MS&E 310 | **Homework #3**
Linear Programming | **2013**

HOMEWORK ASSIGNMENT 3: DUE NOVEMBER 15

Reading: Chapters 5 and 11 of (L&Y), *Linear and Nonlinear Programming*

1. Given an undirected graph $G = (\mathcal{N}, \mathcal{E})$, two special nodes $s, t \in \mathcal{N}$ and cost weights c_e for all $e \in \mathcal{E}$, we would like to find a minimum weight set of edges that intersects every path from s to t. Let $\mathcal{K}$ be the set of all such paths. One way of formulating this problem is as follows

$$\begin{array}{ll} \text{minimize} & \sum_{e \in \mathcal{E}} c_e x_e \\ \text{subject to} & \sum_{e \in K} x_e \geq 1, \quad \forall K \subset \mathcal{K}, \\ & 0 \leq x_e \leq 1, \quad \forall\ e \in \mathcal{E}. \end{array}$$

a) Show that every corner point of the above linear programming problem has 0 or 1 as its entries, so that an optimal basic feasible solution of the problem corresponds to a most economical cut from s to t.

b) This LP problem has explentially many constraints. Prove that the associated separation problem, if the Ellipsoid method is used, can be solved as an LP with polynomially many variables and constraints.

2. Prove that the equilibrium set of the Fisher Market is convex, and the equilibrium price vector is unique.

3. Prove the Eisenberg and Gale theorem.

4. Let

$$\mathcal{F} = \{\mathbf{y} \in R^m : \ \mathbf{a}_j^T \mathbf{y} \leq c_j, \ j = 1, ..., n\}$$

be a bounded polytope with a non-empty interior and $\bar{\mathbf{y}}$ be its analytic center. Now translate $k (< n)$ inequality hyperplanes, say $1, 2, ..., k$, and place them near the analytic center $\bar{\mathbf{y}}$ of $\mathcal{F}$, that is,

$$\mathcal{F}^+ := \{\mathbf{y} : \ \mathbf{a}_j^T \mathbf{y} \leq c_j^+, \ j = 1, ..., n\},$$

where $c_j^+ = c_j$ for $j = k + 1, ..., n$ and $c_j^+ = \gamma \mathbf{a}_j^T \bar{\mathbf{y}} + (1 - \gamma) c_j$ for $j = 1, ..., k$, and constant $0 \leq \gamma \leq 1$. Prove that

$$\frac{AV(\mathcal{F}^+)}{AV(\mathcal{F})} \leq \exp(-k\gamma).$$

5. Do b) of 5.9.7 of (L&Y).

6. Consider the LP problem

$$\begin{array}{lll} \text{minimize} & x_1 + x_2 & \\ \text{subject to} & x_1 + x_2 + x_3 & = 1, \\ & (x_1, x_2, x_3) & \geq 0. \end{array}$$

a) What is the analytic center of the feasible region?

b) Find the central path point $\mathbf{x}(\mu) = (x_1(\mu), x_2(\mu), x_3(\mu))$.

c) Show that as μ decreases to 0, $\mathbf{x}(\mu)$ converges to the unique optimal solution.

d) Let the objective function be just minimizing x_1. Then, find the central path point $\mathbf{x}(\mu)$ again. Which point does the central path converge to now?

e) Draw $\mathbf{x}$ part of the primal-dual potential function level sets:

$$\psi_6(\mathbf{x}, \mathbf{s}) \leq 0 \quad \text{and} \quad \psi_6(\mathbf{x}, \mathbf{s}) \leq -10,$$

and

$$\psi_{12}(\mathbf{x}, \mathbf{s}) \leq 0 \quad \text{and} \quad \psi_{12}(\mathbf{x}, \mathbf{s}) \leq -10;$$

respectively in the primal feasible region (on a plane) for the above two different objective functions.

The last question can be done by forming a team of 1-3 persons and sampling interior points in the primal and dual feasible regions.

Hint: To plot the $\mathbf{x}$ part of the level set of potential function, say $\psi_6(\mathbf{x}, \mathbf{s}) \leq 0$, in primal feasible region F_p, you plot

$$\{\mathbf{x} \in F_p : \min_{\mathbf{s} \in F_d} \psi_6(\mathbf{x}, \mathbf{s}) \leq 0\}$$

where F_d represents the dual feasible region. This can be approximately done by sampling as follows.

You randomly generate N interior feasible points of the primal $\mathbf{x}^p$ and the dual $(y^q, \mathbf{s}^q)$, respectively. For each primal point $\mathbf{x}^p$, you find if it is true that

$$\min_{q=1,\ldots,N} \psi_6(\mathbf{x}^p, \mathbf{s}^q) \leq 0\}.$$

Then, you plot those $\mathbf{x}^p$ who give an "yes" answer.