MS&E 211
Lecture 14

An application to finance: replication and arbitrage

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(Based on slides by Professor Benjamin Van Roy)

Contingent Claims

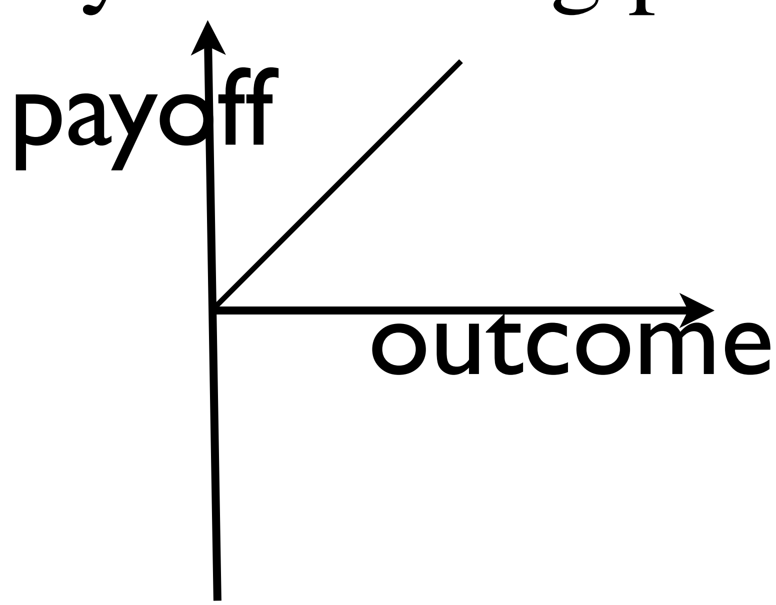
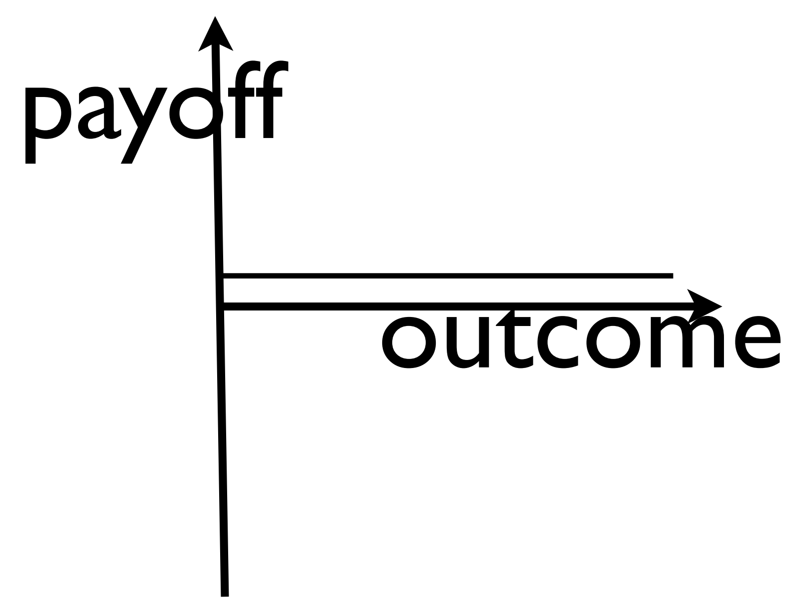
- A context and application area for Linear Programs

- A contingent claim is a contract
  - Receive a payoff depending on an uncertain outcome
  - May pay a price to purchase the contract

- Examples
  - Insurance
  - Negotiated contract with contingencies
  - Stocks, bonds, options, and other derivatives

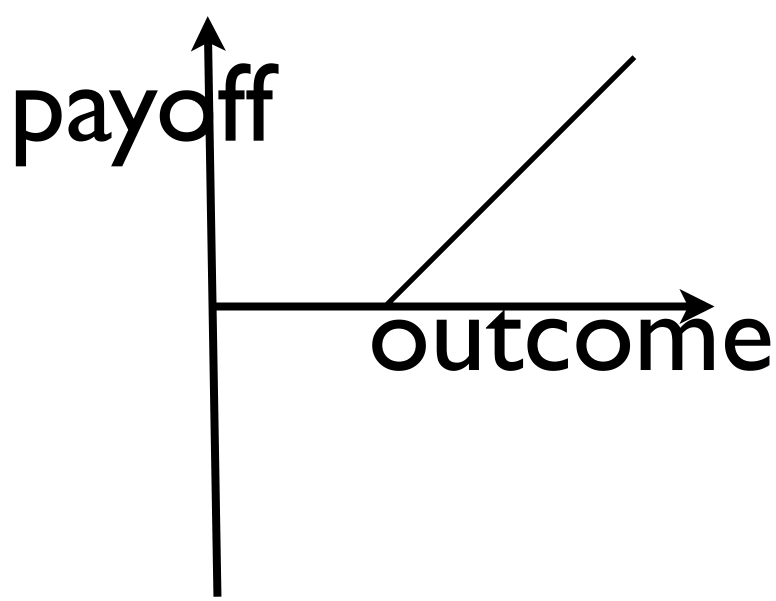
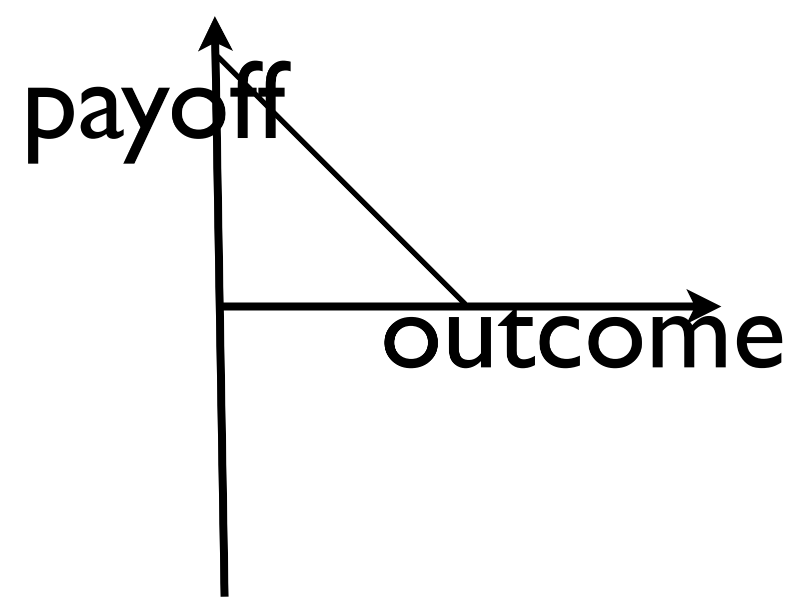
- Mathematical representation $\mathbf{a} \in \Re^M$
  - Enumerate possible outcomes 1, 2, ..., M
  - Specify outcome-contingent payoffs

Stocks and Bonds

	stock	zero-coupon bond
price	p_1	p_2
outcomes	future stock price = 1, ..., M	future stock price = 1, ..., M
payoff vector	$\mathbf{a}^1 \in \Re^M$	$\mathbf{a}^2 \in \Re^M$
illustration		

- Assume one year holding period

European Calls and Puts

	European call option	European put option
price	p_3	p_4
expiration date	1 year	1 year
strike price	\$40	\$60
outcomes	future stock price = 1, ..., M	future stock price = 1, ..., M
payoff vector	$\mathbf{a}^3 \in \Re^M$	$\mathbf{a}^4 \in \Re^M$
illustration		

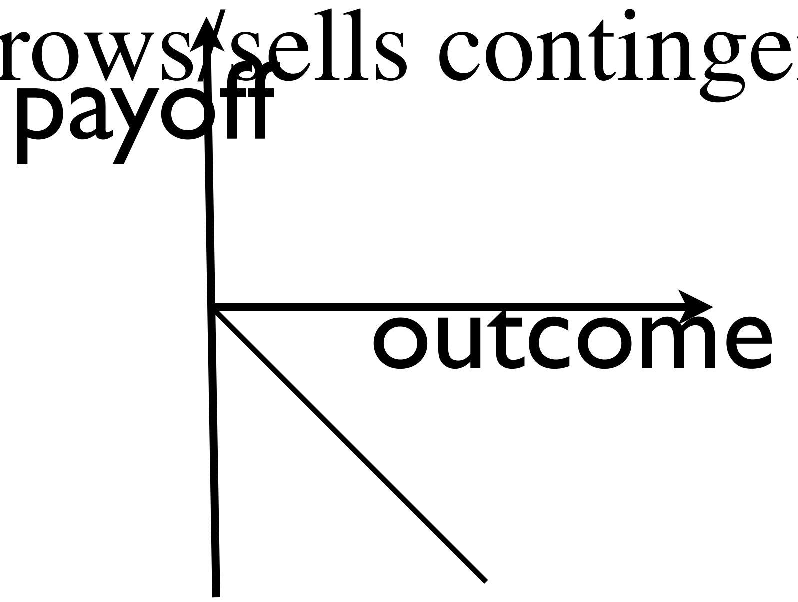
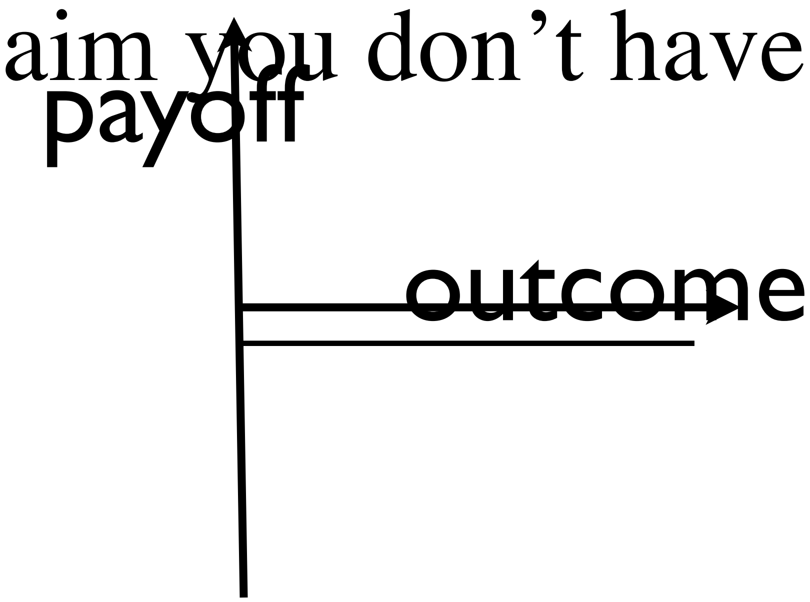
Call and Put Options

- t = Maturity date
- s = Strike Price
- z = future price of stock on maturity date

- Call option: Payoff = $\max\{0, z-s\}$

- Put option: Payoff = $\max\{0, s-z\}$

Short Selling

	short sell stock	short sell zero-coupon bond
price	$-p$	$-p$
outcomes	future stock price = 1, ..., M	future stock price = 1, ..., M
payoff vector	-	-
illustration		

- Broker borrows/sells contingent claim you don't have

Modeling simplifications: no transaction costs or margin

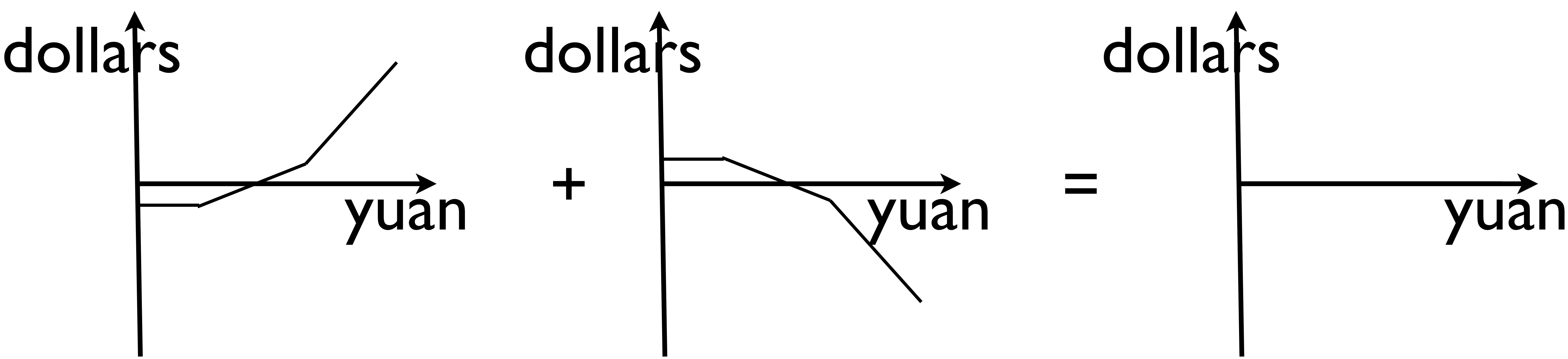
Call and Put Options

- A broker buys a single put option on a stock in corporation XYZ, expiring at time t. She also short-sells a call option on the same stock, expiring at the same time t. The strike price on both options is \$50. She also buys one unit of this stock which she will liquidate at time t. If the future price of the stock is z, what is her payoff at time t?

    - 0
    - \$50 - z
    - z - \$50
    - \$50
    - z

Structured Products

- US manufacturer with offer from retailer in China
  - 100,000 units of telematics system
  - Delivery in three months
- Price of Yuan three months from now influences
  - Shipping and assembly decisions
  - Resulting profit
- Risky project: may or may not be profitable
- Purchase structured product that covers liabilities
  - Can be replicated by trading bonds, Yuan, and options
- Guaranteed positive profit, received immediately



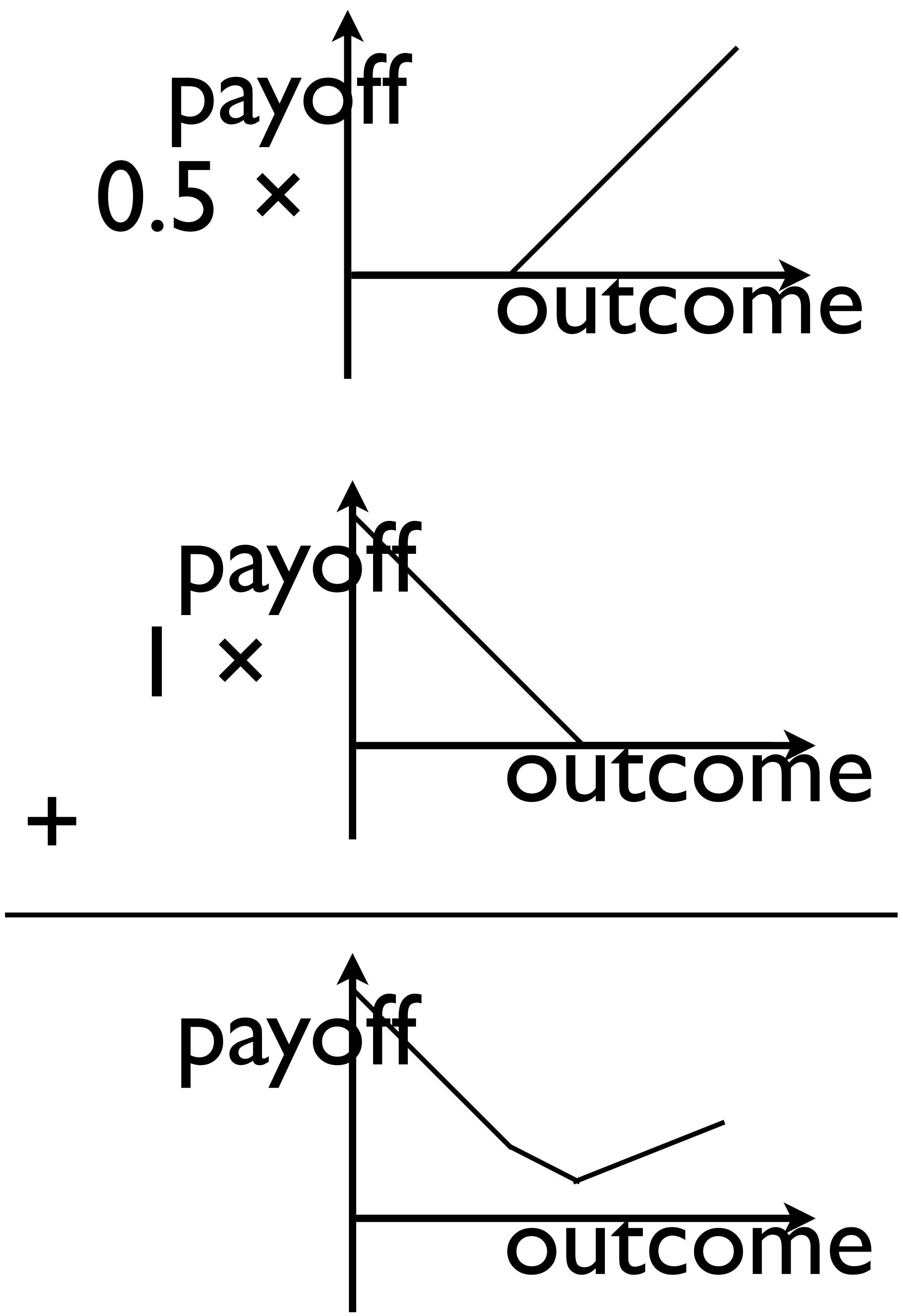
Next

- Show how structured products can be used to “replicate” or “super-replicate” complex financial liabilities

Markets and Portfolios

- N contingent claims, M payoff-relevant outcomes
- Payoff matrix $\mathbf{P} \in \Re^{M \times N}$
- Market prices $\rho \in \Re^N$ (row vector)
- Portfolio vector $\mathbf{x} \in \Re^N$
- Portfolio payoff $\mathbf{Px} \in \Re^M$
- Portfolio price $\rho\mathbf{x}$

Example



$$x = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 8 \\ 0 & 7 \\ 0 & 6 \\ 0 & 5 \\ 0 & 4 \\ 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \\ 4 & 0 \\ 5 & 0 \\ 6 & 0 \\ 7 & 0 \\ 8 & 0 \end{bmatrix} \qquad Px = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2.5 \\ 2 \\ 1.5 \\ 2 \\ 2.5 \\ 3 \\ 3.5 \\ 4 \end{bmatrix}$$

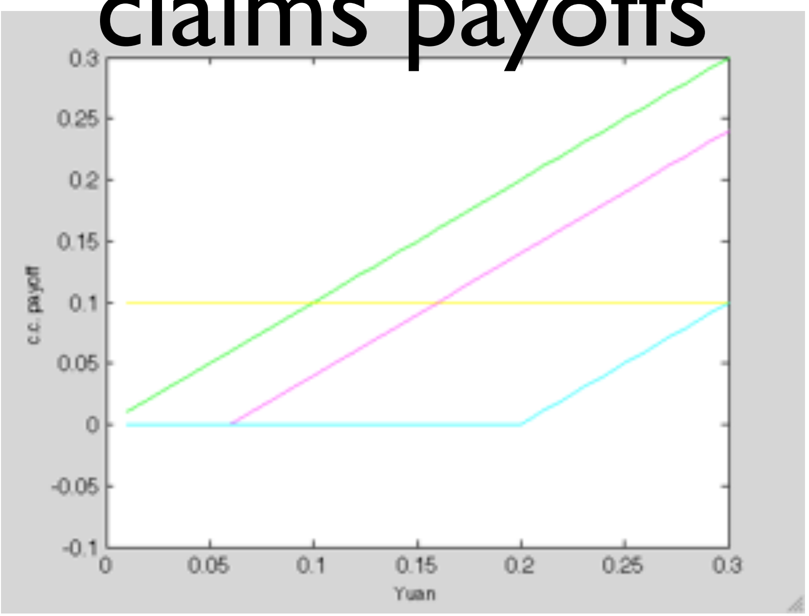
$$\rho = \begin{bmatrix} 3 & 2 \end{bmatrix} \qquad \rho x = 3.5$$

Replication

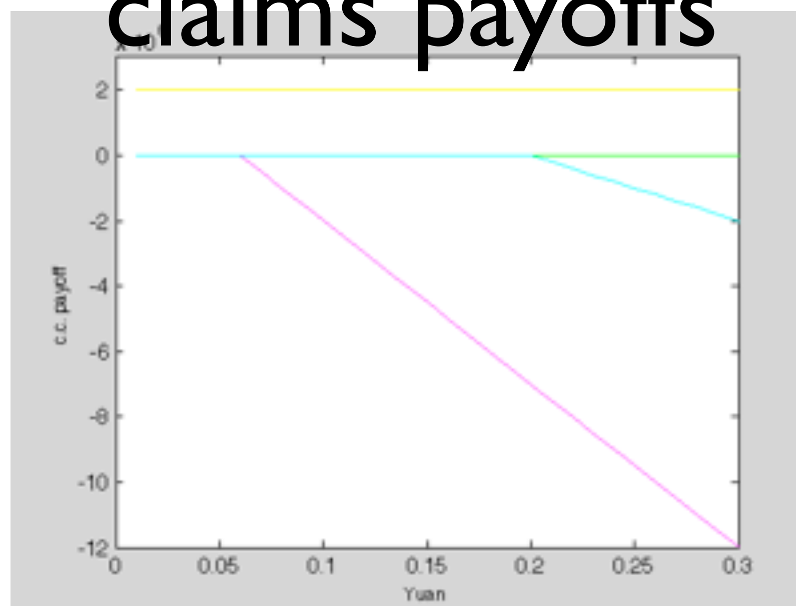
- Liabilities (or desired minimum payoff) $\mathbf{b} \in \Re^M$
- Replicating Portfolio $\mathbf{P}\mathbf{x} = \mathbf{b}$
- Price of Replication $\rho\mathbf{x}$

Example of Replication

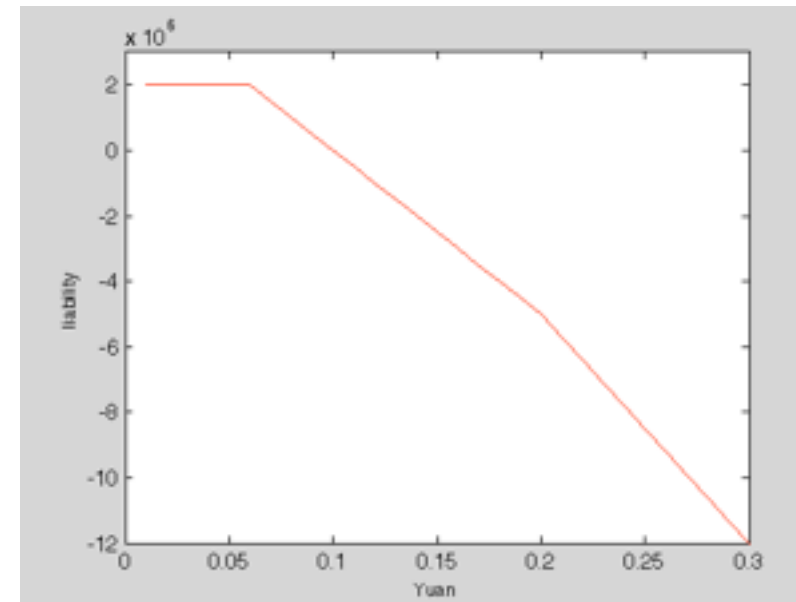
contingent claims payoffs



contingent claims payoffs



liabilities

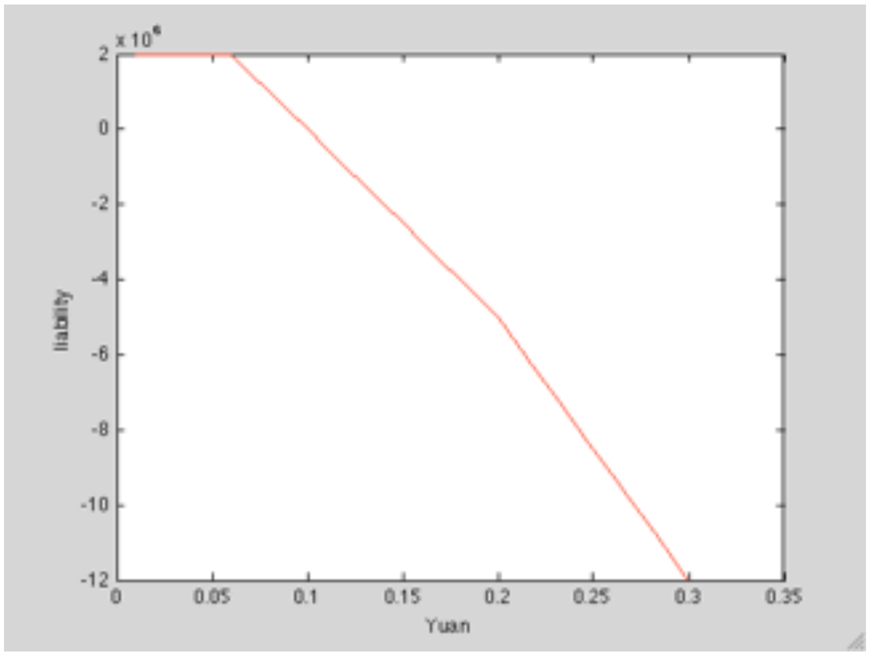


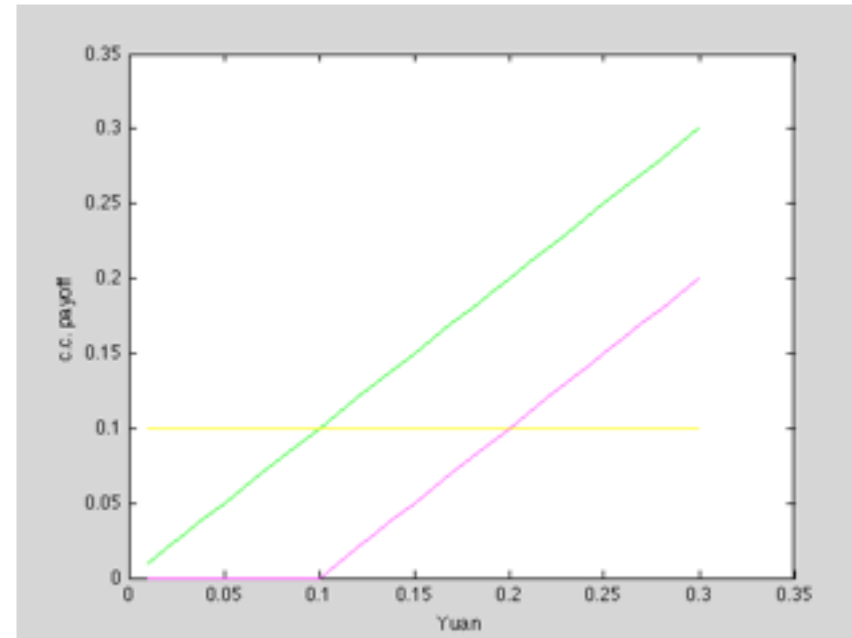
Super-Replication

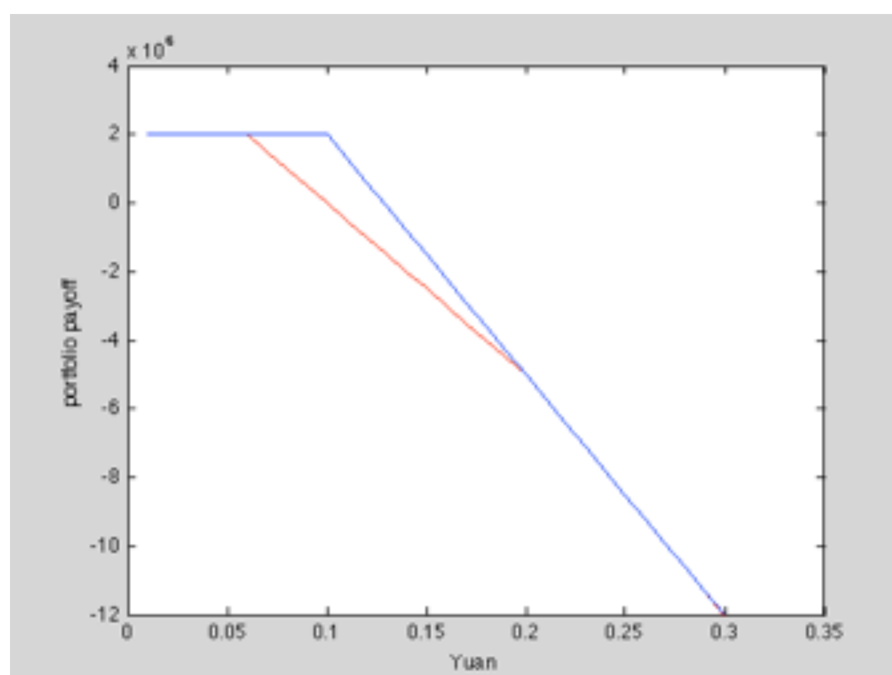
- What if there is no replicating portfolio?
  - Incomplete market
- Super-Replication $\mathbf{Px} \geq \mathbf{b}$
- Minimize price

$$\begin{array}{lrcl} \text{minimize} & \rho\mathbf{x} & & \\ \text{subject to:} & \mathbf{Px} & \geq & \mathbf{b} \end{array}$$

Example of Super-Replication







Arbitrage

- Definition: arbitrage opportunity

$$\mathbf{x} \in \Re^N \quad \text{s.t.} \quad \rho\mathbf{x} < 0 \quad \textit{and} \quad \mathbf{P}\mathbf{x} \geq 0$$

[Positive current profit, and no future risk]

- Most lucrative arbitrage opportunity

$$\begin{array}{lll} \min & \rho\mathbf{x} & \\ \text{s.t.} & \mathbf{P}\mathbf{x} & \geq \quad 0 \end{array}$$

- Is there a problem with this?

Arbitrage

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[Positive current profit, and no future risk]

- Most lucrative arbitrage opportunity

$$\begin{aligned} \min \quad & \rho\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Px} \geq 0 \end{aligned}$$

- Is there a problem with this?

Is it always infeasible?

Does it always has an unbounded optimum?

Both of the above?

Does it not identify an arbitrage opportunity?

Arbitrage

- Def. arbitrage opportunity

$$\mathbf{x} \in \Re^N \quad \text{s.t.} \quad \rho\mathbf{x} < 0 \quad \mathit{and} \quad \mathbf{P}\mathbf{x} \geq 0$$

- Most lucrative arbitrage opportunity

$$\begin{aligned} \min \quad & \rho\mathbf{x} \\ \text{s.t.} \quad & \mathbf{P}\mathbf{x} \geq 0 \end{aligned} \qquad \mathbf{x} \to \infty$$

- Arbitrage opportunity that makes \$1

$$\begin{aligned} \min \quad & \ldots \\ \text{s.t.} \quad & \rho\mathbf{x} = -1 \\ & \mathbf{P}\mathbf{x} \geq 0 \end{aligned}$$

Minimizing Shares Traded

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} |x_i| \\ \text{s.t.} \quad & \rho\mathbf{x} = -1 \\ & \mathbf{P}\mathbf{x} \geq 0 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{N} (x_i^+ + x_i^-) \\ \text{s.t.} \quad & \rho(\mathbf{x}^+ - \mathbf{x}^-) = -1 \\ & \mathbf{P}(\mathbf{x}^+ - \mathbf{x}^-) \geq 0 \\ & \mathbf{x}^+, \mathbf{x}^- \geq 0 \end{aligned}$$