

Orthogonal Projection

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What is a projection?

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Note

In this presentation, projection refers to specifically orthogonal projection, \vec{p} refers to $\text{proj}_S \vec{b}$, and \vec{e} refers to the error vector $\vec{e} = \vec{b} - \vec{p}$. The error vector is also called the rejection.

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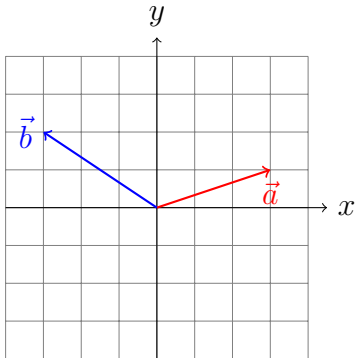
Closest means that the error is orthogonal to S .

Note

The dimension of subspace S can be anything from 0 to the dimension of the space we are working inside.



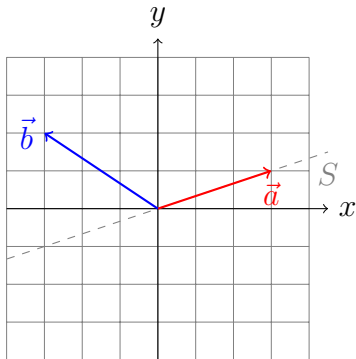
Example



Example

Draw $\text{proj}_a b$.

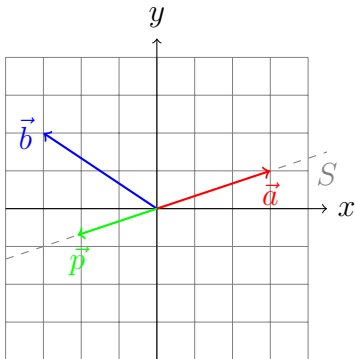
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Note

When projecting a vector onto another vector \vec{a} , the subspace S is the span of \vec{a} .

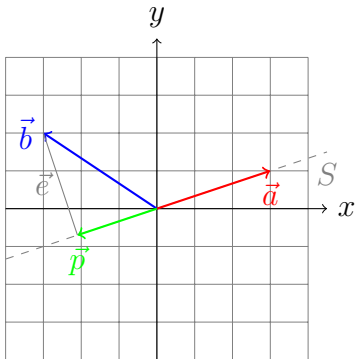
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Project \vec{b} onto S



Project \vec{b} onto S

Suppose that $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are a basis of S , let

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

and let $\vec{p} = A\vec{c}$. We also know:

$$\vec{e} = \vec{b} - \vec{p}$$

$$C(A)^\perp = N(A^T)$$

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Then,

$$\vec{e} = \vec{b} - A\vec{c}$$

$$A^T(\vec{b} - A\vec{c}) = \vec{0}$$

$$A^T \vec{b} - A^T A \vec{c} = \vec{0}$$

$$A^T \vec{b} = A^T A \vec{c}$$

$$(A^T A)^{-1} A^T \vec{b} = \vec{c}$$

$$\boxed{\vec{c} = (A^T A)^{-1} A^T \vec{b}}$$

$$\boxed{\vec{p} = A(A^T A)^{-1} A^T \vec{b}}$$

The Projection Matrix

Recall that for some matrix A , the projection
 $\vec{p} = A(A^T A)^{-1} A^T \vec{b}$.

Observation

For a specific A , \vec{p} is an unchanging linear transformation of \vec{b} . So, we can represent the linear transformation with a matrix P .



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Let $\boxed{P = A(A^T A)^{-1} A^T}$. Then, $\vec{p} = P\vec{b}$.

Note

A must have linearly independent columns.

The Error Matrix

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$$\vec{e} = \vec{b} - \vec{p}$$

$$\vec{e} = I\vec{b} - P\vec{b}$$

$$\vec{e} = (I - P)\vec{b}$$

$$\boxed{E = I - P}$$





Projection Minimizes Error

Prove that for a vector $\vec{x} \in S$ where $\vec{x} \neq \vec{p}$ and $\vec{d} = \vec{b} - \vec{x}$, that $\|\vec{d}\| > \|\vec{e}\|$.

Hint

$$\|\vec{d}\| > \|\vec{e}\| \iff d^2 > e^2$$





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$$d^2 = (\vec{b} - \vec{x})^2$$



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$$\vec{d}^2 = (\vec{b} + \vec{0} - \vec{x})^2$$



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$$\vec{d}^2 = (\vec{b} - \vec{p} + \vec{p} - \vec{x})^2$$

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$$\vec{d}^2 = \vec{e}^2 + 2\vec{e} \cdot (\vec{p} - \vec{x}) + (\vec{p} - \vec{x})^2$$

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$$\begin{aligned}\vec{d}^2 &= \vec{e}^2 + (\vec{p} - \vec{x})^2 \\ \vec{d}^2 - \vec{e}^2 &= (\vec{p} - \vec{x})^2\end{aligned}$$

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$$\begin{aligned}\vec{d}^2 &= \vec{e}^2 + (\vec{p} - \vec{x})^2 \\ \vec{d}^2 - \vec{e}^2 &= (\vec{p} - \vec{x})^2 > 0\end{aligned}$$



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Prove that for a vector $\vec{x} \in S$ where $\vec{x} \neq \vec{p}$ and $\vec{d} = \vec{b} - \vec{x}$, that $\|\vec{d}\| > \|\vec{e}\|$.

$$\vec{d}^2 = \vec{e}^2 + (\vec{p} - \vec{x})^2$$

$$\vec{d}^2 - \vec{e}^2 = (\vec{p} - \vec{x})^2 > 0$$

$$\vec{d}^2 > \vec{e}^2$$

$$P = P^T$$

Prove that $P = P^T$.



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Hint

Start with $P = A(A^T A)^{-1} A^T$.



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Prove that $P = P^T$.

$$P = A(A^T A)^{-1} A^T$$

$$P^T = (A(A^T A)^{-1} A^T)^T$$

$$P^T = (A^T)^T ((A^T A)^{-1})^T A^T$$

$$P^T = A((A^T A)^T)^{-1} A^T$$

$$P^T = A(A^T (A^T)^T)^{-1} A^T$$

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$$P^T = P$$



$$P = P^2$$

Prove that $P = P^2$.



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$$P = A(A^T A)^{-1} A^T$$

$$P^2 = (A(A^T A)^{-1} A^T)^2$$

$$P^2 = (A(A^T A)^{-1} A^T)(A(A^T A)^{-1} A^T)$$

$$P^2 = A(A^T A)^{-1} (A^T A) (A^T A)^{-1} A^T$$

$$P^2 = A(A^T A)^{-1} I A^T$$

$$P^2 = A(A^T A)^{-1} A^T$$

$$P^2 = P$$

Sanity Check: $E = E^2$

Prove that $E = E^2$.

Reminder

E is also a projection matrix and $E = I - P$.

$$E = I - P$$

$$E^2 = (I - P)^2$$

$$E^2 = I - 2IP + P^2$$

$$E^2 = I - 2P + P$$

$$E^2 = I - P$$

- Applications
 - Graphics
 - Least Squares Regression
- Further Learning
 - Practice
 - *Introduction to Linear Algebra* 6th Edition Chapter 4.2 by Gilbert Strang