Orthogonal Projection

Alvin Kim

July 17, 2024

What is a projection?

$\text{proj}_S \vec{b}$, the projection of vector $\vec{b}$ onto subspace S, is the vector inside of S closest to $\vec{b}$.

What is a projection?

$\text{proj}_S \vec{b}$, the projection of vector $\vec{b}$ onto subspace S, is the vector inside of S closest to $\vec{b}$.

Note

In this presentation, projection refers to specifically orthogonal projection, $\vec{p}$ refers to $\text{proj}_S \vec{b}$, and $\vec{e}$ refers to the error vector $\vec{e} = \vec{b} - \vec{p}$.

What is a projection?

$\operatorname{proj}_S \vec{b}$, the projection of vector $\vec{b}$ onto subspace S, is the vector inside of S closest to $\vec{b}$.
Closest means that the error is orthogonal to S.

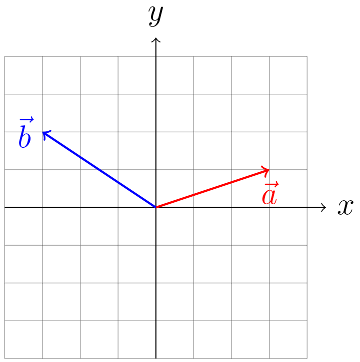
What is a projection?

$\text{proj}_S \vec{b}$, the projection of vector $\vec{b}$ onto subspace S, is the vector inside of S closest to $\vec{b}$.
Closest means that the error is orthogonal to S.

Note

The dimension of subspace S can be anything from 0 to the dimension of the space we are working inside.

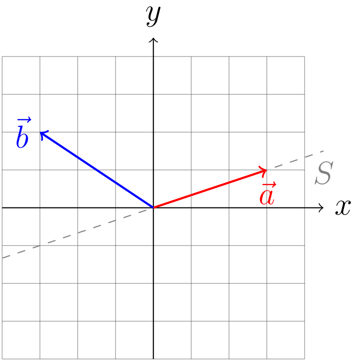
Example



Example

Draw $\operatorname{proj}_a b$.

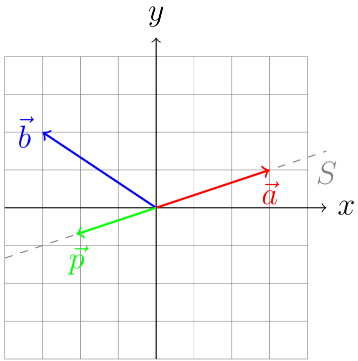
Example



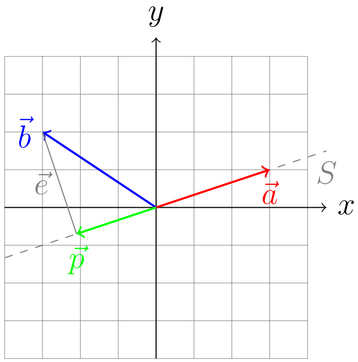
Note

When projecting a vector onto another vector $\vec{a}$, the subspace S is the span of $\vec{a}$.

Example



Example



Projection onto a Vector

Consider projecting $\vec{b}$ onto $\vec{a}$.

Projection onto a Vector

Consider projecting $\vec{b}$ onto $\vec{a}$.
What we know:

$$\begin{aligned} \vec{e} &= \vec{b} - \vec{p} \\ \vec{p} &= c\vec{a} \\ \vec{e} \cdot \vec{a} &= 0 \end{aligned}$$

Projection onto a Vector

Consider projecting $\vec{b}$ onto $\vec{a}$.
What we know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$\vec{p} = c\vec{a}$$
$$\vec{e} \cdot \vec{a} = 0$$

From that:

$$\vec{e} = \vec{b} - c\vec{a}$$
$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} = c\vec{a} \cdot \vec{a}$$
$$c = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$$
$$\boxed{\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}}$$

Projection onto a Vector

Consider projecting $\vec{b}$ onto $\vec{a}$.
What we know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$\vec{p} = c\vec{a}$$
$$\vec{e} \cdot \vec{a} = 0$$

From that:

$$\vec{e} = \vec{b} - c\vec{a}$$
$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} = c\vec{a} \cdot \vec{a}$$
$$c = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$$
$$\boxed{\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}}$$

Projection onto a Vector

Consider projecting $\vec{b}$ onto $\vec{a}$.
What we know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$\vec{p} = c\vec{a}$$
$$\vec{e} \cdot \vec{a} = 0$$

From that:

$$\vec{e} = \vec{b} - c\vec{a}$$
$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} = c\vec{a} \cdot \vec{a}$$
$$c = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$$
$$\boxed{\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}}$$

Projection onto a Vector

Consider projecting $\vec{b}$ onto $\vec{a}$.
What we know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$\vec{p} = c\vec{a}$$
$$\vec{e} \cdot \vec{a} = 0$$

From that:

$$\vec{e} = \vec{b} - c\vec{a}$$
$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} = c\vec{a} \cdot \vec{a}$$
$$c = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$$
$$\boxed{\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}}$$

Projection onto a Vector

Consider projecting $\vec{b}$ onto $\vec{a}$.
What we know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$\vec{p} = c\vec{a}$$
$$\vec{e} \cdot \vec{a} = 0$$

From that:

$$\vec{e} = \vec{b} - c\vec{a}$$
$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} = c\vec{a} \cdot \vec{a}$$
$$c = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$$
$$\boxed{\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}}$$

Projection onto a Vector

Consider projecting $\vec{b}$ onto $\vec{a}$.
What we know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$\vec{p} = c\vec{a}$$
$$\vec{e} \cdot \vec{a} = 0$$

From that:

$$\vec{e} = \vec{b} - c\vec{a}$$
$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$
$$\vec{b} \cdot \vec{a} = c\vec{a} \cdot \vec{a}$$
$$c = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$$
$$\boxed{\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}}$$

Project $\vec{b}$ onto S

Project $\vec{b}$ onto S

Suppose that $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are a basis of S, let

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

and let $\vec{p} = A\vec{c}$. We also know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$C(A)^{\perp} = N(A^T)$$

Project $\vec{b}$ onto S

Suppose that $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are a basis of S, let

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

and let $\vec{p} = A\vec{c}$. We also know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$A^T\vec{e} = \vec{0}$$

Project $\vec{b}$ onto S

Suppose that $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are a basis of S, let

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

and let $\vec{p} = A\vec{c}$. We also know:

$$\vec{e} = \vec{b} - \vec{p}$$
$$A^T\vec{e} = \vec{0}$$

Then,

$$\vec{e} = \vec{b} - A\vec{c}$$
$$A^T(\vec{b} - A\vec{c}) = \vec{0}$$
$$A^T\vec{b} - A^TA\vec{c} = \vec{0}$$
$$A^T\vec{b} = A^TA\vec{c}$$
$$(A^TA)^{-1}A^T\vec{b} = \vec{c}$$
$$\boxed{\vec{c} = (A^TA)^{-1}A^T\vec{b}}$$
$$\boxed{\vec{p} = A(A^TA)^{-1}A^T\vec{b}}$$

The Projection Matrix

Recall that for some matrix A, the projection $\vec{p} = A(A^TA)^{-1}A^T\vec{b}$.

Observation

For a specific A, $\vec{p}$ is an unchanging linear transformation of $\vec{b}$. So, we can represent the linear transformation with a matrix P.

The Projection Matrix

Recall that for some matrix A, the projection $\vec{p} = A(A^TA)^{-1}A^T\vec{b}$.

The Projection Matrix

Recall that for some matrix A, the projection $\vec{p} = A(A^TA)^{-1}A^T\vec{b}$.

Let $\boxed{P = A(A^TA)^{-1}A^T}$. Then, $\vec{p} = P\vec{b}$.

Note

A must have linearly independent columns.

- Applications
  - Graphics
  - Least Squares Regression
- Further Learning
  - Practice
  - *Introduction to Linear Algebra* 6th Edition Chapter 4.2 by Gilbert Strang