Mathina

AN INTERACTIVE STORYBOOK BETWEEN MATHEMATICS AND FANTASY

MATHINA HANDBOOK FOR EDUCATORS

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	4-6 years	7-10 years	11-14 years	15+ years
Logic	The traffic light challenge	The tiled floor trap	Cats and dogs	My gold, please
Symmetry	Mathina wins a lot of new toys!	Mathina, the rosette game and the magic maze	Stamping friezes	The polyhedron carousel
Cryptography	The secret messages	The talking parrot	The lost treasure	The man in the middle
Spatial Visualization	The firebird trainer	Talking to the unidragon	The phoenix race	The mysterious sparkbugs

INTRODUCTORY CHAPTER

1. Development of Mathematical Thinking in the Era of Problem-solving

The 21st century is the century of problem-solving. Children benefit from the early experience of the creative power of mathematical thinking because mathematical thinking skills are necessary in order to understand and grasp our era's complex challenges. However, formal school education proves to be less capable of conveying the broad, everyday applicability of mathematics and the beauty of mathematical thinking.

Mathina's storyworld promotes problem-solving through mathematical thinking. We offer an innovative environment, including engaging stories with digital problem-solving activities about:

- spatial visualization
- logic
- cryptography and
- symmetry

Studies reaffirm that non-formal approaches support learners in building a more sophisticated knowledge base by experimenting with the plurality of math situations, making the links between the mathematics of everyday life and school mathematics explicit. Our goal is to connect motivating non-formal approaches, such as integrating storytelling and interactive apps into formal mathematics learning.

Mathina, in this regard, is in line with the European Union's educational policy goals, which emphasize the effective use of digital learning technologies and resources in education and training. However, a range of surveys and studies, carried out by the European Commission, the OECD, and the World Economic Forum, for example, highlight that there is still a gap in the integration of digital learning technologies and resources in European educational systems. This concerns partly the development of digital skills:

- Digital skills are technical skills required to use digital technologies;
- digital navigation skills are a more comprehensive set of skills needed in order to become successful in the digital world, including finding, prioritizing, and assessing the information's quality and reliability.

Therefore, it is crucial to develop mathematics teaching activities which are easy to integrate into a broad spectrum of educational tools.

2. Bridging the Gap between Formal and Non-formal Mathematics Education

The project Mathina bridges the gap between formal and non-formal mathematics education with its modular and user-friendly system. This system can be accessed and presented in multiple ways (online, offline) and used on various devices (mobile, tablet, computer desktop, smartboard, etc.), depending

[1] Barron, B., Cayton-Hodges, G., Bofferding, L., Copple, C., Darling-Hammond, L., & Levine, M. H. (2011). Take a giant step: A blueprint for teaching young children in a digital age. Joan Ganz Cooney Center at Sesame Workshop.
[2] Trends shaping education 15: A Brave New World: Technology & Education. OECD Publishing, 2016. Source: www.oecd.org/edu/ceri/spotlights-trends-shaping-education.htm

on the location and the intended purpose. Mathina's tools allow educators to guide young learners through a diversity of mathematical challenges and empower them by following their learning paces and paths. It encourages educators to respect learners' individual needs. Furthermore, our project supports new teaching methods, including gamification and storytelling, by offering modular applications aligned with school curricula.

Mathina's innovation lies in bridging non-formal and formal learning by combining four central characteristics:

1. EASY-TO-USE
2. EASY-TO-UNDERSTAND
3. EASY-TO-ADAPT
4. EASY-TO-INCLUDE

3. Mathina's Storyworld for Engagement in Problem-solving

Engagement with Mathina's stories and activities feels more like playing than studying. The fictional characters in Mathina's storyworld motivate children, awaken their interest, and support the integration of problem-solving with storytelling in an emotional and entertaining way.

Mathina implements a transmedia storytelling framework to motivate and engage children in mathematical thinking-based problem-solving in various areas. The primary function of Mathina's problem-based stories is to provide a complex experience of emotional engagement in the thinking development process. An artistic visual environment supports the text-based narrative in the form of images and small animations. Each story offers interactive problems in digital apps, which children and youth can engage with either individually or in groups, supported by their parents or teachers when needed. Apps and the visual experience enhance children and youth's cognitive presence in the thinking development process. The social presence (entering into Mathina's world in small groups / sharing the experiences in a community of learners) and the teaching presence (entering into Mathina's world in the company of a teacher or a parent) can effectively support the learning process, according to the model of Community Inquiry.

Dr. Stavroula Kalogeras, the author of the book Transmedia storytelling and the new era of media convergence in higher education is summarizing the scientific background of this approach in a video presentation below:

https://www.youtube.com/watch?v=MmngfqCKHFo.

Recent results in neuroscience underline the effectiveness of learning through stories (Figure 1). Based on the same model presented above, Mathina offers mostly smaller learning units for shorter-term focused activities, called microlearning. This way the user does not need to go through all the stories in the system in order to gain a comprehensive experience in problem-solving. The Mathina project's stories and tools are available for different age groups.

[3] Kalogeras, S. (2014). Transmedia storytelling and the new era of media convergence in higher education. Springer.

LOGICAL THINKING DEVELOPMENT: MATHINA’S ADVENTURES IN LOGI-CITY

1. Main Mathematical Concepts Implemented in the Stories

Topic: Combinatorics

Preschoolers at the age of 4-6 start to develop logical reasoning and at the same time, their vocabulary is exploding. As part of the development of children's mathematical thinking and reasoning skills, they begin to learn how to use problem-solving and reasoning strategies that are not innate but emerge in their early years, as a core of their logical abilities. This is the age when the endless flow of "why"-questions start. Children get excited about conversations, are interested in short word problems and increasingly, longer narratives. They are busy exploring their environment, sorting items into various sets, excited about patterns and sequences, making connections, and counting.

Children learn to search for structures and regularities to order, predict and create cohesion at this age. Patterns, functions and relations are in the center of their mathematical thinking and reasoning skills. Comparison, classification and seriation surround these core skills, basic analytical thinking and initial problem-solving strategies. Simpler problems in combinatorics support - with the participation of teachers or adults - the simultaneous development of mathematical skills and the recognition of further interconnections between different skill categories (e.g. mathematical thinking and reasoning skills, numerical skills, spatial thinking skil ls).

Topic: Reverse thinking

At the age of 6-7, although children are not yet fully logical thinkers, they start to develop logical thinking. They are able to classify and sort items in multiple ways increasingly independently, and their pattern recognition and pattern-making abilities are tremendously developing. These are essential skills to develop before getting introduced to more complex mathematical problems, which require creative approaches.

From the age of 8, children already apply logic and reasoning to certain events around them. Hypothetical reasoning is an important ability, which requires the development of memory, attention, and creativity. Reverse thinking is a cognitive process, which can play a significant role in coming up with creative ideas, similarly to associative and analogical thinking.

Topic: The art gallery problem

Children between the age of 11-14 enter the world of formal operations. They already think logically and strategically by applying methods and are as well able to implement deductive logic (i.e. reasoning based on one or more statements in order to reach a logical conclusion). Predictive thinking and abstract thinking develop quickly at this age, just like metacognition (self-reflection, awareness and understanding of their own thinking process), which becomes a critical support to problem-solving.

Topic: The divisibility rule

Teenagers' cognitive abilities increase with analytic and argument skills. The logical approach to problem-solving is supported by the advancement in questioning. Probability, statistics, representation of data, and complex calculations help solve complex problems and situations. Equations are understood better as well.

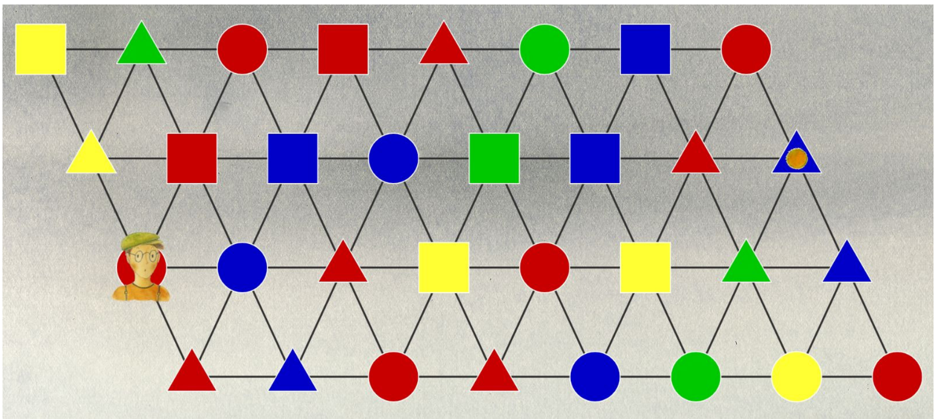
2. Digital Apps for Interactive Problem-solving

The traffic light challenge
Mathematical topic: Combinatorics

Since this age group is not yet ready to read the story without adult help, there are multiple ways to use it in a preschool/primary school setting. Of course, the teacher might read out the story to the children and assist them in using the apps. It is also possible to consider using the app as a model for supporting the thinking process. Those children with a strong visual sense will find their way with your support. Kinesthetic learners might find it helpful to act out the problem by playing with other children or with family members who as an example hold a colorful paper to represent each color. Children can rearrange the actors to represent each case, and associating colors with people can increase the playfulness of the problem-solving process.

The tiled floor trap
Mathematical topic: Reverse thinking

Typical of the age group are the development of new skills such as logic and reasoning. In the story of The tiled floor trap, the problem-solving process is challenged by a creative twist, namely backward thinking, which may motivate children to discover a new, systematic thinking method, while participating in the problem-solving activities.



Cats and dogs
Mathematical topic: art gallery problem
Engaging with the Cats and dogs story requires predictive thinking, abstract thinking, metacognition, the ability to reflect on one's own thinking method, and also the ability to extend logical thinking into a geometrical context.

My gold, please!
Mathematical topic: divisibility rule
To solve the problem introduced in the story of "My gold, please!", analytic and argument skills will be required.

Number of the friend	Remaining amount to give away	Amount to receive	Remaining amoung to give away	Amount to receive	Total amout received
1	123	6			
2	117	9			
3	108	9			
4	99	[redacted]			

3. Ways to continue thinking development

George Polya, pioneer of the didactics of problem-solving, suggested a four-step process for addressing problems in his famous book "How to Solve It":

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Examine the result

Educators and students can implement this scheme on any mathematical problem in Mathina and beyond.

When children give a wrong answer, instead of instantly correcting the solution and reminding them of the mistake, it is usually more crucial to respond positively, by appreciating the effort and examining the thinking process together with the children.

Reflection can be initiated like this:

- Are you saying...?
- As I understood, you concluded that...
- I see, so, you think that...
- What made you think like this?

To develop logical skills in the preschool age group, educators can use sorting games with real objects (everyday objects, toys, geometrical shapes, etc.) or pictures. Sorting can be based e.g. on size, color, dimension and other characteristics. Great games can be invented based on clapping rhythmic patterns, dancing patterns or creating patterns with physical objects. Modeling problems by acting out can be a great fun as well.

For children of the age 5-7, pattern prediction offers exciting opportunities for developing thinking skills. Blocks, colored stones, color paper shapes, fruits, vegetables or any other set of objects can be used to create patterns. Children can challenge each other (and their teachers, too) to continue a pattern. At the age of 6-7, numbers can be assigned to the patterns. It is also possible to create patterns that have more than one answer. One can start to play guessing games, such as 'riddle me', 'twenty questions' - also with numbers, such as connected to number of years, number of objects in the environment, buildings, animals, plants or other everyday objects - to develop questioning, classification and argumentation skills. Participation in homekeeping, such as ordering the toys, everyday objects, making the bed, cooking, baking, sorting silverware, setting the bed can be very useful in developing thinking skills too.

Children of the age 11-14 can identify, explore and explain more complex patterns, also in a mathematical and geometrical context. For example fractions can be represented on the number line, as a portion of a geometrical shape (circle, square, triangle, etc.), on a tape diagram, or using fraction bars. Making systematic lists and identifying subproblems, when working on a complex problem can help to develop thinking skills and get a firm practice in implementing various problem-solving strategies.

Teenagers in the age of 14-18 can explore real-world problems, such as planning shopping at an end of a season sale, planning investments or understanding real-world processes with the help of diagrams and data analytics. They can try and invent logic puzzles, create board games about certain problems related to their interests and studies, analyze the fairness of the voting system, explore complex systems in the environment or systems provided by technology.

Literature used in this chapter:

- Germain-Williams, T. (2017). Teaching Children to Love Problem Solving. A Reference from Birth through Adulthood. World Scientific.
- Parviainen, P. (2019). The Development of Early Mathematical Skills – A Theoretical Framework for a Holistic Model. Journal of Early Childhood Education Research Volume 8 Issue 1 2019, 162–191.
- Polya, G. (1945). How to solve it; a new aspect of mathematical method. Princeton University Press.

SYMMETRY AND POLYHEDRA: MATHINA IN SYMMETRY FAIR

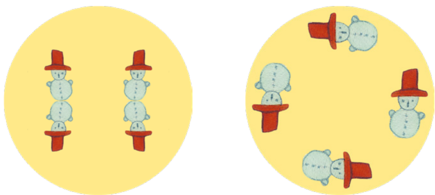
1. Main Mathematical Concepts Implemented in the Stories

Topic: Reflection Symmetries

For the 4-6 age group, the main goal is to explore a symmetry that even small children become familiar with, in an empirical way: the "mirror" symmetry. If we stick to the plane, this means checking whether an image has symmetry axes or not.
For simplicity's sake, only vertical and horizontal symmetry axes are considered.

Topic: Reflection and Rotation Symmetries

In the 7-10 age group, children get better acquainted with reflection symmetry. They also get familiar with rotation symmetry[1]. So, concepts like – the notion of **symmetry**, **properties related to reflection and rotation**, and **classification of rosettes** according to their symmetry – can be explored.
By **symmetry of a figure** we consider an **isometry**, i.e. a function which preserves distances, that maps the figure exactly onto itself, so that *it looks the same before and after the mapping*: it should not be possible to distinguish the initial figure from the final one (either by shape, position, or colour).



The two rosettes above may look similar, but they have a significant difference regarding their symmetry: while the one on the left has reflection symmetries – and it is called **dihedral** – the one on the right doesn't have them – it's a **cyclic** rosette.
And we can distinguish dihedral rosettes according to the number of reflection symmetries they present (for instance, the rosette on the left has 2 symmetry axes, it is described as a D_2). Similarly, we can classify the cyclic rosettes according to the number of rotation symmetries (the one at the right is a C_4).

Topic: Reflection, Rotation, Translation and Glide Reflection Symmetries

In the 11-14 age group, besides reflection and rotation, children are also able to dwell on two less "intuitive" isometries[2]: translation and glide reflection.
As for the notion of symmetry, we consider the one presented in the previous sub-section (Reflection and Rotation Symmetries).
Once the 4 isometries in the plane are known – **Reflection, Rotation, Translation and Glide Reflection**[3] –, children can start exploring the symmetries of friezes.
And there is a very important and striking result concerning frieze symmetry: although we can produce a huge variety of friezes, there are only **7 types of friezes with different symmetries**. The proof of such a surprising result is beyond the reach of 11-14 children. However, understanding the differences

[1] In fact, reflection symmetry is a subject thoroughly addressed in the Portuguese Mathematics curriculum for the "Primary School" (6/9 years). On the other hand, although nowadays rotation symmetry is only included in the 6th degree curriculum (11 year old students), it was included in the Portuguese "Primary School" curriculum some years ago.
[2] Note that all the mentioned isometries are addressed in the Portuguese Mathematics curriculum for the "Lower Secondary" (12/14 years).
[3] It is possible to prove that, in the plane, there are no other isometries.

between the 7 types of friezes and classifying friezes according to their symmetry is a subject that can be explored by children, in a playful way.

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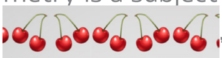
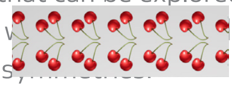
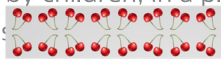
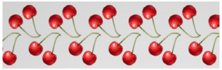

Translations, Vertical reflections


Translations, Horizontal reflection, Glide reflections


Translations, Vertical reflections, Horizontal reflection, Rotations, Glide reflections


Translations, Rotations


Translations, Vertical reflections, Rotations, Glide reflections


Translations, Glide reflections

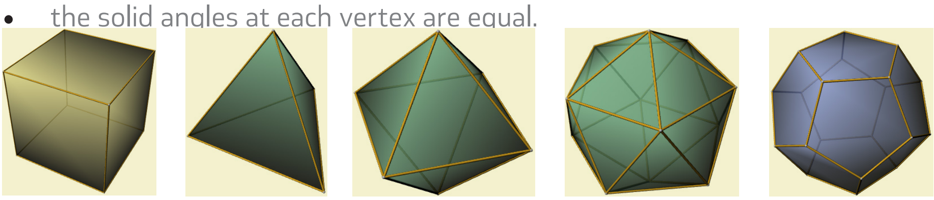

Translations

Topic: Symmetry and Polyhedra

Three-dimensional reasoning is a skill which is acquired with age and, so, this subject is only addressed to the elder age group: 15-19+. And we focused on a specific type of 3D objects familiar to teenagers: **Polyhedra**. For that purpose, we chose two classes of very "symmetrical" polyhedra: **Platonic Solids** and **Uniform Polyhedra**.

A **Platonic Solid** is a polyhedron "as regular as possible". Thus, a Platonic solid should satisfy the following properties:

- their faces are all regular polygons and equal to each other (the same number of edges, all with the same length, and equal angles);
- at each vertex the same number of faces meet;
- the dihedral angles, i.e. the angles between contiguous faces, are equal;
- the solid angles at each vertex are equal.

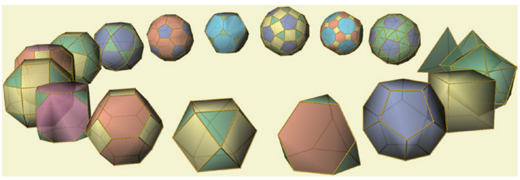


There are only 5 Platonic solids (image above): tetrahedron, cube, octahedron, icosahedron, dodecahedron. And it is possible, in a constructive way, to lead teenagers to obtain all Platonic solids and to conclude that there are no others.[4]

After exploring Platonic solids, we can move on to uniform Polyhedra. A uniform polyhedron is a polyhedron:

- whose faces are all regular polygons, but not all with the same number of sides;
- for each pair of vertices, there is at least one symmetry of the polyhedron that takes one vertex onto the other.

Unlike Platonic solids, the class of uniform polyhedra is not finite – it includes 1) an infinite family of prisms, whose bases are regular polygons and whose side faces are squares; 2) an infinite family of anti-prisms, whose bases are regular polygons and whose side faces are equilateral triangles; 3) 13 other polyhedra (image below).



[4] For more information on the subject, we recommend reading the story "The polyhedron carousel".

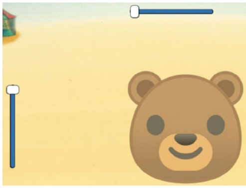
2. Digital Apps for Interactive Problem-solving

Apps for the 4-6 age group
Mathematical topic: Reflection Symmetries

As mentioned before, in this age group the idea is to explore the notion of symmetry axes. All the apps are dedicated to this goal, but they present different levels of difficulty: in Apps 1.1 and 1.2, the child should check if a given image has symmetry axes, while in apps 1.3 and 1.4, the child should finish building symmetric images by himself/herself.

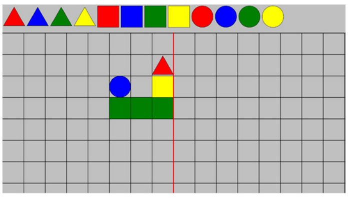
Concerning apps 1.1 and 1.2, the first one can be seen as a training mode for the second one: in fact, in app 1.1, the child only needs to deal with one image at a time, while, in app 1.2, several images are presented simultaneously.

App 1.4 is also more difficult than app 1.3: in the latter, the child only needs to grab pre-existing forms, while in app 1.4, the child should finish drawing symmetrical images.

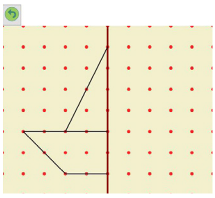


App 1.1

App 1.2



App 1.3



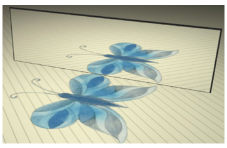
App 1.4

Apps for the 7-10 age group
Mathematical topic: Reflection and Rotation Symmetries

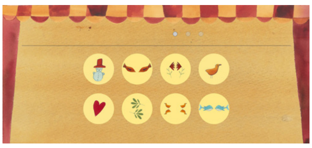
The goal of the first app 2.1 is that children check if a given image has a symmetry axis, by using a mirror.

Apps 2.2 – 2.4 are dedicated to a mathematical subject addressed in this section: the classification of rosettes. These apps were conceived to guide children in this process: through app 2.2, children train how to distinguish dihedral rosettes from cyclic ones. Then, apps 2.3 and 2.4 are dedicated to the classification of, respectively, dihedral and cyclic rosettes.

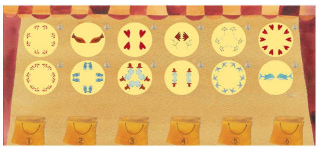
Apps 2.5 and 2.6 cover another mathematical subject addressed in this section – mathematical properties related to reflection and rotation: while app 2.5 is dedicated to properties related to reflections, app 2.6 covers rotations.



App 2.1

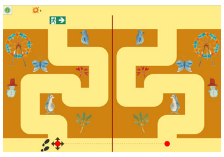


App 2.2

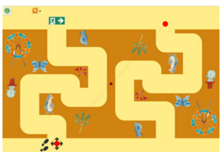


App 2.3

App 2.4



App 2.5



App 2.6

Apps for the 11-14 age group

Mathematical topic: Reflection, Rotation, Translation and Glide Reflection Symmetries. Classification of Friezes

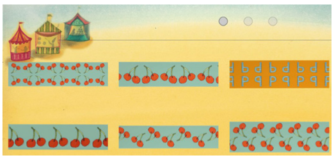
All the apps are dedicated to the classification and stamping of friezes.

Apps 3.1 and 3.3 are more recreational: their goal is to show how a cylinder or a "board" can stamp friezes of a certain type.

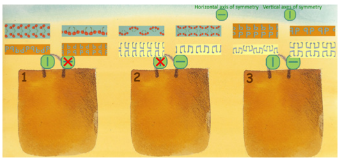
On the other hand, apps 3.2 and 3.4 – 3.8 were conceived to guide children in a process to obtain the **7 types of friezes**: through app 3.2, children start by separating the friezes with symmetry axes. Then, in app 3.3, children come across 3 different classes of friezes, according to the existing symmetry axes (horizontal, vertical or both). Next, in app 3.5, children get a new class of friezes, by separating the friezes with rotation symmetry from the remaining friezes. In app 3.6, children split a previous "class of friezes" into 2 new classes ("vertical reflection symmetry + no rotation symmetry" and "vertical reflection symmetry + rotation symmetry"), getting a total of 5 classes of friezes. Finally, in app 3.7, by separating the friezes with glide reflection symmetry, children discover all possible types of friezes: the 7 ones.

App 3.8 is dedicated to a systematization of the work already carried out: here, children should find, for each of the 7 classes of friezes, the associated symmetries.

App 3.1



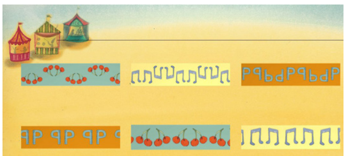
App 3.2



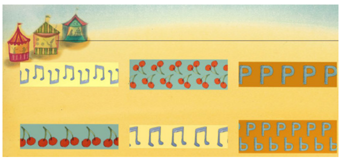
App 3.3

App 3.4

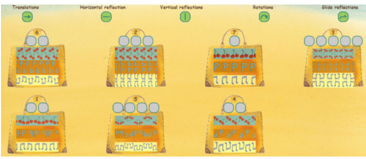
App 3.5



App 3.6



App 3.7



App 3.8

Apps for the 15-19+ age group

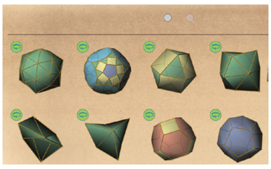
Mathematical topic: Symmetry and Polyhedra

All the apps are dedicated to the main concepts presented in the chapter: classification of polyhedra and creation of good nets for them.

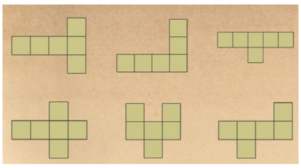
Apps 4.2 and 4.3 are dedicated to finding the good nets for, respectively, the cube and the octahedron.

On the other hand, apps 4.1, 4.4 and 4.5 are dedicated to polyhedra classification: after selecting, from a group of polyhedra, the convex ones, the teenagers can move to "non-platonic polyhedra whose faces are all regular and equal polygons". This app (4.4) aims at showing that, although it is possible to find polyhedra similar to the Platonic Solids, we have little freedom when trying to build Platonic solids.

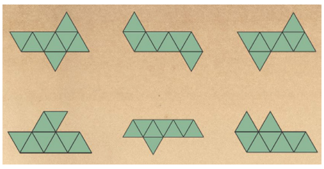
Through the last app, teenagers can get acquainted with uniform polyhedra.



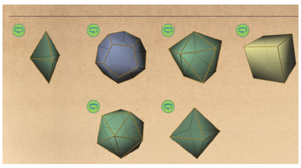
App 4.1



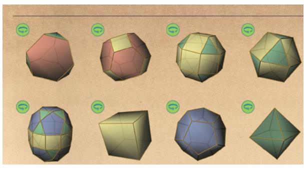
App 4.2



App 4.3



App 4.4



App 4.5

3. Going further

There are several topics which are not covered in Mathina Project, but which are linked to the symmetry part and which can be explored with children/ teenagers. We list some examples below.

For the age group 7-10, in addition to rotation and reflection symmetries, it is possible to start working with translation symmetry. So, further explorations involving "simple" frieze types (for instance without glide reflection symmetry) can be carried out.

For the age group 11-14, more extended experiments can also be developed, involving not only friezes, but also the classification of wallpaper patterns (note that, in this case, translations in different directions should be considered). However, when working with wallpaper symmetry, it is convenient to be careful with the choice of images, so as not to discourage children with overly complex examples.

For the age group 15-19+, in addition to 3D geometry/symmetry, activities related to symmetry in the plane can also be developed, namely activities related to the classification of both friezes and all the wallpaper patterns.

CRYPTOGRAPHY:
MATHINA ON BUCCANEER ISLAND

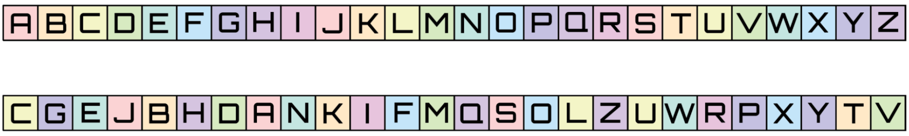
1. Main Mathematical Concepts Implemented in the Stories

Topic: Introduction to cryptography

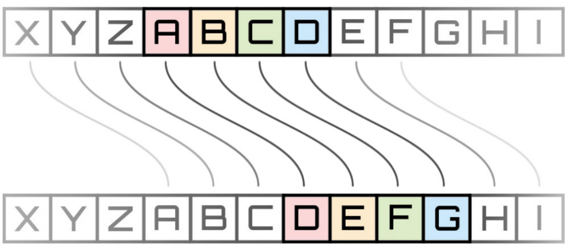
For the 4-6 age group, the main goal is to explore the basics of cryptography, particularly the concept of key and method. Therefore, the focus is on abstraction, particularly the idea of choosing a symbol rather than a specific message. This is since, in this age group, some children did not learn how to read yet. Notice, however, that stories and apps still have text, so the aid of an educator could be needed for completing the story.

Topic: Caesar cypher

In the 7-10 age group, children already learned to read, so it is possible to introduce alphabetical cyphers, like Caesar's one. The focus here is to introduce the concepts of substitution cyphers and the idea that the particular key for the cypher in the story can be reduced to a single letter. How the letters are associated in substitution cyphers is called the **key of the cypher**, so that one encrypts a message simply by replacing each letter with the corresponding one according to the key. The decoding method is the same, but the correspondence is followed in reverse. For example, with the key from the following image, the word "MESSAGE" is encrypted in the word "MBUUCDB".



In its original form, **Caesar cypher** associates to each letter the third letter from it in the alphabet: the letter "A" is associated with the letter "D", "B" to "E" and so on ... It can be generalized by using, instead of 3 as in the original, a different number. To know the key we just need to know the number of "jumps" we have to do.



This topic is further developed for the 11-14 age group. The students in this age group are more familiar with arithmetic operations, so it is possible to deepen the analysis of Caesar Cypher. In particular, we can introduce the concept of **modular arithmetic**. Arithmetic "modulus n" works like the usual arithmetic but, in a sense, it only uses numbers between 0 and n-1. If we have two numbers a and b chosen between 0 and n-1, we can add, subtract or multiply them. To do this, we do the operation normally and then calculate the remainder of the integer division of the number obtained by n: this will be the result of the operation with modular arithmetic (we say, in mathematical language, that I calculate the sum, difference or the product "modulus n").

Topic: Key-exchange systems

The 15-19 age group has acquired enough familiarity with mathematical reasoning related to inverse functions and abstraction, allowing us to introduce a simple key exchange system. We choose to cover the Diffie-Hellman key exchange system, which allows obtaining a shared secret through a public, unencrypted communication channel. The Diffie-Hellman method is based on so-called "One-way functions", i.e. invertible functions which are very easily computable in one direction, but extremely difficult to compute in the other.

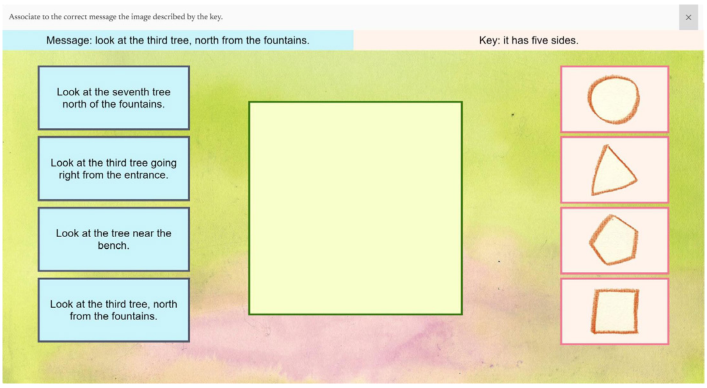
The Diffie-Helmann method uses the discrete logarithm, i.e. the calculation of the logarithm in modular arithmetic, an operation that is extremely more complex from the point of view of computational difficulty than (even discrete) exponentiation.

2. Digital Apps for Interactive Problem-solving

Apps for the 4-6 age group

Mathematical topic: Introduction to cryptography

As mentioned before, in this age group the idea is to explore the concepts of key and method. In App 1.1, the user must select the correct message (matching the one given) and find the key that is given in the form of a hint. App 1.2 and 1.3 further expand on this topic: in App 1.2 more than one image is presented and ambiguity is introduced regarding the key, as the hint can be traced to more than one correct key, highlighting the importance of clarity in mathematical communication. App. 1.3 finally sums up all the concepts: the users should this time find the correct message via the given hint for the key. The Apps are presented to the user in increasing difficulty, fostering comprehension of the topic.



Apps for the 7-10 age group

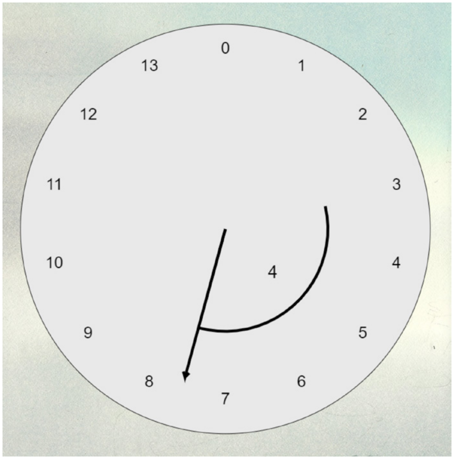
Mathematical topic: Caesar cypher (part 1)

Apps 2.1 and 2.2 are dedicated to the introduction of Caesar cypher. They work in a similar, but opposite manner, App 2.1 being used to cypher and App 2.2 to decipher a hidden message.

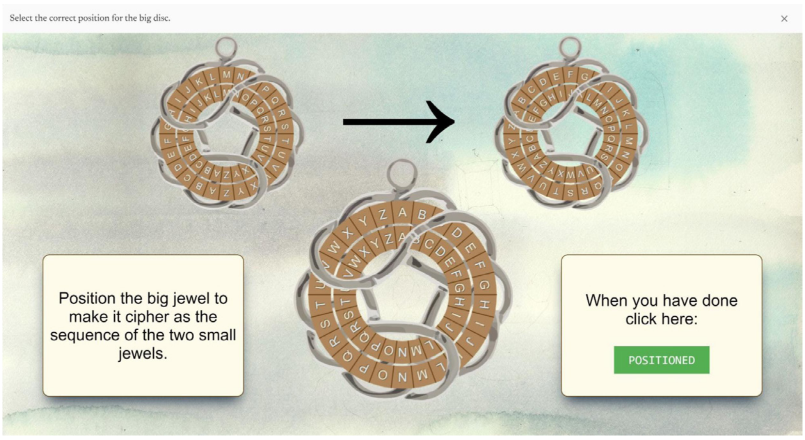
The combined goal of these two apps is to highlight the concept of key and practice in ciphering and deciphering messages. If used in a classroom it is possible to discard the provided messages and use only the middle part of the Apps with custom messages.

Apps for the 11-14 age group
Mathematical topic: Caesar cypher and modular arithmetic

Apps 3.1 and 3.2 are similar in concept to Apps 2.1 and 2.2, providing the same experience, but with an increased difficulty level, reflecting the more advanced age group. In App 3.3 the concept of modular arithmetic is introduced, noting that choosing a key for Caesar cypher means choosing a number from 1 to 25 as the key. This leaves only 25 tries for an intruder to break the message, so brute force attacks are possible.



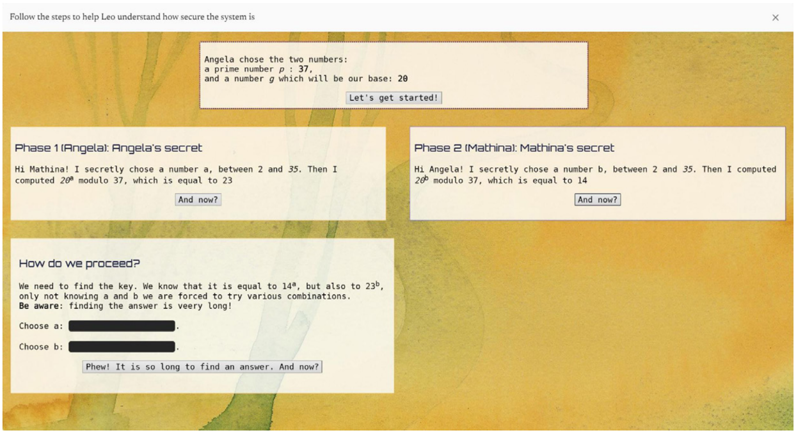
Apps 3.4 and 3.5 show that Caesar cypher does not become more secure if applied two times in a row (this is verified by properties of modular arithmetic), in particular in App 3.4 two subsequent iterations of Caesar cypher are used, whereas in App 3.5 the equivalence to a single iteration is shown.



Building from there, Apps 3.6 and 3.7 present to the student a different kind of cypher, with a message that cannot be deciphered using known methods. Apps 3.8 and 3.8 are used to try some possible reconstruction of both the method and a possible key, applying two Caesar cyphers with different keys to odd-numbered and even-numbered words (App 3.8) to single letters (App 3.9) This last cypher is indeed a particular instance of what is known as Vigénére cypher.

Apps for the 15-19+ age group
Mathematical topic: Key exchange

All the apps are constructed to guide the young learner through the construction of the Diffie-Hellman key exchange method. In particular, App 4.1 presents a more guided experience, in which the arithmetic steps are already provided. App 4.2 and 4.4 replicate a similar experience, but with fewer pre-completed steps, the students have to check if their familiarity with the subject has reached a suitable level. App 4.3 also illustrates how difficult a possible attack on the cypher is, replicating the Man in the middle situation presented in the story.



3. Going further

The Mathina Programme covers a historical part of private-key cryptography, introducing Ceasar's cypher. Further development for the 7-10 age group is that teachers and educators can explore in class how secure this cypher is. In particular, it could be noted that choosing a key for Caesar cypher means choosing a number from 1 to 25 as the key. This leaves only 25 tries for an intruder to break the message, so brute force attacks are possible. This extends the activity to the topics covered for the 11-14 age group.

This is possible given that Cryptography is not usually part of the school curriculum, so it can be used as a bonus topic for after-school activities.

Further development for the 11-14 age group is to ask how secure is Vigénére cypher (note that the name is not given in the Mathina Programme to learners), introducing frequency analysis and similar techniques.

Concerning key exchanges, one-way functions and public cryptography in general, further development can be proposed to learners concerning cyphers such as Kid-RSA (a simplified version of RSA with mathematical concepts suitable for the 15-19 age group).

SPATIAL VISUALIZATION:
THE LAND OF THE FIREBIRDS

By the term "spatial visualization" we refer to relationships between symbolic/algebraic ideas and graphical spatial representations, such as vectors or functions. In the land of the firebirds, we center ourselves specifically around ways to describe and construct curves in the plane. This interplay, fundamental in mathematics, is revisited again and again in mathematical education with different tools and points of view.

It is fundamental in the Mathina stories that mathematical concepts are introduced to children much earlier than expected in the formal education system, although in a more shallow, hidden and intuitive way than what is expected in the formal system. However, stories can be read (or re-read) at any age, in particular at the time when the school system formally introduces that concept, and even re-visiting the story at later stages. With a supportive educator (teacher, parent...) giving hints, new insights can be obtained from the lecture at progressively more mature ages.

1. Main Mathematical Concepts Implemented in the Stories

Parametric curves

In the apps of "The firebird trainer", a unidragon draws a curve in the sky (a planar sky) from the direction that the user gives continuously as a vector (with the magic wand/horn). At the intended age of the story (4-6 years old), children cannot read without assistance, and they are discovering many basic shapes (circles, squares, etc) for the first time.

What defines a shape? Usually, shapes are presented to children as satisfying some conditions:
An equilateral triangle is a shape that has three equal sides. A square has four equal sides and equal angles. Points on a circle are at a fixed distance from its center. But when we ask children to draw a triangle or a circle, they need a constructive method to move the pencil on the paper. How can you construct those basic shapes (triangle, square, circle...), or more complex ones like the shape of an 8? By giving a direction at each time, we are giving a construction method of the shape. For a square, keep the direction "north" for a while, then "east", then "south" and then "west", so at some points we need to turn the direction vector sharply 90 degrees. For an equilateral triangle, the directions form different angles (60 degrees internally, but the direction vector must turn 120 degrees). For a circle, a continuous change of direction is necessary. For an "8", a more involved construction is needed. The child develops thus the ideas of action at a distance, and encoding of information, as well as some spatial intuition and hand-eye coordination required to solve the puzzles.

At a next stage (maybe around 10 years old), the child can work the idea of physical speed (and more precisely, the velocity as a vector). While we all get an intuitive idea by our everyday experience, it is not intuitive at all that we can represent velocity by an arrow.

At a more advanced stage, when the student have full algebraic skills to define functions, and know notions as derivatives and integrals, these apps still offer some useful reflections. The curve that the unidragon flies can be described as a parametric curve: the coordinates of the unidragon are $(x(t), y(t))$ at time

t, and the direction of the curve (the tangent vector) has components (dx/dt , dy/dt).
The tangent vector is thus the vector of the derivatives of the functions defining the curve. Conversely, the curve is obtained from the tangent vector by integration. What the program does is a numerical integration. If the tangent vector is (dx/dt, dy/dt), and the position of the unidragon is currently (x,y), then the program moves the unidragon to the position
$(x,y) + (dx/dt, dy/dt) * dt = (x+dx, y+dy)$
Here dt is a small (but finite) quantity, and the process repeats many times per second.

This numerical understanding of what the program does internally helps to grasp the core ideas of infinitesimal calculus, where dt is a symbolic representation of an infinitely small quantity. The existence of these infinitesimals is counterintuitive and, even sometimes denied in some deductive approaches. Too often, the expression dx/dt is omitted in favour of the prime notation x', and infinitesimals negated in favour of limits, which are historically posterior and didactically more involved.

In summary, the humble unidragon flight can be used in a wide range of learning stages in mathematics, from constructivism in elementary geometry to integration theory.

Vector arithmetic
In the apps of "Talking to the unidragon" we build on top of the intuitive idea of vectors and velocity given in "The firebird trainer", but we add the algebraic layer of representing a vector by a pair of numbers, and not only graphically. The apps allow entering (in sequence) a series of pairs of numbers, that are transformed into arrows that the unidragon flies (see figure 1).

Many concepts can be worked here: First, encoding the graphical information as numbers. This is equivalent to measuring in two dimensions (measuring the vertical and horizontal components). Second, negative numbers. Although the age when children learn about negative numbers in school may vary between 7 and 11 years old, this early exposition can be done by just assuming that the negative sign marks the backward direction. At the upper part of this range (11 years), children can explore arithmetic with negative numbers. Thus, thirdly, children can explore and discover that vectors are linear, that is, we can add and subtract them. Since we are not using more advanced vector calculations (dot products, norms, angles...), we can introduce vectors at the same time that children learn about negative numbers (two stages of education separated by a long period in the formal school system). We do implicitly introduce the idea that a point plus a vector gives another point (affine geometry), and that we can calculate (polygonal) trajectories by this method.

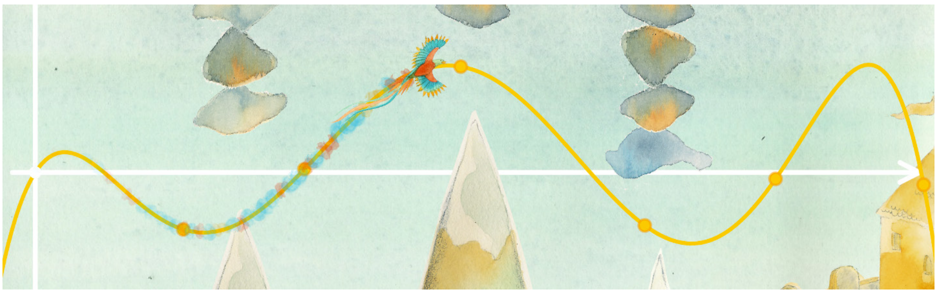


Figure 1

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Those ideas (negative numbers, vector arithmetic...) are not meant to be exposed by the educator on these stories, but just giving a hint and leaving the child to discover by himself, or herself, by exploration.

Functions and calculus

In "The phoenix race", we introduce functions y=f(x). By changing the animal (unidragon by phoenix), we make a distinction on the tools that we have available now to draw explicit curves.
Here we assume that the student (more a teenager than a child) is already

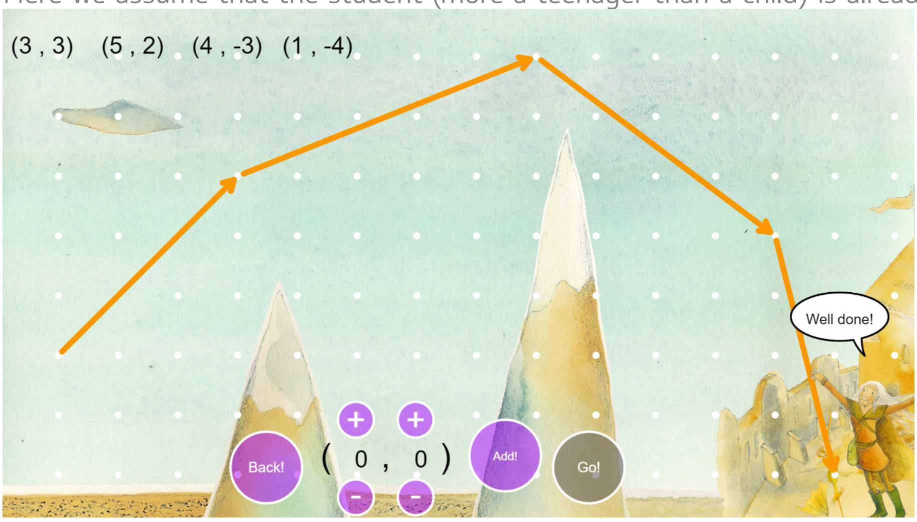


Figure 2

familiar with basic algebraic expressions, first and second degree equations, and can make symbolic manipulations.

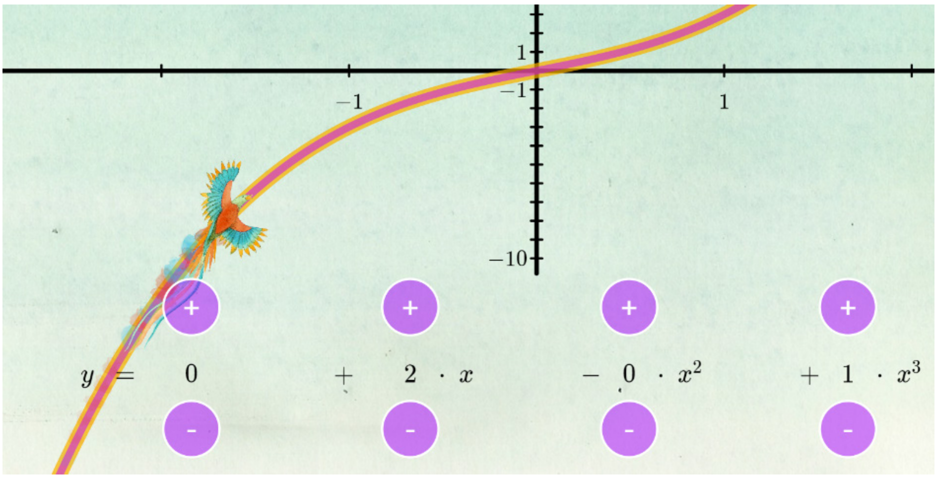


Figure 3

The story offers various apps to draw explicit curves. The first app (see figure 2) is a plotter for polynomial functions. The coefficients of the polynomial can be adjusted to draw a given target curve. The student can learn by experimentation the different roles of the independent term, the first-order term, the leading term...

The second app (see figure 3) gives a polynomial interpolation that passes through seven given points. The user can see the construction that makes that possible. The app asks to make a polynomial that passes through the given points while avoiding certain obstacles. That is not trivial due to Runge's phenomenon.

The third app explores the notion of derivative, similar to the app in "The firebird trainer" but much more explicitly here, that we can see the derivative as the rate of ascension or descent. The students can discover by themselves the relationship between maxima and minima of the function and zeros of the derivative.

The fourth and final app mixes the two techniques we have seen: the user is asked to use a polynomial interpolation to define a curve which is the derivative of our goal function, or in other words, the program integrates the polynomial.

The story explores all those notions graphically, but at this level, the educator can ask the students to take paper and pencil and solve some questions algebraically. For instance:

- How can you generalize the polynomial interpolation to pass through more points?
- How can you make a polynomial that passes through some points with some defined derivatives?
- How can we define "double zeros" of a function? How can we identify them?
- How does the last program find the graph of the function (in other words, the numerical integration)?

Implicit functions and algebraic geometry

In this last story of the firebirds, aimed at the more mature teenage students,

[1] A property of polynomial interpolation according to which an increase in the degree of the interpolation polynomial can lead to a deterioration in the interpolation quality.
(see also: https://en.wikipedia.org/wiki/Runge%27s_phenomenon)

the setting is quite different. The scenario of the enchanted forest, a bit more scary, is more suited to teenagers; the magic is less spectacular but more intriguing; and the idea of sparkbugs as single points of light in space, ideally without any width, requires more abstraction than the big animals flying in the sky. Also, by the end of the story, Flamma breaks the fourth wall, talking to the reader about real-world mathematics and mathematicians. We hope to appeal this way to the maturity of the readers, who can enjoy the story of Mathina and Leo in the forest, but who will find a more stimulating challenge in the apps.

The story uses those microscopic light animals, the sparkbugs, as drawing elements to plot implicit surfaces. For a two-variable function F(x,y), the plot consists of all the points whose coordinates satisfy F(x,y)=0. This idea is straight enough for 15-year-old students, but the implications have a long haul.

After playing a bit with the free-exploration app, it would be appropriate for the educator to compare this method of curve description with the previous ones, even playing a bit with the apps of “The firebird trainer”. The implicit

function method uses properties of the curve to define it, instead of some constructive method of drawing. The example of the circle is particularly illustrative (see the comments and details in the Educators platform).

Next, it would be good to develop some intuition about the equations. The second app offers a guessing game that will help with that. Teachers can add more examples and exercises. Then, some more techniques and insights can be proposed by the educator. We suggest here some of them:

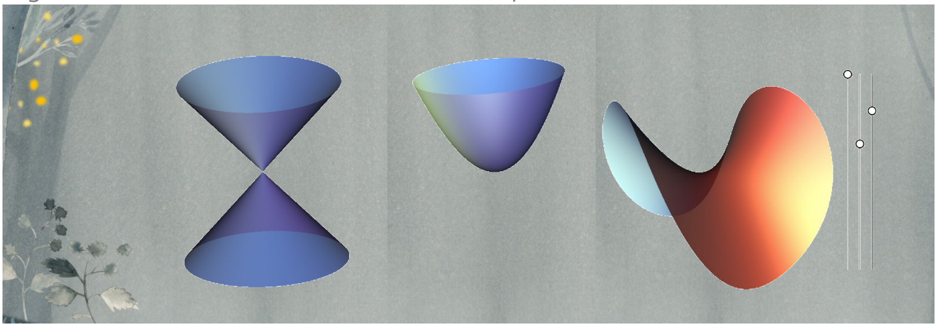
- Show and practice the techniques of deformation, union and intersection of curves (see the Educators platform). These techniques will open the possibilities of combining and altering existing curves in a directed and intentional way, not just exploring the effects by trial and error.
- From the deformation technique (drawing the function F(x,y)-a=0 for small a), draw several curves with different values of a. The educator can then guide the exploration towards the gradient and the normal vector to the curve. In a nutshell, the normal vector to the curve F(x,y)=0 is given by (dF/dx , dF/dy).

The story finishes with the analogous construction in three-dimensions, for implicit functions F(x,y,z)=0. Inside the story, the third app proposes just a bit of free exploration, but an educator can take advantage of this 3D tool in relation to the previous activity:

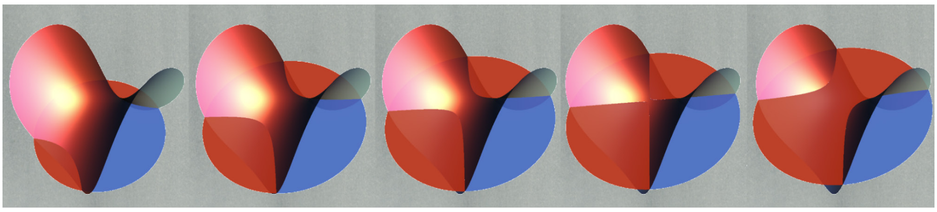
- Use the 3D tool for drawing the surfaces z=F(x,y) in the three-dimensional space. Some good examples are (see figure 5):
  - the cone $z = \mathrm{sqrt}(x^2+y^2)$,
  - the paraboloid $z=x^2 + y^2$,
  - and the saddle $z=x^2-y^2$.
- Use an intersection with a plane at different heights to relate that with the curve F(x,y)=a. For example: enter $(x^2-y^2-z)*(z-a)=0$ and move the first slider in order to change the parameter a and thus the height of the intersection plane (see figure 6).
- Go further and explore the role of partial derivatives for instance with the saddle point.

Figure 5: Cone, Paraboloid, and Saddle

Figure 6: Intersections of the Saddle with planes z=a



Past the stories



With all the Mathina stories we aim to offer the young learners some tools that they can use to explore and to use their creativity further. The stories provide a first entry point to many mathematical ideas, probably earlier than when they are studied at school. However, the synergy between Mathina introductions and the guidance of an educator is what, we believe, will boost the mathematical skills and interest of the learners. By the end of the Mathina stories, the educator would do well in introducing the students to other more versatile tools, for instance, dynamic geometry software such as GeoGebra or Desmos, or even more powerful Computer Algebra Systems such as SageMath. This will empower the students to take their own mathematical adventures beyond Mathina stories.

Mathina

AN INTERACTIVE STORYBOOK BETWEEN MATHEMATICS AND FANTASY