Interior Point Algorithms III

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(LY: Chapter 5)

Termination with Exact Optimizers

- The first is a “cross-over” procedure to find a basic feasible solution (BFS, corner point) whose objective value is at least as good as the current interior point. Let $A, \mathbf{b}, \mathbf{c}$ be integers and L be their bit length, and let a second best BFS solution be $\mathbf{x}^{2nd}$ and the optimal objective value be z^*. Then

$$\mathbf{c}^T\mathbf{x}^{2nd} - z^* > 2^{-L}.$$

  Thus, one can terminate interior-point algorithm when

$$\mathbf{c}^T\mathbf{x}^k - \mathbf{b}^T\mathbf{y}^k \le 2^{-L}.$$

- The second approach is to compute a strictly complementary solution pair. The method uses the primal-dual interior-point pair to identify the strict complementarity partition (P^*, Z^*) and then “purify or project” the primal interior solution onto the primal optimal face and the dual interior solution onto the dual optimal face, based on the following theorem:

Theorem 1 *Given an interior solution* $\mathbf{x}^k$ *and* $\mathbf{s}^k$ *in the solution sequence generated by an interior-point algorithm, define*

$$P^k = \{j : x_j^k \geq s_j^k, \ \forall j\} \quad \textit{and} \quad Z^k = \{1, ..., n\} \setminus P^k.$$

Then, we have $P^k = P^*$ *whenever*

$$\mathbf{c}^T\mathbf{x}^k - \mathbf{b}^T\mathbf{y}^k \leq 2^{-L}.$$

Thus, the worst-case iteration bound for interior-point algorithms is $O(\sqrt{n}L)$ if the initial point pair $(\mathbf{x}^0)^T\mathbf{s}^0 \leq 2^L$.

Initialization

- Combining the primal and dual into a single linear feasibility problem, then applying LP algorithms to find a feasible point of the problem. Theoretically, this approach can retain the currently best complexity result.
- The big M method, i.e., add one or more artificial column(s) and/or row(s) and a huge penalty parameter M to force solutions to become feasible during the algorithm.
- Phase I-then-Phase II method, i.e., first try to find a feasible point (and possibly one for the dual problem), and then start to look for an optimal solution if the problem is feasible and bounded.
- Combined Phase I-Phase II method, i.e., approach feasibility and optimality simultaneously. To our knowledge, the "best" complexity of this approach is $O(n \log(R/\epsilon))$.

Homogeneous and Self-Dual Algorithm

- It solves the linear programming problem without any regularity assumption concerning the existence of optimal, feasible, or interior feasible solutions, while it retains the currently best complexity result
- It can start at any positive primal-dual pair, feasible or infeasible, near the central ray of the positive orthant (cone), and it does not use any big M penalty parameter or lower bound.
- Each iteration solves a system of linear equations whose dimension is almost the same as that solved in the standard (primal-dual) interior-point algorithms.
- If the LP problem has a solution, the algorithm generates a sequence that approaches feasibility and optimality simultaneously; if the problem is infeasible or unbounded, the algorithm will produce an infeasibility certificate for at least one of the primal and dual problems.

Primal-Dual Alternative Systems

A pair of LP has two alternatives

(Solvable)
$$\begin{aligned} A\mathbf{x}-\mathbf{b} &= \mathbf{0} \\ -A^T\mathbf{y}+\mathbf{c} &\geq \mathbf{0}, \\ \mathbf{b}^T\mathbf{y}-\mathbf{c}^T\mathbf{x} &= 0, \\ \mathbf{y}\text{ free},\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

or

(Infeasible)
$$\begin{aligned} A\mathbf{x} &= \mathbf{0} \\ -A^T\mathbf{y} &\geq \mathbf{0}, \\ \mathbf{b}^T\mathbf{y}-\mathbf{c}^T\mathbf{x} &> 0, \\ \mathbf{y}\text{ free},\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

An Integrated Homogeneous System

The two alternative systems can be homogenized as one:

$$
\begin{array}{rrl}
(HP) & A\mathbf{x} - \mathbf{b}\tau & = \mathbf{0} \\
 & -A^T\mathbf{y} + \mathbf{c}\tau & = \mathbf{s} \geq \mathbf{0}, \\
 & \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} & = \kappa \geq 0, \\
 & \mathbf{y} \text{ free}, \ (\mathbf{x}; \tau) & \geq \mathbf{0}
\end{array}
$$

where the two alternatives are

$$\text{(Solvable)}: \ (\tau > 0, \kappa = 0) \quad \text{or} \quad \text{(Infeasible)}: \ (\tau = 0, \kappa > 0)$$

The Homogeneous System is Self-Dual

$$
\begin{array}{lrll}
(HP) & A\mathbf{x} - \mathbf{b}\tau & = \mathbf{0}, & (\mathbf{y}') \\
 & -A^T\mathbf{y} + \mathbf{c}\tau & = \mathbf{s} \geq \mathbf{0}, & (\mathbf{x}') \\
 & \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} & = \kappa \geq 0, & (\tau') \\
 & \mathbf{y} \text{ free},\ (\mathbf{x}; \tau) & \geq \mathbf{0} &
\end{array}
\qquad
\begin{array}{lrl}
(HD) & A\mathbf{x}' - \mathbf{b}\tau' & = \mathbf{0}, \\
 & A^T\mathbf{y}' - \mathbf{c}\tau' & \leq \mathbf{0}, \\
 & -\mathbf{b}^T\mathbf{y}' + \mathbf{c}^T\mathbf{x}' & \leq 0, \\
 & \mathbf{y}' \text{ free},\ (\mathbf{x}'; \tau') & \geq \mathbf{0}
\end{array}
$$

Theorem 2 *System (HP) is feasible (e.g. all zeros) and any feasible solution* $(\mathbf{y}, \mathbf{x}, \tau, \mathbf{s}, \kappa)$ *is self-complementary:*

$$\mathbf{x}^T\mathbf{s} + \tau\kappa = 0.$$

Furthermore, it has a strictly self-complementary feasible solution

$$\begin{pmatrix} \mathbf{x} + \mathbf{s} \\ \tau + \kappa \end{pmatrix} > \mathbf{0},$$

Let's Find Such a Feasible Solution

Given $\mathbf{x}^0 = \mathbf{e} > \mathbf{0}$, $\mathbf{s}^0 = \mathbf{e} > \mathbf{0}$, and $\mathbf{y}^0 = \mathbf{0}$, we formulate

$$
\begin{array}{rrrrrrl}
(HSDP) & \min & & & & \theta & \\
 & \text{s.t.} & & A\mathbf{x} & -\mathbf{b}\tau & +\bar{\mathbf{b}}\theta & = \mathbf{0}, \\
 & & -A^T\mathbf{y} & & +\mathbf{c}\tau & -\bar{\mathbf{c}}\theta & \geq \mathbf{0}, \\
 & & \mathbf{b}^T\mathbf{y} & -\mathbf{c}^T\mathbf{x} & & +\bar{z}\theta & \geq 0, \\
 & & \mathbf{y} \text{ free}, & \mathbf{x} \geq \mathbf{0}, & \tau \geq 0, & \theta \text{ free}, &
\end{array}
$$

where

$$
\bar{\mathbf{b}} = \mathbf{b} - A\mathbf{e}, \quad \bar{\mathbf{c}} = \mathbf{c} - \mathbf{e}, \quad \bar{z} = \mathbf{c}^T\mathbf{e} + 1.
$$

But it may just give us the all-zero solution.

A HSD linear program

Let's try to add one more constraint to prevent the all-zero solution

$$
\begin{array}{lrrrrl}
(HSDP) \quad \min & & & & (n+1)\theta & \\
\text{s.t.} & & A\mathbf{x} & -\mathbf{b}\tau & +\bar{\mathbf{b}}\theta & = \mathbf{0}, \\
& -A^T\mathbf{y} & & +\mathbf{c}\tau & -\bar{\mathbf{c}}\theta & \geq \mathbf{0}, \\
& \mathbf{b}^T\mathbf{y} & -\mathbf{c}^T\mathbf{x} & & +\bar{z}\theta & \geq 0, \\
& -\bar{\mathbf{b}}^T\mathbf{y} & +\bar{\mathbf{c}}^T\mathbf{x} & -\bar{z}\tau & & = -(n+1), \\
& \mathbf{y} \text{ free}, & \mathbf{x} \geq \mathbf{0}, & \tau \geq 0, & \theta \text{ free}. &
\end{array}
$$

Note that the constraints of (HSDP) form a skew-symmetric system and the objective coeffcient vector is the negative of the right-hand-side vector, so that it remains a self-dual linear program.

$(\mathbf{y} = \mathbf{0},\ \mathbf{x} = \mathbf{e},\ \tau = 1,\ \theta = 1)$ is a strictly feasible point for (HSDP).

$$
\begin{array}{lrrrrl}
(HSDP) & \min & & & (n+1)\theta & \\
& \text{s.t.} & A\mathbf{x} & -\mathbf{b}\tau & +\bar{\mathbf{b}}\theta & = \mathbf{0}, \\
& -A^T\mathbf{y} & & +\mathbf{c}\tau & -\bar{\mathbf{c}}\theta & = \mathbf{s} \geq \mathbf{0}, \\
& \mathbf{b}^T\mathbf{y} & -\mathbf{c}^T\mathbf{x} & & +\bar{z}\theta & = \kappa \geq 0, \\
& -\bar{\mathbf{b}}^T\mathbf{y} & +\bar{\mathbf{c}}^T\mathbf{x} & -\bar{z}\tau & & = -(n+1), \\
& \mathbf{y} \text{ free}, & \mathbf{x} \geq \mathbf{0}, & \tau \geq 0, & \theta \text{ free}. &
\end{array}
$$

Denote by $\mathcal{F}_h$ the set of all points $(\mathbf{y}, \mathbf{x}, \tau, \theta, \mathbf{s}, \kappa)$ that are feasible for (HSDP). Denote by $\mathcal{F}_h^0$ the set of interior feasible points with $(\mathbf{x}, \tau, \mathbf{s}, \kappa) > \mathbf{0}$ in $\mathcal{F}_h$. By combining the constraints, we can derive the last (equality) constraint as

$$
\mathbf{e}^T x + \mathbf{e}^T s + \tau + \kappa - (n+1)\theta = (n+1),
$$

which serves indeed as a normalizing constraint for (HSDP) to prevent the all-zero solution.

Theorem 3 *Consider problems (HSDP) and (HSDD).*

i) *(HSDD) has the same form as (HSDP), i.e., (HSDD) is simply (HSDP) with* $(\mathbf{y}, \mathbf{x}, \tau, \theta)$ *being replaced by* $(\mathbf{y}', \mathbf{x}', \tau', \theta')$.

ii) *(HSDP) has a strictly feasible point*

$$\mathbf{y} = \mathbf{0}, \quad x = \mathbf{e} > \mathbf{0}, \quad \tau = 1, \quad \theta = 1, \quad \mathbf{s} = \mathbf{e} > \mathbf{0}, \quad \kappa = 1.$$

iii) *(HSDP) has an optimal solution and its optimal solution set is bounded.*

iv) *The optimal value of (HSDP) is zero, and*

$$(\mathbf{y}, \mathbf{x}, \tau, \theta, \mathbf{s}, \kappa) \in \mathcal{F}_h \quad \text{implies that} \quad (n+1)\theta = \mathbf{x}^T\mathbf{s} + \tau\kappa.$$

v) *There is an optimal solution* $(\mathbf{y}^*, \mathbf{x}^*, \tau^*, \theta^* = 0, \mathbf{s}^*, \kappa^*) \in \mathcal{F}_h$ *such that*

$$\begin{pmatrix} \mathbf{x}^* + \mathbf{s}^* \\ \tau^* + \kappa^* \end{pmatrix} > \mathbf{0},$$

which we call a strictly self-complementary solution. (Similarly, we sometimes call an optimal solution to (HSDP) a self-complementary solution; the strict inequalities above need not hold.)

Theorem 4 *Let* $(\mathbf{y}^*, \mathbf{x}^*, \tau^*, \theta^* = 0, \mathbf{s}^*, \kappa^*)$ *be a strictly self complementary solution for (HSDP).*

i) *(LP) has a solution (feasible and bounded) if and only if* $\tau^* > 0$*. In this case,* $\mathbf{x}^*/\tau^*$ *is an optimal solution for (LP) and* $(\mathbf{y}^*/\tau^*, \mathbf{s}^*/\tau^*)$ *is an optimal solution for (LD).*

ii) *(LP) has no solution if and only if* $\kappa^* > 0$*. In this case,* $\mathbf{x}^*/\kappa^*$ *or* $\mathbf{s}^*/\kappa^*$ *or both are certificates for proving infeasibility: if* $\mathbf{c}^T\mathbf{x}^* < 0$ *then (LD) is infeasible; if* $-\mathbf{b}^T\mathbf{y}^* < 0$ *then (LP) is infeasible; and if both* $\mathbf{c}^T\mathbf{x}^* < 0$ *and* $-\mathbf{b}^T\mathbf{y}^* < 0$ *then both (LP) and (LD) are infeasible.*

Theorem 5 **i)** *For any* $\mu > 0$*, there is a unique* $(\mathbf{y}, \mathbf{x}, \tau, \theta, \mathbf{s}, \kappa)$ *in* $\mathcal{F}_h^0$*, such that*

$$\begin{pmatrix} X\mathbf{s} \\ \\ \tau\kappa \end{pmatrix} = \mu \mathbf{e}.$$

ii) *Let* $(\mathbf{d}_y, \mathbf{d}_x, d_\tau, d_\theta, \mathbf{d}_s, d_\kappa)$ *be in the null space of the constraint matrix of (HSDP) after adding surplus variables* $\mathbf{s}$ *and* κ*, i.e.,*

$$\begin{array}{rrrrrrcl} & A\mathbf{d}_x & -\mathbf{b}d_\tau & +\bar{\mathbf{b}}d_\theta & & & = & \mathbf{0}, \\ -A^T\mathbf{d}_y & & +\mathbf{c}d_\tau & -\bar{\mathbf{c}}d_\theta & -\mathbf{d}_s & & = & \mathbf{0}, \\ \mathbf{b}^T\mathbf{d}_y & -\mathbf{c}^T\mathbf{d}_x & & +\bar{z}d_\theta & & -d_\kappa & = & 0, \\ -\bar{\mathbf{b}}^T\mathbf{d}_y & +\bar{\mathbf{c}}^T\mathbf{d}_x & -\bar{z}d_\tau & & & & = & 0. \end{array} \qquad (1)$$

$$(\mathbf{d}_x)^T\mathbf{d}_s + d_\tau d_\kappa = 0.$$

Endogenous Potential Function and Central Path

$$\psi_{n+\rho}(\mathbf{x},\mathbf{s},\tau,\kappa) := (n+1+\rho)\log(\mathbf{x}^T\mathbf{s}+\tau\kappa)-\sum_{j=1}^{n}\log(x_j s_j)-\log(\tau\kappa),$$

and

$$\mathcal{C}=\left\{(\mathbf{y},\mathbf{x},\tau,\theta,\mathbf{s},\kappa)\in\mathcal{F}_h^0 : \begin{pmatrix} X\mathbf{s} \\ \tau\kappa \end{pmatrix} = \frac{\mathbf{x}^T\mathbf{s}+\tau\kappa}{n+1}\mathbf{e}\right\}.$$

Obviously, the initial interior feasible point proposed in Theorem 3 is on the path with $\mu=1$ or $(\mathbf{x}^0)^T\mathbf{s}^0+\tau^0\kappa^0=n+1$.

Solving (HSDP)

Consider solving the following system of linear equations for $(\mathbf{d}_y, \mathbf{d}_x, d_\tau, d_\theta, \mathbf{d}_s, d_\kappa)$ that satisfies (1) and

$$\begin{pmatrix} X\mathbf{d}_s + S\mathbf{d}_x \\ \tau^k d_\kappa + \kappa^k d_\tau \end{pmatrix} = \gamma\mu\mathbf{e} - \begin{pmatrix} X\mathbf{s} \\ \tau\kappa \end{pmatrix}.$$

Theorem 6 *The $O(\sqrt{n}\log((\mathbf{x}^0)^T\mathbf{s}^0/\epsilon))$ interior-point algorithm, coupled with a termination technique described above, generates a strictly self-complementary solution for (HSDP) in $O(\sqrt{n}(\log(c(A,\mathbf{b},\mathbf{c}))+\log n))$ iterations and $O(n^3(\log(c(A,\mathbf{b},\mathbf{c}))+\log n))$ operations, where $c(A,\mathbf{b},\mathbf{c})$ is a positive number depending on the data $(A,\mathbf{b},\mathbf{c})$. If (LP) and (LD) have integer data with bit length L, then by the construction, the data of (HSDP) remains integral and its length is $O(L)$. Moreover, $c(A,\mathbf{b},\mathbf{c})\le 2^L$. Thus, the algorithm terminates in $O(\sqrt{n}L)$ iterations and $O(n^3L)$ operations.*

Example

Consider the example where

$$A = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}, \quad b = 1, \qquad \text{and} \quad \mathbf{c} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}.$$

Then,

$$y^* = 2, \quad \mathbf{x}^* = (0, 2, 1)^T, \quad \tau^* = 0, \quad \theta^* = 0, \quad \mathbf{s}^* = (2, 0, 0)^T, \quad \kappa^* = 1$$

could be a strictly self-complementary solution generated for (HSDP) with

$$\mathbf{c}^T \mathbf{x}^* = 1 > 0, \quad b y^* = 2 > 0.$$

Thus $(y^*, \mathbf{s}^*)$ demonstrates the infeasibility of (LP), but $\mathbf{x}^*$ doesn't show the infeasibility of (LD). Of course, if the algorithm generates instead $\mathbf{x}^* = (0, 1, 2)^T$, then we get demonstrated infeasibility of both.

Primal-Dual Interior-Point Algorithm for Monotone LCP

Find $(\mathbf{x}, \mathbf{s}) \geq 0$ such that

$$\mathbf{s} = M\mathbf{x} + \mathbf{q}, \; \mathbf{x}^T\mathbf{s} = 0, \; (\mathbf{x}, \mathbf{s}) \geq 0.$$

Once have a pair $(\mathbf{x}, \mathbf{s}) \in \text{int}\, \mathcal{F}$, we compute direction vectors $\mathbf{d}_x$ and $\mathbf{d}_s$ from the Newton system equations:

$$\begin{aligned} S\mathbf{d}_x + X\mathbf{d}_s &= \frac{\mathbf{x}^T\mathbf{s}}{n+\rho}\mathbf{e} - XS\mathbf{e}, \\ M\mathbf{d}_x - \mathbf{d}_s &= \mathbf{0}. \end{aligned} \qquad (2)$$

Note that $\mathbf{d}_x^T\mathbf{d}_s = \mathbf{d}_x^T M\mathbf{d}_x \geq 0$ if M is monotone.

Lemma 1 *Let the direction vector* $\mathbf{d} = (\mathbf{d}_x, \mathbf{d}_s)$ *be generated by equation (2), and let*

$$\theta = \frac{\alpha\sqrt{\min(XS\mathbf{e})}}{\|(XS)^{-1/2}(\frac{\mathbf{x}^T\mathbf{s}}{n+\rho}\mathbf{e} - XS\mathbf{e})\|}\,, \tag{3}$$

where α *is a positive constant less than 1. Let*

$$\mathbf{x}^+ = \mathbf{x} + \theta\mathbf{d}_x \quad \textit{and} \quad \mathbf{s}^+ = \mathbf{s} + \theta\mathbf{d}_s.$$

Then, we have $(\mathbf{x}^+, \mathbf{y}^+, \mathbf{s}^+) \in \operatorname{int}\mathcal{F}$ *and*

$$\psi_{n+\rho}(\mathbf{x}^+, \mathbf{s}^+) - \psi_{n+\rho}(\mathbf{x}, \mathbf{s}) \le -\alpha\sqrt{\min(XS\mathbf{e})}\|(XS)^{-1/2}(\mathbf{e} - \frac{(n+\rho)}{\mathbf{x}^T\mathbf{s}}X\mathbf{s})\|$$

$$+\alpha^2\max\{\frac{(n+\rho)\mathbf{d}_x^T\mathbf{d}_s}{\mathbf{x}^T\mathbf{s}}, \frac{\|X^{-1}\mathbf{d}_s\|^2 + \|S^{-1}\mathbf{d}_x\|^2}{2(1-\alpha)}\}$$

$$\le -\frac{\sqrt{3}}{2}\alpha + \alpha^2\max\{\frac{n+\rho}{2n}, \frac{1}{2(1-\alpha)}\}.$$

Software Implementation

Cplex

SEDUMI: `http://sedumi.mcmaster.ca/`

MOSEK: `http://www.mosek.com/products_mosek.html`

IPOPT: `https://projects.coin-or.org/Ipopt`

sphslfbf: Sparse Linear Programming Solver (Matlabe .m file).