MS&E 311 | Winter 2013-2014
Optimization | February 28, 2014
Prof. Yinyu Ye

Homework Assignment 4
Discussed Wednesday March 12, 2014

Optional Reading. Read selected sections in Luenberger and Ye's *Linear and Nonlinear Programming Third Edition* Chapters 8, 9 and 10.

Solve the following problems:

1. Consider the unconstrained optimization problem:

$$\min_{x} \quad \exp(x) + \exp(-x)$$

   (a) (5pt) Solve the problem by first writing down the first order necessary condition.

   (b) (5pt) Suppose we want to solve the first order necessary condition using Newton's method, write down the iteration formula.

   (c) (5pt) Find the rate of convergence of $\{x_k\}$. i.e. let $x^* = \lim_{k\to\infty} x_k$, find constant $q > 1$, $c > 0$ such that

$$\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^q} = c.$$

   (Hint: use Taylor expansion)

2. Consider the bounded region $A = \{(x, y) : y \geq 0, y \leq 1 - x^2\}$

   (a) (5pt) Find the analytic center of A, which in this case is defined by

$$\arg\min_{(x,y)} \quad -(\log y + \log(1 - x^2 - y)).$$

   Now approximate A with polygon A_n's. In particular, A_n approximates the parabola ($y = 1 - x^2$) with $(2n - 1)$ lines, each line is tangent to the parabola with tangent point $x = \frac{k}{n}$ (and hence $y = 1 - \frac{k^2}{n^2}$), for $k = -(n-1), -(n-2), ..., -1, 0, 1, ..., (n-1)$.

   (b) (5pt) Describe A_n, i.e. write down all the constraints for bounded region A_n.

(c) (5pt) Write down the optimization problem for finding the analytic center for A_n, and the KKT first order necessary condition.

(d) (10pt) What does the analytic center converge to as $n \to \infty$? (Hint: 1. It might be difficult to solve for explicit solutions to the analytic center, but one can still find its limit. 2. use squeeze theorem, which says if $a_n \le x_n \le b_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ (both limits exist and are equal), then $\{x_n\}$ converges to that limit as well.)

(e) (5pt) Does the limit found in part (d) match the answer to part (a)? Is it consistent with what you expected? Why or why not?

3. Consider the unconstrained optimization problem below:

$$\min_{\mathbf{x}} \max_{1\le i\le m} (\mathbf{a}_i^T \mathbf{x} + b_i), \tag{1}$$

given $\mathbf{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$.

(a) (5pt) Derive an equivalent LP problem and write down its dual.

(b) (5pt) Suppose we approximate the objective function $\max_{1\le i\le m}(\mathbf{a}_i^T \mathbf{x} + b_i)$ with a smooth function and consider a different optimization problem:

$$\min_{\mathbf{x}} \log\left(\sum_{i=1}^{m} \exp(\mathbf{a}_i^T \mathbf{x} + b_i)\right) \tag{2}$$

Let z_1 and z_2 be the optimal values of (1) and (2), prove that

$$0 \le z_2 - z_1 \le \log m.$$

(c) (5pt) Suppose we use a different function for approximation:

$$\min_{\mathbf{x}} \frac{1}{\gamma} \log\left(\sum_{i=1}^{m} \exp(\gamma(\mathbf{a}_i^T \mathbf{x} + b_i))\right), \tag{3}$$

for some $\gamma > 0$. Suppose the optimal value to (3) is z_3, derive a bound for $z_3 - z_1$ similar as above. What happens as $\gamma \to \infty$?

4. Consider the LP problem

$$\begin{array}{lll} \text{minimize} & x_1 + x_2 & \\ \text{subject to} & x_1 + x_2 + x_3 & = 1, \\ & (x_1, x_2, x_3) & \ge 0. \end{array}$$

a) What is the analytic center of the feasible region?

b) Find the central path point $\mathbf{x}(\mu) = (x_1(\mu), x_2(\mu), x_3(\mu))$.

c) Show that as μ decreases to 0, $\mathbf{x}(\mu)$ converges to the unique optimal solution.

d) Let the objective function be just minimizing x_1. Then, find the central path point $\mathbf{x}(\mu)$ again. Which point does the central path converge to now?

e) Draw $\mathbf{x}$ part of the primal-dual potential function level sets:

$$\psi_6(\mathbf{x}, \mathbf{s}) \le 0 \quad \text{and} \quad \psi_6(\mathbf{x}, \mathbf{s}) \le -10,$$

and

$$\psi_{12}(\mathbf{x}, \mathbf{s}) \le 0 \quad \text{and} \quad \psi_{12}(\mathbf{x}, \mathbf{s}) \le -10;$$

respectively in the primal feasible region (on a plane) for the above two different objective functions.

The last question can be done by forming a team of 1-3 persons and sampling interior points in the primal and dual feasible regions.

Hint: To plot the $\mathbf{x}$ part of the level set of potential function, say $\psi_6(\mathbf{x}, \mathbf{s}) \le 0$, in primal feasible region F_p, you plot

$$\{\mathbf{x} \in F_p : \min_{\mathbf{s} \in F_d} \psi_6(\mathbf{x}, \mathbf{s}) \le 0\}$$

where F_d represents the dual feasible region. This can be approximately done by sampling as follows.

You randomly generate N interior feasible points of the primal $\mathbf{x}^p$ and the dual $(y^q, \mathbf{s}^q)$, respectively. For each primal point $\mathbf{x}^p$, you find if it is true that

$$\min_{q=1,\ldots,N} \psi_6(\mathbf{x}^p, \mathbf{s}^q) \le 0\}.$$

Then, you plot those $\mathbf{x}^p$ who give an "yes" answer.