Signals and Systems Notes

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1 Convolution

Literally the only important thing for convolution is to know the formula:

$$h[n] * x[n] = y[n] = \sum_{k=-\infty}^{\infty} h[k]x[k-n] \quad (1)$$

Now, if you're not satisfied, let's put a little bit of intuition in here. There's a few ways to look at convolution. First, it's the x[n] added to itself a bunch of times, but each time it's shifted by a value k and multiplied by the value of h at k. This way of looking at it helps when h[n] is very simple, say 1 for some range of values, then the convolution is x[n] added to itself a bunch of times in different phases.

Now, this formula can be extended to continuous time, and it's pretty much the same thing:

$$h(t) * x(t) = y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \quad (2)$$

Here, it might be easier to think of convolution as a the 'impulse response' h(t) 'sliding over' the function x(t), and the value of the convolution is the common area between the two functions. Wait, why is h(t) called the impulse response? Because if x(t) is the impulse function $\delta(t)$, then the output is simply h(t), since impulse function is only nonzero at 0, so the convolution sum reduces to a single value, h(t). So the response to the impulse function is h(t). So h(t) is the impulse response.

Keep in mind, commutativity and associativity hold for convolution (easy to prove with a change of variables $t_2 = t - \tau$), so I might've been hand-waving some interchanges between x and h.

2 Fourier!

2.1 Fourier Series

So here's the thing. Fourier Series are a way to turn functions in the time domain (what we're used to – functions over time) to functions in the frequency domain (essentially, a sum of sines and cosines with different frequencies). The Fourier Series is a formula to transform a function from the time domain to the frequency domain. Don't worry about the proof for this; just know the formula and know how to integrate.

First, we need a definition for the fourier representation of any time-series x(t), as follows:

$$x(t) = \int_0^T a_k e^{jk\omega_0 t} dt \tag{3}$$

for every integer k from $-\infty$ to ∞. Now, how do we transform x(t) to the a_k's? That's the important thing. Intuitively, we know that any periodic function can be represented as a sum of a countably infinite number of sines and cosines; it's just a matter of finding what the scaling factors for these sines and cosines are.

So, the magic formula that converts the time domain to the frequency domain is the following:

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \tag{4}$$

Keep in mind, this only works for PERIODIC functions. In order to get it to work for NONPERIODIC functions, we essentially take the limit of the period to infinity (infinite period means it never repeats), and now instead of having a bunch of discrete a_k values for any integral k, we have a continuum of a(k) values for any real-valued k. But now, since we feel like it, we call it $x(jw_0)$. And now, the formula is:

$$x(jw_0) = \int_{-\infty}^{\infty} x(t) e^{-j\omega_0 t} dt \tag{5}$$

Awesome! Now just one more thing. What happens if we want to go backwards? That is, given the function in the frequency domain (sum of a bunch of sines and cosines), how can we get the function over time back? Well, thanks for asking. Here's the formula.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(jw_0) e^{j\omega_0 t} d\omega_0 \tag{6}$$

2.1.1 Delta Train

Let's take a look at a specific example, the delta train. But first, let's look at a scaled delta function, defined as:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0), \tag{7}$$

which basically means it's a single value at ω_0. Clearly, taking the integral in (6) gives you just a single value, which is $e^{j\omega_0 t}$. Now what if we stacked a bunch of delta's together in a periodic fashion, so now:

$$X(j\omega) = \sum_k 2\pi a_k \delta(\omega - k\omega_0). \tag{8}$$

This basically means that there's a scaled pulse after every period of time, and that's it. Now, clearly since the integral just adds all the values for which the function is nonzero, we get

$$X(j\omega) = \sum_k a_k e^{j\omega_0 t}. \tag{9}$$

2.1.2 What if $x(j\omega)$ is given in terms of magnitude and phase?

Since $x(j\omega)$ is often complex for real x(t), sometimes we will get it in terms of its magnitude and phase. But don't fret, just use this simple formula you probably learned in precalc to get the function in terms of magnitude and phase:

$$x(jw) = |x(jw)|\, e^{j\angle x(j\omega)} \tag{10}$$

Literally just plug that in to the relevant formula, and you're good.

THINGS TO KNOW FOR TEST:

- Convolution (REMEMBER $-\infty$ to ∞)
- Fourier Series
- Fourier Transform

2.2 Fourier Transform Intuition

Know the following Fourier Transforms.

1. $\delta(t) \leftrightarrow 1$
2. $\begin{cases} 1 & -T_1 < t < T_1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow \frac{2sin(\omega T_1)}{\omega}$
3. $sin(t) \leftrightarrow \frac{\pi}{j}\left(\delta(\omega_0 t - 1) - \delta(\omega_0 t + 1)\right)$
4. $cos(t) \leftrightarrow \pi\left(\delta(\omega_0 t - 1) + \delta(\omega_0 t + 1)\right)$

Remember that fourier transforms are linearly combinable, so try to reduce any more complicated functions to these things.

2.3 Using Differentiation

Somehow, the following are equivalent statements:

$$x(t) \leftrightarrow x(j\omega) \tag{11}$$

$$\frac{d}{dt}x(t) \leftrightarrow j\omega \cdot x(j\omega) \tag{12}$$

Also, we have this really cool fourier transform relationship like what:

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) \tag{13}$$

Unfortunately we didn't prove this :/

2.3.1 Time Scaling

The following are equivalent statements, interestingly enough:

$$x(t) \leftrightarrow X(j\omega) \tag{14}$$

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right) \tag{15}$$

You can prove this by taking the integral.