MS&E 311 Winter 2013-2014
Optimization Jan 24, 2014
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Homework Assignment 2
Discussed Friday February 7, 2014

Optional Reading. Read Luenberger and Ye's *Linear and Nonlinear Programming Third Edition* Chapters 4, 11 and 15.9.

Solve the following problems:

1. Find the optimal dual solution of the SOCP problem of 1) of Homework Assignment 1. What about if the primal cone is the p-norm cone for $p \geq 1$?

2. Let $A \in S^n_+$ and $B \in S^n_+$, that is, they are symmetric positive semidefinite matrices. Then, $A \cdot B = 0$ implies that $AB = \mathbf{0} \in S^n$.

3. Let S^n_+ be the cone of all symmetric n-dimensional positive semidefinite matrices. If there is $\bar{y} \in R^m$ such that
$$-\mathcal{A}^T \bar{y} = -\sum_{i=1}^{m} \bar{y}_i A_i \in S^n_{++},$$
where $A_i \in S^n$, $i = 1, ..., m$, are given symmetric n-dimensional matrices, and S^n_{++} denotes the cone of all symmetric n-dimensional positive definite matrices. Then, prove that the set
$$\{\mathcal{A}X \in R^m : \ X \in S^n_+\}$$
is closed, where
$$\mathcal{A}X = \begin{pmatrix} A_1 \cdot X \\ A_2 \cdot X \\ \cdots \\ A_m \cdot X \end{pmatrix}.$$

4. In the pair of conic optimization problems, if one has an interior-point feasible solution, then the other admits a finite optimizer if it is feasible.

5. Consider convex cone

$$C = \{(t; x) : \ t > 0, \ tc(x/t) \leq 0, \ x \in R^2\},$$

where $c(x) \in R$ is a convex function. Construct the dual cone of C for each of the following $c(x)$:

(a) $c(x) = e^{x_1} + e^{x_2} - 1$.

(b) $c(x) = x_1^2 + 2x_2^2 - 2x_1x_2 - 1$

6. Consider the following quadratic program:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 - 2x_2 \\ \text{subject to} & x_1 + x_2 - \kappa \geq 0 \\ & x_1, x_2 \geq 0 \end{array} \tag{1}$$

where $\kappa \in R$ is a constant.

(a) (5pts.) Give a geometric interpretation of an instance of this problem. [Hint: Consider the form of the objective function.]

(b) (5pts.) Why does such a problem always have an optimal solution?

(c) (5pts.) Using the KKT conditions, verify that $(1.5, 2.5)$ is an optimal solution to (1) where $\kappa = 4$.

(d) (10pts.) For certain values of κ, the optimal solution to the problem lies on the boundary of the feasible region. What are those values, and what are the corresponding optimal solutions, Lagrange multipliers, and optimal objective function values?

(e) (5pts.) For the conditions described in (d), compare the value of the Lagrange multiplier corresponding to the constraint $x_1 + x_2 - \kappa \geq 0$ and the derivative of the objective function with respect to κ.

(f) (5pts.) What is the optimal solution for an arbitrary instance of this problem (i.e., for an arbitrary κ) for which the optimal solution does not lie on the boundary of the feasible region?

7. Find the Lagrangian dual of the barrier optimization problem:

$$\begin{array}{lcl} \min & c^T x - \sum_{j=1}^n \ln(x_j), & \\ \text{s.t.} & Ax & = b \\ & x & > 0, \end{array}$$

where we assume that the problem has an interior-point solution, and there exists a vector $y \in R^m$ such that $c - A^T y > 0$.

What are the first-order optimality or KKT conditions?

8. Consider a generalized Arrow–Debreu equilibrium problem in which the market has n agents and m types of commodities. Agent i, $i = 1, ..., n$, has a bundle amount of $w_i = (w_{i1}, w_{i2}, \ldots, w_{im}) \in R^m$ commodities initially and has a linear utility function whose coefficients are $u_i = (u_{i1}, u_{i2}, \ldots, u_{im}) > 0 \in R^m$. The goal is to price each commodity so that the market clears. Note that, given the price vector $p = (p_1, p_2, \ldots, p_m) > 0$, agent i's utility maximization problem is:

$$\begin{array}{ll} \text{maximize} & u_i^T x_i \\ \text{subject to} & p^T x_i \leq p^T w_i \\ & x_i \geq 0 \end{array}$$

   (a) For a given $p \in R^m$, write down the optimality conditions for agent i's utility maximization problem.

   (b) (5pts.) Suppose that $p \in R^m$ and $x_i \in R^m$ satisfy the constraints:

$$\begin{array}{ll} \displaystyle\sum_{i=1}^n x_i = \sum_{i=1}^n w_i & \\ \dfrac{u_i^T x_i}{u_{ij}} \geq \dfrac{p^T w_i}{p_j} & \forall i, j \\ x_{ij} \geq 0,\ p_j > 0 & \forall i, j \end{array}$$

   for $i = 1, \ldots, n$. Show that p is then an equilibrium price vector.

   (c) To find a feasible point, we consider the following minimization problem:

$$\begin{array}{lll} \text{minimize} & \theta & \\ \text{subject to} & \displaystyle\sum_{i=1}^n x_i = \sum_{i=1}^n w_i + e\theta & \\ & \dfrac{u_i^T x_i}{u_{ij}} \geq \dfrac{p^T w_i}{p_j} & \forall i, j \\ & x_{ij} \geq 0,\ p_j > 0 & \forall i, j \end{array} \tag{2}$$

   Here, $e = (1, 1, \ldots, 1)$. Verify that the problem is feasible and that the minimal value of the problem is 0. Is it a convex minimization problem?

(d) By introducing new variables $y_j = \log(p_j)$ for $j = 1, ..., m$, Problem (2) can be written as follows:

$$\begin{array}{rll} \text{minimize} & \theta & \\ \text{subject to} & \sum_{i=1}^n x_i = \sum_{i=1}^n w_i + e\theta & \\ & \log(u_i^T x_i) - \log(u_{ij}) \geq \log\left(\sum_{k=1}^m w_{ik} e^{y_k}\right) - y_j & \forall i, j \\ & x_{ij} \geq 0,\ \ p_j > 0 & \forall i, j \end{array}$$

Show that this problem is convex in x_{ij} and y_j. (Hint: Show that $\log\left(\sum_{k=1}^m w_{ik} e^{y_k}\right)$ is a convex function in the y_k's.)