MS&E 311 Winter 2013-2014
Optimization Jan 10, 2014
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Homework Assignment 1
Discussed Friday Jan 24, 2014

Optional Reading. Read Luenberger and Ye's *Linear and Nonlinear Programming Third Edition* Chapter 1 and Appendices A and B.

Solve the following problems:

1. Consider the SOCP problem

$$\begin{array}{ll} \min & 2x_1 + x_2 + x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 = 1, \\ & x_1 - \sqrt{x_2^2 + x_3^2} \geq 0. \end{array}$$

   Try to find a minimizer and argue why it is a global minimizer.

2. Prove that every local minimizer is also a global minimizer for convex optimization problems. Convex optimization is a problem minimize a continuous convex objective function value over a feasible region that is a convex set

3. Prove the the dual cone of the p-order cone, $p = 1, ..., \infty$, is the q-order cone where $\frac{1}{q} + \frac{1}{p} = 1$.

4. When C is a cone, the dual of C is the closed convex cone

$$C^* = \{y : y \bullet x \geq 0 \quad \forall x \in C\},$$

   where $\bullet$ denotes the inner product. In the following, let all cones be defined in a same metric space, and both C_1 and C_2 are closed convex cones. Show

   (i) $(C_1^*)^* = C_1$.

   (ii) $C_1 \subset C_2 \Longrightarrow C_2^* \subset C_1^*$.

   (iii) $(C_1 + C_2)^* = C_1^* \cap C_2^*$.

   (iv) $C_1^* + C_2^* \subset (C_1 \cap C_2)^*$.

   (v) Let C be the cone of positive semidefinite matrices whose every entry is also nonnegative. What is C^*?

Note: If S and T are subsets of n-dimensional symmetric matrices, then by definition

$$S + T = \{s + t : s \in S,\ t \in T\}.$$

5. (20pts.) Let $f(x) : R^n_{++} \to R$ be a given convex function. Show that the function $g : R^{n+1}_{++} \to R$ given by $g(\tau; x) = \tau \cdot f(x/\tau)$ (called the homogenized version of f) is also a convex function in the domain of $(\tau; x) \in R^{n+1}_{++}$. Now, suppose that $f(x)$ is twice-differentiable. Write out the gradient vector and Hessian matrix of g.

6. (20pts.) Let $g_1, \ldots, g_m$ be a collection of concave functions on R^n such that

$$S = \{x : g_i(x) > 0 \text{ for } i = 1, \ldots, m\} \neq 0.$$

Show that for any positive constant μ and any convex function f on R^n, the function

$$h(x) = f(x) - \mu \sum_{i=1}^{m} \ln g_i(x)$$

is convex over S.

7. Prove that the set $\{Ax : \ x \geq 0 \in R^n\}$ is closed and convex.

8. Farkas' lemma can be used to derive many other (named) theorems of the alternative. This problem concerns a few of these pairs of systems. Using Farkas's lemma, prove each of the following results.

(a) Gordan's Theorem. Exactly one of the following systems has a solution:

$$\begin{array}{rl} \text{(i)} & Ax > 0 \\ \text{(ii)} & y^T A = 0, \quad y \geq 0, \quad y \neq 0. \end{array}$$

(b) Stiemke's Theorem. Exactly one of the following systems has a solution:

$$\begin{array}{rl} \text{(i)} & Ax \geq 0, \quad Ax \neq 0 \\ \text{(ii)} & y^T A = 0, \quad y > 0 \end{array}$$

(c) Gale's Theorem. Exactly one of the following systems has a solution:

$$\begin{array}{rl} \text{(i)} & Ax \leq b \\ \text{(ii)} & y^T A = 0, \quad y^T b < 0, \quad y \geq 0 \end{array}$$