Om Sri Sai Ram

COUNTING: THE ART OF ENUMERATIVE COMBINATORICS.

Chap 1

Ten quickies:
1. 14 pick B1 | B2 | .... G7 | G8
2. 2 pick an orange | an apple
3. 3
4. 3 BB | GG | BG (only pick) $a+b=2$
5. 15
6. 1
7. 6
8. 6
9. 6 $a+b=5$
10. 69 $a, b \geq 1$

$x_1 \in 0, \dots 9$
$x_2 \in 0, \dots 6$
$\therefore 10 \cdot 7 - 1 = 69$
$x_1 + x_2 = k \geq 1$, $x_1 + x_2 = 2$
$\sum_{k=1}^{6} (k+1) +$

Permutations

1. 35
2. 26^2
3. $26 \cdot 25$
4. $21 \cdot 5$
5. $3 \cdot 8$
6. $5 \cdot 4$ ($P_1 : 5, P_2 : 4$)
7. 10 $\neq P_1$
8. 26^4
9. 35
10. mn choose 1 from mn
or choose row : m
then col : n
indep of row
11. indep: 12
12. $12 \cdot 52$
13. 24
14. 13!
15.

§4. one after the other ~ distinguishable way of identical dice thrown.
$\Rightarrow 6 \cdot 6 = 36$. indep
(dice, time of throwing) matters.

simult & indisting (identical dice) : 1,1 - 6,6 aa
$\binom{6}{2}$ ab $\Rightarrow$ 21 ways

$M_2 : \frac{36 - 6}{2} + 6 = 21$

r-permutation : arrangement using exactly r of the n elements (some relevant)
of letters, abcdefg, bcefgda, xyzpqrs all are valid 7-perms of a-z.

§5


A

given a 2 coloring of K_6, $\exists$ monochr Δ

boxes $k+1$

1. $10 + 1 = 11$ pigeons
2. $4 + 1 = 5$ boxes cards
3. boxes $365 \cdot 2 + 1 = 731$
4. $(7+8+9+9) + 1 = 34$
b g r w
5. $n+1$ boxes people (pigeons)
cpl #1 Couple #2 . . . C#n
atleast 2 in a box $\Rightarrow n+1$

6. mutual $\Rightarrow$ a has friend b $(\Leftrightarrow)$ b has friend a. mutual friend.

if all diff $\Rightarrow$ 0 to 19 are possib.
so $\leq$ 20 possibilities
almost all val in 0, .. 19 are covered.
also, if or $\Rightarrow$ 19 x & 19 $\checkmark$ $\Rightarrow$ 0x
$\therefore \leq 19$ possibilities
diff for values of # of mut friends
$\Rightarrow \leq 19$ people if all diff
$\therefore$ for ≥ 20 people, $\exists$ 2 st they have same # of mut friends

1943 pennies ie 1¢ year: 1943

14.

so $P_1\ P_2\ q$ or just q, 2001

1¢ 1¢ 25¢

$0q:\ P_1 + P_2 = \cancel{95}\ 96$

$1:\ 96-25$

$2:\ 96-50$

$3:\ 96-75$

$400-16$

$\# = 96 \cdot 4 - 150$

$= 234$

15.

$a=b=c>d, e$ $\binom{5}{3}$ choose a, b, c

$a > 7.6$

$a+b+c+d+e = 38 < 5a \Rightarrow a \geq 8$

$3a+d+e = 38 \geq 3a \Rightarrow a \leq 12$

$a=8:\ \underset{<8\ <8}{d+e} = 38-24 = 14 \Rightarrow 15 - 2\binom{d}{7} + e = 14-8$ (≥ 0, $=6$)

$= 1$

$(d, e \leq 7\ \ d+e = 14$

$\Rightarrow 7, 7)$

$a = 9:$ ~~38 − 27 + 1~~

~~38 − 30 + 1~~

~~39 − 33~~

~~39 − 36~~

$a=9:\ \underset{\leq 8\ \leq 8}{d+e} = 11$ $e = 3, 4, 5, 6, 7, 8$

6

$a=10:\ \underset{\leq 9\ \leq 9}{d+e} = 8 \Rightarrow 9$

$a = 11:\ d+e = 5 \Rightarrow 6$

$12:\ \ 2 \Rightarrow 3$

$\Rightarrow 1+6+9+6+3 = 25$

$\Rightarrow 25 \cdot \binom{5}{3}$

16. div into grps here

$\boxed{\text{diff}}$ grps.

Sizes: $\underbrace{2\ 2}\ \overbrace{1\ 1 \ldots 1}$

$G_1 G_2 \cdot G_{n-2}$

two boxes 2, n−4 1, 2 0 $\Rightarrow \underbrace{\binom{n}{2}\binom{n-2}{2}}_{\text{choose boxes}}$ arr. $(n-2)!\ \underbrace{\binom{n}{2}\binom{n-2}{2}}_{\text{choose balls}}$ 2 for the 2 ons

the 2 boxes with 2

ab = ba

or $\binom{n}{2}\binom{n-2}{2}\ \frac{n!}{2!\,2!}$ ord grps

or 1+3, n−3 : 1, 2 : 0 boxes balls $\Rightarrow \binom{n}{2}(n-2)\ \frac{n!}{3!}$ $\underbrace{\frac{n!}{3!}}$

or $\binom{n}{2}(n-2)(n-3)!\binom{n}{3}$

17.

$\sqcup\ \sqcup\ \ \sqcup\ \sqcup\ \sqcup\ \sqcup$

$\underbrace{x_1 + x_2}_{\leq 4} + x_3 + x_4 + x_5 + x_6 = 8$

$x_1 + x_2 = k \Rightarrow (k+1)\binom{8-k+3}{3}$

$\# = \sum_{k=0}^{4} (k+1)\binom{11-k}{3}$

Chapter 2 : PIE

$\Rightarrow t - \sum |A| + \sum |A\cap B| - |ABC| = $ all 0, 2+1 counted once $= |A\cup B\cup C|^c$ as req

ie all +

$\Rightarrow |A+B+C| = \sum |A| - \sum |AB| + |ABC|$

n sets $A_1 - A_n$

PIE

claim: $|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{\sigma_k} |\bigcap A_i| \right)$

$= \sum |A_i| - \sum |A_i A_j| + \sum |A_{ijk}| + \dots + (-1)^{n-1} |\bigcap_{i=1}^{n} A_i|$

Proof: for arbitrary $a \in S = \bigcup A_i$, say $a \in$ sets A_1 to A_k.

then a counted on LHS once.

"increases LHS by 1"

& RHS by

$k - \binom{k}{2} + \binom{k}{3} \dots + (-1)^{k-1} \binom{k}{k}$

counted in $|A_i A_j|$ iff $A_i, A_j \in k$ sets with a

$= -\left(-\binom{k}{1} + \dots + (-1)^k \binom{k}{k} \right)$

$= -(-1+0) = 1$ as req.

So RHS ↑ by 1 as well

ie a counted only once on RHS.

for $a \notin S \Rightarrow a \notin A_i$ for any A_i

a counted 0 times anywhere as req.

Problems I

9 choices to leave out 1 digit
0 1 2 ... 8

✓ 1. $9.10^8 - (9! + 9(8.8!))$
9 digit #s, if no 0, for 0

if only 1-9 $\Rightarrow 9^9 - 9!$

if 0 at 1st d allowed, all diff
$10^9 - \binom{10}{9} 9!$

✓ 2. $|A \cup B| = A + B - AB = 5.26^3.2 - 5^2 26^2$

$= 26^2(5.26.2 - 5^2)$

✓ 4. $x_1 + x_2 + x_3 = 25$, ≥ 16

$\binom{27}{2} - 3\left[\binom{11}{2}\right] + 0 - 0$

verif: $\dfrac{\binom{13}{2,2,2}}{\binom{7}{2}} = \langle {5 \atop 9} \rangle \dfrac{9!}{2!}$ $\dfrac{13!}{2^3 3} = \dfrac{6!13!}{3!10!} \dfrac{4!10!}{4!6!}$

$\Leftrightarrow \dfrac{13!}{2^3} \dfrac{2^2}{4!} \quad \dfrac{13!}{4!9!} \dfrac{9!}{2!}$ ✓ $\dfrac{13!}{4!9!}\dfrac{9!}{3!} = \dfrac{6!10!}{3!6!}\dfrac{13!}{3!10!}$ ✓

$\# = \left[\binom{2}{1}\langle {5 \atop 9}\rangle \dfrac{9!}{2!} + \langle {4 \atop 10}\rangle \dfrac{10!}{2!^2}\right] - \left[\langle {7 \atop 7}\rangle 7! + \langle {6 \atop 8}\rangle \dfrac{8!}{2!} + \dfrac{\langle {5 \atop 9}\rangle \frac{9!}{2!}}{\frac{3!}{2!}}\right]$

or C C C O O
C C O O
nice! $\langle {5 \atop 3}\rangle \cdot 1 \cdot \langle {8!} \rangle$
IISS CSCOISO

$+ \langle {8 \atop 6}\rangle 6!$

$= \left[\dfrac{t}{\binom{4}{2}} \cdot \binom{2}{1} + \dfrac{t}{\binom{3}{2}}\right] - \left[\dfrac{t}{\binom{6}{2,2,2}} + \dfrac{t}{\binom{5}{2,2,1}} + \dfrac{t}{\binom{4}{2}\binom{3}{2}}\right]$

$+ \dfrac{t}{\binom{7}{2,2,2,1}}$

just multiplies.
"indep"
just like MISSIP SSIPPI

"ada id is equiv to just one arr. of $a_1 a_2 a_3$"

Problems IV

1. A_i : i followed by $i+1$, $i: 1 \to 7$.

just choose 2
$i, j : 1 \to 7, i \neq j$
$i \; i+1 \; j \; j+1$

$\# = |\cap A_i^c| = |(\cup A_i)^c| = t - |\cup A_i|$

$= 8! - \binom{7}{1} 7! + \left[\binom{6}{1} \cdot 6! + \binom{7}{2} 6!\right]$

$\boxed{i \; i+1}$ rest 6 $i \; i+1 \; i+2$

order fixed $i: 1 \to 6$

$i \; i+1 \; i+2 \; j \; j+1$

$- \left[\binom{5}{1} 5! + \underset{j \to}{6} + \underset{j \leq 6}{\binom{6}{2} \cdot 2}\right]$

$i \; i+1 \; i+2 \; i+3$

$\underbrace{\qquad\qquad\qquad}_{3\ A_i\text{s true.}}$

too long.

first digit = 1: $1 \; \underline{3} \; \_ \; \_ \; \_ \; \_ \; \_$ 3 8   2, 4, 5, 6, 7, 8

$f = g$ iff $f_i = g_i \ \forall i \in \mathbb{N} \cup \{0\}$

$x := (0, 1, 0, 0 \ldots) \sim 0 \cdot x^0 + 1 \cdot x^1 + 0 \cdot x^2 + 0 \cdot x^3 + \ldots$

ie $x\_i = \begin{cases} 0 & i = 0 \text{ or } i \geq 2 \\ 1 & i = 1 \end{cases}$

$= \begin{cases} 1 & i = 1 \\ 0 & \text{otherwise} \end{cases}$

a_i : deg i (imagine $a_i x^i$)

$x^2 := x \otimes x = (0, \overset{\sim\deg 1}{1}, \overset{\sim\deg 2}{0} \ldots) \otimes (0, 1, 0 \ldots)$

$= (0, 0, 1, 0, 0 \ldots)$

$x^k = \begin{cases} 1 & i = k \\ 0 & \text{otherwise} \end{cases} \quad \forall k \in \mathbb{N}$

$x^0 := (1, 0, 0, \ldots) \overset{\text{denoted}}{=} 1.$

$a \in$

$a \otimes x$

so $a := (a, 0, 0, 0 \ldots 0)$

also $0 := (0, 0, 0 \ldots 0)$

note $f \oplus 0 = 0 \oplus f = f$

$f \otimes 1 = 1 \otimes f = f$

$a \otimes x = (a, 0, 0 \ldots) \otimes (0, 1, \ldots)$

$= (0, a, 0, 0 \ldots)$

$:= ax$

$a_k \otimes x^k = (\underbrace{0, 0 \ldots 0}_{\substack{k \\ 0 \to k-1}}, a_k, 0 \ldots 0)$

$:= a_k x^k$

~~f~~ $= a_0$

$(a_0, a_1, a_2 \ldots) = (a_0, 0, 0 \ldots 0) = a_0 \otimes x^0$

$+ (0, a_1, 0 \ldots) \to + a_1 \otimes x^1$

$+ (0, 0, a_2 \ldots) \to + a_2 \otimes x^2$

$+ \ldots$

$\vdots$

$= a_0 + a_1 x + a_2 x^2 + \ldots$

2. $x_1 + \cdots + x_8 = r$

$$[x^r].\left\{\frac{1}{(1-x)^8}\right\} \quad [x^r]\left\{(x^0+x^1+x^2+\cdots)(x^0+x^1+x^2+\cdots)\vdots(x^0+x^1+x^2+\cdots)\right\}$$

3. $[x^r]\left\{(x^0+x^1+x^2)(x^0+x^1+x^2)\underbrace{(x^0+x^1+x^2+\cdots)\cdots}_{n-2}\right\}$

total exp $= r$

1 2 $3-n$

4. $[x^r]\left\{\frac{1}{(1-x)^{n-2}}(1+2x+3x^2)\right\}$

2 ways to put 1 in both log..

00 01 10 20 11 02

tot: 0 1 2

≤ 2

if extended ie total = singly $\frac{1}{(1-x)^2}$ ✓

tot $\leq 2 \Rightarrow$ exponent ≤ 2

# of balls in both

ie the n in $[x^n]\left\{\frac{1}{(1-x)^2}\right\}$

*

only combinations. some As, Bs Cs Ds Es total 10 letters chosen

5. $[x^{10}]\left\{\frac{1}{1-x}\frac{x}{1-x}(1+x)(1+x^2)\frac{1}{1-x^2}\right\}$

*

why? → $\boxed{\text{and} \quad \text{or}}$

[mult rule   addn rule]

Qe

$\{a_r\}$.. $x_1+\cdots+x_6 = r$

all even or all odd.

(+)

# of solns addn.

$x^1 + x^3 + x^5 + \cdots$

$$\Rightarrow [x^r]\left\{\left(\frac{1}{1-x^2}\right)^6\right\} \underset{\text{or}}{+} [x^r]\left\{\left(\frac{x}{1-x^2}\right)^6\right\}$$

$$= [x^r]\left\{\frac{1+x^6}{(1-x^2)^6}\right\}$$

$$\Rightarrow g(x) = (1+x^6)\cdot\left(\frac{1}{1-x^2}\right)^6$$

4. $x^{66}\left\{\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^4}\right\}$

give 2·2·4 balls 8 r8b

5. $x^{n-8}\left\{\frac{1}{(1-x)^8}\right\} = [x^n]\left\{(x^2+x^3+\dots)^8\right\}$

blue balls to P1

$x_1 + x_2 + x_3 + x_4 + \dots + x_8 = n-8$

store red balls to P1, red P2, bP2 ... b4

6. $[x^n]\left\{\frac{1}{(1-x)^{10}}\right\}$

7. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$

$\text{sum} = 35 = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$

close to 3.10

$+ -1x_2 + 0x_3 + 1x_4 + \dots$

$\Rightarrow 5 = -2x_1 - x_2 + x_4 + 2x_5 + 3x_6$

keep all +

$25 = x_2 + 2x_3 + \dots$

$25 = a + 2b + 3c + 4d + 5e$

$e = 5 : (0,0,0,0,0,5) \times \text{sum} \neq 10$

4 :

Sum! sum of exp! oh disting dice!! ie one by one.

$[x^{35}]\left\{(x^1 + x^2 + \dots + x^6)^{10}\right\}$ $(a,b) \neq (b,a)$ ordered ✓

8. indist

$x_1 + \dots x_6 = 10$

$2k = x_1 + 2x_2 + 3x_3 + \dots + 6x_6$

if $2n+1$ indist dice

then

$\underbrace{(\alpha_1 \dots \alpha_{2n+1})}_{\text{even sum}} \leftrightarrow \underbrace{(7-\alpha_1 \dots 7-\alpha_{2n+1})}_{\text{odd sum}}$

$\Rightarrow \dfrac{\binom{2n+1+5}{5}}{2}$ ways to obtain even sum.

so $e^{\alpha z}e^{\beta z} = e^{(\alpha+\beta)z}$ for $\alpha, \beta \in \mathbb{R}$ z not a var!

$$e^{ax} := \sum_{i=0}^{\infty} \frac{(az)^i}{i!} = \left(1, a, \frac{a^2}{2!}, \frac{a^3}{3!}, \dots\right)$$

↳ short form for

→ can use this in egf computations!

$g(x) = (1+x)(1+x^2) \dots$

ways $= [x^r]\, g(x)$

$x_1 + x_2 + x_3 + x_4 + x_5 = 40$

$0 \le \ \ \le 10$

or e_1

e for exponent

"x^{e_i}" common.

$\# = [x^{40}](x^0 + \cdots + x^{10})^5$

$= [x^{40}] \left\{ \underbrace{(1-x^{11})^5}_{\text{contribute the } (-1)^i\binom{5}{i} \text{ in PIE}} \ \underbrace{\sum \binom{r+4}{4} z^r}_{\text{contribute each of the } |A_i \cap \dots| \text{ terms}} \right.$

$= \binom{40+4}{4}\binom{5}{0}$

$- \binom{5}{1}\binom{29+4}{4}$

$+ \binom{5}{2}\binom{18+4}{4}$

$- \binom{5}{3}\binom{7+4}{4}$

$+ \binom{5}{4}\binom{0}{4}$

$- \binom{5}{5}\binom{-4}{4}$

takes care ($A_1 \cap A_2 \cap A_3 \cap A_4$) $= 0$ etc.

20. ALBNYEWORK

A N Y

AAANN, YYK

$$t = \frac{13!}{2!3}$$

$$\# = \frac{t}{\binom{4}{2}} + \frac{t}{\binom{3}{1}} - \frac{t}{\binom{4}{2}\binom{3}{1}}$$

M2: $\left\langle {5 \atop 9} \right\rangle \frac{9!}{2!} + \left\langle {4 \atop 10} \right\rangle \frac{10!}{2!2} - \frac{\left\langle {5 \atop 9} \right\rangle \frac{9!}{2!}}{3}$

A A N N

or

$\left\langle {5 \atop 3} \right\rangle \cdot 1 \cdot \left\langle {8 \atop 6} \right\rangle 6!$ for of AANN, of YYK

21. $|\cap A_i^c|$

$$t = \frac{13!}{2!3}$$

$$\# = t - \frac{12!}{2!2}\binom{3}{1} + \binom{3}{2}\frac{11!}{2!} - \binom{3}{3}10!$$

$$= \sum_{r=0}^{3} (-1)^r \frac{(13-r)!}{2!3-r} \binom{3}{r}$$

22. $4^{20} - \binom{4}{1}3^{20} + \binom{4}{2}2^{20} - \binom{4}{3}1^{20} + 0$

$$= \sum_{r=0}^{4} (-1)^r \binom{4}{r} (4-r)^{20}$$

SUNYATLB

NYA

A

23. $\left[\frac{z^n}{n!}\right] \left\{ (1+x)^5 (1+x+x^2/_2)^2 (1+z+z^2/_2+z^3/_6)^2 \right.$

line pizza

max ⇒ no 3 planes/lines int in a pt.

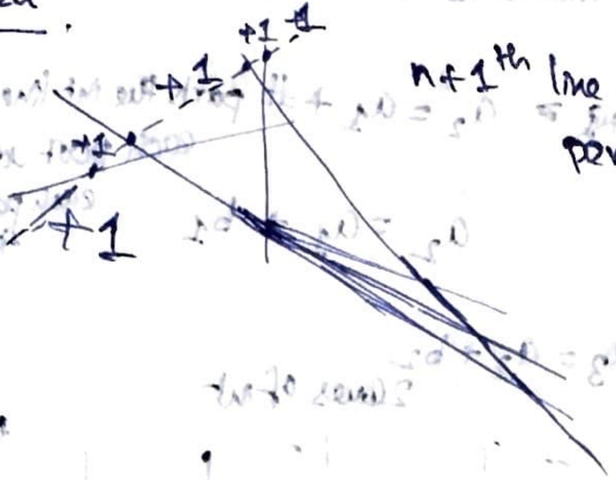
$c_n = c_{n-1} + 1$

$+1$

$\Rightarrow c_1 = 2$

$\Rightarrow c_n = n+1$ $\quad c_0 = 1$ ✓ whole line.

Pl Pizza



$n+1^{th}$ line intersects all n lines. per intersection one pt

aha! #new reg = # pieces of ℓ that int pts cut it into $= c_n$

(n lines int so n ints with $n+1$ th line)

$b_{n+1} = b_n + c_n$

$b_{n+1} = b_n + n + 1$

$\frac{1}{x}(B(x) - b_0) = B(x) + \frac{1}{(1-x)^2}$

$b_{n+1} = b_{n-1} + n + n+1$

$= b_1 + 2 + 3 + \cdots (n+1)$

$= 1 + \frac{(n+1)(n+2)}{2}$

$= \frac{n^2+3n+4}{2}$

$b_n = 1 + \frac{n(n+1)}{2} = \binom{n+1}{2} + 1 = \frac{n^2+n+2}{2}$

$b_0 = 1$ ✓

"the whole plane" 1 portion

$a_0 = 1$ ✓

$a_1 = 2$ ✓

$a_2 = 4$ ✓

$a_3 = 8$ ✓

also, $\deg a_n^i = i$

pol in n

$$a_n^1 = \binom{n+1}{1} + \cancel{\binom{n+1}{-1}}$$

$$a_n^2 = \binom{n+1}{2} + \binom{n+1}{0}$$

$$a_n^3 = \binom{n+1}{3} + \binom{n+1}{1}$$

$$a_n^4 = a_0^4 + \sum_{k=0}^{n-1} a_k^3$$

$$= 1 + \sum_{k=0}^{n-1} \binom{k+1}{3} + \binom{k+1}{1}$$

$$= 1 + \sum_{k=1}^{n} \binom{k}{3} + \binom{k}{1}$$

$$= 1 + \binom{n+1}{4} + \binom{n+1}{2}$$

$$= \binom{n+1}{4} + \binom{n+1}{2} + \binom{n+1}{0}$$

$$a_n^5 = 1 + \sum_{k=1}^{n} \binom{k}{4} + \binom{k}{2} + \binom{k}{0} \to \binom{1}{0} + \dots + \binom{n}{0}$$

add in, $k: 0 \to n$

$$= \binom{n+1}{5} + \binom{n+1}{3} + \binom{n+1}{1} \quad = \binom{n+1}{1} - \binom{0}{0} = n+1-1 = n$$

easy to see

$$\Rightarrow a_n^i = \sum_{k\geq 0} \binom{n+1}{i-2k} \text{ general hyperplanes.}$$

$(i-1)D$

$$\binom{n+1}{3} = \binom{n}{2} + \binom{n}{3}$$

$$\Rightarrow a_n^1 = \binom{n}{1} + \binom{n}{0}$$

$$a_n^2 = \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$$

$$a_n^3 = \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$$

$$a_n^i = \sum_{k\geq 0} \binom{n+1}{i-2k} = \sum_{k\geq 0} \binom{n}{i-k} = \sum_{k=0}^{i} \binom{n}{k}$$