Signals and Systems Notes

Shaya Zarkesh

October 2017

1 Convolution

Literally the only important thing for convolution is to know the formula:

$$h[n] * x[n] = y[n] = \sum_{k=-\infty}^{\infty} h[k]x[k-n] \tag{1}$$

Now, if you're not satisfied, let's put a little bit of intuition in here. There's a few ways to look at convolution. First, it's the x[n] added to itself a bunch of times, but each time it's shifted by a value k and multiplied by the value of h at k. This way of looking at it helps when h[n] is very simple, say 1 for some range of values, then the convolution is x[n] added to itself a bunch of times in different phases.

Now, this formula can be extended to continuous time, and it's pretty much the same thing:

$$h(t) * x(t) = y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{2}$$

Here, it might be easier to think of convolution as a the 'impulse response' h(t) 'sliding over' the function x(t), and the value of the convolution is the common area between the two functions. Wait, why is h(t) called the impulse response? Because if x(t) is the impulse function $\delta(t)$, then the output is simply h(t), since impulse function is only nonzero at 0, so the convolution sum reduces to a single value, h(t). So the response to the impulse function is h(t). So h(t) is the impulse response.

Keep in mind, commutativity and associativity hold for convolution (easy to prove with a change of variables $t_2 = t - \tau$), so I might've been hand-waving some interchanges between x and h.

2 Fourier!

2.1 Fourier Series

So here's the thing. Fourier Series are a way to turn functions in the time domain (what we're used to – functions over time) to functions in the frequency domain (essentially, a sum of sines and cosines with different frequencies). The Fourier Series is a formula to transform a function from the time domain to the frequency domain. Don't worry about the proof for this; just know the formula and know how to integrate.

First, we need a definition for the fourier representation of any time-series x(t), as follows:

$$x(t) = \int_0^T a_k e^{jk\omega_0 t} dt \tag{3}$$

for every integer k from $-\infty$ to ∞. Now, how do we transform x(t) to the a_k's? That's the important thing. Intuitively, we know that any periodic function can be represented as a sum of a countably infinite number of sines and cosines; it's just a matter of finding what the scaling factors for these sines and cosines are.

So, the magic formula that converts the time domain to the frequency domain is the following:

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \tag{4}$$

Keep in mind, this only works for PERIODIC functions. In order to get it to work for NONPERIODIC functions, we essentially take the limit of the period to infinity (infinite period means it never repeats), and now instead of having a bunch of discrete a_k values for any integral k, we have a continuum of a(k) values for any real-valued k. But now, since we feel like it, we call it $x(jw_0)$. And now, the formula is:

$$x(jw_0) = \int_{-\infty}^{\infty} x(t) e^{-j\omega_0 t} dt \tag{5}$$

Awesome! Now just one more thing. What happens if we want to go backwards? That is, given the function in the frequency domain (sum of a bunch of sines and cosines), how can we get the function over time back? Well, thanks for asking. Here's the formula.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(jw_0) e^{j\omega_0 t} d\omega_0 \tag{6}$$

2.1.1 Delta Train

Let's take a look at a specific example, the delta train. But first, let's look at a scaled delta function, defined as:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0), \tag{7}$$

which basically means it's a single value at ω_0. Clearly, taking the integral in (6) gives you just a single value, which is $e^{j\omega_0 t}$. Now what if we stacked a bunch of delta's together in a periodic fashion, so now:

$$X(j\omega) = \sum_k 2\pi a_k \delta(\omega - k\omega_0). \tag{8}$$

This basically means that there's a scaled pulse after every period of time, and that's it. Now, clearly since the integral just adds all the values for which the function is nonzero, we get

$$X(j\omega) = \sum_k a_k e^{j\omega_0 t}. \tag{9}$$

2.1.2 What if $x(j\omega)$ is given in terms of magnitude and phase?

Since $x(j\omega)$ is often complex for real x(t), sometimes we will get it in terms of its magnitude and phase. But don't fret, just use this simple formula you probably learned in precalc to get the function in terms of magnitude and phase:

$$x(jw) = |x(jw)|\, e^{j\angle x(j\omega)} \tag{10}$$

Literally just plug that in to the relevant formula, and you're good.

THINGS TO KNOW FOR TEST:

- Convolution (REMEMBER $-\infty$ to ∞)
- Fourier Series
- Fourier Transform

2.2 Fourier Transform Intuition

Know the following Fourier Transforms.

1. $\delta(t) \leftrightarrow 1$
2. $\begin{cases} 1 & -T_1 < t < T_1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow \frac{2\sin(\omega T_1)}{\omega}$
3. $\sin(\omega_0 t) \leftrightarrow \frac{\pi}{j}\left(\delta(\omega_0 t - 1) - \delta(\omega_0 t + 1)\right)$
4. $\cos(\omega_0 t) \leftrightarrow \pi\left(\delta(\omega_0 t - 1) + \delta(\omega_0 t + 1)\right)$

Remember that fourier transforms are linearly combinable, so try to reduce any more complicated functions to these things.

2.3 Using Differentiation

Somehow, the following are equivalent statements:

$$x(t) \leftrightarrow x(j\omega) \tag{11}$$

$$\frac{d}{dt}x(t) \leftrightarrow j\omega \cdot x(j\omega) \tag{12}$$

Also, we have this really cool Fourier transform integration relationship like what:

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) \tag{13}$$

Unfortunately we didn't prove this :/

2.3.1 Time Scaling

The following are equivalent statements, interestingly enough:

$$x(t) \leftrightarrow X(j\omega) \tag{14}$$

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{j\omega}{a}\right) \tag{15}$$

You can prove this by taking the integral.

2.3.2 Duality

Let's take a look at two functions, one in the time domain and one in frequency domain. We see that there is a duality, or symmetry, defined by the two, as follows:

$$x(t) \leftrightarrow X(j\omega) \tag{16}$$

$$X(t) \leftrightarrow 2\pi x(-j\omega) \tag{17}$$

2.4 Friday, November 3

We went over our homework. Need to know four basic fourier transforms: square wave, Sin/Cos, delta, delta train.

Did a couple problems – gotta know that an even function has no phase?

3 Chapter 5: Discrete Time Fourier Transforms (DTFTs)

Discrete Time Fourier Transforms are the discrete analog to what we have been covering so far, continuous-time Fourier transforms.

$$X(e^{j\omega}) = \sum_{n} x[n]e^{-j\omega n} \tag{18}$$

Why do we need $e^{j\omega}$? Because that's a periodic function, and DTFTs take a discrete signal to a continuous periodic one. Why is that necessary? To maintain informational content on the same order of infinity (@Swapnil).

So, what are the differences between this and continuous? Well, for one, $X(e^{jw})$ is periodic, whereas in continuous that's not neccissarily true.

Properties of Fourier Series:

1. Linearity
2. Time Shift:

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X \tag{19}$$

$$e^{j\omega n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)}) \tag{20}$$

3. Differentiation

$$x[n]-x[n-1] \leftrightarrow (1-e^{-j\omega})X \tag{21}$$

4. Integration

$$y[n] = \sum_{-\infty}^{n} x[m] \leftrightarrow \frac{1}{1-e^{-j\omega}}X + \pi X(e^{j_0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \tag{22}$$

5. Parseval's Relation

$$\sum |x[n]|^2 = \frac{1}{2\pi}\int_{<2\pi>} |X(e^{j\omega})|^2 d\omega \tag{23}$$

6. Convolution
   Convolution is really weird because you have to do something special. Not sure what the special thing was, sorry.

ALL of these properties hold for the continuous case, in some form or another. We showed differentiation and integration above; Parseval's relation is the same, just with an integral instead of a sum on the left side, and the bounds on the right from negative to positive infinity since the FT is no longer periodic, that is:

$$\int_{-\infty}^{\infty} |x(t)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \tag{24}$$

4 Filtering

We're done with Fourier transforms now. YAY! (Well, sad life if you liked them, but who likes intgrals?)

Cool, so now the question is: what's really the point of all this? Think about it this way. What if we want to isolate a certain set of frequencies from a signal?

This is a really common necessity in digital signal processing; say, you want to remove all noise that occurs above 2000 Hz, or remove an echo. How can we do this? We can multiply (dot product) a box-like function in the frequency domain to the signal and easily make the values we want become zero. In the time domain, this would be a convolution of some dampening sinusoidal function. That's really complicated, but at the end of the day, the time domain is what drives the receiver or speaker that the signal is effecting, so we can't stick in the frequency domain forever.

So, intuitively, there's a few kinds of filters we could want.

1. Low-Pass Filter (LPF): These isolate lower frequencies, up to some cutoff. In the frequency domain, it looks like a box centered at 0 with height 1 (since we don't want to change the magnitude of the single) and width corresponding to the interval of signals we want to isolate.

2. High-Pass Filter (HPF): These isolate higher frequencies, above (and below on the negative end) some cutoff. In the frequency domain, these are two infinite boxes, one starting on the positive end starting at some value and moving out to infinity, and one on the negative end starting at the opposite of that value and moving to the negative end.

3. Band Pass Filter (BPF): Isolate a certain range of frequencies. LPF is a form of BPF. We simply put a box around the range of frequencies we want, and we get the exact chunk of the signal we want. Isn't that awesome!

4. Notch: Pretty much the opposite of BPF. We want to isolate everything except a certain range of frequencies, so we have the frequency domain function be at 1, except it's 0 in the range we don't want; it looks like two boxes. HPF is a type of notch.

4.1 Non-ideal Filters

The problem with our idealized vision of a filter is that the time-domain correspondence extends out to infinity in both directions. This means that not only is the system non-causal, it requires values from infinite time ago, so only for an infinite stream can you create the "idealized" boxy band-pass filter.

So, IRL we use non-ideal filters. One form is shown below, with a transition band that goes from 1 to 0 over a nonzero section.

Another form is a ripple band, which is a more sinusoidal curve that stays high for certain values and then goes low. This is not ideal, since we want the same magnitude for all frequencies in the desired range, but we have some tolerance for differing frequencies.

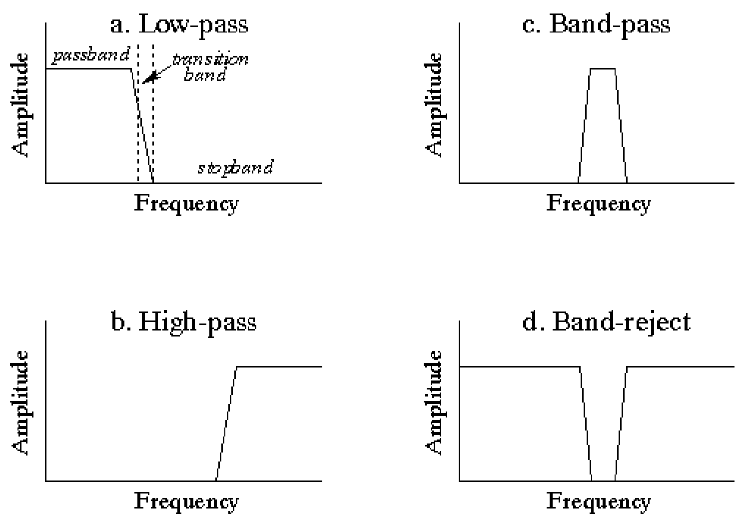

Figure 1: Here's a visualization of these filters in the frequency domain. Note that for our calculations so far, we had no transition band and made it an idealized, straight-edged box.

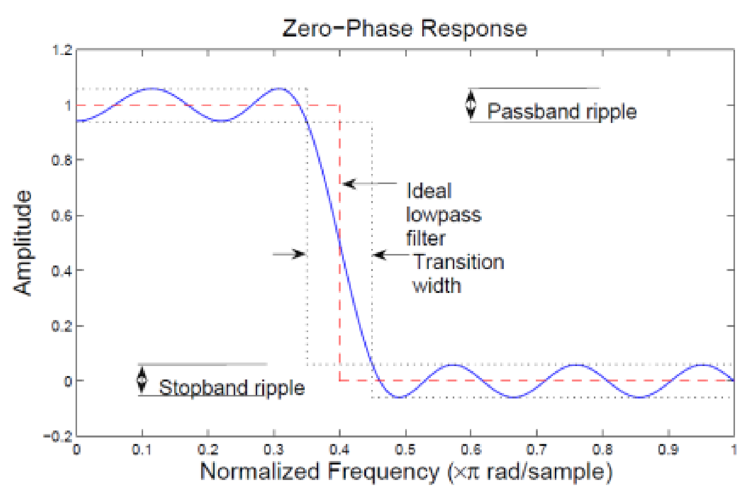

Figure 2: Here's a ripple filter. The red dotted line represents the ideal analog to this filter. Notice how for most part, the filter stays close to the line.

5 Bode Plots

A Bode plot is simply the graph of the magnitude and phase frequency response of a system, logarithmized. This means BOTH axes are logarithmized, eg: you're plotting $\log(\omega)$ vs $\log(|H|)$ except since we feel special about ourselves we make the Y-axis $20\log(|H|)$.

Note: We never got to graphing phase, which is important to see how the sound is being distorted. If the phase is linear, then there's no distortion, but otherwise there is distortion since you're changing the time different frequencies come in at. So for now, we're just gonna assume the filter is pretty good and has negligible distortion, and all that matters now is how loud (amplitude) each frequency comes out as.

Now, the cool thing about Bode plots is that since the graph is logarithmized, you can graph the product of a bunch of terms in the formula as the sum of each individual component when graphing. Let's take a look at an example.

Let's graph the frequency response denoted as $\left(\frac{20}{1+j\omega}\right)\left(\frac{40}{10+j\omega}\right)$. This tends to be really hard but we can use an approximation to convert this to a linear Bode plot. Unfortunately I don't have pictures of these graphs, but it for a component $\left(\frac{20n}{1+aj\omega}\right)$ it is a flat line until $\frac{1}{a}$ and from $\frac{1}{a}$ onward a linearly increasing or decreasing graph with slope n. Again, add the graph of each component of the frequency response (just like adding signals when we were doing convolution) to get the overall approximation for magnitude.

5.1 Using Differential Equations

Here we cover the basics of differential equations in the context of filtering. The main thing we covered here is a second-order differential equation, which appears in many physical applications (eg: the dampening of a spring, RLC circuits). The differential equation for this kind of system is:

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{d^2y(t)}{dt} + \omega_n^2 y(t) = w_n^2 x(t) \tag{25}$$

So, we have a value called zeta, ζ, and if $\zeta < 1$ then it's underdamped (goes up down up down over time with decreasing amplitude) if $\zeta > 1$ then it's overdamped (goes up down maximum once) and if $\zeta = 1$ then it's critically damped.

Note for people who took differential equations: this is exactly what we covered in that class – the relative size of ζ with respect to 1 (eg: is ζ bigger or smaller than 1) corresponds directly to the sign of the discriminant.

5.2 Problems Practice, December 6