Hw # 1 Solutions

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Thurs. Jan. 29, 2015

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Section 1.2 # 1

Determine A_0, A_1, and A_2 such that the function $y(x) = A_0 + A_1x + A_2x^2$ and the function $f(x) = \dfrac{1}{1+x}$ have each of the following sets of properties in common:

(a) $f(0),\ f(\frac{1}{2}),\ f(1)$

$$\begin{aligned}
f(0) &= 1 = y(0) = A_0 \\
f(\tfrac{1}{2}) &= \tfrac{2}{3} = y(\tfrac{1}{2}) = A_0 + \tfrac{1}{2}A_1 + \tfrac{1}{4}A_2 \\
f(1) &= \tfrac{1}{2} = y(1) = A_0 + A_1 + A_2 \\
A_0 = 1 &\implies -\tfrac{2}{3} - \tfrac{1}{2}A_2 = A_1 = -\tfrac{1}{2} - A_2 \implies A_2 = \tfrac{1}{3} \implies A_1 = -\tfrac{5}{6}
\end{aligned}$$

Therefore, $\boxed{y(x) = 1 - \dfrac{5}{6}x + \dfrac{1}{3}x^2}$

(b) $f(0),\ f'(0),\ f''(0)$

$$\begin{aligned}
f(0) &= 1 = y(0) = A_0 \\
f'(0) &= -1 = y'(0) = A_1 \\
f''(0) &= 2 = y''(0) = 2A_2 \implies A_2 = 1
\end{aligned}$$

Therefore, $\boxed{y(x) = 1 - x + x^2}$

(c) $f(\frac{1}{2}),\ f'(\frac{1}{2}),\ f''(\frac{1}{2})$

$$\begin{aligned}
f(\tfrac{1}{2}) &= \tfrac{2}{3} = y(\tfrac{1}{2}) = A_0 + \tfrac{1}{2}A_1 + \tfrac{1}{4}A_2 \\
f'(\tfrac{1}{2}) &= -\tfrac{4}{9} = y'(\tfrac{1}{2}) = A_1 + A_2 \\
f''(\tfrac{1}{2}) &= \tfrac{16}{27} = y''(\frac{1}{2}) = 2A_2 \implies A_2 = \tfrac{8}{27} \implies A_1 = -\tfrac{20}{27} \implies A_0 = \tfrac{26}{27}
\end{aligned}$$

Therefore, $\boxed{y(x) = \dfrac{26}{27} - \dfrac{20}{27}x + \dfrac{8}{27}x^2}$

Section 1.2 # 4

Determine that member $y(x)$ of the set of all linear functions (i.e. $y(x) = a + bx$) which best approximates the function $f(x) = x^2$ over $[0, 1]$ in the sense that each of the following quantities is minimized:

(a) $\int_0^1 [f(x) - y(x)]^2 \, dx$

$$
\begin{aligned}
\hat{f}(a, b) &= \int_0^1 [f(x) - y(x)]^2 \, dx \\
&= \int_0^1 [x^2 - bx - a]^2 \, dx \\
&= \int_0^1 x^4 - 2bx^3 + (b^2 - 2a)x^2 + 2abx + a^2 \, dx \\
&= \left(\tfrac{1}{5}x^5 - \tfrac{b}{2}x^4 + \tfrac{b^2-2a}{3}x^3 + abx^2 + a^2x \right)_0^1 \\
&= \tfrac{1}{5} - \tfrac{b}{2} + \tfrac{b^2-2a}{3} + ab + a^2
\end{aligned}
$$

To find the critical points of $\hat{f}$, we first find where $\hat{f}_a$ and $\hat{f}_b$ equal zero.

$$
\begin{aligned}
\hat{f}_a &= -\tfrac{2}{3} + b + 2a = 0 \implies b = \tfrac{2}{3} - 2a \\
\hat{f}_b &= -\tfrac{1}{2} + \tfrac{2}{3}b + a \implies a = \tfrac{1}{2} - \tfrac{2}{3}b \\
\implies a &= -\tfrac{1}{6} \implies b = 1
\end{aligned}
$$

$(a, b) = (-\frac{1}{6}, 1)$ is a minimum of $\hat{f}$ if:

(I) $\hat{f}_{aa}(-\frac{1}{6}, 1)\hat{f}_{bb}(-\frac{1}{6}, 1) - \left[\hat{f}_{ab}(-\frac{1}{6}, 1)\right]^2 > 0$, and

(II) $\hat{f}_{aa}(-\frac{1}{6}, 1) > 0$

Since $\hat{f}_{aa} = 2$, $\hat{f}_{bb} = \frac{2}{3}$, and $\hat{f}_{ab} = 1$,

$$
\begin{aligned}
\hat{f}_{aa}(-\frac{1}{6}, 1)\hat{f}_{bb}(-\frac{1}{6}, 1) - \left[\hat{f}_{ab}(-\frac{1}{6}, 1)\right]^2 &= (2)(\tfrac{2}{3}) - 1^2 = \tfrac{1}{3} > 0, \text{ and} \\
\hat{f}_{aa}(-\tfrac{1}{6}, 1) &= 2 > 0
\end{aligned}
$$

Thus $(a, b) = (-\frac{1}{6}, 1)$ is a minimum of $\hat{f}$, and $\boxed{y(x) = -\dfrac{1}{6} + x}$

(b) $[f(0) - y(0)]^2 + [f(\frac{1}{2}) - y(\frac{1}{2})]^2 + [f(1) - y(1)]^2$

$$
\begin{aligned}
\hat{f}(a, b) &= [f(0) - y(0)]^2 + [f(\frac{1}{2}) - y(\frac{1}{2})]^2 + [f(1) - y(1)]^2 \\
&= (0 - a)^2 + (\tfrac{1}{4} - a - \tfrac{1}{2}b)^2 + (1 - a - b)^2
\end{aligned}
$$

$$= 3a^2 + \tfrac{5}{4}b^2 + 3ab - \tfrac{5}{2}a - \tfrac{9}{4}b + \tfrac{17}{16}$$

To find the critical points of $\hat{f}$, we first find where $\hat{f}_a$ and $\hat{f}_b$ equal zero.

$$\begin{aligned} \hat{f}_a &= 6a + 3b - \tfrac{5}{2} = 0, \text{ and} \\ \hat{f}_b &= \tfrac{5}{2}b + 3a - \tfrac{9}{4} = 0 \\ \implies a &= -\tfrac{1}{12} \implies b = 1 \end{aligned}$$

As above, $(a, b) = (-\frac{1}{12}, 1)$ is a minimum of $\hat{f}$ if:

(I) $\quad \hat{f}_{aa}(-\frac{1}{12}, 1)\hat{f}_{bb}(-\frac{1}{12}, 1) - \left[\hat{f}_{ab}(-\frac{1}{12}, 1)\right]^2 > 0$, and

(II) $\quad \hat{f}_{aa}(-\frac{1}{12}, 1) > 0$

Since $\hat{f}_{aa} = 6$, $\hat{f}_{bb} = \frac{5}{2}$, and $\hat{f}_{ab} = 3$,

$$\begin{aligned} \hat{f}_{aa}(-\frac{1}{12}, 1)\hat{f}_{bb}(-\frac{1}{12}, 1) - \left[\hat{f}_{ab}(-\frac{1}{12}, 1)\right]^2 &= (6)(\tfrac{5}{2}) - 3^2 = 6 > 0, \text{ and} \\ \hat{f}_{aa}(-\frac{1}{12}, 1) &= 6 > 0 \end{aligned}$$

Thus $(a, b) = (-\frac{1}{12}, 1)$ is a minimum of $\hat{f}$, and $\boxed{y(x) = -\dfrac{1}{12} + x}$

(c) $\quad \displaystyle\max_{x\in[0,1]} |f(x) - y(x)|$

$$\begin{aligned} \hat{f}(a, b) &= \max_{x\in[0,1]} |f(x) - y(x)| \\ &= \max_{x\in[0,1]} |x^2 - bx - a| \end{aligned}$$

Since $g(x) = x^2 - bx - a$ is an upward facing parabola with a leading coefficient of 1, the endpoints of the parabola are potentially maximums on $[0, 1]$. For $|g(x)|$, the vertex $(v, g(v))$ becomes a potential maximum if $g(v) < 0$ because it is reflected over the x-axis, making it at least a local maximum. To minimize $\displaystyle\max_{x\in[0,1]} |g(x)|$, $g(x)$ must have a vertex at $(\frac{1}{2}, -\frac{1}{8})$. This forces the max of $|g(x)|$ to be $\frac{1}{8}$ at $x \in \{0, \frac{1}{2}, 1\}$. Notice the maximum is at both endpoints and at the reflected vertex. We find b by noting that the x-coordinate of the vertex of a parabola of the form $\alpha x^2 + \beta x + \gamma$ is $-\frac{\beta}{2\alpha}$.

$$\begin{aligned} \tfrac{b}{2} = \tfrac{1}{2} &\implies b = 1 \\ g(0) = -a = \tfrac{1}{8} &\implies a = -\tfrac{1}{8} \end{aligned}$$

Thus $\boxed{y(x) = -\dfrac{1}{8} + x}$

Section 1.2 # 5

Determine c_1, c_2, and c_3 in such a way that the formula

$$\int_{-1}^{1} w(x)f(x)dx = c_1 f(-1) + c_2 f(0) + c_3 f(1)$$

yields an exact result when $f(x)$ is 1, x, x^2, and x^3, and hence also when $f(x)$ is any linear combination of those functions, for each of the following weighting functions:

(a) $w(x) = 1$

For $f(x) = 1$,

$$\int_{-1}^{1} dx = c_1 + c_2 + c_3 = 2$$

For $f(x) = x$,

$$\int_{-1}^{1} x\ dx = -c_1 + c_3 = 0 \implies c_1 = c_3$$

For $f(x) = x^2$,

$$\int_{-1}^{1} x^2\ dx = c_1 + c_3 = \frac{2}{3} \implies c_1 = \frac{1}{3} \implies c_3 = \frac{1}{3}$$

For $f(x) = x^3$,

$$\int_{-1}^{1} x^3\ dx = -c_1 + c_3 = 0 \implies c_1 = c_3$$

Thus $c_2 = \frac{4}{3}$. Thus, if $f \in \{f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$, then

$$\boxed{\int_{-1}^{1} f(x)\ dx = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)}$$

Section 1.3 # 8

Suppose that the alternating series

$$S = v_0 - v_1 + v_2 - v_3 + \cdots = \sum_{k=0}^{\infty} (-1)^k v_k$$

converges. Show that the series

$$\tfrac{1}{2}v_0 + \tfrac{1}{2}(v_0 - v_1) - \tfrac{1}{2}(v_1 - v_2) + \cdots = \tfrac{1}{2}v_0 + \tfrac{1}{2}\sum_{k=0}^{\infty} (-1)^k (v_k - v_{k+1})$$

converges to the same sum.

Since $\sum (a_i + b_i) = \sum a_i + \sum b_i$,

$$\begin{aligned}
\tfrac{1}{2}v_0 + \tfrac{1}{2}\sum_{k=0}^{\infty} (-1)^k (v_k - v_{k+1}) &= \tfrac{1}{2}v_0 + \tfrac{1}{2}\sum_{k=0}^{\infty} (-1)^k (v_k) - \tfrac{1}{2}\sum_{k=0}^{\infty} (-1)^k (v_{k+1}) \\
&= \tfrac{1}{2}\sum_{k=0}^{\infty} (-1)^k (v_k) + \tfrac{1}{2}v_0 + \tfrac{1}{2}\sum_{k=0}^{\infty} (-1)^{k+1} v_{k+1} \\
&= \tfrac{1}{2}S + \tfrac{1}{2}v_0 + \tfrac{1}{2}\sum_{k=1}^{\infty} (-1)^k v_k \\
&= \tfrac{1}{2}S + \tfrac{1}{2}\sum_{k=0}^{\infty} (-1)^k v_k \\
&= \tfrac{1}{2}S + \tfrac{1}{2}S \\
&= S
\end{aligned}$$

Thus $\boxed{\displaystyle\sum_{k=0}^{\infty} (-1)^k v_k = S = \frac{1}{2}v_0 + \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k (v_k - v_{k+1})}$

Section 1.3 # 9

Use the transformation of problem 8 to show that

$$\begin{aligned}
S &\equiv 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \\
&= \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+2)} \\
&= \frac{5}{8} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+2)(k+3)} \\
&= \frac{2}{3} + \frac{3}{4}\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+2)(k+3)(k+4)} = \dots
\end{aligned}$$

Show that the retention of five terms in the last sum given ensures that $0.69306 < S < 0.69330$ *or that* $S \approx 0.69318$ *with a maximum error of* ± 12 *units in the place of the fifth digit. About how many terms of the original series would be needed to ensure this accuracy? (The true value is* $S = \log 2 \doteq 0.69315$*)*

To use the result from problem 8, note the definition of S implies v_k is initially $\frac{1}{k+1}$. Thus

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1} = S &= \frac{1}{2}(1) + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\
&= \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{1}{(k+1)(k+2)}
\end{aligned}$$

Since we know this converges, we can use the result of problem 8 again but with v_k equal to $\frac{1}{(k+1)(k+2)}$, and then again with v_k equal to $\frac{1}{(k+1)(k+2)(k+3)}$. Thus

$$\begin{aligned}
S &= \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{1}{(k+1)(k+2)} \\
&= \frac{1}{2} + \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{(k+1)(k+2)} - \frac{1}{(k+2)(k+3)}\right)\right] \\
&= \frac{5}{8} + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{1}{(k+1)(k+2)(k+3)} \\
&= \frac{5}{8} + \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{6}\right) + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{(k+1)(k+2)(k+3)} - \frac{1}{(k+2)(k+3)(k+4)}\right)\right] \\
&= \frac{2}{3} + \frac{3}{4}\sum_{k=0}^{\infty}(-1)^k \frac{1}{(k+1)(k+2)(k+3)(k+4)} = \dots
\end{aligned}$$

Let S_n denote the sum of the first $n+1$ terms (Since we start counting from $n=0$). The fifth ($n=4$) and sixth ($n=5$) partial sums are

$$S_4 = \frac{2}{3} + \frac{3}{4}\left(\frac{1}{24} - \frac{1}{120} + \frac{1}{360} - \frac{1}{840} + \frac{1}{1680}\right) \approx 0.693304, \qquad \text{and}$$
$$S_5 = S_4 + \frac{3}{4}\left(-\frac{1}{3024}\right) \approx 0.693056$$

S is an alternating convergent series and $S_5 < S_4$ imply $0.693056 < S < 0.693304$. Also, alternating convergent series have the property that if $S = S_n + E_n$ where S_n is the n^{th} partial sum and E_n is the remaining error, then $|E_n| < |a_{n+1}|$ where a_{n+1} is the first 'neglected' term not included in the n^{th} partial sum (i.e. the $(n+1)^{\text{st}}$ term in the sequence). Thus

$$|E_4| < \left|\frac{3}{4}\left(-\frac{1}{3024}\right)\right| \approx 0.000248, \qquad \text{and}$$
$$|E_5| < \left|\frac{3}{4}\left(\frac{1}{5040}\right)\right| \approx 0.0001488$$

However, we can further bound the error by nothing that alternating convergent series have the following property: $|S - S_n| < \frac{1}{2}|S_n - S_{n-1}|$. Thus

$$E_5 = |S - S_5| < \frac{1}{2}|S_5 - S_4| = \frac{1}{2}\left|\frac{3}{4}\left(-\frac{1}{3024}\right)\right| \approx 0.000124$$

Thus S_5 is accurate to 12 digits in the place of the fifth digit. Furthermore, since the original series is the alternating harmonic series, we take the reciprocal of the upper bound of the error, $|E_5|$, to get an estimate of how many terms are needed in the original series to achieve the same accuracy. Thus $\frac{1}{0.000124} \approx 8065$ terms are needed in the original series.

Sections 1.4 and 1.5 # 18

Show that the number $(2.46)^{\frac{1}{64}}$ *is known within less than one unit in the place of its* fifth *significant digit if* 2.46 *is known only to be correctly rounded to three digits.*

If $\overline{N} = 2.46$ is known to three digits, then $E(\overline{N}) \leq 5 \times 10^{r-n} = 5 \times 10^{0-3} = 5 \times 10^{-3} = 0.005$ where r is given by $N = N^* \times 10^r$ with $1 \leq N^* < 10$ and n is the number of significant figures for which $\overline{N}$ is accurate. Let $f(x) = x^{\frac{1}{64}} = \sqrt[64]{x}$. Then for $x > 0$, f is differentiable, and $f'(x) = \dfrac{1}{64x^{\frac{63}{64}}}$. Since $f'(x)$ is a decreasing and positive function on $(0, \infty)$, then for $\xi \in [\overline{N} - E(\overline{N}), \overline{N} + E(\overline{N})]$, $|f'(\xi)|$ is maximized at $\overline{N} - E(\overline{N})$. Thus,

$$\begin{aligned}
|E(f(\overline{N}))| &\leq |f'(\xi)|_{\max} \cdot |E(\overline{N})| \\
&\leq |f'(\overline{N} - E(\overline{N}))| \cdot |E(\overline{N})| \\
&= |f'(2.46 - 5 \times 10^{-3})| \cdot 5 \times 10^{-3} \\
&\approx 3.23 \times 10^{-5} = 0.0000323
\end{aligned}$$

Thus $(2.46)^{\frac{1}{64}} \approx 1.01416$ is known within less than four units in its *sixth* significant digit, and certainly within less than one unit in its *fifth* significant digit.

Section 1.6 # 24

Suppose that calculations are to be made in four-digit floating point arithmetic, assuming a double-precision accumulator, but supposing that the computer rounds the number resulting from each operation (addition, multiplication, etc.) to four digits before effecting a subsequent operation on that number. If

$$x_1 = 0.1234 \times 10^3 \qquad x_2 = 0.3456 \times 10^2 \qquad x_3 = 0.5678 \times 10^1$$

are exact numbers, evaluate the results of each of the following machine operations and, in each case, determine the absolute and relative errors associated with the result.

(a) $(x_1 \oplus x_2) \oplus x_3$

$$\begin{aligned}
(x_1 + x_2) &= 0.1234 \times 10^3 + 0.3456 \times 10^2 \\
&= 0.1234 \times 10^3 + 0.03456 \times 10^3 \\
&= 0.15796 \times 10^3 \\
\implies (x_1 \oplus x_2) &= 0.1580 \times 10^3 \\
\implies (x_1 \oplus x_2) + x_3 &= 0.1580 \times 10^3 + 0.5678 \times 10^1 \\
&= 0.1580 \times 10^3 + 0.005678 \times 10^3 \\
&= 0.163678 \times 10^3 \\
\implies (x_1 \oplus x_2) \oplus x_3 &= \boxed{0.1637 \times 10^3} \\
(x_1 + x_2) + x_3 &= 0.15796 \times 10^3 + 0.5678 \times 10^1 \\
&= 0.15796 \times 10^3 + 0.005678 \times 10^3 \\
&= 0.163638 \times 10^3 \\
\implies E &= (0.163638 \times 10^3) - (0.1637 \times 10^3) = -0.000062 \times 10^3 \\
&= \boxed{-0.62 \times 10^{-1}} \\
\implies R &= \frac{-0.62 \times 10^{-1}}{0.163638 \times 10^3} \approx -0.0003788851 \\
&\le \boxed{-0.389 \times 10^{-3}}
\end{aligned}$$

(b) $(x_3 \oplus x_2) \oplus x_1$

$$\begin{aligned}
(x_3 + x_2) &= 0.5678 \times 10^1 + 0.3456 \times 10^2 \\
&= 0.05678 \times 10^2 + 0.3456 \times 10^2 \\
&= 0.40238 \times 10^2 \\
\implies (x_3 \oplus x_2) &= 0.4024 \times 10^2 \\
\implies (x_3 \oplus x_2) + x_1 &= 0.4024 \times 10^2 + 0.1234 \times 10^3
\end{aligned}$$

$$
\begin{aligned}
&= 0.04024 \times 10^3 + 0.1234 \times 10^3 \\
&= 0.16364 \times 10^3 \\
\implies (x_3 \oplus x_2) \oplus x_1 &= \boxed{0.1636 \times 10^3} \\
(x_3 + x_2) + x_1 &= (x_1 + x_2) + x_3 = 0.163638 \times 10^3 \\
\implies E &= (0.163638 \times 10^3) - (0.1636 \times 10^3) = 0.000038 \times 10^3 \\
&= \boxed{-0.38 \times 10^{-1}} \\
\implies R &= \frac{0.38 \times 10^{-1}}{0.163638 \times 10^3} \approx 0.0002322199 \\
&\leq \boxed{-0.233 \times 10^{-3}}
\end{aligned}
$$

(c) $(x_1 \odot x_2) \odot x_3$

$$
\begin{aligned}
(x_1 \cdot x_2) &= 0.1234 \times 10^3 + 0.3456 \times 10^2 \\
&= 0.4264704 \times 10^4 \\
\implies (x_1 \odot x_2) &= 0.4265 \times 10^4 \\
\implies (x_1 \odot x_2) \cdot x_3 &= 0.4265 \times 10^4 \cdot 0.5678 \times 10^1 \\
&= 0.24216670 \times 10^5 \\
\implies (x_1 \odot x_2) \odot x_3 &= \boxed{0.2422 \times 10^5} \\
(x_1 \cdot x_2) \cdot x_3 &= (0.4264704 \times 10^4) \cdot (0.5678 \times 10^1) = 0.24214989312 \times 10^5 \\
\implies E &= (0.24214989312 \times 10^5) - (0.2422 \times 10^5) = -0.00005010688 \times 10^5 \\
&= \boxed{-0.5010688 \times 10^1} \\
\implies R &= \frac{-0.5010688 \times 10^1}{0.24214989312 \times 10^5} \approx -0.000206925 \\
&\leq \boxed{-0.207 \times 10^{-3}}
\end{aligned}
$$

Additional Problem

Find the Taylor Series of the function $f(x, y)$ around the point (x_0, y_0) up to third order terms.

For $f \in C^\infty \times C^\infty$, the Taylor Series of f around (x_0, y_0) is

$$\begin{aligned}
f(x,y) = {} & \frac{1}{0!}\left[\binom{0}{0}(x-x_0)^0(y-y_0)^0 f(x_0,y_0)\right] \\
& + \frac{1}{1!}\left[\binom{1}{0}(x-x_0)^1(y-y_0)^0 f_x(x_0,y_0) + \binom{1}{1}(x-x_0)^0(y-y_0)^1 f_y(x_0,y_0)\right] \\
& + \frac{1}{2!}\Bigg[\binom{2}{0}(x-x_0)^2(y-y_0)^0 f_{xx}(x_0,y_0) + \binom{2}{1}(x-x_0)^1(y-y_0)^1 f_{xy}(x_0,y_0) \\
& \qquad + \binom{2}{2}(x-x_0)^0(y-y_0)^2 f_{yy}(x_0,y_0)\Bigg] \\
& + \frac{1}{3!}\Bigg[\binom{3}{0}(x-x_0)^3(y-y_0)^0 f_{xxx}(x_0,y_0) + \binom{3}{1}(x-x_0)^2(y-y_0)^1 f_{xxy}(x_0,y_0) \\
& \qquad + \binom{3}{2}(x-x_0)^1(y-y_0)^2 f_{xyy}(x_0,y_0) + \binom{3}{3}(x-x_0)^0(y-y_0)^3 f_{yyy}(x_0,y_0)\Bigg] \\
& + \ldots
\end{aligned}$$

Thus the general formula for the Taylor Series of $f \in C^\infty \times C^\infty$ is

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\sum_{k=0}^{n} \binom{k}{n}(x-x_0)^{n-k}(y-y_0)^k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}(x_0,y_0)\right]$$