Collisions of localized patterns in a nonvariational Swift-Hohenberg equation

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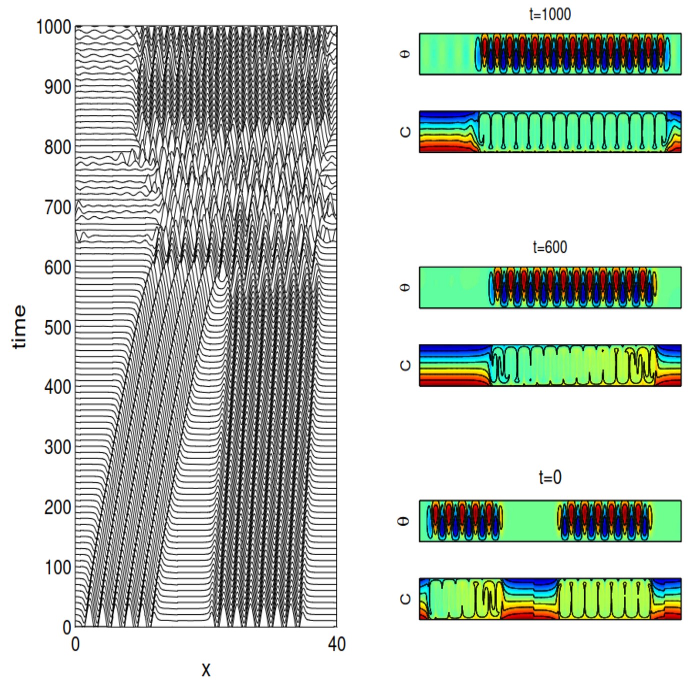
Introduction

Spatially localized structures arise in a wide variety of physical systems, from solitary water waves to neurons, fluid convection, shear flows, and reaction-diffusion systems, to name a few. The bistable Swift-Hohenberg equation (1) is a paradigmatic model of pattern formation, proposed in the context of binary fluid convection [1]. We study [2] the 1D cubic-quintic Swift-Hohenberg equation (SH35) with broken spatial reflection symmetry [3]

Convection cells in experiment [4]



Collision of localized binary convection patterns [1].

$$\partial_t u = ru - (1+\partial_x^2)^2 u + b_3 u^3 - u^5 + \epsilon(\partial_x u)^2. \qquad (1)$$

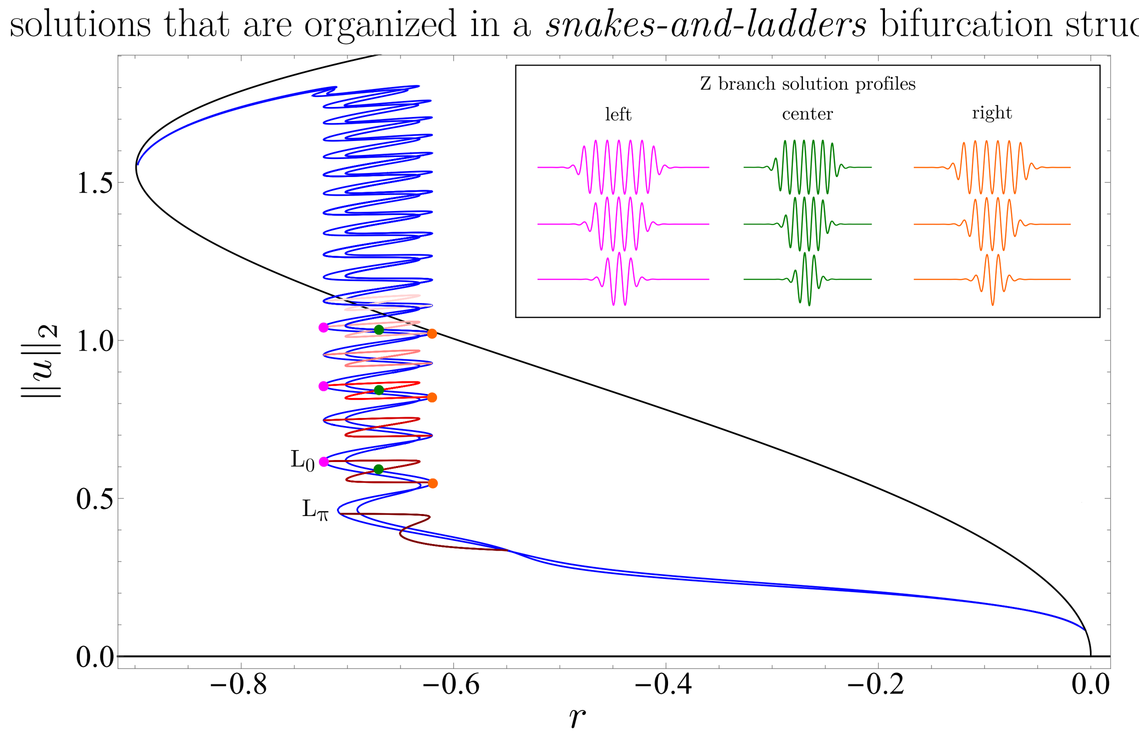
Note that a non-zero ϵ breaks the $u \to -u$ symmetry and the variational structure while maintaining the $x \to -x$ symmetry. To understand the general Swift-Hohenberg behavior ($\epsilon = 0$ case), we also define a free energy functional:

$$\mathcal{F}[u(x)] = \int_0^L \left(-\frac{1}{2}ru^2 + \frac{1}{2}[(1+\partial_x^2)u]^2 - \frac{b_3}{4}u^4 + \frac{u^6}{6} \right) dx$$

with the property that $\partial_t u = -\frac{\delta \mathcal{F}}{\delta u}$. The free energy of spatially periodic patterns equals that of the trivial state at a $r = r_M \approx 0.6775$ in SH35, known as the *Maxwell point* [5].

Bifurcation structure: what types of solutions exist and where?

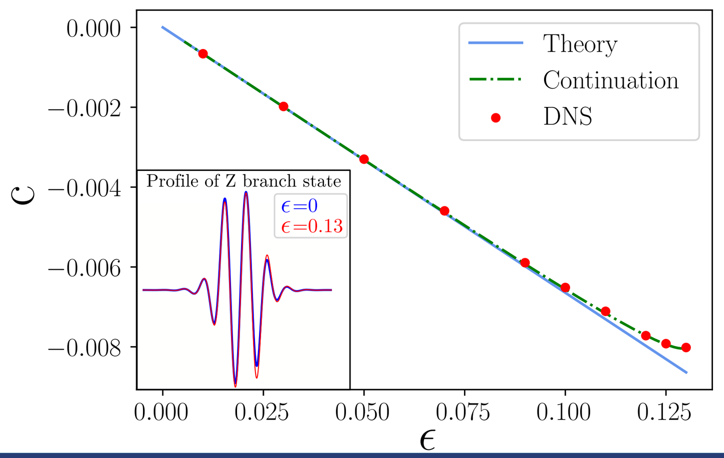
In this project, we consider $b_3 = 2$, $\epsilon = 0.03$, and vary r. This leads to the bifurcation structure which reveals localized solutions that are organized in a *snakes-and-ladders* bifurcation structure [5].



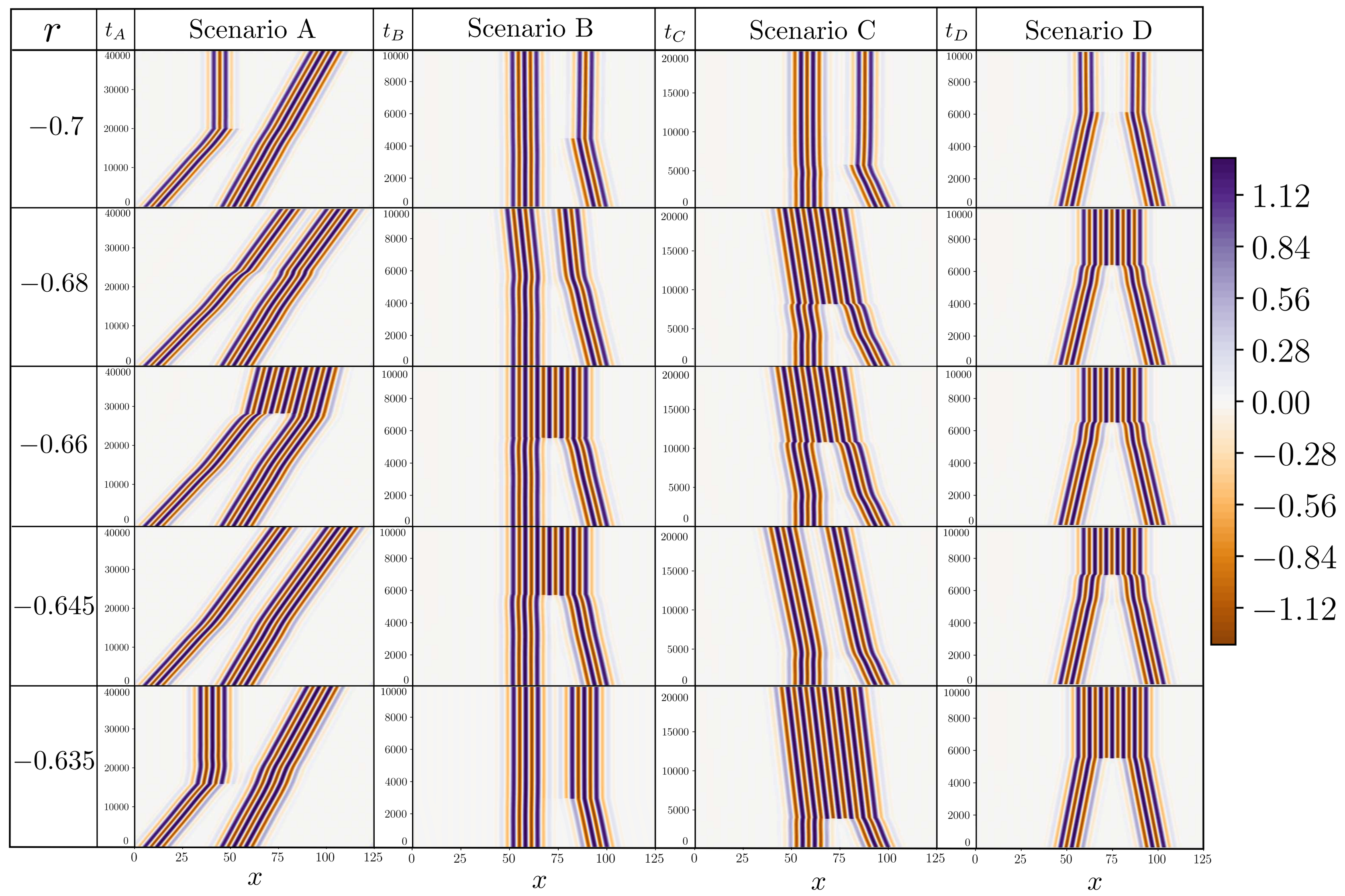
Asymptotic theory of propagation speeds

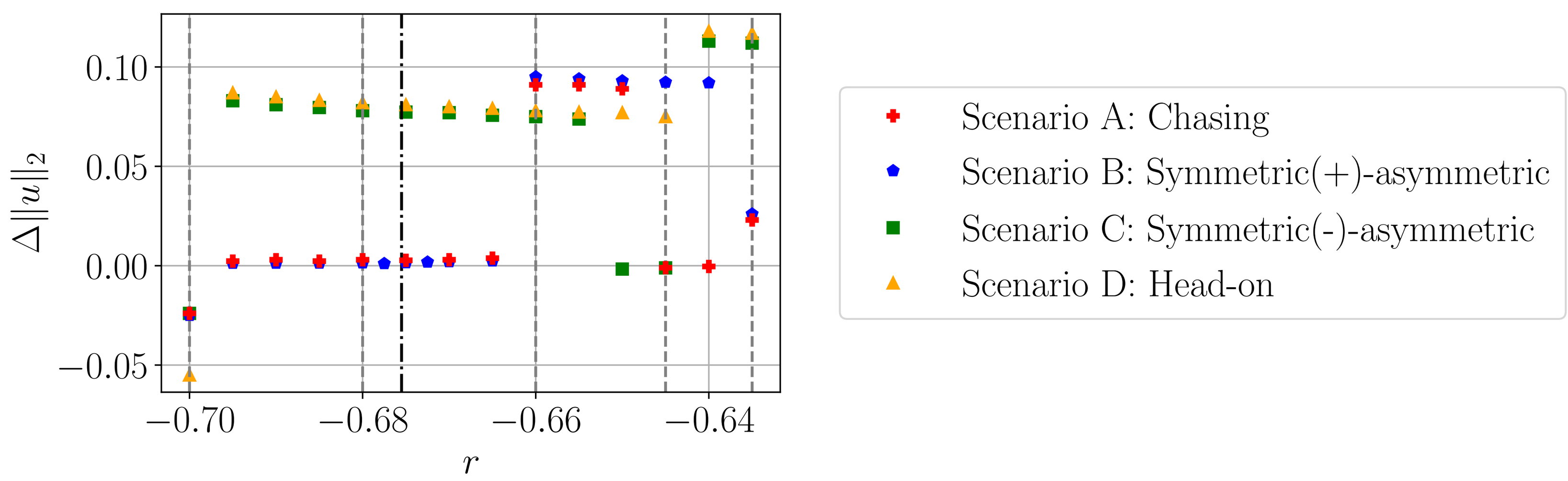
We introduce the fast and slow time variables $\tau = \epsilon^{\frac{1}{2}}t$, $T = \epsilon t$ and denote their spatial phase by $\theta(T)$. Using these, we use multiscale asymptotic theory to arrive at a solvability condition which, for a given profile $U_0(x)$ yields the following propagation speed

$$c \equiv \theta_T = -\epsilon \frac{\int_0^L (U_0'(x))^3 \, dx}{\int_0^L (U_0'(x))^2 \, dx}$$

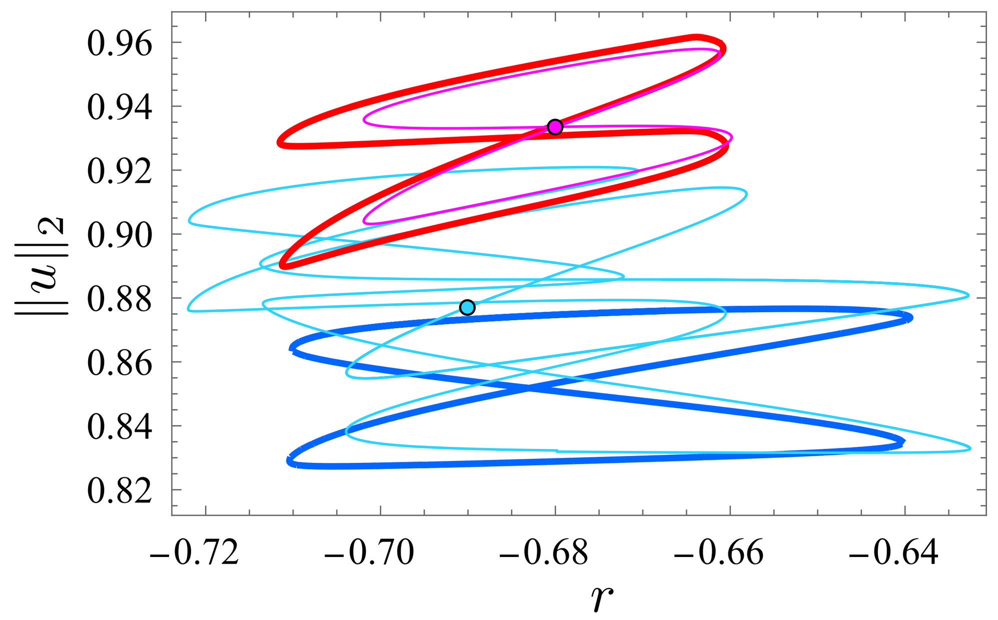


Inelastic Collisions of Localized Structures





Vertical dashed gray lines correspond to r values shown in the table above. Vertical black dash-dotted line shows location of Maxwell point r_M for $\epsilon = 0$ (all remaining results shown are for $\epsilon = 0.03$).



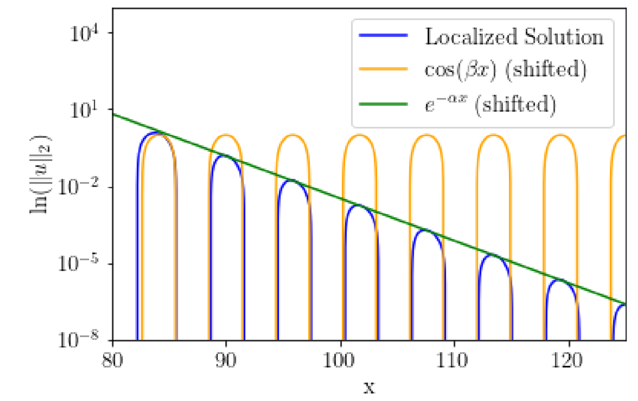
$$\|u\|_2 \equiv \sqrt{\frac{1}{L}\int_0^L u^2(x)dx}$$

Thin pink line: traveling state obtained from scenario A at $r = -0.68$. Thin light blue line: similar traveling state obtained from scenario B at $r = -0.69$. Thick red line: same state as thin pink but continued to $\epsilon = 0$. blue line: same state as light blue continued to $\epsilon = 0$.

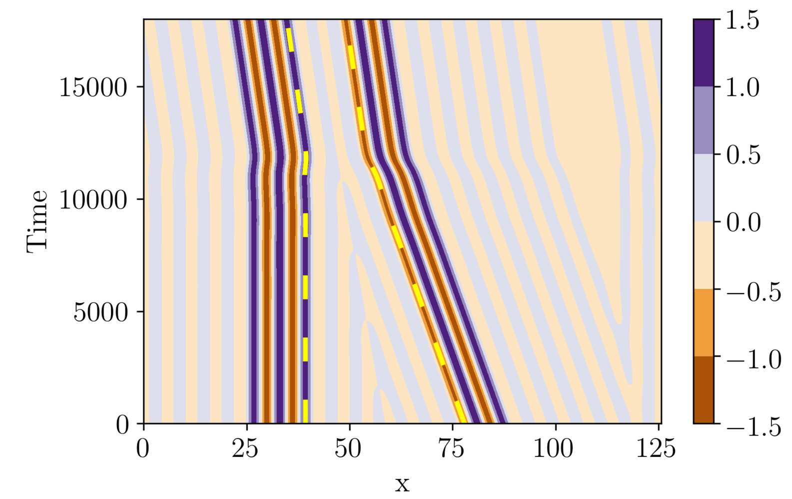
A Reduced Model

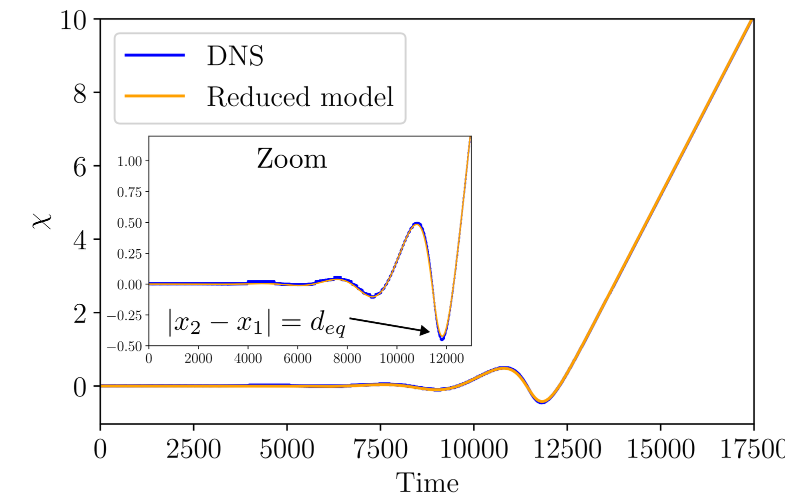
We propose reduced model equations, reminiscent of overdamped particle dynamics and incorporating spatial eigenvalues into α (real part) and β (imaginary part):

$$\frac{dx_1}{dt} = c_1 + g_1 \cos(\beta|x_1 - x_2| - \phi)\, e^{-\alpha|x_1 - x_2|}$$
$$\frac{dx_2}{dt} = c_2 + g_2 \cos(\beta|x_1 - x_2| - \phi)\, e^{-\alpha|x_1 - x_2|}$$



Using a 4th-order Runge-Kutta method and gradient descent to fit unknown model parameters g_i, ϕ (all others known), we validate this model as shown for Scenario B at $r = -0.69$. We use $\chi(t) \equiv x_2(t) - x_1(t) - (c_2 - c_1)t - [x_2(0) - x_1(0)]$ to measure interactions as the deviation from free propagation.



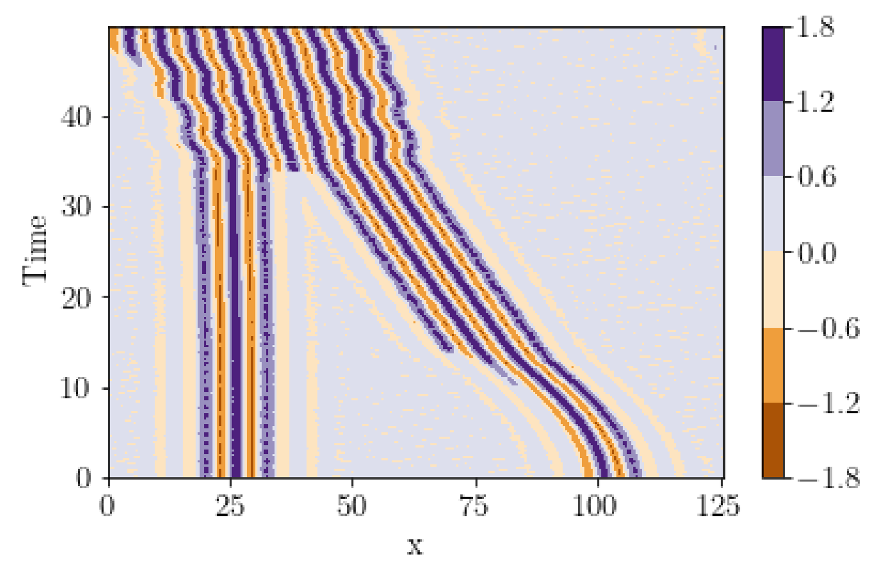


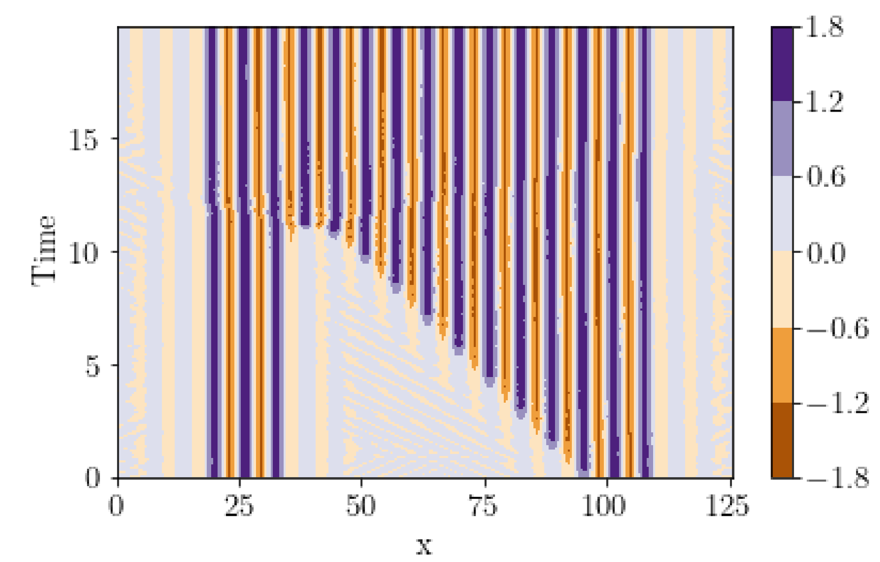
The model works well for pure bound states, but fails to capture the formation/deletion of wavelengths.

Second-order Time Dependence

Introducing wavelike solutions by modifying SH35, we see a rich variety of possibilities such as the following wave-like depinning effect seen at $r = -0.635$ and $\lambda = 1$ (purely second-order time dependence).

$$(1-\lambda)\partial_t u + \lambda \partial_t^2 u = ru - (1+\partial_x^2)^2 u + b_3 u^3 - u^5 + \epsilon(\partial_x u)^2$$





References

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