Dual Interpretations and Duality Applications

Yinyu Ye

Department of Management Science and Engineering

Stanford University

Stanford, CA 94305, U.S.A.

http://www.stanford.edu/~yyye

(LY, Chapters 4.1-4.2, 4.5)

Production Problem I

$$\max\ \mathbf{p}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{r}, \quad \mathbf{x} \ge \mathbf{0}$$

where

- $\mathbf{p}$: profit margin vector
- A: resource consumption rate matrix
- $\mathbf{r}$: available resource vector
- $\mathbf{x}$: production level decision vector

Production Problem II: Liquidation Pricing

- $\mathbf{y}$: the fair price vector
- $A^T\mathbf{y} \geq \mathbf{p}$: competitiveness
- $\mathbf{y} \geq 0$: positivity
- $\min\ \mathbf{r}^T\mathbf{y}$: minimize the total liquidation cost

$$
\text{Primal}: \quad \begin{array}{llll}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 1 \\
 & & x_2 & \leq 1 \\
 & x_1 & +x_2 & \leq 1.5 \\
 & x_1, & x_2 & \geq 0.
\end{array}
$$

$$
\text{Dual}: \quad \begin{array}{lllll}
\text{minimize} & y_1 & +y_2 & +1.5y_3 & \\
\text{subject to} & y_1 & & +y_3 & \geq 1 \\
 & & y_2 & +y_3 & \geq 2 \\
 & y_1, & y_2, & y_3 & \geq 0.
\end{array}
$$

Optimal Value Function

For a fixed matrix A and an objective coefficient vector $\mathbf{c}$, the optimal value is a function of right-hand-side vector $\mathbf{b}$:

$$\begin{array}{rll} z(\mathbf{b}) = & \text{minimize} & \mathbf{c}^T\mathbf{x} \\ & \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Theorem: $z(\mathbf{b})$ is a convex function in $\mathbf{b}$, that is, for any $0 \leq \alpha \leq 1$ we have

$$z(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) \leq \alpha z(\mathbf{b}^1) + (1-\alpha)z(\mathbf{b}^2).$$

Shadow Prices of the Optimal Value

Define a new right-hand-vector $\mathbf{b}^+$ as

$$b_k^+ := b_k + \delta \quad \text{and} \quad b_i^+ := b_i, \ \forall i \neq k.$$

Then the optimal dual solution $\mathbf{y}^*$ satisfies

$$y_k^* = \frac{z(\mathbf{b}^+) - z(\mathbf{b})}{\delta}$$

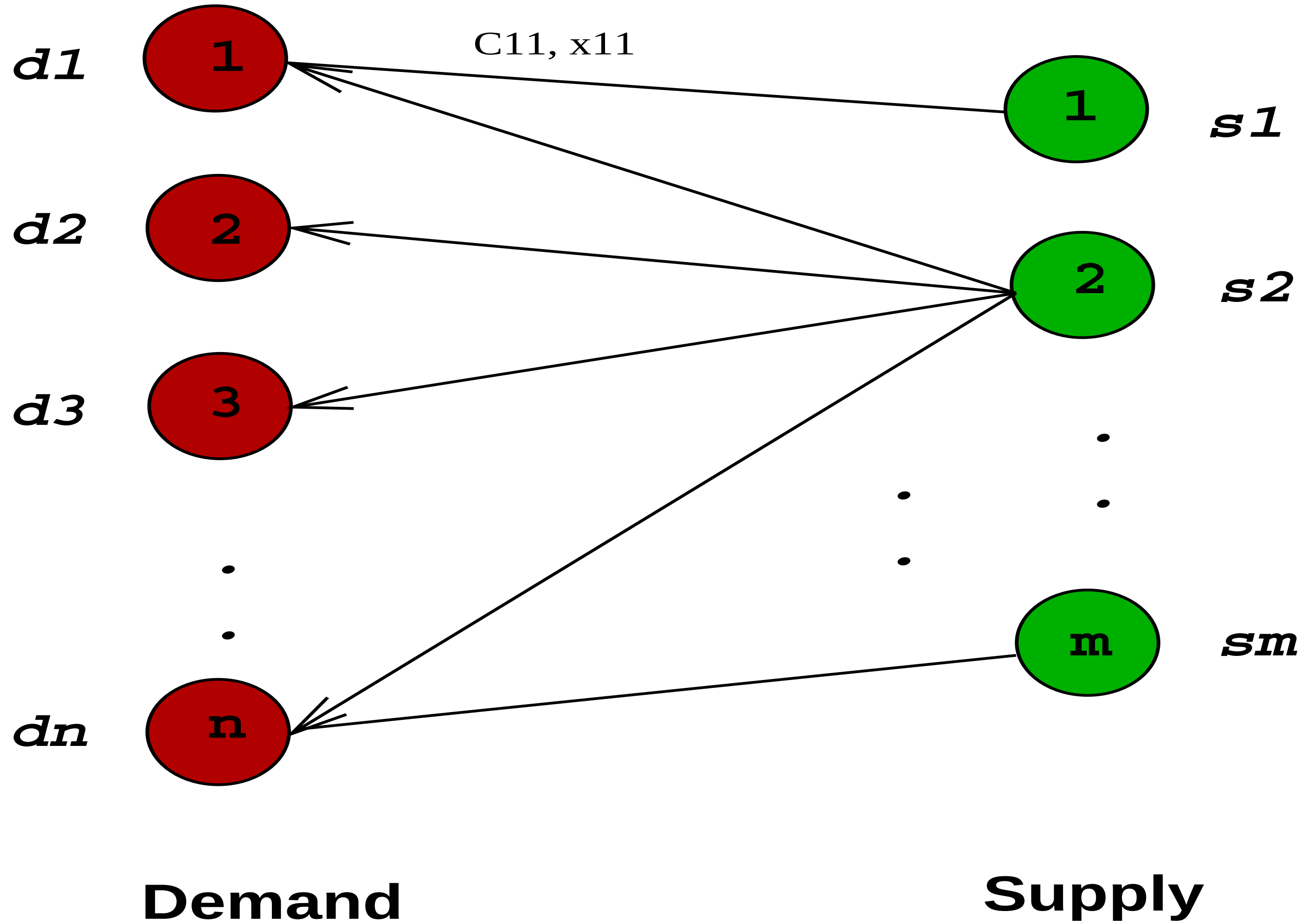
as long as $\mathbf{y}^*$ is the dual optimal solution for $\mathbf{b}^+$, since

$$z(\mathbf{b}^+) = (\mathbf{b}^+)^T \mathbf{y}^* = z(\mathbf{b}) + \delta y_k^*.$$

Thus, the optimal dual solution is the shadow price vector of the right-hand-vector, or the rate of the net change of the optimal objective value over the net change of an entry of the right-hand-vector.

Transportation Problem

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} & = s_i, \ \forall i = 1, ..., m \\
 & \sum_{i=1}^{m} x_{ij} & = d_j, \ \forall j = 1, ..., n \\
 & x_{ij} & \geq 0, \ \forall i, j.
\end{array}
$$


C11, x11
d1
d2
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dn
1
2
3
n
1
2
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s1
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sm
Demand
Supply

Transportation Dual

$$\begin{array}{lll} \max & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j & \\ \text{s.t.} & u_i + v_j & \le c_{ij}, \ \forall i, j. \end{array}$$

u_i: supply site unit price

v_i: demand site unit price

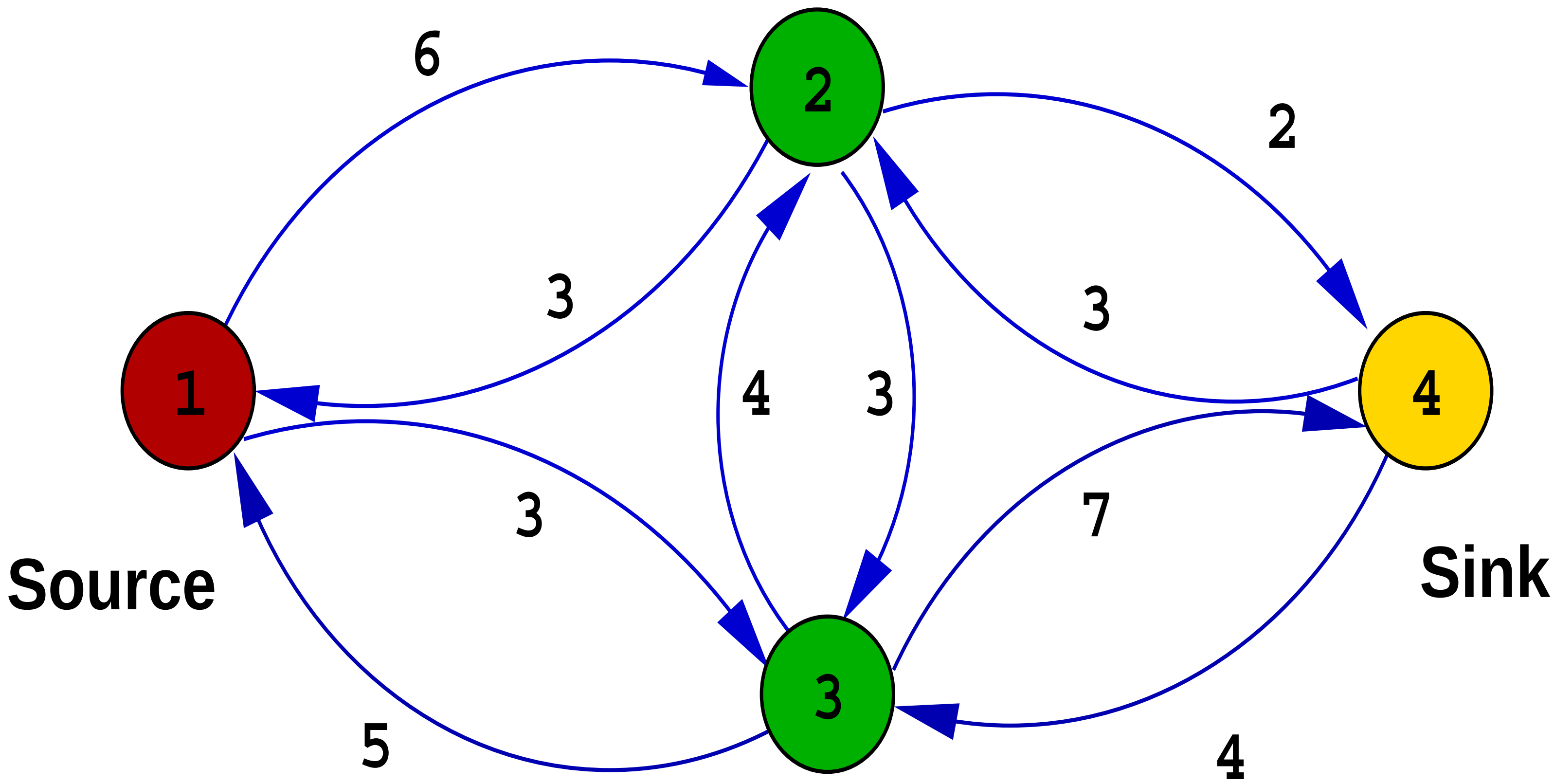
$u_i + v_j \le c_{ij}$: competitiveness

Max-Flow and Min-Cut

Given a directed graph with nodes $1, ..., m$ and edges $\mathcal{A}$, where node 1 is called the source and node m is called the sink, and each edge (i, j) has a flow rate capacity k_{ij}. The Max-Flow problem is to find the largest possible flow rate from source to sink.

Let x_{ij} be the flow rate from node i to node j. Then the problem can be formulated as

$$
\begin{array}{ll}
\text{maximize} & x_{m1} \\
\text{subject to} & \sum_{j:(j,1)\in\mathcal{A}} x_{j1} - \sum_{j:(1,j)\in A} x_{1j} + x_{m1} = 0, \\
& \sum_{j:(j,i)\in\mathcal{A}} x_{ji} - \sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 0, \forall i = 2, ..., m-1, \\
& \sum_{j:(j,m)\in\mathcal{A}} x_{jm} - \sum_{j:(m,j)\in\mathcal{A}} x_{mj} - x_{m1} = 0, \\
& 0 \le x_{ij} \le k_{ij}, \ \forall (i,j) \in \mathcal{A}.
\end{array}
$$



6
2
2
3
3
1
4
4
3
3
7
Source
Sink
3
5
4

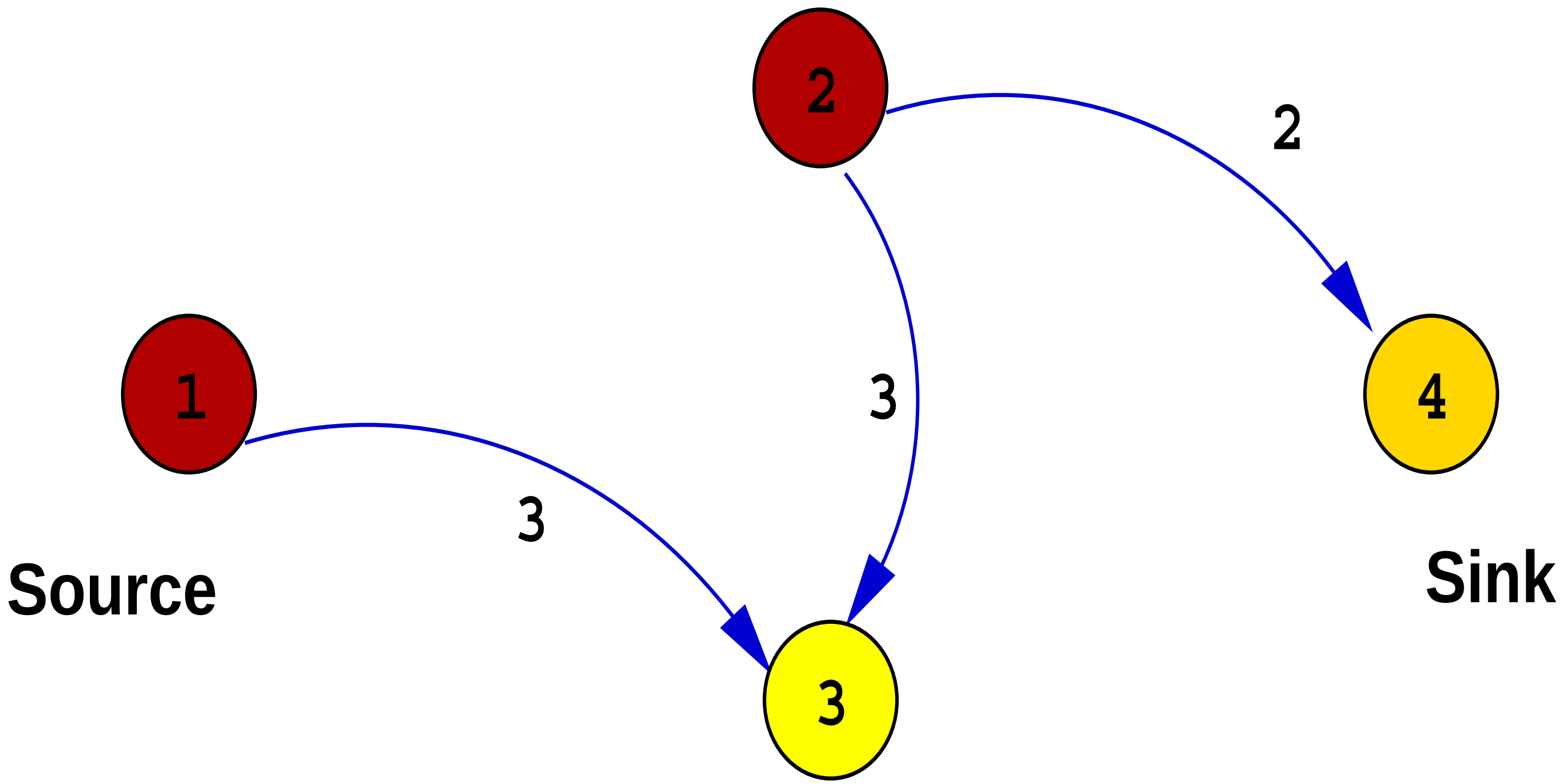
The Dual of the Max-Flow Problem

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\mathcal{A}} k_{ij} z_{ij} \\ \text{subject to} & -y_i + y_j + z_{ij} \geq 0, \ \forall (i,j) \in \mathcal{A}, \\ & y_1 - y_m = 1, \\ & z_{ij} \geq 0, \ \forall (i,j) \in A. \end{array}$$

y_i: node potential value. At an optimal solution, it has the following property. $y_1 = 1, \ y_m = 0$ and for all other i:

$$y_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

This problem is called the Min-Cut problem.


2
2
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Source
Sink
3

Application: Combinatorial Auction Pricing I

Given the m different states that are mutually exclusive and exactly one of them will be true at the maturity. A contract on a state is a paper agreement so that on maturity it is worth a notional $\$1$ if it is on the winning state and worth $\$0$ if is not on the winning state. There are n orders betting on one or a combination of states, with a price limit and a quantity limit.

Combinatorial Auction Pricing II: Order

The j-th order is given as $(\mathbf{a}_j \in R^m_+,\ \pi_j \in R_+,\ q_j \in R_+)$: $\mathbf{a}_j$ is the combination betting vector where each component is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ a_{mj} \end{pmatrix},$$

where 1 is winning and 0 is non-winning; π_j is the price limit for one such a contract, and q_j is the maximum number of contracts the better like to buy.

World Cup Information Market

Order	$\#1$	$\#2$	$\#3$	$\#4$	$\#5$
Argentina	1	0	1	1	0
Brazil	1	0	0	1	1
Italy	1	0	1	1	0
Germany	0	1	0	1	1
France	0	0	1	0	0
Bidding Prize: π	0.75	0.35	0.4	0.95	0.75
Quantity Limit: $\mathbf{q}$	10	5	10	10	5
Order Fill: $\mathbf{x}$	x_1	x_2	x_3	x_4	x_5

Combinatorial Auction Pricing III: Pricing Each State

Let x_j be the number of contracts awarded to the jth order. Then, the jth better will pay the amount

$$\pi_j \cdot x_j$$

and the total collected amount is

$$\sum_{j=1}^{n} \pi_j \cdot x_j = \pi^T \mathbf{x}$$

If the ith state is the winning state, then the auction organizer need to pay back

$$\left(\sum_{j=1}^{n} a_{ij} x_j \right)$$

The question is, how to decide $\mathbf{x} \in R^n$.

Combinatorial Auction Pricing IV: LP model

$$\begin{array}{rrl} \max & \pi^T \mathbf{x} - z & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z & \le \mathbf{0}, \\ & \mathbf{x} & \le \mathbf{q}, \\ & \mathbf{x} & \ge 0. \end{array}$$

$\pi^T \mathbf{x}$: the optimistic amount can be collected.

z: the worst-case amount need to pay back.

Combinatorial Auction V: The Dual

$$
\begin{array}{lrl}
\min & \mathbf{q}^T\mathbf{y} & \\
\text{s.t.} & A^T\mathbf{p}+\mathbf{y} & \geq \pi, \\
& \mathbf{e}^T\mathbf{p} & = 1, \\
& (\mathbf{p},\mathbf{y}) & \geq 0.
\end{array}
$$

$\mathbf{p}$ represents the state price. What is $\mathbf{y}$?

Combinatorial Auction V: Strict Complementarity

$x_j > 0$	$\mathbf{a}_j^T\mathbf{p} + y_j = \pi_j$ and $y_j \geq 0$ so that $\mathbf{a}_j^T\mathbf{p} \leq \pi_j$
$0 < x_j < q_j$	$y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} = \pi_j$
$x_j = q_j$	$y_j > 0$ so that $\mathbf{a}_j^T\mathbf{p} < \pi_j$
$x_j = 0$	$\mathbf{a}_j^T\mathbf{p} + y_j > \pi_j$ and $y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} > \pi_j$

The price is Fair:

$$\mathbf{p}^T(A\mathbf{x} - \mathbf{e}\cdot z) = 0 \quad \text{implies} \quad \mathbf{p}^T A\mathbf{x} = \mathbf{p}^T\mathbf{e}\cdot z = z;$$

that is, the worst case cost equals the worth of total shares. Moreover, if a lower bid wins the auction, so does the higher bid on any same type of bids.

World Cup Information Market Result

Order	$\#1$	$\#2$	$\#3$	$\#4$	$\#5$	State Price
Argentina	1	0	1	1	0	0.2
Brazil	1	0	0	1	1	0.35
Italy	1	0	1	1	0	0.2
Germany	0	1	0	1	1	0.25
France	0	0	1	0	0	0
Bidding Price:π	0.75	0.35	0.4	0.95	0.75	
Quantity Limit:$\mathbf{q}$	10	5	10	10	5	
Order Fill:$\mathbf{x}^*$	5	5	5	0	5	

Combinatorial Auction Pricing VI: Convex Programming Model

$$
\begin{array}{lrl}
\max & \pi^T \mathbf{x} - z + u(\mathbf{s}) & \\
\text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z + \mathbf{s} & = \mathbf{0}, \\
& \mathbf{x} & \leq \mathbf{q}, \\
& \mathbf{x},\ \mathbf{s} & \geq 0.
\end{array}
$$

$u(\mathbf{s})$: a value function for the market organizer on slack shares.

If $u(\cdot)$ is a strictly concave function, then the state price vector is unique.

Constructing the Dual

Obj Coef Vector	RHS
RHS	Obj Coef Vector
A	A^T
Max Model	Min Model
$x_j \geq 0$	j-th constraint $\geq$
$x_j \leq 0$	j-th constraint $\leq$
x_j: free	j-th constraint $=$
i-th constraint $\leq$	$y_i \geq 0$
i-th constraint $\geq$	$y_i \leq 0$
i-th constraint $=$	y_i: free

$$
Primal: \quad \begin{array}{llll}
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 & & x_2 & \le 1 \\
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$$

$$
Dual: \quad \begin{array}{lllll}
\text{minimize} & y_1 & +y_2 & +1.5y_3 & \\
\text{subject to} & y_1 & & +y_3 & \ge 1 \\
 & & y_2 & +y_3 & \ge 2 \\
 & y_1, & y_2, & y_3 & \ge 0.
\end{array}
$$