7. $w(x) = 1 + x^2$

$[a, b]$

$f_1(x) = 1 \qquad f_2(x) = x \qquad f_3(x) = x^2$

$L_1$

$$\|f_1\| = \sqrt{\int_{-1}^{1} (f_1(x))^2 w(x)\,dx} = \sqrt{\int_{-1}^{1} 1 \cdot (x^2+1)\,dx} = \sqrt{\left[\frac{x^3}{3} + x\right]_{-1}^{1}} = \sqrt{\left(\frac{1}{3}+1\right) - \left(-\frac{1}{3}-1\right)} = \sqrt{\frac{8}{3}}$$

$$L_1 = \frac{1}{\sqrt{\frac{8}{3}}} = \sqrt{\frac{3}{8}}$$

$L_2$

$$\|f_2\| = \sqrt{\int_{-1}^{1} x^2 (x^2+1)\,dx} = \sqrt{\int_{-1}^{1} x^4 + x^2\,dx} = \sqrt{\left[\frac{x^5}{5} + \frac{x^3}{3}\right]_{-1}^{1}} = \sqrt{\left(\frac{1}{5}+\frac{1}{3}\right) - \left(-\frac{1}{5}-\frac{1}{3}\right)} = \sqrt{\frac{16}{15}}$$

$$L_2 = \frac{x}{\sqrt{\frac{16}{15}}} = \frac{\sqrt{15}}{4}\,x$$

3] $L_3$: $g_3 = x^2 - \sum_{i=1}^{2} \langle L_i, f_{n+1} \rangle \cdot L_i(x)$

$= x^2 - \left[ \int_{-1}^{1} \sqrt{\frac{3}{8}}\, x^2 (1+x^2)\, dx \cdot \sqrt{\frac{3}{8}} + \int_{-1}^{1} \sqrt{\frac{15}{4}}\, x \cdot (1+x^2)\, dx \cdot \frac{\sqrt{15}}{4} x \right]$

$= x^2 - \left[ \frac{3}{8} \int_{-1}^{1} x^4 + x^2 \, dx + \frac{15}{16} \int_{-1}^{1} x^3 + x \, dx \right]$

$= x^2 - \frac{3}{8} \left[ \frac{x^5}{5} + \frac{x^3}{3} \right]_{-1}^{1} - \frac{15}{16} \left[ \frac{x^4}{4} + \frac{x^2}{2} \right]_{-1}^{1}$

$= x^2 - \frac{3}{8} \left( \frac{2}{5} + \frac{2}{3} \right) - \frac{15}{16} \left( \frac{1}{4} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2} \right)$

$= x^2 - \frac{3}{8} \left( \frac{16}{15} \right)$ (0)

$= x^2 - \frac{2}{5}$

$\| g_3 \| = \sqrt{\int_{-1}^{1} (x^2 - \frac{2}{5})^2 (1+x^2)\, dx}$

$= \sqrt{\int_{-1}^{1} (x^4 - \frac{4}{5}x^2 + \frac{4}{25})(1+x^2)\, dx}$

$= \sqrt{\int_{-1}^{1} x^4 - \frac{4}{5}x^2 + \frac{4}{25} + x^6 - \frac{4}{5}x^4 + \frac{4}{25}x^2 \, dx}$

$$= \sqrt{\int_{-1}^{1} \frac{1}{5}x^4 - \frac{16}{25}x^2 + x^6 + \frac{4}{25}\,dx}$$

$$= \sqrt{\left[\frac{x^7}{7} + \frac{x^5}{25} - \frac{16}{75}x^3 + \frac{4}{25}x\right]_{-1}^{1}}$$

(2)

$$= \sqrt{\frac{1}{7} + \frac{1}{7} + \frac{1}{25} + \frac{1}{25} - \frac{16}{75} - \frac{16}{75} + \frac{4}{25} + \frac{4}{25}}$$

$$= \sqrt{\frac{136}{525}} = \frac{2}{5}\sqrt{\frac{34}{21}}$$

$$L_3 = \frac{5\sqrt{21}}{2\sqrt{34}} \quad (x^2 - \frac{2}{5})$$