Orthogonal Projection

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Note

In this presentation, projection refers to specifically orthogonal projection, $\vec{p}$ refers to $\text{proj}_S \vec{b}$, and $\vec{e}$ refers to the error vector $\vec{e} = \vec{b} - \vec{p}$. The error vector is also called the rejection.

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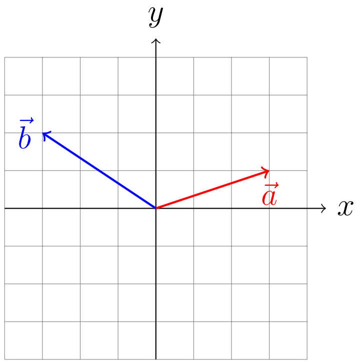
Note

The dimension of subspace S can be anything from 0 to the dimension of the space we are working inside.

Example



Example

Draw $\operatorname{proj}_a b$.

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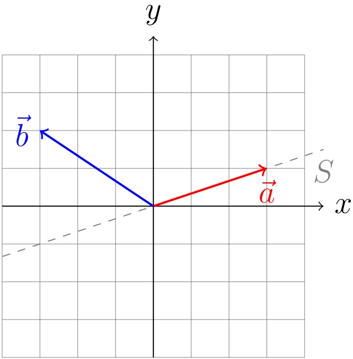

Note

When projecting a vector onto another vector $\vec{a}$, the subspace S is the span of $\vec{a}$.

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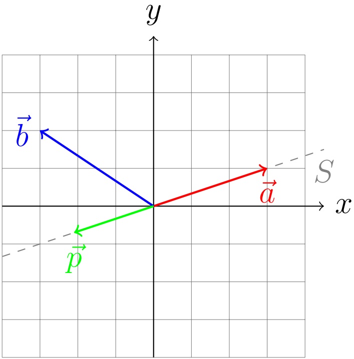

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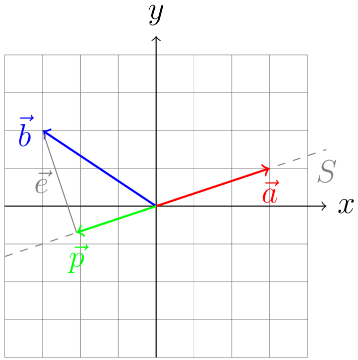

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Suppose that $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are a basis of S, let

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

and let $\vec{p} = A\vec{c}$. We also know:

$$\vec{e} = \vec{b} - \vec{p}$$

$$C(A)^{\perp} = N(A^T)$$

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Then,

$$\vec{e} = \vec{b} - A\vec{c}$$
$$A^T(\vec{b} - A\vec{c}) = \vec{0}$$
$$A^T\vec{b} - A^TA\vec{c} = \vec{0}$$
$$A^T\vec{b} = A^TA\vec{c}$$
$$(A^TA)^{-1}A^T\vec{b} = \vec{c}$$
$$\boxed{\vec{c} = (A^TA)^{-1}A^T\vec{b}}$$
$$\boxed{\vec{p} = A(A^TA)^{-1}A^T\vec{b}}$$

The Projection Matrix

Recall that for some matrix A, the projection $\vec{p} = A(A^T A)^{-1} A^T \vec{b}$.

Observation

For a specific A, $\vec{p}$ is an unchanging linear transformation of $\vec{b}$. So, we can represent the linear transformation with a matrix P.

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Recall that for some matrix A, the projection $\vec{p} = A(A^TA)^{-1}A^T\vec{b}$.

Let $\boxed{P = A(A^TA)^{-1}A^T}$. Then, $\vec{p} = P\vec{b}$.

Note

A must have linearly independent columns.

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The error vector is $\vec{b}$ projected onto $S^{\perp}$.

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$$\vec{e} = \vec{b} - \vec{p}$$
$$\vec{e} = I\vec{b} - P\vec{b}$$
$$\vec{e} = (I - P)\vec{b}$$
$$\boxed{E = I - P}$$

Projection Minimizes Error

Prove that for a vector $\vec{x} \in S$ where $\vec{x} \neq \vec{p}$ and $\vec{d} = \vec{b} - \vec{x}$, that $||\vec{d}|| > ||\vec{e}||$.

Hint

$||\vec{d}|| > ||\vec{e}|| \iff \vec{d}^2 > \vec{e}^2$

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$$\vec{d}^2 = (\vec{b} + \vec{0} - \vec{x})^2$$

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$$\vec{d}^2 = (\vec{b} - \vec{p} + \vec{p} - \vec{x})^2$$

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$$\vec{d}^2 = ({\color{red}\vec{e}} + \vec{p} - \vec{x})^2$$

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$$\vec{d}^2 = \vec{e}^2 + 2\vec{e} \cdot (\vec{p} - \vec{x}) + (\vec{p} - \vec{x})^2$$

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$$\vec{d}^2 - \vec{e}^2 = (\vec{p} - \vec{x})^2$$

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$$\vec{d}^2 = \vec{e}^2 + (\vec{p} - \vec{x})^2$$

$$\vec{d}^2 - \vec{e}^2 = (\vec{p} - \vec{x})^2 > 0$$

$$\vec{d}^2 > \vec{e}^2$$

Prove that $P = P^T$.

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Hint

Start with $P = A(A^TA)^{-1}A^T$.

Prove that $P = P^T$.

$$
\begin{aligned}
P &= A(A^TA)^{-1}A^T \\
P^T &= (A(A^TA)^{-1}A^T)^T \\
P^T &= (A^T)^T((A^TA)^{-1})^TA^T \\
P^T &= A((A^TA)^T)^{-1}A^T \\
P^T &= A(A^T(A^T)^T)^{-1}A^T \\
P^T &= A(A^TA)^{-1}A^T \\
P^T &= P
\end{aligned}
$$

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$$P = A(A^TA)^{-1}A^T$$
$$P^2 = (A(A^TA)^{-1}A^T)^2$$
$$P^2 = (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T)$$
$$P^2 = A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T$$
$$P^2 = A(A^TA)^{-1}IA^T$$
$$P^2 = A(A^TA)^{-1}A^T$$
$$P^2 = P$$

Prove that $E = E^2$.

Reminder

E is also a projection matrix and $E = I - P$.

$$\begin{aligned} E &= I - P \\ E^2 &= (I - P)^2 \\ E^2 &= I - 2IP + P^2 \\ E^2 &= I - 2P + P \\ E^2 &= I - P \end{aligned}$$

Sanity Check: A has One Column

Suppose $A = [\vec{a}]$. Then, $\vec{p} = A(A^T A)^{-1} A^T b$ and $\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$.

A is an Orthonormal Matrix

Note

An orthonormal matrix has unit columns all orthogonal to each other. In other words, an orthonormal matrix Q will satisfy $Q^T Q = I$.

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Note

An orthonormal matrix has unit columns all orthogonal to each other. In other words, an orthonormal matrix Q will satisfy $Q^TQ = I$.

Then, $P = A(A^TA)^{-1}A^T = AIA^T = AA^T$.
Additionally, the coefficients vector, $\vec{c} = (A^TA)^{-1}A^T\vec{b}$, will just be $\vec{c} = A^T\vec{b}$.

Projection from $\mathbb{R}^3$ onto the xz-plane

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Hint

What are a basis of the xz-plane?

Projection from $\mathbb{R}^3$ onto the xz-plane

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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Note

A is orthonormal.

Projection from $\mathbb{R}^3$ onto the xz-plane

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Projection from $\mathbb{R}^3$ onto the xz-plane

$$P = AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$$

More Special Cases (Practice)

- Find the projection matrix from $\mathbb{R}^3$ onto the y-axis.
- What would $\vec{p}$ be if $\vec{b} \perp S$? What if $\vec{b} \in S$?

Key Ideas

- The orthogonal projection of $\vec{b}$ onto subspace S, $\vec{p} = \text{proj}_S \vec{b} \in S$, makes the error $\vec{e} = \vec{b} - \vec{p}$ orthogonal to S.
- The projection also minimizes the error.
- $\boxed{P = A(A^T A)A^T}$
- $P = P^2$
- Other useful equations: $\vec{p} = Pb$, $E = I - P$, $\vec{c} = (A^T A)A^T b$, $P = P^T$, $c = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}$, $\vec{p} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$, $\vec{p} = c\vec{a}$, $\vec{p} = A\vec{c}$

Next Steps

- Applications
  - Graphics
  - Least Squares Regression
  - Orthonormal and Orthogonal Matrices
- Further Learning
  - Practice
  - *Introduction to Linear Algebra* 6th Edition Chapter 4.2 by Gilbert Strang