Mathematical Preliminaries

Yinyu Ye
Department of Management Science and Engineering
Stanford University
Stanford, CA 94305, U.S.A.

http://www.stanford.edu/~yyye

LY, Appendices A, B, and Chapter 1.

Real n-Space; Euclidean Space

- $\mathcal{R}, \quad \mathcal{R}_+, \quad \mathrm{int}\,\mathcal{R}_+$
- $\mathcal{R}^n, \quad \mathcal{R}^n_+, \quad \mathrm{int}\,\mathcal{R}^n_+$
- $\mathbf{x} \geq \mathbf{y}$ means $x_j \geq y_j$ for $j = 1, 2, ..., n$
- $\mathbf{0}$ denotes the zero vector and $\mathbf{e}$ denotes the vector of ones
- Inner-Product:

$$\mathbf{x} \bullet \mathbf{y} := \mathbf{x}^T\mathbf{y} = \sum_{j=1}^{n} x_j y_j$$

- Norm: $\|\mathbf{x}\|_2 := \sqrt{\mathbf{x}^T\mathbf{x}}$, $\|\mathbf{x}\|_\infty := \max\{|x_1|, |x_2|, ..., |x_n|\}$, and $\|\mathbf{x}\|_p := \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$
- The dual of the p norm, denoted by $\|.\|^*$, is the q norm where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p, q < \infty$.

- Column vector:

$$\mathbf{x} = (x_1; x_2; \ldots; x_n)$$

  and row vector:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

- A set of vectors $\mathbf{a}_1, ..., \mathbf{a}_m$ is said to be linearly dependent if there exists some scalars $\lambda_1, ..., \lambda_m$, not all zero, such that the linear combination

$$\sum_{i=1}^{m} \lambda_i \mathbf{a}_i = \mathbf{0}$$

- A linearly independent set of vectors that spans R^n is a basis.

Matrices

- $\mathcal{R}^{m\times n}$, $\mathbf{a}_{i.}$, $\mathbf{a}_{.j}$, a_{ij}
- A_I denotes the submatrix of A whose rows belong to I, A_J denotes the submatrix whose columns belong to J, and A_{IJ} denotes the submatrix whose rows belong to I and whose columns belong to J.
- $\mathbf{0}$ denotes the zero matrix and I denotes the identity matrix
- $\mathcal{N}(A)$, $\mathcal{R}(A)$:

  Theorem 1 *Each linear subspace of $\mathcal{R}^n$ can be generated by finitely many vectors and is also an intersection of finitely many hyperplanes; that is, for each linear subspace of L of $\mathcal{R}^n$ there are matrices A and C such that $L = \mathcal{N}(A) = \mathcal{R}(C)$.*
- $\det(A)$, tr(A)

- Inner Product:

$$A \bullet B := \mathrm{tr}(A^T B) = \sum_{i,j} a_{ij} b_{ij}$$

- The operator norm of A:

$$\|A\|^2 := \max_{\mathbf{0} \neq \mathbf{x} \in \mathcal{R}^n} \frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|^2}$$

- Sometimes we use $X = \mathrm{diag}(\mathbf{x})$
- Eigenvalues and eigenvectors

$$A\mathbf{v} = \lambda \mathbf{v}$$

Symmetric Matrices

- $\mathcal{S}^n$
- The Frobenius norm:

$$\|X\|_f := \sqrt{\text{tr}(X^T X)} = \sqrt{X \bullet X}$$

- Positive Definite (PD): $Q \succ \mathbf{0}$ iff $\mathbf{x}^T Q \mathbf{x} > 0, \quad \text{for all} \quad \mathbf{x} \neq \mathbf{0}$
- Positive SemiDefinite (PSD): $Q \succeq \mathbf{0}$ iff $\mathbf{x}^T Q \mathbf{x} \geq 0, \quad \text{for all} \quad \mathbf{x}$
- The set of PSD matrices: $\mathcal{S}^n_+$, $\quad \text{int}\, \mathcal{S}^n_+$

Known Inequalities

- Cauchy-Schwarz Inequality: given $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, we have $\mathbf{x}^T\mathbf{y} \le \|\mathbf{x}\|\|\mathbf{y}\|$.
- Triangle Inequality: given $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, we have $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.
- Arithmetic Mean-Geometric Mean Inequality: given $\mathbf{x} \in \mathcal{R}^n_+$, we have

$$\frac{\sum x_j}{n} \ge \left(\prod x_j\right)^{1/n}.$$

Hyperplane and Half-spaces

$$H = \{\mathbf{x} : \; \mathbf{a}\mathbf{x} = \sum_{j=1}^{n} a_j x_j = b\}$$

$$H^{+} = \{\mathbf{x} : \; \mathbf{a}\mathbf{x} = \sum_{j=1}^{n} a_j x_j \leq b\}$$

$$H^{-} = \{\mathbf{x} : \; \mathbf{a}\mathbf{x} = \sum_{j=1}^{n} a_j x_j \geq b\}$$

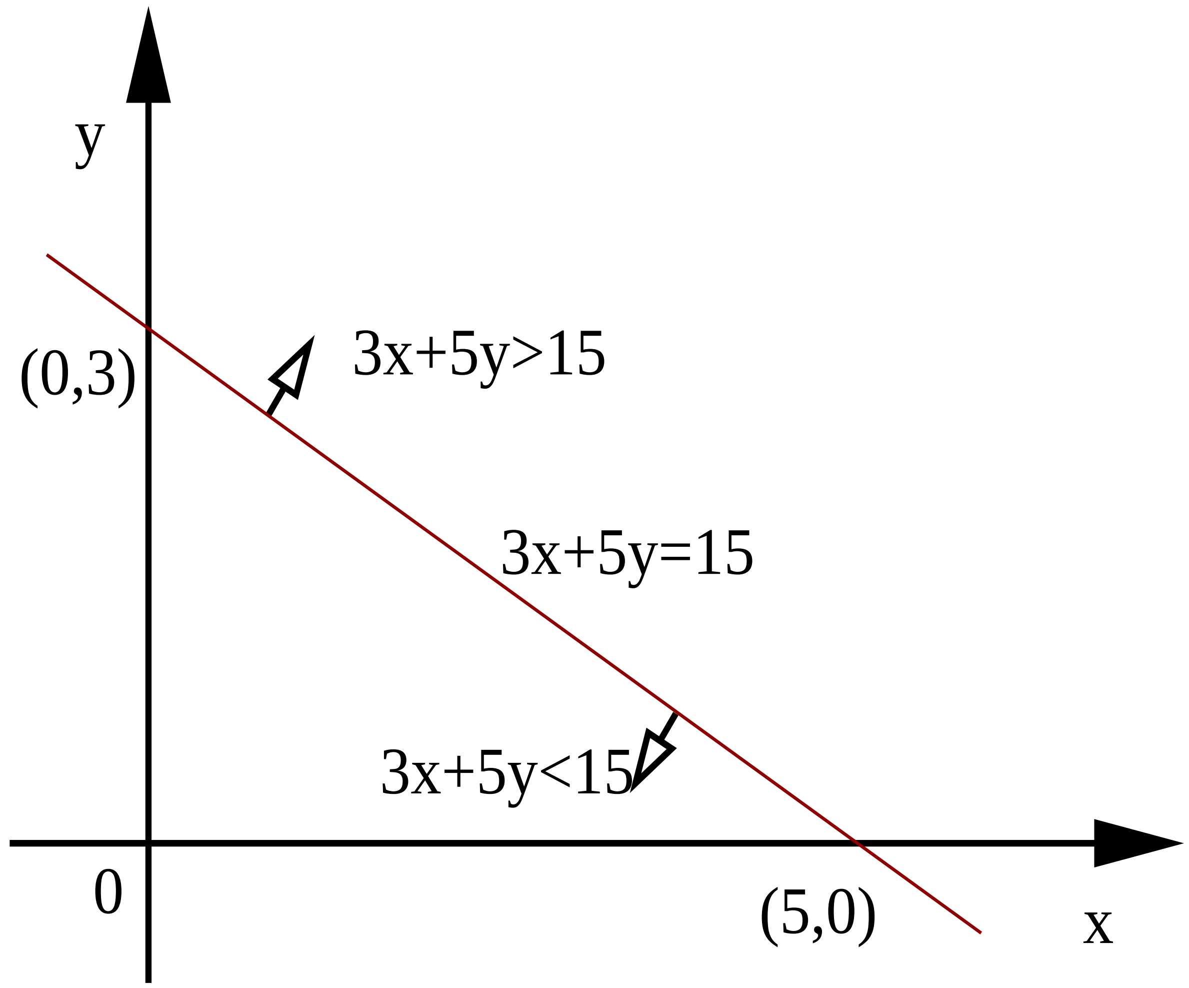

Figure 1: Plane and Half-Spaces

System of Linear Equations

Solve for $\mathbf{x} \in \mathcal{R}^n$ from:

$$\left\{\begin{array}{rcl} \mathbf{a}_1\mathbf{x} & = & b_1 \\ \mathbf{a}_2\mathbf{x} & = & b_2 \\ \ldots & . & . \\ \mathbf{a}_m\mathbf{x} & = & b_m \end{array}\right. \qquad \Rightarrow \qquad A\mathbf{x} = \mathbf{b}$$

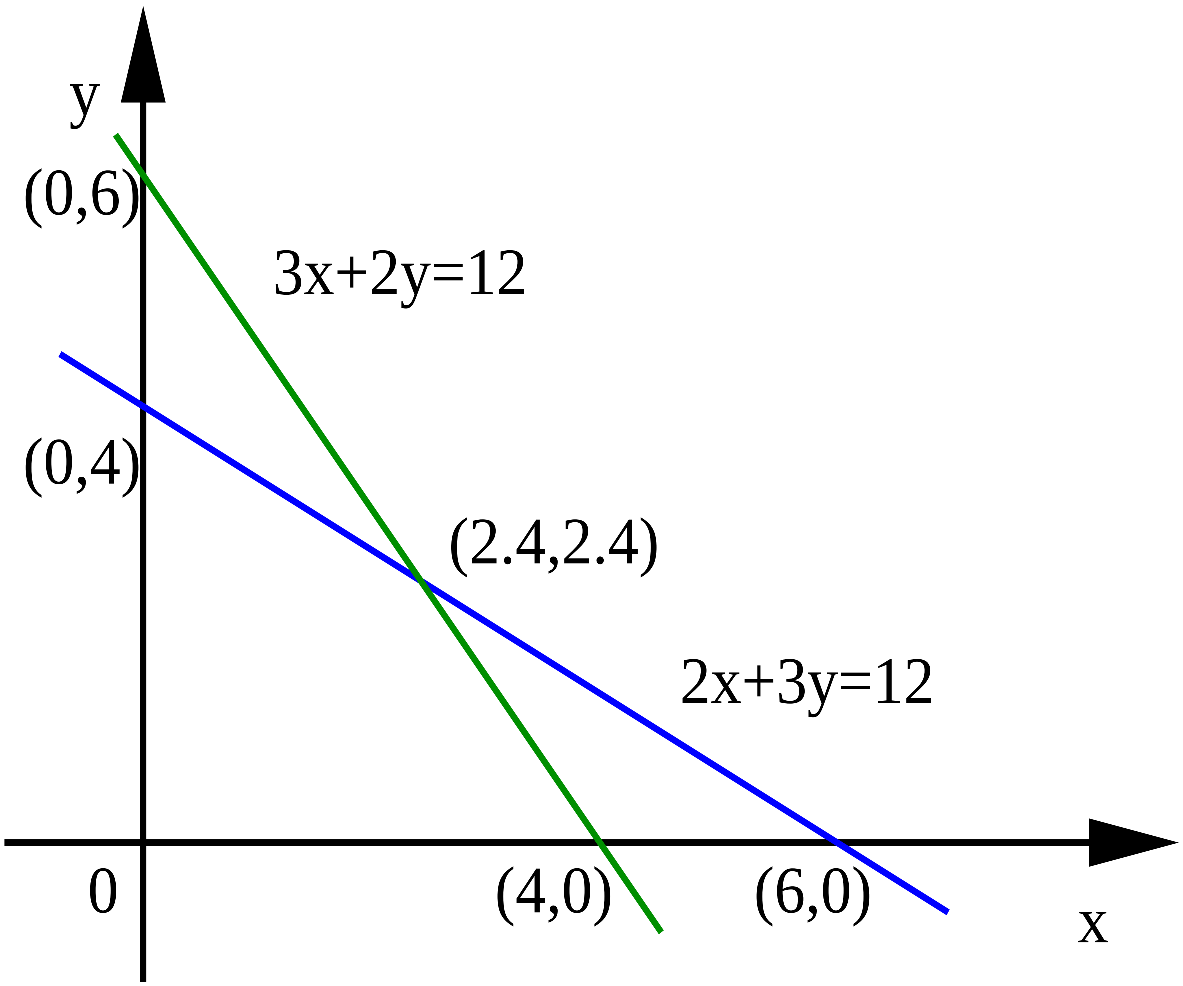

Figure 2: System of Linear Equations

Fundamental Theorem of Linear Equations

Theorem 2 *Given $A \in \mathcal{R}^{m\times n}$ and $\mathbf{b} \in \mathcal{R}^m$, the system $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}\}$ has a solution if and only if that $A^T\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} \neq 0$ has no solution.*

A vector $\mathbf{y}$, with $A^T\mathbf{y} = 0$ and $\mathbf{b}^T\mathbf{y} \neq 0$, is called an infeasibility certificate for the system.

Example Let $A = (1; -1)$ and $\mathbf{b} = (1; 1)$. Then, $\mathbf{y} = (1/2; 1/2)$ is an infeasibility certificate.

Alternative systems: $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}\}$ and $\{\mathbf{y} : \ A^T\mathbf{y} = \mathbf{0}, \ \mathbf{b}^T\mathbf{y} \neq 0\}$.

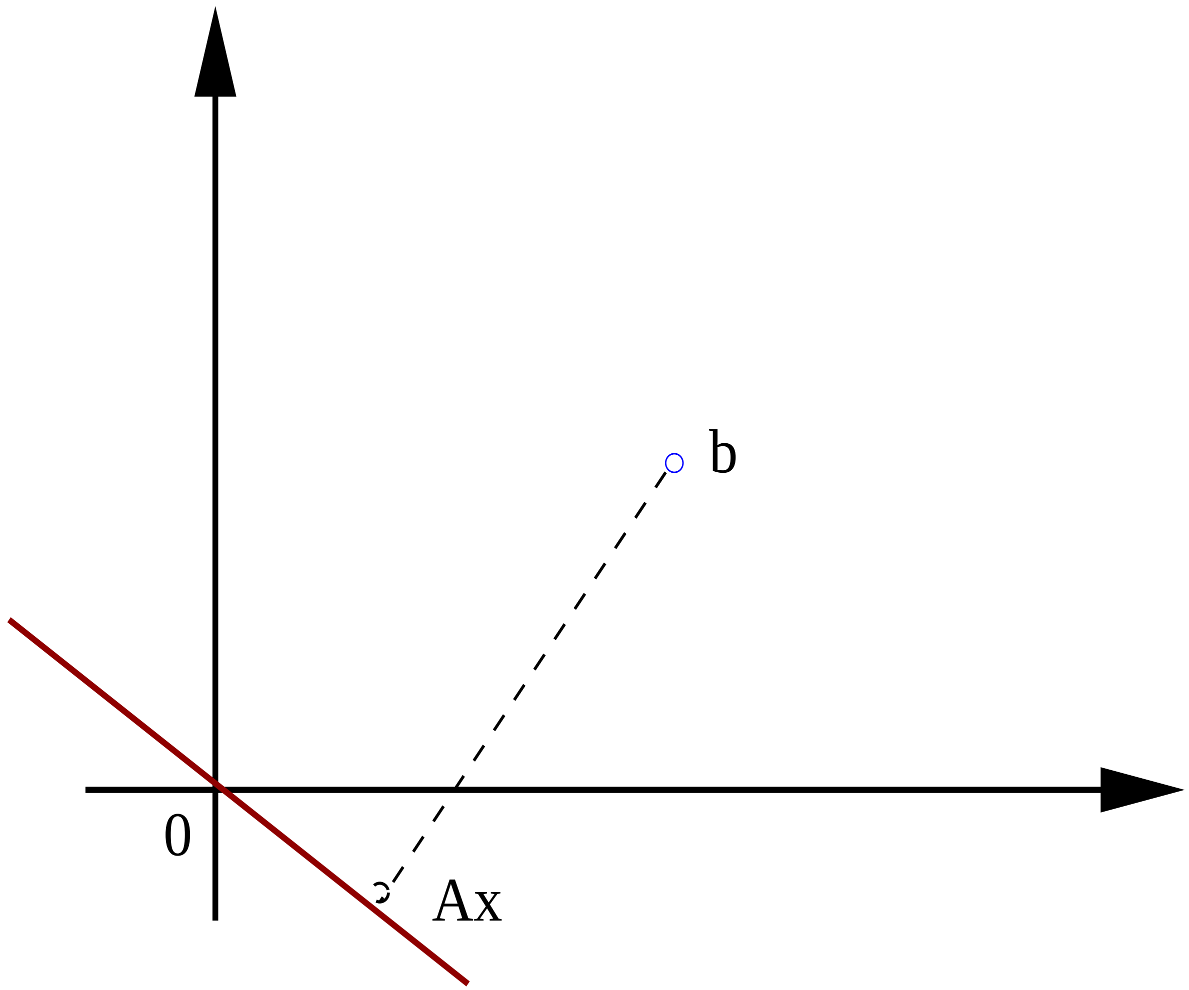

Figure 3: $\mathbf{b}$ is not in the set $\{A\mathbf{x} : \mathbf{x}\}$, and $\mathbf{y}$ is the distance vector from $\mathbf{b}$ to the set.

Affine, Convex, Linear, and Conic Combinations

When $\mathbf{x}$ and $\mathbf{y}$ are two distinct points in R^n and α runs over R,

$$\{\mathbf{z} : \mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}\}$$

is the line connecting $\mathbf{x}$ and $\mathbf{y}$. When $0 \le \alpha \le 1$, it is called the convex combination of $\mathbf{x}$ and $\mathbf{y}$ and it is the line segment between $\mathbf{x}$ and $\mathbf{y}$. Also, the set

$$\{\mathbf{z} : \mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}\},$$

for multipliers α and β is the linear combination of $\mathbf{x}$ and $\mathbf{y}$, and it is the hyperplane containing the origin and $\mathbf{x}$ and $\mathbf{y}$. When $\alpha \ge 0$ and $\beta \ge 0$, such z is called a conic combination.

Convex Sets

- Set notations: $x \in \Omega$, $y \notin \Omega$, $S \cup T$, and $S \cap T$
- Ω is said to be a convex set if for every $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ and every real number $\alpha \in [0, 1]$, the linear combination satisfies $\alpha\mathbf{x}^1 + (1 - \alpha)\mathbf{x}^2 \in \Omega$.
- The convex hull of a set Ω is the intersection of all convex sets containing Ω.
- Any Intersection of convex sets is convex.
- A point in a convex set is an extreme point if and only if it cannot be represented as a convex combination of two distinct points in the set.
- A set is polyhedral if and only if it has finite number of extreme points.

Proof of convex set

- All solutions to the system of linear equations $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}\}$ form a convex set.
- All solutions to the system of linear inequalities $\{\mathbf{x} : \ A\mathbf{x} \leq \mathbf{b}\}$ form a convex set.
- All solutions to the system of linear equations and inequalities $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ form a convex set.
- Ball is a convex set. The ball with a center $\mathbf{y} \in \mathcal{R}^n$ and a radius $r > 0$ is denoted by $B(\mathbf{y}, r) := \{\mathbf{x} : \ \|\mathbf{x} - \mathbf{y}\| \leq r\}$.
- Ellipsoid is a convex set. The ellipsoid with a center $\mathbf{y} \in \mathcal{R}^n$ and a positive definite matrix Q is denoted by $E(\mathbf{y}, Q) = \{\mathbf{x} : \ (\mathbf{x} - \mathbf{y})^T Q(\mathbf{x} - \mathbf{y}) \leq 1\}$.

More Proofs on Convexity

Given a matrix A, let's consider the set B of all $\mathbf{b}$ such that the set $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ is feasible. Show that B is a convex set.

Example:

$$B = \{b : \ \{(x_1, x_2) : \ x_1 + x_2 = b, \ (x_1, x_2) \geq \mathbf{0}\} \text{ is feasible}\}.$$

Convex Cones

- A set C is a cone if $\mathbf{x} \in C$ implies $\alpha\mathbf{x} \in C$ for all $\alpha > 0$.
- A convex cone is a cone which is also convex.
- Dual cone:

$$C^* := \{\mathbf{y} : \ \mathbf{y} \bullet \mathbf{x} \geq 0 \quad \text{for all} \quad \mathbf{x} \in C\}$$

.

Cone Examples

- Example 2.1: The n-dimensional non-negative orthant $\mathcal{R}^n_+ = \{\mathbf{x} \in \mathcal{R}^n : \ \mathbf{x} \geq \mathbf{0}\}$ is a convex cone.
- Example 2.2: The set of all positive semi-definite matrices in $\mathcal{S}^n$, $\mathcal{S}^n_+$, is a convex cone, called the positive semi-definite matrix cone
- Example 2.3: The set $\mathcal{N}^n_2 := \{\mathbf{x} \in \mathcal{R}^n : \ x_1 \geq \|\mathbf{x}_{-1}\|\}$ is a convex cone in $\mathcal{R}^n$ called the second-order cone.
- Example 2.4: The set $\mathcal{N}^n_p := \{\mathbf{x} \in \mathcal{R}^n : \ x_1 \geq \|\mathbf{x}_{-1}\|_p\}$ is a convex cone in $\mathcal{R}^n$ called the p-order cone with $p \geq 1$.

Polyhedral Convex Cones

- A cone C is a (convex) polyhedral if C can be represented as

$$C = \{\mathbf{x} : A\mathbf{x} \leq 0\} \quad \text{or} \quad \{\mathbf{x} : \ \mathbf{x} = A\mathbf{y},\ \mathbf{y} \geq \mathbf{0}\}$$

for some matrix A. In the latter case, C is generated by the columns of A.

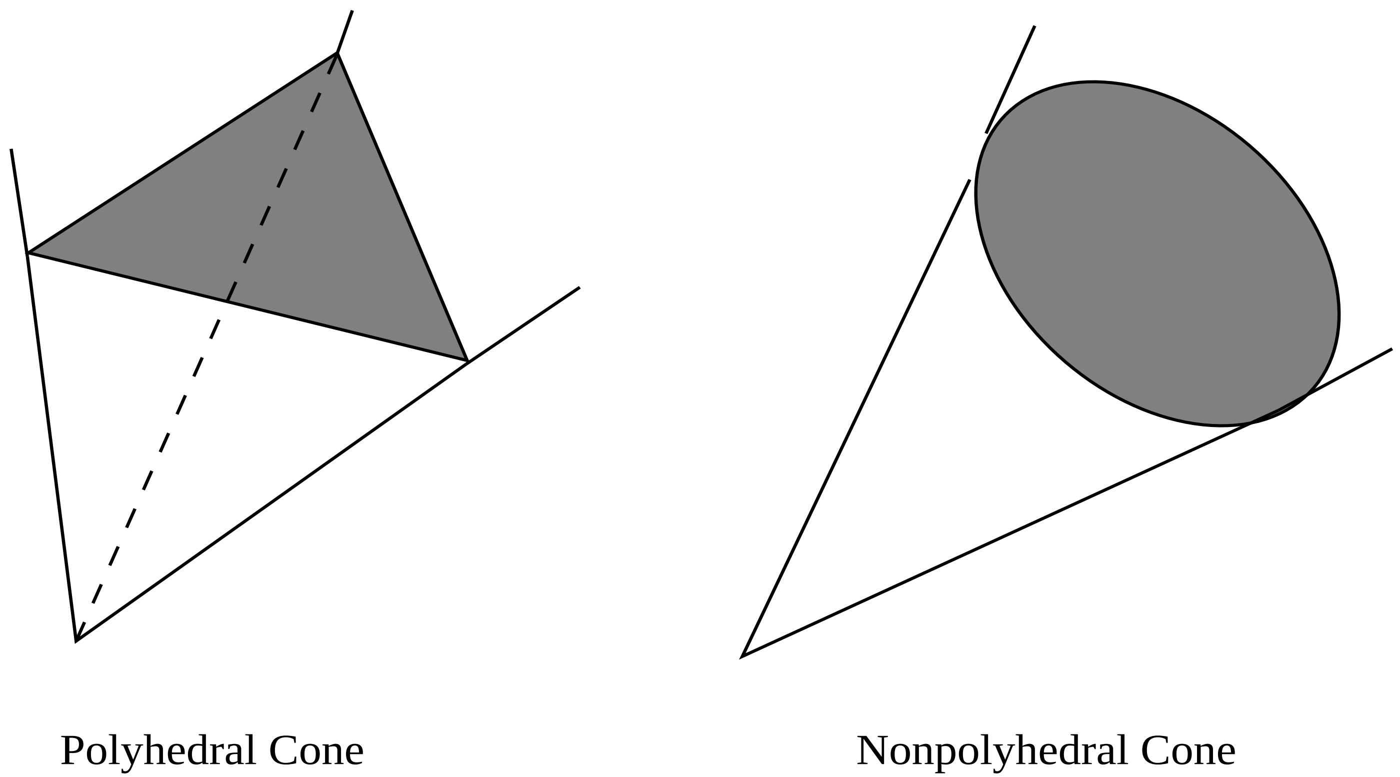

Figure 4: Polyhedral and non-polyhedral cones.

- The nonnegative orthant is a polyhedral cone but the second-order cone is not polyhedral.

Carathéodory's Theorem

The following theorem states that a polyhedral cone can be generated by a set of basic directional vectors.

Theorem 3 *Given matrix* $A \in \mathcal{R}^{m \times n}$ *where* $n > m$*, take a convex polyhedral cone* $C = \{A\mathbf{x} : \mathbf{x} \geq \mathbf{0}\}$*. Then for any* $\mathbf{b} \in C$*,*

$$\mathbf{b} = \sum_{i=1}^{d} \mathbf{a}_{j_i} x_{j_i}, \; x_{j_i} \geq 0, \forall i$$

for some linearly independent vectors $\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_d}$ *chosen from* $\mathbf{a}_1, \ldots, \mathbf{a}_n$*.*

Real Functions

- Continuous functions C
- Weierstrass theorem: a continuous function $f(\mathbf{x})$ defined on a compact set (bounded and closed) $\Omega \subset \mathcal{R}^n$ has a minimizer in Ω.
- The least upper bound or supremum of f over Ω

$$\sup\{f(\mathbf{x}) :\ \mathbf{x} \in \Omega\}$$

and the greatest lower bound or infimum of f over Ω

$$\inf\{f(\mathbf{x}) :\ \mathbf{x} \in \Omega\}$$

- A function $f(\mathbf{x})$ is said to be homogeneous of degree k if $f(\alpha\mathbf{x}) = \alpha^k f(\mathbf{x})$ for all $\alpha \geq 0$.
  Let $\mathbf{c} \in \mathcal{R}^n$ be given and $\mathbf{x} \in \operatorname{int} \mathcal{R}^n_+$. Then $\mathbf{c}^T\mathbf{x}$ is homogeneous of

degree 1 and

$$\phi(\mathbf{x}) = n \log(\mathbf{c}^T \mathbf{x}) - \sum_{j=1}^{n} \log x_j$$

is homogeneous of degree 0.

Let $C \in \mathcal{S}^n$ be given and $X \in \operatorname{int} \mathcal{S}^n_+$. Then $\mathbf{x}^T C \mathbf{x}$ is homogeneous of degree 2, $C \bullet X$ and $\det(X)$ are homogeneous of degree 1 and n, respectively, and

$$\Phi(X) = n \log(C \bullet X) - \log \det(X)$$

is homogeneous of degree 0.

- The gradient vector C^1:

$$\nabla f(\mathbf{x}) = \{\partial f / \partial x_i\}, \quad \text{for} \quad i = 1, ..., n.$$

- The Hessian matrix C^2:

$$\nabla^2 f(\mathbf{x}) := \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} \quad \text{for} \quad i = 1, ..., n;\ j = 1, ..., n.$$

- Vector function: $\mathbf{f} = (f_1; f_2; ...; f_m)$

- The Jacobian matrix of $\mathbf{f}$:

$$\nabla \mathbf{f}(\mathbf{x}) := \begin{pmatrix} \nabla f_1(\mathbf{x}) \\ \vdots \\ \nabla f_m(\mathbf{x}) \end{pmatrix}.$$

Convex Functions

- f is a convex function iff for $0 \leq \alpha \leq 1$,

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}).$$

- The level set of f is convex:

$$L(z) = \{\mathbf{x} : f(\mathbf{x}) \leq z\}.$$

- The convex set $\{(z;\mathbf{x}) : f(\mathbf{x}) \leq z\}$ is called the epigraph of f.
- $tf(\mathbf{x}/t)$ is a convex function of $(t;\mathbf{x})$ for $t > 0$ and it's homogeneous of degree 1.

Proof of convex function

Consider the minimal-objective value function of $\mathbf{b}$ for fixed A and $\mathbf{c}$:

$$\begin{aligned} z(\mathbf{b}) := \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Show that $z(\mathbf{b})$ is a convex function in $\mathbf{b}$ for all feasible $\mathbf{b}$.

Theorems on Functions

Taylor's theorem or the mean-value theorem:

Theorem 4 *Let $f \in C^1$ be in a region containing the line segment $[\mathbf{x}, \mathbf{y}]$. Then there is α with $0 \le \alpha \le 1$ such that*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y})(\mathbf{y} - \mathbf{x}).$$

Furthermore, if $f \in C^2$ then there is α with $0 \le \alpha \le 1$ such that

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x}) + (1/2)(\mathbf{y}-\mathbf{x})^T \nabla^2 f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})(\mathbf{y}-\mathbf{x}).$$

Theorem 5 *Let $f \in C^1$. Then f is convex over a convex set Ω if and only if*

$$f(\mathbf{y}) \ge f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{x},\ \mathbf{y} \in \Omega$.

Theorem 6 *Let* $f \in C^2$*. Then* f *is convex over a convex set* Ω *if and only if the Hessian matrix of* f *is positive semi-definite throughout* Ω*.*

Linear Least Squares Problems

Given $A \in \mathcal{R}^{m\times n}$ and $\mathbf{c} \in \mathcal{R}^n$,

$$\begin{array}{lll} (LS) & \text{minimize} & \|\mathbf{c} - A^T\mathbf{y}\|^2 \\ & \text{subject to} & \mathbf{y} \in \mathcal{R}^m. \end{array}$$

A close form solution:

$$AA^T\mathbf{y} = A\mathbf{c} \quad \text{or} \quad \mathbf{y} = (AA^T)^{-1}A\mathbf{c}.$$

$$\mathbf{c} - A^T\mathbf{y} = \mathbf{c} - A^T(AA^T)^{-1}A\mathbf{c} = \mathbf{c} - P\mathbf{c}$$

Projection matrix: $P = A^T(AA^T)^{-1}A$ or $P = I - A^T(AA^T)^{-1}A$.

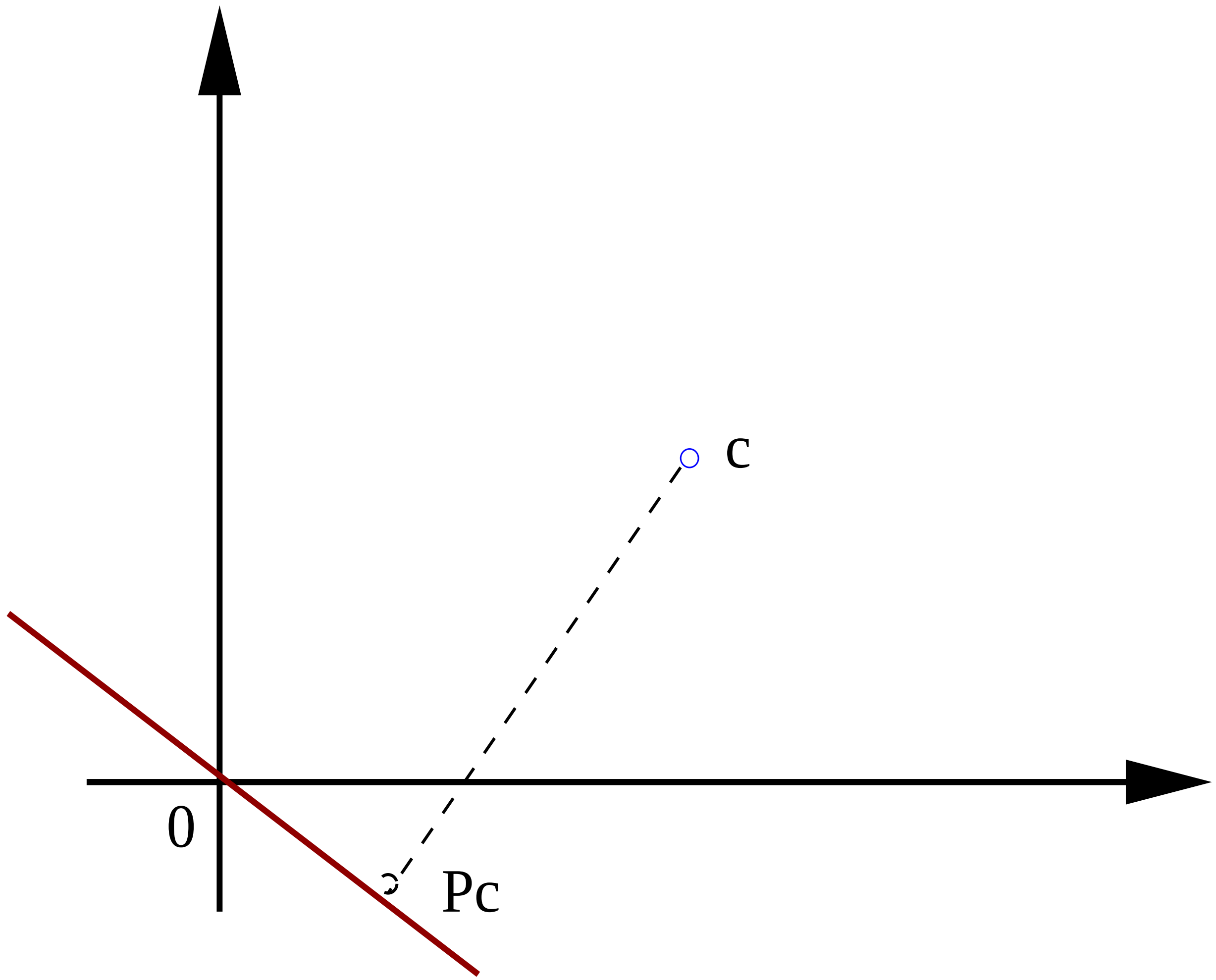

Figure 5: Projection of $\mathbf{c}$ onto a subspace

Choleski decomposition method

$$AA^T = L\Lambda L^T$$

$$L\Lambda L^T \mathbf{y}^* = A\mathbf{c}$$

System of nonlinear equations

Given $\mathbf{f}(\mathbf{x}) : \mathcal{R}^n \rightarrow \mathcal{R}^n$, the problem is to solve n equations for n unknowns:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}.$$

Given a point $\mathbf{x}^k$, Newton's Method sets

$$f(\mathbf{x}) \simeq f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k) = \mathbf{0}.$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k - (\nabla f(\mathbf{x}^k))^{-1} f(\mathbf{x}^k)$$

or solve for direction vector $\mathbf{d}_x$:

$$\nabla f(\mathbf{x}^k)\mathbf{d}_x = -f(\mathbf{x}^k) \quad \text{and} \quad \mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}_x.$$

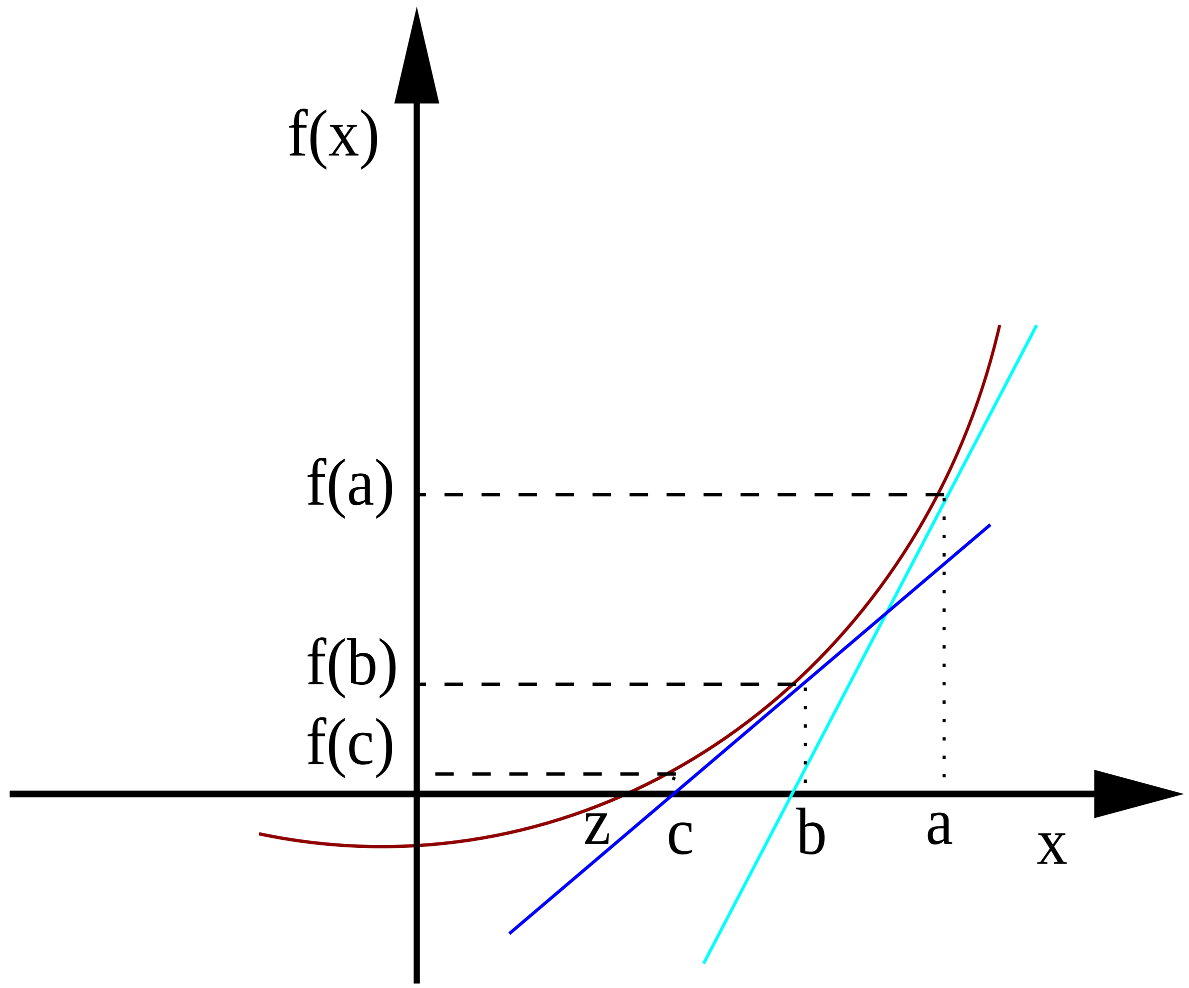

Figure 6: Newton’s method for root finding

The quasi Newton method

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha(\nabla f(\mathbf{x}^k))^{-1} f(\mathbf{x}^k)$$

where scalar $\alpha \geq 0$ is called the step-size. More generally, we may use

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha M^k f(\mathbf{x}^k)$$

where M^k is an $n \times n$ symmetric matrix. In particular, if $M^k = I$, then the method is called the gradient method, where f is viewed as the gradient vector of a real function.

Convergence and Big O

- $\{\mathbf{x}^k\}_0^\infty$ denotes a seqence $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^k, ...$
- We denote $\mathbf{x}^k \to \bar{\mathbf{x}}$ when $\|\mathbf{x}^k - \bar{\mathbf{x}}\| \to 0$
- $g(x) \geq 0$ is a real valued function of a real nonnegative variable, the notation $g(x) = O(x)$ means that $g(x) \leq \bar{c}x$ for some constant $\bar{c}$.
- $g(x) = \Omega(x)$ means that $g(x) \geq \underline{c}x$ for some constant $\underline{c}$.
- $g(x) = \theta(x)$ means that $\underline{c}x \leq g(x) \leq \bar{c}x$.
- $g(x) = o(x)$ means that $g(x)$ goes to zero faster than x does:

$$\lim_{x \to 0} \frac{g(x)}{x} = 0$$