

JMO 2012 Solution Notes

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13 June 2024

This is a compilation of solutions for the 2012 JMO. The ideas of the solution are a mix of my own work, the solutions provided by the competition organizers, and solutions found by the community. However, all the writing is maintained by me.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let \mathbb{R} denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Given a triangle ABC , let P and Q be points on segments \overline{AB} and \overline{AC} , respectively, such that $AP = AQ$. Let S and R be distinct points on segment \overline{BC} such that S lies between B and R , $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.
2. Find all integers $n \geq 3$ such that among any n positive real numbers a_1, a_2, \dots, a_n with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

3. For $a, b, c > 0$ prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2).$$

4. Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \geq 3$, define a sequence of points P_1, P_2, \dots, P_n as follows. First select any point P_1 on the circle, and for $2 \leq k \leq n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α , when travelling counterclockwise around the circle from P_{k-1} to P_k . Suppose that P_a and P_b are the nearest adjacent points on either side of P_n . Prove that $a + b \leq n$.
5. For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \leq k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S .
6. Let P be a point in the plane of $\triangle ABC$, and γ a line through P . Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

§1 Solutions to Day 1

§1.1 JMO 2012/1, proposed by Sungyoon Kim, Inseok Seo

Available online at <https://aops.com/community/p2669111>.

Problem statement

Given a triangle ABC , let P and Q be points on segments \overline{AB} and \overline{AC} , respectively, such that $AP = AQ$. Let S and R be distinct points on segment \overline{BC} such that S lies between B and R , $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P , Q , R , S are concyclic.

Assume for contradiction that (PRS) and (QRS) are distinct. Then \overline{RS} is the radical axis of these two circles. However, \overline{AP} is tangent to (PRS) and \overline{AQ} is tangent to (QRS) , so point A has equal power to both circles, which is impossible since A does not lie on line BC .

§1.2 JMO 2012/2, proposed by Titu Andreescu

Available online at <https://aops.com/community/p2669112>.

Problem statement

Find all integers $n \geq 3$ such that among any n positive real numbers a_1, a_2, \dots, a_n with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

The answer is all $n \geq 13$.

Define (F_n) as the sequence of Fibonacci numbers, by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We will find that Fibonacci numbers show up naturally when we work through the main proof, so we will isolate the following calculation now to make the subsequent solution easier to read.

Claim — For positive integers m , we have $F_m \leq m^2$ if and only if $m \leq 12$.

Proof. A table of the first 14 Fibonacci numbers is given below.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
1	1	2	3	5	8	13	21	34	55	89	144	233	377

By examining the table, we see that $F_m \leq m^2$ is true for $m = 1, 2, \dots, 12$, and in fact $F_{12} = 12^2 = 144$. However, $F_m > m^2$ for $m = 13$ and $m = 14$.

Now it remains to prove that $F_m > m^2$ for $m \geq 15$. The proof is by induction with base cases $m = 13$ and $m = 14$ being checked already. For the inductive step, if $m \geq 15$ then we have

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} > (m-1)^2 + (m-2)^2 \\ &= 2m^2 - 6m + 5 = m^2 + (m-1)(m-5) > m^2 \end{aligned}$$

as desired. \square

We now proceed to the main problem. The hypothesis $\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n)$ will be denoted by (\dagger) .

Proof that all $n \geq 13$ have the property. We first show now that every $n \geq 13$ has the desired property. Suppose for contradiction that no three numbers are the sides of an acute triangle. Assume without loss of generality (by sorting the numbers) that $a_1 \leq a_2 \leq \dots \leq a_n$. Then since a_{i-1}, a_i, a_{i+1} are not the sides of an acute triangle for each $i \geq 2$, we have that $a_{i+1}^2 \geq a_i^2 + a_{i-1}^2$; writing this out gives

$$\begin{aligned} a_3^2 &\geq a_2^2 + a_1^2 \geq 2a_1^2 \\ a_4^2 &\geq a_3^2 + a_2^2 \geq 2a_1^2 + a_1^2 = 3a_1^2 \\ a_5^2 &\geq a_4^2 + a_3^2 \geq 3a_1^2 + 2a_1^2 = 5a_1^2 \\ a_6^2 &\geq a_5^2 + a_4^2 \geq 5a_1^2 + 3a_1^2 = 8a_1^2 \end{aligned}$$

and so on. The Fibonacci numbers appear naturally and by induction, we conclude that $a_i^2 \geq F_i a_1^2$. In particular, $a_n^2 \geq F_n a_1^2$.

However, we know $\max(a_1, \dots, a_n) = a_n$ and $\min(a_1, \dots, a_n) = a_1$, so (\dagger) reads $a_n \leq n \cdot a_1$. Therefore we have $F_n \leq n^2$, and so $n \leq 12$, contradiction!

Proof that no $n \leq 12$ have the property. Assume that $n \leq 12$. The above calculation also suggests a way to pick the counterexample: we choose $a_i = \sqrt{F_i}$ for every i . Then $\min(a_1, \dots, a_n) = a_1 = 1$ and $\max(a_1, \dots, a_n) = \sqrt{F_n}$, so (\dagger) is true as long as $n \leq 12$. And indeed no three numbers form the sides of an acute triangle: if $i < j < k$, then $a_k^2 = F_k = F_{k-1} + F_{k-2} \geq F_j + F_i = a_j^2 + a_i^2$.

§1.3 JMO 2012/3, proposed by Titu Andreescu

Available online at <https://aops.com/community/p2669114>.

Problem statement

For $a, b, c > 0$ prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geq \frac{2}{3}(a^2 + b^2 + c^2).$$

Here are two possible approaches.

¶ **Cauchy-Schwarz approach.** Apply Titu lemma to get

$$\sum_{\text{cyc}} \frac{a^3}{5a + b} = \sum_{\text{cyc}} \frac{a^4}{5a^2 + ab} \geq \frac{(a^2 + b^2 + c^2)^2}{\sum_{\text{cyc}} (5a^2 + ab)} \geq \frac{a^2 + b^2 + c^2}{6}$$

where the last step follows from the identity $\sum_{\text{cyc}} (5a^2 + ab) \leq 6(a^2 + b^2 + c^2)$.

Similarly,

$$\sum_{\text{cyc}} \frac{b^3}{5a + b} = \sum_{\text{cyc}} \frac{b^4}{5ab + b^2} \geq \frac{(a^2 + b^2 + c^2)^2}{\sum_{\text{cyc}} (5ab + b^2)} \geq \frac{a^2 + b^2 + c^2}{6}$$

using the fact that $\sum_{\text{cyc}} 5ab + b^2 \leq 6(a^2 + b^2 + c^2)$.

Therefore, adding the first display to three times the second display implies the result.

¶ **Second linearization approach.** The main magical claim is:

Claim — We have

$$\frac{a^3 + 3b^3}{5a + b} \geq \frac{25}{36}b^2 - \frac{1}{36}a^2.$$

Proof. Let $x = a/b > 0$. The desired inequality is equivalent to

$$\frac{x^3 + 3}{5x + 1} \geq \frac{25 - x^2}{36}.$$

However,

$$\begin{aligned} 36(x^3 + 3) - (5x + 1)(25 - x^2) &= 41x^3 + x^2 - 125x + 83 \\ &= (x - 1)^2(41x + 83) \geq 0. \end{aligned}$$

□

Sum the claim cyclically to finish.

Remark (Derivation of the main claim). The overall strategy is to hope for a constant k such that

$$\frac{a^3 + 3b^3}{5a + b} \geq ka^2 + \left(\frac{2}{3} - k\right)b^2.$$

is true. Letting $x = a/b$ as above and expanding, we need a value k such that the cubic polynomial

$$P(x) := (x^3 + 3) - (5x + 1)\left(kx^2 + \left(\frac{2}{3} - k\right)\right) = (1 - 5k)x^3 - kx^2 + \left(5k - \frac{10}{3}\right)x + \left(k + \frac{7}{3}\right)$$

is nonnegative everywhere. Since $P(1) = 0$ necessarily, in order for $P(1 - \varepsilon)$ and $P(1 + \varepsilon)$ to both be nonnegative (for small ε), the polynomial P must have a double root at 1, meaning the first derivative $P'(1) = 0$ needs to vanish. In other words, we need

$$3(1 - 5k) - 2k + \left(5k - \frac{10}{3}\right) = 0.$$

Solving gives $k = -1/36$. One then factors out the repeated root $(x - 1)^2$ from the resulting P .

§2 Solutions to Day 2

§2.1 JMO 2012/4, proposed by Sam Vandervelde

Available online at <https://aops.com/community/p2669956>.

Problem statement

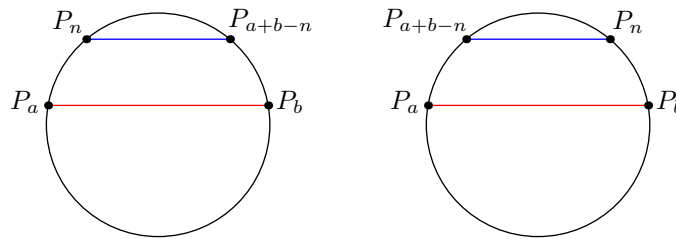
Let α be an irrational number with $0 < \alpha < 1$, and draw a circle in the plane whose circumference has length 1. Given any integer $n \geq 3$, define a sequence of points P_1, P_2, \dots, P_n as follows. First select any point P_1 on the circle, and for $2 \leq k \leq n$ define P_k as the point on the circle for which the length of arc $P_{k-1}P_k$ is α , when travelling counterclockwise around the circle from P_{k-1} to P_k . Suppose that P_a and P_b are the nearest adjacent points on either side of P_n . Prove that $a + b \leq n$.

No points coincide since α is irrational.

Assume for contradiction that $n < a + b < 2n$. Then it follows that

$$\overline{P_n P_{a+b-n}} \parallel \overline{P_a P_b}$$

as shown below.



This is an obvious contradiction since then P_{a+b-n} is contained in the arc $\widehat{P_a P_b}$ of the circle through P_n .

§2.2 JMO 2012/5, proposed by Warut Suksompong

Available online at <https://aops.com/community/p2669967>.

Problem statement

For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \leq k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S .

The answer is $S = 502$ (not 503!).

Claim — If $\gcd(k, 2012) = 1$, then necessarily either k or $2012 - k$ will count towards S .

Proof. First note that both ak, bk are nonzero modulo 2012. Note also that $ak \not\equiv bk \pmod{2012}$.

So if r_a is the remainder of $ak \pmod{2012}$, then $2012 - r_a$ is the remainder of $a(2012 - k) \pmod{2012}$. Similarly we can consider r_b and $2012 - r_b$. As mentioned already, we have $r_a \neq r_b$. So either $r_a > r_b$ or $2012 - r_a > 2012 - r_b$. \square

This implies $S \geq \frac{1}{2}\varphi(2012) = 502$.

But this can actually be achieved by taking $a = 4$ and $b = 1010$, since

- If k is even, then $ak \equiv bk \pmod{2012}$ so no even k counts towards S ; and
- If $k \equiv 0 \pmod{503}$, then $ak \equiv 0 \pmod{2012}$ so no such k counts towards S .

This gives the final answer $S \geq 502$.

Remark. A similar proof works with 2012 replaced by any n and will give an answer of $\frac{1}{2}\varphi(n)$. For composite n , one uses the Chinese remainder theorem to pick distinct a and b not divisible by n such that $\text{lcm}(a - b, a) = n$.

§2.3 JMO 2012/6, proposed by Titu Andreescu, Cosmin Pohoata

Available online at <https://aops.com/community/p2669960>.

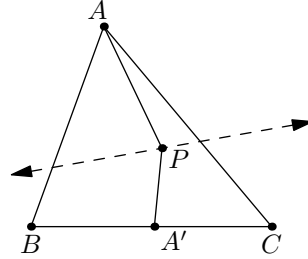
Problem statement

Let P be a point in the plane of $\triangle ABC$, and γ a line through P . Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

We present three solutions.

¶ **First solution (complex numbers).** Let $p = 0$ and set γ as the real line. Then A' is the intersection of bc and $p\bar{a}$. So, we get

$$a' = \frac{\bar{a}(\bar{b}c - b\bar{c})}{(\bar{b} - \bar{c})\bar{a} - (b - c)a}.$$



Note that

$$\bar{a}' = \frac{a(b\bar{c} - \bar{b}c)}{(b - c)a - (\bar{b} - \bar{c})\bar{a}}.$$

Thus it suffices to prove

$$0 = \det \begin{bmatrix} \frac{\bar{a}(\bar{b}c - b\bar{c})}{(\bar{b} - \bar{c})\bar{a} - (b - c)a} & \frac{a(b\bar{c} - \bar{b}c)}{(b - c)a - (\bar{b} - \bar{c})\bar{a}} & 1 \\ \frac{b(\bar{c}a - c\bar{a})}{(\bar{c} - \bar{a})\bar{b} - (c - a)b} & \frac{b(c\bar{a} - \bar{c}a)}{(c - a)b - (\bar{c} - \bar{a})\bar{b}} & 1 \\ \frac{c(\bar{a}b - a\bar{b})}{(\bar{a} - \bar{b})\bar{c} - (a - b)c} & \frac{c(a\bar{b} - \bar{a}b)}{(a - b)c - (\bar{a} - \bar{b})\bar{c}} & 1 \end{bmatrix}.$$

This is equivalent to

$$0 = \det \begin{bmatrix} \bar{a}(\bar{b}c - b\bar{c}) & a(\bar{b}c - b\bar{c}) & (\bar{b} - \bar{c})\bar{a} - (b - c)a \\ \bar{b}(\bar{c}a - c\bar{a}) & b(\bar{c}a - c\bar{a}) & (\bar{c} - \bar{a})\bar{b} - (c - a)b \\ \bar{c}(\bar{a}b - a\bar{b}) & c(\bar{a}b - a\bar{b}) & (\bar{a} - \bar{b})\bar{c} - (a - b)c \end{bmatrix}.$$

This determinant has the property that the rows sum to zero, and we're done.

Remark. Alternatively, if you don't notice that you could just blindly expand:

$$\begin{aligned} & \sum_{\text{cyc}} ((\bar{b} - \bar{c})\bar{a} - (b - c)a) \cdot -\det \begin{bmatrix} b & \bar{b} \\ c & \bar{c} \end{bmatrix} (\bar{c}a - c\bar{a})(\bar{a}b - a\bar{b}) \\ &= (\bar{b}c - c\bar{b})(\bar{c}a - c\bar{a})(\bar{a}b - a\bar{b}) \sum_{\text{cyc}} (ab - ac + \bar{c}a - \bar{b}a) = 0. \end{aligned}$$

¶ **Second solution (Desargues involution).** We let $C'' = \overline{A'B'} \cap \overline{AB}$. Consider complete quadrilateral $ABCA'B'C''C$. We see that there is an involutive pairing τ at P swapping (PA, PA') , (PB, PB') , (PC, PC'') . From the first two, we see τ coincides with reflection about ℓ , hence conclude $C'' = C$.

¶ **Third solution (barycentric), by Catherine Xu.** We will perform barycentric coordinates on the triangle PCC' , with $P = (1, 0, 0)$, $C' = (0, 1, 0)$, and $C = (0, 0, 1)$. Set $a = CC'$, $b = CP$, $c = C'P$ as usual. Since A, B, C' are collinear, we will define $A = (p : k : q)$ and $B = (p : \ell : q)$.

Claim — Line γ is the angle bisector of $\angle APA'$, $\angle BPB'$, and $\angle CPC'$.

Proof. Since $A'P$ is the reflection of AP across γ , etc. □

Thus B' is the intersection of the isogonal of B with respect to $\angle P$ with the line CA ; that is,

$$B' = \left(\frac{p}{k} \frac{b^2}{\ell} : \frac{b^2}{\ell} : \frac{c^2}{q} \right).$$

Analogously, A' is the intersection of the isogonal of A with respect to $\angle P$ with the line CB ; that is,

$$A' = \left(\frac{p}{\ell} \frac{b^2}{k} : \frac{b^2}{k} : \frac{c^2}{q} \right).$$

The ratio of the first to third coordinate in these two points is both $b^2 pq : c^2 k \ell$, so it follows A', B' , and C' are collinear.

Remark (Problem reference). The converse of this problem appears as problem 1052 attributed S. V. Markelov in the book *Geometriya: 9–11 Klassy: Ot Uchebnoy Zadachi k Tvorcheskoy, 1996*, by I. F. Sharygin.