Om Sri Sai Ram!!!

Complex Analysis

De Moivre: for $n \in \mathbb{Z}$ $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof: $n \in \mathbb{N} \Rightarrow$

to show: $(e^{i\theta})^n = e^{in\theta}$

Proof: $e^{i\theta_1} \cdot e^{i\theta_2} \overset{\text{by def}}{=} (\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2)$

$\forall \theta_1, \theta_2 \in \mathbb{R}$

$= \cos\theta_1+\theta_2 + i\sin\theta_1+\theta_2$

$\overset{\text{by def}}{=} e^{i(\theta_1+\theta_2)}$.

The claim follows.

$a^{\alpha+\beta} = a^\alpha \cdot a^\beta$

$n \in \mathbb{Z}$

$n = 0$ let $(\cos\theta + i\sin\theta)^0 = z = (e^{i\theta})^0 = (e^{i\theta})^{1-1} = (e^{i\theta})^1 \cdot (e^{i\theta})^{-1}$

$z = \cancel{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$

$\downarrow$ by def of ^1 $\downarrow$ def of ^-1

$e^{i\theta}$ $\frac{1}{e^{i\theta}}$

$= 1$

$= e^{i0}$, real mult.

$= e^{i0\theta}$

$(e^{i\theta})^0 = e^{i0\theta} = 1$.

$n \in \mathbb{Z}_{<0}$ ie to show: $(e^{i\theta})^{-n} = e^{-in\theta}$

$(e^{i\theta})^{-n} \overset{\text{def of }^{-1}}{=} \frac{1}{(e^{i\theta})^n} = \frac{1}{e^{in\theta}}$ $n \in \mathbb{N}$ $= \frac{1}{e^{i(n\theta)}}$ $|1|=1$ $= \overline{e^{in\theta}}$

$= e^{in(-\theta)}$

$= e^{-in\theta}$

as req.

~~$n \in \mathbb{R}$, one val is~~

$n \in \mathbb{Q}$ $n = p/q$, $p \in \mathbb{Z}, q \in \mathbb{N}, (p,q) = 1$.

$(e^{i\theta})^{p/q} = ((e^{i\theta})^p)^{1/q} = (e^{ip\theta})^{1/q} = z = re^{i\theta'}$ $r \in \mathbb{R}^+$

$e^{ip\theta} = r^q e^{iq\theta'}$

take mod $\Rightarrow r = 1$

$e^{ip\theta} = e^{iq\theta'}$

$\Rightarrow p\theta = q\theta' + 2k\pi$

$\theta' = \frac{p\theta}{q} + \frac{2k\pi}{q}$

$\Rightarrow z = (e^{i\theta})^{p/q} = e^{\frac{ip\theta}{q}}$ (one val)

$= e^{in\theta}$ as req.

true for $n \in \mathbb{R}$ as well.

$c_0 \in \mathbb{C}$ $\displaystyle\int_C c_0 f(z)\,dz = \int_a^b c_0 f(z(t))\, z'(t)\,dt = c_0 \int_a^b f z(t)\, z'(t)\,dt = c_0 \int_a^b f(z)\,dz$

$\displaystyle\int_C (f(z)+g(z))\,dz = \int_a^b (f z(t) + g z(t))\,dt = \int_a^b f z(t)\,dt + \int_a^b g z(t)\,dt = \int_C f(z)\,dz + \int_C g(z)\,dz$

5. $z = e^{i\theta}$, $\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$

$\displaystyle\int_{-i}^{i} z^2+1\,dz = \int z^2\,dz + \int 1\,dz = 2\left(\int_0^i z^2\,dz + \int_0^i 1\,dz\right)$

$= 2\left(i - i/3\right) = \frac{4}{3}i$

6. $\displaystyle I = \int_C f(z)\,dz = \int_{-1}^{1} f z(t)(1+3it^2)\,dt$ $\displaystyle I(z) < 0 \quad = \int_{-1-i}^{0} dz + \int_0^1 4t^3(1+3it^2)\,dt \simeq$

$C: y = x^3$ $\quad z: t + it^3$ ($x = t$)

$x: -1 \to 1$ $\quad t: -1 \to 1$

$f z(t) = \begin{cases} 1 & t^3 < 0 \text{ i.e. } t < 0 \\ 4t^3 & t > 0 \end{cases}$

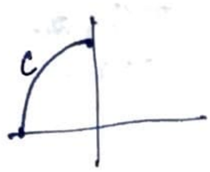
piecewise cont is integrable!

$\quad y<0 \quad y>0$

$(f(z) = f(x,y) = 1,\ 4y)$

$z = x + iy$

243.



$f(z) = z^{1/2} = \sqrt{3}\, e^{i\theta/2 + 0, \pi} = \pm\sqrt{3} e^{i\theta/2}$

principal $\sqrt{z} = \sqrt{|z|}\, e^{i\frac{\operatorname{Arg} z}{2}}$

$= \sqrt{3} e^{i\theta/2}$ here.

$\displaystyle I = \int_{\pi/2}^{\pi} \sqrt{3} e^{i\theta/2}\, 3ie^{i\theta}\,d\theta$

$\displaystyle = 3\sqrt{3}i \int_{\pi/2}^{\pi} e^{i3\theta/2}\,d\theta = 2\sqrt{3}\, e^{i3\theta/2}\Big|_{\pi/2}^{\pi}$

$= 2\sqrt{3}\left(-i - \left(\frac{-1+i}{\sqrt{2}}\right)\right)$

$= \sqrt{6}\left(1 - i(1+\sqrt{2})\right)$

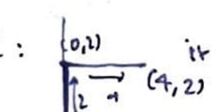
<u>C vs −C</u>

$C: z(t),\ t: a \to b$
$-C: z(t),\ t: b \to a$ $\Big\}$ dt sign opp $\Rightarrow \boxed{I_C = -I_{-C}}$

$\displaystyle I_{-C} = \int_{-C} f(z)\,dz = \int_b^a f(z(t))\,z'(t)\,dt = -\int_a^b (\ \ ) = -I_C$

244.

$C:$



it is a contour

$\displaystyle\int_C = \int_{C_1} + \int_{C_2}$ clearly.

$\displaystyle\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = \int_{C_1+C_2} f(z)\,dz.$

$P: \mathbb{C} \to \mathbb{C}$

let P be st $\frac{dP}{dz}(z) = f(z)\ \forall z$, then $\frac{dP(z(t))}{dt} = P'(z(t))\, z'(t) = f(z(t))\, z'(t) = F'(t)$

~~then $P(z(t))z'(t) = \frac{d}{dt}$~~

So that $P(z(t)) = F(t) + C$.
($\mathbb{R} \to \mathbb{C}$ integration)

~~then $\frac{dz}{dt}$~~

$$\therefore \int_{z_1}^{z_2} f(z)\,dz = P(z(b)) - P(z(a)) = P(z_2) - P(z_1),$$

where $\frac{dP(z)}{dz} = f(z)\ \forall z$

($x \to z$ id to usual one)

$$\Rightarrow \int_a^b z^n\,dz = \left.\frac{z^{n+1}}{n+1}\right|_a^b \left[P = \frac{z^{n+1}}{n+1},\ \frac{dP}{dz} = z^n \right)$$

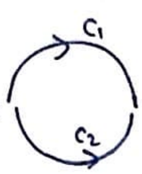
$$\int_C \frac{1}{z^2}\,dz = -\frac{1}{z}\Big|_{-2}^{-2} = 0$$

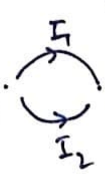
$1 \Rightarrow 2$

if $f(z)$ has an antideriv $F(z)$ in D, then $\int_C f(z)\,dz$ is path independent.

Proof: $\int_C f(z)\,dz = F(z_2) - F(z_1)$ indep of path.

$\uparrow$
id to $V_2 - V_1$ indep of path.

$2 \Rightarrow 3$ choose: z_1  z_2 $\quad \int_{C_1} = \int_{C_2} \Rightarrow \int_{C_1 + (-C_2)} = 0$ ie $\int_{loop} = 0$.

$3 \Rightarrow 2$ $\quad z_1$  z_2 $\quad 3 \Rightarrow I_1 + -I_2 = 0 \Rightarrow I_1 = I_2$

$2 \Rightarrow 1$: todo.

Cauchy-Goursat

ext
un-bdd
not-bdd.

int. bdd

$(\forall r > 0, \exists z \in \text{set st } |z| > r)$

x, y diff on $t: a \to b$. u, v cont.

$$\underbrace{\int_C u\,dx - v\,dy}_{\text{notation}} \overset{\text{def}}{=} \int_a^b \left[u(x(t), y(t))\, x'(t) - v(x(t), y(t))\, y'(t) \right] dt$$

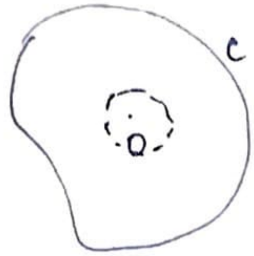
$$\int_C f(z)\,dz = \int_C (u + iv(x,y))\, z'(t)\,dt = \int_C (u+iv)(x' + iy')\,dt = \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$\uparrow$
$t: a \to b$
$(x, y) = (x(t), y(t))$

why analytic req?
inside C

why analytic at all?
piecewise cont not enough?



$$\int_C \frac{1}{z}\,dz = \int_{C_R} \frac{1}{z}\,dz = \int_{-\pi}^{\pi} \frac{1}{re^{i\theta}}\, ire^{i\theta}\,d\theta = \int_{-\pi}^{\pi} i\,d\theta = 2\pi i$$

$dz = iz\,d\theta$, $\frac{dz}{d\theta} = ie^{i\theta}$

($\frac{1}{z}$ analytic in C_R to C ie in $C \cup C_R \cup [\text{ext}\, C_R \cap \text{int}\, C]$

$$\Rightarrow \boxed{\int_C \frac{1}{z}\,dz = 2\pi i \quad \text{as req.}}$$

Any curve not containing 0, $\int_C \frac{1}{z}\,dz = 0$ by CG.

f has to be anal inside as well for Green to hold obviously.

If like for $\frac{1}{z}$ at 0 $f \to \infty$

then $f \cdot dx \cdot dy$ = finite

so $2\pi i$ ✓

Cauchy Integral Theorem

f anal in $C \cup \text{int}\,C$. $z_0 \in \text{int}\,C$, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\,dz$$

$\rightarrow f(z) = 1\ \forall z$, $C_0: z_0 + Re^{i2\pi t}$, $z_0 \in \text{int}\,C$

$(t: 0 \to 1$ or $2\pi t: 0 \to 2\pi)$

$$\int_{C_0} \frac{f(z)}{z - z_0}\,dz = \int_{C_0} \frac{1 \cdot 2\pi i Re^{i2\pi t}}{Re^{i2\pi t}}\,dt = 2\pi i \int_0^1 dt = 2\pi i$$

$$\therefore f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\,dz.$$

$g(z) = \frac{f(z)}{z - z_0}$ anal on $(C \cup \text{int}\,C) \setminus z_0$

by C_1 thm, $\int_C g(z)\,dz = \int_{C_\rho} g(z)\,dz$

by thm ie by Cauchy Goursat $R = C \cup \text{int}\,C$. $\Rightarrow f$ anal in

$C_\rho: z_0 + \rho e^{i2\pi\theta}$ $\theta: 0 \to 1$ +vely or.

now, $\int_C \frac{f(z)}{z - z_0}\,dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)}{z - z_0}\,dz - f(z_0) \int_{C_\rho} \frac{1}{z - z_0}\,dz$

$$= \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

now $z \in C_\rho$

(basically $2\pi i$ is $2\pi i$ because $\int_{C_R} \frac{1}{z}\,dz = 2\pi i$ (or $\int \frac{1}{z - z_0}\,dz$) $C = z_0 + \rho e^{i\theta}$)

Claim: for each $\varepsilon > 0$, $\exists\, \delta > 0$ st $|z - z_0| < \delta \Rightarrow \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz < 2\pi\varepsilon$

take $\delta = 2\rho$ or anything $> \rho$

$|h(z)| = \left|\frac{f(z) - f(z_0)}{z - z_0}\right| < \frac{\varepsilon}{\rho}$

f cont at z_0 $\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ st $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$

choose $\rho < \delta \Rightarrow |h(z)| < \frac{\varepsilon}{\rho}$

hence by ML, $\left|\int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| < 2\pi\rho \cdot \varepsilon/\rho = 2\pi\varepsilon$

Morera's theorem

→ If f is anal in D, ie f' exists in D, then by Cauchy Integral Theorem f is infinitely differentiable in D.
& so all derivs of f are anal *

if $h'(z) = g(z)\ \forall z \in D \Rightarrow h' ✓ \Rightarrow h$ anal $\Rightarrow h'$ anal $\Rightarrow g$ anal as well!

(Morera) f cont on D & $\oint_C f(z)\,dz = 0\ \forall$ closed $C \subset D$ note that D could be multiply connected.

Partial converse to CG Thm ↑ f continuous is assumed

$\Rightarrow f$ is analytic throughout D.

Proof: $\oint_C f(z)\,dz = 0\ \forall C \Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ so int is path indep.

(f cont is req so $\oint_C f(z)\,dz$ exists for any C)

fix $P \in D$

Consider $F(z) = \int_P^z f(z)\,dz$ ie $\int_C f(z)\,dz$ for some $C : P \to z$

F is well defined.
(indep of C)

$$F'(z) = \lim_{\substack{h \to 0\\ h \in \mathbb{C}}} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \frac{\int_z^{z+h} f(z)\,dz}{h} = \lim_{h\to 0} \frac{\int_z^{z+h} f(s)\,ds}{h} \quad s \in \mathbb{C}$$

$\int_P^{z+h} - \int_P^z = \int_z^{z+h}$ as path indep.

$$|F'(z) - f(z)| = \left| \lim_{h \to 0} \int_z^{z+h} \frac{f(s) - f(z)}{h}\,ds \right|$$

$$\int_z^{z+h} (f(s) - f(z))\,ds$$

$s \in [z, z+h]$

for $|s - z| < h \Rightarrow |f(s) - f(z)| < \varepsilon$ (f cont at z)

$\therefore \left| \int_z^{z+h} f(s) - f(z)\,ds \right| < \varepsilon |h|$ (take the straight curve) or if not, take some $d|h|$ dist curve say ie $\varepsilon \to \varepsilon/d$

so $\left| \int_z^{z+h} f(s) - f(z)\,ds \right| < \varepsilon' |h|$

for suitable h, ie $\exists h > 0$ st → $I = \left| \int_z^{z+h} \frac{f(s) - f(z)}{h}\,ds \right| < \varepsilon$ for any $\varepsilon > 0$

($\int^a$ ✓ & one in limits & one inside integrand is finite.)

$$F'(z) - f(z) = \lim_{h \to 0} \int_z^{z+h} \frac{f(s) - f(z)}{h}\,ds$$

Integral Mean Value Theorem $\Rightarrow \left|\oint f(z)\,dz\right| = |f(c)|$
f cont on C

CG: if f is anal in $R = C \cup \text{int } C$, C is a simple closed contour,

then $\oint_C f(z)\,dz = 0$

if f is anal in simply connected D, then for $C \in D$ ↳ any closed contour

$\oint_C f(z)\,dz = 0$

what if mult. connected?

then in  usual CG cannot be appl as f needn't be anal in the inner disk ↄ (it's outside D)

Series Theorems

300. $z_n = x_n + iy_n$ $n \in \mathbb{N}$ & $z = x + iy$

then $z_n \to z \Leftrightarrow x_n \to x$ & $y_n \to y$

Proof: $z_n \to z \Rightarrow$ for any $\varepsilon > 0$ $\exists N$ st $n > N \Rightarrow |z_n - z| < \varepsilon$

$\Rightarrow |x_n - x + i(y_n - y)| < \varepsilon$

$\Rightarrow \varepsilon > |x_n - x + i(y_n - y)| \geq |x_n - x|, |y_n - y|$

so that $x_n \to x$, $y_n \to y$.

$x_n \to x, y_n \to y \Rightarrow$ for any $\varepsilon > 0$ $\exists N_1, N_2$ st $n > N_1 \Rightarrow |x_n - x| < \varepsilon$

$n > N_2 \Rightarrow |y_n - y| < \varepsilon$

let $N = \max(N_1, N_2)$

$\therefore$ for any $\varepsilon > 0$

$\exists N'$, for $n > N' \Rightarrow |x_n - x|, |y_n - y| < \varepsilon/2$

$\Rightarrow |x_n - x + i(y_n - y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

ie for any $\varepsilon > 0$, $\exists N$, $n > N \Rightarrow |z_n - z| < \varepsilon$

$\Rightarrow z_n \to z$ as req!

302. $\lim_{n\to\infty} z_n = z \Rightarrow \lim_{n\to\infty} |z_n| = |z|$ obviously not converse (multiply z_ns & z by diff $e^{i\theta}$s)

$\left| \sqrt{x_n^2 + y_n^2} - \sqrt{x^2 + y^2} \right|$

$\exists N, n > N$ $x_n \in (x - \varepsilon, x + \varepsilon)$ $\sqrt{x_n^2 + y_n^2} \leq |x_n| + |y_n|$

$y_n \in (y - \varepsilon, y + \varepsilon)$ $\sqrt{x_n^2 + y_n^2} \geq \frac{|x_n| + |y_n|}{\sqrt{2}}$ $\sqrt{x_n^2 + y_n^2} - \sqrt{x^2 + y^2}$

$x_n, y_n \in \mathbb{R}$ $|x_n|^2$ $|y_n|^2$ $< |x_n| + |y_n| - \frac{(|x| + |y|)}{\sqrt{2}}$

$|(\sqrt{\ } - \sqrt{\ })| < |x_n - x| + |y_n - y| + c$ $\left|\sqrt{x_n^2 + y_n^2} - \sqrt{x^2 + y^2}\right| < \left| |x_n| - |x| + |y_n| - |y| + (|x| + |y|)(1 - \frac{1}{\sqrt{2}}) \right|$

$< 2\varepsilon + c = \varepsilon'$, oh c could be large, finite $< ||x_n| - |x|| + (\quad) +$ (finite)

319. if $\cancel{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n}$

let $r_0 < R_0$, $C_0 : r_0 e^{i2\pi\theta}$, $|z| < r_0 < R_0$

f anal on $R = C_0 \cup \text{int } C_0$

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s-z} \quad \text{by Cauchy Integral Thm.}$$

$$\cancel{\int_{C_0} \frac{f(s)}{s-z} ds = \int_{C_0} \frac{f(s)}{s} ds \frac{1}{1-z/s} = \int_{C_0} \frac{f(s)}{s} \left(1 + \frac{z}{s} + \left(\frac{z}{s}\right)^2 + \cdots\right) ds}$$

$\cancel{|s| = r_0}$

$\cancel{\left|\frac{z}{s}\right| < 1}$

$$\int_{C_0} \frac{f(s)}{s-z} ds = \int_{C_0} \frac{f(s)\,ds}{s} \frac{1}{1-z/s} \qquad |s| = r_0, \ \left|\frac{z}{s}\right| < 1$$

$$= \int_{C_0} \frac{f(s)\,ds}{s} \left(\frac{1-(z/s)^n}{1-z/s} + \frac{(z/s)^n}{1-z/s} \right)$$

$$= \sum_{r=0}^{n-1} \int_{C_0} \frac{f(s)\,ds}{s} \left(\frac{z}{s}\right)^r + \int_{C_0} \frac{f(s)\, z^n\, ds}{s^n(s-z)}$$

$$= \sum_{n=0}^{N-1} \int_{C_0} \frac{f(s)\,ds\, z^n}{s^{n+1}} + \int_{C_0} \frac{f(s)\, z^N ds}{(s-z)s^N}$$

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} ds = \sum_{n=0}^{N-1} \underbrace{\left[\frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s^{n+1}}\right]}_{\frac{f^{(n)}(0)}{n!}} z^n + \left[\frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{(s-z)s^N}\right] \cdot z^N$$

$$= \sum_{n=0}^{N-1} \frac{f^{n}(0)}{n!} z^n + \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds\, z^N}{s^N(s-z)}$$

$$= S_N + \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \left(\frac{z}{s}\right)^N ds$$

mag. very small as $N \to \infty$

to prove $f(z) = S = \sum_{n=0}^{\infty} \frac{f^n(0) z^n}{n!}$

$$|S_N - f(z)| = \frac{1}{2\pi} \left| \int_{C_0} \frac{f(s)}{(s-z)} \frac{z^N}{s^N} ds \right|$$

$$\le \frac{1}{2\pi} \cdot \left|\frac{f(s)}{s-z} \frac{z^N}{s^N}\right|_{max} \cdot 2\pi r_0 = r_0 \left|\frac{f(s)}{s-z}\right|_{max} \left|\frac{z}{s}\right|^N$$

So $I \to 0$ as $N \to \infty$ $\Rightarrow f(z) = S_\infty$

var. mod; its mod is fixed ($|s| = r_0$)

$g(s) = \frac{f(s)}{s-z}$ is cont. on $C_0 : r_0 e^{i2\pi\theta} = s$, $|z| < r_0$

$\exists M > 0$ st

$\Rightarrow |g(s)| \le M$ for $s \in C_0$.

"bdd on this arc"

$\cancel{\text{so } |S_N - f(z)| \le r_0 \cdot M \left(\frac{z}{s}\right)^N}$, ~~for some ones~~ ~~s is the s st g(s) is max~~

$\left(\frac{f(s)}{s-z}\right)$

or $f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n$ so $|S_N - f(z)| \le r_0 \cdot M \left|\frac{z}{s}\right|^N$

$S_N \to f(z) \Leftarrow$

$= r_0 M q^N$

$= c q^N < \varepsilon$ for large N

So f holomorphic on $|z| < R_0$ $\Rightarrow$ $f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} z^r \quad \forall z \in \mathbb{C} \cap D$
open disk D
(ie $\forall r_0 < R_0$)
not eq

$t - z_0 = z$

$|t - z_0| < R_0 \Rightarrow f(t - z_0) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} (t - z_0)^r$ as f needn't be holomorphic on bdry D ie $|z| = R_0$.

obviously, $f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(z_0)}{r!} (z - z_0)^r$

for $|z - z_0| < R_0$, f anal in D: $|z - z_0| < R_0, z \in \mathbb{C}$

$D = \{z : |z - z_0| < R_0, z \in \mathbb{C}\}$

$$\frac{e^z}{z^3} = z^{-3} + z^{-2} + \frac{z^{-1}}{2} + \sum_{r=0}^{\infty} \frac{z^r}{(r+3)!}$$

& conv when $z \neq 0$ ($z \neq 0$ $\frac{1}{z^3}(1 + z + \cdots) = \frac{e^z}{z^3}$ finite)

~~ie the annular disk~~

let R_1, R_2 be st $R_1, R_2 \in \mathbb{R}$, $0 \le R_1 < R_2$.

annular domain centred at z_0, with inner radius R_1, outer " R_2 $\overset{\text{def}}{=} \{z : R_1 < |z - z_0| < R_2, z \in \mathbb{C}\}$

open connected set

bdry $(D) = \{z : z \notin D, \exists \omega_i \in D \, \forall i \in \mathbb{N}$ st $\lim_{n \to \infty} \omega_n = z\}$

edges of domain.

limits of points in D.

Laurent Series Theorem

Suppose f anal in $D(z_0, R_1, R_2)$ ($R_1 \geq 0$, $R_2 > 0$) & C is any +vely oriented, simple closed contour around z_0 & $C \in \text{int}(\text{bdry}(D))$

so no repititions ie $2\pi i$, $4\pi i$ etc.

Then, at each pt $z \in D$, $f(z)$ can be represented as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where $a_n = \int_C \frac{f(s)\, ds}{(s - z)^{n+1}}$ (needn't be $\frac{f^{(n)}(z_0)}{n!}$ as f needn't be anal on $|z - z_0| \le R_1$)

$b_n = \int_C \frac{f(s)\, ds}{(s - z)^{-n+1}}$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad c_n = \int_C \frac{f(s)\, ds}{(s - z)^{n+1}}, \; n \in \mathbb{Z}$$

ie $\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ converges to $f(z)$ for all $z \in D$.

329.1. $z = 2i \to \infty$

$$\frac{1}{(z-2i)^4} = \cancel{(z-2i)^4} \neq \cancel{\sum_{r=} \binom{r+3}{3} z^r}$$

$$= \frac{1}{16\left(1-\frac{z}{2i}\right)^4} = \sum_{r=0}^{\infty} \frac{1}{16} \binom{r+3}{3} \left(\frac{z}{2i}\right)^r \quad \text{for } z \neq 2i$$

$\frac{z}{2i} \neq 1$ $\Rightarrow \underbrace{0 < |z-2i| < \infty}_{D}$

$$\int_c f(z)dz = \int_{2i+re^{i\theta}} \sum_{r=0}^{\infty} \frac{1}{16} \binom{r+3}{3} \left(\frac{z}{2i}\right)^r dz = 0$$

oh $\underline{\underline{\text{Laurent abt } 2i}} = \frac{1}{(z-2i)^4}$ ← only a b_4 ✓.

$$\int_{\substack{c \\ 2i+re^{i\theta}}} f(z)\,dz = \int_{re^{i\theta}} \frac{1}{t^4}\,dt = 0 \checkmark$$

$\underline{\underline{b_1 = 0}}$.

$f(z) = 1 \ \forall z$

$$\int_c \frac{1}{(s-z)^{n+1}}\,ds = \frac{2\pi i}{n!} f^{(n)}(z) = 0 \ \forall n \in \mathbb{N}, \boxed{z \in \text{int}(C)}^{*}$$

$$\Rightarrow \boxed{\int_c \frac{dz}{(z-z_0)^n} = 0 \ \forall n \in \mathbb{N}, n > 1 \ \forall z_0 \in \mathbb{C}}$$

$$z_0 = 2i,\ n = 4 \Rightarrow \int_c \frac{1}{(z-2i)^4}\,dz = 0$$

$\underline{2i \in \text{int}(C)}$

$\frac{1}{z}$ abt $z=0$ ∴ just $\frac{1}{z}$, $D: \{z : |z| > 0\}$

$\frac{1}{z}$ abt 1 $\quad \frac{1}{z} = \frac{1}{1+z-1} \overset{\text{conv. for } |z-1|<1}{=} \sum_{n=0}^{\infty} (-1)^n (z-1)^n$ ← anal on $|z-1|<1$ (ROC) for Laurent abt $z=1$

$z - z_0$

$a_n = (-1)^n$

$b_n = 0$

ROC: $|z-1| < 1$.

$R < 1$ (inside ROC)

$$\int_c \frac{1}{z}\,dz \overset{CG}{=} 0$$

$$\int_c \frac{1}{z}\,dz \overset{\text{Laurent}}{=} \int_c \sum_{\substack{n \geq 0 \\ n \in \mathbb{Z}}} (-1)^n (z-1)^n\,dz = 0.$$

ord $= m$

$$\phi(z) = f(z)(z-z_0)^m = (\text{some} \geq m \text{ powers}) + b_m + b_{m-1}(z-z_0) + \dots + b_1(z-z_0)^{m-1}$$

and, all derivs ✓

$$\phi^{(m-1)}(z_0) = (\text{some} \geq 1 \text{ power}) + \underbrace{b_1 (m-1)!}_{\text{const}} + 0$$

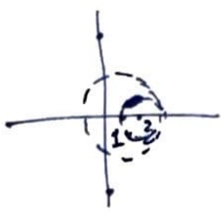
ie $(z-z_0)(\quad)$

$$\therefore \phi^{(m-1)}(z_0) = 0 + b_1(m-1)! + 0$$

$$= b_1 \cdot (m-1)!$$

$$\Rightarrow \boxed{b_1 = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}, \quad \phi(z) = f(z)(z-z_0)^m}$$

352. $1 + 2e^{i2\pi t}$



$$I_{C_1} = \oint_{C = \varepsilon e^{i\theta}+2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=2} f = 2\pi i\,\phi(2) = \frac{17}{20} \cdot 2\pi i = 1.7\pi i$$

ord $= 1$ as $f(z)(z-2)$ is anal at 2.

$$I_{C_2} = 2\pi i \left(\operatorname*{Res}_{z=2} + \operatorname*{Res}_{z=4i} + \operatorname*{Res}_{z=-4i}\right)$$

$$= 2\pi i \left[\frac{17}{20} + \frac{4^4+1}{(4i-2)(8i)} + \frac{4^4+1}{(4i+2)8i}\right]$$

CRT to evaluate improper integrals

$f: \mathbb{R} \to \mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\,dx := \lim_{r_1 \to -\infty} \int_{r_1}^{0} f(x)\,dx + \lim_{r_2 \to \infty} \int_0^{r_2} f(x)\,dx \quad \text{—— (1)}$$

both should exist & be finite then $\int_{-\infty}^{\infty} f(x)\,dx$ exists

$$PV \int_{-\infty}^{\infty} f(x)\,dx := \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx$$

if $\int_{-\infty}^{\infty} f(x)\,dx$ exists by (1)'st def then PV exists & is = the int by (1)

$\lim_{r_1 \to -\infty} \int_{r_1}^{0} f(x)\,dx$, $\lim_{r_2 \to \infty} \int_0^{r_2} f(x)\,dx$ exist and finite

$= l_1$, $= \lim_{r \to \infty} \int_{-r}^{0} f(x)\,dx$

$= l_2$, $= \lim_{r \to \infty} \int_0^{r} f(x)\,dx$

$$l_1 + l_2 = \lim_{r \to \infty} \int_{-r}^{r} f(x)\,dx$$

$= \int_{-\infty}^{\infty} f(x)\,dx$ by def ie $\int_{-\infty}^{\infty} f(x)\,dx$ by (1) $= l_1 + l_2$ integral exists by (1) $\lim_{r \to \infty} \int_{-r}^{r} f(x)\,dx = PV \int_{-\infty}^{\infty} f(x)\,dx$

$$|f(a)-f(b)| < \frac{|a-b|}{2\pi}\oint_C \left|\frac{f(z)\,z'(t)}{(z-a)(z-b)}\right| dt \qquad \underset{\downarrow}{R\cdot 2\pi i\, e^{i2\pi t}}$$

$$= |a-b| \oint_C \frac{|f(z)\,R|}{|(z-a)|\,|z-b|}\, dt$$

(+vely or

$$< |a-b| \oint_C \frac{kR}{|z-a||z-b|}\, dt$$

$$\leq |a-b| \oint_C \frac{kR}{(\underset{\substack{\uparrow\\ R}}{|z|}-|a|)(|z|-|b|)}\, dt$$

$$= |a-b| \oint_C \underbrace{\frac{kR}{(R-|a|)(R-|b|)}}_{\text{const}}\, dt$$

$$= \frac{|a-b|\,kR}{(R-|a|)(R-|b|)} \int_0^1 dt \qquad t: 0\to 1 \text{ +vely or}$$

$$= |a-b|\,k \frac{R}{(R-|a|)(R-|b|)} \quad \boxed{\forall R\in\mathbb{R}^+} *$$

$$|f(a)-f(b)| < |a-b|\,k \frac{R}{(R-|a|)(R-|b|)}$$

$\therefore \exists R\in\mathbb{R}^+$ st $|f(a)-f(b)|<\varepsilon$ for any $\varepsilon>0$

$\Rightarrow f(a)=f(b)\ \forall a,b\in\mathbb{C}$

$\Rightarrow \boxed{f(z)=c\ \forall z\in\mathbb{C}}$

Fundamental Theorem of Algebra

Let $P(x)=\sum_{r=0}^{n} a_r x^r$, $a_i\in\mathbb{C}$. Then $\exists z\in\mathbb{C}$ st $P(z)=0$. $\Rightarrow \exists$ exactly n $z\in\mathbb{C}$ st $P(z)=0$.
& $\boxed{\deg P\neq 0} \to P$ is nonconstant

Proof: FTSOC, $P(z)\neq 0\ \forall z\in\mathbb{C} \Rightarrow g(z)=\frac{1}{P(z)}$ is entire. We try to show g is bdd $\overset{\text{Liouville}}{\Leftrightarrow} g\equiv c \Leftrightarrow f\equiv c$ $\rightarrow\leftarrow$.

$$|g(z)| = \frac{1}{|P(z)|}$$

$$|P(z)| = |a_n z^n + a_{n-1}z^{n-1}+\dots+a_0| = \underset{\substack{\uparrow}}{|z|^n}\left|a_n + \sum_{k=1}^{n}\frac{a_{n-k}}{z^k}\right| \geq |z|^n\left(|a_n| - \left|\sum_{k=1}^{n}\frac{a_{n-k}}{z^k}\right|\right)$$

freedom on $|z|$, $n\neq 0$ as well! choose $|z|>1$ st

$$|z|^k \geq |z| > \left|\frac{2n\,a_{n-k}}{a_n}\right| \forall k$$

$$\Rightarrow \left|\frac{a_n}{2n}\right| > \left|\frac{a_{n-k}}{|z|^k}\right| \forall k$$