Lagrangian Function and Duality

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Lagrangian Function

Consider the constrained problem again

$$
\begin{array}{lll}
(CO) & \inf & f(\mathbf{x}) \\
 & \text{s.t.} & c_i(\mathbf{x}) \le 0, i = 1, 2, ..., m,
\end{array}
$$

Lagrangian Function:

$$L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \mathbf{y}^T \mathbf{c}(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} y_i c_i(\mathbf{x}),$$

where the entries of $\mathbf{y} \in R^m$ are called Lagrange multipliers.

CO Example

Consider a toy problem

$$\begin{array}{ll} \text{minimize} & (x_1-1)^2+(x_2-1)^2 \\ \text{subject to} & x_1+2x_2-1 \le 0, \\ & 2x_1+x_2-1 \le 0. \end{array}$$

$$L(\mathbf{x},\mathbf{y}) = f(\mathbf{x}) + \mathbf{y}^T\mathbf{c}(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} y_i c_i(\mathbf{x}) =$$

$$= (x_1-1)^2+(x_2-1)^2 + y_1(x_1+2x_2-1) + y_2(2x_1+x_2-1).$$

Lagrangian Relaxation Problem

For given $\mathbf{y} \geq \mathbf{0}$

$$
\begin{array}{lll}
(LRP) & \inf & L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \mathbf{y}^T \mathbf{c}(\mathbf{x}) \\
& \text{s.t.} & \mathbf{x} \in R^n.
\end{array}
$$

Here, $\mathbf{y}_i$ can be viewed as a penalty parameter to penalize constraint violation $c_i(\mathbf{x}), i = 1, ..., m$. For the example:

$$
\begin{array}{ll}
\inf & (x_1 - 1)^2 + (x_2 - 1)^2 + y_1(x_1 + 2x_2 - 1) + y_2(2x_1 + x_2 - 1) \\
\text{s.t.} & (x_1; x_2) \in R^2,
\end{array}
$$

and its minimal value is

$$
-1.25y_1^2 - 1.25y_2^2 - 2y_1y_2 + 2y_1 + 2y_2.
$$

Lagrangian Dual Function

For any $\mathbf{y}$

$$\begin{array}{rll} \phi(\mathbf{y}) := & \inf & L(\mathbf{x},\mathbf{y}) = f(\mathbf{x}) + \mathbf{y}^T\mathbf{c}(\mathbf{x}) \\ & \text{s.t.} & \mathbf{x} \in R^n. \end{array}$$

Theorem 1 *The Lagrangian dual function $\phi(\mathbf{y})$ is a concave function.*

$$\begin{aligned} \phi(\alpha\mathbf{y}^1 + (1-\alpha)\mathbf{y}^2) &= f(\mathbf{x}^*) + (\alpha\mathbf{y}^1 + (1-\alpha)\mathbf{y}^2)^T\mathbf{c}(\mathbf{x}^*) \\ &= \alpha f(\mathbf{x}^*) + (1-\alpha)f(\mathbf{x}^*) + (\alpha\mathbf{y}^1 + (1-\alpha)\mathbf{y}^2)^T\mathbf{c}(\mathbf{x}^*) \\ &= \alpha L(\mathbf{x}^*,\mathbf{y}^1) + (1-\alpha)L(\mathbf{x}^*,\mathbf{y}^2) \\ &\geq \alpha\phi(\mathbf{y}^1) + (1-\alpha)\phi(\mathbf{y}^2), \end{aligned}$$

where $\mathbf{x}^*$ is a minimizer of $L(\mathbf{x},\alpha\mathbf{y}^1 + (1-\alpha)\mathbf{y}^2)$.

Dual Function Establishes a Lower Bound

Theorem 2 *(Weak duality theorem) For every $\mathbf{y} \geq \mathbf{0}$, the Lagrangian dual function $\phi(\mathbf{y})$ is less or equal to the infimum value of the original CO problem.*

$$\begin{aligned}\phi(\mathbf{y}) &= \inf\ \{f(\mathbf{x}) + \mathbf{y}^T\mathbf{c}(\mathbf{x})\} \\ &\leq \inf\ \{f(\mathbf{x}) + \mathbf{y}^T\mathbf{c}(\mathbf{x}) \text{ s.t. } \mathbf{c}(\mathbf{x}) \leq \mathbf{0}\} \\ &\leq \inf\ \{f(\mathbf{x}) :\ \text{ s.t. } \mathbf{c}(\mathbf{x}) \leq \mathbf{0}\}.\end{aligned}$$

Recall the toy example:

$$\begin{array}{ll}\text{minimize} & (x_1 - 1)^2 + (x_2 - 1)^2 \\ \text{subject to} & x_1 + 2x_2 - 1 \leq 0, \\ & 2x_1 + x_2 - 1 \leq 0;\end{array}$$

$$\phi(\mathbf{y}) = -1.25y_1^2 - 1.25y_2^2 - 2y_1y_2 + 2y_1 + 2y_2.$$

The Lagrangian Dual Problem

$$
\begin{array}{lll}
(COD) & \sup & \phi(\mathbf{y}) \\
& \text{s.t.} & \mathbf{y} \geq \mathbf{0}.
\end{array}
$$

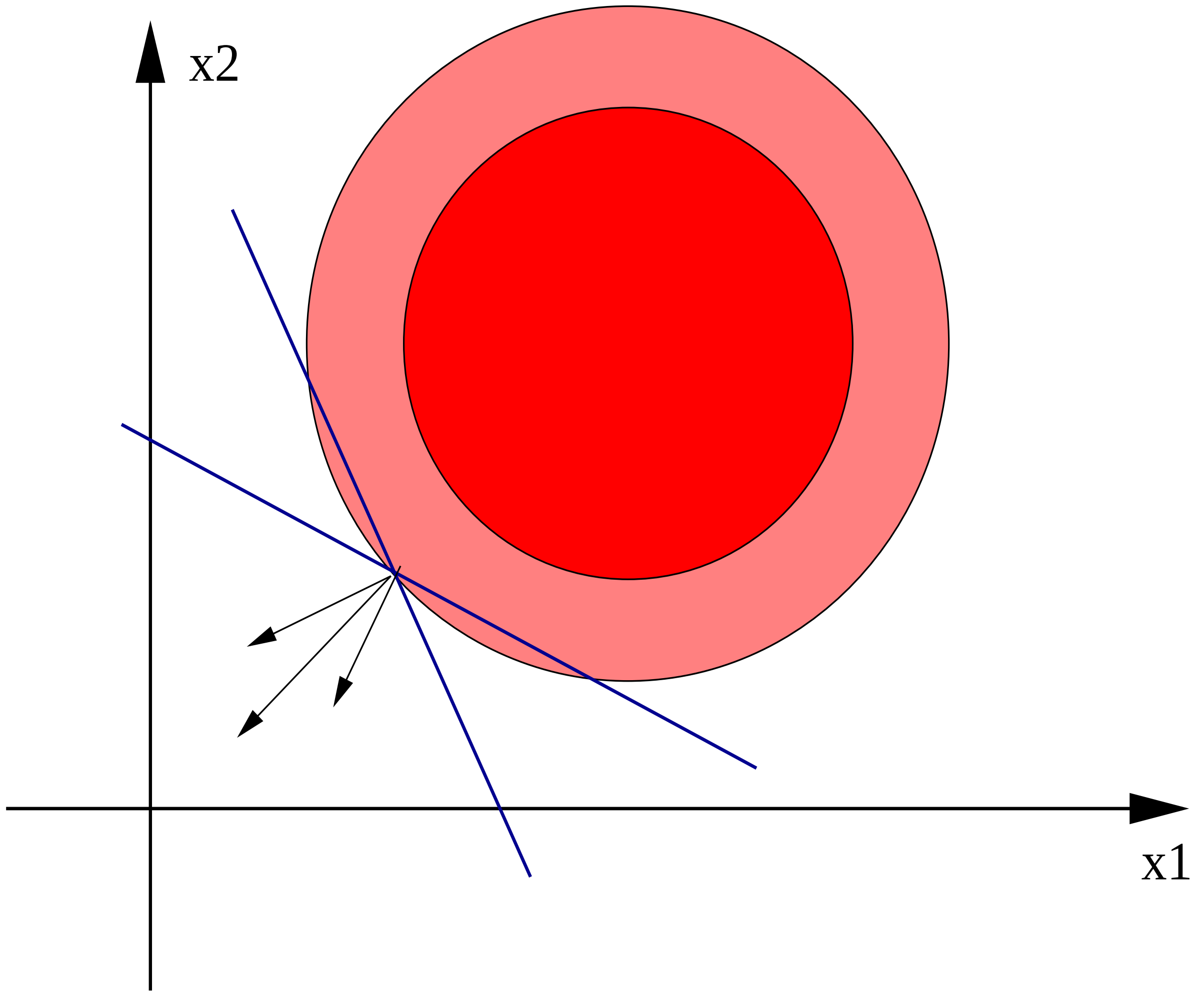
would called the Lagrangian dual of the original CO problem:

$$
\begin{array}{lll}
(COP) & \inf & f(\mathbf{x}) \\
& \text{s.t.} & c_i(\mathbf{x}) \leq 0, i = 1, 2, ..., m.
\end{array}
$$

$$
\begin{array}{ll}
\max & -1.25y_1^2 - 1.25y_2^2 - 2y_1y_2 + 2y_1 + 2y_2 \\
\text{s.t.} & (y_1; y_2) \geq \mathbf{0}.
\end{array}
$$

$$
\begin{array}{ll}
\min & (x_1 - 1)^2 + (x_2 - 1)^2 \\
\text{s.t.} & x_1 + 2x_2 - 1 \leq 0, \\
& 2x_1 + x_2 - 1 \leq 0.
\end{array}
$$

$$\bar{\mathbf{x}} = \left(\frac{1}{3}; \ \frac{1}{3}\right).$$



Lagrangian Strong Duality Theorem

Theorem 3 *Let (COP) be a convex minimization problem, the infimum* f^* *of (COP) be finite, and the suprermum of (COD) be* ϕ^**. In addition, let (COP) have an interior-point feasible solution, that is, there is* $\hat{\mathbf{x}}$ *such that* $\mathbf{c}(\hat{\mathbf{x}}) < \mathbf{0}$*. Then,* $f^* = \phi^*$*, and (COD) admits a maximizer* $\mathbf{y}^*$ *such that*

$$\phi(\mathbf{y}^*) = f^*.$$

Furthermore, if (COP) admits a minimizer $\mathbf{x}^*$*, then*

$$y_i^* c_i(\mathbf{x}^*) = 0, \ \forall i = 1, ..., m.$$

The assumption of "interior-point feasible solution" is called Constraint Qualification condition.

Constraint Qualification

Consider the problem

$$\begin{aligned} \min \quad & x_1 \\ \text{s.t.} \quad & x_1^2 + (x_2 - 1)^2 - 1 \le 0, \\ & x_1^2 + (x_2 + 1)^2 - 1 \le 0, \end{aligned}$$

$$\mathbf{x}^* = (0;\ 0).$$

$$L(\mathbf{x}, \mathbf{y}) = x_1 + y_1(x_1^2 + (x_2 - 1)^2 - 1) + y_2(x_1^2 + (x_2 + 1)^2 - 1).$$

$$\phi(\mathbf{y}) = \frac{-1 - (y_1 - y_2)^2}{y_1 + y_2}.$$

Proof of Lagrangian Strong Duality Theorem

Consider the convex set

$$C := \{(\kappa; \mathbf{s}) : \ \exists \mathbf{x} \text{ s.t. } f(\mathbf{x}) \le \kappa, \ \mathbf{c}(\mathbf{x}) \le \mathbf{s}\}.$$

Then, $(f^*; \mathbf{0})$ is on the closure of C. From the supporting hyper-plane theorem, there exists $(y_0^*; \mathbf{y}^*) \neq \mathbf{0}$ such that

$$y_0^* f^* \le \inf_{(\kappa;\mathbf{s}) \in C} (y_0^* \kappa + (\mathbf{y}^*)^T \mathbf{s}).$$

First, we show $\mathbf{y}^* \ge \mathbf{0}$, since otherwise one can choose $(0; \mathbf{s} > \mathbf{0})$ such that the inequality is violated.

Secondly, we show $y_0^* > 0$, since otherwise one can choose $(\kappa; \mathbf{0})$ or $(0; \mathbf{s} = \mathbf{c}(\hat{\mathbf{x}}) < \mathbf{0})$ such that the above inequality is violated.

Now let us divide both sides by y_0^*, we have

$$f^* \le \inf_{(\kappa;\mathbf{s})\in C}(\kappa + (\mathbf{y}^*)^T\mathbf{s}) = \inf_{\mathbf{x}}(f(\mathbf{x}) + (\mathbf{y}^*)^T\mathbf{c}(\mathbf{x}))$$

$$\le \inf_{\mathbf{x}:\ \mathbf{c}(\mathbf{x})\le\mathbf{0}}(f(\mathbf{x}) + (\mathbf{y}^*)^T\mathbf{c}(\mathbf{x})) = \phi^*.$$

If (COP) admits a minimizer $\mathbf{x}^*$, then $f(\mathbf{x}^*) = f^*$ so that

$$f(\mathbf{x}^*) \le f(\mathbf{x}^*) + (\mathbf{y}^*)^T\mathbf{c}(\mathbf{x}^*) = f(\mathbf{x}^*) + \sum_i^m y_i^* c_i(\mathbf{x}^*),$$

or

$$\sum_i^m y_i^* c_i(\mathbf{x}^*) \ge 0.$$

Since $y_i^* \ge 0$ and $c_i(\mathbf{x}^*) \le 0$ for all i, it must be true $y_i^* c_i(\mathbf{x}^*) = 0$ for all i.

More on Lagrangian Duality

Consider the constrained problem again

$$\begin{array}{lll} (COP) & \inf & f(\mathbf{x}) \\ & \text{s.t.} & \mathbf{h}(\mathbf{x}) = \mathbf{0} \in R^m, \\ & & \mathbf{c}(\mathbf{x}) \leq \mathbf{0} \in R^p, \\ & & \mathbf{x} \in \Omega \subset R^n. \end{array}$$

Typically, Ω is simple set such as a cone or box.

The problem would be a convex optimization problem if $\mathbf{h}(\mathbf{x}) = A\mathbf{x} - \mathbf{b},\ A \in R^{m \times n},\ n \geq m$ (affine functions), all other functions are convex, and Ω is a convex set.

Lagrangian Relaxation Problem

Lagrangian Function:

$$L(\mathbf{x}, \mathbf{y}, \mathbf{s}) = f(\mathbf{x}) + \mathbf{y}^T \mathbf{h}(\mathbf{x}) + \mathbf{s}^T \mathbf{c}(\mathbf{x}),$$

where the entries of $\mathbf{y} \in R^m$ are Lagrange multipliers of the m equality constraints and $\mathbf{s} \in R^p \geq \mathbf{0}$ are Lagrange multipliers of the p inequality equality constraints.

$$\begin{array}{rll} \phi(\mathbf{y}, \mathbf{s}) := & \inf & L(\mathbf{x}, \mathbf{y}, \mathbf{s}) \\ & \text{s.t.} & \mathbf{x} \in \Omega. \end{array}$$

Theorem 4 *The Lagrangian dual function $\phi(\mathbf{y}, \mathbf{s})$ is a concave function.*

Theorem 5 *(Weak duality theorem) For every $\mathbf{s} \geq \mathbf{0}$, the Lagrangian dual function $\phi(\mathbf{y}, \mathbf{s})$ is less or equal to the infimum value of the original CO problem.*

The Lagrangian Dual Problem

$$
\begin{array}{lll}
(COD) & \sup & \phi(\mathbf{y}, \mathbf{s}) \\
 & \text{s.t.} & \mathbf{s} \geq \mathbf{0}.
\end{array}
$$

would called the Lagrangian dual of the original CO problem:

Theorem 6 *(Strong duality theorem) Let (COP) be a convex minimization problem, the infimum f^* of (COP) be finite, and the suprermum of (COD) be ϕ^*. In addition, let (COP) have an interior-point feasible solution, that is, there is $\hat{\mathbf{x}}$ such that $\mathbf{c}(\hat{\mathbf{x}}) < \mathbf{0}$. Then, $f^* = \phi^*$, and (COD) admits a maximizer $\mathbf{y}^*$ such that*

$$\phi(\mathbf{y}^*, \mathbf{s}^*) = f^*.$$

Furthermore, if (COP) admits a minimizer $\mathbf{x}^*$, then

$$y_i^* c_i(\mathbf{x}^*) = 0, \ \forall i = 1, ..., m.$$