Applications of Optimality Condition Theory

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Application: Fisher's Exchange Market

Buyers have money (w_i) to buy goods and maximize their individual utility functions; Producers sell their goods for money. The equilibrium price is an assignment of prices to goods so as when every buyer buys an maximal bundle of goods then the market clears, meaning that all the money is spent and all goods are sold.

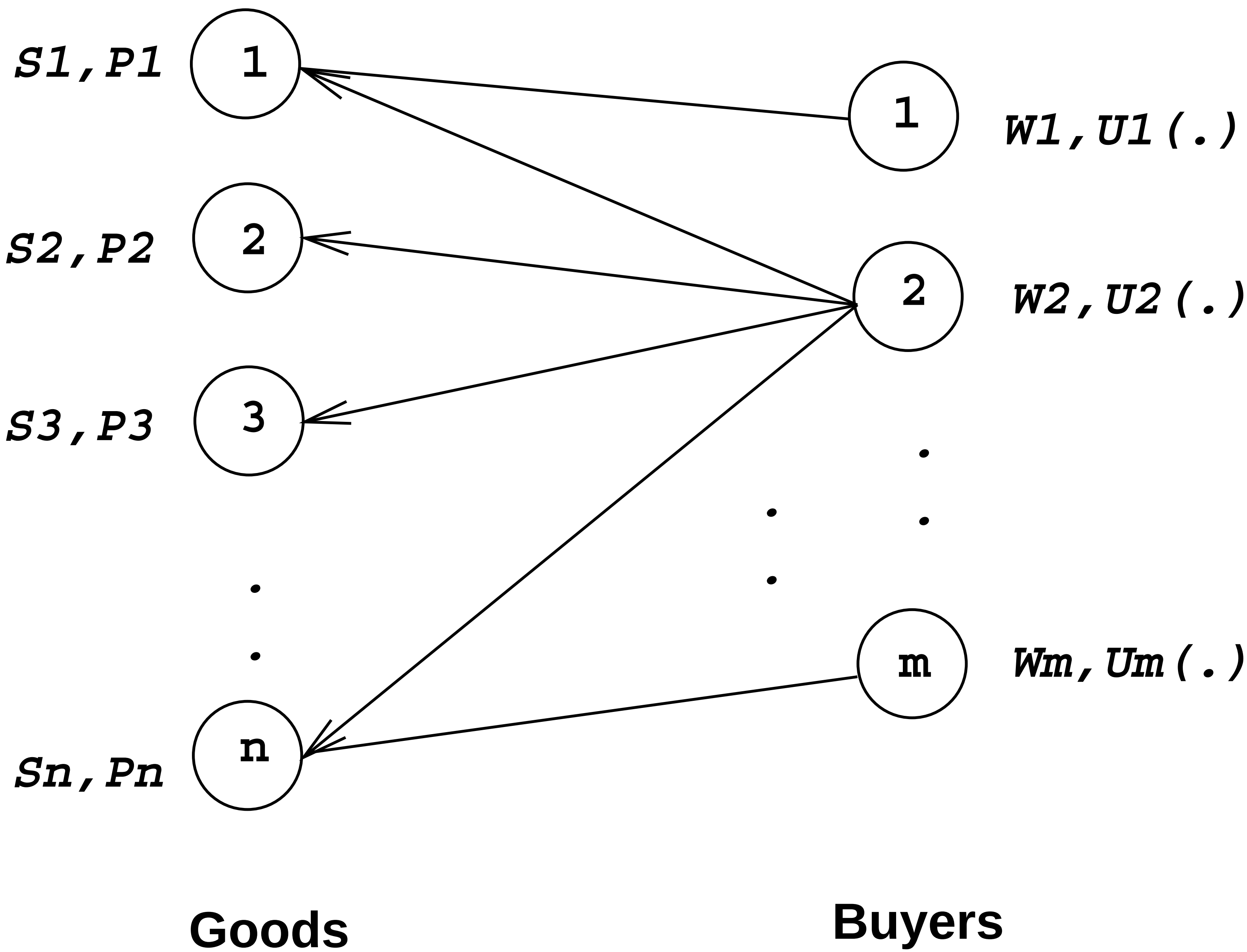

Figure 1: Fisher's Exchange Market Model

Fisher's equilibrium price

Player $i \in B$'s optimization problem for given prices p_j, $j \in G$.

$$
\begin{array}{rl}
\text{maximize} & \mathbf{u}_i^T \mathbf{x}_i := \sum_{j \in G} u_{ij} x_{ij} \\
\text{subject to} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in G} p_j x_{ij} \le w_i, \\
& x_{ij} \ge 0, \quad \forall j,
\end{array}
$$

Assume that the amount of each good is s_j. Then, the equilibrium price vector is the one such that there are maximizers $\mathbf{x}_i^*$s

$$
\sum_{i \in B} x_{ij}^* = s_j, \ \forall j.
$$

We assume u_{ij} has at least one positive coefficient for every j and for every i, respectively.

Example of Fisher's equilibrium price

There are two goods (x and y) of one unit each, and buyers $1, 2$' optimization problems for given prices p_x and p_y are:

$$\begin{array}{ll} \max & 2x_1 + y_1 \\ \text{s.t.} & p_x \cdot x_1 + p_y \cdot y_1 \le w_1 = 5, \\ & x_1, y_1 \ge 0 \end{array} \qquad \begin{array}{ll} \max & 3x_2 + y_2 \\ \text{s.t.} & p_x \cdot x_2 + p_y \cdot y_2 \le w_2 = 8 \\ & x_2, y_2 \ge 0. \end{array}$$

$$p_x = \frac{26}{3},\ p_y = \frac{13}{3}, \quad x_1 = \frac{1}{13},\ y_1 = 1,\ x_2 = \frac{12}{13},\ y_2 = 0$$

Equilibrium price conditions

Player $i \in B$'s dual problem for given prices p_j, $j \in G$.

$$\begin{array}{ll} \text{minimize} & w_i y_i \\ \text{subject to} & \mathbf{p} y_i \geq \mathbf{u}_i, \; y_i \geq 0 \end{array}$$

The necessary and sufficient conditions for a Fisher equilibrium point $\mathbf{x}_i, \mathbf{p}$ (allocations and prices) are:

$$\begin{array}{rl} \mathbf{p}^T \mathbf{x}_i \leq w_i, \; \forall i, & \\ p_j y_i \geq u_{ij}, \; y_i \geq 0, & \forall i, j, \\ \mathbf{u}_i^T \mathbf{x}_i = w_i y_i, & \forall i, \\ \sum_i x_{ij} = s_j, & \forall j, \\ \mathbf{x}_i \geq \mathbf{0}, & \forall i. \end{array}$$

Using $y_i = \mathbf{u}_i^T \mathbf{x}_i / w_i$ to remove y_i, we further simplify them to

Equilibrium price conditions continued

$$\begin{array}{rl}
\sum_j s_j p_j = \sum_i w_i, & \\
\frac{\mathbf{u}_i^T \mathbf{x}_i}{w_i} \cdot p_j \geq u_{ij}, & \forall i, j, \\
\sum_i x_{ij} \leq s_j, & \forall j, \\
\mathbf{x}_i \geq \mathbf{0}, & \forall i.
\end{array}$$

Note that we have removed inequality $\mathbf{p}^T \mathbf{x}_i \leq w_i$. This is because, from the second inequality (after multiplying x_{ij} to both sides and take sum over j) we have

$$\mathbf{p}^T \mathbf{x}_i \geq w_i, \ \forall i.$$

Then, from the rest conditions

$$\sum_i w_i = \sum_j s_j p_j = \sum_i \mathbf{p}^T \mathbf{x}_i \geq \sum_i w_i.$$

Thus, it must be true $\mathbf{p}^T \mathbf{x}_i = w_i, \ \forall i$ and $\sum_i x_{ij} = s_j, \ \forall j$.

Fisher's equilibrium for the example

$$\begin{array}{rl} \max & 2x_1 + y_1 \\ \text{s.t.} & p_x \cdot x_1 + p_y \cdot y_1 \le 5, \\ & x_1, y_1 \ge 0 \end{array} \qquad \begin{array}{rl} \max & 3x_2 + y_2 \\ \text{s.t.} & p_x \cdot x_2 + p_y \cdot y_2 \le 8 \\ & x_2, y_2 \ge 0. \end{array}$$

$$\begin{array}{rl} p_x + p_y = & 5 + 8 = 13, \\ \frac{2x_1+y_1}{5} \cdot p_x \ge 2, & \frac{2x_1+y_1}{5} \cdot p_y \ge 1 \\ \frac{3x_2+y_2}{8} \cdot p_x \ge 3, & \frac{3x_2+y_2}{8} \cdot p_y \ge 1 \\ x_1 + x_2 \le 1, & y_1 + y_2 \le 1. \end{array}$$

Equilibrium price property

Since u_{ij} has at least one positive coefficient for every j, we must have $p_j > 0$ for every j and for every equilibrium price. Moreover, The second inequality can be rewritten as

$$\log(\mathbf{u}_i^T\mathbf{x}_i) + \log(p_j) \geq \log(w_i) + \log(u_{ij}), \ \forall i, j, \ u_{ij} > 0.$$

Thus, we have

Theorem 1 *The equilibrium set of the Fisher Market is convex in allocations and prices. Furthermore, the equilibrium utility values and prices are unique.*

The proof of the first statement is from that all inequalities are convex and all equalities are affine. The second one is based on that $\log$ is strictly concave function.

Aggregate Social Optimization

$$
\begin{array}{ll}
\text{maximize} & \sum_{i \in B} w_i \log(\mathbf{u}_i^T \mathbf{x}_i) \\
\text{subject to} & \sum_{i \in B} x_{ij} \le s_j, \quad \forall j \in G \\
& x_{ij} \ge 0, \quad \forall i, j,
\end{array}
$$

Theorem 2 *(Eisenberg and Gale 1959) Optimal dual (Lagrange) multiplier vector of equality constraints is an equilibrium price vector.*

Optimality Conditions of the aggregated problem

$$
\begin{array}{rcl}
w_i \frac{u_{ij}}{\mathbf{u}_i^T \mathbf{x}_i} & \leq & p_j, \quad \forall i, j \\
w_i \frac{u_{ij} x_{ij}}{\mathbf{u}_i^T \mathbf{x}_i} & = & p_j x_{ij}, \quad \forall i, j \\
\sum_i x_{ij} & \leq & s_j, \quad \forall j \\
p_j \sum_i x_{ij} & = & p_j s_j, \quad \forall j \\
\mathbf{x}_i, \ \mathbf{p} & \geq & \mathbf{0}.
\end{array}
$$

Note that after sum up all complementarity equalities, we have

$$
\sum_i w_i = \sum_j s_j p_j,
$$

so that these necessary and sufficient optimality conditions are identical to the Fisher market equilibrium conditions.

Aggregate model for the example

$$\begin{array}{rl} \max & 2x_1 + y_1 \\ \text{s.t.} & p_x \cdot x_1 + p_y \cdot y_1 \le 5, \\ & x_1, y_1 \ge 0 \end{array} \qquad \begin{array}{rl} \max & 3x_2 + y_2 \\ \text{s.t.} & p_x \cdot x_2 + p_y \cdot y_2 \le 8 \\ & x_2, y_2 \ge 0. \end{array}$$

$$\begin{array}{rl} \text{maximize} & 5\log(2x_1 + y_1) + 8\log(3x_2 + y_2) \\ \text{subject to} & x_1 + x_2 = 1, \\ & y_1 + y_2 = 1, \\ & x_1, x_2, y_1, y_2 \ge 0. \end{array}$$

Application: Arrow-Debreu's Exchange Market

Each trader i, equipped with a good bundle vector $\mathbf{v}_i$, trade with others to maximize its individual utility function. The equilibrium price is an assignment of prices to goods so as when every producer sells his/her own good bundle and buys a maximal bundle of goods then the market clears. Thus, trader i's optimization problem for given prices p_j, $j \in G$, is

$$\begin{array}{ll} \text{maximize} & \mathbf{u}_i^T \mathbf{x}_i := \sum_{j \in P} u_{ij} x_{ij} \\ \text{subject to} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in P} p_j x_{ij} \leq \mathbf{p}^T \mathbf{v}_i, \\ & x_{ij} \geq 0, \quad \forall j, \end{array}$$

Then, the equilibrium price vector is the one such that there are maximizers $\mathbf{x}_i^*$s

$$\sum_i x_{ij}^* = \sum_i v_{ij}, \; \forall j.$$

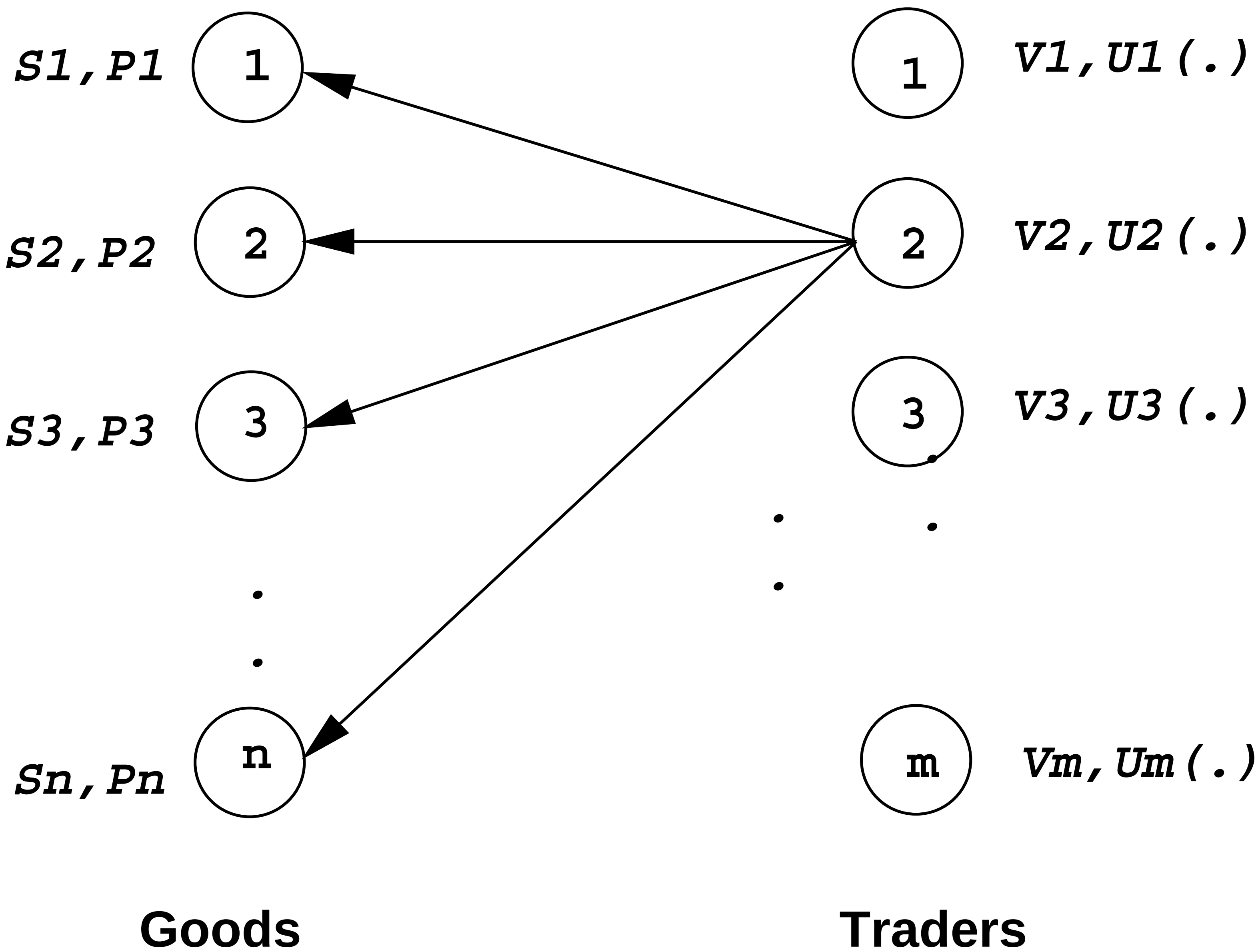

Figure 2: Arrow-Debreu's Exchange Market Model

Example of Arrow-Debreu's Model

Traders $1, 2$ have good bundle

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Their optimization problems for given prices p_x, p_y are:

$$\begin{array}{rl} \max & 2x_1 + y_1 \\ \text{s.t.} & p_x \cdot x_1 + p_y \cdot y_1 \le p_x, \\ & x_1, y_1 \ge 0 \end{array} \qquad \begin{array}{rl} \max & 3x_2 + y_2 \\ \text{s.t.} & p_x \cdot x_2 + p_y \cdot y_2 \le p_y \\ & x_2, y_2 \ge 0. \end{array}$$

$$p_x = 2,\ p_y = 1, \quad x_1 = \frac{1}{2},\ y_1 = 1,\ x_2 = \frac{1}{2},\ y_2 = 0$$

Equilibrium conditions of the Arrow-Debreu market

Similarly, the necessary and sufficient equilibrium conditions of the Arrow-Debreu market are

$$\begin{array}{ll} \frac{\mathbf{u}_i^T \mathbf{x}_i}{\mathbf{p}^T \mathbf{v}_i} \cdot p_j \geq u_{ij}, & \forall i, j, \\ \sum_i x_{ij} = s_j & \forall j, \\ p_j > 0,\ \mathbf{x}_i \geq \mathbf{0}, & \forall i, j. \end{array}$$

In addition, one can fix $\sum_j p_j = 1$ or $p_n = 1$.

Equilibrium conditions of the Arrow-Debreu market continued

Again, the first inequality can be rewritten as

$$\log(\mathbf{u}_i^T\mathbf{x}_i) + \log(p_j) - \log(\mathbf{p}^T\mathbf{v}_i) \geq \log(u_{ij}),\ \forall i, j,\ u_{ij} > 0.$$

Let $y_j = \log(p_j)$. Then, the inequality becomes

$$\log(\mathbf{u}_i^T\mathbf{x}_i) + y_j - \log(\sum_j v_{ij}e^{y_j}) \geq \log(u_{ij}),\ \forall i, j,\ u_{ij} > 0.$$

Note that the function on the left is concave in $\mathbf{x}_i$ and y_j. Thus, the equilibrium set of the Arrow-Debreu Market is convex in $\mathbf{x}_i$ and y_j. That is,

Theorem 3 *The equilibrium set of the Arrow-Debreu Market is convex in allocations and the logarithmic of prices. Furthermore, the equilibrium utility values are unique.*