# Hw # 2 Solutions

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## Contents

# Chapter 2

## Section 2.2

**1.**

$$f(x) \approx \frac{1}{x_1 - x_0} \begin{vmatrix} f(x_0) & x_0 - x \\ f(x_1) & x_1 - x \end{vmatrix} \tag{1}$$

*Use (1) to calculate approximate values of $f(x)$ when $x = 1.1416$, $1.1600$, and $1.2000$ from the following rounded data:*

| $x$ | 1.1275 | 1.1503 | 1.1735 | 1.1972 |
|---|---|---|---|---|
| $f(x)$ | 0.11971 | 0.13957 | 0.15931 | 0.17902 |

For $\hat{x} = 1.1416$, choose $x_0 = 1.1275$ and $x_1 = 1.1503$ so $\hat{x} \in (x_0, x_1)$. Thus

$$\begin{aligned} f(\hat{x}) &\approx \frac{1}{x_1 - x_0} \begin{vmatrix} f(x_0) & x_0 - \hat{x} \\ f(x_1) & x_1 - \hat{x} \end{vmatrix} \\ &= \frac{1}{1.1503 - 1.1275} \begin{vmatrix} 0.11971 & 1.1275 - 1.1416 \\ 0.13957 & 1.1503 - 1.1416 \end{vmatrix} \\ &= \frac{1}{0.0228}[(0.11971)(0.0087) - (-0.0141)(0.13957)] \\ &\approx \boxed{0.1320} \end{aligned}$$

For $\hat{x} = 1.1600$, choose $x_0 = 1.1503$ and $x_1 = 1.1735$ so $\hat{x} \in (x_0, x_1)$. Thus

$$\begin{aligned} f(\hat{x}) &\approx \frac{1}{x_1 - x_0} \begin{vmatrix} f(x_0) & x_0 - \hat{x} \\ f(x_1) & x_1 - \hat{x} \end{vmatrix} \\ &= \frac{1}{1.1735 - 1.1503} \begin{vmatrix} 0.13957 & 1.1503 - 1.1600 \\ 0.15931 & 1.1735 - 1.1600 \end{vmatrix} \\ &= \frac{1}{0.0232}[(0.13957)(0.0135) - (-0.0097)(0.15931)] \\ &\approx \boxed{0.1478} \end{aligned}$$

For $\hat{x} = 1.2000$, choose $x_0 = 1.1735$ and $x_1 = 1.1972$ because those are the two closes data points to $\hat{x}$. However, since $\hat{x} \notin (x_0, x_1)$, we cannot guarantee an upper bound on the error.

$$f(\hat{x}) \approx \frac{1}{x_1 - x_0} \begin{vmatrix} f(x_0) & x_0 - \hat{x} \\ f(x_1) & x_1 - \hat{x} \end{vmatrix}$$

$$
\begin{aligned}
&= \frac{1}{1.1972 - 1.1735} \begin{vmatrix} 0.15931 & 1.1735 - 1.2000 \\ 0.17902 & 1.1972 - 1.2000 \end{vmatrix} \\
&= \frac{1}{0.0237}[(0.15931)(-0.0028) - (-0.0265)(0.17902)] \\
&\approx \boxed{0.1813}
\end{aligned}
$$

**2.**

$$
f(x) \approx f(x_0) + (x - x_0) f[x_0, x_1] \tag{2}
$$

*Calculate the three first divided differences relevant to successive pairs of data in Problem 1, and use (2) to determine approximate values of f(x) for*

$$
\begin{aligned}
x \in \mathcal{X} &= \{x \in [1.1600, 1.1700] \mid x = 1.1600 + 0.0020k,\ k \in \mathbb{Z}\} \\
&= \{1.1600, 1.1620, 1.1640, 1.1660, 1.1680, 1.1700\}
\end{aligned}
$$

The definition of *first divided difference* is

$$
f[x_0, x_1] = \frac{f(x_1) - (x_0)}{x_1 - x_0}
$$

The data points in Problem 1 are $\{(1.1275, 0.11971), (1.1503, 0.13957), (1.1735, 0.15931), (1.1972, 0.17902)\}$. So,

$$
\begin{aligned}
f_1 = f[1.1275, 1.1503] &= \frac{0.13957 - 0.11971}{1.1503 - 1.1275} \approx 0.8711 \\
f_2 = f[1.1503, 1.1735] &= \frac{0.15931 - 0.13957}{1.1735 - 1.1503} \approx 0.8509 \\
f_3 = f[1.1735, 1.1972] &= \frac{0.17902 - 0.15931}{1.1972 - 1.1735} \approx 0.8316
\end{aligned}
$$

Since $\mathcal{X} \subset (1.1503, 1.1735)$, we use $f_2$ to linearly interpolate the values of $x \in \mathcal{X}$. Per equation (2),

$$
\begin{aligned}
f(1.1600) &\approx 0.13957 + (1.1600 - 1.1503)(0.8509) \approx 0.1478 \\
f(1.1620) &\approx 0.13957 + (1.1620 - 1.1503)(0.8509) \approx 0.1495 \\
f(1.1640) &\approx 0.13957 + (1.1640 - 1.1503)(0.8509) \approx 0.1512 \\
f(1.1660) &\approx 0.13957 + (1.1660 - 1.1503)(0.8509) \approx 0.1529 \\
f(1.1680) &\approx 0.13957 + (1.1680 - 1.1503)(0.8509) \approx 0.1546 \\
f(1.1700) &\approx 0.13957 + (1.1700 - 1.1503)(0.8509) \approx 0.1563
\end{aligned}
$$

**3.**

*Prove that $f[x_0, x_1]$ is independent of $x_0$ and $x_1$ if and only if $f(x)$ is a linear function of $x$.*

Suppose $f(x)$ is a linear function. That is, $f(x) = mx + b$ for some constants $m$ and $b$. Then choose two arbitrary points, $x_0$ and $x_1$.

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= \frac{mx_1 + b - mx_0 - b}{x_1 - x_0} \\ &= \frac{m(x_1 - x_0)}{x_1 - x_0} \\ &= m \end{aligned}$$

Since $m$ is a given constant, $f[x_0, x_1]$ is independent of $x_0$ and $x_1$.

Now suppose $f[x_0, x_1]$ is independent of $x_0$ and $x_1$. That is, $f[x_0, x_1]$ is a constant $\forall x_0, x_1 \in \mathbb{R}$. Let $f[x_0, x_1] = m$ be that constant. Then

$$\begin{aligned} m &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ \implies f(x_1) - f(x_0) &= m(x_1 - x_0) \end{aligned}$$

This is the *point-slope form* of a line, and thus $f(x)$ is a linear function of $x$.

**4.**

*If $f(x) = u(x)v(x)$, show that*

$$f[x_0, x_1] = u[x_0]v[x_0, x_1] + u[x_0, x_1]v[x_1]$$

Using the definition of the first divided difference,

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= \frac{u(x_1)v(x_1) - u(x_0)v(x_0)}{x_1 - x_0} \\ &= \frac{u(x_1)v(x_1) - v(x_1)u(x_0) + v(x_1)u(x_0) - u(x_0)v(x_0)}{x_1 - x_0} \\ &= \frac{u(x_0)(v(x_1) - v(x_0)) + v(x_1)(u(x_1) - u(x_0))}{x_1 - x_0} \\ &= u(x_0)\frac{v(x_1) - v(x_0)}{x_1 - x_0} + v(x_1)\frac{u(x_1) - u(x_0)}{x_1 - x_0} \\ &= u[x_0]v[x_0, x_1] + u[x_0, x_1]v[x_1] \end{aligned}$$

**5.**

*If $f'(x)$ is continuous for $x_0 \leq x \leq x_1$, show that*

$$f[x_0, x_1] = f'(\xi)$$

*for some $\xi$ between $x_0$ and $x_1$, and hence also that*

$$f[x_0, x_0] = \lim_{x_1 \to x_0} f[x_0, x_1] = f'(x_0)$$

Let $f'(x)$ be continuous for $x \in [x_0, x_1]$. Then $f(x)$ exists and is continuous on that interval. Then by the *Mean Value Theorem for Derivatives*, $\exists \xi \in (x_0, x_1)$ such that $f(x_1) - f(x_0) = (x_1 - x_0)f'(\xi)$. In other words,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi)$$

Now consider the constant sequence $A = (a_i) = (x_0, x_0, \dots)$ and the sequence $B = (b_i) = (x_{11}, x_{12}, x_{13}, \dots)$ where $x_1 = x_{11}$ and $a_i < b_i \ \forall i \in \mathbb{N}$. Let $B$ converge to $x_0$, and note that $A$ also converges to $x_0$. Using the *Mean Value Theorem for Derivatives* for corresponding intervals $(\min\{a_i, b_i\}, \max\{a_i, b_i\})$ define $C = (\xi_i)$ where $\xi_i$ is the value for which $f(b_i) - f(a_i) = (b_i - a_i)f'(\xi_i)$. By the *Squeeze Theorem*, $C$ converges to $x_0$. Thus,

$$f[x_0, x_0] = \lim_{x_1 \to x_0} f[x_0, x_1] = \lim_{i \to \infty} f[a_i, b_i] = \lim_{i \to \infty} f'(\xi_i) = f'(x_0)$$

## Section 2.3

**7.**

$$\alpha_i^{(k)} = \frac{1}{(x_i - x_0)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_k)} \tag{3}$$

$$f[x_0, \dots, x_k] = \sum_{i=0}^{k} \alpha_i^{(k)} f(x_i) \tag{4}$$

*Suppose that $x_r = x_0 + rh$, $(r = 1, 2, \dots)$, so that the abcissas are at a uniform spacing $h$. Show that (3) then becomes*

$$\alpha_i^{(k)} = \frac{(-1)^{k-i}}{i!(k-i)!}\frac{1}{h^k} = \frac{(-1)^{(k-i)}}{h^k k!}\binom{k}{i}$$

*where $\binom{k}{i}$ is the binomial coefficient. Thus deduce that*

$$f[x_0, \dots, x_k] = \frac{1}{h^k k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(x_i) \tag{5}$$

*in this case.*

Supposing $x_r = x_0 + rh$ for $r = 1, 2, \ldots$, then specifically $x_i = x_0 + ih$, and by (3),

$$\begin{aligned}
\alpha_i^{(k)} &= \frac{1}{\Big([ih][(i-1)h][(i-2)h]\ldots[2h][h]\Big)\Big([-h][-2h]\ldots[-(k-i-1)h][-(k-i)h]\Big)} \\
&= \frac{1}{\Big(i!h^i\Big)\cdot\Big((-1)^{k-i}h^{k-i}(k-i)!\Big)} \\
&= \frac{(-1)^{k-i}}{i!(k-i)!h^k} \\
&= \frac{(-1)^{k-i}}{k!h^k}\cdot\frac{k!}{i!(k-i)!} \\
&= \frac{(-1)^{k-i}}{k!h^k}\binom{k}{i}
\end{aligned}$$

where $\binom{k}{i}$ is the binomial coefficient given by $\frac{k!}{i!(k-i)!}$. Then by (4),

$$\begin{aligned}
f[x_0, \ldots, x_k] &= \sum_{i=0}^{k}\frac{(-1)^{k-i}}{k!h^k}\binom{k}{i}f(x_i) \\
&= \frac{1}{h^k k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x_i)
\end{aligned}$$

**9.**

*If* $f'(x) = \frac{df(x)}{dx}$, *show that*

$$\frac{d}{dx}f[x_0, x] \not\equiv f'[x_0, x]$$

*unless* $f(x)$ *is linear.*

$$\begin{aligned}
f[x_0, x] &= \frac{f(x) - f(x_0)}{x - x_0} \\
\Longrightarrow \quad \frac{d}{dx}f[x_0, x] &= \frac{(x - x_0)(f'(x)) - (f(x) - f(x_0))}{(x - x_0)^2} \\
&= \frac{1}{x - x_0}\left(f'(x) - \frac{f(x) - f(x_0)}{x - x_0}\right)
\end{aligned}$$

$$f'[x_0, x] = \frac{f'(x) - f'(x_0)}{x - x_0}$$

Setting these two equal yields

$$\begin{aligned} f'(x) - \frac{f(x) - f(x_0)}{x - x_0} &= f'(x) - f'(x_0) \\ \implies f[x_0, x] &= f'(x_0) \end{aligned}$$

But $f'(x_0)$ is a constant, and thus $f[x_0, x]$ must be constant in order for the two quantities to be equal. However, $f[x_0, x]$ is only constant if $f(x)$ is a linear function of $x$ (i.e. $f(x) = mx + b$ for some $m, b \in \mathbb{R}$). Thus $\frac{d}{dx} f[x_0, x] \not\equiv f'[x_0, x]$ unless $f(x)$ is linear.

**11.**

*If $f(x) = \frac{ax+b}{cx+d}$, obtain expressions for $f[x,y]$, $f[x,x,y]$, and $f[x,x,y,y]$ in compact forms when $x \neq y$.*

First note that $f[x,x] \equiv f'(x) = \frac{ad - bc}{(cx+d)^2}$. Then,

$$\begin{aligned} f[x,y] &= \frac{\frac{ax+b}{cx+d} - \frac{ay+b}{cy+d}}{x-y} \cdot \frac{(cx+d)(cy+d)}{(cx+d)(cy+d)} \\ &= \frac{(ax+b)(cy+d) - (ay+b)(cx+d)}{(x-y)(cx+d)(cy+d)} \\ &= \frac{adx + bcy - ady - bcx}{(x-y)(cx+d)(cy+d)} \\ &= \boxed{\frac{ad-bc}{(cx+d)(cy+d)}} \\ \implies f[x,x,y] &= \frac{f[x,x] - f[x,y]}{x-y} \\ &= \frac{\frac{ad-bc}{(cx+d)^2} - \frac{ad-bc}{(cx+d)(cy+d)}}{x-y} \\ &= \frac{(ad-bc)(cy+d) - (ad-bc)(cx+d)}{(x-y)(cx+d)^2(cy+d)} \\ &= \boxed{\frac{c(bc-ad)}{(cx+d)^2(cy+d)}} \end{aligned}$$

Similarly, $f[x, y, y] = \dfrac{c(bc - ad)}{(cx + d)(cy + d)^2}$. Thus,

$$\begin{aligned}
f[x, x, y, y] &= \frac{f[x, x, y] - f[x, y, y]}{x - y} \\
&= \frac{\dfrac{c(bc - ad)}{(cx + d)^2(cy + d)} - \dfrac{c(bc - ad)}{(cx + d)(cy + d)^2}}{x - y} \\
&= \frac{c(bc - ad)(cy + d) - c(bc - ad)(cx + d)}{(x - y)(cx + d)^2(cy + d)^2} \\
&= \boxed{\frac{c^2(ad - bc)}{(cx + d)^2(cy + d)^2}}
\end{aligned}$$

## Section 2.5

### 23.

*If $f(x_1)$, $f(x_2)$, and $f(x_3)$ are values of $f(x)$ near a maximum or minimum point at $x = \overline{x}$, obtain the approximation*

$$\overline{x} \approx \frac{x_1 + x_2}{2} - \frac{f[x_1, x_2]}{2f[x_1, x_2, x_3]}$$

*and show that it can also be written in the more symmetrical form*

$$\overline{x} \approx \frac{x_1 + 2x_2 + x_3}{4} - \frac{f[x_1, x_2] + f[x_2, x_3]}{4f[x_1, x_2, x_3]}$$

*Show also that, when the abscissas are equally spaced, it becomes*

$$\overline{x} \approx x_2 - \frac{h}{2}\left(\frac{f_3 - f_1}{f_1 - 2f_2 + f_3}\right)$$

*where $h$ is the common interval. Note $f_i \coloneqq f(x_i)$.*

We can use the following polynomial approximation for $f(x)$:

$$f(x) \approx p_{1,2,3}(x) = f_1 + (x - x_1)f[x_1, x_2] + (x - x_1)(x - x_2)f[x_1, x_2, x_3] \tag{6}$$

$$\implies \frac{d}{dx}p_{1,2,3}(x) = p'_{1,2,3}(x) = f[x_1, x_2] + [(x - x_1) + (x - x_2)]f[x_1, x_2, x_3]$$

Since $\overline{x}$ is a minimum or maximum abcissa, then $f'(\overline{x}) = 0$, so

$$p'_{1,2,3}(\overline{x}) \approx 0$$

$$\implies f[x_1,x_2]+[(\overline{x}-x_1)+(\overline{x}-x_2)]f[x_1,x_2,x_3]\approx 0$$

Solving for $\overline{x}$ yields

$$\overline{x}\approx \boxed{\frac{x_1+x_2}{2}-\frac{f[x_1,x_2]}{2f[x_1,x_2,x_3]}} \tag{7}$$

The definition of $p_{1,2,3}$ in (6) is not unique, however, since the sum of $n$ divided differences up to the $n^{\text{th}}$ divided difference can be summed in $2^n$ ways. So define another polynomial approximation for $f(x)$:

$$f(x)\approx \overline{p_{1,2,3}}(x)=f_3+(x-x_3)f[x_2,x_3]+(x-x_2)(x-x_3)f[x_1,x_2,x_3]$$

$$\implies \frac{d}{dx}\overline{p_{1,2,3}}(x)=\overline{p'_{1,2,3}}(x)=f[x_2,x_3]+[(x-x_2)+(x-x_3)]f[x_1,x_2,x_3]$$

Again, since $\overline{x}$ is a minimum or maximum abscissa, then $f'(\overline{x})=0$. Thus

$$\begin{aligned}\overline{p'_{1,2,3}}(\overline{x})&\approx 0\\ \implies f[x_2,x_3]+[(\overline{x}-x_2)+(\overline{x}-x_3)]f[x_1,x_2,x_3]&\approx 0\end{aligned}$$

Solving for $\overline{x}$ yields

$$\overline{x}\approx \frac{x_2+x_3}{2}-\frac{f[x_2,x_3]}{2f[x_1,x_2,x_3]} \tag{8}$$

Since both (7) and (8) are valid approximations of $\overline{x}$, their arithmetic mean is also valid:

$$\begin{aligned}\overline{x}&\approx \frac{\frac{x_1+x_2}{2}-\frac{f[x_1,x_2]}{2f[x_1,x_2,x_3]}+\frac{x_2+x_3}{2}-\frac{f[x_2,x_3]}{2f[x_1,x_2,x_3]}}{2}\\ &\approx \boxed{\frac{x_1+2x_2+x_3}{4}-\frac{f[x_1,x_2]+f[x_2,x_3]}{4f[x_1,x_2,x_3]}}\end{aligned}$$

Now suppose the abscissas are equally spaced, and $h$ is the common difference $x_{i+1}-x_i$. Thus $x_1=x_2-h$ and $x_3=x_2+h$. Then

$$\frac{x_1+2x_2+x_3}{4}=\frac{x_2-h+2x_2+x_2+h}{4}=x_2$$

and

$$\begin{aligned}f[x_1,x_2]+f[x_2,x_3]&=\frac{f_2-f_1}{h}+\frac{f_3-f_2}{h}\\ &=\frac{f_3-f_1}{h}\end{aligned}$$

and

$$\begin{aligned}
4f[x_1,x_2,x_3] &= 4\cdot\frac{f[x_2,x_3]-f[x_1,x_2]}{2h} \\
&= \frac{2}{h}\left(\frac{f_3-f_2}{h}-\frac{f_2-f_1}{h}\right) \\
&= \frac{2}{h^2}\left(f_1-2f_2+f_3\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{x} &\approx \frac{x_1+2x_2+x_3}{4}-\frac{f[x_1,x_2]+f[x_2,x_3]}{4f[x_1,x_2,x_3]} \\
&= \boxed{x_2-\frac{2}{h}\left(\frac{f_3-f_1}{f_1-2f_2+f_3}\right)}
\end{aligned}$$

## Section 2.6

### 24.

*Show that the truncation error associated with linear interpolation of $f(x)$, using ordinates at $x_0$ and $x_1$ with $x_0 \le x \le x_1$, is not larger in magnitude than*

$$\frac{1}{8}M_2(x_1-x_0)^2$$

*where $M_2$ is the maximum value of $|f''(x)|$ on the interval $[x_0,x_1]$. Does this result hold also for extrapolation?*

For $n=1$, the absolute value of the error is bounded by

$$|E(x)| \le \frac{M_2}{(1+1)!}|\pi(x)|$$

where $M_2$ is the maximum value of $|f''(x)|$ on $[x_0,x_1]$. But $|\pi(x)| = |x-x_0|\cdot|x-x_1|$, so

$$|E(x)| \le \frac{M_2}{2}|x-x_0|\cdot|x-x_1|$$

But since the sum of $x_1-x$ and $x-x_0$ are constant, their product is maximized when they coincide, which is where $x$ is at the midpoint of $x_0$ and $x_1$, and thus the difference is half the distance from $x_0$ to $x_1$, i.e. $\dfrac{x_1-x_0}{2}$. Thus

$$|x-x_0|\cdot|x-x_1| \le \left(\frac{x_1-x_0}{2}\right)^2$$

$$\begin{aligned}\implies |E(x)| &\le \frac{M_2}{2}\left(\frac{x_1-x_0}{2}\right)^2 \\ &= \frac{1}{8}M_2(x_1-x_0)^2\end{aligned}$$

The argument presented above does *not* hold for extrapolation because if $x$ is not between $x_0$ and $x_1$ then $|x-x_0|\cdot|x-x_1|$ is not bounded, and is certainly not bounded by $\left(\frac{x_1-x_0}{2}\right)^2$. As a counterexample, consider $x_0=4$, $x_1=10$, and $x=12$. Then $\left(\frac{x_1-x_0}{2}\right)^2=\frac{36}{4}=9$, but $|12-4|\cdot|12-10|=8\cdot 2=16\not\le 9$.

**29.**

*Obtain the formula*

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+(x-x_0)^2f[x_0,x_0,x_1]+(x-x_0)^2(x-x_1)f[x_0,x_0,x_1,x_1]+E(x)$$

*where*

$$E(x)=\frac{1}{24}(x-x_0)^2(x-x_1)^2f^{\text{iv}}(\xi) \qquad (x_0<x,\xi<x_1)$$

*and show that*

$$|E(x)|\le\frac{h^4}{384}\max_{x_0\le x\le x_1}|f^{\text{iv}}(x)|$$

Newton's Fundamental Formula gives

$$\begin{aligned}f(x)=f[x_0]&+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]\\ &+\cdots+(x-x_0)\dots(x-x_{n-1})f[x_0,\dots,x_n]+E(x)\end{aligned}$$

where

$$E(x)=(x-x_0)\dots(x-x_n)f[x_0,\dots,x_n,x]$$

Suppose $x_0$ and $x_1$ coincide, and $x_2$ and $x_3$ coincide. For ease, let $x_0$ and $x_1$ be those values, respectively. Then

$$\begin{aligned}f(x)=f[x_0]&+(x-x_0)f[x_0,x_0]+(x-x_0)(x-x_0)f[x_0,x_0,x_1]\\ &+(x-x_0)(x-x_0)(x-x_1)f[x_0,x_0,x_1,x_1]+E(x)\end{aligned}$$

where

$$E(x)=(x-x_0)(x-x_0)(x-x_1)(x-x_1)f[x_0,x_0,x_1,x_1,x]$$

Thus,

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2 f[x_0, x_0, x_1] + (x - x_0)^2(x - x_1)f[x_0, x_0, x_1, x_1] + E(x)$$

where

$$E(x) = (x - x_0)^2(x - x_1)^2 f[x_0, x_0, x_1, x_1, x]$$

By *Rolle's Theorem*, $\exists \xi \in [\min(x_0, x_1), \max(x_0, x_1)]$ such that

$$\begin{aligned} f[x_0, x_0, x_1, x_1, x] &= \frac{1}{4!} f^{\text{iv}}(\xi) = \frac{1}{24} f^{\text{iv}}(\xi) \\ \implies E(x) &= \frac{1}{24}(x - x_0)^2(x - x_1)^2 f^{\text{iv}}(\xi) \end{aligned}$$

By the same argument made in Problem 24, $(x - x_0)^2(x - x_1)^2 \leq \left(\frac{x_1 - x_0}{2}\right)^4$. Thus

$$\begin{aligned} E(x) &\leq \frac{1}{24} \cdot \frac{1}{16}(x_1 - x_0)^4 f^{\text{iv}}(\xi) \\ &\leq \frac{1}{384}(x_1 - x_0)^4 \max_{x_0 \leq x \leq x_1} |f^{\text{iv}}(x)| \\ &= \frac{h^4}{384} \max_{x_0 \leq x \leq x_1} |f^{\text{iv}}(x)| \end{aligned}$$

where $h$ is the constant distance between $x_0$ and $x_1$, namely $|x_1 - x_0|$.

**30.**

$$E(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\pi(x) \tag{9}$$

$$E(x) = \pi(x) f[x_0, \ldots, x_n, x] \tag{10}$$

*If* $f(x) = \dfrac{1}{x+1}$ *and* $y(x)$ *is the polynomial approximation of degree* $n$ *which agrees with* $f(x)$ *when* $x = 0, 1, 2, \ldots, n$*, show that the use of (9) leads to the error bound*

$$|E(x)| < |x(x-1)\ldots(x-n)|$$

*whereas (10) permits the less conservative bound*

$$|E(x)| < \frac{1}{(n+1)!}|x(x-1)\ldots(x-n)|$$

*when* $x \geq 0$

First, note the following:

$$\begin{aligned} f^{(n+1)}(x) &= \frac{(-1)^{n+1}(n+1)!}{(x+1)^{n+2}} \\ \implies f^{(n+1)}(\xi) &= \frac{(-1)^{n+1}(n+1)!}{(\xi+1)^{n+2}} \end{aligned}$$

Thus, by (9),

$$\begin{aligned} |E(x)| &= \left| \frac{1}{(n+1)!} \cdot \frac{(-1)^{n+1}(n+1)!}{(\xi+1)^{n+2}} \cdot \pi(x) \right| \\ &= \left| \frac{1}{(\xi+1)^{n+2}} \cdot (x-x_0)(x-x_1)\dots(x-x_n) \right| \end{aligned}$$

$x_i = i$ for $i = 0, 1, 2, \dots, n \implies \xi \in (0, n) \implies \xi + 1 \geq 1 \implies \left| \frac{1}{(\xi+1)^{n+2}} \right| < 1$. Thus,

$$|E(x)| < |x(x-1)\dots(x-n)|$$

An inductive attempt is made to prove the 'less conservative' bound

$$|E(x)| < \frac{1}{(n+1)!} |x(x-1)\dots(x-n)|$$

For $n = 0$, $|f[x_0, x]| = \left| \frac{f(x) - f(x_0)}{x - x_0} \right| = \left| \frac{1 - (x+1)}{x(x+1)} \right| = \frac{1}{x+1}$. If $x = 0$, the error is guaranteed to be 0, which is certainly less than $\frac{1}{(0+1)!} = 1$. If $x > 0$, then $\frac{1}{x+1} < 1 = \frac{1}{(0+1)!}$. Then by (10),

$$|E(x)| < \frac{1}{(0+1)!}|x| = |x|$$

Now assume $\forall k \in \{0, \dots, n-1\}$, $|f[x_0, \dots, x_k, x]| < \frac{1}{(k+1)!}$, which would, by (10), show that $|E(x)| < \frac{1}{(k+1)!}|x(x-1)\dots(x-k)|$. Also note that because our data points are equally spaced, (5) holds (with $h = 1$), and thus $f[x_0, \dots, x_n] = \frac{1}{n!} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x_i)$. Thus,

$$f[x_0, \dots, x_n, x] = \frac{f[x_0, \dots, x_{n-1}, x] - f[x_0, \dots, x_n]}{x - x_n}$$

$$
\begin{aligned}
&< \frac{\frac{1}{n!} - \frac{1}{n!}\sum_{i=0}^{n}\left[(-1)^{n-i}\binom{n}{i}f(i)\right]}{x-n} \\
&= \frac{\frac{1}{n!} - \frac{1}{n!}\sum_{i=0}^{n}\left[(-1)^{n-i}\frac{n!}{i!(n-i)!}\frac{1}{i+1}\right]}{x-n} \\
&= \frac{\frac{1}{n!} - \sum_{i=0}^{n}\left[(-1)^{n-i}\frac{1}{(i+1)!(n-i)!}\right]}{x-n} \\
&= \frac{\frac{1}{n!} - \frac{1}{(n+1)!}\sum_{i=0}^{n}\left[(-1)^{n-i}\binom{n+1}{i+1}\right]}{x-n}
\end{aligned}
$$

At this time, I am unable to adequately bound this error function.

# Chapter 3

## Section 3.2

### 1.

*By noticing that the zeroth Lagrangian coefficient function of degree $n$ takes on the value unity when $x = x_0$ and the value zero when $x = x_1, \ldots, x_n$, and by considering the associated divided difference table (or otherwise), show that*

$$
l_0(x) = 1 + \frac{x-x_0}{x_0-x_1} + \frac{(x-x_0)(x-x_1)}{(x_0-x_1)(x_0-x_2)} + \cdots + \frac{(x-x_0)\ldots(x-x_{n-1})}{(x_0-x_1)\ldots(x_0-x_n)} \tag{11}
$$

*and that similar expressions can be written down by symmetry for the other coefficient functions.*

I will show this by induction. First, note the base case. For $n = 1$,

$$
\begin{aligned}
1 + \frac{x-x_0}{x_0-x_1} &= \frac{x_0-x_1+x-x_0}{x_0-x_1} \\
&= \frac{x-x_1}{x_0-x_1}
\end{aligned}
$$

Now suppose the equation holds for and for $n = k$. For $n = k+1$,

$$
1 + \frac{x-x_0}{x_0-x_1} + \cdots + \frac{(x-x_0)\ldots(x-x_{n-1})}{(x_0-x_1)\ldots(x_0-x_n)} + \frac{(x-x_0)\ldots(x-x_n)}{(x_0-x_1)\ldots(x_0-x_{n+1})}
$$

$$
\begin{aligned}
&\qquad\qquad \textit{(by the induction assumption)}\\
&= \frac{(x-x_1)\dots(x-x_n)}{(x_0-x_1)\dots(x_0-x_n)} + \frac{(x-x_0)\dots(x-x_{n-1})(x-x_n)}{(x_0-x_1)\dots(x_0-x_n)(x_0-x_{n+1})}\\
&= \frac{[(x-x_1)\dots(x-x_{n-1})]\cdot[x_0-x_{n+1}+x-x_0]}{(x_0-x_1)\dots(x_0-x_n)(x_0-x_{n+1})}\\
&= \frac{(x-x_1)\dots(x-x_n)}{(x_0-x_1)\dots(x_0-x_{n+1})}
\end{aligned}
$$

Thus (11) holds for all natural numbers. The corresponding formulas for $i = 1, \dots, n$ are:

$$
\begin{aligned}
l_1(x) &= 1 + \frac{x-x_1}{x_1-x_0} + \frac{(x-x_0)(x-x_1)}{(x_1-x_0)(x_1-x_2)} + \dots + \frac{(x-x_0)\dots(x-x_{n-1})}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}\\
l_2(x) &= 1 + \frac{x-x_2}{x_2-x_0} + \frac{(x-x_0)(x-x_2)}{(x_2-x_0)(x_2-x_1)} + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\\
&\quad + \dots + \frac{(x-x_0)\dots(x-x_{n-1})}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)}\\
&\ \vdots\\
l_i(x) &= 1 + \frac{x-x_i}{x_i-x_0} + \frac{(x-x_i)(x-x_0)}{(x_i-x_0)(x_i-x_1)} + \frac{(x-x_i)(x-x_0)(x-x_1)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)}\\
&\quad + \dots + \frac{(x-x_0)\dots(x-x_i)}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})}\\
&\quad + \dots + \frac{(x-x_0)\dots(x-x_{n-1})}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}\\
&\ \vdots\\
l_n(x) &= 1 + \frac{x-x_n}{x_n-x_0} + \frac{(x-x_n)(x-x_0)}{(x_n-x_0)(x_n-x_1)} + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-2})(x-x_n)}{(x_n-x_0)\dots(x_n-x_{n-1})}
\end{aligned}
$$

**4.**

$$
\begin{vmatrix}
y & 1 & x & x^2 & \dots & x^n\\
f(x_0) & 1 & x_0 & x_0^2 & \dots & x_0^n\\
\vdots & \vdots & \vdots & \vdots & & \vdots\\
f(x_n) & 1 & x_n & x_n^2 & \dots & x_n^n
\end{vmatrix} = 0 \tag{12}
$$

*Show that*

$$\begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)$$

*and use this fact to express the result of expanding the left-hand member of (12) with respect to the elements of the first column, and equating the result to zero, in Lagrangian form when* $n = 2$.

In the second line of the following calculation, the term $a_1a_2a_3$ is added and subtracted so that the correct factoring can occur.

$$\begin{aligned}
\begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} &= a_2a_3^2 - a_2^2a_3 - a_1a_3^2 + a_1^2a_3 + a_1a_2^2 - a_1^2a_2 \\
&= a_2a_3^2 - a_2^2a_3 - a_1a_2a_3 + a_1a_2^2 - a_1a_3^2 + a_1a_2a_3 + a_1^2a_3 - a_1^2a_2 \\
&= (a_2a_3 - a_1a_2 - a_1a_3 + a_1^2)(a_3 - a_2) \\
&= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)
\end{aligned}$$

Consider (12) for $n = 2$:

$$\begin{vmatrix} y & 1 & x & x^2 \\ f(x_0) & 1 & x_0 & x_0^2 \\ f(x_1) & 1 & x_1 & x_1^2 \\ f(x_2) & 1 & x_2 & x_2^2 \end{vmatrix} = 0$$

$$y \cdot \begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} - f(x_0) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} + f(x_1) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} - f(x_2) \cdot \begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \end{vmatrix} = 0$$

Each of the four 3x3 matrices can be expanded in a similar fashion as above, so

$$\begin{gathered}
y[(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)] - f(x_0)[(x_1 - x)(x_2 - x)(x_2 - x_1)] \\
+f(x_1)[(x_0 - x)(x_2 - x)(x_2 - x_0)] - f(x_2)[(x_0 - x)(x_1 - x)(x_1 - x_0)] = 0
\end{gathered}$$

Solving for $y$ yields

$$\begin{aligned}
y &= f(x_0)\frac{(x_1 - x)(x_2 - x)}{(x_1 - x_0)(x_2 - x_0)} - f(x_1)\frac{(x_0 - x)(x_2 - x)}{(x_1 - x_0)(x_2 - x_1)} + f(x_2)\frac{(x_0 - x)(x_1 - x)}{(x_2 - x_0)(x_2 - x_1)} \\
&= f(x_0)\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f(x_1)\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f(x_2)\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \\
&= l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)
\end{aligned}$$

**6.**

*By considering the limit of the three-point Lagrangian interpolation formula relative to $x_0$, $x_0+\epsilon$, and $x_1$, as $\epsilon \to 0$, obtain the formula*

$$f(x) = \frac{(x_1-x)(x+x_1-2x_0)}{(x_1-x_0)^2} f(x_0) + \frac{(x-x_0)(x_1-x)}{x_1-x_0} f'(x_0) + \frac{(x-x_0)^2}{(x_1-x_0)^2} f(x_1) + E(x)$$

*where*

$$E(x) = \frac{1}{6}(x-x_0)^2(x-x_1)f'''(\xi)$$

The Lagrangian coefficient of $f(x_k)$ in $p_{0,1,\ldots,n}$ is

$$l_k(x) = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$

We will denote $x_0$, $x_1$, and $x_2$ as $x_0$, $x_0+\epsilon$, and $x_1$, respectively. Thus,

$$\begin{aligned}
l_0(x) &= \frac{(x-x_0-\epsilon)(x-x_1)}{(x_0-x_0-\epsilon)(x_0-x_1)} = \frac{-1}{\epsilon}\cdot\frac{(x-x_0-\epsilon)(x-x_1)}{x_0-x_1} \\
&= \frac{-1}{\epsilon}\cdot\left[\frac{(x-x_1)(x-x_0)}{x_0-x_1} - \frac{\epsilon(x-x_1)}{x_0-x_1}\right] \\
l_{0,\epsilon}(x) &= \frac{(x-x_0)(x-x_1)}{(x_0+\epsilon-x_0)(x_0+\epsilon-x_1)} = \frac{1}{\epsilon}\cdot\frac{(x-x_0)(x-x_1)}{x_0+\epsilon-x_1} \\
&= \frac{1}{\epsilon}\cdot\left[\frac{(x-x_0)(x-x_1)}{x_0+\epsilon-x_1} + \frac{(x-x_1)(x-x_0)}{x_0-x_1} - \frac{(x-x_1)(x-x_0)}{x_0-x_1}\right] \\
l_1(x) &= \frac{(x-x_0)(x-x_0-\epsilon)}{(x_1-x_0)(x_1-x_0-\epsilon)}
\end{aligned}$$

Using these coefficients, we can write

$$\begin{aligned}
p_{0,1,2}(x) &= l_0(x)f(x_0) + l_{0,\epsilon}(x)f(x_0+\epsilon) + l_1(x)f(x_1) \\
&= \frac{-f(x_0)}{\epsilon}\left[\frac{(x-x_1)(x-x_0)}{x_0-x_1} - \frac{\epsilon(x-x_1)}{x_0-x_1}\right] \\
&+ f(x_0+\epsilon)\left[\frac{(x-x_0)(x-x_1)}{\epsilon(x_0+\epsilon-x_1)} + \frac{(x-x_1)(x-x_0)}{\epsilon(x_0-x_1)} - \frac{(x-x_1)(x-x_0)}{\epsilon(x_0-x_1)}\right] \\
&+ f(x_1)\frac{(x-x_0)(x-x_0-\epsilon)}{(x_1-x_0)(x_1-x_0-\epsilon)} \\
&= \frac{(x-x_1)(x-x_0)}{x_0-x_1}\left[\frac{f(x_0+\epsilon)-f(x_0)}{\epsilon}\right] + f(x_0)\frac{x-x_1}{x_0-x_1}
\end{aligned}$$

$$+ f(x_0+\epsilon)(x-x_0)(x-x_1)\left[\frac{g(x_0+\epsilon)-g(x_0)}{\epsilon}\right] + f(x_1)\frac{(x-x_0)(x-x_0-\epsilon)}{(x_1-x_0)(x_1-x_0-\epsilon)}$$

where

$$g(x) = \frac{1}{x-x_1}$$

Since $f'(x) \coloneqq \lim_{\epsilon\to 0}\dfrac{f(x+\epsilon)-f(x)}{\epsilon}$, $g'(x) \coloneqq \lim_{\epsilon\to 0}\dfrac{g(x+\epsilon)-g(x)}{\epsilon}$, and $g'(x) = \dfrac{-1}{(x-x_1)^2}$

$$\begin{aligned}
\lim_{\epsilon\to 0} p_{0,1,2} &= f'(x_0)\frac{(x-x_1)(x-x_0)}{x_0-x_1} + f(x_0)\left(\frac{x-x_1}{x_0-x_1} - \frac{(x-x_0)(x-x_1)}{(x_0-x_1)^2}\right) + f(x_1)\frac{(x-x_0)^2}{(x_1-x_0)^2} \\
&= \frac{(x_1-x)(x+x_1-2x_0)}{(x_1-x_0)^2}f(x_0) + \frac{(x-x_0)(x_1-x)}{x_1-x_0}f'(x_0) + \frac{(x-x_0)^2}{(x_1-x_0)^2}f(x_1) \\
f(x) &= p_{0,1,2} + E(x) \\
&= \frac{(x_1-x)(x+x_1-2x_0)}{(x_1-x_0)^2}f(x_0) + \frac{(x-x_0)(x_1-x)}{x_1-x_0}f'(x_0) + \frac{(x-x_0)^2}{(x_1-x_0)^2}f(x_1) + E(x)
\end{aligned}$$