)^n_{i,j}
\end{aligned}$$

We would now like to solve this equation for $u^{n+1}_{i,j}$

$$\begin{aligned}
u^{n+1}_{i,j}-2u^n_{i,j}+u^{n-1}_{i,j}+\frac{b\Delta t}{2}\left(u^{n+1}_{i,j}-u^{n-1}_{i,j}\right) = {} & \\
& \frac{\Delta t^2}{2\Delta x^2}\left(\left(q(x,y)^n_{i+1,j}+q(x,y)^n_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right) \\
& + \frac{\Delta t^2}{2\Delta y^2}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right) \\
& + \Delta t^2 f(x,y,t)^n_{i,j}
\end{aligned}$$

$$\begin{aligned}
u^{n+1}_{i,j}\left(1+\frac{b\Delta t}{2}\right) = {} & \\
& \frac{\Delta t^2}{2\Delta x^2}\left(\left(q(x,y)^n_{i+1,j}+q(x,y)^n_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right) \\
& + \frac{\Delta t^2}{2\Delta y^2}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right) \\
& + \Delta t^2 f(x,y,t)^n_{i,j}+2u^n_{i,j}+u^{n-1}_{i,j}\left(\frac{b\Delta t}{2}-1\right)
\end{aligned}$$

We end up with the discretized equation

$$\begin{aligned}
u^{n+1}_{i,j} = {} & \\
& \tfrac{\Delta t^2}{2\Delta x^2\left(1+\frac{b\Delta t}{2}\right)}\left(\left(q(x,y)^n_{i+1,j}+q(x,y)^n_{i,j}\right)\left(u^n_{i+1,j}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i-1,j}\right)\left(u^n_{i,j}-u^n_{i-1,j}\right)\right) \\
& + \tfrac{\Delta t^2}{2\Delta y^2\left(1+\frac{b\Delta t}{2}\right)}\left(\left(q(x,y)^n_{i,j+1}+q(x,y)^n_{i,j}\right)\left(u^n_{i,j+1}-u^n_{i,j}\right)-\left(q(x,y)^n_{i,j}+q(x,y)^n_{i,j-1}\right)\left(u^n_{i,j}-u^n_{i,j-1}\right)\right) \\
& + \tfrac{\Delta t^2}{1+\frac{b\Delta t}{2}} f(x,y,t)^n_{i,j}+2u^n_{i,j}+u^{n-1}_{i,j}\tfrac{1+\frac{b\Delta t}{2}}{\frac{b\Delta t}{2}-1}
\end{aligned} \tag{10}$$

2 Boundary condition

If we take a look at out numerical equation, we can see that well get a problem when we get to the boundary $i = 0, \quad i = L_x, \quad j = 0$ and $j = L_y$
We have been given the boundary condition $\frac{\partial u}{\partial \eta} = 0$

$$\frac{\partial u}{\partial \eta} = 0 \rightarrow [D_{2x}u]^n_{i,j} = 0 \tag{11}$$

When we are on the boundary $x = 0 \Rightarrow i = 0$ we get

$$[D_{2x}u]^n_{i,j} = \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta x} = 0 \tag{12}$$

$$\frac{u^n_{1,j} - u^n_{-1,j}}{2\Delta x} = 0$$
$$\rightarrow u^n_{1,j} = u^n_{-1,j}$$

When we are on the boundary $y = 0 \Rightarrow j = 0$ we get

$$[D_{2x}u]^n_{i,j} = \frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta x} = 0 \tag{13}$$

$$\frac{u^n_{1,j} - u^n_{-1,j}}{2\Delta x} = 0$$
$$\rightarrow u^n_{1,j} = u^n_{-1,j}$$

3 Initial condition

If we take a look at eq.(**??**) we can see that we are going to get a problem with $u^{n-1}_{i,j}$ in our first time step. We can solve this by looking at our initial condition. If we assume that the velocity at $t = 0$ is zero, $u'(x, y, 0) = 0$ then we get

$$\frac{\partial u}{\partial t} = 0 \rightarrow [D_t u]^n_{i,j} = V(x, y) \tag{14}$$

$$\frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2\Delta t} = V(x, y) \tag{15}$$

For $t = 0 \rightarrow n = 0$ we get

$$\frac{u^1_{i,j} - u^{-1}_{i,j}}{2\Delta t} = V(x, y) \tag{16}$$
$$u^1_{i,j} = u^{-1}_{i,j} + 2V(x, y)\Delta t \tag{17}$$
$$u^{-1}_{i,j} = u^1_{i,j} - 2V(x, y)\Delta t \tag{18}$$

This solves our problem. If we now use $n = 0$ in eq.(**??**) we get

$$\begin{aligned}
u^1_{i,j}\left(1 + \frac{b\Delta t}{2}\right) = \\
&\frac{\Delta t^2}{2\Delta x^2}\left(\left(q(x,y)^0_{i+1,j} + q(x,y)^0_{i,j}\right)\left(u^0_{i+1,j} - u^0_{i,j}\right) - \left(q(x,y)^0_{i,j} + q(x,y)^0_{i-1,j}\right)\left(u^0_{i,j} - u^0_{i-1,j}\right)\right) \\
&+ \frac{\Delta t^2}{2\Delta y^2}\left(\left(q(x,y)^0_{i,j+1} + q(x,y)^0_{i,j}\right)\left(u^0_{i,j+1} - u^0_{i,j}\right) - \left(q(x,y)^0_{i,j} + q(x,y)^0_{i,j-1}\right)\left(u^0_{i,j} - u^0_{i,j-1}\right)\right) \\
&+ \Delta t^2 f(x,y,t)^0_{i,j} + 2u^0_{i,j} + u^{-1}_{i,j}\left(\frac{b\Delta t}{2} - 1\right)
\end{aligned}$$

If we now use $u^1_{i,j} = u^{-1}_{i,j}$ we get

$$
\begin{aligned}
&u^1_{i,j} = \\
&\quad \frac{\Delta t^2}{4\Delta x^2}\left(\left(q(x,y)^0_{i+1,j} + q(x,y)^0_{i,j}\right)\left(u^0_{i+1,j} - u^0_{i,j}\right) - \left(q(x,y)^0_{i,j} + q(x,y)^0_{i-1,j}\right)\left(u^0_{i,j} - u^0_{i-1,j}\right)\right) \\
&\quad + \frac{\Delta t^2}{4\Delta y^2}\left(\left(q(x,y)^0_{i,j+1} + q(x,y)^0_{i,j}\right)\left(u^0_{i,j+1} - u^0_{i,j}\right) - \left(q(x,y)^0_{i,j} + q(x,y)^0_{i,j-1}\right)\left(u^0_{i,j} - u^0_{i,j-1}\right)\right) \\
&\quad + \frac{\Delta t^2}{2} f(x,y,t)^0_{i,j} + u^0_{i,j} - V(x,y)\Delta t\left(\frac{b\Delta t}{2} - 1\right)
\end{aligned}
$$

4 Stability criterion

5 Manufactured solution

5.1 Finding f(x,y,t)

$$
u(x,y,t) = e^{-bt}\cos\left(\frac{m_x\pi}{L_x}\right)\cos\left(\frac{m_y\pi}{L_y}\right)\cos(\omega t) \tag{19}
$$

Choose some $q(x,y) = A$, $A \neq 0$ eq.(1) becomes

$$
\frac{\partial^2 u}{\partial t^2} + v\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t) \tag{20}
$$

If we now insert eq.(19) into eq.(20) we get

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -bu(x,y,t) - \omega\underbrace{\frac{\sin(\omega t)}{\cos(\omega t)}}_{\tan(\omega t)}u(x,y,t) \\
\frac{\partial^2 u}{\partial t^2} &= (b^2 - \omega^2)u(x,y,t) + 2\omega b\underbrace{\frac{\sin(\omega t)}{\cos(\omega t)}}_{\tan(\omega t)}u(x,y,t) \\
\frac{\partial^2 u}{\partial x^2} &= -\left(\frac{m_x\pi}{L_x}\right)^2 u(x,y,t) \\
\frac{\partial^2 u}{\partial y^2} &= -\left(\frac{m_y\pi}{L_y}\right)^2 u(x,y,t)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \left[(b^2 - \omega^2) + 2\omega b\tan(\omega t) - b^2 - b\omega\tan(\omega t)\right]u(x,y,t) &= -A\left[\left(\frac{m_x\pi}{L_x}\right)^2 + \left(\frac{m_y\pi}{L_y}\right)^2\right]u(x,y,t) + f(x,y,t) \\
\Rightarrow f(x,y,t) &= \left[\omega b\tan(\omega t) - \omega^2 + A\pi^2\left(\left(\frac{m_x}{L_x}\right)^2 + \left(\frac{m_y}{L_y}\right)^2\right)\right]u(x,y,t)
\end{aligned}
$$

5.2 Finding I(x,y)