Hw # 4 Solutions

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Chapter 3

Section 3.6

29.

A Riemann sum associated with an integral $\int_a^b f(x)dx$ *is an approximation of the form*

$$S_n = \sum_{k=0}^{n} f(t_k)(s_{k+1} - s_k)$$

where

$$a = s_0 \le t_0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_n \le t_n \le s_{n+1} = b$$

Any sequence of such sums in which the subdivision of $[a,b]$ *is refined in such a way that* $\max(s_{k+1} - s_k) \to 0$ *tends to the (Riemann) integral* I *if it exists.*

(a) *Show that the approximations afforded by the repeated midpoint rule, the trapezoidal rule, and the parabolic rule are Riemann sums. (Display the values of* $s_1, s_2, \ldots, s_n$ *in each case.)*

MIDPOINT RULE

The repeated midpoint rule is the following approximation:

$$\int_a^b f(x)dx \approx h\Big(f_{\frac{1}{2}} + f_{\frac{3}{2}} + \cdots + f_{n-\frac{1}{2}}\Big)$$

where $f_{k+\frac{1}{2}} = f(a + (k + \frac{1}{2})h)$ and $b = a + nh$. Rearranging the terms gives

$$\begin{aligned} h\Big(f_{\frac{1}{2}} + f_{\frac{3}{2}} + \cdots + f_{n-\frac{1}{2}}\Big) &= \sum_{k=0}^{n-1} f(a + (k + \tfrac{1}{2})h)(h) \\ &= \sum_{k=0}^{n-1} f(t_k)(s_{k+1} - s_k) \end{aligned}$$

where $f(t_k) = f_{k+\frac{1}{2}} = f(a + (k + \frac{1}{2})h)$ and $s_k = a + kh$. Note $\max(s_{k+1} - s_k) = \max(h) = h \to 0$ as $h \to 0$. Since

$$a = s_0 \le t_0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_{n-1} \le t_{n-1} \le s_n = b$$

and k ranges from 0 to $n-1$, the repeated midpoint rule is a Riemann sum.

TRAPEZOIDAL RULE

The trapezoidal rule is the following approximation:

$$\int_a^b f(x)dx \approx h\Big(\tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-2} + f_{n-1} + \tfrac{1}{2}f_n\Big)$$

where $f_k = f(a+kh)$ and $b = a + nh$. Rearranging the terms gives

$$\begin{aligned} h\Big(\tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-2} + f_{n-1} + \tfrac{1}{2}f_n\Big) &= \sum_{k=0}^{n-1} \tfrac{1}{2}(f_k + f_{k+1})(h) \\ &= \sum_{k=0}^{n-1} f(t_k)(s_{k+1} - s_k) \end{aligned}$$

where $t_k \in (s_k, s_{k+1})$ such that $f(t_k) = \frac{1}{2}(f_k + f_{k+1})$ and $s_k = a + kh$. Note there always exists such a t_k if f is continuous. Note $\max(s_{k+1} - s_k) = \max(h) = h \to 0$ as $h \to 0$. Since

$$a = s_0 \le t_0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_{n-1} \le t_{n-1} \le s_n = b$$

and k ranges from 0 to $n-1$, the repeated trapezoidal rule is a Riemann sum.

PARABOLIC RULE

Let n be an even integer. The parabolic rule is the following approximation:

$$\int_a^b f(x)dx \approx \frac{h}{3}\Big(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 4f_{n-3} + 2f_{n-2} + 4f_{n-1} + f_n\Big)$$

where $f_k = f(a+kh)$ and $b = a + nh$. Rearranging the terms gives

$$\begin{aligned} &\frac{h}{3}\Big(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 4f_{n-3} + 2f_{n-2} + 4f_{n-1} + f_n\Big) \\ &\qquad = \sum_{k=0}^{\frac{n}{2}-1} \tfrac{1}{6}(f_{2k} + 4f_{2k+1} + f_{2k+2})(2h) \\ &\qquad = \sum_{k=0}^{\frac{n}{2}-1} f(t_k)(s_{k+1} - s_k) \end{aligned}$$

where $t_k \in (s_k, s_{k+1})$ such that $f(t_k) = \frac{1}{6}(f_{2k} + 4f_{2k+1} + f_{2k+2})$ and $s_k = a + 2kh$. Note there always exists such a t_k if f is continuous. Note $\max(s_{k+1} - s_k) = \max(h) = h \to 0$ as $h \to 0$. Since

$$a = s_0 \le t_0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_{n/2-1} \le t_{n/2-1} \le s_{n/2} = b$$

and k ranges from 0 to $\frac{n}{2} - 1$, the repeated parabolic rule is a Riemann sum.

(b) *The relation*

$$\int_a^b f(x)dx \approx \frac{h}{4}\Big(5f_0 + f_1 + f_2 + 10f_3 + f_4 + f_5 + 10f_6 + \dots$$
$$\dots + 10f_{n-3} + f_{n-2} + f_{n-1} + 5f_n\Big)$$

with the notations of Sec. 3.6 is an equality when $f(x)$ is any linear function. Prove that the approximation is not *a Riemann sum.*

In order to be a Riemann Sum, $\displaystyle\int_a^b f(x)dx = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ must have the property that $\displaystyle\sum_{i=1}^{n}\Delta x_i = b-a$ where $b = x_{n+1}$ and $a = x_0$, and thus $b-a = (n+1)h$. However,

$$\frac{h}{4}\left(5f_0 + f_1 + f_2 + 10f_3 + \dots + 10f_{n-3} + f_{n-2} + f_{n-1} + 5f_n\right)$$
$$=\frac{5h}{4}f_0 + \frac{h}{4}f_1 + \frac{h}{4}f_2 + \frac{10h}{4}f_3 + \dots + \frac{10h}{4}f_{n-3} + \frac{h}{4}f_{n-2} + \frac{h}{4}f_{n-1} + \frac{5h}{4}f_n$$

Therefore,

$$\begin{aligned}\sum_{i=0}^{n}\Delta x_i &= \left[2\left(\frac{5}{4}\right) + \left(\frac{n}{3}-1\right)\frac{10}{4} + \left(2\frac{n}{3}\right)\frac{1}{4}\right]h\\ &= \left[\frac{10}{4} + \frac{10n}{12} - \frac{10}{4} + \frac{2n}{12}\right]h\\ &= nh\\ &\neq (n+1)h\end{aligned}$$

Thus $\dfrac{h}{4}\left(5f_0 + f_1 + f_2 + 10f_3 + \dots + 10f_{n-3} + f_{n-2} + f_{n-1} + 5f_n\right)$ is not a Riemann Sum.

30.

Convergence of composite rules *Suppose that $[a,b]$ is divided into r equal parts by $a = X_0 < X_1 < \dots < X_{r-1} < X_r = b$, and let $\frac{b-a}{r} = H$. If an m-point formula which yields exact results when integrating a constant is used to approximate the integral of $f(x)$ over each subinterval $[X_i, X_{i+1}]$, prove that the sum converges to the integral over $[a,b]$ as the spacing $H \to 0$. (If the result of applying the m-point formula to $[X_0, X_1]$ is of the form*

$$\int_{X_0}^{X_1} f(x)dx \approx H\sum_{k=0}^{m-1} w_k f(X_0 + c_k) \qquad (0 \le c_k \le H)$$

show that the total approximation is given by

$$\int_a^b f(x)dx \approx \sum_{k=0}^{m-1} w_k \left(H \sum_{i=0}^{r-1} f(X_i + c_k) \right)$$

and that the inner sum is a Riemann sum for $f(x)$ over $[a,b]$. Then let $r \to \infty$ and complete the proof. See also Davis and Rabinowitz [1967], Sec. 2.4)

Since each interval (X_i, X_{i+1}) is approximated by

$$\int_{X_i}^{X_{i+1}} f(x)dx \approx H \sum_{k=0}^{m-1} w_k f(X_i + c_k) \quad (0 \le c_k \le H)$$

then

$$\begin{aligned}
\int_a^b f(x)dx &= \int_{X_0}^{X_r} f(x)dx \\
&= \sum_{i=0}^{r-1} \int_{X_i}^{X_{x+1}} f(x)dx \\
&\approx \sum_{i=0}^{r-1} \left[H \sum_{k=0}^{m-1} w_k f(X_i + c_k) \right] \\
&= \sum_{i=0}^{r-1} \left[H \left(w_0 f(X_i + c_0) + \cdots + w_{m-1} f(X_i + c_{m-1}) \right) \right] \\
&= \left[H \left(w_0 f(X_0 + c_0) + \cdots + w_{m-1} f(X_0 + c_{m-1}) \right) \right] \\
&\quad + \left[H \left(w_0 f(X_1 + c_0) + \cdots + w_{m-1} f(X_1 + c_{m-1}) \right) \right] \\
&\quad + \left[H \left(w_0 f(X_2 + c_0) + \cdots + w_{m-1} f(X_2 + c_{m-1}) \right) \right] \\
&\quad \vdots \\
&\quad + \left[H \left(w_0 f(X_{r-1} + c_0) + \cdots + w_{m-1} f(X_{r-1} + c_{m-1}) \right) \right] \\
&= w_0 H (f(X_0 + c_0) + f(X_1 + c_0) + \cdots + f(X_{r-1} + c_0)) \\
&\quad + w_1 H (f(X_0 + c_1) + f(X_1 + c_1) + \cdots + f(X_{r-1} + c_1)) \\
&\quad + w_2 H (f(X_0 + c_2) + f(X_1 + c_2) + \cdots + f(X_{r-1} + c_2)) \\
&\quad \vdots \\
&\quad + w_{Hm-1} (f(X_0 + c_{m-1}) + f(X_1 + c_{m-1}) + \cdots + f(X_{r-1} + c_{m-1})) \\
&= \sum_{k=0}^{m-1} w_k \left(H \sum_{i=0}^{r-1} f(X_i + c_k) \right)
\end{aligned}$$

31.

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80} f^{\text{iv}}(\xi) \tag{1}$$

Show that the composite rule corresponding to the repeated use of Newton's three-eights *rule (1) is of the form*

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\big(f_0 + 3f_1 + 3f_2 + 2f_3 + \cdots + 2f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n\big) - \frac{nh^5}{80} f^{\text{iv}}(\xi)$$

where n is to be an integral multiple of 3. Also, by considering the case when n is a multiple of 6, so that both this fule and the parabolic rule can be used with the same spacing h, account for the fact that the parabolic rule is nearly always preffered.

Let 3 divde the integer n. By the linearality of integration,

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \cdots + \int_{x_{n-6}}^{x_{n-3}} f(x)dx + \int_{x_{n-3}}^{x_n} f(x)dx$$

By using Newton's three-eighths rule on each of the subintervals,

$$\begin{aligned}
\int_{x_0}^{x_n} f(x)dx &= \left[\frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80} f^{\text{iv}}(\xi_0)\right] + \left[\frac{3h}{8}(f_3 + 3f_4 + 3f_5 + f_6) - \frac{3h^5}{80} f^{\text{iv}}(\xi_1)\right] \\
&\qquad + \cdots + \left[\frac{3h}{8}(f_{n-6} + 3f_{n-5} + 3f_{n-4} + f_{n-3}) - \frac{3h^5}{80} f^{\text{iv}}(\xi_{\frac{n-6}{3}})\right] \\
&\qquad\qquad + \left[\frac{3h}{8}(f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n) - \frac{3h^5}{80} f^{\text{iv}}(\xi_{\frac{n-3}{2}})\right] \\
&\qquad\qquad\qquad \textit{(for some } \xi_k \in (x_k, x_{k+3})\textit{)} \\
&= \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + 2f_3 + \cdots + 2f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n\right) \\
&\qquad\qquad - \frac{3h^5}{80}\Big(f^{\text{iv}}(\xi_0) + \cdots + f^{\text{iv}}(\xi_{\frac{n-3}{3}})\Big) \\
&= \frac{3h}{8}\big(f_0 + 3f_1 + 3f_2 + 2f_3 + \cdots + 2f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n\big) - \frac{nh^5}{80} f^{\text{iv}}(\xi)
\end{aligned}$$

where $\xi \in (x_0, x_n)$ such that $f^{\text{iv}}(\xi) = f^{\text{iv}}(\xi_0) + \cdots + f^{\text{iv}}(\xi_{\frac{n-3}{3}})$. Note that if f is continuous, than such a ξ always exists.

Section 3.7

32.

Given the following rounded values of the function

$$f(x) = \sqrt{\frac{2}{\pi}} \exp\left[-\frac{x^2}{2}\right]$$

calculate approximate values of the integral

$$P(1) = \sqrt{\frac{2}{\pi}} \int_0^1 \exp\left[-\frac{t^2}{2}\right] dt \doteq 0.6826895$$

by use of the trapezoidal rule with $h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$, and compare the results with the rounded true value:

x	0.000	0.125	0.250	0.375	0.500
$f(x)$	0.7978846	0.7916754	0.7733362	0.7437102	0.7041307

x	0.625	0.750	0.875	1.000
$f(x)$	0.6563219	0.6022749	0.5441100	0.4839414

$h = 1$**:**

$$\begin{aligned} P(1) &\approx \frac{1}{2}(f_0 + f_1)(h) \\ &= \frac{1}{2}(0.7978846 + 0.4839414)(1) \\ &\approx 0.640913 \\ \implies |E| &= |0.6826895 - 0.640913| \\ &\approx 0.0417765 \end{aligned}$$

$h = \frac{1}{2}$**:**

$$\begin{aligned} P(1) &\approx \tfrac{1}{2}(f_0 + 2f_{\frac{1}{2}} + f_1)(h) \\ &= \tfrac{1}{2}(0.7978846 + 2(0.7041307) + 0.4839414)(\tfrac{1}{2}) \\ &\approx 0.67252185 \\ \implies |E| &= |0.6826895 - 0.67252185| \\ &\approx 0.01016765 \end{aligned}$$

$h = \frac{1}{4}$**:**

$$\begin{aligned} P(1) &\approx \tfrac{1}{2}(f_0 + 2f_{\frac{1}{4}} + 2f_{\frac{1}{2}} + 2f_{\frac{3}{4}} + f_1)(h) \\ &= \tfrac{1}{2}(0.7978846 + 2(0.7733362) + 2(0.7041307) + 2(0.6022749) + 0.4839414)(\tfrac{1}{4}) \\ &\approx 0.6801637 \\ \implies |E| &= |0.6826895 - 0.6801637| \\ &\approx 0.0025258 \end{aligned}$$

$h = \frac{1}{8}$**:**

$$
\begin{aligned}
P(1) &\approx \tfrac{1}{2}(f_0 + 2f_{\frac{1}{8}} + 2f_{\frac{1}{4}} + 2f_{\frac{3}{8}} + 2f_{\frac{1}{2}} + 2f_{\frac{5}{8}} + 2f_{\frac{3}{4}} + 2f_{\frac{7}{8}} + f_1)(h) \\
&= \tfrac{1}{2}\Big(0.7978846 + 2(0.7916754) + 2(0.7733362) + 2(0.7437102) + 2(0.7041307) \\
&\qquad + 2(0.6563219) + 2(0.6022749) + 2(0.5441100) + 0.4839414\Big)(\tfrac{1}{8}) \\
&\approx 0.6820590375 \\
\implies |E| &= |0.6826895 - 0.6820590375| \\
&\approx 0.0006304625
\end{aligned}
$$

h	$\lvert E\rvert$
1	0.0417765
$\frac{1}{2}$	0.01016765
$\frac{1}{4}$	0.0025258
$\frac{1}{8}$	0.0006304625

33.

Repeat the calculations of **32.** *using instead the repeated midpoint rule with* $h = 1, \frac{1}{2}, \frac{1}{4}$.

$h = 1$**:**

$$
\begin{aligned}
P(1) &\approx (f_{\frac{1}{2}})(h) \\
&= (0.7041307)(1) \\
&\approx 0.7041307 \\
\implies |E| &= |0.6826895 - 0.7041307| \\
&\approx 0.0214412
\end{aligned}
$$

$h = \frac{1}{2}$**:**

$$
\begin{aligned}
P(1) &\approx (f_{\frac{1}{4}} + f_{\frac{3}{4}})(h) \\
&= (0.7733362 + 0.6022749)(\tfrac{1}{2}) \\
&\approx 0.68780555 \\
\implies |E| &= |0.6826895 - 0.68780555| \\
&\approx 0.00511605
\end{aligned}
$$

$h = \frac{1}{4}$**:**

$$\begin{aligned}
P(1) &\approx (f_{\frac{1}{8}} + f_{\frac{3}{8}} + f_{\frac{5}{8}} + f_{\frac{7}{8}})(h) \\
&= (0.7916754 + 0.7437102 + 0.6563219 + 0.5441100)(\tfrac{1}{4}) \\
&\approx 0.683954375 \\
\implies |E| &= |0.6826895 - 0.683954375| \\
&\approx 0.001264875
\end{aligned}$$

h	$\lvert E\rvert$
1	0.0214412
$\frac{1}{2}$	0.00511605
$\frac{1}{4}$	0.001264875

40.

By a double application of Simpson's rule, derive the formula

$$\int_{x_0}^{x_2}\int_{y_0}^{y_2} f(x,y)dydx = \frac{hk}{9}\Big[f_{0,0} + f_{0,2} + f_{2,0} + f_{2,2} \\ + 4(f_{0,1} + f_{1,0} + f_{1,2} + f_{2,1}) + 16f_{1,1}\Big] + E$$

where $x_r \equiv x_0 + rh$, $y_s \equiv sk$, and $f_{r,s} \equiv f(x_r, y_s)$, and show that

$$E = -\frac{hk}{45}\left[h^4\frac{\partial^4 f(\xi_1,\eta_1)}{\partial x^4} + k^4\frac{\partial^4 f(\xi_2,\eta_2)}{\partial y^4}\right]$$

where ξ_1, ξ_2 lie in (x_0, x_2) and η_1, η_2 in (y_0, y_2). [More elaborate formulas for two-way integration over a rectangle ("cubature formulas") are obtainable by double application of other one-dimensional integration formulas.]

First, we approximate the inner integral using Simpson's rule:

$$\int_{y_0}^{y_2} f(x,y)dy \approx \frac{k}{3}\Big(f(x,y_0) + 4f(x,y_1) + f(x,y_2)\Big) - \frac{h^5}{90}\frac{\partial^4 f}{\partial y^4}(x,\eta)$$

for some $\eta \in (y_0, y_2)$. Thus,

$$\int_{x_0}^{x_2}\int_{y_0}^{y_2} f(x,y)dydx \\ \approx \int_{x_0}^{x_2}\left[\frac{k}{3}\Big(f(x,y_0) + 4f(x,y_1) + f(x,y_2)\Big) - \frac{k^5}{90}\frac{\partial^4 f}{\partial y^4}(x,\eta)\right]dx$$

$$\approx \underbrace{\frac{k}{3}\left[\int_{x_0}^{x_2} f(x,y_0)dx + 4\int_{x_0}^{x_2} f(x,y_1)dx + \int_{x_0}^{x_2} f(x,y_2)dx\right]}_{=A} - \frac{k^5}{90}\int_{x_0}^{x_2} \frac{\partial^4 f}{\partial y^4}(x,\eta)dx$$

$$\int_{x_0}^{x_2} f(x,y_0)dx \approx \frac{h}{3}\Big(f(x_0,y_0) + 4f(x_1,y_0) + f(x_2,y_0)\Big) - \frac{h^5}{90}\ \frac{\partial^4 f}{\partial x^4}(\overline{\xi_0},y_0)$$

for some $\overline{\xi_0} \in (x_0,x_2)$.

$$\int_{x_0}^{x_2} f(x,y_1)dx \approx \frac{h}{3}\Big(f(x_0,y_1) + 4f(x_1,y_1) + f(x_2,y_1)\Big) - \frac{h^5}{90}\ \frac{\partial^4 f}{\partial x^4}(\overline{\xi_1},y_1)$$

for some $\overline{\xi_1} \in (x_0,x_2)$.

$$\int_{x_0}^{x_2} f(x,y_2)dx \approx \frac{h}{3}\Big(f(x_0,y_2) + 4f(x_1,y_2) + f(x_2,y_2)\Big) - \frac{h^5}{90}\ \frac{\partial^4 f}{\partial x^4}(\overline{\xi_2},y_2)$$

for some $\overline{\xi_2} \in (x_0,x_2)$. Thus,

$$A = \frac{hk}{9}\Big[(f_{0,0}+f_{0,2}+f_{2,0}+f_{2,2}) + 4(f_{1,0}+f_{0,1}+f_{2,1}+f_{1,2}) + 16f_{1,1}\Big]$$
$$-\frac{kh^5}{270}\left(\frac{\partial^4 f}{\partial x^4}(\overline{\xi_0},y_0) + 4\frac{\partial^4 f}{\partial x^4}(\overline{\xi_1},y_1) + \frac{\partial^4 f}{\partial x^4}(\overline{\xi_2},y_2)\right)$$

By theorem 2 on page 32, $\exists \xi_1 \in (x_0,x_2)$, $\eta_1 \in (y_0,y_2)$ such that

$$\left(\frac{\partial^4 f}{\partial x^4}(\overline{\xi_0},y_0) + 4\frac{\partial^4 f}{\partial x^4}(\overline{\xi_1},y_1) + \frac{\partial^4 f}{\partial x^4}(\overline{\xi_2},y_2)\right) = 6\frac{\partial^4 f}{\partial x^4}(\xi_1,\eta_1)$$

Thus,

$$A = \frac{hk}{9}\Big[(f_{0,0}+f_{0,2}+f_{2,0}+f_{2,2}) + 4(f_{1,0}+f_{0,1}+f_{2,1}+f_{1,2}) + 16f_{1,1}\Big] - \frac{kh^5}{45}\frac{\partial^4 f}{\partial x^4}(\xi_1,\eta_1)$$

Also, by the First Law of the Mean, $\exists \xi_2 \in (x_0,x_2)$, $\eta_2 \in (y_0,y_2)$ such that

$$\int_{x_0}^{x_2} \frac{\partial^4 f}{\partial y^4}(x,\eta)dx = 2k\frac{\partial^4 f}{\partial y^4}(\xi_2,\eta_2)$$

Thus,

$$\begin{aligned}
&\int_{x_0}^{x_2}\int_{y_0}^{y_2} f(x,y)dydx \\
&\quad = A - \frac{k^5}{90}\left[2k\frac{\partial^4 f}{\partial y^4}(\xi_2,\eta_2)\right] \\
&\quad = \frac{hk}{9}\Big[f_{0,0}+f_{0,2}+f_{2,0}+f_{2,2} + 4(f_{0,1}+f_{1,0}+f_{1,2}+f_{2,1}) + 16f_{1,1}\Big] \\
&\qquad - \frac{hk}{45}\left[h^4\frac{\partial^4 f(\xi_1,\eta_1)}{\partial x^4} + k^4\frac{\partial^4 f(\xi_2,\eta_2)}{\partial y^4}\right]
\end{aligned}$$