Midterm Review

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(LY Chapters 1- 4 and Appendices)

Separating Hyperplane Theorem

The most important theorem about convexity is the following theorem:

Theorem 1 *(Separating Hyperplane Theorem)*[a] *Let $C \subset \mathcal{E}$, where $\mathcal{E}$ is either $\mathcal{R}^n$ or $\mathcal{M}^n$, be a closed convex set and let y be a point exterior to C. Then there is a vector $a \in \mathcal{E}$ such that*

$$a \bullet y < \inf_{x \in C} a \bullet x.$$

[a] Appendix B.3

Farkas' Lemma

The following results are Farkas' lemma and its variants.

Theorem 2 *Let $A \in \mathcal{R}^{m\times n}$ and $b \in \mathcal{R}^m$. Then, the system $\{x : \ Ax = b, \ x \geq 0\}$ has a feasible solution x if and only if that $A^T y \leq 0$ implies $b^T y \leq 0$.*

A vector y, with $A^T y \leq 0$ and $b^T y = 1$, is called a (primal) infeasibility certificate for the system $\{x : \ Ax = b, \ x \geq 0\}$.

Geometrically, Farkas' lemma means that if a vector $b \in \mathcal{R}^m$ does not belong to the cone generated by $a_{.1}, ..., a_{.n}$, then there is a hyperplane separating b from cone$(a_{.1}, ..., a_{.n})$.

Example

Let $A = (1, 1)$ and $b = -1$. Then $y = -1$ is an infeasibility certificate for $\{x : \ Ax = b, \ x \geq 0\}$.

Theorem 3 *Let* $A \in \mathcal{R}^{m\times n}$ *and* $c \in \mathcal{R}^n$*. Then, the system* $\{y : \ A^T y \le c\}$ *has a solution* y *if and only if that* $Ax = 0$ *and* $x \ge 0$ *imply* $c^T x \ge 0$*.*

Again, a vector $x \ge 0$, with $Ax = 0$ and $c^T x = -1$, is called a (dual) infeasibility certificate for the system $\{y : \ A^T y \le c\}$.

Example

Let $A = (1; -1)$ and $c = (1; -2)$. Then, $x = (1; 1)$ is an infeasibility certificate for $\{y : \ A^T y \le c\}$.

Duality Theory

Consider the linear program (LP) in standard form, called the Primal Problem,

$$
\begin{array}{lll}
(LP) & \text{minimize} & c^T x \\
 & \text{subject to} & Ax = b, \; x \geq 0,
\end{array}
$$

where $x \in \mathcal{R}^n$. The Dual Problem (LD) can be written as:

$$
\begin{array}{lll}
(LD) & \text{maximize} & b^T y \\
 & \text{subject to} & A^T y + s = c, \; s \geq 0,
\end{array}
$$

where $y \in \mathcal{R}^m$ and $s \in \mathcal{R}^n$. The components of s are called dual slacks.

Duality Theory

Theorem 4 *(Weak Duality Theorem) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty. Then*

$$c^T x \geq b^T y \quad \textit{where} \quad x \in \mathcal{F}_p,\ (y, s) \in \mathcal{F}_d.$$

This theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $(c^T x - b^T y)$ the Duality Gap.

Theorem 5 *(Strong Duality Theorem) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty. Then, x^* is optimal for (LP) if and only if the following conditions hold:*

i) $x^* \in \mathcal{F}_p$,

ii) *There is a pair $(y^*, s^*) \in \mathcal{F}_d$,*

iii) $c^T x^* = b^T y^*$.

Theorem 6 *(LP Duality Theorem) If (LP) and (LD) both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal.*

If one of (LP) and (LD) has no feasible solution, then the other is either unbounded or has no feasible solution. If one of (LP) and (LD) is unbounded then the other has no feasible solution.

Note: these theorems show that if a pair of feasible solutions can be found to the primal and dual problems with equal objective values, then these solutions are both optimal. The converse is also true; i.e., there is no “gap.”

For some feasible x and (y, s),

$$x^T s = x^T (c - A^T y) = c^T x - b^T y$$

is called the Complementarity Gap.

If $x^T s = 0$, then we say x and s are complementary to each other.

Since both x and s are nonnegative, $x^T s = 0$ implies that $x_j s_j = 0$ for all $j = 1, \ldots, n$.

$$\left\{ \begin{array}{rcl} Xs & = & 0 \\ Ax & = & b \\ -A^T y - s & = & -c. \end{array} \right.$$

This system has total $2n + m$ unknowns and $2n + m$ equations including n nonlinear equations.

Constructing the Dual

Obj Coeff Vector	RHS
RHS	Obj Coeff Vector
A	A^T
Max Model	Min Model
$x_j \geq 0$	j-th constraint $\geq$
$x_j \leq 0$	j-th constraint $\leq$
x_j: free	j-th constraint $=$
i-th constraint $\leq$	$y_i \geq 0$
i-th constraint $\geq$	$y_i \leq 0$
i-th constraint $=$	y_i: free

Basic Feasible Solution (BFS)

In the LP standard form, select m linearly independent columns, denoted by the index set B, from A. Then

$$A_B x_B = b$$

for the m-vector x_B. By setting the variables, x_N, of x corresponding to the remaining columns of A equal to zero, we obtain a solution x such that

$$Ax = b.$$

Then, x is said to be a (Primal) Basic Solution to (LP) with respect to the Basis A_B. The components of x_B are called Basic Variables.

If a basic solution $x \geq 0$, then x is called a Basic Feasible Solution (BFS).

If one or more components in x_B has value zero, that basic feasible solution x is said to be (Primal) Degenerate.

A dual vector y satisfying

$$A_B^T y = c_B$$

is said to be the corresponding Dual Basic Solution. If the dual basic solution is also feasible, that is

$$s = c - A^T y \geq 0.$$

and if one or more slacks in $c_N - A_N^T y$ has value zero, that dual basic feasible solution y is said to be Dual Degenerate.

Theorem 7 *(The Fundamental Theorem of LP) Given (LP) and (LD) where A has full row rank m,*

i) *if there is a feasible solution, there is a BFS;*

ii) *if there is an optimal solution, there is an optimal basic solution.*

Sample Problem 1

Let $A_1 \in R^{m\times n}, A_2 \in R^{m\times p}$ be two given matrices, and let $c_1 \in R^n, c_2 \in R^p$ be two given *non-negative vectors*. Consider the problem

$$\begin{array}{ll} \min & c_1^T x_1 + c_2^T x_2 \\ \text{s.t.} & A_1 x_1 + A_2 x_2 = b \\ & x_1, \quad x_2 \geq 0, \end{array}$$

and assume it is *feasible*.

(a) The problem has an optimal solution. Why?

(b) Let (x_1, x_2) be a feasible solution to the problem and its objective value equals $b^T y$ where y satisfies

$$A_1^T y \le \alpha_1 c_1$$

$$A_2^T y \le \alpha_2 c_2,$$

where α_1 and α_2 are two scalars greater than or equal to 1, then

$$b^T y \le \alpha_1 \cdot c_1^T x_1^* + \alpha_2 \cdot c_2^T x_2^*.$$

(α_1, α_2) is usually called the bi-factor approximation ratio and used in approximating algorithms for combinatorial optimization.

(a) Consider the dual problem:

$$\begin{array}{ll} \max & b^T y \\ \text{subject to} & A_1^T y \leq c_1, \\ & A_2^T y \leq c_2. \end{array}$$

Since $c_1 \geq 0$ and $c_2 \geq 0$, $y = 0$ is a feasible point for the dual. By LP duality, since the primal and dual problems are feasible, both must have optimal solutions.

(b) Let (x_1^*, x_2^*) be a primal optimal solution, and let (x_1, x_2) be a primal solution with value $c_1^T x_1 + c_2^T x_2 = b^T y$, where y satisfies

$$A_1^T y \leq \alpha_1 c_1,$$

$$A_2^T y \leq \alpha_2 c_2.$$

Since (x_1^*, x_2^*) is primal feasible, we must have $A_1 x_1^* + A_2 x_2^* = b$, and $x_1^*, x_2^* \geq 0$. Then,

$$(A_1^T y)^T x_1^* \leq \alpha_1 c_1^T x_1^*,$$
$$(A_2^T y)^T x_2^* \leq \alpha_2 c_2^T x_2^*.$$

Finally,

$$b^T y = (A_1 x_1^* + A_2 x_2^*)^T y = (A_1^T y)^T x_1^* + (A_2^T y)^T x_2^* \leq \alpha_1 c_1^T x_1^* + \alpha_2 c_2^T x_2^*.$$

Sample Problem 2

Assume that all basic feasible solutions (BFS) of a standard LP problem are non degenerate (that is, every basic variable has a positive value at every BFS). Then consider using the Simplex method to solve the problem. Prove that, if at a pivot step there is exactly one negative reduced cost coefficient, then the corresponding entering variable will remain as a basic variable for the remaining steps of the Simplex method.