Orthogonal Projection

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- 1 Introduction
 - Definitions
 - Example
- 2 Formulas
 - Projection onto a Vector
 - Projection onto a Subspace
 - The Projection Matrix
- 3 Properties
 - Projection Minimizes Error
 - $\blacksquare P^T$
 - $\blacksquare P^2$

4 Special Cases

- Sanity Check: A Has One Column
- Projection from R³ onto y-axis
- Projection from R^3 onto xz-plane
- A is an Orthonormal Matrix
- $\vec{b} \perp S$
- $b \in S$
- 5 Conclusion
 - Formulas Recapped
 - Properties Recapped
 - Conclusion



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 $\operatorname{proj}_{S} \vec{b}$, the projection of vector \vec{b} onto subspace S, is the vector inside of S closest to \vec{b} .



Introduction

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Note

In this presentation, projection refers to specifically orthogonal projection, \vec{p} refers to proj_s \vec{b} , and \vec{e} refers to the error vector $\vec{e} = \vec{b} - \vec{p}$.



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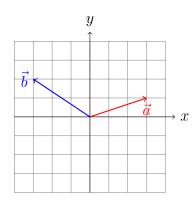
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Note

The dimension of subspace S can be anything from 0 to the dimension of the space we are working inside.



Example



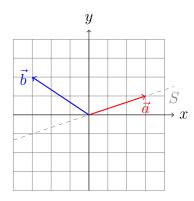


Draw $\operatorname{proj}_a b$.



Example

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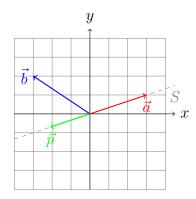


Note

When projecting a vector onto another vector \vec{a} , the subspace S is the span of \vec{a} .



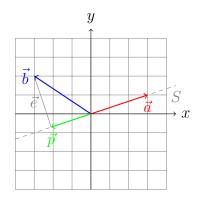
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Project \vec{b} onto S



Project \vec{b} onto S

Suppose that $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are a basis of S, let

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

and let $\vec{p} = A\vec{c}$. We also know:

$$\vec{e} = \vec{b} - \vec{p}$$

$$C(A)^{\perp} = N(A^T)$$



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Then.

$$\vec{e} = \vec{b} - A\vec{c}$$

$$A^{T}(\vec{b} - A\vec{c}) = \vec{0}$$

$$A^{T}\vec{b} - A^{T}A\vec{c} = \vec{0}$$

$$A^{T}\vec{b} = A^{T}A\vec{c}$$

$$(A^{T}A)^{-1}A^{T}\vec{b} = \vec{c}$$

$$\vec{c} = (A^T A)^{-1} A^T \vec{b}$$

$$\vec{\vec{p}} = A(A^T A)^{-1} A^T \vec{b}$$



The Projection Matrix

Recall that for some matrix A, the projection $\vec{p} = A(A^T A)^{-1} A^T \vec{b}$.

Observation

For a specific A, \vec{p} is an unchanging linear transformation of \vec{b} . So, we can represent the linear transformation with a matrix P.



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The Projection Matrix

Recall that for some matrix A, the projection $\vec{p} = A(A^T A)^{-1} A^T \vec{b}$.

Let
$$P = A(A^T A)^{-1} A^T$$
. Then, $\vec{p} = P\vec{b}$.

Note

A must have linearly independent columns.



- Applications
 - Graphics
 - Least Squares Regression
- Further Learning
 - Practice
 - Introduction to Linear Algebra 6th Edition Chapter 4.2 by Gilbert Strang

