MS&E 211, Autumn 2014-15, Lecture 12

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(Based on slides by Benjamin Van Roy)

Rock Paper Scissors

- Pay off matrix: Player 1, Payoff is given by matrix P

P =

ROW = PLAYER 2 COL= PLAYER 1	Rock	Paper	Scissors
Rock	0	1	-1
Paper	-1	0	1
Scissors	1	-1	0

- Player 2, pay off Q = -P
  - **Zero-Sum Game**, so just use P
    - Player 1 wants to choose a strategy to maximize the payoff P
    - Player 2 wants to choose a strategy to minimize the payoff P

- Nash Equilibrium: A strategy for player 1 and a strategy for player 2 such that no player has the incentive to unilaterally deviate

Two-Player Zero-Sum Games and Duality

Battle of Wits

Is there an equilibrium?

primal

$$\begin{array}{ll} \max & cx \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array}$$

dual

$$\begin{array}{ll} \min & yb \\ \text{s.t.} & yA \ge c \\ & y \ge 0 \end{array}$$

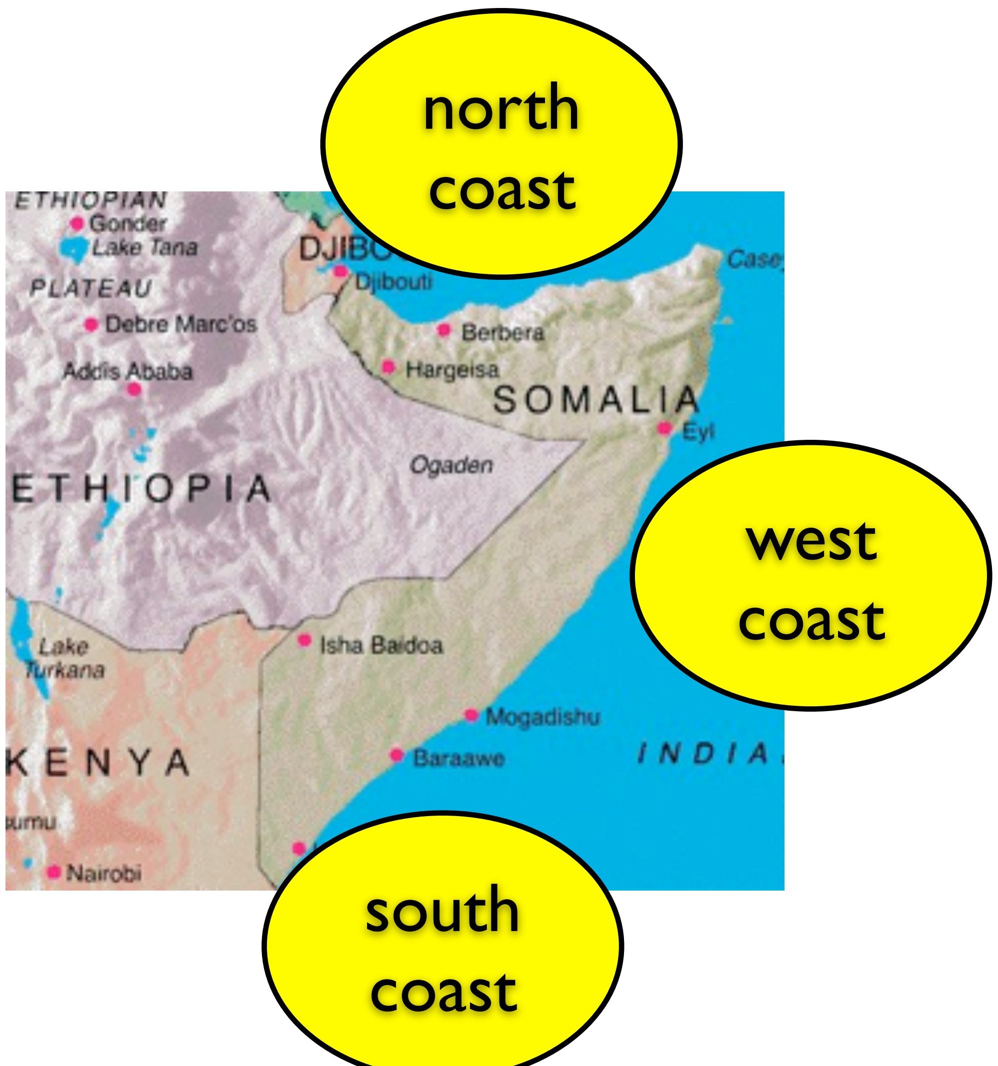
(y is a row vector)

Duality Theorem

Primal has an optimal solution if and only if dual does.

If x^* and y^* are optimal then $cx^* = y^*b$

CTF-150 versus the Somali Pirates



area	intercept probability
north	1/3
west	1/2
south	3/4

Strategies

area	**intercept probability**
north	1/3
west	1/2
south	3/4

pirates

CTF-150

success probability	north	west	south
north	2/3	1	1
west	1	1/2	1
south	1	1	1/4

$= P$

payoff matrix

- Pirates assume CTF to be “clairvoyant”
- Pirate strategy A: north
  - CTF-150 strategy: north
  - Success probability: 2/3
- Pirate strategy B: equiprobable actions
  - CTF-150 strategy: south
  - Success probability: 3/4

Can the pirates do better?

Optimal Strategy against a Clairvoyant Opponent

- Possible strategies for pirates and coast guard

$$\boxed{\mathcal{X} = \left\{ x \in \Re^3 \middle| x \geq 0, \sum_{n=1}^{3} x_n = 1 \right\}} \qquad \boxed{\mathcal{Y} = \left\{ y \in \Re^3 \middle| y \geq 0, \sum_{n=1}^{3} y_n = 1 \right\}}$$

- Pirate's probability of success $= yPx$
- Pirate selects a strategy (assuming optimal countermeasure)

$$\boxed{\max_{x \in \mathcal{X}} \left(\min_{y \in \mathcal{Y}} yPx \right)}$$

- What if coast guard strategizes similarly?

$$\boxed{\min_{y \in \mathcal{Y}} \left(\max_{x \in \mathcal{X}} yPx \right)}$$

This is an equilibrium!

General Two-Player Zero-Sum Game

- Actions
  - Player 1: {1, 2, 3, ..., N}
  - Player 2: {1, 2, 3, ..., M}

- Payoff matrix: $P \in \Re^{M \times N}$

- Strategies

  - Player 1: $\mathcal{X} = \left\{ x \in \Re^N \;\middle|\; x \geq 0, \sum_{n=1}^{N} x_n = 1 \right\}$

  - Player 2: $\mathcal{Y} = \left\{ y \in \Re^M \;\middle|\; y \geq 0, \sum_{m=1}^{M} y_m = 1 \right\}$

- Expected payoff: $yPx = \sum_{m=1}^{M} \sum_{n=1}^{N} x_n y_m P_{mn}$

Conservative Strategies via Linear Programming

Observation: for a clairvoyant opponent, a pure counterstrategy suffices

- Player 1 $\quad \max_{x\in\mathcal{X}} \underbrace{\min_{y\in\mathcal{Y}} yPx}_{u} \quad \Rightarrow$

$$\begin{array}{ll} \max & u \\ \text{s.t.} & ue \leq Px \\ & \sum_{n=1}^{N} x_n = 1 \\ & x \geq 0 \end{array}$$

- Player 2 $\quad \min_{y\in\mathcal{Y}} \underbrace{\max_{x\in\mathcal{X}} yPx}_{v} \quad \Rightarrow$

$$\begin{array}{ll} \min & v \\ \text{s.t.} & ve \geq yP \\ & \sum_{m=1}^{M} y_m = 1 \\ & y \geq 0 \end{array}$$

(e is a vector of all 1's, and is a row or column vector as needed)

Equilibrium

- **Def.** An equilibrium is a pair of strategies (x^*, y^*) such that
  - y^* is optimal for player 2 if player 1 uses x^*
  - x^* is optimal for player 1 if player 2 uses y^*

- No incentives to deviate: $y^* P x \leq y^* P x^* \leq y P x^* \qquad \forall x \in \mathcal{X}, y \in \mathcal{Y}$

- Conservative strategies attain an equilibrium

Minimax Theorem

$$\max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} yPx = \min_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} yPx$$

$$y^* P x^* \geq \min_{y \in \mathcal{Y}} y P x^* = \max_{x \in \mathcal{X}} y^* P x \geq y^* P x^*$$

$$y^* P x \leq y^* P x^* \leq y P x^* \qquad \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

Proof of Minimax Theorem

$$\boxed{\max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} yPx}$$

$$\Downarrow$$

$$\boxed{\begin{array}{ll} \max & u \\ \text{s.t.} & ue \leq Px \\ & \sum_{n=1}^{N} x_n = 1 \\ & x \geq 0 \end{array}}$$

$$\boxed{\min_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} yPx}$$

$$\Downarrow$$

$$\boxed{\begin{array}{ll} \min & v \\ \text{s.t.} & ve \geq yP \\ & \sum_{m=1}^{M} y_m = 1 \\ & y \geq 0 \end{array}}$$

duals

$$u^* = v^*$$