Orthogonal Projection

Alvin Kim

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What is a projection?

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Note

In this presentation, projection refers to specifically orthogonal projection, $\vec{p}$ refers to $\text{proj}_S \vec{b}$, and $\vec{e}$ refers to the error vector $\vec{e} = \vec{b} - \vec{p}$. The error vector is also called the rejection.

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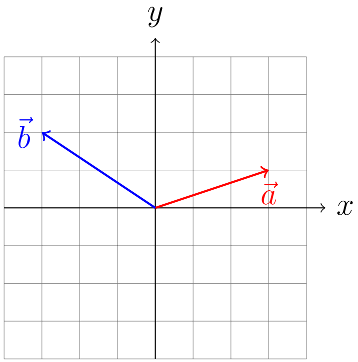

Example

Draw $\text{proj}_a\, b$.

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└Introduction
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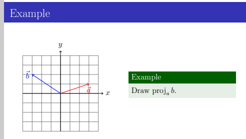

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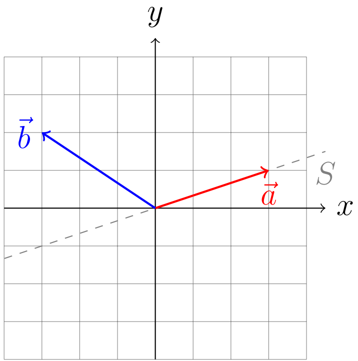

Note

When projecting a vector onto another vector $\vec{a}$, the subspace S is the span of $\vec{a}$.

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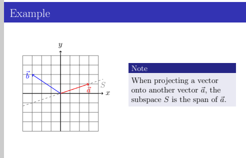

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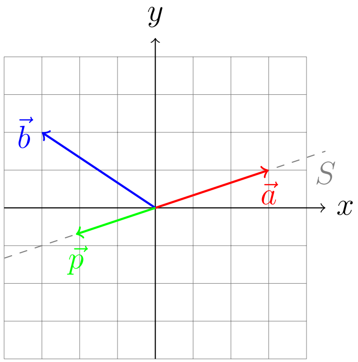

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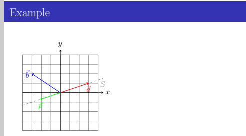

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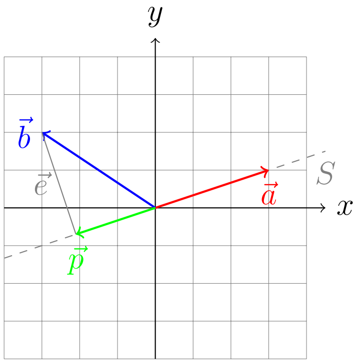

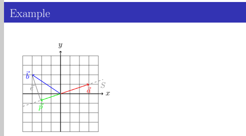

Projection onto a Vector

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The Projection Matrix

Recall that for some matrix A, the projection
$\vec{p} = A(A^TA)^{-1}A^T\vec{b}$.

Observation

For a specific A, $\vec{p}$ is an unchanging linear transformation of $\vec{b}$. So, we can represent the linear transformation with a matrix P.

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To multiply some vector by a matrix is to apply a linear transformation to the vector.

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Let $\boxed{P = A(A^TA)^{-1}A^T}$. Then, $\vec{p} = P\vec{b}$.

Note

A must have linearly independent columns.

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$$\begin{aligned} \vec{e} &= \vec{b} - \vec{p} \\ \vec{e} &= I\vec{b} - P\vec{b} \\ \vec{e} &= (I - P)\vec{b} \\ &\boxed{E = I - P} \end{aligned}$$

The Error Matrix
The error vector is $\vec{b}$ projected onto $S^{\perp}$. Since finding the error vector is also a projection, there should be an error matrix E for some P such that $\vec{e} = E\vec{b}$.
$\vec{e} = \vec{b} - \vec{p}$
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Projection Minimizes Error

Prove that for a vector $\vec{x} \in S$ where $\vec{x} \neq \vec{p}$ and $\vec{d} = \vec{b} - \vec{x}$, that $||\vec{d}|| > ||\vec{e}||$.

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$||\vec{d}|| > ||\vec{e}|| \iff \vec{d}^2 > \vec{e}^2$

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Prove that $P = P^T$.

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Start with $P = A(A^TA)^{-1}A^T$.

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$$
\begin{aligned}
P &= A(A^TA)^{-1}A^T \\
P^T &= (A(A^TA)^{-1}A^T)^T \\
P^T &= (A^T)^T((A^TA)^{-1})^TA^T \\
P^T &= A((A^TA)^T)^{-1}A^T \\
P^T &= A(A^T(A^T)^T)^{-1}A^T \\
P^T &= A(A^TA)^{-1}A^T \\
P^T &= P
\end{aligned}
$$

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\begin{aligned}
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P^2 &= (A(A^TA)^{-1}A^T)^2 \\
P^2 &= (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) \\
P^2 &= A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T \\
P^2 &= A(A^TA)^{-1}IA^T \\
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Prove that $E = E^2$.

Reminder

E is also a projection matrix and $E = I - P$.

$$
\begin{aligned}
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E^2 &= (I - P)^2 \\
E^2 &= I - 2IP + P^2 \\
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Sanity Check: A has One Column

Suppose $A = [\vec{a}]$. Then, $\vec{p} = A(A^TA)^{-1}A^Tb$ and $\vec{p} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$.

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This is an informal sanity checkmention inner product = dot productIt is easy to see that A^TA in the former equation corresponds to $\vec{a}\cdot\vec{a}$ and since it is inverted, it is the divisor in the latter equation. A^Tb corresponds to $\vec{a}\cdot\vec{b}$. the orders do seem off, but since many of the terms in the latter equation are scalars, the order is flexible.

A is an Orthonormal Matrix

Note

An orthonormal matrix has unit columns all orthogonal to each other. In other words, an orthonormal matrix Q will satisfy $Q^TQ = I$.

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Then, $P = A(A^TA)^{-1}A^T = AIA^T = AA^T$.

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Orthogonal Projection
Special Cases
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Projection from $\mathbb{R}^3$ onto the xz-plane

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Hint

What are a basis of the xz-plane?

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$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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Example

$$P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$$

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More Special Cases (Practice)

- Find the projection matrix from $\mathbb{R}^3$ onto the y-axis.
- What would $\vec{p}$ be if $\vec{b} \perp S$? What if $\vec{b} \in S$?

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emphasize that these are just the special, interesting cases, and that these problems don't represent most problems.

Key Ideas

- The orthogonal projection of $\vec{b}$ onto subspace S, $\vec{p} = \text{proj}_S \vec{b} \in S$, makes the error $\vec{e} = \vec{b} - \vec{p}$ orthogonal to S.
- The projection also minimizes the error.
- $\boxed{P = A(A^T A)A^T}$
- $P = P^2$
- Other useful equations: $\vec{p} = Pb$, $E = I - P$, $\vec{c} = (A^T A)A^T b$, $P = P^T$, $c = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$, $\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$, $\vec{p} = c\vec{a}$, $\vec{p} = A\vec{c}$

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A must have LI cols

Next Steps

- Applications
  - Graphics
  - Least Squares Regression
  - Orthonormal and Orthogonal Matrices
- Further Learning
  - Practice
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