Hw # 3 Solutions

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Chapter 10

Section 10.10

34.

Suppose that the equation $f(x) = x^2 + a_1x + a_2 = 0$ possesses real roots α and β. Show that the iteration $z_{k+1} = -\dfrac{a_1z_k + a_2}{z_k}$ is stable at $x = \alpha$ if $|\alpha| > |\beta|$, the iteration $z_{k+1} = -\dfrac{a_2}{z_k + a_1}$ is stable at $x = \alpha$ if $|\alpha| < |\beta|$, and the iteration $z_{k+1} = -\dfrac{z_k^2 + a_2}{a_1}$ is stable at $x = \alpha$ if $2|\alpha| < |\alpha + \beta|$.

Suppose α and β are the roots of $f(x) = 0$ such that $\alpha > \beta$. We know that $\alpha + \beta = -a_1$ and $\alpha\beta = a_2$. We can rearrange $f(x) = 0$ to be $x = F(x)$ in the following ways:

1) $x = F_1(x) = -\dfrac{a_1x + a_2}{x}$. The iterative method that arises from this is $z_{k+1} = -\dfrac{a_1z_k + a_2}{z_k}$. But $F_1'(x) = \dfrac{a_2}{x^2} = \dfrac{\alpha\beta}{x^2}$. If $|\alpha| > |\beta|$, then $\left|\dfrac{\beta}{\alpha}\right| < 1$. But $|F_1'(\alpha)| = \left|\dfrac{\beta}{\alpha}\right| < 1$, proving the iterative method is stable.

2) $x = F_2(x) = -\dfrac{a_2}{x + a_1}$. The iterative method that arises from this is $z_{k+1} = -\dfrac{a_2}{z_k + a_1}$. But $F_2'(x) = \dfrac{a_2}{(x + a_1)^2} = \dfrac{\alpha\beta}{(x - \alpha - \beta)^2}$. If $|\alpha| < |\beta|$, then $\left|\dfrac{\alpha}{\beta}\right| < 1$. But $|F_2'(\alpha)| = \left|\dfrac{\alpha}{\beta}\right| < 1$, proving the iterative method is stable.

3) $x = F_3(x) = -\dfrac{x^2 + a_2}{a_1}$. The iterative method that arises from this is $z_{k+1} = -\dfrac{z_k^2 + a_2}{a_1}$. But $F_3'(x) = -\dfrac{2x}{a_1} = \dfrac{2x}{\alpha + \beta}$. If $2|\alpha| < |\alpha + \beta|$, then $\left|\dfrac{2\alpha}{\alpha + \beta}\right| < 1$. But $|F_3'(\alpha)| = \left|\dfrac{2\alpha}{\alpha + \beta}\right| < 1$, proving the iterative method is stable.

37.

The real root α of the equation $x + \log x = 0$ lies between 0.56 and 0.57. Show that the iteration $z_{k+1} = -\log z_k$ is unstable at $x = \alpha$ and verify this fact by calculation. Then show that the iteration $z_{k+1} = \exp[-z_k]$ is stable at $x = \alpha$, and determine α to five places.

Let $F_1(x) = -\log x$, which produces the iterative method $z_{k+1} = -\log z_k$. Note $F_1'(x) = -\dfrac{1}{x}$ and $|F_1'(\alpha)| > \dfrac{1}{0.57} > 1$, proving the iterative method is unstable. However, set $F_2(x) = \exp[-x]$, which produces the iterative method $z_{k+1} = \exp[-z_k]$. Note $F_2'(x) = -\exp[-x]$ and $|F_2'(\alpha)| < \exp[-0.56] < 1$, proving the iterative method is stable. To compute the actual value of α, I set $z_0 = 0.56$, and used Python to find the following values:

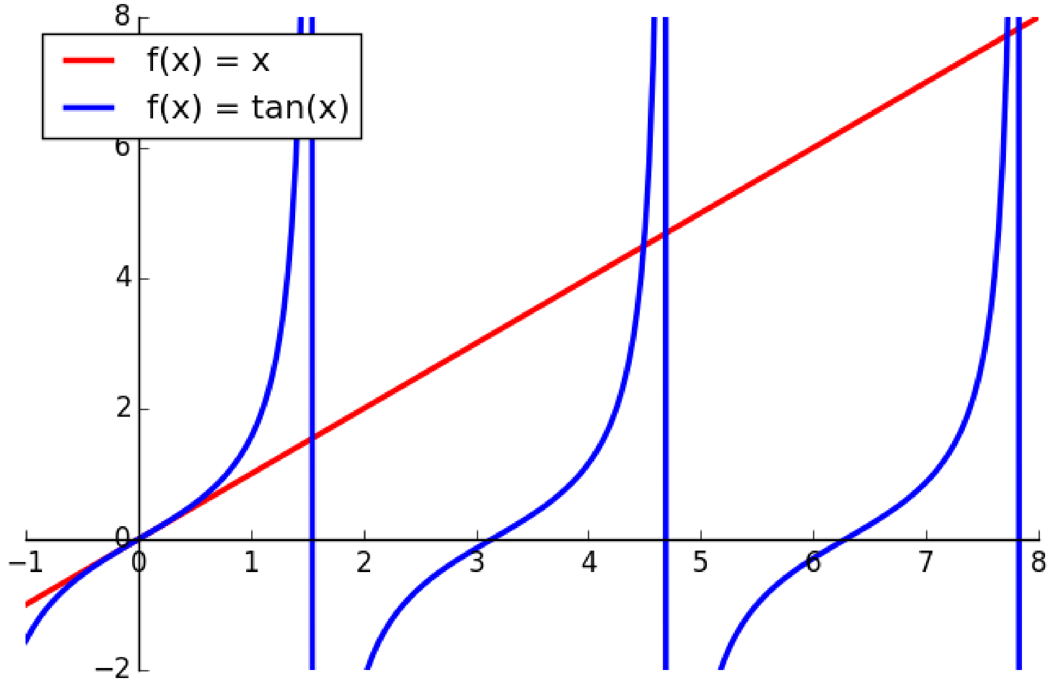
$$\begin{aligned}
z_0 &= 0.56 \\
z_1 = \exp\left[-z_0\right] &= 0.571209063849 \\
z_2 = \exp\left[-z_1\right] &= 0.564842095522 \\
z_3 = \exp\left[-z_2\right] &= 0.568449900456 \\
z_4 = \exp\left[-z_3\right] &= 0.5664027392 \\
z_5 = \exp\left[-z_4\right] &= 0.567563444613 \\
z_6 = \exp\left[-z_5\right] &= 0.566905052824 \\
z_7 = \exp\left[-z_6\right] &= 0.567278421354 \\
z_8 = \exp\left[-z_7\right] &= 0.567066656979 \\
z_9 = \exp\left[-z_8\right] &= 0.567186754211 \\
z_{10} = \exp\left[-z_9\right] &= 0.567118640742 \\
z_{11} = \exp\left[-z_{10}\right] &= 0.567157270476 \\
z_{12} = \exp\left[-z_{11}\right] &= 0.567135361765 \\
z_{13} = \exp\left[-z_{12}\right] &= 0.567147787106 \\
z_{14} = \exp\left[-z_{13}\right] &= 0.567140740145 \\
z_{15} = \exp\left[-z_{14}\right] &= 0.567144736777 \\
z_{16} = \exp\left[-z_{15}\right] &= 0.567142470113 \\
z_{17} = \exp\left[-z_{16}\right] &= 0.567143755636 \\
z_{18} = \exp\left[-z_{17}\right] &= 0.56714302656
\end{aligned}$$

Clearly this is converging in an oscillatory fashion to $\boxed{\alpha \approx 0.567143}$.

40.

Consider the iterative solution of the equation $\tan x = x$.

(a) *By superimposing the graphs of* $y = x$ *and* $y = \tan x$*, show that the* r^{th} *positive root of this equation is in the interval* $[r\pi, (r + \frac{1}{2})\pi]$.



Clearly, the first intersection of $\tan x$ and x occurs between π and $(1 + \frac{1}{2})\pi$, and the second occurs between 2π and $(2 + \frac{1}{2})\pi$. We can deduce that the r^{th} intersection occurs in the interval $[r\pi, (r + \frac{1}{2})\pi]$.

(b) *Show that the iteration $z_{k+1} = r\pi + \tan^{-1} z_k$ is stable for the determination of the r^{th} positive root.*

Setting $F(x) = r\pi + \tan^{-1} x$ yields $F'(x) = \dfrac{1}{1 + x^2}$. Since we are finding positive roots, we are only passing positive values to $F'(x)$, thus $F'(x) < 1 \ \forall x > 0$, proving the iteration is stable for the r^{th} positive root.

(c) *With $[a, b] = [r\pi, (r + \frac{1}{2})\pi]$ and $F(x) = r\pi + \tan^{-1} x$, show that when $a \leq x \leq b$ it is true that both $a < F(x) < b$ and $0 < F'(x) < 1$. Hence deduce in two ways that convergence to α_r is assured if $a \leq z_0 \leq b$.*

Note that $F(x) = r\pi + \tan^{-1} x$ is an increasing function that thus reaches its minimum on $[a, b]$ at a and its maximum at b. Note $F(a) = r\pi + \tan^{-1}(a) = r\pi = a$ and $F(b) = r\pi + \tan^{-1} b < r\pi + \frac{1}{2}\pi = (r + \frac{1}{2})\pi = b$. Thus $a < F(x) < b$. Also, $F'(x) = \dfrac{1}{1 + x^2} < 1$ for $x > 0$. Thus $0 < F'(x) < 1$. Thus we know that convergence to α_r is assured if $a \leq z_0 \leq b$ in two ways. On the one hand, we know the interval is contracting interval. We also know that $|F'(x)| < 1$ for all values close to α_r.

(d) *Use the iteration of part (b) to determine both α_1 and α_2 to five decimal places.*

I used Python to compute the following: set $z_0 = \pi$. Then,

$$\begin{aligned} z_0 &= 3.14159 \\ z_1 = F(z_0) &= 4.40421966514 \\ z_2 = F(z_1) &= 4.48911944313 \\ z_3 = F(z_2) &= 4.49320682586 \\ z_4 = F(z_3) &= 4.4933998952 \\ z_5 = F(z_4) &= 4.49340900664 \\ z_6 = F(z_5) &= 4.49340943661 \end{aligned}$$

Clearly, $\boxed{\alpha_1 \approx 4.49341}$. I used Python to compute the following: set $z_0 = 2\pi$. Then,

$$\begin{aligned} z_0 &= 6.28318 \\ z_1 = F(z_0) &= 7.69615031257 \\ z_2 = F(z_1) &= 7.7247704596 \\ z_3 = F(z_2) &= 7.72524390334 \\ z_4 = F(z_3) &= 7.72525170619 \\ z_5 = F(z_4) &= 7.72525183478 \\ z_6 = F(z_5) &= 7.7252518369 \end{aligned}$$

Clearly, $\boxed{\alpha_2 \approx 7.72525}$.

41.

Consider the polynomial $f(x) = x^5 + 5x - 1$.

(a) *Prove that $f(x)$ has exactly one real zero α and that $0.1 < \alpha < 0.2$.*

Note $f'(x) = 5x^4 + 5$, which is positive for all x, and thus if there is a zero, there can only be one. However, since f is an odd-degree polynomial, it has an odd number of zeros. Therefore, there is exactly one zero, α. Since $f(0.1) < 0$ and $f(0.2) > 0$, $\alpha \in (0.1, 0.2)$.

(b) *Without more closely locating α, prove that the iteration $z_{k+1} = z_k - cf(z_k)$ will converge to α if $0 < c < \frac{1}{5.008}$ and if $0.1 \leq z_0 \leq 0.2$.*

Suppose $F(x) = x - cf(x)$ and $z_0 \in [0.1, 0.2]$. Then $F'(x) = 1 - cf'(x) = 1 - c(5x^4 + 5) < 1 - c(5(0.1)^4 + 5) = -5.0005c + 1$. Thus $|F'(x)| < 1 \iff -1 < -5.0005c <$

$1 \iff 0 < c < \dfrac{2}{5.0005}$. But $c < \dfrac{1}{5.008}$ certainly implies $c < \dfrac{2}{5.0005}$, and thus the iteration converges to α.

(c) *With the choice $c = \dfrac{1}{5.01}$, show that the asymptotic convergence factor is between 4×10^{-4} and 2×10^{-3}, so that ultimately each iteration will provide three or four additional correct decimal places.*

For $c = \dfrac{1}{5.01}$, $F'(x) = 1 - \dfrac{1}{5.01}(5x^4 + 5)$.

$$F'(0.1) = 1 - \frac{1}{5.01}(5.0005) \approx 1 - 0.99810379 \approx 0.0018962 \approx 2 \times 10^{-3}$$
$$F'(0.2) = 1 - \frac{1}{5.01}(5.008) \approx 1 - 0.999600798 \approx 0.0003992 \approx 4 \times 10^{-4}$$

Since the asymptotic convergence factor ρ_k is approximately $F'(\alpha)$ and $\alpha \in (0.1, 0.2)$, we deduce that $4 \times 10^{-4} < \rho_k < 2 \times 10^{-3}$, and thus each iteration will proved three or four additional correct decimal places.

(d) *Verify that with $c = \dfrac{1}{5.01}$, two iterations provide 10-place accuracy when $z_0 = 0.2$, while three are needed when $z_0 = 0.1$.*

Using $z_{k+1} = z_k - cf(z_k)$ with $z_0 = 0.2$ yields the following:

$$\begin{aligned} z_0 &= 0.2 \\ z_1 = z_0 - \frac{1}{5.01} f(z_0) &\approx 0.19993612774451097 \\ z_2 = z_1 - \frac{1}{5.01} f(z_1) &\approx 0.199936102181 \\ z_3 = z_2 - \frac{1}{5.01} f(z_2) &\approx 0.199936102171 \\ z_4 = z_3 - \frac{1}{5.01} f(z_3) &\approx 0.199936102171 \end{aligned}$$

z_2 is accurate to 10 place values. Using $z_{k+1} = z_k - cf(z_k)$ with $z_0 = 0.1$ yields the following:

$$\begin{aligned} z_0 &= 0.1 \\ z_1 = z_0 - \frac{1}{5.01} f(z_0) &\approx 0.199798403194 \\ z_2 = z_1 - \frac{1}{5.01} f(z_1) &\approx 0.199936046618 \end{aligned}$$

$$z_3 = z_2 - \frac{1}{5.01} f(z_2) \approx 0.199936102149$$
$$z_4 = z_3 - \frac{1}{5.01} f(z_3) \approx 0.199936102171$$
$$z_5 = z_4 - \frac{1}{5.01} f(z_4) \approx 0.199936102171$$

z_3 is accurate to 10 place values. Note we do not round the tenth value based on the eleventh value. We truncate after 10 values.

Section 10.11

46.

Repeat the determination of problems 37 ~~and 38~~ using the Newton-Raphson iteration both with $f(x) = x + \log x$ and with $f(x) = x - e^{-x}$.

The Newton-Raphson iterative method for $f(x) = 0$ is $z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}$. For $z_0 = 0.56$, and $f(x) = x + \log x$ $\left(\text{and so } f'(x) = 1 + \frac{1}{x}\right)$, the following values are computed:

$$z_0 = 0.56$$
$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} = 0.567114331629$$
$$z_2 = z_1 - \frac{f(z_1)}{f'(z_1)} = 0.567143289938$$
$$z_3 = z_2 - \frac{f(z_2)}{f'(z_2)} = 0.56714329041$$
$$z_4 = z_3 - \frac{f(z_3)}{f'(z_3)} = 0.56714329041$$
$$z_5 = z_4 - \frac{f(z_4)}{f'(z_4)} = 0.56714329041$$

This clearly converges much faster than the iterative method used in problem 37. For $z_0 = 0.56$, and $f(x) = x - e^{-x}$ (and so $f'(x) = 1 + e^{-x}$), the following values are computed:

$$z_0 = 0.56$$
$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} = 0.567134037161$$
$$z_2 = z_1 - \frac{f(z_1)}{f'(z_1)} = 0.567143290394$$

$$z_3 = z_2 - \frac{f(z_2)}{f'(z_2)} = 0.56714329041$$
$$z_4 = z_3 - \frac{f(z_3)}{f'(z_3)} = 0.56714329041$$
$$z_5 = z_4 - \frac{f(z_4)}{f'(z_4)} = 0.56714329041$$

This also clearly converges much faster than the iterative method used in problem 37.

48.

By applythin the Newton-Raphson procedure to $f(x) = 1 - \dfrac{1}{ax}$, obtain the recurrence formula $z_{k+1} = z_k(2 - az_k)$ for the iterative determination of the reciprocal of a without effecting division, and show that if ϵ_k denotes the error in z_k, then there follows $\epsilon_{k+1} \approx a\epsilon_k^2$ when $z_k \approx \dfrac{1}{a}$. Also show that the iteration will converge to $\dfrac{1}{a}$ if $0 < z_0 < \dfrac{2}{a}$. Does it converge when $z_0 = 0$ or $z_0 = \dfrac{2}{a}$?

Suppose $f(x) = 1 - \dfrac{1}{ax}$. This clearly has one root, namely $\alpha = \dfrac{1}{a}$. Also, $f'(x) = \dfrac{1}{ax^2}$. Define a classical Newton-Raphson iterative method by $z_{k+1} = z_k - \dfrac{f(z_k)}{f'(z_k)}$. Then

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} = z_k - \frac{1 - \dfrac{1}{az_k}}{\dfrac{1}{az_k^2}} = z_k - (az_k^2 - z_k) = z_k(2 - az_k)$$

Let ϵ_k denote the error of z_k, that is, $|\alpha - z_k|$, or $\left|\dfrac{1}{a} - z_k\right|$. Then

$$\epsilon_{k+1} = |\alpha - z_{k+1}| = |\alpha - z_k(2 - az_k)| = |a(\alpha^2 - 2\alpha z_k + z_k^2)| = |a(\alpha - z_k)^2| = a\epsilon_k^2$$

Pick z_0 such that $0 < z_0 < \dfrac{2}{a}$. Then $|z_0 - \alpha| < \alpha$. Thus since $\epsilon_{k+1} = a\epsilon_k^2$,

$$\begin{aligned} & |\alpha - z_1| = a(\alpha - z_0)^2 < a\alpha^2 = \alpha \\ \implies & |\alpha - z_2| = a(\alpha - z_1)^2 < a\alpha^2 = \alpha \\ & \vdots \\ \implies & |\alpha - z_{k+1}| = a(\alpha - z_k)^2 < a\alpha^2 = \alpha \end{aligned}$$

Thus the iterative method $F(x) = x - \dfrac{f(x)}{f'(x)}$ is a contractive mapping on $(0, 2\alpha)$, and thus converges to α for any $z_0 \in (0, 2\alpha)$. Since f and f' are undefined at 0, we can say that this iteration does not converge for $z_0 = 0$. Note $F(2\alpha) = 2\alpha(2 - a(2\alpha)) = 2\alpha \cdot (0) = 0$. Then the iteration diverges, and so this iteration does not converge for $z_0 = 2\alpha$.

55.

Suppose that $f(x)$ possesses two zeros α_1 and α_2 which are nearly coincident, so that $f'(x)$ vanishes at a point β between α_1 and α_2. By making use of the relation

$$f(\alpha) = f(\beta) + (\alpha - \beta)f'(\beta) + \frac{(\alpha-\beta)^2}{2}f''(\beta) + \dots$$

show that if β is determined first, then initial approximations to the nearby zeros of $f(x)$ are given by

$$\alpha_{1,2} \approx \beta \pm \left[-\frac{2f(\beta)}{f''(\beta)}\right]^{\frac{1}{2}}$$

if $f''(\beta) \neq 0$, and are real if $f(\beta)$ and $f''(\beta)$ are of opposite sign, after which improved values may be obtained by an appropriate iterative method. [Note that the case of a double root α is also included since then $f(\beta) = 0$ and $\beta = \alpha$.] Also use this procedure to determine the two real roots of the equation

$$3x^4 + 8x^3 - 6x^2 - 25x + 19 = 0$$

(which are near $x = 1$) to five places.

Supposing $f(x)$ has two real zeros α_1 and α_2 which are nearly coincident so that $f'(\beta) = 0$ for some $\beta \in [\alpha_1, \alpha_2]$, then we can use the following relation to determine α_1 and α_2.

$$f(\alpha) \approx f(\beta) + (\alpha - \beta)f'(\beta) + \frac{(\alpha-\beta)^2}{2}f''(\beta)$$

Since $f'(\beta) = 0$ and $f(\alpha) = 0$,

$$\begin{aligned}
0 &\approx f(\beta) + \frac{(\alpha-\beta)^2}{2}f''(\beta) \\
&\approx \left(\frac{f''(\beta)}{2}\right)\alpha^2 + \left(-\beta f''(\beta)\right)\alpha + \left(f(\beta) + \frac{\beta^2 f''(\beta)}{2}\right) \\
\implies \alpha_{1,2} &\approx \frac{\beta f''(\beta) \pm \sqrt{\beta^2\left[f''(\beta)\right]^2 - 2f(\beta)f''(\beta) - \beta^2\left[f''(\beta)\right]^2}}{f''(\beta)} \\
&\approx \beta \pm \sqrt{-2\frac{f(\beta)}{f''(\beta)}}
\end{aligned}$$

This is only true if $f''(x) \neq 0$ and if $f(\beta)$ and $f''(\beta)$ are of opposite sign. Now consider

$$\begin{aligned}
f(x) &= 3x^4 + 8x^3 - 6x^2 - 25x + 19 = 0 \\
f'(x) &= 12x^3 + 24x^2 - 12x - 25
\end{aligned}$$

$$f''(x) = 36x^2 + 48x - 12$$

and note that $f'(x)$ has a zero near $x = 1$. Then let $\beta = 1$, and so $f(\beta) = -1$ and $f''(\beta) = 72 \neq 0$. We can use the previous result since $f(\beta)$ and $f''(\beta)$ are of opposite sign. So,

$$\begin{aligned}
\alpha_{1,2} &\approx 1 \pm \sqrt{-2 \cdot \frac{-1}{72}} \\
&\approx 1 \pm \frac{1}{6} \approx 0.8\overline{3} \text{ and } 1.1\overline{6}
\end{aligned}$$

Using the Newton-Raphson iterative method, $z_{k+1} = z_k - \dfrac{f(z_k)}{f'(z_k)}$, with $z_0 = 0.833$, we attain

$$\begin{aligned}
z_0 &= 0.833 \\
z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} &= 0.840030007503 \\
z_2 = z_1 - \frac{f(z_1)}{f'(z_1)} &= 0.840149213603 \\
z_3 = z_2 - \frac{f(z_2)}{f'(z_2)} &= 0.840149248228 \\
z_4 = z_3 - \frac{f(z_3)}{f'(z_3)} &= 0.840149248228 \\
z_5 = z_4 - \frac{f(z_4)}{f'(z_4)} &= 0.840149248228
\end{aligned}$$

Thus $\alpha_1 \approx 0.840149$. Using the Newton-Raphson iterative method, $z_{k+1} = z_k - \dfrac{f(z_k)}{f'(z_k)}$, with $z_0 = 1.167$, we attain

$$\begin{aligned}
z_0 &= 1.167 \\
z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} &= 1.17229382387 \\
z_2 = z_1 - \frac{f(z_1)}{f'(z_1)} &= 1.17219516283 \\
z_3 = z_2 - \frac{f(z_2)}{f'(z_2)} &= 1.17219512836 \\
z_4 = z_3 - \frac{f(z_3)}{f'(z_3)} &= 1.17219512836 \\
z_5 = z_4 - \frac{f(z_4)}{f'(z_4)} &= 1.17219512836
\end{aligned}$$

Thus $\alpha_2 \approx 1.172195$.

57.

Proceed as in problem 55 with the root pair of the equation

$$f(x) = x^6 - 16x^3 + x^2 + 59 = 0$$

which is near $x = 2$.

Note the following:

$$\begin{aligned} f'(x) &= 6x^5 - 48x^2 + 2x \\ f''(x) &= 30x^4 - 96x + 2 \end{aligned}$$

Note $f'(x)$ has a zero near $x = 2$, so let $\beta = 2$. Then $f(\beta) = -1 < 0$ and $f''(\beta) = 290 > 0$. We can use the previous result since $f(\beta)$ and $f''(\beta)$ are of opposite sign. So,

$$\begin{aligned} \alpha_{1,2} &\approx 2 \pm \sqrt{-2 \cdot \frac{-1}{290}} \\ &\approx 2 \pm \frac{1}{12} \approx 1.91\overline{6} \text{ and } 2.08\overline{3} \end{aligned}$$

Using the Newton-Raphson iterative method, $z_{k+1} = z_k - \dfrac{f(z_k)}{f'(z_k)}$, with $z_0 = 1.9167$, we attain

$$\begin{aligned} z_0 &= 1.9167 \\ z_1 &= z_0 - \frac{f(z_0)}{f'(z_0)} = 1.89314133146 \\ z_2 &= z_1 - \frac{f(z_1)}{f'(z_1)} = 1.895837569 \\ z_3 &= z_2 - \frac{f(z_2)}{f'(z_2)} = 1.89587198372 \\ z_4 &= z_3 - \frac{f(z_3)}{f'(z_3)} = 1.89587198937 \\ z_5 &= z_4 - \frac{f(z_4)}{f'(z_4)} = 1.89587198937 \\ z_6 &= z_5 - \frac{f(z_5)}{f'(z_5)} = 1.89587198937 \end{aligned}$$

Thus $\alpha_1 \approx 1.89587$. Using the Newton-Raphson iterative method, $z_{k+1} = z_k - \dfrac{f(z_k)}{f'(z_k)}$, with $z_0 = 2.083$, we attain

$$z_0 = 2.083$$

$$
\begin{aligned}
z_1 &= z_0 - \frac{f(z_0)}{f'(z_0)} = 2.06965633786 \\
z_2 &= z_1 - \frac{f(z_1)}{f'(z_1)} = 2.06843314443 \\
z_3 &= z_2 - \frac{f(z_2)}{f'(z_2)} = 2.06842295626 \\
z_4 &= z_3 - \frac{f(z_3)}{f'(z_3)} = 2.06842295555 \\
z_5 &= z_4 - \frac{f(z_4)}{f'(z_4)} = 2.06842295555 \\
z_6 &= z_5 - \frac{f(z_5)}{f'(z_5)} = 2.06842295555
\end{aligned}
$$

Thus $\alpha_2 \approx 2.068423$.