The 4th US Ersatz Math Olympiad

Solutions and Results

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30 June 2024

Contents

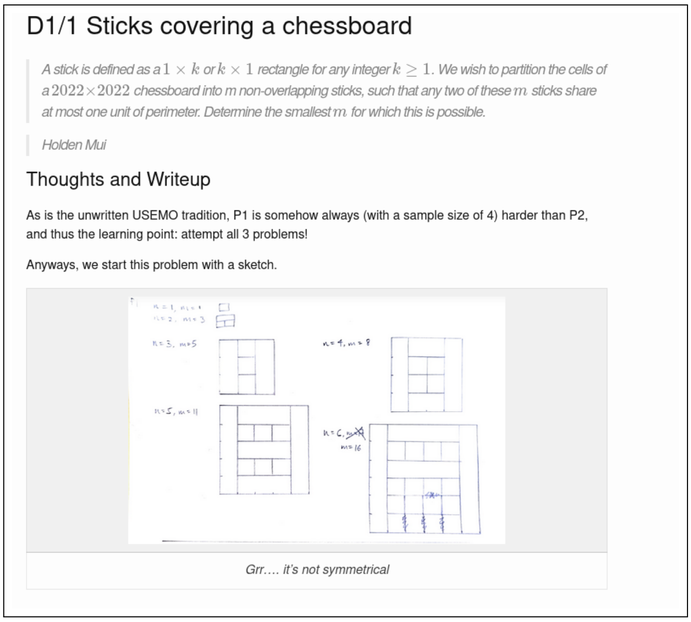
1 Summary

The fourth USEMO was held on October 22 – 23, 2022. A total of 56 students submitted at least one paper.

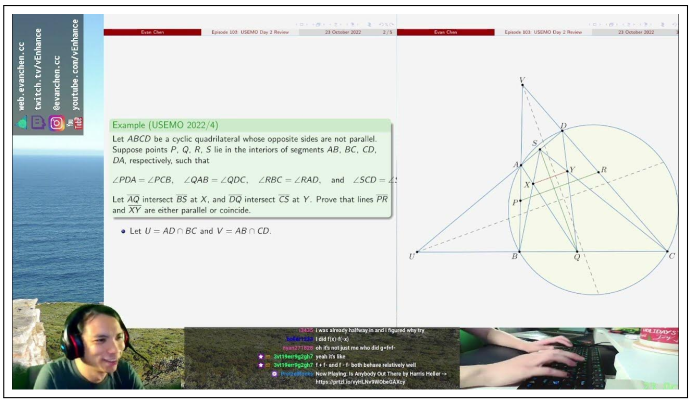
The test was generally too difficult. See the statistics at the end, but note that these may be misleading due to students who ended up not submitting anything because they did not feel like they had meaningful progress on any of the problems.

§1.1 Spectator commentary

- Hu Man Keat wrote a blog post covering the solve process and grading for the first two problems. You can read it at: https://potatostealer.github.io/opinions/2022/11/05/some-musings-for-october.html.



- I held a casual discussion of the day 2 problems on Twitch. The video is uploaded at https://youtu.be/dC9VpiGVRqs.



If you know of any other commentary (video, blog, etc.) that you'd like to have featured here, send it along and I will eventually add it.

§1.2 Thanks

I am once again grateful to many individuals who helped make this competition possible.

§1.2a Proposers of problems

I thank Aayam Mathur, Anant Mudgal, Ankan Bhattacharya, Arjun Gupta, Boon Qing Hong, Debayu Chakrabarti, Elbert Benedict, Holden Mui, Jovan Vuković, Krutarth Shah, Lincoln Liu, Munmun Bhadra, Nikolai Beluhov, Santiago Rodriguez Sierra, Sutanay Bhattacharya, Tee Jin Seng, Tilek Askerbekov, Tran Quang Hung, Valentio Iverson, for contributing 32 problem proposals.

§1.2b Reviewers

Thank you to Nikolai Beluhov, who single-handedly reviewed every submitted proposal and greatly helped shape the exam.

§1.2c Graders

Thanks to everyone who graded at least one paper: Aarav Gupta, Abdullahil Kafi, Akash, Aleksij Tasikj, Ana Boiangiu, Andrei Chirita, Arghadeep Deb, Atul Shatavart Nadig, Axel Dobloug, Bakhtier, Bhabanishankar Rath, Dan, Debayu Chakrabarti, Evan Chen, Félix Moreno Peñarrubia, Gvozden Lapčević, Hu Man Keat, Imad Uddin Ahmad Hasin, Immanuel Josiah Balete, Jayden Pan, Kang Taeyoung, Lasitha Vishwajith Jayasinghe, Leon Lau, Lincoln Liu, Manasseh Ahmed, Marin Hristov, Max Chornyi, Orestis Lignos, Pedro Henrique de Almeida Ursino, Petko Valeriev Lazarov, Pranav Choudhary, Rushil Mathur, S M A Nahian, Sanjana Das, Trung Nguyen, Valentio Iverson, Victor Kostadinov.

§1.2d Other supporters

I would like to thank the Art of Problem Solving for offering the software and platform for us to run the competition. Special thanks to Deven who was my main contact for this iteration.

§1.3 Boya Zhang (2007-2022)

I am greatly saddened to report that one of the contestants, Boya Zhang, passed away before the grading of the USEMO could be completed. He was supposed to have received a Distinction award.

I was informed by one of his classmates, who wrote the following:

> Boya was such an amazing, kind-hearted person. He excelled at math, video games, and making us all laugh. He had such a positive impact on every person who knew him, and on this community as well as the broader math community. You might have seen him on the 2022 Mathcounts Nationals Countdown Round, or you might have met him online on OTIS or somewhere else, or you might have even knew him personally like I did. No matter what it is, let us all respect his memory by having a moment of silence.
>
> Thank you Boya, for everything you've given to us, for everything you accomplished, and for showing us what it means to be a good person.

This short section is thus dedicated in Boya's memory.

2 Results

If you won one of the seven awards, please reach out to `usemo@evanchen.cc` to claim your prize!

§2.1 Top Scores

Congratulations to the top three scorers, who win the right to propose problems to future instances of USEMO.

1st place JunWen Huang, 33 points

2nd place Maximus Lu, 29 points

3rd place Krishna Pothapragada, 22 points

§2.2 Special awards

See the Rules for a description of how these are awarded. For the purposes of awarding monetary prizes, ties are broken more or less arbitrarily by considering the presentation of elegance of solutions (which is obviously subjective). When this occurs, the names of tied students are noted as well.

Youth prize Allan Yuan

Top female Angela Liu

Top day 1 Alexander Wang

Top day 2 Srinivas Arun

§2.3 Honorable mentions

This year we award Honorable Mention to anyone scoring at least 17 points (who is not in the top three already). The HM's are listed below in alphabetical order.

Alexander Wang

Allan Yuan

Alston Xu

Bora Olmez

Christopher Qiu

Eduardo Aragon

Henrick Rabinovitz

Lerchen Zhong

Neal Yan

Wilbert Chu

§2.4 Distinction

Normally, we award Distinction to anyone scoring at least 14 points (two fully solved problems). Because this would be too restrictive this year due to a difficult exam (it would have resulted in only five awards being given), we decreased the cutoff to just 7 points. In the future, the wording for the Distinction criteria will be edited to be either 14 points or top 25 students, whichever is more inclusive.

The Distinction awards are listed below in alphabetical order.

Advaith Avadhanam

Angela Liu

Aprameya Tripathy

Boya Zhang

Carlos Rodriguez

Christopher Lu

Elliott Liu

Ethan R Lee

Feodor Yevtushenko

Isaac Chen

Jacopo Rizzo

Owen Zhou

Razzi Masroor

Rohan Das

Ruilin Wang

Soham Bhadra

Srinivas Arun

3 Solutions and marking schemes

§3.1 USEMO 1 — proposed by Holden Mui

Problem statement

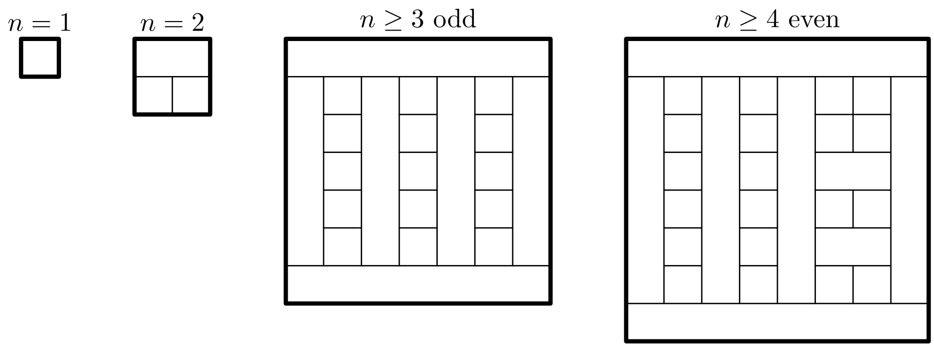
A *stick* is defined as a $1 \times k$ or $k \times 1$ rectangle for any integer $k \geq 1$. We wish to partition the cells of a 2022×2022 chessboard into m non-overlapping sticks, such that any two of these m sticks share at most one unit of perimeter. Determine the smallest m for which this is possible.

§3.1a Solution

In general, with 2022 replaced by n, we will prove the answer is

$$m = \begin{cases} 1 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ \frac{1}{2}(n^2 - 2n + 7) & \text{if } n \geq 3 \text{ and } n \text{ is odd} \\ \frac{1}{2}(n^2 - 2n + 8) & \text{if } n \geq 3 \text{ and } n \text{ is even,} \end{cases}$$

with the following construction. For $n = 2022$ this gives 2042224 as the answer.



The optimality for $n \in \{1, 2\}$ is easy to check, so assume $n \geq 3$.

The main idea is to view the problem as taking an $n \times n$ grid of squares and deleting some of the edges until the resulting figure is a set of sticks. Since the number of sticks is equal to n^2 minus the number of removed edges in the square, it suffices to maximize the number of removed edges. However:

Claim — No two of the deleted edges may share an endpoint.

Proof. Obvious. □

As a consequence of this claim:

- At most $\frac{1}{2}(n-3)^2$ edges can be removed within the central $n - 3$ by $n - 3$ grid of lattice points.

- Additionally, at most $4(n-2)$ of the outer edges can be removed since there are $4(n-2)$ lattice points within one unit of the boundary.

Hence, the minimum number of removed edges is at least

$$n^2 - \left(\left\lfloor \frac{1}{2}(n-3)^2 \right\rfloor + 4(n-2) \right)$$

which equals the claimed minimum.

> **Remark** (Torus variant)**.** One can ask the same problem on an $n \times n$ torus. The answer is $\frac{1}{2}n^2$ for even n and $\frac{1}{2}(n^2-1)$ for odd n; the proof is analogous to the one above, but without the outer edge consideration.

§3.1b Marking scheme

For solutions which are not complete, the following items are available and are additive:

(i) **+1 point** is awarded for stating the correct answer which is 2,042,224 (or the answer for general n), even with no justification.

(ii) **+1 point** is awarded for giving a working construction.

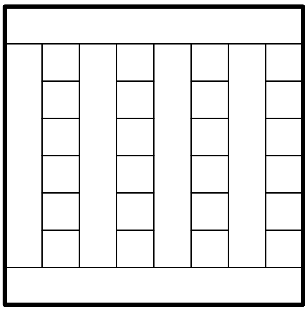
(iii) **+1 point** for the observation that there is at most one deleted edge per vertex.

For weaker bounds of m (if the value of m is not correct), the following items are available, which are not additive with (i)-(iii) or each other:

(iv) **1 point** for constructions which use at most

$$\frac{1}{2}(n^2 - n) + 2 = 2043233$$

sticks; an example of such a construction is shown below for $n = 8$.



Note that, for odd n, this construction is actually optimal, so the student would have gotten 2 points in an odd year!

(v) **0 points** for constructions which use more than 2043233 sticks.

(vi) **1 point** for proving that $m \geq cn^2$ is *necessary* to fulfill the condition (i.e. a bound, not a construction), for any positive constant $c \geq 1/3$.

(vii) **2 points** for proving that $m \geq \frac{1}{2}n^2 - cn$ is *necessary* to fulfill the condition (i.e. a bound, not a construction), for any positive constant $c \geq 1$.

And obviously, **7 points** for a perfectly working solution. The following deductions can apply to a correct solution; they are additive.

(viii) **-1 point** if whole solution is correct but there are calculation/algebraic errors in answer extraction (e.g. calculating for $n = 2022$, algebraic errors)

(ix) **-1 point** if the construction given (if any) is wrong or missing.

No deduction for not stating the answer for $n = 2022$ explicitly (if they did the problem by considering a general n and there are no other errors).

§3.2 USEMO 2 — proposed by Sutanay Bhattacharya

Problem statement

A function $\psi\colon \mathbb{Z} \to \mathbb{Z}$ is said to be *zero-requiem* if for any positive integer n and any integers $a_1, \dots, a_n$ (not necessarily distinct), the sums $a_1 + a_2 + \cdots + a_n$ and $\psi(a_1) + \psi(a_2) + \cdots + \psi(a_n)$ are not both zero.

Let f and g be two zero-requiem functions for which $f \circ g$ and $g \circ f$ are both the identity function (that is, f and g are mutually inverse bijections). Given that $f + g$ is *not* a zero-requiem function, prove that $f \circ f$ and $g \circ g$ are both zero-requiem.

§3.2a Solution

We give three solutions. The first one explicitly classifies all zero-requiem functions after which the problem becomes fairly routine. The second solution is more indirect but short. The third solution is the author's original submission.

¶ First solution (Nikolai Beluhov). First we describe all zero-requiems.

Lemma

A function $f\colon \mathbb{Z} \to \mathbb{Z}$ is a zero-requiem if and only if either

- there exists some real constant C such that $f(n) > Cn$ for all integers n; or
- there exists some real constant C such that $f(n) < Cn$ for all integers n.

Proof. Sufficiency is clear. We move on to necessity.

For all integers n, let P_n be the point $(n, f(n))$, and let $\mathcal{P} = \{P_n \mid n \in \mathbb{Z}\}$.

Suppose, for the sake of contradiction, that the origin O is in the convex hull of $\mathcal{P}$. Then it is also in the convex hull of some finite subset of $\mathcal{P}$. Let $O = \sum_{i=1}^{k} \alpha_i P_{n_i}$, where the α_i are positive and $\sum_i \alpha_i = 1$. Without loss of generality, all of the α_i are rational. Choose N so that $\alpha_i N$ is a positive integer for all i. Then the multiset comprised of $\alpha_i N$ copies of n_i, over all $1 \le i \le k$, shows that f is not a zero-requiem, and we arrive at a contradiction.

Thus O must be outside of the convex hull of $\mathcal{P}$. Consequently, there exists a line ℓ through O such that all of the P_i are strictly on the same side of ℓ. But that is exactly what we wanted to prove. □

The rest is not too difficult.

Let f and f^{-1} satisfy the Lemma with constants C and D, respectively. (But we have not fixed the directions of the inequalities yet.) Note that $C \neq 0$ and $D \neq 0$ since both of f and f^{-1} attain both positive and negative values.

Case 1 $C > 0$.

If $f(n) > Cn$ for all n, then also $f(f(n)) > Cf(n) > C^2 n$ for all n, and so $f \circ f$ satisfies the Lemma with constant C^2. The sub-case when $f(n) < Cn$ is analogous.

Case 2 $D > 0$.

If $f^{-1}(n) > Dn$ for all n, then also $n = f^{-1}(f(n)) > Df(n)$ and $f(n) < \frac{1}{D}n$ for all n, and we finish as in Case 1. The sub-case when $f^{-1}(n) < Dn$ is analogous.

Case 3 $C < 0$ and $D < 0$.

If $f(n) > Cn$ and $f^{-1}(n) > Dn$ for all n, then clearly $f + f^{-1}$ satisfies the Lemma with constant $C+D$. The sub-case when $f(n) < Cn$ and $f^{-1}(n) < Dn$ is analogous.

What remains is $f(n) < Cn$ and $f^{-1}(n) > Dn$ for all n. (The sub-case when $f(n) > Cn$ and $f^{-1}(n) < Dn$ is analogous. We simply swap f and f^{-1}.) By the same reasoning as in Case 2, it follows that $f(n) > \frac{1}{D}n$ for all n. (The direction of the inequality is reversed because this time around $D < 0$.) But then $f(0) < C \cdot 0 = 0$ and $f(0) > \frac{1}{D} \cdot 0 = 0$, and we arrive at a contradiction. Thus this final sub-case cannot occur.

¶ Second solution (rephrased from several contestants). Assume for contradiction that neither $f + g$ nor $g \circ g$ is zero-requiem, meaning that there exist $c_1, \dots, c_m$ and $w_1, \dots, w_n$ such that

$$0 = \sum_{i=1}^{m} c_i = \sum_{i=1}^{m} f(c_i) + \sum_{i=1}^{m} g(c_i)$$
$$0 = \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} g(g(w_i)).$$

Set $x_i = g(w_i)$ and discard w_i; then we can rewrite the two equations as

$$0 = \sum_{i=1}^{m} c_i = \underbrace{\sum_{i=1}^{m} f(c_i)}_{\neq 0 \text{ as } f \text{ is ZR}} + \underbrace{\sum_{i=1}^{m} g(c_i)}_{\neq 0 \text{ as } g \text{ is ZR}}$$
$$0 = \sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} g(x_i).$$

Since f and g were given to be zero-requiem, the sums $\sum_{i=1}^{m} f(c_i)$, $\sum_{i=1}^{m} g(c_i)$, and $\sum_{i=1}^{n} x_i$ are all nonzero. So let's say WLOG that $\sum_{i=1}^{m} g(c_i)$ and $\sum_{i=1}^{m} x_i$ have opposite sign. That means we have integers $p, q > 0$ such that

$$p \sum_{i=1}^{m} (c_i) + q \sum_{i=1}^{n} x_i = 0.$$

But

$$p \sum_{i=1}^{m} c_i + q \sum_{i=1}^{n} f(x_i) = p \cdot 0 + q \cdot 0 = 0.$$

This is a contradiction to f being zero-requiem if we feed in the sequence constructed by taking $(c_1, \dots, c_m)$ each p times and $(x_1, \dots, x_n)$ each q times.

¶ Third solution (author's). For comedic value, we retain the Code Geass flavortext suggested by the authors.

We say a function $f \colon \mathbb{Z} \to \mathbb{Z}$ is

- *Nunnally* if $a_1 + \cdots + a_n \geq 0$ implies $f(a_1) + \cdots + f(a_n) > 0$;
- *Marianne* if $a_1 + \cdots + a_n \geq 0$ implies $f(a_1) + \cdots + f(a_n) < 0$;
- *Jeremiah* if $a_1 + \cdots + a_n \leq 0$ implies $f(a_1) + \cdots + f(a_n) > 0$;
- *Charles* if $a_1 + \cdots + a_n \leq 0$ implies $f(a_1) + \cdots + f(a_n) < 0$;

for any finite sequence $a_1, \ldots, a_n$ of integers.

Claim — A function is zero-requiem if and only if it is at least one of these four categories.

Proof. Using the condition on the sequence $(0,0)$ we see that $f(0) \neq 0$. Thus there are two cases, $f(0) > 0$ and $f(0) < 0$. These cases are essentially the same, so we start with $f(0) > 0$.

When $f(0) > 0$, there are three sub-cases:

- Suppose there exists a positive integer x so that $f(x) \leq 0$, say $f(x) = -y$. We claim that f is a Jeremiah. Indeed, if not, then there exist $a_1, \ldots, a_n$ with non-positive sum $-p$, so that $f(a_1) + \cdots + f(a_n) = -S$ is non-positive. Consider the following sequence:
$$\underbrace{\boxed{a_1, \ldots, a_n}, \ldots, \boxed{a_1, \ldots, a_n}}_{xf(0) \text{ blocks}}, \underbrace{x, \ldots, x}_{pf(0)\ x\text{'s}}, \underbrace{0, \ldots, 0}_{xS+py\ 0\text{'s}}.$$
Clearly this has sum $xf(0) \cdot (-p) + pf(0) \cdot x + 0 = 0$. The sum of the sequence of the $f-$values of these is
$$\begin{aligned} &xf(0) \cdot (f(a_1) + \cdots + f(a_n)) + pf(0) \cdot f(x) + (xS + py) \cdot f(0) \\ &= xf(0)(-S) + pf(0)(-y) + (xS + py)f(0) = 0 \end{aligned}$$
This is a contradiction, proving our claim.
- Suppose there exists a negative integer x so that $f(x) \leq 0$. Now the function f_1 defined by $f_1(x) = f(-x)$ satisfies the hypotheses of the previous case, so f_1 is a Jeremiah. Then clearly f is a Nunnally.
- Otherwise $f(x) > 0$ for all x, so f is trivially a Nunnally.

Now say $f(0) < 0$. Then $f_2 = -f$ is also a zero-requiem and satisfies $f_2(0) > 0$, so it's either a Jeremiah or a Nunnally, whence clearly f is either a Charles or Marianne. This proves our claim. $\square$

Claim — The following four statements are true.

- If f is a Nunnally or a Jeremiah, f^{-1} cannot be Nunnally or a Marianne.
- If f is a Marianne or a Charles, f^{-1} cannot be a Jeremiah or a Charles.
- If f^{-1} is a Nunnally or a Jeremiah, f cannot be Nunnally or a Marianne.
- If f^{-1} is a Marianne or a Charles, f cannot be a Jeremiah or a Charles.

Proof. For the first statement, note that

$$\begin{aligned} a_1 + \cdots + a_n = 0 &\implies f(a_1) + \cdots + f(a_n) > 0 \\ \implies f^{-1}(f(a_1)) + \cdots + f^{-1}(f(a_n)) \neq 0 \\ &\implies a_1 + \cdots + a_n \neq 0. \end{aligned}$$

The second statement is proven in the same way. The third and fourth statements follow by swapping the roles of f and f^{-1} in the first two statements. □

Back to the main problem.

Claim — If f is a Nunnally or a Charles, then $f \circ f$ is a zero-requiem.

Proof. Indeed, in the first case, we have

$$\begin{aligned} a_1 + \cdots + a_n = 0 &\implies f(a_1) + \cdots + f(a_n) > 0 \\ &\implies f(f(a_1)) + \cdots + f(f(a_n)) > 0 \\ &\implies (f \circ f)(a_1) + \cdots (f \circ f)(a_n) \neq 0, \end{aligned}$$

proving the claim. The other case follows similarly. □

The only remaining cases are when f and f^{-1} are both Mariannes or they are both Jeremiahs. In the first case,

$$\begin{aligned} a_1 + \cdots + a_n = 0 &\implies f(a_1) + \cdots + f(a_n) < 0 \text{ and } f^{-1}(a_1) + \cdots f^{-1}(a_n) < 0 \\ \implies (f(a_1) + f^{-1}(a_1)) + \cdots + (f(a_n) + f^{-1}(a_n)) < 0 \\ &\implies (f + f^{-1})(a_1) + \cdots + (f + f^{-1})(a_n) \neq 0, \end{aligned}$$

which means $f + f^{-1}$ is zero-requiem. A similar argument holds when f, f^{-1} are both Jeremiahs, so we are done.

§3.2b Marking scheme

For incomplete solutions, the following points are not additive.

(i) **4 points**: Proving the first lemma in the first solution

(ii) **1 point**: Conjecturing the first lemma in the first solution

(iii) **0 points**: Neither $\sum_{i=1}^{k} f(a_i)$ nor $\sum_{i=1}^{k} g(a_i)$ is zero.

(iv) **1 point**: For showing that for all $(a_n)_n$ such $a_1 + \cdots + a_n = 0$, $f(a_1) + \cdots + f(a_n)$ are either all positive or all negative.

(v) **2 points**: For proving $f(x) > 0$ must hold for either all positive x or all negative x

(vi) **1 point**: For considering $f(a_1) + \cdots + f(a_k) = g(f \circ f(a_1)) + \cdots + g(f \circ f(a_k))$ and f, g being opposite signs

(vii) **3 Points****: Proof of existence of $\sum_{i=1}^{\ell} b_i$ that has opposite sign with either $\sum_{i=1}^{k} f(a_i)$ or $\sum_{i=1}^{k} g(a_i)$ (Only for proofs by contradiction)

(viii) **5 points**: Failing to finish in the case $f(x) > 0$ for $x < 0$, but otherwise complete

(ix) **5 points**: Failing to finish in the case $f(x) > 0$ for $x > 0$, but otherwise complete

§3.3 USEMO 3 — proposed by Nikolai Beluhov

Problem statement

Point P lies in the interior of a triangle ABC. Lines AP, BP, and CP meet the opposite sides of triangle ABC at points A', B', and C', respectively. Let P_A be the midpoint of the segment joining the incenters of triangles BPC' and CPB', and define points P_B and P_C analogously. Show that if

$$AB' + BC' + CA' = AC' + BA' + CB',$$

then points P, P_A, P_B, and P_C are concyclic.

§3.3a Solution

We present two approaches, one completely synthetic, and the other arguably more straightforward length computation.

¶ First solution (author's). We will need a couple of lemmas.

Lemma 3.3.1 ("Sparrow lemma")

Let I be the incenter of triangle ABC. Point U lies on ray $AB^{\rightarrow}$ and point V lies on ray $CA^{\rightarrow}$ beyond A so that $AU - AV = AB + AC - BC$. Then points A, I, U, and V are concyclic.

Proof. Let the incircle of triangle ABC touch sides AB and AC at points K and L, respectively. Since $AK = AL = \frac{1}{2}(AB + AC - BC)$, it follows that $KU = LV$. Hence, right triangles IKU and ILV are congruent, and so $\angle AUI = \angle IUK = \angle IVL = \angle AVI$. □

Lemma 3.3.2 (IMO 1979/3)

Let P be a common point of the two circles Γ_1 and Γ_2. A variable line ℓ through P meets Γ_1 and Γ_2 again at points T_1 and T_2, respectively. Then, as line ℓ varies, the perpendicular bisectors of all segments T_1T_2 pass through a constant point.

Proof. Let D_1 and D_2 be the points diametrically opposite P in circles Γ_1 and Γ_2, respectively. Then both lines D_1T_1 and D_2T_2 are perpendicular to segment T_1T_2. Thus the perpendicular bisector of segment T_1T_2 coincides with the mid-line of the strip formed by lines D_1T_1 and D_2T_2, and so it always passes through the midpoint of segment D_1D_2. □

We are ready to tackle the problem.

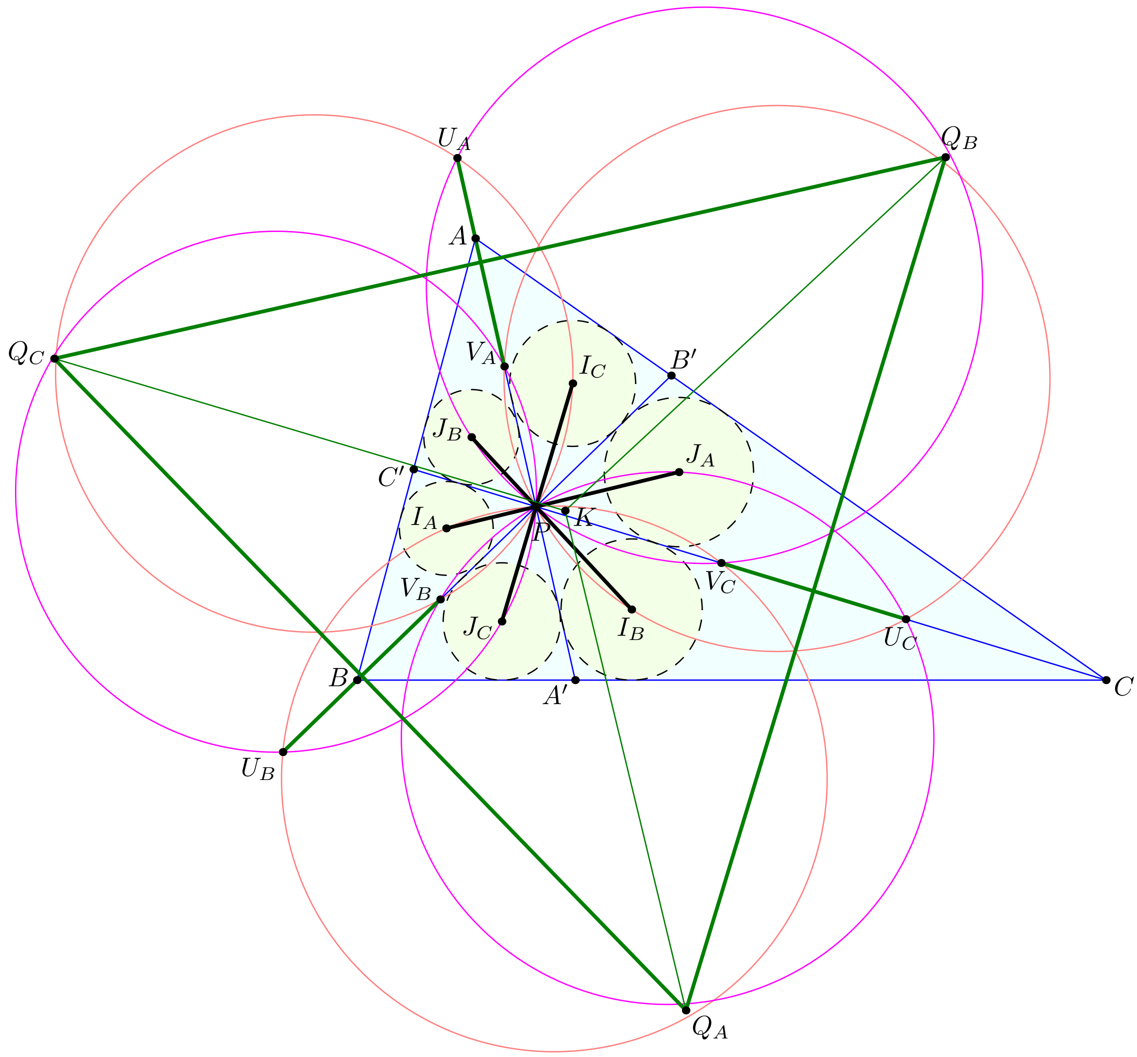
Since $AB' + BC' + CA' = AC' + BA' + CB'$, we can find points U_A and V_A on ray $PA^{\rightarrow}$, U_B and V_B on ray $PB^{\rightarrow}$, and U_C and V_C on ray $PC^{\rightarrow}$ simultaneously satisfying all six of the equations

$$PU_A - PV_B = PA + PB' - AB',$$

$$\begin{aligned}
PU_A - PV_C &= PA + PC' - AC', \\
PU_B - PV_C &= PB + PC' - BC' \\
PU_B - PV_A &= PB + PA' - BA', \\
PU_C - PV_A &= PC + PA' - CA', \\
PU_C - PV_B &= PC + PB' - CB',
\end{aligned}$$

because any five of the equations implies the sixth one.

Denote by I_A, I_B, I_C, J_A, J_B, J_C the incenters of $\triangle BPC'$, $\triangle CPA'$, $\triangle APB'$, $\triangle CPB'$, $\triangle APC'$, $\triangle BPA'$, respectively, as shown below.



By Lemma 1 applied to triangle APB', we get that P, I_C, U_A, V_A are concyclic. Similarly, the other five incenters lie on the analogous circles through P.

Let s_A be the perpendicular bisector of segment I_AJ_A, let t_A be the perpendicular bisector of segment U_AV_A, and define lines s_B, t_B, s_C, and t_C analogously.

Consider the circumcircles $(PI_AU_BU_C)$ and $(PJ_AV_BV_C)$. Applying Lemma 2 to this shows that lines s_A, t_B, and t_C are concurrent at a point Q_A. Analogously, lines s_B, t_C, and t_A meet at some point Q_B and lines s_C, t_A, and t_B meet at some point Q_C.

Observe that lines t_A, t_B, and t_C are perpendicular to lines AA', BB', and CC', respectively, whereas lines s_A, s_B, and s_C are perpendicular to lines $I_{B,C}I_{C,B}$, $I_{C,A}I_{A,C}$, and $I_{A,B}I_{B,A}$, respectively. Since the latter three lines bisect the pairwise angles between the former three lines, we conclude that lines s_A, s_B, and s_C bisect the interior angles of the triangle formed by lines t_A, t_B, and t_C.

Therefore, lines s_A, s_B, and s_C meet at the incenter K of triangle $Q_AQ_BQ_C$. Or, equivalently, all four points P, P_A, P_B, and P_C lie on the circle with diameter $\overline{PK}$. The solution is complete.

Remark. There are at least three notable points P which satisfy the conditions of the problem:

- The centroid, when A', B', and C' are the midpoints of the sides;
- the Gergonne point, when A', B', and C' are the tangency points of the incircle with the sides;
- and the Nagel point, when A', B', and C' are the tangency points of the corresponding excircles with the sides.

These special cases are all in Alexander Skutin, Tran Quang Hung, Antreas Hatzipolakis, and Kadir Altintas, *Cosmology of Plane Geometry*, 2019; revised edition 2021. The case of the Gergonne point is Theorem 1.1.1 in the 2019 edition and Theorem 1.1.1 (1) in the 2021 edition; the case of the Nagel point is Theorem 1.1.1 (2) in the 2021 edition; and the case of the centroid is Theorem 5.1.1 in the 2019 edition and Theorem 2.1.1 in the 2021 edition. Thus the problem generalises all of these theorems.

Remark. Both lemmas are well-known. Lemma 1 is, for example, in Alexander Polyanskiy, *Vorobyami po Pushkam!*, Kvant 02/2012. Lemma 2 is IMO 1979, problem 3 by Nikolay Vasilyev and Igor Sharygin; reprinted as Kvant 12/1979, problem M600, part (a).

¶ Second solution (sent by Arjun Gupta). The proof hinges on the following lemma, which is essentially a restatement of Ptolemy's theorem.

Lemma (Trigonometric Ptolemy)

Let ℓ_A, ℓ_B, ℓ_C be three lines concurrent at a point P. Let θ_A denote value of non-obtuse angle between ℓ_B and ℓ_C. Define θ_B, θ_C similarly. Impose a sign convention for lengths on ℓ_A, ℓ_B, ℓ_C such that among the three rays formed by positive direction through P, no ray lies inside the angle formed by the other two rays. Let P_A be any point on ℓ_A, and define ℓ_B, ℓ_C similarly. Then

$$\sin\theta_A \cdot \overrightarrow{PP_A} + \sin\theta_B \cdot \overrightarrow{PP_B} + \sin\theta_C \cdot \overrightarrow{PP_C} = 0$$

holds if and only if points P, P_A, P_B, P_C are concyclic.

Proof. Suppose first that P, P_A, P_B, P_C are concyclic. WLOG points P, P_B, P_A, P_C lie on a circle in that order. Ptolemy's Theorem gives

$$PP_A \cdot P_BP_C = PP_B \cdot P_CP_A + PP_C \cdot P_AP_B.$$

By law of sines,

$$P_BP_C : P_CP_A : P_AP_B = \sin\theta_A : \sin\theta_B : \sin\theta_C$$

So we obtain

$$\sin\theta_A \cdot PP_A = \sin\theta_B \cdot PP_B + \sin\theta_C \cdot PP_C.$$

Now P_A lies inside $\angle P_B P P_C$, so ℓ_A passes through interior of $\angle P_B P P_C$. This means $\overrightarrow{PP_B}$, $\overrightarrow{PP_C}$ have the same sign, WLOG both are non-negative. Now $\overrightarrow{PP_A}$ must be non-positive, otherwise P lies inside $\triangle P_A P_B P_C$. It follows

$$\sin\theta_A(-\overrightarrow{PP_A}) = \sin\theta_B(\overrightarrow{PP_B}) + \sin\theta_C(\overrightarrow{PP_C})$$
$$\implies \sin\theta_A \cdot \overrightarrow{PP_A} + \sin\theta_B \cdot \overrightarrow{PP_B} + \sin\theta_C \cdot \overrightarrow{PP_C} = 0.$$

This proves one direction. Now the converse direction just follows from a phantom point argument. $\square$

Let I_A, J_A denote the incenters of triangles BPC' and CPB', respectively. Define I_B, J_B, I_C, I_Cb similarly. Let ℓ_A, ℓ_B, ℓ_C denote the lines I_AJ_A, I_BJ_B, I_CJ_C, respectively. Clearly $P \in \ell_A, \ell_B, \ell_C$.

We will write $\overrightarrow{AB}$ to denote the directed length of segment AB. Let the lengths on ℓ_A, ℓ_B, ℓ_C are directed, with lengths $\overrightarrow{PI_A}$, $\overrightarrow{PI_B}$, $\overrightarrow{PI_C}$ being positive.

For a triangle PXY, let $s(PXY)$ denote the value $\frac{PX+PY-XY}{2}$. Note if I is incenter of $\triangle PXY$, then

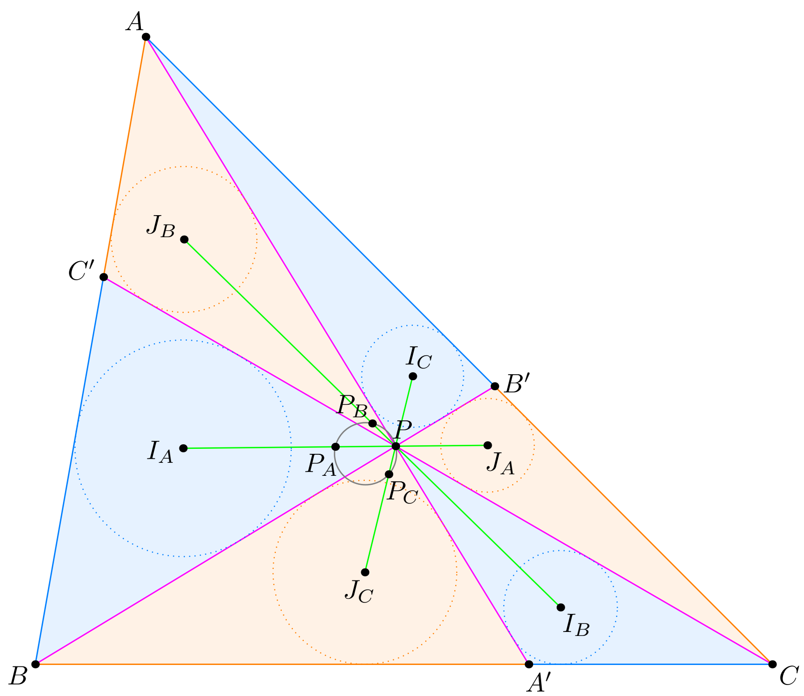
$$PI = \frac{s(PXY)}{\cos\frac{\angle XPY}{2}}.$$

Since $AB' + BC' + CA' = AC' + BA' + CB'$, so we obtain

$$s(PBC') + s(PCA') + s(PAB') = s(PCA') + s(PCB') + s(PAC').$$

Let θ_A be acute angle between lines ℓ_B and ℓ_C. Define θ_B, θ_C similarly. Observe that

$$\theta_A = \frac{\angle BPC}{2} = 90^\circ - \frac{\angle BPC'}{2} = 90^\circ - \frac{\angle CPB'}{2}.$$



So we obtain

$$\overrightarrow{PI_A} = \frac{s(PBC')}{\sin\theta_A}, \quad \overrightarrow{PJ_A} = \frac{-s(PCA')}{\sin\theta_A}$$

$$\implies \overrightarrow{PP_A} = \frac{\overrightarrow{PI_A} + \overrightarrow{PJ_A}}{2} = \frac{1}{2}\left(\frac{s(PBC') - s(PCA')}{\sin\theta_A}\right)$$

We analogously obtain

$$\overrightarrow{PP_B} = \frac{1}{2}\left(\frac{s(PCA') - s(PCB')}{\sin\theta_B}\right)$$
$$\overrightarrow{PP_C} = \frac{1}{2}\left(\frac{s(PAB') - s(PAC')}{\sin\theta_C}\right)$$

It follows that

$$\sin\theta_A \cdot \overrightarrow{PP_A} + \sin\theta_B \cdot \overrightarrow{PP_B} + \sin\theta_C \cdot \overrightarrow{PP_C} = 0$$

so our earlier Ptolemy-based lemma applies and the problem is solved.

§3.3b Marking scheme

As usual, incomplete computational approaches earn partial credits only based on the amount of synthetic progress which is made. No points are awarded for just drawing a diagram or simple observations.

Solution 1 Rubric

(i) **2 points** are awarded for proving the points P, I_C, U_A, V_A are concyclic (and the symmetric ones)

(ii) **2 points** are awarded for using proving s_A, t_B, t_C meet at Q_A (and the symmetric ones)

(iii) **3 points** for finishing the problem

Solution 2 Rubric

(iv) **1 point** is awarded for stating the Lemma

(v) **1 point** is awarded for proving the Lemma

(vi) **4 points** for:

$$\sin\left(90^\circ + \frac{\psi_A}{2}\right) \cdot \overrightarrow{PP_A} = \frac{PB - PC + PC' - PB' - BC' + CB'}{4}$$

(vii) **1 point** is awarded for finishing the problem

Outside the two solutions

(viii) If there is any non-trivial linear relation for the tangents from P to the incircles, **1 point** is awarded.

If the student has approaches from both the solutions, they get the maximum of the two possible markings.

§3.4 USEMO 4 — proposed by Tilek Askerbekov

Problem statement

Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points P, Q, R, S lie in the interiors of segments AB, BC, CD, DA, respectively, such that

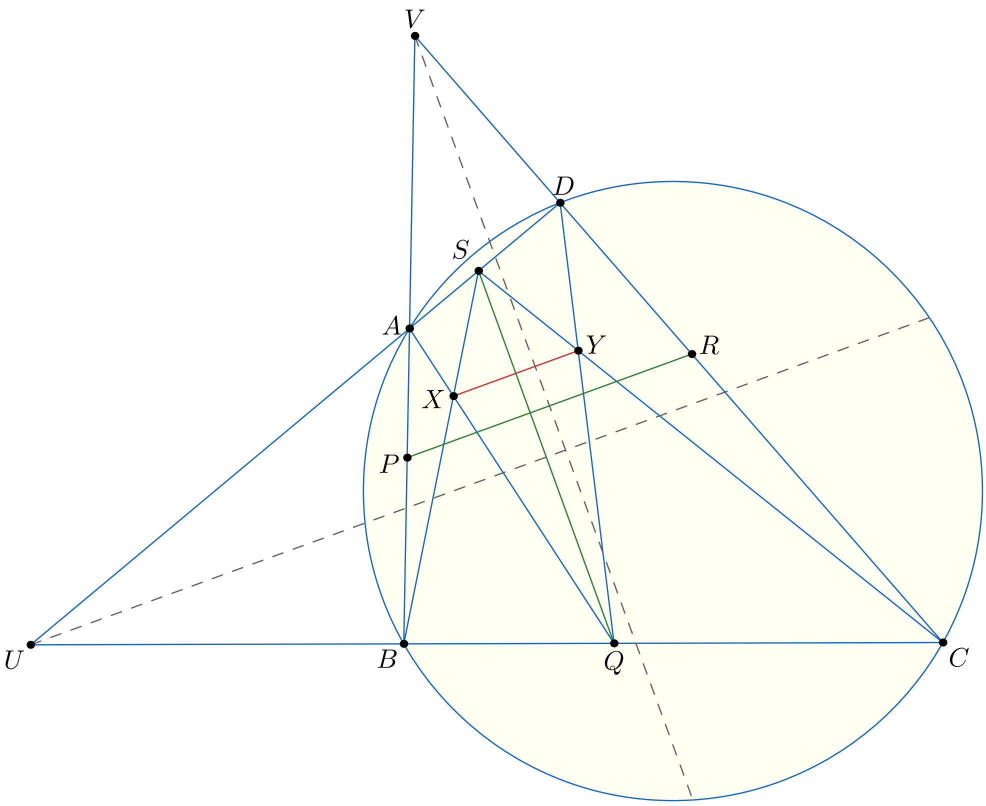
$$\angle PDA = \angle PCB, \quad \angle QAB = \angle QDC, \quad \angle RBC = \angle RAD, \quad \text{and} \quad \angle SCD = \angle SBA.$$

Let $\overline{AQ}$ intersect $\overline{BS}$ at X, and $\overline{DQ}$ intersect $\overline{CS}$ at Y. Prove that lines $\overline{PR}$ and $\overline{XY}$ are either parallel or coincide.

§3.4a Solution

We present two approaches. The first is based on the points $U = \overline{AD} \cap \overline{BC}$ and $V = \overline{AB} \cap \overline{CD}$. The latter is based on $E = \overline{AC} \cap \overline{BD}$.

¶ First solution (author's). Let $U = \overline{AD} \cap \overline{BC}$ and $V = \overline{AB} \cap \overline{CD}$.



Claim — We have $US = UQ$ and $VP = VR$.

Proof. We have

$$\measuredangle BSA = \measuredangle BAS + \measuredangle SBA = \measuredangle BCD + \measuredangle DCS = \measuredangle BCS$$

hence

$$US^2 = UB \cdot UC.$$

Similarly, $UQ^2 = UA \cdot UD = UB \cdot UC$. So, $US = UQ$; similarly $VP = VR$. $\square$

Claim — Quadrilateral $SXQY$ is a kite (with $SX = SY$ and $QX = QY$).

Proof. We have

$$\measuredangle BSQ = \measuredangle USQ - \measuredangle USB = \measuredangle SQU - \measuredangle SCB = \measuredangle QSC$$

so $\overline{SQ}$ bisects $\angle BSC$; similarly it bisects $\angle AQD$. $\square$

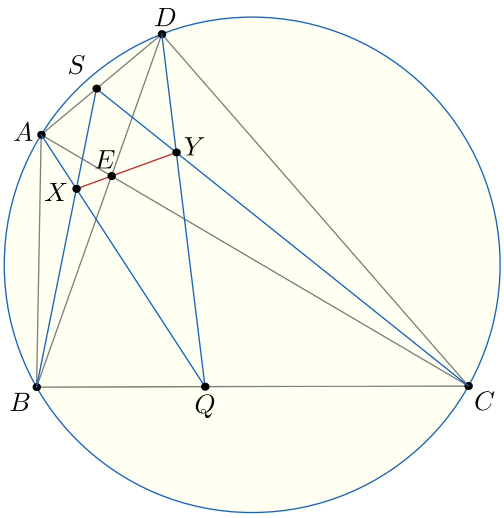
Claim — The internal bisectors of $\angle U$ and $\angle V$ are perpendicular.

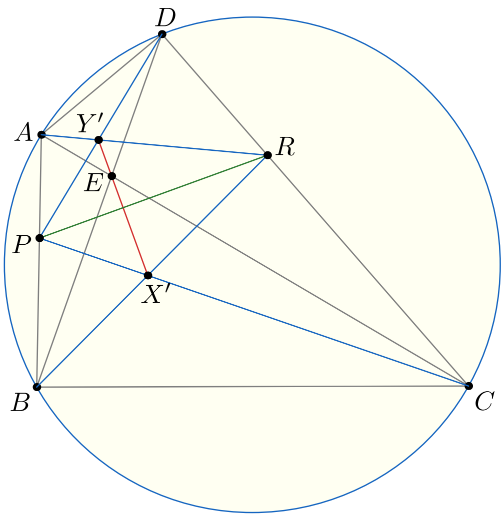
Proof. The angle between these angle bisectors equals

$$\begin{aligned}&\frac{1}{2}\angle DUC + \angle DAV + \frac{1}{2}\angle BVC \\ &= 90^\circ - \frac{\angle ADC}{2} - \frac{\angle DCB}{2} + \angle BCD + 90^\circ - \frac{\angle ABC}{2} - \frac{\angle DCB}{2} \\ &= 90^\circ.\end{aligned}$$ $\square$

As $\overline{SQ}$ and $\overline{PR}$ are perpendicular to the internal bisectors of $\angle U$ and $\angle V$ by the first claim, so by the third claim $\overline{QS} \perp \overline{PR}$. Meanwhile the second claim gives that $\overline{XY}$ is perpendicular to $\overline{SQ}$, completing the problem.

¶ Second solution due to Nikolai Beluhov. Let $E = \overline{AC} \cap \overline{BD}$. Then E lies on $\overline{XY}$ by Pappus's theorem.





Claim — Line XEY is the interior bisector of $\angle AEB$ and $\angle CED$.

Proof. The angle conditions imply that X and Y are corresponding points in the two similar triangles AEB and DEC. Hence, $\angle AEX = \angle DEY$ and $\angle BEX = \angle CEY$. Since segments EX and EY are collinear, we're done. $\square$

Introduce the points

$$X' = \overline{BR} \cap \overline{CP} \qquad \text{and} \qquad Y' = \overline{AR} \cap \overline{DP}.$$

By the same argument as before, line $X'EY'$ is the internal angle bisector of angles $\angle AED$ and $\angle BEC$.

Claim — Quadrilateral $PX'RY'$ is a kite (with $PX' = PY'$ and $RX' = RY'$).

Proof. Because X' and Y' are corresponding points in $\triangle BEC$ and $\triangle AED$,

$$\angle RX'Y' = 180^\circ - \angle BX'E = 180^\circ - \angle AY'E = \angle RY'X',$$

and so $RX' = RY'$. Similarly, $PX' = PY'$. $\square$

Thus, $\overline{PR}$ is perpendicular to $\overline{X'EY'}$, hence parallel to the interior bisector of $\angle AEB$ and $\angle CED$. Together with the first claim, we're done.

Remark. It's possible to write up this solution without ever defining X' and Y'. The idea is to instead prove $SXQY$ is a kite (which is natural since X and Y are already marked) and hence obtain the sentence "$\overline{SQ}$ is parallel to the internal angle bisector of $\angle AED$ and $\angle BEC$" (using the first claim). Then cyclically shift the labels in to get the sentence "$\overline{PR}$ is parallel to the internal angle bisector of $\angle DEC$ and $\angle AEB$".

§3.4b Marking scheme

As usual, incomplete computational approaches earn partial credits only based on the amount of synthetic progress which is made. No points are awarded for just drawing a diagram or simple observations.

There are two major paths a solution can follow:

- One which introduces and works with $AB \cap CD$ and $AD \cap BC$ (the first official solution)
- One which works around $AC \cap BD$ (the second official solution).

Marks are to be given as the maximum of the scores obtained across either of the approaches.

Rubric for solution 1

The following partial items are available and are additive:

- **1 point** for showing that $US = UQ$ or something similar
- **3 points** for showing that SQ bisects $\angle BSC$ and $\angle AQD$ or any claim analogous to this, and concluding that SXQ is congruent to SYQ (or even just directly using this to claim $XY \perp SQ$).
- **2 points** for proving that $PR \perp SQ$
- **1 point** for completing the solution

For solutions that achieve at most 1 point from the above scheme the following items (additive) are available too.

- **+1 point** for making the correct conjecture that $SXQY$ is a kite, or equivalently that SXQ is congruent to SYQ (with no proof attached)

Rubric for solution 2

The following partial items are available and are additive:

- **0 points** for just claiming E lies on XY by Pappus' theorem.
- **2 points** for the claim that X and Y are corresponding points in similar triangles AEB and DEC.
- **2 points** for proving that line XY bisects $\angle AEB$ and/or $\angle CED$.
- **2 points** for showing that $XY \perp QS$
- **1 point** for completing the solution

For solutions that achieve at most 2 points from the above scheme the following items (additive) are available too.

- **+1 point** for making the correct conjecture that $XY \perp SQ$
- **+1 point** for making the correct conjecture that line XY bisects $\angle AEB$ and/or $\angle CED$.

Common items for both solutions

- **No deduction** for configuration issues (such as not using directed angles) or small typos in angle chasing
- **No deduction** for making claims (without written proof) that have reasoning analogous to a claim already proven. (As an example, after showing SQ bisects $\angle BSC$ there is no need to prove that QS bisects $\angle AQD$).
- Of course, a solution that uses a mixture of both approaches but ends up proving the required condition and is mathematically accurate receives full credit too.
- **-1 point** for skipping the angle chase for showing the angle bisectors of angle U and angle V are perpendicular (in case that approach has been taken, and the solution is complete otherwise)

§3.5 USEMO 5 — proposed by Jovan Vuković

Problem statement

Let $\tau(n)$ denote the number of positive integer divisors of a positive integer n (for example, $\tau(2022) = 8$). Given a polynomial $P(X)$ with integer coefficients, we define a sequence $a_1, a_2, \dots$ of nonnegative integers by setting

$$a_n = \begin{cases} \gcd\left(P(n),\ \tau(P(n))\right) & \text{if } P(n) > 0 \\ 0 & \text{if } P(n) \le 0 \end{cases}$$

for each positive integer n. We then say the sequence *has limit infinity* if every integer occurs in this sequence only finitely many times (possibly not at all).

Does there exist a choice of $P(X)$ for which the sequence $a_1, a_2, \dots$ has limit infinity?

§3.5a Solution

We claim the answer is no, such P does not exist.

Clearly we may assume P is nonconstant with positive leading coefficient. Fix P and fix constants $n_0, c > 0$ such that $c = P(n_0) > 0$. We are going to prove that infinitely many terms of the sequence are at most c.

We start with the following lemma.

Claim — For each integer $n \ge 2$, there exists an integer $r = r(n)$ such that

- For any prime p which is at most n, we have $\nu_p(P(r)) = \nu_p(c)$.
- We have
$$c \cdot \prod_{\text{prime } p \le n} \le r \le 2c \cdot \prod_{\text{prime } p \le n} p.$$

Proof. This follows by the Chinese remainder theorem: for each $p \le n$ we require $r \equiv n_0 \pmod{p^{\nu_p(c)+1}}$, which guarantees $\nu_p(P(r)) = \nu_p(P(n_0)) = \nu_p(c)$. Then there exists such an r modulo $\prod_{p \le n} p^{\nu_p(c)+1}$ as needed. $\square$

Assume for contradiction that all a_i are eventually larger than c. Take n large enough that $n > c$ and $r = r(n)$ has $a_r > c$. Then consider the term a_r:

- Using the conditions in the lemma it follows there exists a prime $p_n > n$ which divides $a_r = \gcd(P(r), \tau(P(r)))$ (otherwise a_r, which divides $P(r)$, is at most c).
- As p_n divides $\tau(P(r))$, this forces $P(r)$ to be divisible by (at least) $q_n^{p_n - 1}$ for some prime q_n.
- For the small primes p at most n, we have $\nu_p(P(r)) = \nu_p(c) < c < n \le p_n - 1$. It follows that $q_n > n$.
- Ergo,
$$P(r) \ge q_n^{p_n - 1} > n^n.$$

In other words, for large enough n, we have the asymptotic estimate

$$\begin{aligned} n^n < P(r) &= O(1) \cdot r^{\deg P} \\ &= O(1) \cdot c^{\deg P} \cdot \prod_{\text{prime } p \le n} p^{\deg p} \\ &< O(1) \cdot n^{\deg P \cdot \pi(n)} \end{aligned}$$

where $\pi(n)$ denotes the number of primes less than n. For large enough n this is impossible since the primes have zero density:

$$\lim_{n\to\infty} \frac{\pi(n)}{n} = 0.$$

Remark. For completeness, we outline a short elementary proof that $\lim_{n\to\infty} \frac{\pi(n)}{n} = 0$. For integers $M > 0$ define

$$\delta(M) := \prod_{p \le M} \left(1 - \frac{1}{p}\right).$$

Then $\pi(n) < \delta(M)n + \prod_{p \le M} p$, so it suffices to check that $\lim_{M\to\infty} \delta(M) = 0$. But

$$\frac{1}{\delta(M)} = \prod_{p \le M} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \le M} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right) \ge 1 + \frac{1}{2} + \dots + \frac{1}{M}$$

which diverges for large M.

§3.5b Marking scheme

None of the following items are additive.

(i) **0 points** for claiming the answer is no.

(ii) **0 points** for solving the problem for linear polynomials.

(iii) **1 point** for claiming infinitely many terms of the sequence have value c for some constant c in terms of P whose ν_p's are "well-behaved".

(iv) **1 point** for creating a useful sequence $r(n)$ but not finishing.

(v) **2 points** for proving that for any constant $C > 0$, there exists some prime $p > C$ and index r, for which $p \mid a_r$.

(vi) **4 points** if, in the previous item, it is also proved that r is bounded by a reasonable function of p like $p^{O(p)}$.

(vii) **6 points** for finishing the problem apart from observing primes have zero density.

(viii) No points deducted for stating that $\pi(n) < Cn$ for big enough n, for any value of C, even without proof.

(ix) No points deducted for small errors caused by $P(n) \le 0$ for finitely many n.

(x) **7 points** for a complete solution.

§3.6 USEMO 6 — proposed by Evan Chen

Problem statement

Find all positive integers k for which there exists a *nonlinear* function $f\colon \mathbb{Z} \to \mathbb{Z}$ which satisfies

$$f(a) + f(b) + f(c) = \frac{f(a-b) + f(b-c) + f(c-a)}{k}$$

for any integers a, b, c with $a + b + c = 0$.

§3.6a Solution

The complete set of solutions is given by

- For $k = 1$, $f(x) \equiv C_1 x + C_2 (x \mod 2) + C_3$.
- For $k = 3$, $f(x) \equiv C_1 x + C_2 x^2$.
- For $k = 9$, $f(x) \equiv C_1 x + C_2 x^4$.
- For all other k, only $f(x) \equiv C_1 x$.

Here C_1, C_2, C_3 are arbitrary integers. We can check they work, so now we just want to show they are the only ones.

We will solve the functional equation for $f\colon \mathbb{Z} \to \mathbb{C}$, claiming that the above solution set remains the only one. If $k = 1$, we can shift by constants to get $f(0) = 0$; if $k \neq 1$ apply $a = b = c = 0$ to get $f(0) = 0$ anyways. Now note that $f(x) \equiv x$ is a solution, so we may shift by the identity to assume $f(-1) = f(1)$.

We will prove in this case, $f \equiv 0$ unless $k = 1, 3, 9$.

Now plug in $(a, b, c) = (n + 1, -n, -1)$ and $(a, b, c) = (1, n, -(n + 1))$ gives

$$\begin{aligned} f(2n + 1) + f(-n + 1) + f(-n - 2) &= k \left(f(n + 1) + f(-n) + f(-1) \right) \\ &= k \left(f(-n - 1) + f(n) + f(1) \right). \end{aligned}$$

The last two by induction imply f is even. Now, by using this and $(a, b, c) = (n, -n, 0)$ we obtain

$$\begin{aligned} f(2n + 1) + f(n + 2) + f(n - 1) &= k \left(f(n + 1) + f(n) + f(1) \right) \\ f(2n) + 2f(n) &= k \left(2f(n) + f(0) \right) \implies f(2n) = (2k - 2) f(n). \end{aligned}$$

Thus f is determined recursively by $f(1)$ (by induction). In particular, if $f(1) = 0$ then $f(n) \equiv 0$ by induction.

Now, let us assume $f(1) \neq 0$, and hence by scaling $f(1) = 1$. We can then compute:

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 2k - 2 \\ f(3) &= k^2 \\ f(4) &= 4k^2 - 8k + 4 \\ f(5) &= k^3 - 2k^2 + 7k - 5 \end{aligned}$$

$$\begin{aligned} f(6) &= 2k^3 - 2k^2 \\ f(7) &= 4k^3 - 6k^2 - 4k + 7 \\ f(8) &= 8k^3 - 24k^2 + 24k - 8. \end{aligned}$$

Plug in $(a, b, c) = (5, -3, -2)$ and compute $f(8) + f(1) + f(7) = k\left(f(5) + f(3) + f(2)\right)$, which simplifies to give

$$k^4 - 13k^3 + 39k^2 - 27k = 0 \implies k(k-1)(k-3)(k-9) = 0$$

so $k = 1$, $k = 3$, or $k = 9$. In these cases it is easy to check by induction now that $f(n) = n \mod 2$, $f(n) = n^2$, and $f(n) = n^4$.

§3.6b Marking scheme

In this rubric, a student earns up to 2 points for giving valid constructions, and up to 5 points for proving those solutions are the only ones. These points for the construction are additive with those for the proof.

The construction has three steps:

- Proving $k = 3$ works (writing something like: just expand and use $(a + b + c)^2 = 0$, counts as a fine proof).
- Proving $k = 1$ works (writing something like: check cases according to parity and use $a + b + c = 0$, counts as a fine proof).
- Proving $k = 9$ works (writing something like: put $c = -(a + b)$ and just expand the 4th powers, counts as a fine proof).

They are scored as follows:

(i) **1 point** for any two of the three constructions

(ii) **2 points** for obtaining all three constructions.

For the rest of the proof (not additive with each other):

(iii) **0 points** for just $f(0) = 0$.

(iv) **0 points** for relating $f(x)$ to $f(-x)$, say by shifting.

(v) **2 points** for obtaining a recursion for $f(0), f(1), \dots$ in terms of k, such as $f(2x) = (2k-2)f(x)$ and $f(2x + 1) = \dots$, which in principle allows the computation of any $f(N)$ for any $N > 0$.

In rare cases, it's possible to award **1 point** for more generally showing that if $f(x)$ has certain "property" for enough small values of $|x|$, then $f(x)$ has that property for every value of x. This is in the same spirit as recursion, but less rigidly defined. The criteria for a "property" should be consulted with the problem captain, case-by-case.

(vi) **3 points** for computing $f(N)$ in terms of k for two odd values of $N \geq 5$. It's okay even if the polynomials is not completely correct (i.e. arithmetic errors).

(vii) **4 points** for obtaining a nonzero polynomial in k which equals 0, say by plugging in $(a, b, c) = (5, -3, -2)$. It's okay even if the polynomials is not completely correct (i.e. arithmetic errors), or the degree of the polynomial is greater than 3.

(viii) **5 points** for concluding $k \in \{1, 3, 9\}$ correctly.

4 Statistics

§4.1 Summary of scores for USEMO 2022

N	56	1st Q	2	Max	33
μ	9.09	Median	7	Top 3	22
σ	8.19	3rd Q	16	Top 12	17

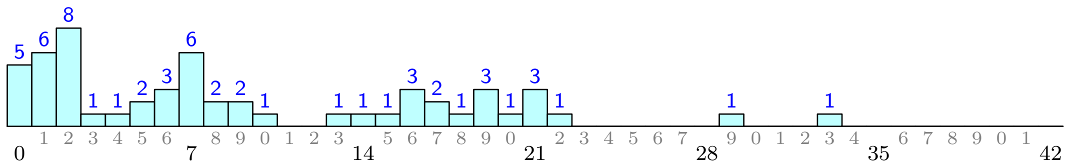
§4.2 Problem statistics for USEMO 2022

	P1	**P2**	**P3**	**P4**	**P5**	**P6**
0	14	43	51	29	45	43
1	10	3	0	4	3	6
2	11	0	0	0	1	2
3	1	1	0	0	0	0
4	1	2	0	3	0	0
5	1	0	1	0	0	1
6	1	0	1	4	3	1
7	17	7	3	16	4	3
Avg	3.02	1.12	0.57	2.71	0.91	0.75
QM	4.17	2.63	1.93	4.18	2.36	1.99
#5+	**19**	**7**	**5**	**20**	**7**	**5**
%5+	%33.9	%12.5	%8.9	%35.7	%12.5	%8.9

§4.3 Rankings for USEMO 2022

Sc	Num	Cu	Per	Sc	Num	Cu	Per	Sc	Num	Cu	Per
42	0	0	0.00%	28	0	2	3.57%	14	1	18	32.14%
41	0	0	0.00%	27	0	2	3.57%	13	1	19	33.93%
40	0	0	0.00%	26	0	2	3.57%	12	0	19	33.93%
39	0	0	0.00%	25	0	2	3.57%	11	0	19	33.93%
38	0	0	0.00%	24	0	2	3.57%	10	1	20	35.71%
37	0	0	0.00%	23	0	2	3.57%	9	2	22	39.29%
36	0	0	0.00%	22	1	3	5.36%	8	2	24	42.86%
35	0	0	0.00%	21	3	6	10.71%	7	6	30	53.57%
34	0	0	0.00%	20	1	7	12.50%	6	3	33	58.93%
33	1	1	1.79%	19	3	10	17.86%	5	2	35	62.50%
32	0	1	1.79%	18	1	11	19.64%	4	1	36	64.29%
31	0	1	1.79%	17	2	13	23.21%	3	1	37	66.07%
30	0	1	1.79%	16	3	16	28.57%	2	8	45	80.36%
29	1	2	3.57%	15	1	17	30.36%	1	6	51	91.07%
								0	5	56	100.00%

§4.4 Histogram for USEMO 2022



§4.5 Full stats for USEMO 2022

Rank	P1	P2	P3	P4	P5	P6	Σ
1.	7	0	6	7	7	6	33
2.	7	7	0	7	7	1	29
3.	7	1	0	7	0	7	22
4.	7	7	0	7	0	0	21
4.	7	0	7	7	0	0	21
4.	7	0	0	7	0	7	21
7.	7	7	0	6	0	0	20
8.	7	4	7	0	0	1	19
8.	6	7	0	0	6	0	19
8.	5	7	0	0	7	0	19
11.	7	3	0	7	1	0	18
12.	7	4	0	6	0	0	17
12.	7	0	0	4	6	0	17
14.	7	0	0	7	2	0	16
14.	7	0	0	1	7	1	16
14.	1	0	0	7	1	7	16
17.	7	0	0	7	0	1	15
18.	0	0	7	6	1	0	14
19.	1	0	0	6	6	0	13
20.	2	7	0	0	0	1	10
21.	7	0	0	0	0	2	9
21.	2	0	0	7	0	0	9
23.	1	0	0	7	0	0	8
23.	1	0	0	7	0	0	8
25.	7	0	0	0	0	0	7
25.	7	0	0	0	0	0	7
25.	0	7	0	0	0	0	7
25.	0	0	0	7	0	0	7
25.	0	0	0	7	0	0	7
25.	0	0	0	7	0	0	7
31.	2	0	0	4	0	0	6
31.	0	1	0	0	0	5	6
31.	0	0	5	1	0	0	6
34.	4	0	0	1	0	0	5
34.	0	1	0	4	0	0	5
36.	3	0	0	1	0	0	4
37.	1	0	0	0	0	2	3
38.	2	0	0	0	0	0	2

Rank	P1	P2	P3	P4	P5	P6	Σ
38.	2	0	0	0	0	0	2
38.	2	0	0	0	0	0	2
38.	2	0	0	0	0	0	2
38.	2	0	0	0	0	0	2
38.	2	0	0	0	0	0	2
38.	2	0	0	0	0	0	2
38.	2	0	0	0	0	0	2
46.	1	0	0	0	0	0	1
46.	1	0	0	0	0	0	1
46.	1	0	0	0	0	0	1
46.	1	0	0	0	0	0	1
46.	1	0	0	0	0	0	1
46.	0	0	0	0	0	1	1
52.	0	0	0	0	0	0	0
52.	0	0	0	0	0	0	0
52.	0	0	0	0	0	0	0
52.	0	0	0	0	0	0	0
52.	0	0	0	0	0	0	0