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Lab session # 2
ALTEGRAD 2023

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1 Question 1

I see three ways to enumerate the maximum number of edges.

- for every vertex we count the number of edges we can add without doubling them. Starting with the first, we count n possible edges, then (n-1) because the second vertex is already linked to the first, etc... which lead to the equation
$$\sum_{i=1}^{n} i = \frac{n(n-1)}{2} \tag{1}$$
- we make every edges between every pairs of edges. Each of the n edges can be linked to (n-1) other edeges, but we need to divide n(n-1) by 2 because we counted every edges two times. So we also get
$$\frac{n(n-1)}{2} \tag{2}$$
- we can just use the binomial coefficient
$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} \tag{3}$$

To count the maximum number of triangles we just need to use the binomial coefficient
$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6} \tag{4}$$

2 Question 2

G_1 and G_2 having identical degree distribution does not imply that $G_1 \cong G_2$. We can construct a counter example by remembering that two isomorphic graphs has the same number of cycles. The simplest think to make a graph without cycle and another with one. Here it is:

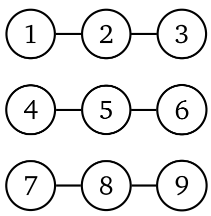

Figure 1: G_1

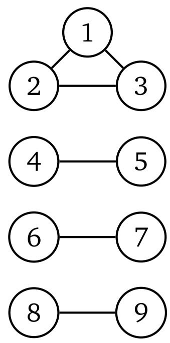

Figure 2: G_1

An interesting thing to think about is that even if two graphs has the same number of cycles, if the cycles are not the same length then the graphs are not isomorphic. We could then think of another meaningful example that shows this property.

3 Question 3

We distingish two cases.

- if $n = 3$, C_3 is indeed a closed triangle so $gcc(C_3) = \frac{1}{1} = 1$

- if $n \geq 4$, C_n does not contain any triangle then $gcc(C_n) = 0$

Let's briefly proove that for $n \geq 4$, C_n does not contain any closed triangle. Suppose there is a closed triangle is C_n, then each of the 3 vertices of the closed triangle has 2 neighbours. But the graph is connex and has at least 4 vertices so we need to link a fourth vertex to one of the 3 vertices of the closed triangle, then one of the 3 vertices of the closed triangle get a third neighbour which contradicts the fact that every vertex of a cycle graph has exactly 2 neighbours.

4 Question 4

We know that L_{rw} is semi-definite positive then its eigenvalues are ≥ 0. We also know that, as every line of L_{rw} sums to 0, $(1 \dots 1)^T$ is the eigenvector associated to the eigenvalue 0 which is the smallest. Then in question 4., $u_1 = (1 \dots 1)$. Therefore,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}([u_1]_i - [u_1]_j)^2 = 0 \tag{5}$$

5 Question 5

Here are the calculations of Q_1 for the left graph modularity and Q_2 for the right one:

$$Q_1 = \left(\frac{6}{14} - \left(\frac{14}{2 \times 14} \right)^2 \right) + \left(\frac{6}{14} - \left(\frac{14}{2 \times 14} \right)^2 \right) \approx 0.357 \tag{6}$$

$$Q_2 = \left(\frac{5}{14} - \left(\frac{17}{2 \times 14} \right)^2 \right) + \left(\frac{2}{14} - \left(\frac{11}{2 \times 14} \right)^2 \right) \approx -0,023 \tag{7}$$

6 Question 6

Here are the shortest path kernel for the pairs (P4,P4), (P4,S4) and (S4,S4):

- $\phi(P_4) = (3, 2, 1, 0)$ then,

$$k(P_4, P_4) = < \phi(P_4), \phi(P_4) >= 3^2 + 2^2 + 1^2 + 0 = 14 \tag{8}$$

- $\phi(S_4) = (3, 3, 0, 0)$ then,

$$k(P_4, S_4) = < \phi(P_4), \phi(S_4) >= 9 + 6 = 15 \tag{9}$$

- And,

$$k(S_4, S_4) = < \phi(S_4), \phi(S_4) >= 9 + 9 = 18 \tag{10}$$

7 Question 7

$$k(G, G') = f_G^T f_{G'} = < f_G, f_{G'} >= < \phi(G), \phi(G') >$$
$$\phi(G) = f_G = \left(f_G^{(1)}, f_G^{(2)}, f_G^{(3)}, f_G^{(4)} \right) \in \mathbb{N}^4$$

where $f_G^{(i)} = \# \{\text{subgraphs of } G \text{ that are isomorphic to } graphlet_i\}$.
$f_G^{(i)}$ are cardinals so they obviously are ≥ 0. That means

$$< f_G^T, f_{G'} >= \sum_{i=1}^{4} f_G^{(i)} f_{G'}^{(i)} = 0 \implies \forall i = 1, \dots, 4, f_G^{(i)} f_{G'}^{(i)} = 0$$

$\iff \forall i = 1, \dots, 4, f_G^{(i)} = 0$ or $f_{G'}^{(i)} = 0$
$\iff \forall i = 1, \dots, 4,\ graphlet_i$ cannot be isomorphic to a subgraph of G and to a subgraph of G'
Here is an example:

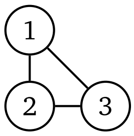

Figure 3: G

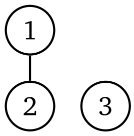

Figure 4: G'

$f_G = (1, 0, 0, 0)$ and $f_{G'} = (0, 0, 1, 0)$ then, $k(G, G') = f_G^T f_{G'} = 1.0 + 0.0 + 0.1 + 0.0 = 0$
In fact we can take any two graphlets of the four.

References