PRIMAL	maximize	minimize	DUAL
constraints	$\leq b_i$ $\geq b_i$ $= b_i$	≥ 0 ≤ 0 *unconstrained*	*variables*
variables	≥ 0 ≤ 0 *unconstrained*	$\geq c_j$ $\leq c_j$ $= c_j$	*constraints*

Note, using the rules in the above table, the dual of

$$
\begin{array}{ll}
\text{minimize} & cx \\
\text{subject to} & A^1 x \leq b^1 \\
& A^2 x \geq b^2 \\
& A^3 x = b^3
\end{array}
$$

becomes

$$
\begin{array}{ll}
\text{maximize} & y^1 b^1 + y^2 b^2 + y^3 b^3 \\
\text{subject to} & y^1 A^1 + y^2 A^2 + y^3 A^3 = c \\
& y^1 \leq 0 \\
& y^2 \geq 0
\end{array}
$$

Since the dual of the dual is the primal, reorganizing the above table yields an alternative procedure for converting primals that involve minimization to their duals.

PRIMAL	minimize	maximize	DUAL
constraints	$\leq b_i$ $\geq b_i$ $= b_i$	≤ 0 ≥ 0 *unconstrained*	*variables*
variables	≥ 0 ≤ 0 *unconstrained*	$\leq c_j$ $\geq c_j$ $= c_j$	*constraints*

4.4 Two-Player Zero-Sum Games

In this section, we consider games in which each of two opponents selects a strategy and receives a payoff contingent on both his own and his opponent's selection. We restrict attention here to zero-sum games – those in which a payoff to one player is a loss to his opponent. Let us consider an example that illustrates the nature of such problems.

Example 4.4.1. (drug running) *A South American drug lord is trying to get as many of his shipments across the border as possible. He has a fleet of boats available to him, and each time he sends a boat, he can choose one of three ports at which to unload. He could choose to unload in San Diego, Los Angeles, or San Francisco.*

The USA Coastguard is trying to intercept as many of the drug shipments as possible but only has sufficient resources to cover one port at a time. Moreover, the chance of intercepting a drug shipment differs from port to port. A boat arriving at a port closer to South America will have more fuel with which to evade capture than one arriving farther away. The probabilities of interception are given by the following table:

Port	Probability of interception
San Diego	*1/3*
Los Angeles	*1/2*
San Francisco	*3/4*

The drug lord considers sending each boat to San Diego, but the coastguard realizing this would always choose to cover San Diego, and only $2/3$ *of his boats would get through. A better strategy would be to pick a port at random (each one picked with* $1/3$ *probability). Then, the coastguard should cover port 3, since this would maximize the number of shipments captured. In this scenario,* $3/4$ *of the shipments would get through, which is better than* $2/3$*. But is this the best strategy?*

Clearly, the drug lord should consider randomized strategies. But what should he optimize? We consider as an objective maximizing the probability that a ship gets through, assuming that the Coastguard knows the drug lord's choice of randomized strategy. We now formalize this solution concept for general two-person zero-sum games, of which our example is a special case.

Consider a game with two players: player 1 and player 2. Suppose there are N alternative decisions available to player 1 and M available to player 2. If player 1 selects decision $j \in \{1, \ldots, N\}$ and player 2 selects decision $i \in \{1, \ldots, M\}$, there is an expected payoff of P_{ij} to be awarded to player 1 at the expense of player 2. Player 1 wishes to maximize expected payoff, whereas player 2 wishes to minimize it. We represent expected payoffs for all possible decision pairs as a matrix $P \in \Re^{M \times N}$.

A randomized strategy is a vector of probabilities, each associated with a particular decision. Hence, a randomized strategy for player 1 is a vector $x \in \Re^N$ with $ex = 1$ and $x \geq 0$, while a randomized strategy for player 2 is a vector $y \in \Re^M$ with $ye = 1$ and $y \geq 0$. Each x_j is the probability that player 1 selects decision j, and each y_i is the probability that player 2 selects decision

i. Hence, if the players apply randomized strategies x and y, the probability of payoff P_{ij} is $y_i x_j$ and the expected payoff is $\sum_{i=1}^{M} \sum_{j=1}^{N} y_i x_j P_{ij} = yPx$.

How should player 1 select a randomized policy? As a solution concept, we consider selection of a strategy that maximizes expected payoff, assuming that player 2 knows the strategy selected by player 1. One way to write this is as

$$\max_{\{x \in \Re^N | ex=1, x \geq 0\}} \min_{\{y \in \Re^M | ye=1, y \geq 0\}} yPx.$$

Here, y is chosen with knowledge of x, and x is chosen to maximize the worst-case payoff. We will now show how this optimization problem can be solved as a linear program.

First, consider the problem of optimizing y given x. This amounts to a linear program:

$$\begin{array}{ll} \text{minimize} & y(Px) \\ \text{subject to} & e^T y = 1 \\ & y \geq 0. \end{array}$$

It is easy to see that the basic feasible solutions of this linear program are given by $e^1, \ldots, e^M$, where each e^i is the vector with all components equal to 0 except for the ith, which is equal to 1. It follows that

$$\min_{\{y \in \Re^M | ye=1, y \geq 0\}} yPx = \min_{i \in \{1, \ldots, M\}} (Px)_i.$$

This minimal value can also be expressed as the solution to a linear program:

$$\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & ve \leq Px, \end{array}$$

where $v \in \Re$ is the only decision variable and x is fixed. In particular, the optimal value v^* resulting from this linear program satisfies

$$v^* = \min_{\{y \in \Re^M | ye=1, y \geq 0\}} yPx.$$

To determine an optimal strategy for player 1, we find the value of x that maximizes v^*. In particular, an optimal strategy is delivered by the following linear program:

$$\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & ve \leq Px \\ & ex = 1 \\ & x \geq 0, \end{array}$$

where $v \in \Re$ and $x \in \Re^N$ are decision variables. An optimal solution to this linear program provides a stochastic strategy x that maximizes the payoff v, assuming that player 2 knows the randomized strategy of player 1 and selects a payoff-minimizing counter-strategy. We illustrate application of this linear program through a continuation of Example 4.4.2.

Example 4.4.2. (linear programming for drug running) *To determine an optimal drug running strategy, we formulate the problem in the terms we have introduced. The drug lord's strategy is represented as a vector $x \in \Re^3$ of three probabilities. The first, second, and third components represent the probabilities that a ship is sent to San Diego, Los Angeles, or San Francisco, respectively. The payoff is 1 if a ship gets through, and 0 otherwise. Hence, the expected payoff P_{ij} is the probability that a ship gets through if player 1 selects decision j and player 2 selects decision i. The payoff matrix is then*

$$P = \begin{bmatrix} 2/3 & 1 & 1 \\ 1 & 1/2 & 1 \\ 1 & 1 & 1/4 \end{bmatrix}.$$

The optimal strategy for the drug lord is given by a linear program:

$$\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & ve \leq Px \\ & ex = 1 \\ & x \geq 0. \end{array}$$

Suppose that the drug lord computes an optimal randomized strategy x^* by solving the linear program. Over time, as this strategy is used to guide shipments, the drug lord can estimate the Coastguard's strategy y. Given y, he may consider adjusting his own strategy in response to y, if that will increase expected payoff. But should it be possible for the drug lord to improve his expected payoff after learning the Coastguard's strategy? Remarkably, if the coastguard selects a randomized strategy through an approach analogous to that we have described for the drug lord, neither the drug lord nor the Coastguard should ever need to adjust their strategies. We formalize this idea in the context of general two-player zero-sum games.

Recall from our earlier discussion that player 1 selects a randomized strategy x^* that attains the maximum in

$$\max_{\{x \in \Re^N | ex=1, x \geq 0\}} \min_{\{y \in \Re^M | ye=1, y \geq 0\}} yPx,$$

and that this can be done by solving a linear program

$$\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & ve \leq Px \\ & ex = 1 \\ & x \geq 0. \end{array}$$

Consider determining a randomized strategy for player 2 through an analogous process. An optimal strategy will then be a vector y^* that attains the minimum in

$$\min_{\{y\in\Re^M | ye=1, y\geq 0\}} \max_{\{x\in\Re^N | ex=1, x\geq 0\}} yPx.$$

Similarly with the case of finding a strategy for player 1, this new problem can be converted to a linear program:

$$\begin{array}{ll} \text{minimize} & u \\ \text{subject to} & ue \geq yP \\ & ye = 1 \\ & y \geq 0. \end{array}$$

A remarkable fact is that – if player 1 uses x^* and player 2 uses y^* – neither player should have any reason to change his strategy after learning the strategy being used by the other player. Such a situation is referred to as an *equilibrium*. This fact is an immediate consequence of the minimax theorem:

Theorem 4.4.1. (Minimax) *For any matrix $P \in \Re^{M\times N}$,*

$$\max_{\{x\in\Re^N | ex=1, x\geq 0\}} \min_{\{y\in\Re^M | ye=1, y\geq 0\}} yPx = \min_{\{y\in\Re^M | ye=1, y\geq 0\}} \max_{\{x\in\Re^N | ex=1, x\geq 0\}} yPx.$$

The minimax theorem is a simple corollary of strong duality. In particular, it is easy to show that the linear programs solved by players 1 and 2 are duals of one another. Hence, their optimal objective values are equal, which is exactly what the minimax theorem states.

Suppose now that the linear program solved by player 1 yields an optimal solution x^*, while that solved by player 2 yields an optimal solution y^*. Then, the minimax theorem implies that

$$y^*Px \leq y^*Px^* \leq yPx^*,$$

for all $x \in \Re^N$ with $ex = 1$ and $x \geq 0$ and $y \in \Re^M$ with $ye = 1$ and $y \geq 0$. In other words, the pair of strategies (x^*, y^*) yield an equilibrium.

4.5 Allocation of a Labor Force

Our economy presents a network of interdependent industries. Each both produces and consumes goods. For example, the steel industry consumes coal to manufacture steel. Reciprocally, the coal industry requires steel to support its own production processes. Further, each industry may be served