MS&E 310 **Homework #2**
Linear Programming **2013**

HOMEWORK ASSIGNMENT 2: DUE OCTOBER 18

Reading: Chapters 3-4 of L&Y, *Introduction to Linear and Nonlinear Programming.*

1. Farkas' lemma can be used to derive many other (named) theorems of the alternative. This problem concerns a few of these pairs of systems. Using Farkas's lemma, prove each of the following results.

(a) Gordan's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad A\mathbf{x} > \mathbf{0}$$
$$\text{(ii)} \quad \mathbf{y}^T A = \mathbf{0}, \quad \mathbf{y} \geq \mathbf{0}, \quad \mathbf{y} \neq \mathbf{0}.$$

(b) Stiemke's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad A\mathbf{x} \geq \mathbf{0}, \quad A\mathbf{x} \neq \mathbf{0}$$
$$\text{(ii)} \quad \mathbf{y}^T A = \mathbf{0}, \quad \mathbf{y} > \mathbf{0}$$

(c) Gale's Theorem. Exactly one of the following systems has a solution:

$$\text{(i)} \quad A\mathbf{x} \leq \mathbf{b}$$
$$\text{(ii)} \quad \mathbf{y}^T A = \mathbf{0}, \quad \mathbf{y}^T\mathbf{b} < 0, \quad \mathbf{y} \geq \mathbf{0}$$

2. Given that the dual of a linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

in standard form is

$$\begin{array}{ll} \text{maximize} & \mathbf{y}^T\mathbf{b} \\ \text{subject to} & \mathbf{y}^T A \leq \mathbf{c}^T \ , \\ & (\mathbf{y} \text{ free}) \end{array}$$

develop an appropriate dual for each of the following LPs:

(a)

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

(b)

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

(c)

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \bar{A}\mathbf{x} \geq \bar{\mathbf{b}} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

3. Consider the auction problem in Lecture note #4. The LP pricing problem has an objective

$$\pi^T\mathbf{x} - z$$

where the scalar

$$z = \max[A\mathbf{x}]$$

is the maximum number of contracts among all states (recall that $A\mathbf{x} \in R^m$ is a vector representing the number of contracts in each state). Thus, z represents the *worst-case* payback amount. Now assuming that the auction organizer knows the discrete probability distribution, say $\mathbf{v} \in R^m_+$, for each state to win. Then the *expected* payback amount would be

$$\left(\sum_{i=1}^{n} v_i \cdot [Ax]_i\right) = \mathbf{v}^T A\mathbf{x}$$

Develop an LP model to decide the contract award vector $\mathbf{x}$ and to price each state using the expected payback rather than the worst-case payback, that is, using the objective function

$$\pi^T\mathbf{x} - \mathbf{v}^T A\mathbf{x}$$

in the LP setting. How to solve the problem faster? Moreover, explain the price properties using duality and/or complementarity.

4. Strict Complementarity Theorem:

- Read the proof of the strict complementarity theorem for the LP standard form in Lecture note #3.

- Consider the LP problem

$$\begin{array}{lll} (LP) & \text{maximize} & \mathbf{c}^T\mathbf{x} = \sum_{j=1}^n c_j x_j \\ & \text{subject to} & \sum_{j=1}^n \mathbf{a}_j x_j = A\mathbf{x} \le \mathbf{b},\ \mathbf{0} \le \mathbf{x} \le \mathbf{e}; \end{array}$$

where data $A \in R^{m\times n}$, $\mathbf{a}_j \in R^m$, $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$ and $\mathbf{e}$ is the vector of all ones, and variables $\mathbf{x} \in R^n$. You may interpret this is a linear program to sell the items of inventory $\mathbf{b}$ to n customers such that the revenue is maximized.

Suppose the problem is feasible and bounded.

1. Write down the dual of the problem. What are the interpretations of the dual price vector associated with the constraints $A\mathbf{x} \le \mathbf{b}$ and the dual price vector associated with the constraints $\mathbf{x} \le \mathbf{e}$) ?
2. What properties does a strictly complementary solution have for this linear program pair?
3. Suppose the linear program pair has a strictly complementary primal solution $\mathbf{x}^*$ such that $x_j^* = 0$ or $x_j^* = 1$ for all j, and let $\mathbf{y}^*$ be a strictly complementary dual price vector associated with the constraints $A\mathbf{x} \le \mathbf{b}$. Now consider a on-line linear program where customer $(c_j, \mathbf{a}_j)$ comes sequentially, and the seller have to make a decision $x_j = 0$ or $x_j = 1$ as soon as the customer arrives. Prove that the following mechanism or decision rule, given $\mathbf{y}^*$ being known, is optimal: If $c_j > \mathbf{a}_j^T\mathbf{y}^*$ then set $x_j = 1$; otherwise, set $x_j = 0$.

5. Consider a system of m linear equations in n nonnegative variables, say

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \ge \mathbf{0}.$$

Assume the right-hand side vector $\mathbf{b}$ is nonnegative. Now consider the (related) linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{e}^T\mathbf{y} \\ \text{subject to} & A\mathbf{x} + I\mathbf{y} = \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0},\ \mathbf{y} \ge \mathbf{0} \end{array}$$

where $\mathbf{e}$ is the m-vector of all ones, and I is the $m \times m$ identity matrix. This linear program is called a *Phase One Problem.*

(a) Write the dual of the Phase One Problem.

(b) Show that the Phase One Problem always has a basic feasible solution.

(c) Using theorems proved in class, show that the Phase One Problem always has an optimal solution.

(d) Write the complementary slackness conditions for the Phase One Problem.

(e) Prove that $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}\} \neq \emptyset$ if and only if the optimal value of the objective function in the corresponding Phase One Problem is zero.

6. Exercise 4.8-7 of of L&Y.

7. Exercise 4.8-8 of of L&Y.

8. Exercise 4.8-10 of of L&Y.

9. Let A be an m by n matrix and let $\mathbf{b}$ be a vector in R^m. We consider the problem of minimizing $\|A\mathbf{x} - \mathbf{b}\|_\infty$ over all $\mathbf{x} in R^n$. Let v be the value of the optimal cost.

(a) Let $\mathbf{p}$ be any vector in R^m that satisfies $\|\mathbf{p}\|_1 = \sum_{i=1}^m |p_i| \leq 1$ and $A^T\mathbf{p} = \mathbf{0}$. Show that $\mathbf{b}^T\mathbf{p} \leq v$

(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{array}{ll} \text{maximize} & \mathbf{b}^T\mathbf{p} \\ \text{subject to} & A^T\mathbf{p} = \mathbf{0} \\ & \|\mathbf{p}\|_1 \leq 1. \end{array}$$

Show that the optimal cost on this problem is equal to v.