

# Orthogonal Projection

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## 1 Introduction

- Definitions
- Example

## 2 Formulas

- Projection onto a Vector
- Projection onto a Subspace
- The Projection Matrix
- The Error Matrix

## 3 Properties

- Projection Minimizes Error
- $P^T$
- $P^2$
- Sanity Check:  $E^2$

## 4 Special Cases

- Sanity Check:  $A$  Has One Column
- Projection from  $R^3$  onto y-axis
- Projection from  $R^3$  onto xz-plane
- $A$  is an Orthonormal Matrix
- $\vec{b} \perp S$
- $\vec{b} \in S$

## 5 Conclusion

- Formulas Recapped
- Properties Recapped
- Conclusion



# What is a projection?

$\text{proj}_S \vec{b}$ , the projection of vector  $\vec{b}$  onto subspace  $S$ , is the vector inside of  $S$  closest to  $\vec{b}$ .



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## Note

In this presentation, projection refers to specifically orthogonal projection,  $\vec{p}$  refers to  $\text{proj}_S \vec{b}$ , and  $\vec{e}$  refers to the error vector  $\vec{e} = \vec{b} - \vec{p}$ . The error vector is also called the rejection.



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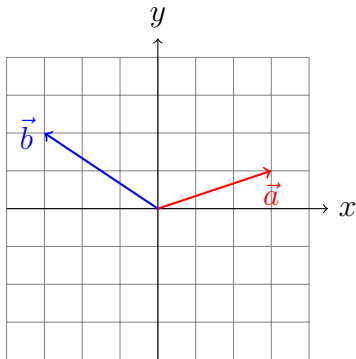
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## Note

The dimension of subspace  $S$  can be anything from 0 to the dimension of the space we are working inside.



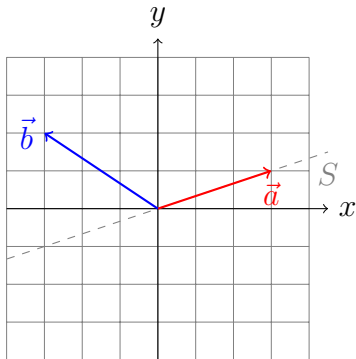
# Example



## Example

Draw  $\text{proj}_a b$ .

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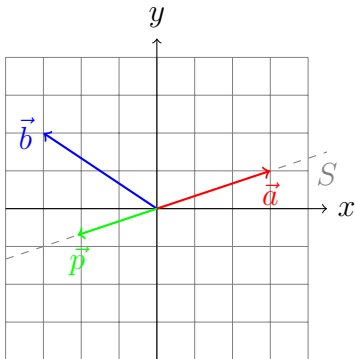


## Note

When projecting a vector onto another vector  $\vec{a}$ , the subspace  $S$  is the span of  $\vec{a}$ .

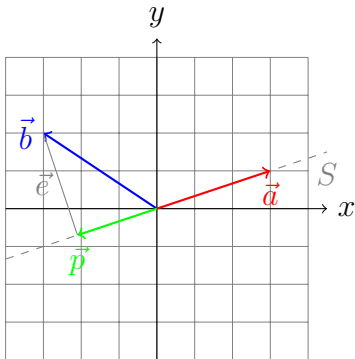


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Suppose that  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$  are a basis of  $S$ , let

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

and let  $\vec{p} = A\vec{c}$ . We also know:

$$\vec{e} = \vec{b} - \vec{p}$$

$$C(A)^\perp = N(A^T)$$

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$$\begin{aligned}\vec{e} &= \vec{b} - \vec{p} \\ A^T \vec{e} &= \vec{0}\end{aligned}$$

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$$A^T \vec{e} = \vec{0}$$

Then,

$$\vec{e} = \vec{b} - A\vec{c}$$

$$A^T(\vec{b} - A\vec{c}) = \vec{0}$$

$$A^T \vec{b} - A^T A \vec{c} = \vec{0}$$

$$A^T \vec{b} = A^T A \vec{c}$$

$$(A^T A)^{-1} A^T \vec{b} = \vec{c}$$

$$\boxed{\vec{c} = (A^T A)^{-1} A^T \vec{b}}$$

$$\boxed{\vec{p} = A(A^T A)^{-1} A^T \vec{b}}$$



# The Projection Matrix

Recall that for some matrix  $A$ , the projection  
 $\vec{p} = A(A^T A)^{-1} A^T \vec{b}$ .

## Observation

For a specific  $A$ ,  $\vec{p}$  is an unchanging linear transformation of  $\vec{b}$ . So, we can represent the linear transformation with a matrix  $P$ .

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Let  $\boxed{P = A(A^T A)^{-1} A^T}$ . Then,  $\vec{p} = P\vec{b}$ .

## Note

$A$  must have linearly independent columns.

# The Error Matrix

The error vector is  $\vec{b}$  projected onto  $S^\perp$ .



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$$\vec{e} = \vec{b} - \vec{p}$$

$$\vec{e} = I\vec{b} - P\vec{b}$$

$$\vec{e} = (I - P)\vec{b}$$

$$\boxed{E = I - P}$$

- Applications
  - Graphics
  - Least Squares Regression
- Further Learning
  - Practice
  - *Introduction to Linear Algebra* 6th Edition Chapter 4.2 by Gilbert Strang