Rouché's Theorem

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1 Introduction/Preliminaries

Argument Principle

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \sum_a n(C,a) - \sum_p n(C,p) \tag{1}$$

where $n(C, z)$ signifies the **winding number** of C around z and we sum over all zeros a (with multiplicity) and sum over all poles p (with multiciplicity) [1].

Definition: We define the *winding number* $n(C, z)$ to be the number of times a curve C travels around a point z.

Note: we can interpret the argument principle as $\oint_C \frac{f'(z)}{f(z)}\,dz$ representing $2\pi i$ times the winding number of $f(C)$ around the origin, so in in Eq. 1, the left hand side simply denotes $n(f(C), 0)$ [1]. (This representation will be used more often throughout the rest of the paper)

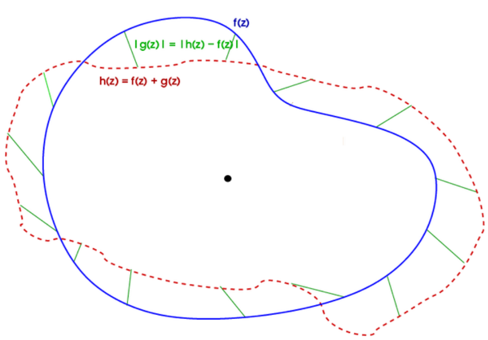
1.1 Rouché's Theorem

Rouché's Theorem, a theorem used to easily find the location of zeros of analytic functions, also uses the argument principle in its proof and also can be used to prove the Fundamental Theorem of Algebra as we will seee.

Rouché's Theorem: *Given some simple closed contour C, for any two complex-valued functions $f(z)$ and $g(z)$ that are analytic inside and on C, if $|f(z)| > |g(z)|$ at every point on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros (including multiplicity) inside C.*

1.2 An Intuitive Explanation of Rouché's Theorem

While a formal proof of Rouché's theorem can be found on the internet quite easily, we can also use the argument principle to give an intuitive interpretation of Rouché's theorem.



If we let $f(z)$ and $g(z)$ be two complex-valued analytic functions inside and on some closed curve C such that $h(z) = f(z) + g(z)$, as we follow z along some curve C, $f(z)$ and $g(z)$ draw closed curves in the complex plane (example curves are shown in the figure [2] to the left). If we require that $f(z)$ is closer to $g(z)$ than the origin at all times as seen in the figure (also required by Rouché's theorem via the condition $|f(z)| > |g(z)|$), then the curves will have the same winding number and by the argument principle, $f(z)$ and $f(z) + g(z) = h(z)$ will have the same number of zeros inside C.

2 Applications

2.1 A Short Example of Bounding Roots

Calculating exact roots can many times be very difficult, but even if we cannot calculate them exactly, deriving a bound on the roots helps immensely with approximating and understanding the behavior of polynomials. Rouché's theorem can help us bound the roots as such which can be clearly seen using the following example:

Example 2.1.1 How many roots does $z^5 + 3z^2 + 1$ have in the open annulus defined by $1 < |z| < 2$ [3]?

Attempting to use Rouché's theorem, if we can try to split our polynomial into two functions $f(z)$ and $g(z)$ where $|f(z)| > |g(z)|$ so that we can apply Rouché's theorem. Looking at cases:

1. Take C such that $|z| = 1$: we separate it into $f(z) = 3z^2$ and $g(z) = z^5 + 1$. Also note that on C, $f(z) = |3z^2| = 3 > 2 = |z|^5 + 1 \geq |z^5 + 1| = |g(z)|$, so we can apply Rouché's theorem in this curve to see that since $f(z)$ has a trivial zero at the origin with multiplicity two, $f(z) + g(z)$ also has two roots.

2. If we take C such that $|z| = 2$: we separate it into $g(z) = 3z^2 + 1$ and $f(z) = z^5$. Also note that $f(z) = |z^5| \geq 32 > 13 = 3|z|^2 + 1 \geq |3z^2 + 1| = |g(z)|$, so we can apply Rouché's theorem in this curve to see that since $f(z)$ has a trivial zero at the origin with multiplicity 5 now, $f(z) + g(z)$ also has 5 roots.

3. Checking to make sure $|z| = 1$ doesn't have roots, we simply plug it into our polynomial and see $5 \neq 0$ so it doesn't have a root on $|z| = 1$.

Thus looking at all the cases, we since our polynomial has 5 roots within the open disk of radius 2 centered at the origin, only has 2 roots within the open disk of radius 1 centered at the origin, and has no roots on the circle of radius 1 centered at the origin, it has exactly 3 roots in the annulus defined by $1 < |z| < 2$.

2.2 Proof of the Fundamental Theorem of Algebra

In class and in homework 8/11, we briefly discussed the Fundamental Theorem of Algebra, but here, we will prove an equivalent formulation stated as such:

The Fundamental Theorem of Algebra: If there exists a degree-n polynomial, $P(z) = a_0 + a_1 z + a_2 z^2 \cdots a_n z^n$ with complex coefficients, $a_n \neq 0$, and $n \in \mathbb{Z}^+$, then $P(z)$ has n complex roots (including multiplicities).

We begin our proof by letting $P(z) = f(z) + g(z)$ where $f(z) = a_n z^n$, $g(z) = \sum_{i=0}^{n-1} a_i z^i$, and C be a circle in the complex plane such that $|z| = R >> 1$ [4]. So, parametrizing along the curve C, we see that

$$|f(z)| = |a_n| R^n \text{ and } |g(z)| = \left| \sum_{i=0}^{n-1} a_i z^i \right| \leq \sum_{i=0}^{n-1} |a_i z^i| = \sum_{i=0}^{n-1} |a_i||z^i|$$

using the triangle inequality. Since we are traveling along C, we can substitute $|z| = R$ to find that we get $|g(z)| \leq \sum_{i=0}^{n-1} |a_i| R^i$. Now, since $|f(z)|$ is always positive ($R >> 1$), we can divide our inequality for $|g(z)|$ by $|f(z)|$ to get (writing out the terms):

$$\frac{|g(z)|}{|f(z)|} \leq \frac{|a_0| + |a_1|R + |a_2|R^2 + \cdots + |a_{n-1}|R^{n-1}}{|a_n|R^n}$$

We can also take R to be a large value such that $R > \frac{\sum_{i=0}^{n-1} |a_i|}{|a_n|}$, which if we rearrange, $|a_n|R > \sum_{i=0}^{n-1} |a_i|$ so $\frac{|a_0| + |a_1|R + |a_2|R^2 + \cdots + |a_{n-1}|R^{n-1}}{|a_n|R^n} < 1$, so $\frac{|g(z)|}{|f(z)|} < 1$ meaning that $|f(z)| > |g(z)|$ and we have thus fulfilled the conditions for Rouché's theorem to be applied:

- C is a simple closed curve
- $f(z)$ and $g(z)$ are analytic on and inside C (since they are represented using power series).
- $|f(z)| > |g(z)|$ at every point on C.

Applying Rouché's theorem, we see that this means $f(z)$ and $f(z) + g(z)$ have the same number of zeros (with multiplicities). Looking at the form of $f(z) = a_n z^n$, we trivially have a zero with multiplicity n at the origin. Thus, $f(z)$ has n zeros meaning that $f(z) + g(z) = P(z)$ must have the same number of zeros, so $P(z)$ has n zeros and we have proved the Fundamental Theorem of Algebra.

References

[1] Peter Duren, Walter Hengartner, and Richard S. Laugesen. The argument principle for harmonic functions. *The American Mathematical Monthly*, 103(5):411–415, 1996.

[2] Rouché's Theorem. Rouché's theorem — Wikipedia, the free encyclopedia, 2023. [Online; accessed 3-May-2023].

[3] P. Martin. A note on rouche's theorem. *International Journal of Mathematical Education in Science and Technology*, 27(6):931–948, 1996.

[4] Faculty of Khan. Rouche's Theorem and the Fundamental Theorem of Algebra, September 2019.