Orthogonal Projection

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What is a projection?

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Note

In this presentation, projection refers to specifically orthogonal projection, $\vec{p}$ refers to $\text{proj}_S \vec{b}$, and $\vec{e}$ refers to the error vector $\vec{e} = \vec{b} - \vec{p}$. The error vector is also called the rejection.

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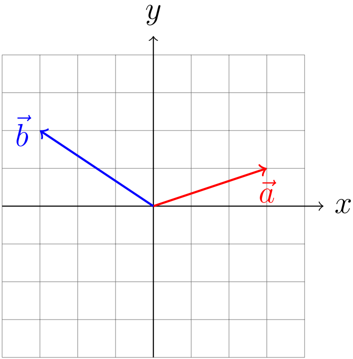
Note

The dimension of subspace S can be anything from 0 to the dimension of the space we are working inside.

Example



Example

Draw $\text{proj}_a\, b$.

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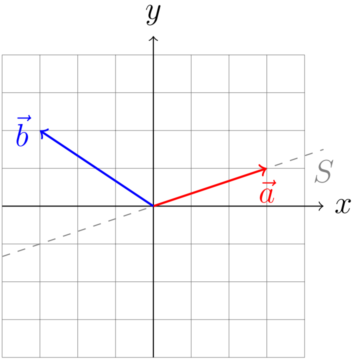

Note

When projecting a vector onto another vector $\vec{a}$, the subspace S is the span of $\vec{a}$.

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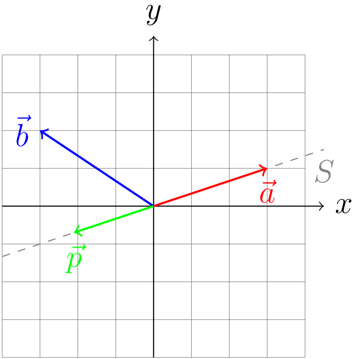

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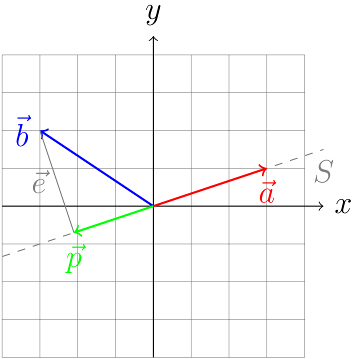

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Suppose that $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are a basis of S, let

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

and let $\vec{p} = A\vec{c}$. We also know:

$$\vec{e} = \vec{b} - \vec{p}$$

$$C(A)^{\perp} = N(A^T)$$

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Then,

$$\vec{e} = \vec{b} - A\vec{c}$$
$$A^T(\vec{b} - A\vec{c}) = \vec{0}$$
$$A^T\vec{b} - A^TA\vec{c} = \vec{0}$$
$$A^T\vec{b} = A^TA\vec{c}$$
$$(A^TA)^{-1}A^T\vec{b} = \vec{c}$$
$$\boxed{\vec{c} = (A^TA)^{-1}A^T\vec{b}}$$
$$\boxed{\vec{p} = A(A^TA)^{-1}A^T\vec{b}}$$

The Projection Matrix

Recall that for some matrix A, the projection $\vec{p} = A(A^TA)^{-1}A^T\vec{b}$.

Observation

For a specific A, $\vec{p}$ is an unchanging linear transformation of $\vec{b}$. So, we can represent the linear transformation with a matrix P.

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Let $\boxed{P = A(A^TA)^{-1}A^T}$. Then, $\vec{p} = P\vec{b}$.

Note

A must have linearly independent columns.

The Error Matrix

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$$\begin{aligned} \vec{e} &= \vec{b} - \vec{p} \\ \vec{e} &= I\vec{b} - P\vec{b} \\ \vec{e} &= (I - P)\vec{b} \\ &\boxed{E = I - P} \end{aligned}$$

Projection Minimizes Error

Prove that for a vector $\vec{x} \in S$ where $\vec{x} \neq \vec{p}$ and $\vec{d} = \vec{b} - \vec{x}$, that $||\vec{d}|| > ||\vec{e}||$.

Hint

$||\vec{d}|| > ||\vec{e}|| \iff \vec{d}^2 > \vec{e}^2$

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$$\vec{d}^2 = (\vec{b} + \vec{0} - \vec{x})^2$$

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$$\vec{d}^2 = \vec{e}^2 + (\vec{p} - \vec{x})^2$$
$$\vec{d}^2 - \vec{e}^2 = (\vec{p} - \vec{x})^2 > 0$$
$$\vec{d}^2 > \vec{e}^2$$

- Applications
  - Graphics
  - Least Squares Regression
- Further Learning
  - Practice
  - *Introduction to Linear Algebra* 6th Edition Chapter 4.2 by Gilbert Strang